

ON THE ALGEBRAIC FUNDAMENTAL GROUP
 OF SMOOTH VARIETIES IN CHARACTERISTIC $p > 0$

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ABSTRACT. We define an analog in characteristic $p > 0$ of the proalgebraic completion of the topological fundamental group of a complex manifold.

1. INTRODUCTION

Let X be a smooth algebraic variety defined over a field k endowed with a rational point $x \in X(k)$.

If k is the field of complex numbers \mathbb{C} , the proalgebraic completion $\pi^{\text{alg,rs}}(X, x)$ of the topological fundamental group $\pi_1^{\text{top}}(X, x)$ is defined as the prosystem $\varprojlim H$, where $H \subset GL(n, \mathbb{C})$ runs over the Zariski closures of the monodromy groups $\rho(\pi^{\text{top}}(X, x))$ of complex linear representations $\rho : \pi_1^{\text{top}}(X, x) \rightarrow GL(n, \mathbb{C})$. The profinite completion $\varprojlim H$, where H runs over the finite quotients of $\pi_1^{\text{top}}(X, x)$, is, via the Riemann existence theorem, identified with Grothendieck's étale fundamental group $\pi_1^{\text{ét}}(X, x)$. Since any finite group is embeddable in $GL(n, \mathbb{C})$ for some n , this defines, thinking of $\pi_1^{\text{ét}}(X, x)$ as a complex (constant) proalgebraic group, a surjective homomorphism $\varphi_{\mathbb{C}}^{\text{rs}} : \pi^{\text{alg,rs}}(X, x) \rightarrow \pi_1^{\text{ét}}(X, x)$, and in fact $\pi_1^{\text{ét}}(X, x)$ is the profinite quotient of $\pi^{\text{alg,rs}}(X, x)$. By the Riemann-Hilbert correspondence, $\pi^{\text{alg,rs}}(X, x)$ is the Tannaka group-scheme of the category of \mathcal{O}_X -coherent regular singular \mathcal{D}_X -modules, which is a full subcategory of the category of \mathcal{O}_X -coherent \mathcal{D}_X -modules. We denote by $\pi^{\text{alg}}(X, x)$ the corresponding Tannaka group-scheme and by $\varphi_{\mathbb{C}} : \pi^{\text{alg}}(X, x) \rightarrow \pi^{\text{alg,rs}}(X, x) \xrightarrow{\varphi_{\mathbb{C}}^{\text{rs}}} \pi_1^{\text{ét}}(X, x)$ the composite morphism. It is surjective as well, and since any flat connection with finite monodromy is regular singular, $\pi_1^{\text{ét}}(X, x)$ is the profinite quotient of $\pi^{\text{alg}}(X, x)$.

If k is a characteristic 0 field, $\pi^{\text{alg}}(X, x)$ is defined as the Tannaka group-scheme of the k -linear tensor category of \mathcal{O}_X -coherent \mathcal{D}_X -modules equipped with the fiber functor defined as the restriction of the module on x . The full subcategory of *finite objects*, that is, objects with finite monodromy group-scheme or, said differently, objects which have the property that the full Tannaka subcategory which is spanned by it has a finite Tannaka group-scheme and defines a profinite k -group-scheme $\pi^{\text{ét}}(X, x)$. Since $\pi^{\text{ét}}(X, x)(\bar{k}) = \pi_1^{\text{ét}}(X, x)$ ([5, Remark 2.10]) and both $\pi^{\text{alg}}(X, x)$ and $\pi^{\text{ét}}(X, x)$ satisfy base change for finite extensions $k \subset L$ ([6, Property 2.54]), we see that the surjection $\varphi : \pi^{\text{alg}}(X, x) \rightarrow \pi^{\text{ét}}(X, x)$ is a k -form of $\varphi_{\mathbb{C}}$ for any complex embedding $k \subset \mathbb{C}$. Moreover, by definition, φ induces the profinite quotient homomorphism.

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If k is a characteristic $p > 0$ field, the category of \mathcal{O}_X -coherent \mathcal{D}_X -modules is again a k -linear abelian tensor rigid category. It is part of Katz' theorem asserting that this category is equivalent to the category of stratified \mathcal{O}_X -coherent sheaves (see [9, Theorem 1.3], [3, Theorem 8], where it is shown over $k = \bar{k}$). If $k = \bar{k}$, its Tannaka group-scheme $\pi^{\text{alg}}(X, x)$ is shown to be prosmooth in [3, Corollary 12] (strictly speaking, it is shown there only for the profinite part, but dos Santos' proof applies more generally as mentioned in [4, Corollary 7]). The homomorphism φ is then defined by the full embedding of the subcategory of objects with finite monodromy group-scheme. So by definition, φ induces the profinite quotient homomorphism.

On the other hand, if X is a reduced connected scheme over a characteristic $p > 0$ field k , endowed with a rational point $x \in X(k)$, Nori [10, Chapter II] constructed a fundamental group-scheme $\pi^N(X, x)$ as the projective system of finite k -group-schemes G for which there is a G -torsor $h : Y \rightarrow X$ under G with trivialization at x . The proétale quotient of $\pi^N(X, x)$ is precisely $\pi^{\text{ét}}(X, x)$.

Summarizing, one has a diagram

$$(1.1) \quad \begin{array}{ccc} \pi^{\text{alg}}(X, x) & \xrightarrow{\text{surj}} & \pi^{\text{ét}}(X, x) \\ & & \uparrow \text{surj} \\ & & \pi^N(X, x). \end{array}$$

The aim of our article is to define a Tannaka category $\text{Strat}(X, \infty)$ over a perfect field k , which contains the category of \mathcal{O}_X -coherent \mathcal{D}_X -modules as a full subcategory, in such a way that its Tannaka group-scheme $\pi^{\text{alg}, \infty}(X, x)$, which thus surjects onto $\pi^{\text{alg}}(X, x)$, also surjects onto $\pi^N(X, x)$. In other words, we complete (1.1) to

$$(1.2) \quad \begin{array}{ccc} \pi^{\text{alg}}(X, x) & \xrightarrow{\text{surj}} & \pi^{\text{ét}}(X, x) \\ \uparrow \text{surj} & & \uparrow \text{surj} \\ \pi^{\text{alg}, \infty}(X, x) & \xrightarrow{\text{surj}} & \pi^N(X, x). \end{array}$$

As a byproduct, we obtain a purely Tannakian geometric description of $\pi^N(X, x)$ (see Corollary 4.9). Recall that we assume that X is smooth. If in addition X is proper, Nori himself described his fundamental group-scheme $\pi^N(X, x)$ as the Tannaka group-scheme of the category of essentially finite bundles [10, Chapter I]. He extends in [10, Chapter III] his construction to non-proper curves by using parabolic bundles. Lacking desingularization in characteristic $p > 0$ makes it difficult to generalize his construction to the higher dimensional case. If k has characteristic 0, then, as already mentioned, $\pi^N(X, x) = \pi^{\text{ét}}(X, x)$ is the Tannaka group-scheme of the category of finite flat connections [6, Section 2] or, equivalently, of the category of \mathcal{O}_X -coherent \mathcal{D}_X -modules with finite monodromy group-scheme.

Our construction (see Section 3, most particularly Definition 3.2) generalizes on a smooth variety defined over a perfect characteristic $p > 0$ field k the construction of the category of flat connections (*loc. cit.*) in characteristic 0 and the construction of the stratified bundles (*loc. cit.*) in characteristic $p > 0$. We now explain the main idea.

For $i \in \mathbb{N}$, let us define inductively the relative Frobenius $F^{(i)} : X^{(i)} \rightarrow X^{(i+1)}$ over k in the usual manner. As k is assumed to be perfect, one defines $X^{(-1)} =$

$X \otimes_{k, F_k^{-1}} k$, where $F_k : \text{Spec } k \rightarrow \text{Spec } k$ is the absolute Frobenius of k , together with the relative Frobenius $F^{(-1)} : X^{(-1)} \rightarrow X^{(0)}$. Then one iterates to define inductively $F^{(i)} : X^{(i)} \rightarrow X^{(i+1)}$ for $i \in \mathbb{Z}, i < 0$. For $a, b \in \mathbb{Z}, a < b$ we define $F^{(a,b)} : X^{(a)} \xrightarrow{F^{(a)} \circ \dots \circ F^{(b-1)}} X^{(b)}$.

Recall that a stratified bundle is a sequence $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$, where $E^{(i)}$ is a bundle on $X^{(i)}$ and $\sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*} E^{(i+1)}$ is a $\mathcal{O}_{X^{(i)}}$ -isomorphism. For $t \in \mathbb{N}, t \neq 0$, we define an object of $\mathbf{Strat}(X, t)$ to be a sequence $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$, where $E^{(i)}$ is a bundle on $X^{(i)}$ and $\sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*} E^{(i+1)}$ is a $\mathcal{O}_{X^{(i)}}$ -isomorphism for all $i \geq 1$, but for $i = 0$, $\sigma_0 : F^{(-t,0)*} E^{(0)} \xrightarrow{\cong} F^{(-t,1)*} E^{(1)}$ is a $\mathcal{O}_{X^{(-t)}}$ -isomorphism. The morphisms are the ones between the bundles which respect all the structures. We show (Theorem 3.4) that the obvious functor $\mathbf{Strat}(X, t) \subset \mathbf{Strat}(X, t+1)$, which assigns $(E_i, F^{(-t-1)*} \sigma_0, \sigma_i, i \geq 1)$ to $(E_i, \sigma_0, \sigma_i, i \geq 1)$, induces a full embedding of Tannaka categories, where the fiber functor is simply the restriction of $E^{(0)}$ to the rational point x . Then $\mathbf{Strat}(X, \infty)$ is defined as the inductive limit over $t \rightarrow \infty$ of the categories $\mathbf{Strat}(X, t)$ (Corollary 3.5). In order to show that the Tannaka group-scheme $\pi^{\text{alg}, \infty}(X, x)$ of $\mathbf{Strat}(X, \infty)$ surjects onto $\pi^N(X, x)$, we use a slight modification of Nori’s reconstruction theorem [10, Chapter I, Proposition 2.9] of a torsor $h : Y \rightarrow X$ under a finite group scheme G out of the induced functor $h^\# : \text{Rep}_k(G) \rightarrow \text{Coh}(X)$ which assigns to a finite dimensional k -linear representation V of G the vector bundle on X which is defined by flat descent for h on $\mathcal{O}_Y \times_k V$ (Theorem 2.4).

This allows us to define the group-scheme homomorphism $\pi^{\text{alg}, \infty}(X, x) \rightarrow \pi^N(X, x)$ (Theorem 4.5). In order to show that this map induces the profinite quotient, we in particular use the categorical translation of injectivity and surjectivity of homomorphisms of Tannaka group-schemes ([2, Proposition 2.12]).

2. NORI’S FUNDAMENTAL GROUP-SCHEME

Let k be a field of characteristic $p > 0$ and X be a k -scheme. Let $x \in X(k)$ be a rational point and $i_x : x \rightarrow X$ be the closed embedding.

Nori [10, Chapter II] defines the category $\mathbf{N}(X, x)$ of triples $(Y \xrightarrow{f} X, G, y)$ where

- (a) G/k is a finite group scheme,
- (b) $f : Y \rightarrow X$ is a G -torsor,
- (c) y is a k -point of Y lying above x .

A morphism between two such triples $(Y_i \xrightarrow{f_i} X, G_i, y_i), i = 1, 2$, is a pair $(\phi : G_1 \rightarrow G_2, \psi : Y_1 \rightarrow Y_2)$ such that ψ is an X -morphism which is ϕ -equivariant and $\psi(y_1) = y_2$. Nori shows [10, Chapter II, Proposition 2] that if X is reduced and geometrically connected, then the projective limit $\varprojlim_{\mathbf{N}(X,x)} G$ exists. He defines

Definition 2.1. Let X be a reduced geometrically connected k -scheme. Then its Nori fundamental group-scheme is the profinite k -group-scheme

$$\pi^N(X, x) = \varprojlim_{\mathbf{N}(X,x)} G.$$

Since giving a rational point $y \in f^{-1}(x)$ is the same as giving a trivialization $f^{-1}(x) \cong_k G$, $\mathbf{N}(X, x)$ is equivalent to the category of triples $(h : Y \rightarrow X, G, f^{-1}(x) \cong_k G)$, where the morphisms between two such objects are defined by

torsor morphisms which respect the trivialization. We will not need this slightly different phrasing.

Definition 2.2. Let G be a finite k -group-scheme, and let $h : Y \rightarrow X$ be a G -torsor. Then it induces a functor $h^\# : \text{Rep}_k(G) \rightarrow \text{Coh}(X)$ which assigns to a finite dimensional k -representation V the bundle on X which comes by flat descent from $\mathcal{O}_Y \otimes_k V$.

- Properties 2.3.**
- 1) The functor $h^\#$ defined in Definition 2.2 is exact, k -linear and compatible with the tensor structure. Thus it is a *fiber functor* in the sense of Deligne [1, 1.9]. Since $\text{Rep}_k(G)$ is a Tannaka category, it follows [1, Corollaire 2.10] that $h^\#$ is faithful.
 - 2) The functor $i_x^* : \text{Coh}(X) \rightarrow \text{Vec}_k$ defined as the restriction to the rational point, with values in the category of finite dimensional k -vector spaces, is a fiber functor on the subcategory of vector bundles. The composite functor $i_x^* \circ h^\# : \text{Rep}_k(G) \rightarrow \text{Vec}_k$ is a fiber functor.
 - 3) Let $h_i : Y_i \rightarrow X$ be G_i -torsors where $i = 1, 2$. Let $\phi : G_1 \rightarrow G_2$ be a group homomorphism and $\psi : Y_1 \rightarrow Y_2$ be an equivariant map with respect to ϕ . We denote by ϕ^* the induced functor $\text{Rep}_k(G_2) \rightarrow \text{Rep}_k(G_1)$. Then one has the equality $h_2^\# = h_1^\# \circ \phi^*$ of functors. Indeed, if V is a G_2 -representation, $\psi^* : \mathcal{O}_{Y_2} \otimes_k V \rightarrow \psi_*(\mathcal{O}_{Y_1} \otimes_k \phi^*(V))$ induces a \mathcal{O}_X -linear map $h_2^\#(V) \rightarrow h_1^\#(V)$ between those two vector bundles, which, after composing with i_x^* , is the identity on V . So $h_2^\#(V) = h_1^\# \circ \phi^*(V)$.
 - 4) Let $h : Y \rightarrow X$ be a G -torsor, let $b : X' \rightarrow X$ be a morphism, and let $x' \in X'(k)$ be a rational point with $b(x') = x$. Let $Y' = Y \times_X X' \rightarrow X'$ and $h' : Y' \rightarrow X'$ denote the projection. Then one has the equality $b^* \circ h^\# = h'^\#$ of functors. Indeed, denoting by $b' : Y' \rightarrow Y$ the induced morphism, if V is a G -representation, $(b')^* : \mathcal{O}_Y \otimes_k V \rightarrow (b')_* \mathcal{O}_{Y'} \otimes_k V$ induces $\mathcal{O}_{X'}$ -linear map $b^* \circ h^\#(V) \rightarrow (h')^\#(V)$ between vector bundles, which is the identity on V after composing with $i_{x'}$. So $b^* \circ h^\# = (h')^\#$.

The following is a direct consequence of [10, Proposition 2.9].

Theorem 2.4. *Let G be a finite k -group-scheme and let $F : \text{Rep}_k(G) \rightarrow \text{Coh}(X)$ be a fiber functor such that $i_x^* \circ F$ is the forgetful functor $F_G : \text{Rep}_k(G) \rightarrow \text{Vec}_k$. Then there exists a unique object $(Y \xrightarrow{h} X, G, y)$ of $\mathbf{N}(X, x)$ such that $F = h^\#$ and $(h^{-1}(x), y) = (G, 1)$. For any other object $(Y' \xrightarrow{h'} X, G, y') \in \mathbf{N}(X, x)$ such that $F = h'^\#$, there exists a unique isomorphism in $\mathbf{N}(X, x)$ between $(Y \xrightarrow{h} X, G, y)$ and $(Y' \xrightarrow{h'} X, G, y')$.*

Proof. By Nori's reconstruction theorem [10, Proposition 2.9], $F(k[G])$, where $k[G]$ is the regular representaton of G , is a finite \mathcal{O}_X -algebra. The G -torsor $h : Y \rightarrow X$ is defined to be $\text{Spec}_X F(k[G])$. By Property 2.3 2), $i_x^* \circ F(k[G]) = F_G(k[G]) = k[G]$. Said differently, $h^{-1}(x) = \text{Spec}_x k[G] = G$. Then y is the rational point of $h^{-1}(x)$ which is $1 \in G$. By the unicity in *loc. cit.*, h is uniquely defined. If $y' = g \in h^{-1}(x)(k)$ is another rational point, then multiplication $g : Y \rightarrow Y$ by g , together with the conjugation $G \rightarrow G, h \mapsto ghg^{-1}$, defines an isomorphism $(h : Y \rightarrow X, G, y) \rightarrow (h : Y \rightarrow X, G, y')$ in $\mathbf{N}(X, x)$. \square

3. THE CATEGORY OF GENERALIZED STRATIFIED BUNDLES

The aim of this section is to define the category of *generalized stratified bundles*. We start with some notation.

Notation 3.1. Let k be a perfect field of characteristic $p > 0$ and let X be a *smooth* scheme over k which is geometrically irreducible.

For $i \in \mathbb{N}$, we define inductively the relative Frobenius $F^{(i)} : X^{(i)} \rightarrow X^{(i+1)}$ over k in the usual manner, by defining $X^{(0)} = X$, $X^{(i+1)}$ to be the fiber product of $X^{(i)} \otimes_{k, F_k} k$ over the absolute Frobenius $F_k : \text{Spec } k \rightarrow \text{Spec } k$ of k and defining $F^{(i)}$ to be the factorization of the absolute Frobenius $F_{X^{(i)}} : X^{(i)} \rightarrow X^{(i)}$ morphism.

For $i \in \mathbb{Z}, i < 0$, we define inductively $F^{(i)} : X^{(i)} \rightarrow X^{(i+1)}$ over k as follows. First we set $X^{(-1)} = X \otimes_{F_k^{-1}} k$. Then we define $F^{(-1)} : X^{(-1)} \rightarrow X$ to be the relative Frobenius. Similarly, we define $X^{(-i-1)} = X^{(-i)} \otimes_{F_k^{-1}} k$, together with the relative Frobenius $F^{(-i-1)} : X^{(-i-1)} \rightarrow X^{(-i)}$ over k .

For $a, b \in \mathbb{Z}, a < b$ we define $F^{(a,b)} : X^{(a)} \xrightarrow{F^{(a)} \circ \dots \circ F^{(b-1)}} X^{(b)}$.

Recall that a *stratified bundle* (see [9, Section 1]) is a sequence $(E^{(i)}, \sigma^{(i)}), i \in \mathbb{N}$, where $E^{(i)}$ is a \mathcal{O}_X -coherent sheaf on $X^{(i)}$ and $\sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*} E^{(i+1)}$ is a $\mathcal{O}_{X^{(i)}}$ -isomorphism. One defines the *category* $\text{Strat}(X)$ of *stratified bundles* by defining

$$\text{Hom}((D^{(i)}, \tau^{(i)}), (E^{(i)}, \sigma^{(i)}))$$

to be a set of sequences $f_i : D^{(i)} \rightarrow E^{(i)}$ of morphisms of $\mathcal{O}_{X^{(i)}}$ -coherent sheaves, which commute with all the σ_i and τ_i . It is a fact (*loc. cit.*) that if $(E^{(i)}, \sigma^{(i)}), i \in \mathbb{N}$ is a stratified sheaf, the $E^{(i)}$ are all locally free, and if $f = (f)_i, i \in \mathbb{N}$, is a morphism of stratified sheaves, then f_i are vector bundle maps (i.e. locally split), so the category is abelian, rigid, and monoidal. Moreover, the Hom-sets are finite dimensional k -vector spaces. As X is geometrically irreducible, the unit object $\mathbb{I} = (\mathcal{O}_X, \text{Id}), i \in \mathbb{N}$, fulfills $\text{End}(\mathbb{I}) = k$. If now X is endowed with a rational point $x \in X(k)$, then $\omega_x : \text{Strat}(X) \rightarrow \text{Vec}_k, (E^{(i)}, \sigma^{(i)}) \mapsto E_0|_x$ is a fiber functor in the sense of Deligne [1, 1.9] and thus yields the structure of a Tannaka category on $\text{Strat}(X)$. A fundamental property due to dos Santos is that the corresponding Tannaka k -group-scheme $\text{Aut}^\otimes(\omega_x)$ is prosmooth ([3, Corollary 12], [4, Corollary 7]).

Definition 3.2. Let $t \geq 0$ be an integer. A *t-stratified bundle* is a sequence

$$(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}),$$

where $E^{(i)}$ is a \mathcal{O}_X -coherent sheaf on $X^{(i)}$,

$$\sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*} E^{(i+1)}$$

is a $\mathcal{O}_{X^{(i)}}$ -isomorphism for $i \geq 1$, and for $i = 0$,

$$\sigma^{(0)} : F^{(-t,0)*} E^{(0)} \xrightarrow{\cong} F^{(-t,1)*} E^{(1)}$$

is a $\mathcal{O}_{X^{(-t)}}$ -isomorphism.

One defines the *category* $\text{Strat}(X, t)$ of *t-stratified bundles* by defining

$$\text{Hom}((D^{(i)}, \sigma^{(i)}), (E^{(i)}, \tau^{(i)}))$$

to be a set of sequences $f_i : D^{(i)} \rightarrow E^{(i)}$ of morphisms of \mathcal{O}_X -coherent sheaves, which commute with all the σ_i and τ_i .

In particular, $\text{Strat}(X, 0) = \text{Strat}(X)$.

Example 3.3. We now give an example of a non-trivial 1-stratified bundle on $X = \mathbb{A}_k^1 = \text{Spec}(k[[x]])$. Thus $X^{(i)} = \text{Spec}(k[[x_i]])$, where the relative Frobenius $X^{(i)} \rightarrow X^{(i+1)}$ is induced by $x_{i+1} \rightarrow x_i^p$. For simplicity let us assume $p = \text{char}(k) = 2$. Let V be a 2 dimensional vector space over k with basis e_1, e_2 . Define

$$E^{(i)} = \mathcal{O}_{X^{(i)}} \otimes_k V \quad \forall i \geq 0$$

and

$$\sigma^{(i)} : E^{(i)} \rightarrow F^{(i)*} E^{(i+1)}, \quad i \geq 1,$$

to be the isomorphism induced by the identity on V . We define

$$\sigma^{(0)} : F^{(-1,0)*} E^{(0)} \rightarrow F^{(-1,1)*} E^{(1)}$$

to be the isomorphism defined by sending

$$e_1 \rightarrow e_1, \quad e_2 \rightarrow x_{-1}e_1 + e_2.$$

We claim that the -1 -stratified bundle thus defined is not isomorphic to the trivial stratified bundle of rank 2. If indeed this were the case, then we would have a $k[x]$ -module automorphism $\phi : k[x] \otimes_k V \rightarrow k[x] \otimes_k V$ such that

$$\phi \otimes_{k[x]} k[x_{-1}] = \sigma^{(0)}.$$

This is impossible since x_{-1} is not contained in $k[x]$. It can be shown (see (4.3)) that this -1 -stratified bundle “arises” from the non-trivial α_p -torsor on \mathbb{A}_k^1 defined by the relative Frobenius of \mathbb{A}_k^1 .

Theorem 3.4. *The notation is as in Notation 3.1.*

- 1) For every integer $t \geq 0$, $\text{Strat}(X, t)$ is a k -linear, abelian, rigid, tensor category.
- 2) The functor

$$(+): \text{Strat}(X, t) \subset \text{Strat}(X, t + 1),$$

$$(E_i, \sigma_0, \sigma_i, i \geq 1) \mapsto (E_i, F^{(-t-1)*} \sigma_0, \sigma_i, i \geq 1)$$

induces a full faithful embedding of k -linear, abelian, rigid, tensor categories.

- 3) If $x \in X(k)$ is a rational point, the functor

$$\omega_x : \text{Strat}(X, t) \rightarrow \text{Vec}_k,$$

$$(E^{(i)}, \sigma^{(i)}) \mapsto E_0|_x$$

is a fiber functor, which makes $(\text{Strat}(X, t), \omega_x)$ a Tannaka category.

Proof. We show 1). Since $\text{Strat}(X, 0) = \text{Strat}(X)$, we assume $t > 0$. If $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$ is an object in $\text{Strat}(X, t)$, then $(E_+^{(i)} = E^{(i+1)}, \sigma_+^{(i)} = \sigma^{(i+1)}, i \in \mathbb{N})$ is an object $\text{Ver}(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \in \text{Strat}(X^{(1)})$. Since $E^{(i)}$ is locally free, by the isomorphism $\sigma^{(0)}$, $F^{(-t,0)*} E^{(0)}$ is locally free. Since X is smooth, the relative Frobenius is flat; thus by flat descent, $E^{(0)}$ is locally free as well. So $\text{Strat}(X)$ is rigid and monoidal. On the other hand,

$$(3.1) \quad \text{Hom}((D^{(i)}, \tau^{(i)}, i \in \mathbb{N}), (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}))$$

$$\subset \text{Hom}(\text{Ver}(D^{(i)}, \tau^{(i)}, i \in \mathbb{N}), \text{Ver}(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}))$$

and is obviously a k -vector space. So the Hom-sets are finite dimensional k -vector spaces. Moreover, any morphism $f = (f^{(i)}, i \in \mathbb{N})$ is such that $f^i, i \geq 1$, is a

morphism of vector bundles. Thus by the isomorphisms $\tau^{(0)}$ and σ^0 , Ker, Im and Coker of $f^{(0)}$ are pulled back to vector bundles on $X^{(-t)}$ via $F^{(-t,0)}$. Thus by flat descent again, there are vector bundles on X . We conclude that $\mathbf{Strat}(X, t)$ is an abelian category. This shows 1).

2) follows immediately from the factorization of (3.1) through (+).

We show 3): the point $x \in X(k)$ maps to $x^{(1)} \in X^{(1)}(k)$, and the map $x \rightarrow x^{(1)}$ is the identity on the residue fields $k(x) = k(x^{(1)}) = k$. If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence in $\mathbf{Strat}(X, t)$, then $0 \rightarrow \text{Ver}(A) \rightarrow \text{Ver}(B) \rightarrow \text{Ver}(C) \rightarrow 0$ is an exact sequence in $\mathbf{Strat}(X^{(1)})$. Thus $0 \rightarrow \omega_{x^{(1)}}(\text{Ver}(A)) \rightarrow \omega_{x^{(1)}}(\text{Ver}(B)) \rightarrow \omega_{x^{(1)}}(\text{Ver}(C)) \rightarrow 0$ is an exact sequence in Vec_k . But

$$(3.2) \quad \omega_{x^{(1)}}(\text{Ver}(A)) = \omega_x(A).$$

This shows that ω_x is exact. Furthermore, ω_x is obviously k -linear and compatible with the tensor structure. This finishes the proof. \square

Corollary 3.5. *Let the notation be as in Theorem 3.4. The category*

$$\mathbf{Strat}(X, \infty) = \varinjlim_{+, t \in \mathbb{N}} \mathbf{Strat}(X, t)$$

is a k -linear, abelian, rigid tensor category on which, if X has a rational point $x \in X(k)$, the functor ω_x is a fiber functor.

Definition 3.6. The notation is as in Theorem 3.4.

- 1) We define $\pi^{\text{alg}}(X, x)$ to be the Tannaka k -group scheme $\text{Aut}^{\otimes}(\omega_x)$ of $(\mathbf{Strat}(X), \omega_x)$.
- 2) We define $\pi^{\text{alg}, \infty}(X, x)$ to be the Tannaka k -group scheme $\text{Aut}^{\otimes}(\omega_x)$ of $(\mathbf{Strat}(X, \infty), \omega_x)$.

The functor $(+) : \mathbf{Strat}(X) \rightarrow \mathbf{Strat}(X, \infty)$ defines the homomorphism

$$(3.3) \quad (+)^* : \pi^{\text{alg}, \infty}(X, x) \rightarrow \pi^{\text{alg}}(X, x).$$

Lemma 3.7. *The homomorphism $(+)^*$ in (3.3) is faithfully flat.*

Proof. We apply [2, Proposition 2.21]. As $(+)$ is fully faithful, the lemma is equivalent to saying that if A is an object on $\mathbf{Strat}(X)$ and $B \subset (+)A$ is a subobject in $\mathbf{Strat}(X, \infty)$, then there is a subobject $B' \subset A$ in $\mathbf{Strat}(X)$ such that $B = (+)B'$. One has that $\text{Ver}(B) \subset \text{Ver}(A)$ is a subobject in $\mathbf{Strat}(X^{(1)})$. Thus $F^{(0)*}B^{(1)} \subset A^{(0)}$ is a subvector bundle with the property that $F^{(-t,0)*} \circ F^{(0)*}B^{(1)} = F^{(-t,1)*}B^{(1)} = F^{(-t,0)*}B^{(0)}$. Thus $B' = (F^{(0)*}B^{(1)}, B^{(i)}, i \geq 1, F^{(0)*}, \sigma^{(i)}, i \geq 1) \subset A$ is a subobject of A such that $(+)B' = B$. This finishes the proof. \square

4. COMPARISON OF $\pi^{\text{alg}, \infty}(X, x)$ WITH $\pi_1^N(X, x)$

In order to achieve the comparison, we start with a construction.

Construction 4.1. The notation is as in Notation 3.1, and $x \in X(k)$ is a rational point. Let $(h : Y \rightarrow X, G, y)$ be an object of $\mathbf{N}(X, x)$. Using this object, we construct a tensor functor

$$h^* : \text{Rep}_k(G) \rightarrow \mathbf{Strat}(X, \infty)$$

together with a factorization of functors

$$(4.1) \quad \begin{array}{ccc} \text{Rep}_k(G) & \xrightarrow{h^*} & \text{Strat}(X, \infty) \\ & \searrow F_G & \downarrow \omega_x \\ & & \text{Vec}_k. \end{array}$$

Here $F_G : \text{Rep}_k(G) \rightarrow \text{Vec}_k$ is the forgetful functor.

Recall that if G is a finite k -group-scheme, there is an exact sequence of finite k -group schemes $1 \rightarrow G_0 \rightarrow G \rightarrow G_{\text{ét}} \rightarrow 1$, where G_0 is the 1-component of G and $G_{\text{ét}}$ is étale. Furthermore, as k is perfect, $G_{\text{red}} \subset G$ is a closed subgroup-scheme and the composite $G_{\text{red}} \xrightarrow{\iota} G \rightarrow G_{\text{ét}}$ is an isomorphism. Thus ι yields on G the structure of a semi-direct product of $G_{\text{ét}}$ by G_0 . The construction of h^* will be such that the image of h^* is contained in $\text{Strat}(X, t)$, where t is a natural number such that the image of the k -group-scheme homomorphism $G^{(-t)} \rightarrow G$ is equal to $G_{\text{ét}}$.

Let V be a finite dimensional k -representation of G . We set

$$(4.2) \quad E^{(0)} = h^\#(V).$$

For $i \in \mathbb{N} \setminus \{0\}$, the relative Frobenius is an isomorphism of the étale k -group-schemes

$$(4.3) \quad F^{(0,i)} : G_{\text{ét}} \xrightarrow{\cong} G_{\text{ét}}^{(i)}.$$

Thus $\iota(G) \circ F^{(0,i)-1} : G_{\text{ét}}^{(i)} \subset G$ is a closed embedding, and composing with it defines a $G_{\text{ét}}^{(i)}$ -action on V . Since $h : Y \rightarrow X$ is a G -torsor, for $i \geq 0$, $h^{(i)} : Y^{(i)} \rightarrow X^{(i)}$ is also a $G^{(i)}$ -torsor. Let $h_{\text{ét}}^{(i)} : Y_{\text{ét}}^{(i)} \rightarrow X^{(i)}$ be the induced $G_{\text{ét}}^{(i)}$ -torsor obtained by moding out by $G_0^{(i)}$. We define

$$(4.4) \quad E^{(i)} = (h_{\text{ét}}^{(i)})^\#(V).$$

One has

$$(4.5) \quad \sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*} E^{(i+1)}, \quad i \in \mathbb{N} \setminus \{0\}.$$

The object $h^*(V) \in \text{Strat}(X, t)$ which we wish to construct will have the property

$$(4.6) \quad \text{Ver}(h^*(V)) = (E^{(i)}, \sigma^{(i)}, i \geq 1).$$

It remains to define $\sigma^{(0)}$. By definition,

$$(4.7) \quad F^{(0)*} E^{(1)} = (h_{\text{ét}}^{(0)})^\#(V) = (h_{\text{ét}})^\#(V).$$

Let t be a natural number such that the image of $G^{(-t)} \rightarrow G$ is equal to $G_{\text{ét}}$. One has the following commutative diagram of k -varieties:

$$(4.8) \quad \begin{array}{ccccc} Y^{(-t)} & \xrightarrow{F^{(-t,0)}} & Y & & \\ & \searrow & \exists! \lambda \nearrow & & \\ & & Y_{\text{ét}}^{(-t)} & \xrightarrow{F^{(-t,0)}} & Y_{\text{ét}} \\ & \searrow h^{(-t)} & & & \downarrow h \\ X^{(-t)} & \xrightarrow{F^{(-t,0)}} & X & & \end{array}$$

The morphism $F^{(-t,0)} : Y^{(-t)} \rightarrow Y$ is equivariant under $F^{(-t,0)} : G^{(-t)} \rightarrow G$. Likewise, the morphism $F^{(-t,0)} : Y_{\text{ét}}^{(-t)} \rightarrow Y_{\text{ét}}$ is equivariant under $F^{(-t,0)} : G_{\text{ét}}^{(-t)} \rightarrow G_{\text{ét}}$. The commutativity of the diagram implies that

$$(4.9) \quad \lambda^*(\mathcal{O}_Y \otimes_k V) = F^{(-t,0)*}(\mathcal{O}_{Y_{\text{ét}}} \otimes_k V) = F^{(-t,1)*}(\mathcal{O}_{Y_{\text{ét}}^{(1)}} \otimes_k V)$$

equivariantly for the action of $G_{\text{ét}}^{(-t)}$. Thus

$$(4.10) \quad (h_{\text{ét}}^{(-t)})\#(V) = F^{(-t,0)*}E^{(0)} = F^{(-t,1)*}E^{(1)}.$$

We define $\sigma^{(0)} : F^{(-t,0)*}E^{(0)} = F^{(-t,1)*}E^{(1)}$ to be the equality of (4.10).

Thus, starting with $V \in \text{Rep}_k(G)$, we have constructed an object $h^*(V) = (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \in \text{Strat}(X, t)$. Clearly, any $\phi \in \text{Hom}_{\text{Rep}_k(G)}(V, W)$ induces $h^*(\phi) \in \text{Hom}_{\text{Strat}(X,t)}(h^*(V), h^*(W))$. This defines the functor

$$(4.11) \quad h^* : \text{Rep}_k(G) \rightarrow \text{Strat}(X, \infty)$$

by composing with (+). Moreover, one has

$$(4.12) \quad h^*(V)_x = (\mathcal{O}_Y \otimes_k V)_y = V.$$

This shows the commutativity of (4.1).

Remark 4.2. In the above construction we use the fact that for a finite flat group scheme G over a perfect field k , the epimorphism $G \rightarrow G_{\text{ét}}$ admits a section (necessarily unique). In other words $G_{\text{ét}}$ can be canonically thought of as a subgroup scheme of G via the identification $G_{\text{red}} = G_{\text{ét}}$. When k is not a perfect field, G_{red} may not be a subgroup scheme (for example, $G = \text{Spec } k[t]/(t^{p^2} - at^p)$, $a \in k \setminus k^p$; see [8, Chapter III, Exercice (3.2)]) and the above construction of h^* does not make sense. This is the reason why we assume throughout that k be perfect. We thank Nguyễn Duy Tân for this important remark.

Example 4.3. Let $p = \text{char}(k) = 2$ for simplicity and let $G = \alpha_2 = \text{Spec}(k[t]/t^2)$. Let $X = \mathbb{A}_k^1 = \text{Spec}(k[x])$. Let $P = \text{Spec}(k[u])$ and $h : P \rightarrow X$ be the relative Frobenius defined by $x \rightarrow u^2$. Then h is a G -torsor. Thus by Construction 4.1, one has a functor

$$h^* : \text{Rep}_k(G) \rightarrow \text{Strat}(X, -1).$$

We now compute that $h^*(k[G])$ is nothing but the -1 -stratified bundle defined in Example 3.3. Here $k[G] = k[v]/(v^2)$ is the regular representation of G . As in Example (3.3), let $X^{(i)} = k[x_i]$. Let $h^*(k[G]) = (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$. As all schemes are affine, we confuse coherent sheaves with corresponding modules. Since $G_{\text{ét}}$ is trivial, by definition of h^* we see that

$$E^{(i)} = k[x_i] \otimes_k k[v]/(v^2) \quad \forall i \geq 1$$

with

$$\sigma^{(i)} : E^{(i)} \rightarrow F^{(i)*}E^{(i+1)}, \quad i \geq 1,$$

induced by the identity map on $k[v]/(v^2)$. Then $E^{(0)}$ is by definition the $k[x]$ -module of invariants of $k[u] \otimes_k k[v]/(v^2)$, where the action of $G = \text{Spec } k[t]/(t^2)$ is defined by

$$u \rightarrow u + t, \quad v \rightarrow v + t.$$

Since $(u + v)^2 = u^2 = x$, one has $E^{(0)} = k[x] \cdot 1 \oplus k[x] \cdot (u + v)$. On P we have an identification

$$h^*E^{(0)} = k[u] \otimes_k k[v]/(v^2)$$

defined by $\tau : 1 \mapsto 1 \otimes 1, u + v \mapsto u \otimes 1 + 1 \otimes v$. The map $\sigma^{(0)}$ is nothing but the pull back of τ via the isomorphism $X^{(-1)} \rightarrow P$ defined by

$$k[u] \rightarrow k[x_{-1}], \quad u \mapsto x_{-1}.$$

We thus see that

$$\sigma^{(0)} : k[x_{-1}] \cdot 1 \oplus k[x_1] \cdot (u + v) \longrightarrow k[x_{-1}] \otimes k[v]/(v^2)$$

is defined by $1 \mapsto 1 \otimes 1, (u + v) \mapsto u \otimes 1 + 1 \otimes v$. It is then an elementary exercise to see that the stratified bundle $h^*(k[G])$ is isomorphic to the -1 -stratified bundle defined in Example 3.3.

Lemma 4.4. *The functor h^* defined in (4.11) is k -linear, exact, compatible with the tensor structure and faithful.*

Proof. As already recalled in the Properties 2.3 1), faithfulness follows from the remaining properties. On the other hand, k -linearity and compatibility with the tensor structures are straightforward. Exactness is proven by using Ver as in Theorem 3.4 3). Indeed, $\text{Ver} \circ h^*$ with values in $\text{Strat}(X^{(1)})$ is obviously exact, while a sequence in $\text{Strat}(X, \infty)$ is exact if and only if it remains exact after applying Ver . \square

If $(h_i : Y_i \rightarrow X, G_i, y_i)$ are objects in $\mathbf{N}(X, x)$ for $i = 1, 2$ and $(\psi : Y_1 \rightarrow Y_2, \phi : G_1 \rightarrow G_2, y_1 \rightarrow y_2)$ is a morphism in $\mathbf{N}(X, x)$, then Property 2.3 3) implies that $h_2^* = h_1^* \circ \phi^*$. On the other hand, the projective system of ϕ in $\mathbf{N}(X, x)$ induces an inductive system $\varinjlim_{\mathbf{N}(X, x), \phi^*} \text{Rep}_k(G)$ which is a Tannaka category, with the forgetful functor F_G as the fiber functor. The Tannaka k -group-scheme $\text{Aut}^\otimes(F_G)$ is simply $\varprojlim_{\mathbf{N}(X, x), \phi} G$, which is precisely Nori’s fundamental group-scheme $\pi^N(X, x)$. As in addition the construction is obviously functorial in h , we conclude:

Theorem 4.5. *Let the notation be as in Construction 4.1. The functor h^* defined in (4.11) for one object $(h : Y \rightarrow X, G, y)$ of $\mathbf{N}(X, x)$ induces a functor of Tannakian categories*

$$\mathfrak{h}^* : \left(\varinjlim_{\mathbf{N}(X, x), \phi^*} \text{Rep}_k(G), F_G \right) \rightarrow \left(\text{Strat}(X, \infty), \omega_x \right)$$

and the Tannaka-dual homomorphism of k -group-schemes

$$\mathfrak{h}^{*\vee} : \pi^{\text{alg}, \infty}(X, x) \rightarrow \pi^N(X, x),$$

which is functorial in X .

The aim of the rest of the section is to show that the homomorphism $\mathfrak{h}^{*\vee}$ is faithfully flat and induces the profinite quotient homomorphism.

Proposition 4.6. *Let $(Y \xrightarrow{h} X, G, y)$ be an object of $\mathbf{N}(X, x)$. The following conditions are equivalent:*

- 1) *The induced map $\pi^{\text{alg}, \infty}(X, x) \rightarrow G$ (see (4.11)) is an epimorphism.*
- 2) *The induced map $\pi^N(X, x) \rightarrow G$ is an epimorphism.*
- 3) *The functor h^* in (4.11) is fully faithful, and its image is closed under taking subquotients in $\text{Strat}(X, \infty)$.*

Proof. The equivalence (1) \Leftrightarrow (3) follows from [2, Proposition 2.21]. Moreover, since by construction the map $\pi^{\text{alg},\infty}(X, x) \rightarrow G$ factors through $\pi^N(X, x)$, (1) \Rightarrow (2) is obvious.

We show (2) \Rightarrow (3). Let \mathcal{C} denote the full subcategory of $\text{Strat}(X, \infty)$ generated by subquotients in $\text{Strat}(X, \infty)$ of objects which are in the image of $h^* : \text{Rep}_k(G) \rightarrow \text{Strat}(X, \infty)$. Property 3) is equivalent to saying that $h^* : \text{Rep}_k(G) \rightarrow \mathcal{C}$ is an equivalence of categories. By standard Tannaka formalism, \mathcal{C} itself is a k -linear, abelian, rigid tensor subcategory of $\text{Strat}(X, \infty)$. Thus (\mathcal{C}, ρ_x) is a Tannaka subcategory of $(\text{Strat}(X, \infty), \omega_x)$, where $\rho_x = \omega_x|_{\mathcal{C}}$.

We now show that $h^* : \text{Rep}_k(G) \rightarrow \mathcal{C}$ is an equivalence of categories. Let $H = \text{Aut}(\rho_x)$ be the Tannaka k -group-scheme of (\mathcal{C}, ρ_x) . We claim that the induced homomorphism $H \rightarrow G$ is a closed immersion. This is equivalent ([2, Proposition 2.21]) to saying that every object of \mathcal{C} is a subquotient in \mathcal{C} of an object in $h^*(\text{Rep}_k(G))$, which is true since by definition of \mathcal{C} a subquotient in \mathcal{C} of objects in $h^*(\text{Rep}_k(G))$ is the same as a subquotient in $\text{Strat}(X, \infty)$ of objects in $h^*(\text{Rep}_k(G))$. We conclude in particular that H is a finite group-scheme.

The fiber functor (in the sense of Deligne [1, 1.9]; see Properties 2.3 1)) $\omega_X : \text{Strat}(X, \infty) \rightarrow \text{Coh}(X)$ defined by $(E_i, \sigma_i, i \in \mathbb{N}) \mapsto E_0$ restricts to the fiber functor $\rho_X : \mathcal{C} \rightarrow \text{Coh}(X)$. One has a commutative diagram of functors

$$(4.13) \quad \begin{array}{ccc} \text{Rep}_k(G) & \xrightarrow{h^*} & \mathcal{C} \\ & \searrow h^\# & \downarrow \rho_x \\ & & \text{Coh}(X) \end{array}$$

and, upon applying i_x , (4.1) implies that $i_x \circ h^\# = F_G$. By applying Theorem 2.4, we obtain a morphism

$$(4.14) \quad (h_H : Y_H \rightarrow X, H, y_H) \rightarrow (h : Y \rightarrow X, G, y)$$

in $\mathbf{N}(X, x)$. This in turn induces a factorization of $\pi^N(X, x) \rightarrow G$ as

$$(4.15) \quad \begin{array}{ccc} \pi^N(X, x) & \longrightarrow & G \\ \downarrow & \nearrow & \\ H & & \end{array}$$

But $\pi^N(X, x) \rightarrow G$ is assumed to be an epimorphism. Thus $H \rightarrow G$ must be an epimorphism. Since it is also a closed immersion, we conclude that

$$(4.16) \quad H \xrightarrow{\cong} G.$$

In other words,

$$(4.17) \quad h^* : \text{Rep}_k(G) \xrightarrow{\cong} \mathcal{C}.$$

This finishes the proof. □

Recall that k is perfect.

Lemma 4.7. *Let G be a finite k -group-scheme, and let $h : Y \rightarrow X$ be a G -torsor. Then the following conditions are equivalent:*

- (i) *h admits a reduction (necessarily unique) of structure group to $G_{\text{red}} = G_{\text{ét}} \subset G$.*

(ii) For every natural number n , there is a G -torsor $h_n : Y_n \rightarrow X^{(n)}$ which pulls back via $X \xrightarrow{F^{(0,n)}} X^{(n)}$ to h .

Proof. We show (i) \Rightarrow (ii). Let $h_{\acute{e}t} : Y_{\acute{e}t} \rightarrow X$ be a $G_{\acute{e}t}$ -torsor which is a reduction of structure of h for the closed embedding $G_{\acute{e}t} \subset G$. Thus $Y = Y_{\acute{e}t} \times_{G_{\acute{e}t}} G$. The isomorphism (4.3) induces a cartesian diagram

$$(4.18) \quad \begin{array}{ccc} Y_{\acute{e}t} & \xrightarrow{F^{(0,n)}} & (Y_{\acute{e}t})^{(n)} \\ h_{\acute{e}t} \downarrow & \square & \downarrow (h_{\acute{e}t})^{(n)} \\ X & \xrightarrow{F^{(0,n)}} & X^{(n)} \end{array}$$

We set $Y_n = (Y_{\acute{e}t})^{(n)} \times_{G_{\acute{e}t}} G$ and $h_n = (h_{\acute{e}t})^{(n)} \times_{G_{\acute{e}t}} G$.

We show (ii) \Rightarrow (i). For a large enough positive integer n , we consider the commutative diagram similar to (4.8):

$$(4.19) \quad \begin{array}{ccc} Y_n^{(-n)} & \xrightarrow{\gamma} & Y_n \\ & \searrow & \downarrow h_n \\ & & (Y_n^{(-n)})_{\acute{e}t} \\ & \swarrow & \downarrow \\ X & \xrightarrow{\quad} & X^{(n)} \end{array}$$

$\exists !$ (dashed arrow from $(Y_n^{(-n)})_{\acute{e}t}$ to Y_n)

We explain the terms in the diagram: with Notation 3.1, one has $Y_n^{(-n)} = Y_n \otimes_{F_k^{-n}} k$. Thus h_n induces $h_n \otimes_{F_k^{-n}} k : Y_n^{(-n)} \rightarrow (X^{(n)})^{(-n)} = X$, which is a principal $G^{(-n)}$ bundle. The top horizontal map γ is equivariant with respect to $G^{(-n)} \xrightarrow{F^{(-n,0)}} G$. Since n is large, the image of $G^{(-n)} \rightarrow G$ is precisely $G_{\acute{e}t} \subset G$. Therefore, γ factors uniquely through $(Y_n^{(-n)})_{\acute{e}t}$. Via the identification $G_{\acute{e}t}^{(-n)} \xrightarrow{F^{(-n,0)}} G_{\acute{e}t}$, the morphism $(Y_n^{(-n)})_{\acute{e}t} \rightarrow X^{(n)}$ is a $G_{\acute{e}t}$ -torsor. The above commutative diagram shows the existence of an equivariant map $(Y_n^{(-n)})_{\acute{e}t} \rightarrow Y_n \times_{X^{(n)}} X$. We conclude that the G -torsor $Y_n \times_{X^{(n)}} X \rightarrow X$ has a reduction of structure group to $G_{\acute{e}t}$. \square

Theorem 4.8. *Let the notation be as in Notation 3.1 and let $x \in X(k)$ be a rational point. Then the homomorphism $\mathfrak{h}^{*\vee} : \pi^{\text{alg},\infty}(X, x) \rightarrow \pi^N(X, x)$ is the profinite quotient homomorphism.*

Proof. We have already shown in Proposition 4.6 that the homomorphism $\mathfrak{h}^{*\vee}$ is surjective. In order to show that $\mathfrak{h}^{*\vee}$ is the profinite completion homomorphism, we need to show that any epimorphism

$$\phi : \pi^{\text{alg},\infty}(X, x) \rightarrow G,$$

where G is a k -finite group-scheme, factors through $\pi^N(X, x)$. This is equivalent to showing that given any finite Tannaka subcategory $\mathcal{T} \subset \text{Strat}(X, \infty)$ (i.e. with $G = \text{Aut}^{\otimes}(\mathcal{T}, \rho_x)$ finite), where $\rho_x = \omega_x|_{\mathcal{T}}$, there exists an object $(h : Y \rightarrow X, G, y)$

in $\mathbf{N}(X, x)$ such that \mathcal{T} is the image of the functor h^* constructed in (4.11). We do this in two steps.

Step 1. For each $n \geq 0$, we consider the fiber functor

$$(4.20) \quad \omega_{X^{(n)}} : \mathbf{Strat}(X, \infty) \rightarrow \mathbf{Coh}(X^{(n)}), \quad (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \mapsto E^{(n)}.$$

It restricts to a fiber functor

$$P_n : \mathcal{T} \rightarrow \mathbf{Coh}(X^{(n)}).$$

Let $\delta : \mathbf{Rep}_k(G) \rightarrow \mathcal{T}$ be the equivalence of Tannaka categories defined by the inverse functor to the equivalence induced by ρ_x . Consider

$$P_n \circ \delta : \mathbf{Rep}_k(G) \rightarrow \mathbf{Coh}(X^{(n)}).$$

By Theorem 2.4, we obtain G -torsors $(h_n : Y_n \rightarrow X^{(n)})$ for every n such that

$$(4.21) \quad h_n^\# = P_n \circ \delta.$$

Since the G -torsors thus obtained are unique up to isomorphism, the equality

$$P_n = F^{(n)*} \circ P_{n+1}, \quad \forall n \geq 1,$$

implies that the torsor h_{n+1} pulls back to h_n . Thus by Lemma 4.7, each Y_n admits a reduction of structure group to $G_{\text{ét}} \subset G$ for all $n \geq 1$.

Step 2. Composing δ with the inclusion $\mathcal{T} \hookrightarrow \mathbf{Strat}(X, \infty)$ we obtain a functor from $\mathbf{Rep}_k(G) \rightarrow \mathbf{Strat}(X, \infty)$. We also have the functor $h_0^* : \mathbf{Rep}_k(G) \rightarrow \mathbf{Strat}(X, \infty)$ (see (4.11)) defined by the G -torsor $h_0 : Y_0 \rightarrow X$. In order to finish the proof we have to show that these two functors coincide. This is equivalent to saying that the following diagram of functors commutes:

$$(4.22) \quad \begin{array}{ccc} \mathbf{Rep}_k(G) & \xrightarrow{\delta} & \mathcal{T} \\ & \searrow h_0^* & \downarrow \text{incl.} \\ & & \mathbf{Strat}(X, \infty). \end{array}$$

Let V be an object of $\mathbf{Rep}_k(G)$. We will show that there is an isomorphism between $i(V)$ and $h_0^*(V)$, which is functorial in V . This will finish the proof. Let $\delta(V) = (\delta(V)^{(n)}, \sigma^{(n)}, n \in \mathbb{N})$ and $h_0^*(V) = (E^{(n)}, \tau^{(n)}, n \in \mathbb{N})$.

We let $h_{n,\text{ét}} : Y_{n,\text{ét}} \rightarrow X^{(n)}$ be the $G_{\text{ét}}$ -torsor induced by h_n for $n \geq 1$. Note that by Construction 4.1 of the functor h_0^* , one has

$$(4.23) \quad E^{(n)} = h_{n,\text{ét}}^\#(V) \quad \forall n \geq 1 \quad \text{and} \quad E^{(0)} = h_0^\#(V).$$

On the other hand, by definition of the functors P_n ,

$$P_n(i(V)) = i(V)^{(n)}.$$

Thus by (4.21), one has

$$(4.24) \quad i(V)^{(n)} = h_n^\#(V) \quad \forall n \geq 0.$$

But as explained before, for every $n \geq 1$, $h_n : Y_n \rightarrow X^{(n)}$ admits a reduction of structure group to $G_{\text{ét}}$. Thus by Proposition 2.3 (3),

$$(4.25) \quad h_n^\#(V) = h_{n,\text{ét}}^\#(V) \quad \forall n \geq 1.$$

Thus we conclude that

$$(4.26) \quad i(V) = h_0^*(V). \quad \square$$

If \mathcal{T} is any k -linear, abelian, rigid tensor category, together with a neutral fiber functor $\omega : \mathcal{T} \rightarrow \text{Vec}_k$, we denote by \mathcal{T}^{fin} the full subcategory spanned by objects E which have the property that the full tensor subcategory $\langle E \rangle \subset \mathcal{T}$ spanned by E and its dual E^\vee has a finite Tannaka group scheme $\text{Aut}^\otimes(\langle E \rangle, \omega|_{\langle E \rangle})$. So by construction, Theorem 4.8 has the following consequence:

Corollary 4.9. *With the notation as in Theorem 4.8, the full embedding*

$$\text{Strat}(X, x)^{\text{fin}} \subset \text{Strat}(X, x)$$

induces via the fiber functor ω_x the quotient homomorphism

$$\pi^{\text{alg}, \infty}(X, _) \rightarrow \pi^N(X, x).$$

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