TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 364, Number 11, November 2012, Pages 5693–5713
S 0002-9947(2012)05583-8
Article electronically published on June 12, 2012

WAVE EQUATION WITH DAMPING AFFECTING ONLY A SUBSET OF STATIC WENTZELL BOUNDARY IS UNIFORMLY STABLE

MARCELO M. CAVALCANTI, IRENA LASIECKA, AND DANIEL TOUNDYKOV

Abstract. A stabilization/observability estimate and asymptotic energy decay rates are derived for a wave equation with nonlinear damping in a portion of the interior and Wentzell condition on the boundary: $\partial_\nu u + u = \Delta_T u$. The dissipation does not affect a full collar of the boundary, thus leaving out a portion subjected to the high-order Wentzell condition.

Observability of wave equations with damping supported away from the *Neumann* boundary is known to be intrinsically more difficult than the corresponding Dirichlet problem because the uniform Lopatinskii condition is not satisfied by such a system. In the case of a Wentzell boundary, the situation is more difficult since the "natural" energy now includes the H^1 Sobolev norm of the solution on the boundary. To establish uniform stability it is necessary not only to overcome the presence of the Neumann boundary operator, but also to establish an inverse-type coercivity estimate on the H^1 trace norm of the solution. This goal is attained by constructing multipliers based on a refinement of *nonradial* vector fields employed for "unobserved" Neumann conditions. These multipliers, along with a suitable geometry (local convexity), allow reconstruction of the high-order part of the potential energy from the damping that is supported only in a far-off region of the domain.

1. Introduction

The stability of the dynamics generated by hyperbolic PDEs critically depends on the location of the controls and the geometry of the underlying manifold (see e.g. a comprehensive overview in [GLLT04]). A feedback mechanism imposed on an entire system is not practical, as one strives to minimize the direct interference; more optimal design requires verifying whether "restricted" actuators can affect the state of the system on its portions where controls are absent. The case when controls leave out a subset of the domain *along with a portion of its boundary* are more difficult, especially in the presence of higher-order boundary conditions. Of current interest is a semilinear wave model with Wentzell ("Ventcel" or "Venttsel")-Robin boundary conditions. The order of (tangential) derivatives on the boundary

Received by the editors July 31, 2010.

2010 *Mathematics Subject Classification.* Primary 35L05; Secondary 93B07, 93D15.

Key words and phrases. Wentzell, Ventcel, Venttsel, wave equation, nonlinear damping, localized damping, energy decay, multipliers.

The research of the first author was partially supported by the CNPq under Grant 300631/2003-0.

The research of the second author was partially supported by the National Science Foundation under Grant DMS-0606682 and by AFOSR Grant FA9550-09-1-0459.

The research of the third author was partially supported by the National Science Foundation under Grant DMS-0908270.

matches the order of the principal operator:

$$\tag{1.1} \left\{ \begin{array}{rcll} \Box u + a(x) g(u_t) & = & 0 & \text{in } \;]0,T[\times\Omega, \\ \partial_\nu u + u & = & \Delta_T u & \text{on } \;]0,T[\times\Gamma, \\ u(0,x) = u_0(x), u_t(0,x) & = & u_1(x) & \text{in } \;\Omega\,. \end{array} \right.$$

Here Ω denotes a bounded class-C^2 domain in $\mathbb{R}^3$ with boundary Γ; Δ_T is the Laplace-Beltrami operator on Γ. The feedback map g is continuous monotone increasing, $g(0) = 0$, and $a(x) \in L^\infty(\Omega)$ is a nonnegative cutoff function restricting the effect of g to some subset of the domain.

1.1. **Previous work and new challenges.** Boundary conditions of the type imposed in (1.1) were studied in 1959 by A. D. Wentzell as the most general type of admissible boundary condition for a second-order elliptic PDE [Ven59]. In diffusion problems, Wentzell conditions model heat transfer between the environment and a solid whose boundary is coated with a layer of high conductivity [SA76, pp. 169–170]. In the context of dynamic elasticity such conditions model a high-rigidity layer coating the boundary [Lem87]. Many results have emerged on semigroup generation by elliptic operators with Wentzell conditions, and associated parabolic problems. To this end see, for instance [War06, War10] and [CGG08, CGG09], which incorporate some of the latest results and overview preceding developments.

Focusing on hyperbolic problems one must mention dynamic Wentzell boundary conditions when the interior dynamics is coupled to another wave that propagates through the boundary, for instance:

$$\tag{1.2} y_{tt} - \Delta_T y + \frac{\partial u}{\partial \nu} + ky = F \quad \text{on} \quad]0,T[\times\Gamma.$$

Such models for wave equations and elastodynamics were addressed in [Hem00]. Subsequently [Hem01] proved boundary controllability of the linear elasticity model with two controls: in both the tangential and normal components. Stability of wave equations with dynamic Wentzell conditions, subject to both localized interior damping *and* full boundary damping (e.g. $F = -y_t$ in (1.2)), was established in [KM04]. Later these results were extended to the variable-coefficient case [CKM07]. For transmission problems for wave equations with dynamic Wentzell condition on the interface, see [KZ06] and the references therein. Dynamic boundary conditions of Wentzell-type lead to a hybrid-cascade formulation of the resulting evolution where the equation on the boundary becomes an independent evolution of the process. This leads to a natural formulation as a system of evolutionary PDEs [LM88].

Instead, static Wentzell-type conditions, as in (1.1), cannot be incorporated into an independent evolution. The second-order differential operator (the Laplace-Beltrami operator) is a genuine unbounded perturbation of the boundary operator, hence falls beyond the scope of the classical elliptic theory. In addition, the stability properties of the corresponding models are more intricate (as explained below) requiring new approaches and techniques. For this reason this class of problems is of particular interest from both the mathematical and applied points of view.

Without interior damping ($a(x) \equiv 0$), even linear full-boundary dissipation

$$\left\{ \begin{array}{rcl} \Box u & = & 0 \\ \partial_\nu u + u + u_t & = & \Delta_T u \end{array} \right.$$

does not guarantee exponential stability. In [Hem00], the author shows that elasticity and wave models with static Wentzell conditions are strongly stable; however, at least in the case of the wave equation, *not uniformly exponentially stable*. In pursuit of a minimal region occupied by a stabilizing feedback, we are, therefore, prompted to localize to patches of the interior, as in (1.1).

Localized interior damping (1.1) was considered (still under dynamic boundary conditions) in [KM08], without restrictions on the geometry of the domain, but with the feedback effective on a *full collar* of the boundary. Later [CDCFT09] considered a mixed Dirichlet-(static) Wentzell boundary value problem for a wave equation where the Dirichlet boundary did not possess a collar affected by a dissipative feedback; however, a neighborhood of the Wentzell boundary was still fully covered by the damping. In such a configuration the damping placed on a neighborhood of the Wentzel boundary provides a sufficient dissipative force to control and reconstruct the high-order part of potential energy arising from the boundary dynamics. The question whether interior damping as in (1.1) supported *away from a segment of static Wentzell boundary* ensures uniform stability has been, to our knowledge, open. On the other hand, any result allowing reduction of damping's support is of central interest in technological applications.

The possibility of removing damping from star-shaped boundary segments subject to Dirichlet conditions was observed in [QR77] and followed later in [Lag89, LL88, Kom94] and many works cited therein. The technique goes back to radial vector fields introduced for waves on exterior domains ([LMP62] and [Str]).

However, hyperbolic problems with Neumann conditions proved intrinsically more difficult due to the failure of the Kreiss-Sakamoto (the uniform Lopatinskii condition), which Dirichlet problems possessed. Now damping had to control the "boundary Lagrangian", the trace of $u_t^2 - |\nabla u|^2$, which was neither bounded by finite energy, nor had a dissipative sign. Thus, the well-known "radial" multipliers in the literature had no chance to succeed. The resolution of this problem was proved possible for domains with convexity-like properties, via constructions of special *nonradial* vector fields and corresponding weighted multipliers. Such fields were first introduced (with impetus from [Tat]) in observability estimates of [LTZ00, Appendix A], and subsequently applied to stabilization of nonlinear wave equations in [LT06, Tou07] and the theory of attractors [BT10]. See also [IY00] for related work on inverse problems.

Wentzell boundary conditions retain the difficulties of Neumann ones, but at the same time increase the energy level on the boundary, so the damping placed away from a portion of Γ must be capable of controlling the "heavy" part of the potential component while acting from far away. Specifically, propagation of the dissipation enabled by local convexity should also reconstruct all tangential derivatives of the solution on that segment. We note that in Neumann problems the geometric restrictions suitable for localized interior damping sufficed for localized boundary damping [LT06, DLT09]; given this similarity, the insufficiency of the full-boundary dissipation, as discovered in [Hem00], does not leave much hope that localized interior damping in Wentzell models provides uniform stability.

This latter issue prompted the current article. Via analysis of nonradial multipliers, we establish that damping can be propagated to control the Wentzell boundary, without any additional restrictions on geometry beyond those employed for waves with *unobserved portion of Neumann boundary*. A particular consequence is that for

$g(u_t) = u_t$ the system (1.1) is exponentially stable when the uncontrolled segment of the boundary satisfies suitable geometric (convexity-type) conditions. The method introduced in this article, in addition to being able to handle the undamped part of the Wentzell boundary, also provides a much streamlined and simplified treatment of the result given in [CDCFT09] when the full collar around the Wentzell boundary is dissipated. It is thus hoped that the method developed here is "transcendental" and may be applied to other models where Wentzell boundary conditions are prescribed.

1.2. **Notation.** The forms $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ will denote respectively the $L^2(\Omega)$ and $L^2(\Gamma)$ inner products. We let $(a) = (a, 1)$ and $\langle a \rangle = \langle a, 1 \rangle$, i.e. just integration over Ω or over Γ. Doubling of parentheses indicates an additional integration in time, e.g. $\langle\!\langle a, b \rangle\!\rangle = \int_0^T \int_\Gamma a \cdot b \, dx$, etc.

Given the above product notation, the norm in $L^2(\Omega)$ will be denoted either by $\|u\|_\Omega$ or $\sqrt{(u^2)}$. Analogously $\|u\|_\Gamma$ will stand for the $L^2(\Gamma)$ norm of u; the same symbol could be replaced with $\sqrt{\langle u^2 \rangle}$. The notation $\| \cdot \|_{1,X}$ will indicate the Sobolev space norm in $H^1(X)$.

The relation $a(s) \lesssim b(s)$ indicates that $a \leq Cb$ for some constant $C > 0$, independent of s, but possibly dependent on: (i) the properties of the domain Ω (including all associated Sobolev constants, various smooth cutoff functions and vector fields constructed on this domain or its boundary Γ), (ii) the feedback map $g(s)$, and (iii) the cutoff function $a(x)$. When $a(s) \lesssim b(s)$ and $b(s) \lesssim C_2 a(s)$ we will write $a \sim b$.

Placing a circle over a relation, e.g. $\overset{\circ}{=}$, $\overset{\circ}{\leq}$, or $\overset{\circ}{\lesssim}$, indicates that the estimate holds up to a perturbation by so-called "lower-order" terms. This notation will be explained in Section 6.2.

1.3. **Well-posedness.** The well-posedness of the system (1.1) can be shown via monotone operator theory. Let

$$V := H^1(\Omega) \oplus H^1(\Gamma)$$

denote the subset of $H^1(\Omega)$ functions whose traces belong to $H^1(\Gamma)$. Define the (weak) Wentzell-Robin Laplacian $A : V \to V'$ via

$$(Af)(g) = (\nabla f \cdot \nabla g) + \langle f, g \rangle + \langle \nabla_T f \cdot \nabla_T g \rangle.$$

The Nemytski operator $B : v \hookrightarrow a(\cdot) g(v)$ is interpreted as the subgradient of a convex functional, $v \mapsto \int_\Omega \int_0^{v(x)} a(x) g(s) ds dx$, and the generator $\mathbb{A}$ of the flow is defined on the finite-energy space

$$\mathscr{H} := V \times L^2(\Omega), \tag{1.3}$$

$$\mathscr{D}(\mathbb{A}) = \left\{ \{u, v\} \in \mathscr{H} : a(x) g(v) \in V', Au + Bv \in L^2(\Omega) \right\}. \tag{1.4}$$

The square of the norm in $\mathscr{H}$ is equivalent to the quadratic energy functional:

$$E(t) = E(u(t), u_t(t)) := \tfrac{1}{2} \|\nabla u\|_\Omega^2 + \tfrac{1}{2} \|\nabla u\|_\Gamma^2 + \tfrac{1}{2} \|u\|_\Gamma^2 + \tfrac{1}{2} \|u_t\|_\Omega^2. \tag{1.5}$$

For further details on the proof of existence, see [CDCFT09] and [Bar93]; we will only summarize the relevant results.

Theorem 1.1 ([CDCFT09, Thm. 2.1])**.** *The PDE* (1.1) *generates a nonlinear nonexpansive semigroup flow* $t \mapsto \{u(t), u_t(t)\}$ *on the state space* $\mathscr{H} \cong [H^1(\Omega) \oplus H^1(\Gamma)] \times L^2(\Omega)$.

- *For any* $T > 0$, $\{u, u_t\} \in C([0,T]; \mathscr{H})$*; in addition,* $a(x)g(u_t)u_t \in L^\infty(\mathbb{R}^+; L^1(\Omega))$ *and for all* $t \geq s \geq 0$,

$$E(t) + \int_s^t \big(a(x)g(u_t), u_t\big) = E(s). \tag{1.6}$$

- *If* $\{u_0, u_1\} \in \mathscr{D}(\mathbb{A})$*, then such a solution, called a* strong *solution, possesses additional regularity:* $\{u, u_t\} \in L^\infty(\mathbb{R}^+; \mathscr{D}(\mathbb{A}))$.

2. Assumptions

2.1. **Geometry of Ω.** The set of geometric assumptions is typical for a problem with uncontrolled Neumann conditions on the boundary ([LTZ00, p. 302] and [LT06]).

Assumption 2.1 (Geometry and cutoff $a(x)$)**.**

(a) There is an open set $\Gamma_a \subset \Gamma$ and some $\theta > 0$ such that $a(x)$ has an a.e. positive lower bound on a θ-collar of Γ_a:

$$\operatorname*{essinf}_{\Omega_a} a(x) > 0 \quad \text{for} \quad \Omega_a := \big\{x \in \Omega \;:\; \operatorname{dist}(x, \Gamma_a) \leq \theta\big\}. \tag{2.1}$$

(b) There is an open set $\Gamma_{geom} \supset \overline{\Gamma \setminus \Gamma}_a$ (i.e. it includes $\Gamma \setminus \Gamma_a$ and overlaps Γ_a), which satisfies:

i) There is a point $x_0 \in \mathbb{R}^3$ such that

$$(x - x_0) \cdot \nu(x) \leq 0 \quad \text{on} \quad \Gamma_{geom} \tag{2.2}$$

with ν being the outward normal field on Γ.

ii) Γ_{geom} is a level set of a smooth function $\ell(x)$:

$$\Gamma_{geom} = \{x \in \mathbb{R}^3 : \ell(x) = 0\}, \quad \ell \in C^4(\mathbb{R}^3), \quad \nabla\ell \neq 0 \quad \text{on} \quad \Gamma_{geom}. \tag{2.3}$$

iii) The Hessian matrix of ℓ is nonnegative definite on Γ_{geom}:

$$\mathbf{H}_\ell\big|_{\Gamma_{geom}} \geq 0. \tag{2.4}$$

Remark 2.1. Alternatively, in place of (2.2) and (2.4) we can assume $(x - x_0) \cdot \nu \geq 0$ and $\mathbf{H}_\ell\big|_{\Gamma_{geom}} \leq 0$. See [LTZ00, p. 302].

To enforce Assumption 2.1 it suffices for the boundary away from $\operatorname{supp}(a)$ to be star-shaped and convex as shown in Figure 1; however, more general scenarios are also possible: see [LTZ00, Appendix A] for a detailed discussion. An important consequence of the above technical condition is the existence of a special vector field described in the next proposition.

Proposition 2.1 ([LTZ00, pp. 301–303] Vector field $\mathfrak{h}^\lambda$)**.** *Suppose Assumption* 2.1 *is satisfied. There exists* $\lambda_1 > 0$ *such that for all* $\lambda \geq \lambda_1$ *there is a vector field* $\mathfrak{h}^\lambda \in C^2(\overline{\Omega})$ *with the following properties:*

(i) $\mathfrak{h}^\lambda \cdot \nu = 0$ *on* Γ_{geom}.

(ii) The Jacobi matrix $[\nabla \mathfrak{h}^\lambda]$ *is strictly positive definite on* Ω*:* $[\nabla \mathfrak{h}^\lambda] \geq \rho I$ *for some* $\rho > 0$.

(iii) *Moreover, in some $\mathbb{R}^3$ neighborhood of Γ_{geom} the field $\mathfrak{h}^\lambda$ is given by the gradient of a (strictly convex) function:*

$$\mathfrak{h}^\lambda = \nabla d_\lambda, \qquad d_\lambda(x) := \tfrac{1}{2}|x-x_0|^2 - (x-x_0)\cdot \nu \frac{\ell(x)}{|\nabla \ell(x)|} + \lambda \ell(x)^2, \tag{2.5}$$

where the normal field is locally transversely extended: $\nu = \nu_{ext} = \nabla\ell/|\nabla\ell|$.

Next, we define auxiliary smooth cutoffs whose job is to split the domain into regions where damping is effective and where it is not. These functions will be used in the proof of the main result.

Definition 2.1 (Cutoffs ψ, ϕ)**.** Construct $\psi, \phi \in C^2(\overline{\Omega}; [0,1])$ with the following properties:

(i) $\operatorname{supp}(\phi) \subset \Omega_a$ for Ω_a as in (2.1),
(ii) $\psi(\Gamma \setminus \Gamma_{geom}) = \{0\}$,
(iii) $\max\{\psi(x), \phi(x)\} = 1$ for each $x \in \Omega$.

Figure 1 depicts a possible domain, and its partition by cutoffs ψ and ϕ.

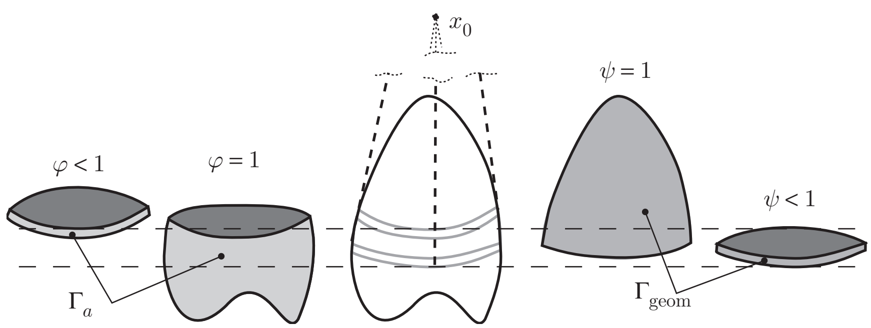

FIGURE 1. A sample domain Ω and its possible decomposition into overlapping segments by the cutoffs ϕ and ψ described in Definition 2.1. The boundary Γ consists of two overlapping regions: Γ_a and Γ_{geom}.

2.2. **Nonlinear damping.** For the wellposedness of (1.1) it suffices to have g continuous, monotone increasing on $\mathbb{R}$, $g(0) = 0$. However, the system may not be uniformly stable on the finite energy space $\mathscr{H}$ if $g(s)$ is *not linearly bounded at infinity.* To make the latter notion more precise the following definition will be employed.

Definition 2.2 (Order at infinity)**.** A monotone increasing map $f : \mathbb{R} \to \mathbb{R}$, $f(0) = 0$ is of the order $r := \mathcal{O}(f) \geq 0$ at infinity if there exists $c > 0$ such that

$$|s|^{r+1} \sim f(s)s \quad \text{whenever} \quad |s| \geq c. \tag{2.6}$$

When the order r exceeds, falls below, or equals 1 we say that the map f is, respectively, superlinear, sublinear, or linearly bounded at infinity.

The counterexample constructed in [VM00] proved that solutions to a wave equation with sublinear boundary damping do not decay exponentially, and uniform decay rates necessitate more regular solutions. A requirement for additional regularity with damping that is sublinear at infinity was employed in [Nic03]. Sublinear and superlinear at infinity feedback maps were extensively studied in [LT06, DLT09] for wave equations. The regularity conditions below will only be needed when $\mathcal{O}(g) \neq 1$.

Assumption 2.2 (Damping)**.** Function g is a continuous monotone increasing function with $g(0) = 0$.

- If g is sublinear at infinity, i.e. $\mathcal{O}(g) < 1$, then assume that there exists $p > 2$ such that
$$\|u_t\|_{L^\infty(\mathbb{R}^+;L^p(\Omega))} < \infty.$$
- If g is superlinear at infinity, $\mathcal{O}(g) > 1$, then assume there is $p \geq 1 + \mathcal{O}(g)$ such that
$$\|\nabla u\|_{L^\infty(\mathbb{R}^+;L^p(\Omega))} < \infty.$$

For instance, if $\mathcal{O}(g) \leq 3$, then $\mathscr{D}(\mathbb{A}) \subset H^2(\Omega) \times H^1(\Omega)$, so strong initial data $\{u_0, u_1\} \in \mathscr{D}(\mathbb{A})$ produces a trajectory where u_t is uniformly bounded in $H^1(\Omega) \hookrightarrow L^{p=6}(\Omega)$, and $\nabla u \in H^1(\Omega) \hookrightarrow L^{p=6}(\Omega)$. See [LT06] for more examples.

3. Observability and energy decay rates

The first and the main technical result of the paper is the perturbed observability inequality from which the decay rates will readily follow. It is sufficient to establish the estimate for strong solutions only. The decay results will subsequently be extended to any initial data in $\mathscr{H}$ as in [LT93, CDCFT09].

Theorem 3.1 (**Observability estimate**)**.** *Suppose Assumptions* 2.1 *and* 2.2 *hold. Then there exists $T > 0$, dependent only on Ω and $a(x)$, and a constant $C_T > 0$, dependent on T and possibly on $\mathcal{O}(g)$ (see the corollary below), such that for any strong solution $\{u, u_t\}$ of* (1.1),

$$E(T) + \int_0^T E(t)dt \leq C_T \Big[\big(\!\big(a(x)g(u_t),\ u_t + |u| + |\nabla u| \big)\!\big) + \big(\!\big(a(x)u_t^2 \big)\!\big) \Big]. \tag{3.1}$$

Remark 3.1 (Regularity). Note that when $\mathcal{O}(g) = r > 3$, then from the structure (1.4) of the generator's domain one can only infer $a\, g(u_t) \in [H^1(\Omega) \oplus H^1(\Gamma)]'$ and $u \in H^1(\Omega)$, which does not imply that the pairing $a\, g(u_t)|\nabla u|$ is well defined. However, the monotonicity of g and $a\, g(u_t)u_t \in L^\infty(\mathbb{R}^+; L^1(\Omega))$ imply that $\sqrt[r]{a}\, g(u_t) \in L^{(r+1)/r}(\Omega)$, and (since $a(x)$ is bounded) $ag(u_t) \in L^{(r+1)/r}(\Omega)$. The regularity Assumption 2.2 imposes $|\nabla u| \in L^{r+1}(\Omega) = [L^{(r+1)/r}(\Omega)]'$. Hence the products in (3.1) are well defined for strong solutions satisfying Assumption 2.2.

The estimate (3.1) incorporates analysis of trace dynamics, and from this point on the study of energy decay follows exactly the same algorithm as for a wave equation with non-Wentzell boundary conditions, via an argument developed in [LT93] and further developed in [LT06].

Corollary 3.1.1 ([LT06])**.** *Suppose Assumptions* 2.1 *and* 2.2 *hold. Let p be as in Assumption* 2.2*, and let $T > 0$ be given by Theorem* 3.1*. Let $h_0(s)$ be a concave monotone increasing function, such that $h_0(0) = 0$ and*

$$s^2 + g(s)^2 \leq h_0\big(g(s)s\big) \quad \forall |s| < 1 \tag{3.2}$$

(h_0 can always be constructed since g is continuous and monotone, 0 at the origin). Next, define $h_1(s)$ as follows:

- *if g is sublinear at infinity, i.e. $r = \mathcal{O}(g) < 1$, then let $h_1(s) := s^{\frac{p-2}{p-r-1}}$,*
- *if g is superlinear at infinity, i.e. $r = \mathcal{O}(g) > 1$, set $h_1(s) := s^{\frac{r(p-2)}{r(p-1)-1}}$,*
- *otherwise ($\mathcal{O}(g) = 1$) define $h_1(s) = s$.*

CONCLUSION: Then

$$E(t) \le S\big((t/T) - 1\big), \quad \forall t \ge T$$

with $\lim_{t\to\infty} S(t) = 0$. Moreover, S is the solution to the following nonlinear ODE:

$$\dot{S} + q(S) = 0, \qquad S(0) = E(0), \tag{3.3}$$

where for some $C > 0$,

$$q := I - \Big(I + \big[C(h_0 + h_1)\big]^{-1}\Big)^{-1} = \big(I + C(h_0 + h_1)\big)^{-1}. \tag{3.4}$$

- *If $\mathcal{O}(g) = 1$, then the constant C is independent of the initial energy.*
- *If $\mathcal{O}(g) < 1$, then C is proportional to $\|u_t\|^{\frac{p(1-r)}{p-1-r}}_{L^\infty(\mathbb{R}^+;L^p(\Omega))}$.*
- *If $\mathcal{O}(g) > 1$, then C is proportional to $\|\nabla u\|^{\frac{p(r-1)}{(p-1)r-1}}_{L^\infty(\mathbb{R}^+;L^p(\Omega))}$.*

Proof. If (3.1) is established it can be directly used as a starting point in the proof of [LT06, Lemma 2]. The argument is identical. The only extra step is to split the space-time integrals into sets where $|u_t| < 1$ and $|u_t| > 1$ and apply Schwartz' inequality: $C_\varepsilon \int_{X_0} \big[g(u_t)^2 + u_t^2\big] + \frac{\varepsilon}{2}(\!(|\nabla u|^2 + u^2)\!)$. Then proceed to estimate the RHS of (3.1) as in the proof of [LT06, Lemma 2], wherefrom the decay rates follow. $\square$

As a simple consequence, if $g(s)$ is linearly bounded everywhere on $\mathbb{R}$, then h_0, h_1 are linear, and $q(s)$ in (3.4) is just a constant multiple of s. Thus the energy decays exponentially (without any extra assumptions on the regularity of the solutions):

Example 3.1 (Exponential decay)**.** If there exist $c_1, c_2 > 0$ such that

$$c_1 s^2 \le g(s)s \le c_2 s^2 \tag{3.5}$$

for all $s \in \mathbb{R}$, then for some constant $\omega > 0$ (dependent on c_1 and c_2, but independent of $E(0)$), and some $T > 0$ the energy satisfies:

$$E(t) \le E(0)e^{-\omega((t/T)-1)} \quad \forall t \ge T\,.$$

For a more general damping g, however, equation (3.3) is nonlinear and is difficult to solve in closed form. But the energy decay can be closely approximated by a simpler ODE because near the origin the identity map I yields much smaller values than concave functions whose derivatives blow up near the origin. It can be shown that when g is nonlinear the inverse in (3.4) is asymptotically determined by either h_0 or h_1.

Corollary 3.1.2 ([LT06, Corollary 1])**.** *Suppose that the assumptions of Corollary* 3.1.1 *are satisfied. Let h_0, h_1, T, and C be as in Corollary* 3.1.1. *If it is possible to pick $i, j \in \{0, 1\}$ so that*

$$\lim_{s\searrow 0} \frac{h_i(s)}{h_j(s)} = 0, \tag{3.6}$$

then the function with the faster growth near the origin, namely h_j, governs the asymptotic decay rate of the energy. Specifically, for any positive $\gamma < 1$, chosen arbitrarily close to 1, there exists time $T_0 = T_0(\gamma) \geq T$ (T_0 may increase as γ gets closer to 1) such that

$$E(t) \leq \tilde{S}\big((t/T) - 1\big), \quad \forall t \geq T_0(\gamma) \geq T$$

and $\tilde{S}$ solves

$$\frac{d}{dt}\tilde{S} + h_j^{-1}\left(\frac{\gamma}{C}\tilde{S}\right) = 0, \quad \tilde{S}(0) = E(0)\,. \tag{3.7}$$

This result permits us to examine much more general cases, a few samples of which we provide below to make this exposition self-contained. For a more detailed discussion, see [LT06].

Example 3.2 (Sublinear damping at infinity)**.** Suppose g is linearly bounded near 0, namely satisfies (3.5) for $|s| < 1$, whereas at infinity $\mathcal{O}(g) < 1$, which means that for a given positive $\theta < 1$,

$$|g(s)| \sim |s|^{\theta}, \quad \forall |s| > 1\,.$$

Then h_0, as determined by (3.2), is linear: $h_0(s) = (c_1^{-1} + c_2)s$, whereas $h_1(s) = s^{\frac{p-2}{p-\theta-1}}$, as dictated by Corollary 3.1.1 assuming there is $p > 2$ so that u_t belongs to $L^\infty(\mathbb{R}^+; L^p(\Omega))$. Assert first that such a $p > 2$ exists. Then (3.6) holds with h_1 in the denominator, and (3.7) reads

$$\tilde{S}_t + \left(\alpha\tilde{S}\right)^{\rho} = 0, \quad \tilde{S}(0) = E(0),$$

where $\rho = \frac{p-\theta-1}{p-2}$ and $\alpha = \frac{\gamma}{C}$ for any positive $\gamma < 1$. The constant C is proportional to $\|u_t\|^{\frac{p(1-\theta)}{p-1-\theta}}_{L^\infty(\mathbb{R}^+;L^p(\Omega))}$ as stated in Corollary 3.1.1. The solution to this ODE is

$$\tilde{S}(t) = \big(\alpha^{\rho}(\rho - 1)(t + c_0)\big)^{-\frac{1}{\rho-1}}, \quad c_0 = \frac{E(0)^{1-\rho}}{\alpha^{\rho}(\rho - 1)}. \tag{3.8}$$

Thus, for large t ($\gg c_0$), we have $E(t) \leq C_1((t/T) - 1)^{-\frac{p-2}{1-\theta}}$, where C_1 is proportional to $\|u_t\|^p_{L^\infty(\mathbb{R}^+;L^p(\Omega))}$. Note that as p increases, or as the exponent θ approaches 1 from below (the linear case), the decay rate improves, whereas as $\theta \searrow 0$, or as the regularity $p \searrow 2$, the rate diminishes.

To ensure availability of such a $p > 2$, we could pick the initial data from the domain of the generator $\mathscr{D}(\mathbb{A})$; then for sublinear damping at infinity we get $\{u, u_t\} \in L^\infty(\mathbb{R}_+; H^2(\Omega) \times H^1(\Omega))$. Consequently, in 3D the Sobolev embeddings guarantee available uniform regularity $u_t \in L^{p=6}(\Omega)$, whence

$$E(t) \lesssim t^{-\frac{4}{1-\theta}}.$$

Thus, for instance, in 3D, under saturated damping ($\theta = 0$), the finite energy of strong solutions decays as t^{-4}.

Example 3.3 (Superlinear damping at infinity)**.** Suppose again that g in (3.5) holds for $|s| < 1$, and at infinity $\mathcal{O}(g) > 1$: for a given $r > 1$,

$$|g(s)| \sim |s|^{r}, \quad \forall |s| > 1\,.$$

The discussion is identical to the previous example, and the decay rate is still determined by h_1. This time, however, $\alpha = \frac{\gamma}{C}$ with C proportional to $\|\nabla u\|_{L^\infty(\mathbb{R}^+;L^p(\Omega))}^{\frac{p(r-1)}{(p-1)r-1}}$, $\rho = \frac{r(p-1)-1}{r(p-2)}$, and p must not fall below $1+r$.

From (3.8) conclude that for t large (neglecting c_0), $E(t) \leq C_1((t/T)-1)^{-\frac{r(p-2)}{r-1}}$ with C_1 proportional to $\|\nabla u\|_{L^\infty(\mathbb{R}^+;L^p(\Omega))}^p$. If the available regularity p increases or as r tends to 1 (from above), which models linear growth, the rate of energy decay improves.

If, for example, the damping is cubic, $r = 3$, then in three dimensions the domain of the generator $\mathscr{D}(\mathbb{A})$ still embeds into $H^2(\Omega) \times H^1(\Omega)$. Consequently $\nabla u \in H^1(\Omega) \hookrightarrow L^6(\Omega)$, providing the rate of decay

$$E(t) \lesssim t^{-6}.$$

In 2 dimensions for any $r > 1$ strong solutions satisfy $\nabla u \in L^\infty(\mathbb{R}_+; H^1(\Omega))$ and the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ holds for any $1 \leq p < \infty$. Hence in 2D the rate of decay is "subexponential": asymptotically faster than any fixed algebraic rate, since one can choose any $p < \infty$ (but at the expense of the larger embedding constant $H^1(\Omega) \hookrightarrow L^p(\Omega)$).

Example 3.4 (Exponential superlinear damping near the origin)**.** Now let the damping $g(s)$ at infinity, for $|s| > 1$, be either linearly bounded $\mathcal{O}(g) = 1$, or behave as in any of the above examples. Whereas, near the origin, let it decay exponentially as $s \searrow 0$,

$$g(s) = s^3 e^{-1/s^2}, \quad |s| < 1$$

(the odd factor is attached to ensure monotone increasing growth). In order to construct h_0 via (3.2), it suffices to have $h_0(g(s)s) = 2s^2$ leading to

$$h_0^{-1}(s) = \tfrac{1}{4}s^2 \exp(-2/s).$$

Observing that for $0 < s < 1$, $h_0^{-1}(s) < e^{-2/s}$, whence $h_0(s) > -2/\ln(s)$, so the estimate (3.6) holds with h_0 in the denominator and (linear or polynomial) h_1 in the numerator. The ODE (3.7) reads

$$\tilde{S}_t + \tfrac{1}{4}(\alpha\tilde{S})^2 \exp[-2/(\alpha\tilde{S})] = 0, \quad \tilde{S}(0) = E(0)\,,$$

for $\alpha = \frac{\gamma}{C}$, $0 < \gamma < 1$, and C as in (3.1.1). Then for $t \geq T_0(\gamma) \geq T$ the energy decays logarithmically, as $E(t) \leq \tilde{S}((t/T)-1) = \frac{2}{\alpha}\big(\ln\big[\frac{\alpha}{2}\big(\frac{t}{T} - 1 + c_0\big)\big]\big)^{-1}$, where c_0 depends on the initial energy $E(0)$.

This time, under sublinear or superlinear damping at infinity as in the previous examples, having smoother solutions, i.e. larger p, will not substantially impact the decay because low frequencies dissipate logarithmically, much slower than the algebraic rates discussed in the above examples. However, it is still necessary to have that additional regularity to ensure that parameter C is finite, and the high frequencies decay to begin with (unless g is linearly bounded at infinity, $\mathcal{O}(g) = 1$, in which case the state space regularity of weak solutions is sufficient).

The rest of the paper will be devoted to the proof of the new technical result: Theorem 3.1.

4. Decomposition of the solution

Let ψ and ϕ be as in Definition 2.1. The supports of these maps split the domain Ω into a part where the damping is "properly supported" (the nonnegative coefficient $a(x)$ is bounded away from 0), and the complement, whose boundary intersected with Γ satisfies suitable geometric restrictions. The supports of ψ and ϕ, however, overlap on a small region where one can take advantage of both the damping and the geometry of the exterior boundary. Such a transition strip will help to accommodate the commutators originating from the cutoffs.

Now introduce

$$w := \psi u \quad \text{and} \quad v := \phi u, \tag{4.1}$$

which, according to (1.1), satisfy

$$\Box w = R_\psi(u) \quad \text{and} \quad \Box v = R_\phi(u)\,, \tag{4.2}$$

where R includes the cutoff commutator and the damping:

$$R_\theta(u) := [\![M_\theta,\, \Delta]\!] u - \theta(x) a(x) g(u_t)\,.$$

Operator M_θ denotes the pointwise (a.e.) multiplication by $\theta(x)$.

Subsequent analysis will apply different multipliers to the two equations in (4.2), ultimately combining the ensuing identities to get an estimate on the original solution u.

5. Strategy

The observability inequality (3.1) reconstructs the total energy of the system from the data available through the dissipation. First, let's analyze a much simpler situation when the damping is actively supported on the entire collar near the boundary Γ (in the present notation: $\Gamma_{geom} = \emptyset$). In this case the potential energy is first estimated via the kinetic energy (observed and controlled by the damping) in a layer near the boundary. This is achieved by using the so-called equipartition of energy multipliers represented by weighted multiples of the solution v as in (4.1). Having obtained a bound on v, one can proceed to reconstruct the energy for the internal part of the region represented by w. That can be done by using "flux multipliers" $(\nabla w \cdot \mathfrak{q})$ based on a gradient of the solution and a "standard" radial vector field $\mathfrak{q}$. This procedure takes advantage of already reconstructed energy in the collar: since w in this scenario is compactly supported within Ω (as $\Gamma_{geom} = \emptyset$), the full energy for w coincides with internal energy, *with no high-order trace terms.* A version of this program has been successfully implemented in [CDCFT09].

The problem is *very different* when a part of a collar near the boundary is left *undissipated* : $\Gamma_{geom} \neq \emptyset$. As before, the equipartition equation reconstructs full energy in terms of the damping only on Γ_a. However, when the flux-multiplier is applied, in the absence of information from the collar neighboring Γ_{geom}, various unbounded trace terms emerge. Since the boundary conditions involve Neumann traces, classical geometric conditions (star-shaped domain) are no longer sufficient to control the energy in the unobserved collar. This is a well-recognized difficulty due to the fact that the uniform Lopatinskii condition is not satisfied by such a system. This situation has been dealt with in the case of a wave equation with unobserved Neumann boundary. The problem was resolved by constructing special multipliers—an idea inspired by [Tat]— based on nonradial vector fields, which depend on the geometry of the domain near the unobserved boundary [LTZ00]. Thus, it is not surprising that these multipliers may prove useful and pave the way to a resolution of the present situation.

However, Wentzell configuration poses another obstacle. The energy to be reconstructed also involves the boundary energy at the H^1 Sobolev level. Such energy is intrinsically “unbounded” with respect to the available interior regularity. Thus, the role of the corresponding multiplier becomes twofold: not only to rebuild the internal energy with no information on the values of the “Lagrangian” on the boundary, but also to reconstruct the full H^1 energy on Γ_{geom}. Note that v is supported away from Γ_{geom}; thus the equipartition of energy plays no role in this process.

As we shall see, this task will be accomplished by refining the construction of multipliers introduced in [LTZ00]. The idea is to add more convexity to the function generating the vector field by increasing the second parameter λ in (2.5). Details are given below and the argument will proceed as follows:

- Apply multipliers to equations (4.2) to obtain fundamental identities, which connect the boundary conditions with the interior dynamics of the problem.
- Use the structure of the multipliers (based on the shape of the domain) to establish that from the trace terms collected over Γ_{geom} the unbounded terms cancel, while the $H^1(\Gamma)$ norm of the energy emerges with a suitable (dissipative) sign.
- Estimate the remaining terms in the fundamental identities and add the results to obtain (3.1), possibly polluted by lower-order terms: norms of the solution in topologies coarser than the finite-energy level. The latter quantities can be shown to be controlled by the damping via a now-standard compactness-uniqueness strategy.

6. The fundamental identities

We start with general multipliers equalities (flux and equipartition) which are now well known in the literature. Let $\mathfrak{q} \in C^1(\overline{\Omega}; \mathbb{R}^n)$. Then any w in the space $\bigcap_{j=0}^{2} W^{j,\infty}(0,T; H^{2-j}(\Omega))$ (more precisely, its unique version in $C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^1(\Omega))$ obeys

$$
\begin{aligned}
(\!(\Box w, \mathfrak{q}\cdot\nabla w)\!) &= (w_t, \mathfrak{q}\cdot\nabla w)\Big|_0^T + (\!(\nabla w\cdot[\nabla\mathfrak{q}]\nabla w)\!) + \tfrac{1}{2}(\!(\operatorname{div}\mathfrak{q}, w_t^2 - |\nabla w|^2)\!) \\
&\quad - \tfrac{1}{2}\left\langle\!\!\left\langle (\mathfrak{q}\cdot\nu), w_t^2 - |\nabla w|^2 \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \frac{\partial w}{\partial\nu}, \mathfrak{q}\cdot\nabla w \right\rangle\!\!\right\rangle .
\end{aligned}
\tag{6.1}
$$

For any $\xi \in C(\overline{\Omega})$ and any $w \in \bigcap_{j=0,1} W^{j,\infty}(0,T; H^{1-j}(\Omega))$:

$$
(\!(\Box w, \xi w)\!) = (w_t, \xi w)\big|_0^T - (\!(\xi, w_t^2 - |\nabla w|^2)\!) + (\!(\nabla w\cdot\nabla\xi, w)\!) - \left\langle\!\!\left\langle \frac{\partial w}{\partial\nu}, \xi w \right\rangle\!\!\right\rangle . \tag{6.2}
$$

These identities valid for strong solutions are, by now, standard; see, for instance, [Tou07, Section 7.2].

Add (6.1) with $\mathfrak{q} = \mathfrak{h}^\lambda$ to equation (6.2) with $\xi = \frac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)$ for constant $\rho > 0$ as given by Proposition 2.1:

(6.3)
$$
\begin{aligned}
&\left(\!\left(\nabla w \cdot \left[\nabla \mathfrak{h}^\lambda - \tfrac{1}{2}\rho I\right] \nabla w\right)\!\right) + \tfrac{1}{2}\left(\!\left(\rho, w_t^2\right)\!\right) - \left\langle\!\!\left\langle \frac{\partial w}{\partial \nu}, \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w + \mathfrak{h}^\lambda \cdot \nabla w \right\rangle\!\!\right\rangle \\
=&\tfrac{1}{2}\left\langle\!\left\langle (\mathfrak{h}^\lambda \cdot \nu), w_t^2 - |\nabla w|^2 \right\rangle\!\right\rangle - \tfrac{1}{2}\left(\!\left(\nabla w \cdot \nabla(\operatorname{div}\mathfrak{h}^\lambda), w\right)\!\right) \\
&+ \left(\!\left(R_\psi(u), \mathfrak{h}^\lambda \cdot \nabla w - \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w\right)\!\right) - \left(w_t, \mathfrak{h}^\lambda \cdot \nabla w + \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w\right)\Big|_0^T.
\end{aligned}
$$

Invoke (6.2) for v, add $\int_0^T \int_\Omega 2v_t^2$ to both sides of the result, and apply the boundary condition in (1.1):

$$
\boxed{\left(\!\left(|\nabla v|^2 + v_t^2\right)\!\right) + \left\langle\!\left\langle v^2 + |\nabla_T v|^2 \right\rangle\!\right\rangle = \left(\!\left(2, v_t^2\right)\!\right) + \left(\!\left(R_\phi(u), v\right)\!\right) - (v_t, v)\Big|_0^T}. \tag{6.4}
$$

When $\Gamma_{geom} = \emptyset$ the above equalities along with the known stabilizability argument [LT93] lead to the observability estimate stated in Theorem 3.1.

6.1. **What is new when $\Gamma_{geom} \neq \emptyset$?** Note that when a full collar is observed, the boundary integrals in (6.3) vanish, since w would be supported away from $\Gamma_a = \Gamma$ in such a scenario. Thus, any radial flux multiplier, e.g. $\nabla w \cdot (x - x_0)$, where essentially $\mathfrak{h}^\lambda = \nabla \frac{1}{2}|x - x_0|^2$, could provide reconstruction of the energy for w, while the equipartition of energy relation would suffice in the collar and would account for the boundary energy.

Instead, when Γ_{geom} is nonempty, further challenges arise from the presence of the two boundary integrals in (6.3). The quantities represented by these integrals are highly unbounded with respect to finite energy, and it is the handling of these integrals that necessitates the introduction of nonradial vector fields whose structure depends on the level sets defining the boundary. Below we outline the analysis of the terms appearing in both identities, with the detailed argument to take place in Section 7. It should be noted that the arguments given there not only handle the case $\Gamma_{geom} \neq \emptyset$, but also provide a more concise and independent proof of the same result when $\Gamma_{geom} = \emptyset$ originally established in [CDCFT09].

In (6.4) the control of the full energy of the variable v is guaranteed once the terms on the RHS of that inequality are appropriately bounded: either by the dissipation, or by so-called lower levels of the energy, or else by the energy evaluated discretely at the terminal points $t = 0$ and $t = T$. Indeed, the first term and the damping present in R_ψ on the RHS of (6.4) are controlled by the dissipation (the support of v_t is in the dissipative region), while the second term involves a product of the first-order commutator $[\![M_\theta, \Delta]\!]$ applied to u, and the zero-order (in derivatives) term v, which altogether will be shown as an inessential or “lower-order” product. The very last quantity in (6.4) is a discrete term controlled by a scalar multiple of $E(0) + E(T)$.

On the other hand, the analysis of (6.3) is more involved and critical to the entire argument. While the first two products provide (via Proposition 2.1) control of the interior energy for w, there is still the need of extracting the boundary energy on Γ_{geom} from unstructured boundary products. Moreover, from the terms on the RHS of this identity, $\left(\!\left(R_\psi(u), \mathfrak{h}^\lambda \cdot \nabla w\right)\!\right)$ is precisely at the energy level (first-order commutator term multiplied by the gradient), whereas the boundary terms are unbounded with respect to the state space topology. Thus the goal is (i) to extract some coercivity estimates from the boundary terms, (ii) to eliminate any other unstructured unbounded traces, and (iii) to handle the energy-level product involving the commutator within R_ψ. These steps will be accomplished through a careful analysis of the traces, and by adjusting the field $\mathfrak{h}^\lambda$ (with respect to its original definition in [LTZ00, p. 302]). We shall begin by dispensing with the products which are "easy", i.e. of a lower order with respect to the energy of the problem, leaving the most demanding boundary analysis for the subsequent section.

6.2. **Lower-order terms.** The rest of the proof below will be devoted to an analysis of (6.3) and (6.4). To make the presentation more streamlined we note that a number of terms in those identities can be classified as "inessential" in the following sense:

Definition 6.1 (Lower-order terms)**.** Let $\{u, u_t\} \in C([0,T]; \mathscr{H})$ be a solution of (1.1). A quantity X is said to be of a lower order if for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|X| \le \varepsilon \int_0^T E(t)dt + C_\varepsilon \int_0^T \|u(t)\|^2_\Omega dt\,. \tag{6.5}$$

Any such lower-order term or an algebraic combination thereof will be denoted by l.o.t.(u). If $Y - Z = $ l.o.t.(u) we will write $Y \overset{\circ}{=} Z$. Similarly, inequalities of the form $Y \le Z + $ l.o.t.(u) will be written as $Y \overset{\circ}{\le} Z$.

Proposition 6.1. *Let $\{u, u_t\} \in C([0,T]; \mathscr{H})$ be a solution of* (1.1)*.*

(i) For any constant $C > 0$, $\quad l.o.t.(u) \overset{\circ}{=} C l.o.t.(u)$.
(ii) $(\!(a, b)\!) = l.o.t.(u)$ if there is $\delta > 0$ such that for a.e. t, $\|a(t)\|_\Omega \lesssim \|u(t)\|_{1-\delta,\Omega}$, and $\|b(t)\|_\Omega \lesssim \|u(t)\|_{1,\Omega}$.
(iii) $\langle\!\langle a, b\rangle\!\rangle = l.o.t.(u)$, provided there is $\delta > 0$ so that for a.e. t, $\|a(t)\|_\Gamma \lesssim \|u(t)\|_{1/2-\delta,\Gamma}$ and $\|b(t)\|_\Gamma \lesssim \|u(t)\|_{1/2,\Gamma}$.

Proof. The up-to-a-constant comparison $\lesssim$ indicates that the result is independent of multiplication by L^∞ functions, such as the cutoffs ψ, ϕ, $\operatorname{div}\mathfrak{h}^\lambda$, etc. This conclusion follows since sup norms can be factored out and absorbed into C_ε in (6.5). For the same reason claim (i) holds: for any given $\varepsilon > 0$ work with $\varepsilon_1 = \varepsilon/C$ in (6.5).

To verify (ii) use Schwartz' inequality, interpolation of Sobolev spaces, and Young's estimate:

$$|(\!(a,b)\!)| \lesssim \frac{1}{\varepsilon}\int_0^T \|u\|^2_{1-\delta,\Omega} + \frac{\varepsilon}{4}\int_0^T \|u\|^2_{1,\Omega} \lesssim C_\varepsilon \int_0^T \|u\|^2_\Omega + \Big(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\Big)\int_0^T \|u\|^2_{1,\Omega},$$

which confirms (6.5) since $(1/2)\|u\|^2_{1,\Omega} \le E(t)$. The analysis for a Γ-based product follows similarly via the continuous trace map $H^{1-\delta}(\Omega) \to H^{1/2-\delta}(\Gamma)$. □

Proposition 6.2 (Control of lower-order terms)**.** *There exists $T > 0$ and a constant $L_T > 0$ dependent on T such that for any solution $\{u, u_t\}$,*

$$(\!(u^2)\!) \le L_T (\!(a(x)g(u_t), u_t)\!)\,.$$

Proof. The result can be established with the standard compactness-uniqueness argument and proceeds exactly as the proof of [CDCFT09, Lemma 4.3] (the fact that the RHS here is of the form $g(u_t)u_t$ vs. $g(u_t)^2 + u_t^2$ in the latter reference plays no role since $g(s)s$ also vanishes only at $s = 0$, which is the requisite feature). The key is that any solution that remains stationary on $\operatorname{supp} a(x)$ for a sufficiently long time must be identically 0. $\square$

7. Coercivity on the unobserved boundary Γ_{geom}

The primary challenge is that besides stabilizing the "boundary Lagrangian" induced by the Neumann-type boundary conditions, we must prove that the left-hand sides of (6.3) and (6.4) together control the $H^1(\Gamma)$ norm of the solution. Let $\lambda_1 > 0$ be given by Proposition 2.1.

Lemma 7.1. *Adopt the hypotheses of Theorem* 3.1. *With reference to the vector field* $\mathfrak{h}^\lambda$ *introduced in Proposition* 2.1, *there exists* $\lambda_2 > \lambda_1$ *such that for all* $\lambda > \lambda_2$, *the following estimate holds:*

$$\left\langle\!\!\left\langle -\frac{\partial w}{\partial \nu},\ \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w + \mathfrak{h}^\lambda\cdot\nabla w \right\rangle\!\!\right\rangle \overset{\circ}{\geq} \langle\!\langle \rho, |\nabla_T w|^2 + w^2 \rangle\!\rangle. \tag{7.1}$$

Proof. Let $\mathfrak{h}^\lambda$ and $\lambda_1 > 0$ be as defined in Proposition 2.1. Then according to the latter, for $\lambda > \lambda_1$, the vector field $\mathfrak{h}^\lambda$ has only tangential components on Γ_{geom}; since $w\big|_\Gamma$ is only supported on Γ_{geom}, then

$$(\mathfrak{h}^\lambda\cdot\nabla w)\big|_\Gamma = (\mathfrak{h}^\lambda\cdot\nabla_T w)\big|_\Gamma. \tag{7.2}$$

Apply the boundary condition of (1.1) to ultimately obtain, modulo some lower-order terms:

$$\begin{aligned}
&\left\langle\!\!\left\langle -\frac{\partial w}{\partial \nu},\ \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w + \mathfrak{h}^\lambda\cdot\nabla w \right\rangle\!\!\right\rangle \\
&= \overbrace{\left\langle\!\!\left\langle \nabla_T w,\ \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)\nabla_T w + \nabla_T(\mathfrak{h}^\lambda\cdot\nabla w) \right\rangle\!\!\right\rangle}^{I} + \overbrace{\left\langle\!\!\left\langle \nabla_T w,\ \tfrac{1}{2}w\nabla_T(\operatorname{div}\mathfrak{h}^\lambda - \rho) \right\rangle\!\!\right\rangle}^{II} \\
&\quad + \overbrace{\left\langle\!\!\left\langle w,\ \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w + \mathfrak{h}^\lambda\cdot\nabla w \right\rangle\!\!\right\rangle}^{III}.
\end{aligned} \tag{7.3}$$

To the three terms on the RHS of the preceding identity we apply the following:

- The terms II and III, according to Proposition 6.1, are of a lower order.
- Term I requires more work, and before addressing it we recall some properties of the Levi-Civita connection on a (Riemannian) manifold $\mathcal{M}$ with metric g, and inner-product denoted "$\cdot_g$". E.g. see [Doc, pp. 53–56 and 141–142]:
  a) For vector fields X, Y, Z on $\mathcal{M}$ the following identity holds:
  $$X(Y\cdot_g Z) = \mathbb{D}_X Y\cdot_g Z + Y\cdot_g \mathbb{D}_X Z\,,$$
  and, in particular,
  $$\mathbb{D}_X Y\cdot_g Y = \frac{1}{2}X(Y\cdot_g Y) = \frac{1}{2}X(|Y|_g^2)\,.$$
  b) Any $H^2(\mathcal{M})$ function f induces a symmetric bilinear map
  $$\{X, Y\}\mapsto (\mathbb{D}_X\nabla_g f)\cdot_g Y = (\mathbb{D}_Y\nabla_g f)\cdot_g X,$$

where ∇_g indicates the gradient on $\mathcal{M}$. The bilinear map is the Hessian of f, denoted $\mathbb{D}^2[f]$.

With the above identities in mind, recall that (7.2) permits us to replace $\mathfrak{h}^\lambda \cdot \nabla w$ by $\mathfrak{h}^\lambda \cdot \nabla_T w$; i.e., we can work exclusively with vector fields on Γ. When dealing with tangential fields the Euclidean inner product $\cdot$ induces a Riemannian metric on the embedded manifold Γ.

Function d_λ will be the one that produces $\mathfrak{h}^\lambda$ (see (2.5)). Next, let W indicate the vector field corresponding to $\nabla_T w$, meaning that given $x \in \Gamma$, $W(x) \in T_x\Gamma$ is a linear functional such that for $f \in H^1(\Gamma)$, $W : f \mapsto \nabla_T w(x) \cdot \nabla_T f(x)$. Similarly, let F be the vector field identified with $\mathfrak{h}^\lambda$. Then

$$\begin{aligned}
\nabla_T w \cdot \nabla_T(\mathfrak{h}^\lambda \cdot \nabla_T w) &= W\left(\mathfrak{h}^\lambda \cdot \nabla_T w\right) = (\mathbb{D}_W \mathfrak{h}^\lambda) \cdot \nabla_T w + \mathfrak{h}^\lambda \cdot \mathbb{D}_W W \\
&= (\mathbb{D}_W \nabla_T d_\lambda) \cdot W + (\mathbb{D}_W \nabla_T w) \cdot \mathfrak{h}^\lambda \\
&= \mathbb{D}^2[d_\lambda](W, W) + \mathbb{D}^2[w](F, W) \\
&= \mathbb{D}^2[d_\lambda](W, W) + (\mathbb{D}_F W) \cdot W \\
&= \mathbb{D}^2[d_\lambda](W, W) + \frac{1}{2} F\left(|W|^2\right) \\
&= \nabla_T w \cdot \mathbb{D}^2[d_\lambda] \nabla_T w + \frac{1}{2} \mathfrak{h}^\lambda \cdot \nabla_T |\nabla_T w|^2.
\end{aligned}$$

At this point we may also invoke Green's identity $\langle\!\langle \mathfrak{h}^\lambda \cdot \nabla_T |\nabla_T w|^2 \rangle\!\rangle = -\langle\!\langle \operatorname{div}_T \mathfrak{h}^\lambda, |\nabla_T w|^2 \rangle\!\rangle$.

Given the above considerations of the quantities I, II and III in (7.3), the latter itentity can be rewritten as

(7.4)
$$\begin{aligned}
&\left\langle\!\!\left\langle -\frac{\partial w}{\partial \nu},\ \tfrac{1}{2}(\operatorname{div} \mathfrak{h}^\lambda - \rho) w + \mathfrak{h}^\lambda \cdot \nabla w \right\rangle\!\!\right\rangle \\
&= \left\langle\!\!\left\langle \nabla_T w \cdot \left\{ \mathbb{D}^2[d_\lambda] + \tfrac{1}{2}(\operatorname{div} \mathfrak{h}^\lambda - \operatorname{div}_T \mathfrak{h}^\lambda - \rho) I \right\} \nabla_T w \right\rangle\!\!\right\rangle \\
&\quad + \left\langle\!\!\left\langle \nabla_T w,\ \tfrac{1}{2} w \nabla_T (\operatorname{div} \mathfrak{h}^\lambda - \rho) \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \tfrac{1}{2}(\operatorname{div} \mathfrak{h}^\lambda - \rho), w^2 \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle w, \mathfrak{h}^\lambda \nabla_T w \right\rangle\!\!\right\rangle \\
&\overset{\circ}{=} \left\langle\!\!\left\langle \nabla_T w \cdot \left\{ \mathbb{D}^2[d_\lambda] + \tfrac{1}{2}(\operatorname{div} \mathfrak{h}^\lambda - \operatorname{div}_T \mathfrak{h}^\lambda - \rho) I \right\} \nabla_T w \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \tfrac{1}{2}(\operatorname{div} \mathfrak{h}^\lambda - \rho), w^2 \right\rangle\!\!\right\rangle,
\end{aligned}$$

where in the last step we just suppressed some (but, for convenience, not all) of the lower-order terms.

Henceforth, to further analyze (7.4), it will be convenient to work in geodesic frames. Given $x \in \Gamma_{geom}$ there exist a Γ-neighborhood of x, with a smooth geodesic frame $\{\tau_1, \tau_2\}$, which extends to a local orthonormal frame $\{\tau_1, \tau_2, \nu\}$ on some $\mathbb{R}^3$ neighborhood x. Moreover, we can arrange it so that

$$\nu = \nu_{ext}\big|_{\Gamma_{geom}} = \nabla \ell / |\nabla \ell|$$

for ℓ as in (2.3). The fields τ_1, τ_2 and ν are orthonormal on such a neighborhood and the tangential vector field brackets $[\tau_i, \tau_j]$ vanish. Operators D_{τ_1}, D_{τ_2} and D_ν

will stand for the action of these vectors, i.e. indicate the corresponding directional derivatives. In this frame we decompose $\mathfrak{h}^\lambda = \mathfrak{h}_1^\lambda \tau_1 + \mathfrak{h}_2^\lambda \tau_2 + \mathfrak{h}_3^\lambda \nu$.

The following two observations are crucial (for additional details on their proofs see [CLT, pp. 16–18]):

(1) Let X be a vector field on $\mathbb{R}^n$ that is tangent to Γ_{geom}. Then $\mathbb{D}^2[d_\lambda](X,X) = [\nabla\mathfrak{h}^\lambda](X,X) + \Pi(X)(\nabla d_\lambda \cdot \nu)$ where Π is the second fundamental form of the manifold Γ. But since $\nabla d_\lambda = \mathfrak{h}^\lambda$ is tangential to Γ_{geom} then $\Pi(X)(\nabla d_\lambda \cdot \nu) = 0$, so the actions of $\mathbb{D}^2[d_\lambda]$ and $[\nabla\mathfrak{h}^\lambda]$ coincide on tangential vector fields. Thus, due to part (ii) of Proposition 2.1,

$$\rho|\nabla_T w|^2 \le \nabla_T w \cdot \mathbb{D}^2[d_\lambda]\nabla_T w\,, \tag{7.5}$$

which implies that the hessian $\mathbb{D}^2[d_\lambda]$ is positive definite on the manifold Γ with the same constant $\rho > 0$ as for $[\nabla\mathfrak{h}^\lambda]$.

(2) Using the definition (2.5) of $\mathfrak{h}^\lambda$ near Γ_{geom} we have (recall also $\nu = \nabla\ell/|\nabla\ell|$ in a small tubular neighborhood of Γ_{geom})

$$\frac{d}{d\lambda}D_\nu({\mathfrak{h}^\lambda}_3) = \frac{d}{d\lambda}\nabla(\mathfrak{h}^\lambda\cdot\nu)\cdot\nu = 2\nabla\Big\{\ell|\nabla\ell|\Big\}\cdot\nu = 2|\nabla\ell|^2 + \ell\frac{\nabla(|\nabla\ell|^2)}{|\nabla\ell|}.$$

Thus, on Γ_{geom}, where $\ell \equiv 0$,

$$\frac{d}{d\lambda}D_\nu({\mathfrak{h}^\lambda}_3)\Big|_{\Gamma_{geom}} = 2|\nabla\ell|^2 > 0,$$

where strict inequality results from the nondegeneracy condition (2.3). Since $|\nabla\ell|$ is bounded away from 0 on Γ_{geom}, then by choosing λ large enough the term $D_\nu(\mathfrak{h}^\lambda)$ can be made arbitrarily large. A similar analysis yields

$$\frac{d}{d\lambda}D_{\tau_j}(\mathfrak{h}_j^\lambda)\Big|_{\Gamma_{geom}} = 2(\nabla\ell(x)\cdot\tau_j)^2\Big|_{\Gamma_{geom}} = 0 \quad\text{and}\quad \frac{d}{d\lambda}\mathfrak{h}_j^\lambda\Big|_{\Gamma_{geom}} = 0\,,\quad j=1,2,3\,.$$

So $\frac{d}{d\lambda}(\operatorname{div}\mathfrak{h}^\lambda - \operatorname{div}_T\mathfrak{h}^\lambda) = 2|\nabla\ell|^2 > 0$ on $\overline{\Gamma}_{geom}$, whence for λ large enough

$$\operatorname{div}\mathfrak{h}^\lambda - \operatorname{div}_T\mathfrak{h}^\lambda = \ge \rho, \tag{7.6}$$

and

$$\tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)\big|_{\Gamma_{geom}} = \tfrac{1}{2}\left(D_{\tau_1}({\mathfrak{h}^\lambda}_1) + D_{\tau_2}({\mathfrak{h}^\lambda}_2) + D_\nu({\mathfrak{h}^\lambda}_3) - \rho\right) \ge \rho > 0. \tag{7.7}$$

Combine the results of (7.5) and (7.6) to conclude that

$$\rho|\nabla_T w|^2 + \rho w^2 \le \Big\{\mathbb{D}^2[d_\lambda] + \frac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \operatorname{div}_T\mathfrak{h}^\lambda - \rho)I\Big\}\nabla_T w\cdot\nabla_T w + \frac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w^2\,. \tag{7.8}$$

So with the help of (7.7), the boundary products on the LHS of (6.3) control full potential energy on the boundary, as claimed in (7.1). □

Having verified Lemma 7.1, the analysis of the boundary terms on the RHS of (6.3) is almost complete. The only remaining boundary product vanishes due to the fact that (independently of any large enough λ) $\mathfrak{h}^\lambda\cdot\nu \equiv 0$ on Γ_{geom}, which supports the trace of w, so

$$\tfrac{1}{2}\langle\!\langle (\mathfrak{h}^\lambda\cdot\nu), w_t^2 - |\nabla w|^2\rangle\!\rangle = 0. \tag{7.9}$$

8. Concluding recovery estimates

Let $\lambda > \lambda_2 > \lambda_1$ with λ_1 given by Proposition 2.1 and λ_2 by Lemma 7.1 . We construct $\mathfrak{h}^\lambda$ as in Proposition 2.1 with a given λ. Apply (7.1), (7.9) to (6.3):

$$\begin{aligned}
&(\!(\tfrac{1}{2}\rho,\ w_t^2 + |\nabla w|^2)\!) + \langle\!\langle \tfrac{1}{2}\rho,\ |\nabla_T w|^2 + w^2\rangle\!\rangle \\
&\overset{\circ}{\leq} (\!([\![M_\psi,\,\Delta]\!], \mathfrak{h}^\lambda\cdot\nabla w)\!) - (\!(\psi(x)a(x)g(u_t), \mathfrak{h}^\lambda\cdot\nabla w - \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w)\!) \\
&\quad - \big(w_t, \mathfrak{h}^\lambda\cdot\nabla w + \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w\big)\Big|_0^T .
\end{aligned} \tag{8.1}$$

Identity (6.4) modulo lower-order terms becomes

$$(\!(|\nabla v|^2 + v_t^2)\!) + \langle\!\langle v^2 + |\nabla_T v|^2\rangle\!\rangle = (\!(2, v_t^2)\!) - (\!(\phi(x)a(x)g(u_t), v)\!) - \big(v_t, v\big)\Big|_0^T. \tag{8.2}$$

The commutator $[\![M_\psi,\,\Delta]\!]$ in (8.1) is a first-order operator supported only on the set $\overline{\{x:\psi\in(0,1)\}}$ which by assumption falls into the set $\{x:\phi=1\}$, where $v=u$. Hence for any $t>0$,

$$\big([\![M_\psi,\,\Delta]\!], \mathfrak{h}^\lambda\cdot\nabla w\big) \lesssim \int_{\operatorname{supp}[\![M_\psi,\,\Delta]\!]} (u^2 + |\nabla u|^2)| \lesssim \|v\|^2_{1,\Omega}\,.$$

Accordingly, for some constant $K>0$, independent of T,

$$(\!([\![M_\psi,\,\Delta]\!], \mathfrak{h}^\lambda\cdot\nabla w)\!) \leq K\big[(\!(|\nabla v|^2)\!) + \langle\!\langle v^2\rangle\!\rangle\big]. \tag{8.3}$$

Hence to be rid of this energy-level commutator we can multiply (8.2) by any $K_1 > K$, add the resulting equation to (8.1), apply (8.3) and absorb $K\big[(\!(|\nabla v|^2)\!) + \langle\!\langle v^2\rangle\!\rangle\big]$ into the LHS (getting the coefficient $K_1 - K > 0$ on the left). Afterwards we may also handle the products at 0 and T, which are controlled by energy norms, as follows from Sobolev embeddings:

$$\begin{aligned}
\big(w_t, \mathfrak{h}^\lambda\cdot\nabla w + \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w\big)\Big|_0^T - K_1\big(v_t, v\big)\Big|_0^T &\lesssim E(T) + E(0) \\
&\overset{(1.6)}{=} 2E(T) + (\!(a(x)g(u_t), u_t)\!)\,.
\end{aligned} \tag{8.4}$$

Note that even accounting for the suppressed constants, the coefficient of $E(T)$ (hidden by $\lesssim$) on the RHS is independent of T.

Finally, since $(\psi+\phi)\geq 1$ on Ω, then the outcome of the above procedures is:

(8.5)

$$\begin{aligned}
\int_0^T E(t)dt &\lesssim (\!(|\nabla v|^2 + v_t^2)\!) + \langle\!\langle v^2 + |\nabla_T v|^2\rangle\!\rangle + (\!(w_t^2 + |\nabla w|^2)\!) + \langle\!\langle |\nabla_T w|^2 + w^2\rangle\!\rangle \\
&\overset{\circ}{\lesssim} -(\!(\psi(x)a(x)g(u_t), \mathfrak{h}^\lambda\cdot\nabla w - \tfrac{1}{2}(\operatorname{div}\mathfrak{h}^\lambda - \rho)w)\!) \\
&\quad + (\!(v_t^2)\!) + (\!(\phi(x)a(x)g(u_t), v)\!) \\
&\quad + E(T) + (\!(a(x)g(u_t), u_t)\!).
\end{aligned}$$

Since the energy is nonincreasing, as (1.6) shows, it follows that

$$\int_0^T E(t)dt - CE(T) \geq \frac{1}{2}\int_0^T E(t)dt + (\tfrac{1}{2}T - C)E(T)$$

and T can be chosen large enough to ensure that $T - 2C > 0$. Since C is affected only by Sobolev embeddings, and the choice of functions $\phi, \psi, \mathfrak{h}^\lambda, a(x)$, then the magnitude of T is effectively a function of the geometry of the domain, and the cutoff $a(x)$.

The remaining terms on the RHS of (8.5) are compatible with the desired result if we expand $\nabla w = (\nabla\psi)w + \psi(\nabla w)$, and factor out supremums of ψ, ϕ $|\mathfrak{h}^\lambda|$, $\operatorname{div}\mathfrak{h}^\lambda$, etc. from all the products. The quantity v_t^2 is dominated pointwise a.e. by $C_\phi a(x) u_t^2$ according to the definition of the cutoff $\phi(x)$. These observations conclude the proof of Theorem 3.1, up to the lower-order terms (Definition 6.1). As for the latter, the energy integral multiplied by ε on the RHS of (6.5) can be absorbed into $\frac{1}{2}\int_0^T E(t)dt$ for ε sufficiently small, while the lower-order $L^2(\Omega)$ norm of the solution is inessential as follows from Proposition 6.2.

The proof of Theorem 3.1 is thus completed. $\square$

References

[Bar93] V. Barbu. *Analysis and control of nonlinear infinite-dimensional systems*, volume 190. Academic Press, Inc., Boston, MA, 1993. MR1195128 (93j:49002)

[BT10] F. Bucci and D. Toundykov. Finite-dimensional attractors for systems of composite wave/plate equations with localized damping. *Nonlinearity*, 23(9):2271–2306, 2010. MR2672645

[CDCFT09] M.M. Cavalcanti, V.N.D. Cavalcanti, R. Fukuoka, and D. Toundykov. Stabilization of the damped wave equation with Cauchy-Ventcel boundary conditions. *J. Evol. Equ.*, 9(1):143–169, 2009. MR2501356 (2010f:35256)

[CGG08] G.M. Coclite, G.R. Goldstein, and J.A. Goldstein. Stability estimates for parabolic problems with Wentzell boundary conditions. *J. Differential Equations*, 245(9):2595–2626, 2008. MR2455779 (2009j:35195)

[CGG09] G.M. Coclite, G.R. Goldstein, and J.A. Goldstein. Stability of parabolic problems with nonlinear Wentzell boundary conditions. *J. Differential Equations*, 246(6):2434–2447, 2009. MR2498847 (2010d:35193)

[CKM07] M.M. Cavalcanti, A. Khemmoudj, and M. Medjden. Uniform stabilization of the damped Cauchy-Ventcel problem with variable coefficients and dynamic boundary conditions. *J. Math. Anal. Appl.*, 328(2):900–930, 2007. MR2290021 (2008b:93080)

[CLT] M. M. Cavalcanti, I. Lasiecka, and D. Toundykov. Geometrically constrained stabilization of wave equation with Wentzell boundary conditions. *Appl. Anal.*, 2012. In press. DOI 10.1080/00036811.2011.647910.

[DLT09] M. Daoulatli, I. Lasiecka, and D. Toundykov. Uniform energy decay for a wave equation with partially supported nonlinear boundary dissipation without growth restrictions. *Discrete Contin. Dyn. Syst.*, 2(1):67–94, 2009. MR2481581 (2010f:35227)

[Doc] M. P. do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty. MR1138207 (92i:53001)

[GLLT04] R. Gulliver, I. Lasiecka, W. Littman, and R. Triggiani. *The case for differential geometry in the control of single and coupled PDEs: the structural acoustic chamber*, volume 137 of *Geometric methods in inverse problems and PDE control; IMA Vol. Math. Appl.*, pages 73–181. Springer, 2004. MR2169903 (2006g:93065)

[Hem00] A. Heminna. Stabilisation frontière de problèmes de Ventcel. *ESAIM Control Optim. Calc. Var.*, 5:591–622 (electronic), 2000. MR1799332 (2001k:93102)

[Hem01] A. Heminna. Contrôlabilité exacte d'un problème avec conditions de Ventcel évolutives pour le système linéaire de l'élasticité. *Rev. Mat. Complut.*, 14(1):231–270, 2001. MR1851730 (2002k:93010)

[IY00] V. Isakov and M. Yamamoto. *Carleman estimate with the Neumann boundary condition and its applications to the observability inequality and inverse hyperbolic problems*, volume 268 of *Differential geometric methods in the control of partial differential equations (Boulder, CO, 1999); Contemp. Math.*, pages 191–225. Amer. Math. Soc., Providence, RI, 2000. MR1804796 (2002e:35150)

[KM04] A. Khemmoudj and M. Medjden. Exponential decay for the semilinear Cauchy-Ventcel problem with localized damping. *Bol. Soc. Parana. Mat. (3)*, 22(2):97–116, 2004. MR2190136

[KM08] A. Kanoune and N. Mehidi. Stabilization and control of subcritical semilinear wave equation in bounded domain with Cauchy-Ventcel boundary conditions. *Appl. Math. Mech. (English Ed.)*, 29(6):787–800, 2008. MR2423224 (2009b:93124)

[KZ06] H. Koch and E. Zuazua. A hybrid system of PDE's arising in multi-structure interaction: coupling of wave equations in n and $n-1$ space dimensions. In *Recent trends in partial differential equations*, volume 409 of *Contemp. Math.*, pages 55–77. Amer. Math. Soc., Providence, RI, 2006. MR2243949 (2009d:35212)

[Kom94] V. Komornik. *Exact controllability and stabilization*. Masson, Paris, 1994. MR1359765 (96m:93003)

[Lag89] J. E. Lagnese. *Boundary stabilization of thin plates*, volume 10. Society for Industrial and Applied Mathematics, SIAM; Philadelphia, PA, 1989. MR1061153 (91k:73001)

[LT93] I. Lasiecka and D. Tataru. Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential Integral Equations*, 6(3):507–533, 1993. MR1202555 (94c:35129)

[LMP62] P. D. Lax, C. S. Morawetz, and R. S. Phillips. The exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle. *Bull. Amer. Math. Soc.*, 68:593–595, 1962. MR0142890 (26:457)

[LL88] J. Lagnese and J.L. Lions. "Modeling analysis and control of thin plates," Masson, Paris, 1988. MR953313 (89k:73001)

[Lem87] K. Lemrabet. Le problème de Ventcel pour le système de l'élasticité dans un domaine de $\mathbf{R}^3$. *C. R. Acad. Sci. Paris Sér. I Math.*, 304(6):151–154, 1987. MR880120 (88e:35168)

[LM88] W. Littman and L. Markus, Stabilization of hybrid system of elasticity by feedback boundary damping. *Ann. Math. Pura Appl.* 152:281–330, 1988. MR980985 (90c:93082)

[LT06] I. Lasiecka and D. Toundykov. Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms. *Nonlinear Anal.*, 64(8):1757–1797, 2006. MR2197360 (2006k:35189)

[LTZ00] I. Lasiecka, R. Triggiani, and X. Zhang. *Nonconservative wave equations with unobserved Neumann B.C.: global uniqueness and observability in one shot*, volume 268 of *Differential geometric methods in the control of partial differential equations (Boulder, CO, 1999); Contemp. Math.*, pages 227–325. Amer. Math. Soc, 2000. MR1804797 (2002b:93018)

[Nic03] S. Nicaise. Stability and controllability of an abstract evolution equation of hyperbolic type and concrete applications. *Rend. Mat. Appl. (7)*, 23(1):83–116, 2003. MR2044992 (2005a:93024)

[QR77] J.P. Quinn and D.L. Russell. Asymptotic stability and energy decay rates for solutions of hyperbolic equations with boundary damping. *Proc. Roy. Soc. Edinburgh Sect. A*, 77(1-2):97–127, 1977. MR0473539 (57:13205)

[SA76] A. Samarski and V. Andréev. *Razrostnye metody dlya ellipticheskikh uravnenij (Difference methods for elliptic equations)*. Nauka, Moscow, 1976. MR0502017 (58:19209a)

[Str] W. Strauss, Dispersal of waves vanishing on the boundary of an exterior domain. *Comm. Pure Appl. Math.* 28:265–278, 1975. MR0367461 (51:3703)

[Tat] D. Tataru. Private Communication.

[Tou07] D. Toundykov. Optimal decay rates for solutions of a nonlinear wave equation with localized nonlinear dissipation of unrestricted growth and critical exponent source terms under mixed boundary conditions. *Nonlinear Anal.*, 67(2):512–544, 2007. MR2317185 (2008f:35257)

[VM00] J. Vancostenoble and P. Martinez. Optimality of energy estimates for the wave equation with nonlinear boundary velocity feedbacks. *SIAM J. Control Optim.*, 39(3):776–797 electronic, 2000. MR1786329 (2001i:35193)

[War06] M. Warma. The Robin and Wentzell-Robin Laplacians on Lipschitz domains. *Semigroup Forum*, 73(1):10–30, 2006. MR2277314 (2007g:35032)

[War10] M. Warma. Analyticity on L_1 of the heat semigroup with Wentzell boundary conditions. *Arch. Math. (Basel)*, 94(1):85–89, 2010. MR2581338 (2011b:35185)

[Ven59] A. D. Ventcel'. On boundary conditions for multi-dimensional diffusion processes. *Theor. Probability Appl.*, 4:164–177, 1959. MR0121855 (22:12585)

Department of Mathematics, State University of Maringá, 87020-900, Maringá, PR, Brazil.

E-mail address: mmcavalcanti@uem.br

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22904 – and – Department of Mathematics, King Fahd University of Petroleum and Minerals, Dhahram, 31261, Saudi Arabia

E-mail address: il2v@virginia.edu

Department of Mathematics, University of Nebraska-Lincoln, Lincoln, Nebraska 68588

E-mail address: dtoundykov@math.unl.edu