TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 364, Number 5, May 2012, Pages 2443–2460
S 0002-9947(2012)05462-6
Article electronically published on January 20, 2012

# ON NON-LOCAL REFLECTION FOR ELLIPTIC EQUATIONS OF THE SECOND ORDER IN $\mathbb{R}^2$ (THE DIRICHLET CONDITION)

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ABSTRACT. Point-to-point reflection holding for harmonic functions subject to the Dirichlet or Neumann conditions on an analytic curve in the plane almost always fails for solutions to more general elliptic equations. We develop a non-local, point-to-compact set, formula for reflecting a solution of an analytic elliptic partial differential equation across a real-analytic curve on which it satisfies the Dirichlet conditions. We also discuss the special cases when the formula reduces to the point-to-point forms.

## 1. INTRODUCTION

The Schwarz symmetry principle is one of the celebrated tools in analysis and mathematical physics that has been attracting the attention of many mathematicians [1]– [14], [17]–[20], [22]–[28]. From the point of view of applications it is important to have an explicit reflection formula for a specific problem ([7], [10], [22]). One of the open questions is the following: for what partial differential equations, boundary conditions and spatial dimensions does such a formula exist and what is the structure of this formula, in other words, whether it is a point-to-point formula (see, for example [8]) or it has a more complicated structure, for example, a point to a finite set [20] or a point to a continuous set (see, for example, [2] and the references therein).

In this paper, we derive a reflection formula for solutions of elliptic equations in $\mathbb{R}^2$ with respect to a non-singular real-analytic curve and study the obtained formula. We call this formula non-local, since unlike the classical point-to-point reflection (see Theorem 1.1 below) this is a point to compact set reflection, generalizing the following celebrated Schwarz reflection principle for harmonic functions.

**Theorem 1.1** ([17], Chapter 9, p. 51; [28], Chapter 1, p. 4)**.** *Let $\Gamma = \{(x,y) : f(x,y) = 0\} \subset \mathbb{R}^2$ be a non-singular real-analytic curve and $P' \in \Gamma$. Then, there exists a neighborhood $U$ of $P'$ and an anti-conformal mapping $R : U \to U$ which is the identity on $\Gamma$, permutes the components $U_1, U_2$ of $U \setminus \Gamma$ and relative to which any harmonic function $u(x,y)$ defined near $\Gamma$ and vanishing on $\Gamma$ is odd; i.e.,*

$$u(x_0, y_0) = -u(R(x_0, y_0)) \tag{1.1}$$

*for any point $(x_0, y_0)$ sufficiently close to $\Gamma$. Note that if the point $(x_0, y_0) \in U_1$, then the "reflected" point $R(x_0, y_0) \in U_2$.*

Received by the editors April 21, 2009 and, in revised form, April 14, 2010.
2010 *Mathematics Subject Classification.* Primary 35J15; Secondary 32D15.
*Key words and phrases.* Elliptic equations, reflection principle, analytic continuation.

Here the mapping $R$ can be described by considering a complex domain $U_{\mathbb{C}}$ in the space $\mathbb{C}^2$, such that $U_{\mathbb{C}} \cap \mathbb{R}^2 = U$, to which the function $f$, defining the curve $\Gamma$, is continued analytically. After the transformation of the variables, $z = x + iy$, $\zeta = x - iy$, the equation of the complexified curve $\Gamma_{\mathbb{C}}$ can be rewritten in the form

$$f\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right) = 0. \tag{1.2}$$

If $\operatorname{grad} f(x,y) \neq 0$ on $\Gamma$, (1.2) can be solved with respect to $z$ or $\zeta$; the corresponding solutions we denote as $\zeta = S(z)$ and $z = \widetilde{S}(\zeta)$. The function $S(z)$ is called the *Schwarz function* of the curve $\Gamma$ ([6], Chapter 5, p. 21). The mapping $R$ is given by

$$R(x,y) = R(z) = \overline{S(z)}. \tag{1.3}$$

Formula (1.1) has been generalized to cover several other situations including the Helmholtz equation and wave equation, and the polyharmonic functions (see, for example, [25], [1], [20] and the references therein). The purpose of this paper is to obtain an explicit reflection formula for solutions to the elliptic equation

$$Lu \equiv \Delta_{x,y} u + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = 0 \tag{1.4}$$

with respect to a real analytic curve in $\mathbb{R}^2$, where the solution vanishes, and to investigate the properties of the mapping induced by this formula. Here $a(x,y)$, $b(x,y)$ and $c(x,y)$ are real-analytic functions in the domain $U \subset \mathbb{R}^2$.

In what follows, a formula, expressing the value of a function $u(x,y)$ at an arbitrary point $(x_0, y_0) \in U_1$ in terms of its values at points in $U_2$, is called a reflection formula. It is more often an integro-differential operator than a point-to-point reflection (1.1), which seems to be quite rare for solutions of partial differential equations. In particular, for solutions of the Helmholtz equation $(\Delta_{x,y} + \lambda^2)u(x,y) = 0$ vanishing on a curve $\Gamma$, point-to-point reflection holds only when $\Gamma$ is a line, while for harmonic functions in $\mathbb{R}^3$ it holds only when $\Gamma$ is either a plane or a sphere [8], [18]. The paper by P. Ebenfelt and D. Khavinson [8] is devoted to the further study of point-to-point reflection for harmonic functions. There it was shown that point-to-point reflection in the sense of the Schwarz reflection principle for $n > 2$ is very rare in $\mathbb{R}^n$ when $n$ is even, and that it never holds when $n$ is odd, unless $\Gamma$ is a sphere or a hyperplane. Reflection properties of solutions of the Helmholtz equation have also been considered in [9], [23], [25]. Two later papers are devoted to the derivation of non-local formulas for the Helmholtz equation subject to Dirichlet and Neumann conditions respectively. Recently a reflection formula for harmonic functions subject to the Robin condition, $\alpha\, \partial_n u + \beta\, u = 0$, on a real-analytic curve was derived in [2], and it was shown that the obtained (non-local) formula reduces to well-known point-to-point reflection laws corresponding to the Dirichlet and Neumann boundary conditions when one of the coefficients, $\alpha$ or $\beta$, vanishes.

The structure of the paper is as follows: in Section 2 we describe some preliminaries; in Section 3 we formulate the main theorem, which is proven in Section 4. Conclusions and the special cases, when the point-to-point reflections hold, are discussed in Section 5.

## 2. Preliminaries

We are starting this section by recalling a classical B. Riemann result for hyperbolic equations (see [13], Chapter 2, p. 65 or [12], Chapter 4, p. 127 for detailed explanations; here we follow a short version [17], Chapter 9, p. 55).

Consider a hyperbolic differential equation with entire coefficients

$$Hu \equiv \frac{\partial^2 u}{\partial x \partial y} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = 0; \tag{2.1}$$

its adjoint equation is

$$H^* u \equiv \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial (au)}{\partial x} - \frac{\partial (bu)}{\partial y} + cu = 0. \tag{2.2}$$

The Riemann function $\mathfrak{R}_H(x, y; x_0, y_0)$ of the operator $H$ is defined as the solution to the Goursat problem:

$$\begin{cases} H^* \mathfrak{R}_H = 0 \quad \text{ near } (x_0, y_0), \\ \mathfrak{R}_H(x_0, y; x_0, y_0) = \exp\Big\{ \int\limits_{y_0}^{y} a(x_0, \tau) d\tau \Big\}, \\ \mathfrak{R}_H(x, y_0; x_0, y_0) = \exp\Big\{ \int\limits_{x_0}^{x} b(t, y_0) dt \Big\}. \end{cases} \tag{2.3}$$

Note that $\mathfrak{R}_H$ is an entire function of all four variables; moreover, $\mathfrak{R}_H(x, y; x_0, y_0) = \mathfrak{R}_{H^*}(x_0, y_0; x, y)$, $\mathfrak{R}_H(x_0, y_0; x_0, y_0) = 1$ and the following Riemann's lemma holds.

**Lemma 2.1.** *Let $\Gamma := \{(x, y) | \, y = s(x)\}$ be a non-characteristic with respect to $H$ real-analytic curve that divides a domain $U \subset \mathbb{R}^2$ into two connected components $U_1$ and $U_2$, and let $u(x, y)$ be a solution of* (2.1) *near $\Gamma$. For all points $P(x_0, y_0) \in U$ sufficiently close to $\Gamma$ we have*

$$u(P) = \frac{1}{2}u(M)\mathfrak{R}_H(M) + \frac{1}{2}u(N)\mathfrak{R}_H(N) - \int\limits_{M}^{N} (\mathfrak{U} dy - \mathfrak{V} dx), \tag{2.4}$$

*where $M = (s^{-1}(y_0), y_0)$, $N = (x_0, s(x_0))$ and*

$$\mathfrak{U} = a\mathfrak{R}_H + \frac{1}{2}\mathfrak{R}_H \frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial \mathfrak{R}_H}{\partial y} u,$$
$$\mathfrak{V} = b\mathfrak{R}_H + \frac{1}{2}\mathfrak{R}_H \frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial \mathfrak{R}_H}{\partial x} u.$$

If in addition the solution to the equation $Hu = 0$ vanishes on $\Gamma$, formula (2.4) reduces to

$$u(P) = \frac{1}{2}\int\limits_{M}^{N} \mathfrak{R}_H (\frac{\partial u}{\partial x} dx - \frac{\partial u}{\partial y} dy). \tag{2.5}$$

*Remark* 2.2. For the wave equation, $a = b = c = 0$ in (2.1), the Riemann function equals 1 identically. Consider a point $P(x_0, y_0) \in U_1$ and a solution of the wave equation vanishing on $\Gamma$. Let's allow the path of integration $MN$ in (2.5) to degenerate to a pair of segments (in $U_2$) of a vertical and a horizontal characteristic through points $M$ and $N$, which intersect at a point $Q(s^{-1}(y_0), s(x_0)) \in U_2$. Then formula (2.5) becomes

$$u(P) = -u(Q).$$

Since the points $P$ and $Q$ are located on the opposite sides of the curve $\Gamma$, the latter formula states a point-to-point reflection law for the wave equation.

*Remark* 2.3. If for a solution of the wave equation vanishing on $\Gamma$ we allow the path of integration $MN$ in (2.5) to degenerate to a polygonal line consisting of vertical and horizontal segments with vertices $M = Q_1, Q_2, Q_3, \ldots, Q_n = N$ such that $Q_{2k+1} \in \Gamma$ and $Q_{2k} \in U_2 \setminus \Gamma$, $k = 1, \ldots, (n-1)/2$, then a version of a point to finite set reflection will be obtained [20]

$$u(P) = -\sum_{k=1}^{(n-1)/2} u(Q_{2k})$$

(see [20] for other examples of point to finite set formulas).

If we consider the elliptic equation (1.4) in the complex domain $U_{\mathbb{C}} \subset \mathbb{C}^2$, then the equation and its adjoint in characteristic variables $(z, \zeta)$ become similar to the hyperbolic equation (2.1) and its adjoint (2.2),

$$L_{\mathbb{C}} u \equiv \frac{\partial^2 u}{\partial z \partial \zeta} + A\frac{\partial u}{\partial z} + B\frac{\partial u}{\partial \zeta} + Cu = 0, \tag{2.6}$$

$$L_{\mathbb{C}}^* u \equiv \frac{\partial^2 u}{\partial z \partial \zeta} - \frac{\partial (Au)}{\partial z} - \frac{\partial (Bu)}{\partial \zeta} + Cu = 0, \tag{2.7}$$

where the coefficients in (1.4) are replaced with

$$A(z,\zeta) = \frac{1}{4}\big[a(x,y) + ib(x,y)\big], \quad B(z,\zeta) = \frac{1}{4}\big[a(x,y) - ib(x,y)\big],$$
$$C(z,\zeta) = \frac{1}{4}c(x,y).$$

Analogously, the Riemann function of $L$ is defined as the solution to the Goursat problem in $\mathbb{C}^2$:

$$\begin{cases} L_{\mathbb{C}}^* \mathfrak{R} \equiv \frac{\partial^2}{\partial z \partial \zeta}\mathfrak{R} - \frac{\partial}{\partial z}(A\mathfrak{R}) - \frac{\partial}{\partial \zeta}(B\mathfrak{R}) + C\mathfrak{R} = 0, \\ \mathfrak{R}_{|z=z_0} = \exp\Big\{\int\limits_{\zeta_0}^{\zeta} A(z_0,\tau)d\tau\Big\}, \\ \mathfrak{R}_{|\zeta=\zeta_0} = \exp\Big\{\int\limits_{z_0}^{z} B(t,\zeta_0)dt\Big\}. \end{cases} \tag{2.8}$$

By a *fundamental solution* of operator $L$ we understand a solution of the equation $L^* G(x_0, y_0, x, y) = \delta(x_0, y_0)$, where $L^*$ is the adjoint to $L$ differential operator. Thus, function $G$ written in the characteristic variables $z = x + iy$ and $\zeta = x - iy$ is a solution to the equation

$$L_{\mathbb{C}}^* u = \frac{\partial^2 u}{\partial z \partial \zeta} - \frac{\partial Au}{\partial z} - \frac{\partial Bu}{\partial \zeta} + Cu = \delta(z_0, \zeta_0). \tag{2.9}$$

The following formula (see [13], Chapter 3, p. 72) shows that the Riemann function is a factor of the logarithm in an expression for the fundamental solution of the operator $L_{\mathbb{C}}$:

$$G(z,\zeta; z_0,\zeta_0) = -\frac{1}{4\pi}\mathfrak{R}(z,\zeta;z_0,\zeta_0)\ln[(z-z_0)(\zeta-\zeta_0)] + g_0(z,\zeta,z_0,\zeta_0), \tag{2.10}$$

where $g_0(z, \zeta, z_0, \zeta_0)$ is an entire function.

Note that the fundamental solution exists (see [15], Chapter 3, p. 50) and is uniquely determined up to the kernel of the operator $L$.

There are different representations of the fundamental solution, for example, [5]; [15], Chapter 3, p. 76; [16]. However, for what follows we need a special representation as a sum of two functions, each of which has a logarithmic singularity on a single characteristic in $\mathbb{C}^2$. This representation is given by the following theorem.

**Theorem 2.4** ([26]). *There exists a fundamental solution of $L$ that can be represented in the form*

$$G = -\frac{1}{4\pi}(G_1 + G_2), \tag{2.11}$$

$$G_j = \sum_{k=0}^{\infty} \alpha_k^j(x_0, y_0; x, y) f_k(\psi_j), \qquad j = 1, 2\,, \tag{2.12}$$

$$f_k(\xi) = \begin{cases} (-1)^{-k-1}(-k-1)!\xi^k, & k \le -1, \\ \frac{\xi^k}{k!}(\ln \xi - C_k), & k = 0, 1, \ldots\,, \end{cases} \tag{2.13}$$

$$C_0 = 0, \qquad C_k = \sum_{l=1}^{k} \frac{1}{l}, \qquad k = 1, 2, \ldots\,,$$

$$\psi_1 = (x - x_0) + i(y - y_0) = z - z_0, \qquad \psi_2 = (x - x_0) - i(y - y_0) = \zeta - \zeta_0. \tag{2.14}$$

*Here the coefficients $\alpha_k^j$ are uniquely determined by the recursive transport equations*

$$\begin{gathered} \mathfrak{L}\alpha_0^j = 0, \qquad \mathfrak{L}\alpha_{k+1}^j = -L_{\mathbb{C}}^*\alpha_k^j, \\ \mathfrak{L} = \frac{\partial \psi_j}{\partial z} \cdot \Big[\frac{\partial}{\partial \zeta} - A\Big] + \frac{\partial \psi_j}{\partial \zeta} \cdot \Big[\frac{\partial}{\partial z} - B\Big] \end{gathered} \tag{2.15}$$

*subject to the initial conditions*

$$\begin{cases} \alpha^1_{0|_{\zeta=\zeta_0}} = \exp\Big\{\int\limits_{z_0}^{z} B(t, \zeta_0)dt\Big\},\ \alpha^1_{0|_{\zeta=\zeta_0}} = 0,\ k = 1, 2, \ldots\,, \\ \alpha^2_{0|_{z=z_0}} = \exp\Big\{\int\limits_{\zeta_0}^{\zeta} A(z_0, \tau)d\tau\Big\}, \quad \alpha^2_{k|_{z=z_0}} = 0, \quad k = 1, 2, \ldots\,. \end{cases} \tag{2.16}$$

Note that (2.15) and (2.16), in particular, imply that

$$\alpha_0^1 = \exp\Big(\int_{\zeta_0}^{\zeta} A(z, \tau)d\tau + \int_{z_0}^{z} B(t, \zeta_0)dt\Big), \quad \alpha_0^2 = \exp\Big(\int_{\zeta_0}^{\zeta} A(z_0, \tau)d\tau + \int_{z_0}^{z} B(t, \zeta)dt\Big). \tag{2.17}$$

Taking into account (2.10), one can interpret $\alpha_k^j$ as coefficients in the following series representations for the Riemann function (2.8) [26]:

$$\mathfrak{R}(z_0, \zeta_0, z, \zeta) = \sum_{k=0}^{\infty} \alpha_k^1(z_0, \zeta_0, z, \zeta)\frac{(z - z_0)^k}{k!} = \sum_{k=0}^{\infty} \alpha_k^2(z_0, \zeta_0, z, \zeta)\frac{(\zeta - \zeta_0)^k}{k!}. \tag{2.18}$$

*Remark* 2.5. For the Laplace equation, $a = b = c = 0$, and, therefore, $A = B = C = 0$. Thus, $\alpha_0^1 = \alpha_0^2 = 1$ and $\alpha_j^1 = \alpha_j^2 = 0$, $j \ge 1$, and

$$G_1^L = \ln(z - z_0) \quad \text{and} \quad G_2^L = \ln(\zeta - \zeta_0), \tag{2.19}$$

respectively, which leads to a standard fundamental solution:

$$G = -\frac{1}{4\pi}\ln[(z-z_0)(\zeta-\zeta_0)] = -\frac{1}{4\pi}\ln[(x-x_0)^2+(y-y_0)^2)]. \tag{2.20}$$

*Remark* 2.6. In the case of the Helmholtz equation, $a = b = 0$ and $c = \lambda^2$, that is, $\frac{\partial^2}{\partial z\partial\zeta}u^H + \frac{\lambda^2}{4}u^H = 0$. Here $\lambda$ is a real number, and the functions $G_1$ and $G_2$ reduce to the form used in [23], [25]:

$$\begin{aligned} G_1^H &= \sum_{k=0}^{\infty}\frac{[-\lambda^2(z-z_0)(\zeta-\zeta_0)]^k}{4^k(k!)^2}\Big(\ln(z-z_0)-C_k\Big),\\ G_2^H &= \sum_{k=0}^{\infty}\frac{[-\lambda^2(z-z_0)(\zeta-\zeta_0)]^k}{4^k(k!)^2}\Big(\ln(\zeta-\zeta_0)-C_k\Big). \end{aligned} \tag{2.21}$$

Summing up $G_1^H$ and $G_2^H$ and multiplying by $-\frac{1}{4\pi}$ one obtains the well-known fundamental solution of the Helmholtz equation:

$$-\frac{1}{4\pi}(G_1^H+G_2^H) = \frac{c+\ln\lambda/2}{2\pi}J_0\Big(\lambda\sqrt{(z-z_0)(\zeta-\zeta_0)}\Big) - \frac{1}{4}N_0\Big(\lambda\sqrt{(z-z_0)(\zeta-\zeta_0)}\Big), \tag{2.22}$$

where $c$ is the Euler constant, and $J_0$ and $N_0$ are the Bessel and the Neumann functions of zero order respectively.

## 3. The main result

Consider a solution of the homogeneous linear elliptic differential equation, written in its canonical form [12], Chapter 5, p. 136 (with the Laplace operator, $\Delta_{x,y}$, in the principal part), in a domain $U \subset \mathbb{R}^2$ vanishing on an algebraic curve $\Gamma$,

$$\begin{cases} Lu \equiv \Delta_{x,y}u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0 \text{ near } \Gamma,\\ u(x,y)_{|\Gamma} = 0; a,\ b,\ c \ \text{ are real-analytic functions of } x,\ y. \end{cases} \tag{3.1}$$

**Theorem 3.1.** *Under the above assumptions, the following reflection formula holds in $U$:*

$$\begin{aligned} u(P) \;=\; &-c_0(P,\Gamma)\,u(Q)\\ &+\frac{1}{2i}\int_\Gamma^Q\Big(\big\{u\frac{\partial V}{\partial x} - V\frac{\partial u}{\partial x} - auV\big\}dy - \big\{u\frac{\partial V}{\partial y} - V\frac{\partial u}{\partial y} - buV\big\}dx\Big), \end{aligned} \tag{3.2}$$

*where $P = (x_0, y_0)$ and $Q = R(P)$ (see (1.3)), and the integral is computed along any curve joining $\Gamma$ with $Q$. Here*

$$\begin{aligned} c_0(P,\Gamma) = \frac{1}{2}\Big\{&\exp\Big[\int_{z_0}^{\widetilde{S}(\zeta_0)} B(t, S(z_0))dt + \int_{\zeta_0}^{S(z_0)} A(z_0,\tau)d\tau\Big]\\ &+\exp\Big[\int_{\zeta_0}^{S(z_0)} A(\widetilde{S}(\zeta_0),\tau)d\tau + \int_{z_0}^{\widetilde{S}(\zeta_0)} B(t,\zeta_0)dt\Big]\Big\}, \end{aligned} \tag{3.3}$$

*where*

$$A(z,\zeta)=\frac{1}{4}\big[a(x,y)+ib(x,y)\big],\quad B(z,\zeta)=\frac{1}{4}\big[a(x,y)-ib(x,y)\big],$$
$$C(z,\zeta)=\frac{1}{4}c(x,y),\quad V=V(x_0,y_0,x,y)=V_1(x_0,y_0,x,y)-V_2(x_0,y_0,x,y).$$

*The functions $V_j$ are solutions of the Cauchy-Goursat problems:*

$$\begin{cases} L^*_{\mathbb{C}}V_j=0, & j=1,2\,,\\ V_{j}{}_{|_{\Gamma_{\mathbb{C}}}}=\mathfrak{R}_{|_{\Gamma_{\mathbb{C}}}}, & j=1,2\,,\\ V_1=\exp\Big\{\displaystyle\int_{\zeta_0}^{\zeta}A(\widetilde{S}(\zeta),\tau)d\tau+\int_{z_0}^{z}B(t,\zeta)dt\Big\}, & \textit{on the char. } \widetilde{l}_1=\{\widetilde{S}(\zeta)=z_0\},\\ V_2=\exp\Big\{\displaystyle\int_{\zeta_0}^{\zeta}A(z,\tau)d\tau+\int_{z_0}^{z}B(t,S(z))dt\Big\}, & \textit{on the char. } \widetilde{l}_2=\{S(z)=\zeta_0\},\end{cases}$$

*where $L^*_{\mathbb{C}}$ is the adjoint operator to $L_{\mathbb{C}}$ and $\mathfrak{R}(z_0,\zeta_0,z,\zeta)$ is the Riemann function of $L$.*

## 4. Proof of Theorem 3.1

4.1. **Sketch of the proof.** We begin with Green's formula expressing a solution of the equation $Lu=0$ at a point $P$ via its values on a contour $\gamma\subset U_1$ surrounding the point $P$ [11]:

$$u(P)=\int_{\gamma}\omega[u,G], \tag{4.1}$$

where

$$\omega[u,G]=\big\{u\frac{\partial G}{\partial x}-G\frac{\partial u}{\partial x}-auG\big\}\,dy-\big\{u\frac{\partial G}{\partial y}-G\frac{\partial u}{\partial y}-buG\big\}\,dx. \tag{4.2}$$

Here $G=G(x,y,x_0,y_0)$ is an arbitrary fundamental solution of $L$, that is, a solution to the equation $L^*G(x_0,y_0,x,y)=\delta(x_0,y_0)$. It is well known that $G$ is a real-analytic function in $\mathbb{R}^2$ except at the point $P(x_0,y_0)$. Its continuation to the complex space $\mathbb{C}^2$ has logarithmic singularities on the complex characteristics passing through this point, i.e., on $K_P:=\{(x-x_0)^2+(y-y_0)^2=0\}$. Our proof is based on the idea suggested by Garabedian [11] to deform contour $\gamma$ across the curve $\Gamma$ from the domain $U_1$ to the domain $U_2$. To be able to realize this deformation, first, we use a special representation for a fundamental solution, that is, a sum of two functions, each of which has a singularity on a single characteristic only. This representation is given by Theorem 2.4 above. Next, we replace the fundamental solution $G$ with a so-called *reflected fundamental solution.* After proving the existence and uniqueness of the reflected fundamental solution, we describe the deformation of $\gamma$ and obtain the desired reflected formula. Finally, we simplify the formula and discuss the cases for which it reduces to the simplest point-to-point form.

4.2. **The reflected fundamental solution.** This section is devoted to the construction of the reflected fundamental solution $\widetilde{G}$, which plays a key role by enabling us to deform the contour $\gamma$ across the boundary. $\widetilde{G}$ depends on the operator $L$ and the curve $\Gamma$.[1] As will be shown in the next two sections, the reflected fundamental solution determines whether the corresponding reflection formula can be reduced to the point-to-point form.

Function $\widetilde{G}$ is a solution of the equation $L^*_{\mathbb{C}}\widetilde{G} = 0$ subject to the boundary condition $G = \widetilde{G}$ on $\Gamma_{\mathbb{C}}$ and has singularities only on the "reflected" characteristic lines $\widetilde{l}_1$ and $\widetilde{l}_2$ (see Figure 1) intersecting the real space at the reflected point $Q = R(P)$ in the domain $U_2$ and intersecting $\Gamma_{\mathbb{C}}$ at $K_P \cap \Gamma_{\mathbb{C}}$.

We seek the reflected fundamental solution in the form

$$\widetilde{G}(z_0,\zeta_0,z,\zeta) = -\frac{1}{4\pi}(\widetilde{G}_1(z_0,\zeta_0,z,\zeta) + \widetilde{G}_2(z_0,\zeta_0,z,\zeta)), \tag{4.3}$$

where the functions $\widetilde{G}_j$, $j = 1,2$, are defined as the solutions to the following Cauchy-Goursat problems with prescribed singularities:

$$\begin{cases} L^*_{\mathbb{C}}\widetilde{G}_j = 0, \qquad j = 1,2, \\ \widetilde{G}_{j_{|\Gamma_{\mathbb{C}}}} = G_{j_{|\Gamma_{\mathbb{C}}}}, \\ \widetilde{G}_j \text{ has singularities only on the char. } \widetilde{l}_j =: \{\widetilde{\psi}_j(z,\zeta) = 0\}, \end{cases} \tag{4.4}$$

where $\widetilde{\psi}_1 = \widetilde{S}(\zeta) - z_0$ and $\widetilde{\psi}_2 = S(z) - \zeta_0$ are solutions of the Hamilton-Jacobi equation

$$\frac{\partial\widetilde{\psi}_j}{\partial z}\cdot\frac{\partial\widetilde{\psi}_j}{\partial\zeta} = 0. \tag{4.5}$$

First, we construct the solutions to the problems (4.4) as some formal expansions. Then we justify their convergence.

We seek this expansion in the form [21],

$$\widetilde{G}_1 = \sum_{k=0}^{\infty}\beta^1_k(z_0,\zeta_0,z,\zeta)f_k(\widetilde{\psi}_1), \tag{4.6}$$

$$\widetilde{G}_2 = \sum_{k=0}^{\infty}\beta^2_k(z_0,\zeta_0,z,\zeta)f_k(\widetilde{\psi}_2), \tag{4.7}$$

where $f_k(\xi)$ is defined by (2.13).

Substituting (4.6) and (4.7) into (4.4), we obtain the following recursion for the coefficients $\beta^1_k$ and $\beta^2_k$:

$$\begin{aligned} &\Big(\frac{\partial\beta^1_0}{\partial z} - B\,\beta^1_0\Big)\widetilde{S}'(\zeta) = 0, \quad \Big(\frac{\partial}{\partial z}\,\beta^1_{k+1} - B\,\beta^1_{k+1}\Big)\widetilde{S}'(\zeta) = -L^*_{\mathbb{C}}\beta^1_k, \quad k \ge 0, \\ &\Big(\frac{\partial\beta^2_0}{\partial\zeta} - A\,\beta^2_0\Big)S'(z) = 0, \quad \Big(\frac{\partial}{\partial\zeta}\,\beta^2_{k+1} - A\,\beta^2_{k+1}\Big)S'(z) = -L^*_{\mathbb{C}}\beta^2_k, \quad k \ge 0 \end{aligned} \tag{4.8}$$

subject to the following initial conditions:

$$\beta^1_{k_{|\Gamma_{\mathbb{C}}}} = \alpha^1_{k_{|\Gamma_{\mathbb{C}}}}, \qquad \beta^2_{k_{|\Gamma_{\mathbb{C}}}} = \alpha^2_{k_{|\Gamma_{\mathbb{C}}}}, \qquad k = 0,1,2,\dots. \tag{4.9}$$

[1] $\widetilde{G}$ depends on the boundary condition as well, but the latter is beyond the scope of this paper; see [2], [25] and [27] for some relevant results.

Note that both $S'(z)$ and $\widetilde{S}'(\zeta)$ do not vanish on $\Gamma_{\mathbb{C}}$ ([6], Chapter 7, p. 42). Indeed, functions $S(z)$ and $\widetilde{S}(\zeta)$ are inverses of each other (see (1.2)), so $\widetilde{S}(S(z)) = z$. Differentiating the latter equation and taking into account that $S(z) = \zeta$ on $\Gamma_{\mathbb{C}}$, we obtain $\widetilde{S}'(\zeta) \cdot S'(z) = 1$. In $\mathbb{R}^2$, $\widetilde{S}'(\zeta) = \overline{S'(z)}$; therefore, $|S'(z)| = |\widetilde{S}'(\zeta)| = 1$ on $\Gamma$. Thus, both functions $S'(z)$ and $\widetilde{S}'(\zeta)$ are non-zero throughout some neighborhood of $\Gamma$ as continuous functions.

Thus, functions $\beta_k^1$ and $\beta_k^2$ are uniquely determined near $\Gamma_{\mathbb{C}}$, specifically

$$\begin{aligned} \beta_0^1 &= \exp\Big(\int\limits_{\zeta_0}^{\zeta} A(\widetilde{S}(\zeta),\tau)d\tau + \int\limits_{z_0}^{z} B(t,\zeta)dt + \int\limits_{z_0}^{\widetilde{S}(\zeta)} [B(t,\zeta_0) - B(t,\zeta)]dt\Big), \\ \beta_0^2 &= \exp\Big(\int\limits_{\zeta_0}^{\zeta} A(z,\tau)d\tau + \int\limits_{\zeta_0}^{S(z)} [A(z_0,\tau) - A(z,\tau)]d\tau + \int\limits_{z_0}^{z} B(t,S(z))dt\Big). \end{aligned} \tag{4.10}$$

Hence, the formal expansions for the functions $\widetilde{G}_1$, $\widetilde{G}_2$ satisfying conditions (4.4) are constructed.

**Lemma 4.1.** *The series* (4.6) *and* (4.7) *converge near* $\Gamma_{\mathbb{C}}$.

*Proof of Lemma* 4.1. Let us prove the convergence of the series (4.7) by considering an auxiliary family of problems depending on the parameter $\xi$:

$$\begin{cases} L_{\mathbb{C}}^* V_\xi(z_0,\zeta_0,z,\zeta,\xi) = 0, \\ V_\xi(z_0,\zeta_0,z,S(z),\xi) = \Phi(z_0,\zeta_0,z,S(z),\xi), \\ V_\xi(z_0,\zeta_0,\widetilde{S}(\zeta_0-\xi),\zeta,\xi) = 0. \end{cases} \tag{4.11}$$

Here $\Phi$ is a given analytic function that has Taylor expansion

$$\Phi(z_0,\zeta_0,z,S(z),\xi) = \sum_{k=0}^{\infty} \alpha_k^2(z_0,\zeta_0,z,S(z)) \frac{(S(z)-\zeta_0+\xi)^{k+1}}{(k+1)!},$$

where the coefficients $\alpha_k^2(z_0,\zeta_0,z,\zeta)$ are the same as in (2.18) [26].

The Taylor expansion of the solution to the problem (4.11) (if it exists) has the form

$$V_\xi(z_0,\zeta_0,z,\zeta,\xi) = \sum_{k=0}^{\infty} \beta_k^2(z_0,\zeta_0,z,\zeta) \frac{(S(z)-\zeta_0+\xi)^{k+1}}{(k+1)!}, \tag{4.12}$$

where the coefficients $\beta_k^2$ are the same as the coefficients in series (4.7). Convergence of the latter, therefore, followed from convergence (4.12). To show existence and uniqueness of the solution to the problem (4.11) in the class of analytic functions we use the substitution

$$V_\xi(z_0,\zeta_0,z,\zeta,\xi) = \int\limits_{S(z)}^{\zeta} d\tau \int\limits_{\widetilde{S}(\zeta_0-\xi)}^{z} \mu(z_0,\zeta_0,t,\tau,\xi)dt + \Phi(z_0,\zeta_0,z,S(z),\xi) \tag{4.13}$$

with unknown density $\mu$, which reduces the problem (4.11) to the Volterra integral equation

$$\begin{aligned}
&\mu(z_0,\zeta_0,z,\zeta,\xi)+A(z,\zeta)S'(z)\int\limits_{\widetilde{S}(\zeta_0-\xi)}^{z}\mu(z_0,\zeta_0,t,S(z),\xi)dt\\
&-A(z,\zeta)\int\limits_{S(z)}^{\zeta}\mu(z_0,\zeta_0,z,\tau,\xi)d\tau-B(z,\zeta)\int\limits_{\widetilde{S}(\zeta_0-\xi)}^{z}\mu(z_0,\zeta_0,t,\zeta,\xi)dt\\
&-F(z,\zeta)\int\limits_{S(z)}^{\zeta}d\tau\int\limits_{\widetilde{S}(\zeta_0-\xi)}^{z}\mu(z_0,\zeta_0,t,\tau,\xi)d\tau=\Psi(z_0,\zeta_0,z,\zeta,\xi),
\end{aligned}\tag{4.14}$$

where

$$F(z,\zeta)=\frac{\partial}{\partial z}A(z,\zeta)+\frac{\partial}{\partial\zeta}B(z,\zeta)-C(z,\zeta),\tag{4.15}$$

$$\Psi(z_0,\zeta_0,z,\zeta,\xi)=F(z,\zeta)\Phi(z_0,\zeta_0,z,\xi)+A(z,\zeta)\frac{\partial}{\partial z}\Phi(z_0,\zeta_0,z,\xi).\tag{4.16}$$

The existence and uniqueness of the analytic solution of equation (4.14) can be proven by the iteration technique described in [29], Chapter 1, p. 11. Thus, there exists a unique solution of (4.11), which has a unique Taylor expansion with respect to the variable $\xi$ at the point $\xi=-(S(z)-\zeta_0)$; this expansion coincides with the expansion (4.12). Thus, series (4.12) converges in the neighborhood of $\Gamma$, and so does (4.7).

Analogously, considering the following auxiliary problem depending on the parameter $\eta$:

$$\begin{cases}
L^*_{\mathbb{C}}V_\eta(z_0,\zeta_0,z,\zeta,\eta)=0,\\
V_\eta(z_0,\zeta_0,\widetilde{S}(\zeta),\zeta,\eta)=\sum\limits_{k=0}^{\infty}\alpha^1_k(z_0,\zeta_0,\widetilde{S}(\zeta),\zeta)\frac{(\widetilde{S}(\zeta)-z_0+\eta)^{k+1}}{(k+1)!},\\
V_\eta(z_0,\zeta_0,z,S(z_0-\eta)),\eta)=0,
\end{cases}\tag{4.17}$$

whose solution has the Taylor expansion $V_\eta=\sum_{k=0}^{\infty}\beta^1_k\frac{(\widetilde{S}(\zeta)-z_0+\eta)^{k+1}}{(k+1)!}$, one can show convergence of (4.6). That finishes the proof. □

4.3. **The reflected fundamental solution as a multiple-valued function.** As was conjectured in [4]: "Perhaps looking-glass milk isn't good to drink". In this section we show that the reflected fundamental solution (the looking-glass fundamental solution), except for some special cases, does not inherit all of the properties of a "true" fundamental solution; in particular, the representation (2.10) with the Riemann function as a factor of the logarithm does not hold. Moreover, as we are about to show, the factors of the logarithms in $\widetilde{G}_1$ and $\widetilde{G}_2$ (see (4.6) and (4.7)) are not the same, which makes the reflected fundamental solution a multiple-valued function even in $\mathbb{R}^2$. The latter explains (see Section 4.4) why point-to-point reflection almost always fails.

Indeed, consider a point moving along a continuous curve $\gamma$ surrounding either the branch line $z=z_0$ of $G_1$ or the branch line $\zeta=\zeta_0$ of $G_2$ (2.11). As the point makes a complete cycle around the line, it passes to the next sheet of the Riemann surface; while going around the cyclic path surrounding both characteristics at once, it remains on the same sheet of the Riemann surface. For the reflected fundamental solution $\widetilde{G}$, the point passes to the next sheet even if the curve $\gamma$ lays in $\mathbb{R}^2$ and surrounds both intersecting branch lines, $\widetilde{S}(\zeta) = z_0$ and $S(z) = \zeta_0$.

To show this, let us compute the increment of the function $\widetilde{G}$ when a curve $\gamma \subset \mathbb{R}^2$ is a circle of a small radius $\rho$ centered at the point $Q\big(\widetilde{S}(\bar{z}_0), S(z_0)\big)$:

$$2\pi i(-\frac{1}{4\pi})\sum_{j=0}^{\infty}\Big(\beta_j^1\frac{(\widetilde{S}(\zeta)-z_0)^j}{j!} - \beta_j^2\frac{(S(z)-\zeta_0)^j}{j!}\Big). \tag{4.18}$$

Taking into account that $\zeta = \bar{z}$ in $\mathbb{R}^2$, let us set $z = \widetilde{S}(\bar{z}_0) + \rho e^{i\phi}$ and $\zeta = S(z_0) + \rho e^{-i\phi}$, and expand the Schwarz function and its inverse into Taylor series at the point $Q$: $S(z) = \bar{z}_0 + C_1\rho e^{i\phi} + o(\rho)$, $\widetilde{S}(\bar{z}) = z_0 + \bar{C}_1\rho e^{-i\phi} + o(\rho)$.

Without loss of generality assume that the coefficients $A$, $B$ and $C$ in (2.6) are constants (otherwise we should use their Taylor expansions in this analysis); then $\beta_0^1 = \beta_0^2$ (see (4.10)), and

$$\beta_1^1 = (AB - C)e^{(A(\zeta-\zeta_0)+B(z-z_0))}\big((z - \widetilde{S}(\zeta))/\widetilde{S}'(\zeta) + \zeta - \zeta_0\big), \tag{4.19}$$

$$\beta_2^1 = (AB - C)e^{(A(\zeta-\zeta_0)+B(z-z_0))}\big((\zeta - \widetilde{S}(z))/S'(z) + z - z_0\big). \tag{4.20}$$

Thus, the increment (4.18) becomes

$$\begin{aligned}\frac{\rho}{2i}(AB - C)\Big(&[C_1\widetilde{S}(\bar{z}_0) - C_1 z_0 + S(z_0) - \bar{z}_0]\bar{C}_1 e^{-i\phi} \\ &-[\bar{C}_1 S(z_0) - \bar{C}_1\bar{z}_0 + \widetilde{S}(\bar{z}_0) - z_0]C_1 e^{i\phi}\Big) + o(\rho).\end{aligned} \tag{4.21}$$

Formula (4.21) shows that the increment can be equal to zero only in two cases: either when (i) $AB - C = 0$ or (ii) expressions in the brackets equal to zero. The latter happens if boundary $\Gamma$ is a segment of a straight line, while (i), for example, holds if operator $L$ is the Laplacian.

Having the detailed description of the reflected fundamental solution we are ready to derive the reflection formula by explaining how the contour $\gamma$ in (4.1) can be deformed from one side of the reflecting surface $\Gamma_{\mathbb{C}}$ to the other.

4.4. **Deformation of the contour.** Formula (4.1) involves integration over a contour $\gamma \subset U_1$ surrounding both characteristics on which the functions $G_1$ and $G_2$ have singularities (lines $l_1$ and $l_2$ in Figure 1). To express the value $u(P)$ in terms of the values of $u(x, y)$ in $U_2$, that is, to construct a reflection formula, it is sufficient to deform the contour $\gamma$ from the domain $U_1$ to the domain $U_2$. Note that since the integrand in (4.1) is a closed form, $d\omega = 0$, the value of the integral will not change while we are deforming the contour $\gamma$ homotopically.

First, the contour is deformed to the complexified curve $\Gamma_{\mathbb{C}}$. Taking into account that the characteristics of $G$ passing through the point $P$ intersect $\Gamma_{\mathbb{C}}$ at two different points in $\mathbb{C}^2$, assume that the point $P$ lies so close to the curve $\Gamma$ that there exists a connected domain $\Omega \subset \Gamma_{\mathbb{C}}$, univalently projected onto a plane, that contains both points of intersections [23].

We start the deformation with stretching the contour $\gamma$ (see (4.1)) in the real plane until its small arc reaches the curve $\Gamma$ (it becomes a mirror image of $\widetilde{\gamma}$ in Figure 1). Then we substitute a sum of $G_1$ and $G_2$ for $G$ in (4.1) and split the

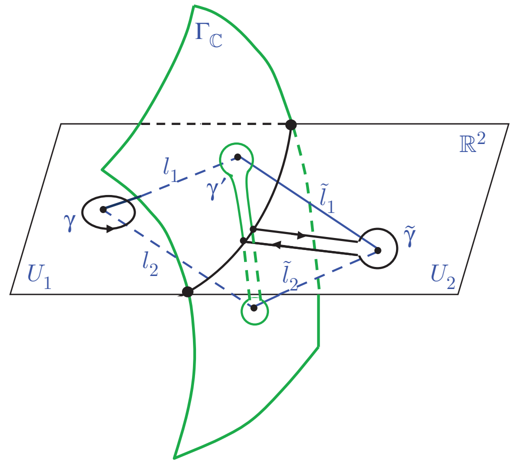


FIGURE 1. Contour deformation

integral:

$$u(P) = \int_{\gamma} \omega[u, G_1] + \int_{\gamma} \omega[u, G_2]. \tag{4.22}$$

Note that contour $\gamma$ is not closed on the Riemann surfaces of each $G_1$ and $G_2$ (see Section 4.3). As a point of disconnection (one of two endpoints) let us choose a point $K \in \gamma \cap \Gamma$. Then in the first integral in (4.22) we "lift" the contour $\gamma$ to $\gamma'$ (solid line above the plane in Figure 1) such that we do not move some points of $\gamma \cap \Gamma$ in the neighborhood of the point $K$. Then we do the symmetric (with respect to plane $\mathbb{R}^2$) deformation in the second integral in (4.22).

Taking into account that $u_{|\Gamma_{\mathbb{C}}} = 0$, the differential form $\omega$ (4.2) on $\Gamma_{\mathbb{C}}$ becomes

$$\omega'[u, G_j] = G_j \frac{\partial u}{\partial y}\, dx - G_j \frac{\partial u}{\partial x}\, dy, \quad j = 1, 2. \tag{4.23}$$

Now we can replace $G_j$ with $\widetilde{G}_j$ (see formula (4.3)). Indeed, according to (4.4),

$$\int_{\gamma'} \omega'[u, G_1] = \int_{\gamma'} \omega'[u, \widetilde{G}_j]. \tag{4.24}$$

In order to deform contour $\gamma'$ from $\Gamma_{\mathbb{C}}$ to the domain $U_2$, it is necessary to apply the "mirror" deformation procedure. Note that during this deformation the point $K$ is fixed and the contour surrounds one of the "reflected" characteristic lines $\widetilde{l}_1$ or $\widetilde{l}_2$ (see Figure 1) intersecting the real space at the reflected point $Q = R(P)$ in the domain $U_2$ and intersecting $\Gamma_{\mathbb{C}}$ at $K_P \cap \Gamma_{\mathbb{C}}$.

Finally, we have

$$u(P) = \int_{\widetilde{\gamma}} \omega[u, \widetilde{G}_1] + \int_{\widetilde{\gamma}} \omega[u, \widetilde{G}_2]. \tag{4.25}$$

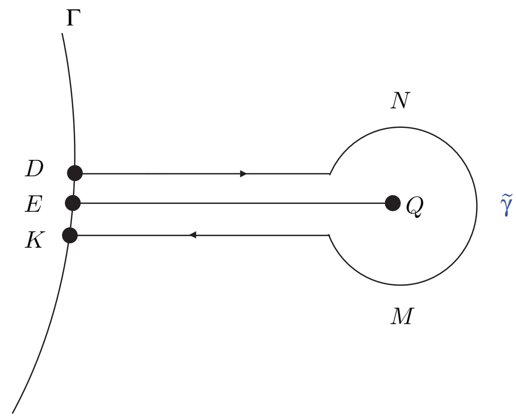


FIGURE 2. Contour transformation

This formula can be rewritten as a single integral

$$u(P) = \int\limits_{\widetilde{\gamma}} \omega[u, \widetilde{G}], \tag{4.26}$$

but as was discussed in Section 4.3 the contour $\widetilde{\gamma}$, generally, is not closed on the Riemann surface of $\widetilde{G}$, so in most of the cases we do not expect to be able to move the point $K$ (see Figure 2) from the curve $\Gamma$. Formula (4.26) is a version of a desired reflection formula. In the next section we simplify it and show that it holds in the large.

4.5. **The reflection formula in the large.** Formula (4.26) in $(z, \zeta)$ variables has the form

$$u(P) = \int\limits_{\widetilde{\gamma}} \widetilde{\omega}[u, \widetilde{G}], \tag{4.27}$$

where

$$\widetilde{\omega}[u, \widetilde{G}] = i\Big(\big\{u\frac{\partial \widetilde{G}}{\partial \zeta} - \widetilde{G}\frac{\partial u}{\partial \zeta} - 2Au\widetilde{G}\big\}d\zeta - \big\{u\frac{\partial \widetilde{G}}{\partial z} - \widetilde{G}\frac{\partial u}{\partial z} - 2Bu\widetilde{G}\big\}dz\Big). \tag{4.28}$$

Here $\widetilde{G}$ is the reflected fundamental solution and the contour $\widetilde{\gamma} \subset U_2$ surrounds the point $Q$ (see Figure 2). Recall that $\widetilde{G}$ is a sum of two series (with certain radii of convergence). Now we are going to show that the formula holds in the large.

Let us rewrite the functions $\widetilde{G}_l$ in the form:

$$\widetilde{G}_l = V_l \ln \widetilde{\psi}_l + \widetilde{V}_l, \qquad l = 1, 2, \tag{4.29}$$

where

$$V_l = \sum_{j=0}^{\infty} \beta_j^l \frac{(\widetilde{\psi}_l)^j}{j!}, \qquad \widetilde{V}_l = \sum_{j=0}^{\infty} \beta_j^l \frac{(\widetilde{\psi}_l)^j}{j!} C_j, \tag{4.30}$$

and

$$\widetilde{\psi}_1 = \widetilde{S}(\zeta) - z_0, \qquad \widetilde{\psi}_2 = S(z) - \zeta_0. \tag{4.31}$$

Substituting (4.29) and (4.30) into (4.27) and letting the radius of the arc NM go to zero (see Figure 2) results in vanishing integrals of the terms involving products of the function $\widetilde{V}_l$ and derivatives of the function $u$ as integrals of holomorphic functions over a closed contour. Combining terms in (4.27) involving derivatives of logarithms and separating them from the terms involving logarithmic functions yields

$$u(P) = \mathbb{Q} + \mathbb{I}, \tag{4.32}$$

where $\mathbb{Q}$ and $\mathbb{I}$ in the characteristic variables have the form

$$\mathbb{Q} = -\frac{i}{4\pi}\sum_l \int_{\widetilde{\gamma}} \Big(uV_l \frac{\partial}{\partial \zeta}(\ln \widetilde{\psi}_l)d\zeta - uV_l \frac{\partial}{\partial z}(\ln \widetilde{\psi}_l)dz\Big), \tag{4.33}$$

$$\mathbb{I} = -\frac{1}{4\pi}\int_{\widetilde{\gamma}} \widetilde{\omega}[u, \widetilde{G}_1]\ln \widetilde{\psi}_1 \; - \frac{1}{4\pi}\int_{\widetilde{\gamma}} \widetilde{\omega}[u, \widetilde{G}_2]\ln \widetilde{\psi}_2\,. \tag{4.34}$$

Substituting series (4.30) for $V_l$ into (4.33) and computing the residues at the point $Q$ where the integrand has a simple pole, we have

$$\begin{aligned}\mathbb{Q} = &-\frac{1}{2}u(Q)\Big(\exp\Big(\int_{z_0}^{\widetilde{S}(\zeta_0)} B(t, S(z_0))dt + \int_{\zeta_0}^{S(z_0)} A(z_0, \tau)d\tau\Big) \\ &+ \exp\Big(\int_{\zeta_0}^{S(z_0)} A(\widetilde{S}(\zeta_0), \tau)d\tau + \int_{z_0}^{\widetilde{S}(\zeta_0)} B(t, \zeta_0)dt\Big)\Big),\end{aligned} \tag{4.35}$$

which holds in the large.

Using properties of the logarithmic function and replacing the contour $\widetilde{\gamma}$ with a segment EQ, the second integral can be rewritten as

$$\mathbb{I} = 2\pi i(-\frac{1}{4\pi})\Big(\int_E^Q \widetilde{\omega}[u, V_1] - \int_E^Q \widetilde{\omega}[u, V_2]\Big) = \frac{1}{2i}\int_E^Q \widetilde{\omega}[u, V], \tag{4.36}$$

where $V = V_1 - V_2$. Note that the logarithms in (4.34) have complex conjugated arguments (4.31) in $\mathbb{R}^2$; however, they cancel each other only if the factors $V_1$ and $V_2$ are equal, which generally is not the case.

Even though the latter formula involves the series $V_1$ and $V_2$, it also holds in the large, since these expansions can be interpreted as solutions of the following Cauchy problems:

$$\begin{cases} L^*_{\mathbb{C}} V_j = 0, \qquad j = 1, 2, \\ V_{j|_{\Gamma_{\mathbb{C}}}} = \mathfrak{R}_{|\Gamma_{\mathbb{C}}}, \\ V_j = \exp\Big\{\int_{\zeta_0}^{\zeta} A(\theta_j, \tau)d\tau + \int_{z_0}^{z} B(t, \eta_j)dt\Big\} \text{ on the characteristic } \widetilde{l}_j =: \{\widetilde{\psi}_j(z, \zeta) = 0\}, \end{cases} \tag{4.37}$$

where $\theta_1 = \widetilde{S}(\zeta)$, $\theta_2 = z$, $\eta_1 = \zeta$ and $\eta_2 = S(z)$. Problem (4.37) by a substitution with unknown density $\mu$, for example, for $j = 2$,

$$V_2(z_0, \zeta_0, z, \zeta) = \int_{S(z)}^{\zeta} d\tau \int_{\widetilde{S}(\zeta_0)}^{z} \mu(z_0, \zeta_0, t, \tau) dt + \mathfrak{R}(z_0, \zeta_0, z, S(z))\, e^{\int_{\zeta_0}^{\zeta} A(z,\tau)\, d\tau - \int_{\zeta_0}^{S(z)} A(z,\tau)\, d\tau}, \tag{4.38}$$

can be reduced to the Volterra integral equation, whose solution as a function of four complex variables exists and is unique in some cylindrical domain near $\Gamma$ (see [29], Chapter 1, p. 11). Thus, the solutions of (4.37) exist in $\mathbb{C}^4$ as multiple-valued analytic functions, whose singularities coincide with those of $S(z)$ and $\widetilde{S}(\zeta)$.

Combining (4.32), (4.35) and (4.36) we arrive at the formula (3.2), which proves the theorem.

## 5. Conclusions and remarks

5.1. **Equations with constant coefficients.** We have obtained a reflection formula for elliptic equations with analytic coefficients subject to homogeneous Dirichlet conditions on a real-analytic curve. This is a point to compact set reflection, which in some cases can be essentially simplified.

Consider the case when the coefficients $a$, $b$ and $c$ in equation (1.4) are constants,

$$\Delta_{x,y} u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0, \tag{5.1}$$

and, therefore, $A$, $B$ and $C$ are constants as well. In this case solutions $\alpha_k^j$ to problems (2.15)–(2.16) can be written explicitly, and the Riemann function (2.18) has the form

$$\mathfrak{R}(z_0, \zeta_0, z, \zeta) = \sum_{k=0}^{\infty} \frac{\big((z-z_0)(\zeta-\zeta_0)(AB-C)\big)^k}{(k!)^2} \exp\left(A(\zeta-\zeta_0) + B(z-z_0)\right). \tag{5.2}$$

Our main conclusion confirms the fact that the point-to-point reflection is quite rare.

**Theorem 5.1.** *For non-trivial solutions of elliptic equation* (5.1) *with constant coefficients vanishing on a real-analytic curve* $\Gamma$, *there is no point-to-point reflection unless one of the following conditions holds:*

*(i)* $\Gamma$ *is a line,*

*(ii)* $a^2 + b^2 - 4c = 0$.

*Proof.* The proof immediately follows from the fact that the integral term $\mathbb{I} \neq 0$ in (4.36). Indeed, formula (4.21) implies that $V \neq 0$. Thus, for $\mathbb{I}$ to be zero, function $u$ and its first derivative must vanish on a path joining the curve $\Gamma$ with the reflected point, which contradicts the assumption that $u$ is not equal to zero identically. □

**Theorem 5.2.** *Let* $\Gamma := \{\alpha x + \beta y + \delta = 0\}$ *be a line. Then for any solution of the equation* $\Delta u + au_x + bu_y + cu = 0$ *with constant coefficients vanishing on* $\Gamma$ *the* *following point-to-point reflection formula holds in* $\mathbb{R}^2$*:*

$$u(P) = -\exp\Big(-\frac{(\alpha x_0 + \beta y_0 + \delta)(a\alpha + b\beta)}{\alpha^2 + \beta^2}\Big) u(Q). \tag{5.3}$$

*Proof.* Under the assumptions of the theorem, the Schwarz function is $S(z) = mz + q$, where

$$m = \frac{\beta^2 - \alpha^2 + i\,2\alpha\beta}{\alpha^2 + \beta^2}, \qquad q = \frac{-2\alpha\delta + i\,2\beta\delta}{\alpha^2 + \beta^2}.$$

Functions $V_1$ and $V_2$ are equal (see (4.37)),

$$V_1 = V_2 = \sum_{k=0}^{\infty} \frac{\big((mz + q - \zeta_0)(\bar{m}\zeta + \bar{q} - z_0)(AB - C)\big)^k}{(k!)^2}\, e^{(A(\zeta - \zeta_0) + B(z - z_0))},$$

and therefore, $V = V_1 - V_2 = 0$, and the integral $\mathbb{I} = 0$ (see (4.36)). Formula (4.35) can be simplified, and $\mathbb{Q} = -u(Q)\, e^{A(mz_0 + q - \zeta_0) + B(\bar{m}\zeta + \bar{q} - z_0)}$. The latter in variables $(x, y)$ gives (5.3). $\square$

**Corollary 5.3.** *Let* $\Gamma$ *be a line with equation* $y = 0$*. Then for any solution of* (5.1) *vanishing on* $\Gamma$ *the following reflection formula holds:*

$$u(x_0, y_0) = -e^{-by_0} u(x_0, -y_0). \tag{5.4}$$

**Corollary 5.4.** *Let* $\Gamma$ *be a line with equation* $x = 0$*. Then for any solution of* (5.1) *vanishing on* $\Gamma$ *the reflection formula has the form*

$$u(x_0, y_0) = -e^{-ax_0} u(-x_0, y_0). \tag{5.5}$$

**Corollary 5.5.** *If* $a = b = 0$ *formula* (5.3) *recovers known point-to-point reflection for solutions of the Helmholtz equation vanishing on a line*

$$u(P) = -u(Q).$$

*Remark* 5.6. Note that in the case of the Helmholtz equation, $a = b = 0$ and $c = \lambda^2$, when $\Gamma$ is a real-analytic curve, formula (4.35) can be simplified, and $\mathbb{Q} = -u(Q)$, but $\mathbb{I} \neq 0$ in (4.36) unless $\Gamma$ is a line ([17], Chapter 9, p. 59; [18]; [23]).

**Theorem 5.7.** *Let* $\Gamma$ *be a real-analytic curve. Then for any solution of the equation* $\Delta u + au_x + bu_y + (a^2 + b^2)/4\, u = 0$ *vanishing on* $\Gamma$ *the following point-to-point reflection formula holds in* $\mathbb{R}^2$*:*

$$u(P) = -e^{A(S(z_0) - \zeta_0) + B(\widetilde{S}(\zeta_0) - z_0)} u(Q). \tag{5.6}$$

*Proof.* In characteristic variables, condition $c = (a^2 + b^2)/4$ is equivalent to $AB - C = 0$. Then the Riemann function (5.2) has the simplest form

$$\mathfrak{R}(z_0, \zeta_0, z, \zeta) = e^{A(\zeta - \zeta_0) + B(z - z_0)}, \tag{5.7}$$

and $V_1 = V_2 = \mathfrak{R}$ for any analytic curve $\Gamma$. Thus, the reflection formula has the point to point form (5.6). $\square$

*Remark* 5.8. Equation $\Delta u + au_x + bu_y + (a^2 + b^2)/4\, u = 0$ can be transformed into the Laplace equation using the substitution $u(x, y) = v(x, y)e^{-(ax+by)/2}$, where $v$ is a harmonic function, and, therefore, $v$ enjoys the celebrated Schwarz symmetry principle (1.1).

**Example 5.9.** Formula (5.6) for the unit circle centered at the origin can be rewritten in $(x, y)$ variables as follows:

$$u(x_0, y_0) = -\exp\Big(\frac{2(ax_0 + by_0)(1 - x_0^2 - y_0^2)}{x_0^2 + y_0^2}\Big) u\Big(\frac{x_0}{x_0^2 + y_0^2}, \frac{y_0}{x_0^2 + y_0^2}\Big). \tag{5.8}$$

5.2. **A final remark.** Thus, for elliptic equations of the second order with real-analytic coefficients in $\mathbb{R}^2$, there is no point-to-point reflection with respect to a real-analytic curve $\Gamma$ unless $\Gamma$ is a line or the following constraint $a^2 + b^2 - 4c = 0$ for the coefficients of the equation holds.

As follows from [20] for elliptic equations in $\mathbb{R}^2$, there is no point to finite set reflection as well.

Point to compact set reflection is always possible. This set is a curve having one of its endpoints on a reflecting curve. The other endpoint is located at the reflected point itself.

## Acknowledgments

The research of the author was supported in part by OU Research Challenge Program, award # RC-09043. The author is especially grateful to the anonymous referee, whose comments have improved the paper.

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