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L^p-THEORY FOR STRONG SOLUTIONS TO FLUID-RIGID BODY INTERACTION IN NEWTONIAN AND GENERALIZED NEWTONIAN FLUIDS

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Abstract. Consider the system of equations describing the motion of a rigid body immersed in a viscous, incompressible fluid of Newtonian or generalized Newtonian type. The class of fluids considered includes in particular shear-thinning or shear-thickening fluids of power-law type of exponent $d \geq 1$. We develop a method to prove that this system admits a unique, local, strong solution in the L^p-setting. The approach presented in the case of generalized Newtonian fluids is based on the theory of quasi-linear evolution equations and requires that the exponent p satisfies the condition $p > 5$.

1. Introduction

The study of the motions of rigid bodies immersed in a fluid is a classical problem of fluid mechanics. Rigorous mathematical studies on this coupled system were initiated in the works of Weinberger [46] and Sauer [39], who investigated the stationary problem. In [40] it was shown by Serre that for every geometry of the rigid body, at least one stationary solution exists.

Starting from these investigations, Galdi and several of his coworkers considered the stationary and instationary problem for prescribed, steady, self-propelled and free movements of the rigid body. They proved existence theorems for weak and strong solutions to this system under various assumptions; see e.g. [17], [21], [19] and [20]. For a detailed discussion of the problem in the Newtonian case, we refer to [18].

Different approaches to global *weak* solutions for the instationary problem are due to Conca, San Martín and Tucsnak [4], to Gunzburger, Lee and Seregin [25] and to Desjardins and Esteban [9] and [8]. The case of several bodies, regardless of possible contacts, was considered in the weak setting by Feireisl [12], Hoffmann and Starovoitov [27], and San Martín, Staravoitov and Tucsnak [38].

In this paper, we develop an *L^p-theory* for strong solutions to the fluid-rigid body interaction problem, for Newtonian and generalized Newtonian fluids. The L^p-theory is of central importance in the case of generalized Newtonian fluids, where the assumption $p > 5$ will be needed.

In order to describe our approach, denote the bounded domain occupied by the rigid body by $\mathcal{B}(t)$ and the exterior domain filled by the fluid by $\mathcal{D}(t) := \mathbb{R}^3 \backslash \overline{\mathcal{B}(t)}$. The interface between body and fluid is denoted by $\Gamma(t)$ and the outer normal at

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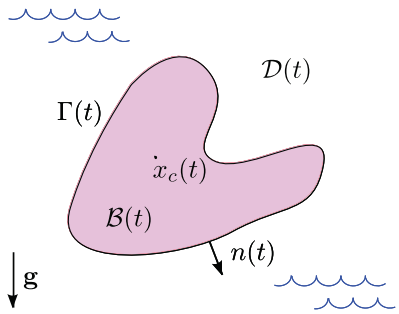

$\Gamma(t)$ is denoted by $n(t)$. We write $\mathcal{Q}_{\mathcal{D}} := \{(t,x) \in \mathbb{R}^4 : t \in \mathbb{R}_+, x \in \mathcal{D}(t)\}$ and similarly, $\mathcal{Q}_\Gamma$. The fluid's motion is governed by the equations

$$\begin{cases} v_t + \operatorname{div} \mathbf{T}(v,q) + (v \cdot \nabla)v &= f, & \text{in } \mathcal{Q}_{\mathcal{D}}, \\ \operatorname{div} v &= 0, & \text{in } \mathcal{Q}_{\mathcal{D}}, \\ v &= v_{\mathcal{B}}, & \text{on } \mathcal{Q}_\Gamma, \\ v(0) &= v_0, & \text{in } \mathcal{D}, \end{cases} \tag{1.1}$$

where v and q denote the velocity and pressure of the fluid, $\mathbf{T}(v,q)$ its stress tensor, and $\mathcal{D} = \mathcal{D}(0)$. In the case of a Newtonian fluid, the stress tensor $\mathbf{T}(v,q)$ is given by

$$\mathbf{T}(v,q) = \mu_0 \mathcal{E}^{(v)} - q\mathrm{Id},$$

where $\mathcal{E}^{(v)} := \frac{1}{2}(\nabla v + (\nabla v)^T)$ denotes the rate of strain tensor and $\mu_0 > 0$ the viscosity of the fluid. For generalized Newtonian fluids, we allow the stress tensor to be of the form

$$\mathbf{T}^\mu(v,q) := \mu(|\mathcal{E}^{(v)}|_2^2)\mathcal{E}^{(v)} - q\mathrm{Id},$$

where $\mu \in C^{1,1}(\mathbb{R}_+;\mathbb{R})$ is a viscosity function satisfying $\mu(s) > 0$ and $\mu(s) + 2s\mu'(s) > 0$ for all $s \geq 0$. Here, $|\mathcal{E}^{(v)}|_2$ denotes the Hilbert-Schmidt norm of $\mathcal{E}^{(v)}$.

The fluid equations are coupled to the balance equations for the momentum and the angular momentum of the rigid body,

$$\begin{cases} \mathrm{m}\eta'(t) + \int_{\Gamma(t)} \mathbf{T}(v,q)(t,x)n(t,x)\,\mathrm{d}\sigma &= \mathbf{F}(t), \quad t \in \mathbb{R}_+, \\ (J\omega)'(t) + \int_{\Gamma(t)} (x - x_c(t)) \times \mathbf{T}(v,q)(t,x)n(t,x)\,\mathrm{d}\sigma &= \mathbf{M}(t), \quad t \in \mathbb{R}_+, \\ \eta(0) &= \eta_0, \\ \omega(0) &= \omega_0, \end{cases} \tag{1.2}$$

which contain the drag force and the torque exerted by the fluid onto the body. The constants m and J are the body's mass and inertia tensor, x_c is the position of its center of gravity, and η and ω denote its translational and angular velocity, so that

$$v_{\mathcal{B}}(t,x) := \eta(t) + \omega(t) \times (x - x_c(t))$$

is its full velocity. The functions $f, \mathbf{F}$ and $\mathbf{M}$ denote external forces and torques. To model the free fall of the body under the influence of gravitation, we set $f = \mathbf{g}$, $\mathbf{F} = \mathrm{m}\mathbf{g}$, and $\mathbf{M} = 0$ for a constant vector $\mathbf{g}$.

Since the domain $\mathcal{Q}_{\mathcal{D}}$ depends on the motion of the rigid body, we are dealing with a moving boundary problem. It is hence natural to transform equation (1.1) to a fixed domain $\mathcal{D} := \mathcal{D}(0)$.

There are several possibilities for this transform. We will illustrate them in the case of Newtonian fluids. The first one introduced by Galdi [18] is linear, meaning that the whole space is rigidly rotated and shifted back to its original position at every time $t > 0$. This is only possible if the fluid fills the unbounded domain exterior to the obstacle or a sphere. In the fluid equations, this transform generates a drift term with unbounded coefficients, i.e. the new fluid operator has the form $Lu := P(\Delta u + (\Omega \times x \cdot \nabla)u - \Omega \times u)$ in the purely rotational case. Here P denotes the Helmholtz projection and Ω the rotational velocity. It was shown by Hishida [26] that L generates a C_0-semigroup on $L^2_\sigma(\mathcal{D})$, which is, however, *not* analytic. One of the fundamental difficulties of this approach thus arises as the transformed problem is no longer parabolic. In [19], Galdi and Silvestre used this approach to show the existence of strong L^2-solutions to the coupled system (1.1) and (1.2). They apply a Faedo-Galerkin technique to the transformed system, where the underlying Hilbert space has to be constructed to respect the body's geometric properties and restrict the movement of the interface to rigid motions.

A second approach is characterized by a non-linear, "local" change of coordinates which only acts in a suitable bounded neighborhood of the obstacle. Tucsnak, Cumsille and Takahashi used this transform, initially introduced by Inoue and Wakimoto [28], and showed the existence of a unique, local strong L^2-solution to the coupled problem on bounded and unbounded fluid domains in two and three space dimensions, cf. [43], [44], [6], [5] and [37].

The approach we will be using is based on this second change of variables. This transform is defined in a way as to preserve the solenoidal condition on the fluid velocity and not to change the regularity of the solutions. The rigid body equations also change and become non-linear.

After this transformation of the problem onto a fixed domain, our approach is based on maximal regularity estimates for the linearized transformed problem. Our argument relies on the extension of the classical maximal regularity estimates for the Stokes problem to the fluid-rigid body interaction situation, a local pressure estimate, and on an appropriate representation of the forces which act on the rigid body. We then rewrite the full non-linear transformed problem as a fixed point problem in the space of maximal regularity and show in this way the existence of a unique strong solution to the coupled problem in the L^p-setting.

The advantage of our approach is not only that it allows us to prove the existence of an L^p-solution, but that it can easily be modified to fit the cases of unbounded or bounded fluid domains and different types of external forces. Moreover, it allows us to investigate the interaction of a rigid body with a *generalized Newtonian fluid*.

The investigation of power-law type fluids started with the pioneering work of Ladyzhenskaya [30]. It was developed further by Málek, Nečas, Ruţička, Rajagopal, Frehse and Steinhauer; see e.g. [31], [34], [32], [14], [15], and the survey article [33]. The existence of strong solutions for power-law type fluids of exponent $d > \frac{7}{5}$ was shown in [10]. Note that all these results deal with the situation of pure fluid flow on a *fixed* domain.

In all of these works, the integrability index p of the solution (v, q) of the pure fluid equation is determined by the power-law exponent d, due to the use of monotone operator techniques for the d-coercive fluid operator

$$\operatorname{div} \mu(|\mathcal{E}^{(v)}|)\mathcal{E}^{(v)}.$$

On the other hand, using different techniques, Bothe and Prüss [3] recently showed that under the conditions described in (2.2), the generalized Navier-Stokes problem governed by the stress tensor $\mathbf{T}^\mu$, as defined above, yields a local L^p-strong solution for all $p > 5$. In particular, this includes a large class of physically reasonable power-law fluids of exponent $d \geq 1$.

In view of these results, the development of an L^p-theory for the fluid-rigid body interaction equations is essential in the case of generalized Newtonian fluids.

There exists only a few works concerning rigorous mathematical analysis of the coupled problem. The existence of *weak* solutions was proved recently in [13]. There it was shown that a weak solution exists globally in time for shear-thickening power-law type fluids of exponent $d \geq 4$. The result also includes the case of several moving obstacles, and it is shown that in this system, contact of obstacles does not occur. In [22], the problem of orientation of rigid bodies in steady fall is studied in second-order liquids.

We prove the existence of a unique local strong L^p-solution for the coupled generalized Newtonian model. In this case, the dependence of the viscosity on the shear rate implies that the operator A given by

$$\begin{aligned} A(v)_i &:= (\operatorname{div}(\mu(|\mathcal{E}^{(v)}|_2^2)\mathcal{E}^{(v)}))_i \\ &= \mu(|\mathcal{E}^{(v)}|_2^2)\Delta v_i + 2\mu'(|\mathcal{E}^{(v)}|_2^2) \sum_{j,k,l=1}^{3} \varepsilon_{ij}^{(v)} \varepsilon_{kl}^{(v)} \partial_j \partial_l v_k, \end{aligned}$$

which replaces the Laplacian in the fluid equations, is quasi-linear. Freezing A at a reference solution v_*, we obtain the linear operator A_* given by

$$(A_* v)_i = \mu(|\mathcal{E}^{(v_*)}|_2^2)\Delta v_i + 2\mu'(|\mathcal{E}^{(v_*)}|_2^2) \sum_{j,k,l=1}^{3} \varepsilon_{ij}^{(v_*)} \varepsilon_{kl}^{(v_*)} \partial_j \partial_l v_k$$

and investigate the above fluid-rigid body interaction problem as a *quasi-linear evolution equation.* In [3], Bothe and Prüss showed that the fluid equations governed by A_* still yield maximal L^p-regularity.

The coarse structure of the paper is the following. In the next section, we will state the main results for both the Newtonian and the generalized Newtonian cases. We will then first focus on the proof for Newtonian fluids. The crucial step is to solve and obtain maximal regularity estimates for the transformed and linearized system by giving a suitable reformulation of the problem as a *linear* fixed point problem in $W^{1,p}(0,T;\mathbb{R}^6)$. The reformulation and the solution rely on classical maximal regularity estimates of the Stokes problem and on good pressure estimates near the boundary. In the second part of the paper, starting from Section 8, we consider the generalized Newtonian case. We show how the proof for Newtonian fluids may be extended to work in this more general case.

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2. Main results and strategy of the proof

For future reference, we rewrite (1.1) and (1.2) into one system of equations in the unknowns v, q, η and ω,

$$
\begin{cases}
v_t + (v\cdot\nabla)v &= \mathbf{g} - \operatorname{div}\mathbf{T}(v,q), & \text{in } \mathcal{Q}_{\mathcal{D}},\\
\operatorname{div} v &= 0, & \text{in } \mathcal{Q}_{\mathcal{D}},\\
v(t,x) &= \omega(t)\times(x-x_c(t)) + \eta(t), & \text{on } \mathcal{Q}_{\Gamma},\\
v(0) &= v_0, & \text{on } \mathcal{D},\\
\mathrm{m}\eta' &= \mathrm{m}\mathbf{g} - \int_{\Gamma(t)} \mathbf{T}(v,q)n(t)\,d\sigma, & t>0,\\
(J\omega)' &= -\int_{\Gamma(t)}(x-x_c(t))\times\mathbf{T}(v,q)n(t)\,d\sigma, & t>0,\\
\eta(0) &= \eta_0,\\
\omega(0) &= \omega_0,
\end{cases}
\tag{2.1}
$$

where $\mathbf{T}$ may be of the form

$$\mathbf{T}(v,q) = \mu_0\mathcal{E}^{(v)} - q\mathrm{Id} \tag{N}$$

or

$$\mathbf{T}^{\mu}(v,q) := \mu(|\mathcal{E}^{(v)}|_2^2)\mathcal{E}^{(v)} - q\mathrm{Id}, \tag{G}$$

where

$$\mathcal{E}^{(v)} := \frac{1}{2}(\nabla v + (\nabla v)^T), \qquad \varepsilon_{ij}^{(v)} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$$

denotes the deformation tensor and

$$|\mathcal{E}^{(v)}|_2^2 = \sum_{i,j=1}^{n} (\varepsilon_{ij}^{(v)})^2$$

its Hilbert-Schmidt norm. For the viscosity function, we assume that $\mu \in C^{1,1}(\mathbb{R}_+;\mathbb{R})$ and that the assumptions

$$\mu(s) > 0 \quad \text{and} \quad \mu(s) + 2s\mu'(s) > 0 \quad \text{for all } s \geq 0 \tag{2.2}$$

are satisfied. Note that the second growth condition follows provided the function $s \mapsto \mu(s)\sqrt{s}$ is strictly increasing. Physically, this is reasonable because it means that the viscous stress $\mu(|\mathcal{E}|_2^2)|\mathcal{E}|_2$ is increasing with increasing rate of shear $|\mathcal{E}|_2$. In particular, (2.2) includes shear-thinning ($d<2$) and shear-thickening ($d>2$) power-law type fluids of the kind

$$\mu(|\mathcal{E}^{(v)}|_2^2) = \mu_0(1+|\mathcal{E}^{(v)}|_2^2)^{d/2-1}. \tag{2.3}$$

For information on the modeling of non-Newtonian fluids, we refer for example to [29] and [36].

We use the following function spaces. For a domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, the Sobolev spaces of order $m \in \mathbb{N}$ are denoted by $W^{m,q}(D)$. The inhomogeneous Sobolev space of order one is given by the equivalence classes of functions in $D^{1,q} := \{f \in L^1_{\mathrm{loc}}(D) : \partial^\alpha f \in L^q(\Omega), |\alpha| = 1\}$ with respect to constants. It is denoted by $\widehat{W}^{1,q}(D)$. We denote by $H^{s,p}(D)$ the scale of Bessel potential spaces. On a domain of class C^1 and for $0 < s \leq m$, they can be given by complex interpolation of Sobolev spaces,

$$H^{s,q}(D) := [L^q(D), W^{m,q}(D)]_{\frac{s}{m}}$$

and they are compatible with the Sobolev spaces of integer order on the above type of domains. For every $0 < s < m, 1 \leq q < \infty, 1 \leq p \leq \infty$, we define Besov spaces

on domains by real interpolation of Sobolev spaces,

$$B^s_{q,p}(D) := (L^q(D), W^{m,q}(D))_{s/m,p},$$

and we denote the Sobolev-Slobodeckiĭ spaces $W^{s,q}(D)$ by

$$W^{s,q}(D) := \begin{cases} W^{s,q}(D) & \text{if } s \in \mathbb{N}, \\ B^s_{q,q}(D) & \text{if } s \notin \mathbb{N}. \end{cases}$$

They coincide with the Bessel potential spaces if $q = 2$ and we often use the fact that $H^{s+\varepsilon,q}(D) \hookrightarrow W^{s+\varepsilon/2,q}(D) \hookrightarrow H^{s,q}(D)$ for every $\varepsilon > 0$. For more information on these function spaces, we refer to [45]. Finally, we say that $v_0 \in B^{2-2/p}_{q,p}(\mathcal{D})$ satisfies the compatibility condition (CN) if

$$v_0 - z_0 \in (L^q_\sigma(\mathcal{D}), D(A_q))_{1-\frac{1}{p},p}, \tag{2.4}$$

for some $z_0 \in C^2(\mathcal{D})$ satisfying $z_0(x) = \omega_0 \times x + \eta_0$ for $x \in \Gamma$ and $\operatorname{div} z_0 = 0$. Here, $D(A_q)$ denotes the domain of the Stokes operator in $L^q(\mathcal{D})$. Note that condition (2.4) does not depend on the choice of z_0 and that a function $z_0 \in C^2(\mathcal{D})$ satisfying $z_0(x) = \omega_0 \times x + \eta_0$ for $x \in \Gamma$ and $\operatorname{div} z_0 = 0$ always exists; see Section 4.

We may now state our two main results on the existence of a unique, local strong solution for problem (2.1) under natural assumptions on the data.

Theorem 2.1. *Let* $\mathbf{T}$ *be of type* (N). *Assume that* $\frac{3}{2q} + \frac{1}{p} \le \frac{3}{2}$, $\eta_0, \omega_0 \in \mathbb{R}^3$ *and that* $\mathcal{D}$ *is an exterior domain of class* $C^{2,1}$. *Let* $v_0 \in B^{2-2/p}_{q,p}(\mathcal{D})$ *be such that the compatibility condition (CN) is satisfied. Then there exists a maximal time interval* $[0,T)$, *such that problem* (2.1) *admits a unique strong solution*

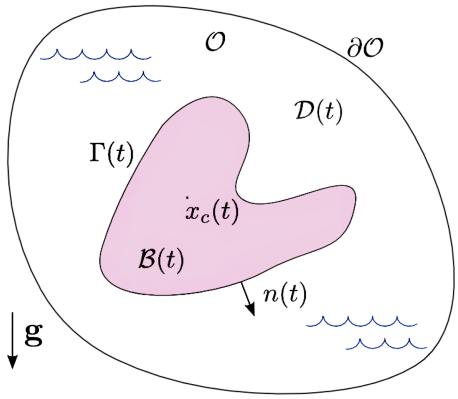
$$\begin{aligned} v &\in L^p(0,T; W^{2,q}(\mathcal{D}(\cdot))) \cap W^{1,p}(0,T; L^q(\mathcal{D}(\cdot))), \\ q &= q_0 + \mathbf{g} \cdot Y, \quad q_0 \in L^p(0,T; \widehat{W}^{1,q}(\mathcal{D}(\cdot))), Y \in C^1(0,T; C^\infty(\mathcal{D}(\cdot))), \\ \eta &\in W^{1,p}(0,T), \\ \omega &\in W^{1,p}(0,T). \end{aligned}$$

Theorem 2.2. *Let* $\mathbf{T}^\mu$ *be of type* (G). *Assume that* $p > 5$, $\eta_0, \omega_0 \in \mathbb{R}^3$, μ *satisfies* (2.2) *and that* $\mathcal{D}$ *is an exterior domain of class* $C^{2,1}$. *Let* $v_0 \in W^{2-2/p,p}(\mathcal{D})$ *be such that* $\operatorname{div} v_0 = 0$ *and* $v_0|_\Gamma(x) = \eta_0 + \omega_0 \times x$. *Then there exists a maximal interval* J_T, $T > 0$, *such that problem* (2.1) *admits a unique strong solution*

$$\begin{aligned} v &\in L^p(J_T; W^{2,p}(\mathcal{D}(\cdot))) \cap W^{1,p}(J_T; L^p(\mathcal{D}(\cdot))), \\ q &= q_0 + \mathbf{g} \cdot Y, \quad q_0 \in L^p(J_T; \widehat{W}^{1,p}(\mathcal{D}(\cdot))),\ Y \in C^1(J_T; C^\infty(\mathcal{D}(\cdot))), \\ \eta &\in W^{1,p}(J_T), \\ \omega &\in W^{1,p}(J_T). \end{aligned}$$

Note that the stress tensor of type (G) includes the stress tensor of type (N) as the special case $\mu = \text{const}$. In this sense, Theorem 2.2 generalizes Theorem 2.1. On the other hand, Theorem 2.1 shows that in the Newtonian case, less restrictive assumptions on p, q are needed. The following remark regards both theorems. In the generalized Newtonian case, this means that $p = q > 5$ must be assumed.

Remark 2.3. a) Some function Y in the condition for the pressure q is needed since the constant gravity vector $\mathbf{g}$ is not integrable.



b) The maximal time T of existence of the solution can be characterized as follows. Either T can be arbitrarily large or one of the functions

$$t \mapsto \|v(t)\|_{B^{2-2/p}_{q,p}(\mathcal{D}(t))}, \quad t \mapsto |\eta(t)|, \quad t \mapsto |\omega(t)|$$

is unbounded on $[0,T)$, because otherwise the solution could be extended.

c) A characterization of $(L^q_\sigma(\mathcal{D}), D(A_q))_{1-\frac{1}{p},p}$ was given by Amann in [1, Thm. 3.4], where it is shown that

$$(L^q_\sigma(\mathcal{D}), D(A_q))_{1-\frac{1}{p},p} = \{g \in B^{2-2/p}_{q,p}(\mathcal{D}) : \operatorname{div} g = 0, g|_\Gamma = 0\}, \quad \text{if } \frac{1}{q} < 2 - \frac{2}{p} \leq 2$$

and

$$(L^q_\sigma(\mathcal{D}), D(A_q))_{1-\frac{1}{p},p} = \{g \in B^{2-2/p}_{q,p}(\mathcal{D}) : \operatorname{div} g = 0, g \cdot N|_\Gamma = 0\}, \quad \text{if } 0 \leq 2 - \frac{2}{p} < \frac{1}{q}.$$

We obtain similar results in the case of a bounded fluid domain. More precisely, consider (2.1) on a bounded domain $\mathcal{O} \subset \mathbb{R}^3$, which contains an obstacle $\mathcal{B}$ and a liquid filling $\mathcal{D} := \mathcal{O} \backslash \mathcal{B}$.

Setting Dirichlet boundary conditions on $\partial\mathcal{O}$, the new initial-boundary value problem reads

$$\left\{\begin{array}{rcll} v_t + (v \cdot \nabla)v &=& \mathbf{g} - \operatorname{div} \mathbf{T}(v,q), & \text{in } \mathcal{Q}_\mathcal{D}, \\ \operatorname{div} v &=& 0, & \text{in } \mathcal{Q}_\mathcal{D}, \\ v(t,x) &=& \omega(t) \times (x - x_c(t)) + \eta(t), & \text{on } \mathcal{Q}_\Gamma, \\ v &=& 0, & \text{on } \partial\mathcal{O}, \\ v(0) &=& v_0, & \text{in } \mathcal{D}, \\ \mathrm{m}\eta' &=& \mathrm{m}\mathbf{g} - \int_{\Gamma(t)} \mathbf{T}(v,q)n(t)\, d\sigma, & t > 0, \\ (J\omega)' &=& -\int_{\Gamma(t)} (x - x_c(t)) \times \mathbf{T}(v,q)n(t)\, d\sigma, & t > 0, \\ \eta(0) &=& \eta_0, & \\ \omega(0) &=& \omega_0. & \end{array}\right. \tag{2.5}$$

In Section 7, we prove that this problem can be solved by using the same approach as described for system (2.1). However, it furnishes the additional difficulty of possible contacts of body and wall. The body needs to start at some distance from the boundary, and we need to restrict the lifespan of the solution to a time interval in which no contact occurs. The following result then follows in analogy to Theorem 2.1.

Theorem 2.4. *Let $\frac{3}{2q}+\frac{1}{p}\leq\frac{3}{2}$, and let $\mathcal{O}\subset\mathbb{R}^3$ be a bounded domain with boundary of class $C^{2,1}$ and $\overline{\mathcal{B}}\subset\mathcal{O}$ as above. Let η_0,ω_0 be in $\mathbb{R}^3$ and $v_0\in B^{2-2/p}_{q,p}(\mathcal{D})$, such that* (2.4) *and the corresponding conditions on $\partial\mathcal{O}$ are satisfied. If*

$$\operatorname{dist}(\mathcal{B}(0),\partial\mathcal{O})>d \qquad \text{for some } d>0,$$

then there exists an interval $[0,T)$, such that problem (2.5) *admits a unique strong solution*

$$\begin{aligned} v &\in L^p(0,T;W^{2,q}(\mathcal{D}(\cdot)))\cap W^{1,p}(0,T;L^q(\mathcal{D}(\cdot))),\\ q &= q_0+\mathbf{g}\cdot Y,\quad q_0\in L^p(0,T;\widehat{W}^{1,q}(\mathcal{D}(\cdot))), Y\in C^1(0,T;C^\infty(\mathcal{D}(\cdot))),\\ \eta &\in W^{1,p}(0,T),\\ \omega &\in W^{1,p}(0,T). \end{aligned}$$

Remark 2.5. Theorem 2.2 can be modified to fit the case of a bounded domain $\mathcal{O}$ holding both the fluid and the rigid body.

The proof of Theorem 2.1 is organized as follows. In a first step, in Section 3, we transform the equations in (2.1) to the fixed cylindrical domain $(0,T)\times\mathcal{D}$. Following the approach of Inoue and Wakimoto [28], this transform preserves the solenoidal condition on the fluid velocity and does not change the regularity of the solutions; cf. also [6], [11] and [43]. The rigid body equations also change. It is important to note that in our situation, the transform is an unknown part of the solution.

Section 4 is devoted to finding a solution for the linearized transformed problem and proving maximal regularity estimates. Our argument depends on an extension of the maximal regularity of the Stokes problem to the fluid-rigid body interaction situation, on suitable local pressure estimates and on an appropriate representation of the forces which act on the rigid body, as in [18].

In Section 5, we rewrite the full non-linear transformed problem as a fixed point problem in the space of maximal regularity. Our strategy relies on the linear estimates established in Section 4 and on embedding properties of the underlying spaces.

The sixth section is devoted to estimates on the coordinate transform and on the right-hand sides appearing in the transformed systems, which make the fixed point mapping contractive on small time intervals.

In Section 7, we prove Theorems 2.1 and 2.4.

Starting from Section 8, we show how this argument may be modified to prove Theorem 2.2. The crucial preliminary ingredients are the maximal L^p-regularity of the generalized Stokes operator and local pressure estimates for this problem, given in Section 10. In Sections 9 and 11, we show how the change of coordinate and the fixed point argument apply in this situation.

3. Coordinate transform

Let $m(t)$ denote the skew-symmetric matrix satisfying

$$m(t)x=\omega(t)\times x. \tag{3.1}$$

We consider the differential equation

$$\left\{\begin{aligned} \partial_t X_0(t,y) &= m(t)(X_0(t,y)-x_c(t))+\eta(t), && (0,T)\times\mathbb{R}^3,\\ X_0(0,y) &= y, && y\in\mathbb{R}^3. \end{aligned}\right. \tag{3.2}$$

Its solution is of the form $X_0(t,y) = Q(t)y + x_c(t)$, with some matrix $Q(t) \in SO(3)$, where $Q \in W^{2,p}(0,T;\mathbb{R}^{3\times 3})$, if $\eta, \omega \in W^{1,p}(0,T)$. The corresponding inverse $Y_0(t)$ of $X_0(t)$ is given by

$$Y_0(t,x) = Q^T(t)(x - x_c(t))$$

and satisfies the differential equation

$$\left\{ \begin{array}{rcll} \partial_t Y_0(t,x) & = & -M(t)Y_0(t,x) - \xi(t), & (0,T)\times\mathbb{R}^3, \\ Y_0(0,x) & = & x, & x \in \mathbb{R}^3, \end{array} \right. \tag{3.3}$$

where

$$M(t) := Q^T(t)m(t)Q(t), \quad \xi(t) := Q^T(t)\eta(t). \tag{3.4}$$

We modify the diffeomorphisms X_0, Y_0 of $\mathcal{D}(t)$ and $\mathcal{B}(t)$ such that they rotate space only on a suitable open neighborhood of the rotating and translating body in order to avoid rotation at infinity. We may define this new diffeomorphism X implicitly, using an ODE of the type (3.2),

$$\left\{ \begin{array}{rcll} \partial_t X(t,y) & = & b(t, X(t,y)), & (0,T)\times\mathbb{R}^3, \\ X(0,y) & = & y, & y \in \mathbb{R}^3, \end{array} \right. \tag{3.5}$$

where the right-hand side b determines the modified velocity of this change of coordinates. We require that close to the rigid body, b should be equal to the velocity of the body and that further away, it should be equal to zero. In addition, b should be smooth in the space variables and divergence free, in order to preserve the divergence-free condition on the fluid velocity. We choose open balls $B_1, B_2 \subset \mathbb{R}^3$ such that $\overline{\mathcal{B}} \subset B_1 \subset \overline{B_1} \subset B_2$, and we define a cut-off function $\chi \in C^\infty(\mathbb{R}^3;[0,1])$,

$$\chi(y) := \left\{ \begin{array}{ll} 1 & \text{if } y \in \overline{B_1}, \\ 0 & \text{if } y \in \mathcal{D} \setminus B_2, \end{array} \right. \tag{3.6}$$

and a time-dependent vector field $b : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ by

$$b(t,x) := \chi(x - x_c(t))[m(t)(x - x_c(t)) + \eta(t)] - B_K(\nabla_x \chi(\cdot - x_c(t))m(t)(\cdot - x_c(t)))(x). \tag{3.7}$$

Here, $B_K : C_c^\infty(K;\mathbb{R}) \to C_c^\infty(K;\mathbb{R}^3)$ indicates the Bogovskiĭ operator corresponding to an open set K containing the annulus $\overline{B_2}\backslash B_1$. It is a bounded operator yielding $\operatorname{div} B_K g = g$ if $\int_K g = 0$ (cf. Proposition 4.2). The function $b(t)$ belongs to $C^\infty(\mathbb{R}^3)$ and it is bounded. Due to

$$\int_{\overline{B_2}\backslash B_1} \nabla\chi(y)m(t)(y - x_c(t))\,\mathrm{d}y = \int_{\overline{B_2}\backslash B_1} \chi(y - x_c(t))\operatorname{tr} m(t)\,\mathrm{d}y = 0,$$

the correction by the Bogovskiĭ term yields $\operatorname{div} b(t) = 0$ for all $t \in [0,T]$, as

$$\operatorname{div} b = \chi \operatorname{tr} m + (\nabla\chi)m(\cdot - x_c) - (\nabla\chi)m(\cdot - x_c) = 0.$$

It follows that $b \in W^{1,p}(0,T;C_{c,\sigma}^\infty(\mathbb{R}^3))$ and $b|_\Gamma = m(x - x_c) + \eta$.

Given $\eta, \omega \in W^{1,p}(0,T)$, the equation (3.5) admits a unique solution $X \in C^1((0,T);C^\infty(\mathbb{R}^n))$, by the Picard-Lindelöf theorem. The solution has continuous mixed partial derivatives $\frac{\partial^{|\alpha|+1}X}{\partial t(\partial y_j)^\alpha}$, $\frac{\partial^{|\alpha|}X}{(\partial y_j)^\alpha}$, where $\alpha \in \mathbb{N}_0^3$ denotes a multi-index. By uniqueness, the function $X(t,\cdot)$ is bijective, and we denote its inverse by $Y(t,\cdot)$. Since $\operatorname{div} b = 0$, Liouville's theorem implies that X and Y are volume-preserving, i.e.

$$J_X(t,y)J_Y(t,X(t,y)) = \mathrm{Id} \text{ and } \det J_X(t,y) = \det J_Y(t,x) = 1,$$

for the Jacobians J_X and J_Y. Given X, the inverse transform Y satisfies the differential equation

$$\left\{\begin{array}{rcll} \partial_t Y(t,x) & = & b^{(Y)}(t,Y(t,x)), & (0,T)\times\mathbb{R}^3, \\ Y(0,x) & = & x, & x\in\mathbb{R}^3, \end{array}\right. \tag{3.8}$$

where

$$b^{(Y)}(t,y) := -J_X^{-1}(t,y)b(t,X(t,y)). \tag{3.9}$$

Note that by this definition, $b^{(Y)}$ and Y possess the same space and time regularity as b and X. Within the ball B_1, X,Y coincide with X_0,Y_0; whereas in the complement of $K\cup\overline{B_1}$, $\partial_t X(t,y)=\partial_t Y(t,x)=0$.

For $(t,y)\in[0,T)\times\mathbb{R}^3$ we now define

$$\begin{array}{rcl} u(t,y) & := & J_Y(t,X(t,y))v(t,X(t,y)), \\ p(t,y) & := & q(t,X(t,y)), \\ \Omega(t) & := & Q^T(t)\omega(t), \\ \xi(t) & := & Q^T(t)\eta(t), \\ \mathbf{G}(t,y) & := & J_Y(t,X(t,y))\mathbf{g}, \\ \mathcal{T}(u(t,y),p(t,y)) & := & Q^T(t)\mathbf{T}(Q(t)u(t,y),p(t,y))Q(t). \end{array} \tag{3.10}$$

In particular, it follows from (3.4), (3.1), and the fact that for matrices $Q\in SO(3)$, $Q(a\times b)=Qa\times Qb$, that

$$M(t)x = Q^T(t)m(t)Q(t)x = Q^T(t)(\omega(t)\times Q(t)x) = (Q^T(t)\omega(t))\times x = \Omega(t)\times x.$$

In addition, the outer normal at $\mathcal{B}$ is denoted by N and it satisfies $N=Q^T(t)n(t)$. The transformed inertia tensor $I=Q^T(t)J(t)Q(t)$ no longer depends on time since for all $a,b\in B_1$,

$$a\cdot I\cdot b = \frac{|\mathcal{B}|}{m}\int_{\mathcal{B}(0)}(a\times y)\cdot(b\times y). \tag{3.11}$$

The transforms X,Y satisfy the assumptions in order to calculate the fluid part of the transformed system in u and p as in the original work by Inoue and Wakimoto and to show that the two systems of equations are equivalent, i.e. the fluid part of the original problem (2.1) admits a strong solution (v,q) if and only if there exists a corresponding solution $(u,p)\in X^T_{p,q}\times Y^T_{p,q}$ to the fluid part of the transformed problem below [28, Proposition 2.1, Theorem 2.5]. In order to obtain the transformed rigid body equations, we use that

$$\int_{\Gamma(t)}\mathbf{T}(v,q)n(t)\,\mathrm{d}\sigma = Q\int_\Gamma \mathcal{T}(u,p)N\,\mathrm{d}\sigma$$

and

$$\int_{\Gamma(t)}(x-x_c(t))\times\mathbf{T}(v,q)n(t)\,\mathrm{d}\sigma = Q\int_\Gamma y\times\mathcal{T}(u,p)N\,\mathrm{d}\sigma.$$

On the cylindrical domain $(0,T)\times\mathcal{D}$ we obtain

(3.12)
$$\begin{cases} u_t+(\mathcal{M}-\mathcal{L})u &= \mathbf{G}-\mathcal{N}(u)-\mathcal{G}p, & \text{in } (0,T)\times\mathcal{D},\\ \operatorname{div} u &= 0, & \text{in } (0,T)\times\mathcal{D},\\ u(t,y) &= \Omega(t)\times y+\xi(t), & \text{on } (0,T)\times\Gamma,\\ u(0) &= v_0 & \text{in } \mathcal{D},\\ \mathrm{m}\xi' &= \mathrm{m}\mathbf{G}(\cdot,0)-\mathrm{m}(\Omega\times\xi)-\int_\Gamma \mathcal{T}(u,p)N\,\mathrm{d}\sigma, & t\in(0,T),\\ I\Omega' &= \Omega\times(I\Omega)-\int_\Gamma y\times\mathcal{T}(u,p)N\,\mathrm{d}\sigma, & t\in(0,T),\\ \xi(0)=\eta_0 \quad\text{and} & \Omega(0)=\omega_0. \end{cases}$$

In this system, the operator, $\mathcal{L}$ denotes the transformed Laplace operator and it is given by

$$\begin{aligned}(\mathcal{L}u)_i &:= \sum_{j,k=1}^n \partial_j(g^{jk}\partial_k u_i)+2\sum_{j,k,l=1}^n g^{kl}\Gamma^i_{jk}\partial_l u_j\\ &\quad+\sum_{j,k,l=1}^n\left(\partial_k(g^{kl}\Gamma^i_{kl})+\sum_{m=1}^n g^{kl}\Gamma^m_{jl}\Gamma^i_{km}\right)u_j.\end{aligned}\tag{3.13}$$

The convection term is transformed into

$$(\mathcal{N}u)_i := \sum_{j=1}^n u_j\partial_j u_i+\sum_{j,k=1}^n \Gamma^i_{jk}u_ju_k,$$

and the transformed time derivative and the transformed gradient are given by

$$(\mathcal{M}u)_i := \sum_{j=1}^n \dot{Y}_j\partial_j u_i+\sum_{j,k=1}^n\left(\Gamma^i_{jk}\dot{Y}_k+(\partial_k Y_i)(\partial_j\dot{X}_k)\right)u_j.$$

Finally,

$$(\mathcal{G}p)_i := \sum_{j=1}^n g^{ij}\partial_j p.$$

The coefficients are given by the metric contravariant tensor

$$g^{ij}=\sum_{k=1}^n(\partial_k Y_i)(\partial_k Y_j),$$

the metric covariant tensor

$$g_{ij}=\sum_{k=1}^n(\partial_i X_k)(\partial_j X_k),$$

and the Christoffel symbol

$$\Gamma^i_{jk}=\sum_{l=1}^n g^{jk}(\partial_i g_{jl}+\partial_j g_{lk}-\partial_k g_{ij}).$$

We note that a solution (u,p,ξ,Ω) to (3.12) yields a solution (v,q,η,ω) by the definitions in (3.10).

4. The solution of the linear problem

The transformed problem (3.12) is now linearized as follows. We add $(\mathcal{L}-\Delta)u$ and $(\nabla-\mathcal{G})p$ to the first line and $\int_\Gamma \mathbf{T}(u,p)N\,\mathrm{d}\sigma$, $\int_\Gamma y\times\mathbf{T}(u,p)N\,\mathrm{d}\sigma$ to the equations on the rigid body. Then all non-linear terms are moved to the right-hand side and fixed. The remaining linear system has the following form:

$$
\left\{\begin{array}{rcll}
u_t-\Delta u+\nabla p &=& f_0, & \text{in } (0,T)\times\mathcal{D},\\
\operatorname{div} u &=& 0, & \text{in } (0,T)\times\mathcal{D},\\
u-\Omega\times y+\xi &=& 0, & \text{on } (0,T)\times\Gamma,\\
u(0) &=& u_0, & \text{in } \mathcal{D},\\
\mathrm{m}\xi'+\int_\Gamma \mathbf{T}(u,p)N\,\mathrm{d}\sigma &=& f_1, & t\in(0,T),\\
I\Omega'+\int_\Gamma y\times\mathbf{T}(u,p)N\,\mathrm{d}\sigma &=& f_2, & t\in(0,T),\\
\xi(0)=\xi_0 & \text{and} & \Omega(0)=\Omega_0. &
\end{array}\right. \tag{4.1}
$$

We show that for given $f_0\in L^p(0,T;L^q(\mathcal{D};\mathbb{R}^3)), f_1,f_2\in L^p(0,T;\mathbb{R}^3)$, there is a solution

$$
\begin{array}{rcl}
u &\in& X^T_{p,q}:=W^{1,p}(0,T;L^q(\mathcal{D}))\cap L^p(0,T;W^{2,q}(\mathcal{D})),\\
p &\in& Y^T_{p,q}:=L^p(0,T;\widehat{W}^{1,q}(\mathcal{D})),\\
(\xi,\Omega) &\in& W^{1,p}(0,T;\mathbb{R}^6)
\end{array}
$$

to this system. In the following, we use the notation $J_T:=(0,T), T>0$ and $\|\cdot\|_{p,q}:=\|\cdot\|_{L^p(J_T;L^q(\mathcal{D}))}$. The main result of this section is the following maximal regularity theorem for the linearized Newtonian fluid-rigid interaction problem.

Theorem 4.1. *Let $\mathcal{D}$ be an exterior domain with boundary of class $C^{2,1}$ and $p,q\in(1,\infty)$. Let $\xi_0,\Omega_0\in\mathbb{R}^3$ and $u_0\in B^{2-2/p}_{q,p}(\mathcal{D})$ such that the compatibility condition* (2.4) *is satisfied. Moreover, let $f_0\in L^p(J_T;L^q(\mathcal{D};\mathbb{R}^3)), f_1,f_2\in L^p(J_T;\mathbb{R}^3)$. Then for every $T>0$, problem* (4.1) *admits a unique solution*

$$
u\in X^T_{p,q},\quad p\in Y^T_{p,q},\quad (\xi,\Omega)\in W^{1,p}(J_T;\mathbb{R}^6),
$$

which satisfies the estimate

$$
\begin{aligned}
&\|u\|_{X^T_{p,q}}+\|p\|_{Y^T_{p,q}}+\|\xi\|_{W^{1,p}(J_T)}+\|\Omega\|_{W^{1,p}(J_T)}\\
&\quad\le C(\|f_0\|_{p,q}+\|f_1\|_p+\|f_2\|_p+\|u_0\|_{B^{2-2/p}_{q,p}(\mathcal{D})}+|\xi_0|+|\Omega_0|),
\end{aligned}
$$

where the constant C depends only on the geometry of the rigid body and on T.

The proof of this theorem is divided into two parts. In subsection 4.1, we recall known results on maximal regularity of the Stokes problem and summarize embedding properties of the spaces $X^T_{p,q}$ and $W^{1,p}(J_T)$. We also introduce properties of the Bogovskiĭ operator in order to deal with the Stokes problem with inhomogeneous Dirichlet boundary conditions.

The crucial part of the proof is in subsection 4.2, where we give a reformulation of (4.1) as a linear equation in the unknowns ξ and Ω only, which allows us to deal with the coupling between fluid and rigid body.

4.1. Estimates for velocity and pressure.

For $1<p,q<\infty$, the Stokes operator A_q with Dirichlet boundary conditions in $L^q_\sigma(\mathcal{D})$ is defined by

$$
\left\{\begin{array}{rcl}
A_q u &:=& P_q\Delta u,\\
D(A_q) &:=& W^{2,q}(\mathcal{D})\cap W^{1,q}_0(\mathcal{D})\cap L^q_\sigma(\mathcal{D})),
\end{array}\right.
$$

where P_q denotes the Helmholtz projection on $L^q(\mathcal{D})$. In the following, we denote the corresponding space of maximal regularity by

$$X^T_{p,q,\sigma} := W^{1,p}(J_T; L^q(\mathcal{D})) \cap L^p(J_T; D(A_q)),$$

and the space for the associated pressure is $Y^T_{p,q}$, as above. The following classical result is due to Solonnikov [42].

Proposition 4.1. *Let $T_0 > 0$, $T \in J_{T_0}$, $f \in L^p(J_T; L^q(\mathcal{D}))$, $f_\sigma = P_q f$ and $u_0 \in (L^q_\sigma(\mathcal{D}), D(A_q))_{1-\frac{1}{p},p}$. Then there exists a unique solution $u \in X^T_{p,q,\sigma}$ to the inhomogeneous Stokes problem*

$$\begin{cases} u'(t) - Au(t) &= f_\sigma(t), \quad t \in J_T, \\ u(0) &= u_0, \end{cases}$$

and there exists a constant $C > 0$ independent of T, u_0 and f_σ, such that

$$\|u\|_{X^T_{p,q,\sigma}} \le C(\|f_\sigma\|_{p,q} + \|u_0\|_{(L^q_\sigma(\mathcal{D}),D(A_q))_{1-\frac{1}{p},p}}).$$

Moreover, setting $\nabla p := (\mathrm{Id} - P_q)(\Delta u + f)$, it follows that $(u,p) \in X^T_{p,q,\sigma} \times Y^T_{p,q}$ solves

$$\begin{cases} u'(t) - \Delta u(t) + \nabla p(t) &= f(t), \quad t \in J_T, \\ u(0) &= u_0, \end{cases} \tag{4.2}$$

and satisfies the estimate

$$\|u\|_{X^T_{p,q,\sigma}} + \|p\|_{Y^T_{p,q}} \le C(\|f\|_{p,q} + \|u_0\|_{(L^q_\sigma(D),D(A_q))_{1-1/p,p}}). \tag{4.3}$$

Concerning the Bogovskiĭ operator, we cite the following properties from [2], [16], and [23].

Proposition 4.2. *Let $1 < p < \infty$, and let $D \subset \mathbb{R}^n$ be a bounded domain with locally Lipschitz boundary. Then there exists an operator*

$$B_D : C^\infty_c(D) \to C^\infty_c(D)^n$$

such that

$$\operatorname{div} B_D g = g, \qquad \text{if } \int_D g = 0.$$

Moreover, for $s > -2 + \frac{1}{p}$, B_D can be continuously extended to a bounded operator from $W^{s,p}(D)$ to $W^{s+1,p}(D)^n$.

In our situation, the Bogovskiĭ operator is used to deal with inhomogeneous Dirichlet boundary conditions for the Stokes problem. We consider the special case when h is a function on the boundary Γ with an extension H of h onto $\mathcal{D}$ satisfying

$$H|_\Gamma = h, \quad H \in W^{1,p}(J_T; C^2(\mathcal{D})) \quad \text{and} \quad \operatorname{div} H = 0. \tag{4.4}$$

Of special interest are the functions h and H given by a rigid motion $\Omega \times y + \xi$ on $\mathbb{R}^3$. Then we may define

$$b_h := \chi H - B_K((\nabla\chi)H), \tag{4.5}$$

where $b_h(y) = \Omega \times y + \xi$ on $\mathcal{B}$ and where χ is the cut-off function from (3.6), to obtain $b_h \in W^{1,p}(J_T; C^2_{c,\sigma}(\mathbb{R}^3))$. The Stokes problem

$$\begin{cases} u_t - \Delta u + \nabla p &= f, \quad \text{in } J_T \times \mathcal{D}, \\ \operatorname{div} u &= 0, \quad \text{in } J_T \times \mathcal{D}, \\ u &= h, \quad \text{on } J_T \times \Gamma, \\ u(0) &= u_0, \quad \text{in } \mathcal{D} \end{cases} \tag{4.6}$$

is thus equivalent to

$$\begin{cases} \partial_t u_b - \Delta u_b + \nabla p &= f + \Delta b_h - \partial_t b_h, & \text{in } J_T \times \mathcal{D}, \\ \operatorname{div} u_b &= 0, & \text{in } J_T \times \mathcal{D}, \\ u_b &= 0, & \text{on } J_T \times \partial\mathcal{D}, \\ u_b(0) &= u_0 - b_h(0), & \text{in } \mathcal{D}, \end{cases}$$

so that by (4.3), we have a solution $(u = u_b + b_h, p)$ of (4.6) which satisfies

$$\|u\|_{X^T_{p,q}} + \|p\|_{Y^T_{p,q}} \le C(\|f\|_{p,q} + \|u_0 - b_h(0)\|_{(L^q_\sigma(D), D(A_q))_{1-1/p,p}} + \|b_h\|_{X^T_{p,q}}). \tag{4.7}$$

This motivates the definition of the solution operators

$$\mathcal{U}(f, h, u_0) := u \in X^T_{p,q}, \qquad \mathcal{P}(f, h, u_0) := p \in Y^T_{p,q} \tag{4.8}$$

for problem (4.6).

The following proposition yields embedding properties of $X^T_{p,q}$ which will be needed later on.

Proposition 4.3. *Let $D \subset \mathbb{R}^n$ be a $C^{1,1}$-domain with compact boundary, let $p, q \in (1, \infty)$, $\alpha \in (0, 1)$ and $T_0 > 0$. Then*

$$X^{T_0}_{p,q} \hookrightarrow H^{\alpha,p}(J_{T_0}; H^{2(1-\alpha),q}(D)).$$

In particular, if $r, s \in (1, \infty) \cup \{\infty\}$, $\mu \in \{0, 1\}$ and

$$\frac{2-\mu}{2} + \frac{n}{2r} - \frac{n}{2q} \ge \frac{1}{p} - \frac{1}{s},$$

then for all $T \in J_{T_0}$,

$$X^T_{p,q} \hookrightarrow L^s(J_T; W^{\mu,r}(D)).$$

Moreover, there exists a constant $C_0 = C(T_0)$, independent of $T \in J_{T_0}$, such that the estimate

$$\|u\|_{L^s(J_T; W^{\mu,r}(D))} \le C_0 \|u\|_{X^T_{p,q}}$$

holds true for all $u \in X^T_{p,q,0} := \{w \in X^T_{p,q} : w|_{t=0} = 0\}$.

For a proof of this proposition by the mixed derivatives theorem (cf. [7]), we refer to [11, Lem. 4.2]. At this point we also note the more elementary embedding constants

$$\|f\|_p \le T^{1/p-1/q} \|f\|_q \qquad \text{for all } f \in L^q(J_T), \qquad q > p,$$

and

$$\|f\|_\infty \le T^{1/p'} \|f\|_{W^{1,p}(0,T)} \qquad \text{for all } f \in W^{1,p}_0(J_T), \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

which will be used frequently. As a consequence of Proposition 4.3, we may improve the estimate (4.3) on the Stokes equation in the special case of solenoidal body forces.

The following lemma is a parabolic variant of [35, Lemma 13].

Lemma 4.2. *Let $\mathcal{D}$ be an exterior domain, $f \in L^p(J_T; L^q_\sigma(\mathcal{D}))$ and $u_0 = 0$. Then, if the pressure $p = \mathcal{P}(f, 0, 0) \in Y^T_{p,q}$ solving (4.6) with homogeneous boundary and initial conditions is chosen in such a way that $p \in L^p(J_T; L^q(\mathcal{D}_R))$ and $\int_{\mathcal{D}_R} p = 0$ for some $R > 0$ where $\mathcal{D}_R = \mathcal{D} \cap B_R$, then*

$$\|p\|_{L^p(J_T; L^q(\mathcal{D}_R))} \le C T^{\alpha/p} \|f\|_{p,q}, \qquad 0 \le \alpha < \frac{1}{2q}.$$

Proof. Choose $\mathcal{D}_R$ and p from $L^p(J_T;\widehat{W}^{1,q}(\mathcal{D}))$ such that $\partial\mathcal{D}\subset\partial\mathcal{D}_R$ and $p\in L^p(J_T;L^q(\mathcal{D}_R))$, $\int_{\mathcal{D}_R}p=0$. Then

$$\nabla p(t,x)=((\mathrm{Id}-P_{\mathcal{D}})\Delta u(x))(t)\quad\text{for a.a. } t\in J_T. \tag{4.9}$$

Using a similar argument as in [35, Lem. 13], for every $p(t)$, we obtain

$$\|p(t)\|_{L^q(\mathcal{D}_R)}\le C\,\|u(t)\|_{H^{2-2\alpha,q}(\mathcal{D})}\,,\quad\text{for a.a. } t\in J_T, \tag{4.10}$$

for all $0\le\alpha<\frac{1}{2q}$. The argument relies on the fact that by testing and using the selfadjointness of the fractional powers of the Dirichlet-Laplacian, we obtain $\|p(t)\|_{L^q(\mathcal{D}_R)}\lesssim\|(-\Delta)^{1-\alpha}u(t)\|_{L^q(\mathcal{D}_R)}$ for $0<\alpha<\frac{1}{2q}$. Indeed, the domain $D((-\Delta)^{1-\alpha})$ of $(-\Delta)^{1-\alpha}$ is given by complex interpolation, $D((-\Delta)^{1-\alpha})=[L^q,D(\Delta)]_{1-\alpha}$. If $\alpha<\frac{1}{2q}$, we obtain $D((-\Delta)^{1-\alpha})=H^{2-2\alpha,q}(D)$, as the boundary values no longer matter. From (4.10), by interpolation, Proposition 4.3, and by (4.3) we get that

$$\begin{aligned}\|p\|_{L^p(J_T;L^q(\mathcal{D}_R))}&\le C\,\|u\|^{1-\alpha}_{L^p(J_T;W^{2,q}(\mathcal{D}))}\,\|u\|^{\alpha}_{p,q}\le CT^{\alpha/p}\,\|u\|^{1-\alpha}_{X^T_{p,q,\sigma}}\,\|u\|^{\alpha}_{L^\infty(J_T;L^q(\mathcal{D}))}\\&\le CT^{\alpha/p}\,\|f\|_{p,q}\,,\end{aligned} \tag{4.11}$$

where the constant C may be chosen independently of T. □

4.2. **Reformulation of** (4.1)**.** In this subsection, we give a reformulation of (4.1) in the unknowns ξ,Ω. We obtain homogeneous initial conditions for the problem by subtracting the solution $u^*=\mathcal{U}(f_0,\Omega_0\times y+\xi_0,u_0),p^*=\mathcal{P}(f_0,\Omega_0\times y+\xi_0,u_0)$ of

$$\begin{cases} u^*_t-\Delta u^*+\nabla p^* &= f_0, & \text{in } J_{T_0}\times\mathcal{D},\\ \operatorname{div}u^* &= 0, & \text{in } J_{T_0}\times\mathcal{D},\\ u^* &= \Omega_0\times y+\xi_0, & \text{on } J_{T_0}\times\Gamma,\\ u^*(0) &= u_0, & \text{in } \mathcal{D}\end{cases}$$

from u,p and ξ_0,Ω_0 from ξ,Ω, respectively. Note that as in (4.7), we have the estimate

$$\|u^*\|_{X^T_{p,q}}+\|p^*\|_{Y^T_{p,q}}\le C(\|u_0\|_{B^{2-2/p}_{q,p}(\mathcal{D})}+\|f_0\|_{p,q}+|\xi_0|+|\Omega_0|). \tag{4.12}$$

The next modification of the problem concerns the normal component of the fluid boundary condition only. It is a splitting of the body velocity into the tangential component, which allows good estimates of the corresponding fluid pressure by Lemma 4.2, and can be treated by separating fluid and body part, and the normal component, which has to be fully inverted in order to solve the problem.

Let $e_1=(1,0,0),e_2=(0,1,0),e_3=(0,0,1)$, and let $v^{(i)},V^{(i)}$ be solutions of the weak Neumann problems

$$\begin{cases}\Delta v^{(i)} &= 0, & \text{in } \mathcal{D},\\ \frac{\partial v^{(i)}}{\partial N} &= N\cdot e_i, & \text{on } \Gamma,\end{cases}$$

$$\begin{cases}\Delta V^{(i)} &= 0, & \text{in } \mathcal{D},\\ \frac{\partial V^{(i)}}{\partial N} &= N\cdot(e_i\times y), & \text{on } \Gamma.\end{cases}$$

Using a localization argument and elliptic regularity estimates (see [24, Proposition 7.1] for details), we obtain the estimate

$$\|\nabla v^{(i)}\|_{W^{2,q}(\mathcal{D})}+\|\nabla V(i)\|_{W^{2,q}(\mathcal{D})}\le C. \tag{4.13}$$

For given $\xi, \Omega \in W_0^{1,p}(J_T)$, let

$$v_{\xi,\Omega}(t) := \sum_i \Omega_i(t) V^{(i)} + \xi_i(t) v^{(i)} \qquad \text{for all } t \in J_T, \tag{4.14}$$

which implies that

$$\begin{cases} \Delta v_{\xi,\Omega}(t) &= 0, & \text{in } \mathcal{D}, \\ \frac{\partial v_{\xi,\Omega}(t)}{\partial N} &= (\Omega(t) \times y + \xi(t)) \cdot N, & \text{on } \Gamma, \end{cases}$$

is satisfied. Then $\nabla v_{\xi,\Omega} \in X^T_{p,q,0}$ and

$$\|\nabla v_{\xi,\Omega}\|_{W^{1,p}(J_T;W^{2,q}(\mathcal{D}))} + \|\partial_t v_{\xi,\Omega}\|_{L^p(J_T;\widehat{W}^{1,q}(\mathcal{D}))} \leq C(\|\xi\|_{W^{1,p}(J_T)} + \|\Omega\|_{W^{1,p}(J_T)}).$$

Writing

$$\begin{aligned} u &= u^* + \hat{u} + \nabla v_{\xi,\Omega}, \\ p &= p^* + \hat{p} - \partial_t v_{\xi,\Omega}, \\ \xi &= \xi_0 + \hat{\xi}, \\ \Omega &= \Omega_0 + \hat{\Omega}, \end{aligned}$$

the problem (4.1) is equivalent to the set of equations

$$\begin{cases} \hat{u}_t - \Delta \hat{u} + \nabla \hat{p} = 0, & \text{in } J_T \times \mathcal{D}, \\ \operatorname{div} \hat{u} = 0, & \text{in } J_T \times \mathcal{D}, \\ \hat{u} = h(\hat{\xi}, \hat{\Omega}) & \text{on } J_T \times \Gamma, \\ \hat{u}(0) = 0, & \text{in } \mathcal{D}, \\ \mathrm{m}\hat{\xi}' + \int_\Gamma \partial_t v_{\hat{\xi},\hat{\Omega}} N \,\mathrm{d}\sigma & \\ \qquad + \int_\Gamma \mathbf{T}(\hat{u} + u^* + \nabla v_{\hat{\xi},\hat{\Omega}}, \hat{p} + p^*) N \,\mathrm{d}\sigma = f_1, & t \in J_T, \\ I\hat{\Omega}' + \int_\Gamma y \times (\partial_t v_{\hat{\xi},\hat{\Omega}} N) \,\mathrm{d}\sigma & \\ \qquad + \int_\Gamma y \times \mathbf{T}(\hat{u} + u^* + \nabla v_{\hat{\xi},\hat{\Omega}}, \hat{p} + p^*) N \,\mathrm{d}\sigma = f_2, & t \in J_T, \\ \hat{\xi}(0) = 0 \text{ and } \hat{\Omega}(0) = 0, & \end{cases} \tag{4.15}$$

where

$$h(\hat{\xi}, \hat{\Omega}) := \hat{\Omega} \times y + \hat{\xi} - \nabla v_{\hat{\xi},\hat{\Omega}}|_\Gamma.$$

Due to the correction by $v_{\hat{\xi},\hat{\Omega}}$, we get $\hat{u}|_\Gamma N = 0$ on the boundary. Given ξ, Ω, the fluid part of this system can be solved by $\mathcal{U}_h(\xi, \Omega) := \mathcal{U}(0, h(\xi, \Omega), 0) \in X^T_{p,q,0}$ and $\mathcal{P}_h(\xi, \Omega) := \mathcal{P}(0, h(\xi, \Omega), 0) \in Y^T_{p,q}$, where $\mathcal{U}, \mathcal{P}$ are taken from (4.8). This solution depends linearly and continuously on $\xi, \Omega \in W_0^{1,p}(J_T)$.

Now let $\mathbb{I} := \begin{pmatrix} \mathrm{m}\mathrm{Id}_3 & 0 \\ 0 & I \end{pmatrix}$ be the constant momentum matrix for the problem. For all $0 < \varepsilon < 1 - \frac{1}{q}$, we define the operator

$$\mathcal{J} : W^{\varepsilon+1/q,q}_{\mathrm{loc}}(\mathcal{D};\mathbb{R}^{3\times 3}) \to \mathbb{R}^6, \quad h \mapsto \begin{pmatrix} \int_\Gamma hN \,\mathrm{d}\sigma \\ \int_\Gamma y \times hN \,\mathrm{d}\sigma \end{pmatrix}.$$

From the boundedness of the trace operator $\gamma : H^{\varepsilon/2+1/q,q}_{\mathrm{loc}}(\mathcal{D}) \to L^q(\partial\mathcal{D})$, it follows that

$$|\mathcal{J}(g)| \le C\|g\|_{W^{\varepsilon+1/q,q}(\mathcal{D})}, \quad g \in W^{\varepsilon+1/q,q}(\mathcal{D}). \tag{4.16}$$

Furthermore, we define an *added mass* $\mathbb{M}$ of $\mathcal{B}$ (cf. [18, p. 685]) in the following way. Let $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ be given by

$$a_{ij} = \int_\Gamma v^{(i)} n_j \,\mathrm{d}\sigma, \quad b_{ij} = \int_\Gamma V^{(i)}(e_j \times y)\cdot n \,\mathrm{d}\sigma,$$
$$c_{ij} = \int_\Gamma v^{(i)}(e_j \times y)\cdot n \,\mathrm{d}\sigma, \quad d_{ij} = \int_\Gamma V^{(i)} n_j \,\mathrm{d}\sigma$$

and

$$\mathbb{M} := \begin{pmatrix} a_{11} & a_{12} & a_{13} & c_{11} & c_{12} & c_{13} \\ a_{21} & a_{22} & a_{23} & c_{21} & c_{22} & c_{23} \\ a_{31} & a_{32} & a_{33} & c_{31} & c_{32} & c_{33} \\ d_{11} & d_{12} & d_{13} & b_{11} & b_{12} & b_{13} \\ d_{21} & d_{22} & d_{23} & b_{21} & b_{22} & b_{23} \\ d_{31} & d_{32} & d_{33} & b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

Note that, by definition, it follows that $\mathcal{J}(\partial_t v_{\xi,\Omega}) = \mathbb{M}\begin{pmatrix} \xi' \\ \Omega' \end{pmatrix}$ and a direct computation shows that $\mathcal{J}(\mathcal{E}^{(\nabla v_{\xi,\Omega})}) = 0$ for all $\xi, \Omega \in W^{1,p}_0(J_T)$. Hence the fifth, sixth, and seventh lines in (4.15) can be rewritten as

$$(\mathbb{I} + \mathbb{M})\begin{pmatrix} \hat{\xi}' \\ \hat{\Omega}' \end{pmatrix} = -\mathcal{J}\mathbf{T}(\hat{u}, \hat{p}) - \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \mathcal{J}\mathbf{T}(u^*, p^*),$$
$$\hat{\xi}(0) = 0 \quad \text{and} \quad \hat{\Omega}(0) = 0.$$

Setting $\hat{u} := \mathcal{U}_h(\xi, \Omega)$ and $\hat{p} := \mathcal{P}_h(\xi, \Omega)$, we see that this is equivalent to problem (4.15). For every choice of the body's density and mass $\rho_{\mathcal{B}}, \mathrm{m} > 0$, $\mathbb{I}$ is strictly positive, so the following lemma implies that $\mathbb{I} + \mathbb{M}$ is invertible.

Lemma 4.3. *The matrix $\mathbb{M}$ is symmetric and semipositive-definite.*

Proof. The matrix $\mathbb{M}$ is symmetric since by Gauss' theorem,

$$\begin{aligned} a_{ij} &= \int_\Gamma v^{(i)} N_j \,\mathrm{d}\sigma \\ &= \int_\Gamma v^{(i)} \frac{\partial v^{(j)}}{\partial N} \,\mathrm{d}\sigma = \int_{\mathcal{D}} \mathrm{div}\,(v^{(i)}\nabla v^{(j)}) = \int_{\mathcal{D}} \nabla v^{(i)} \cdot \nabla v^{(j)} = \sum_{l=1}^{3} \int_{\mathcal{D}} \partial_l v^{(i)} \partial_l v^{(j)}, \end{aligned}$$

and similarly,

$$b_{ij} = \sum_{l=1}^{3} \int_{\mathcal{D}} \partial_l V^{(i)} \partial_l V^{(j)} = b_{ji} \quad \text{and} \quad c_{ij} = \sum_{l=1}^{3} \int_{\mathcal{D}} \partial_l v^{(i)} \partial_l V^{(j)} = d_{ji}.$$

For any vector $z = (x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6$, we obtain

$$\begin{aligned} z^T \mathbb{M} z &= \sum_{i,j=1}^{3} a_{ij} x_i x_j + \sum_{i,j=1}^{3} c_{ij} x_i y_j + \sum_{i,j=1}^{3} d_{ij} y_i x_j + \sum_{i,j=1}^{3} b_{ij} y_i y_j \\ &= \sum_{l=1}^{3} \int_{\mathcal{D}} (\sum_{i=1}^{3} \partial_l v^{(i)} x_i + \sum_{i=1}^{3} \partial_l V^{(i)} y_i)^2 \geq 0. \end{aligned}$$

□

4.3. **Proof of Theorem 4.1.** Lemma 4.3 allows us to rewrite (4.15) in the form

$$\begin{pmatrix} \hat{\xi} \\ \hat{\Omega} \end{pmatrix} = \mathcal{R} \begin{pmatrix} \hat{\xi} \\ \hat{\Omega} \end{pmatrix} + f^*, \tag{4.17}$$

where $\mathcal{R} : W_0^{1,p}(J_T; \mathbb{R}^6) \to W_0^{1,p}(J_T; \mathbb{R}^6)$ is given by

$$\mathcal{R}(\xi, \Omega)(t) := \int_0^t (\mathbb{I} + \mathbb{M})^{-1} \mathcal{J} \left[\mathbf{T}(\mathcal{U}_h(\xi, \Omega), \mathcal{P}_h(\xi, \Omega)) \right](s) \, \mathrm{d}s$$

and

$$f^*(t) := \int_0^t (\mathbb{I} + \mathbb{M})^{-1} (\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \mathcal{J} \mathbf{T}(u^*, p^*))(s) \, \mathrm{d}s.$$

In the following lemma, we show that for sufficiently small $T > 0$ there exists a unique $(\hat{\xi}, \hat{\Omega}) \in W_0^{1,p}(J_T; \mathbb{R}^6)$ satisfying

$$\begin{pmatrix} \hat{\xi} \\ \hat{\Omega} \end{pmatrix} = (\mathrm{Id} - \mathcal{R})^{-1} f^*. \tag{4.18}$$

This is a consequence of the maximal regularity properties of $\mathcal{U}_h, \mathcal{P}_h$, the improved time regularity of $\mathcal{P}_h(\hat{\xi}, \hat{\Omega}) \in Y_{p,q}^T$ shown in Lemma 4.2, and the boundedness of $\mathcal{J}$.

Lemma 4.4. *The map $\mathcal{R}$ is bounded and $L := \|\mathcal{R}\|_{\mathcal{L}(W_0^{1,p}(J_T;\mathbb{R}^6))} < 1$ for sufficiently small $T > 0$. Moreover, $f^* \in W_0^{1,p}(J_T; \mathbb{R}^6)$.*

Proof. Let $(\xi, \Omega) \in W_0^{1,p}(J_T; \mathbb{R}^6)$. The functions $H = \xi + \Omega \times \cdot - \nabla v_{\xi,\Omega}$ and $h(\xi, \Omega) = \xi + \Omega \times \cdot|_\Gamma - \nabla v_{\xi,\Omega}|_\Gamma$ satisfy the conditions (4.4), so that we get

$$\|\mathcal{U}_h(\xi, \Omega)\|_{X_{p,q,\sigma}^T} + \|\mathcal{P}_h(\xi, \Omega)\|_{Y_{p,q}^T} \leq C \left\| b_{h(\xi,\Omega)} \right\|_{X_{p,q}^T} \leq C(\|\xi\|_{W^{1,p}(J_T)} + \|\Omega\|_{W^{1,p}(J_T)}),$$

where $b_{h(\xi,\Omega)}$ is the auxiliary function from (4.5). Moreover, by Lemma 4.2 we obtain

$$\begin{aligned} \|\mathcal{P}_h(\xi, \Omega)\|_{L^p(0,T;L^q(\mathcal{D}_R))} &\leq \|\mathcal{P}(-\partial_t b_{h(\xi,\Omega)} + \Delta b_{h(\xi,\Omega)}, 0, 0)\|_{X_{p,q}^T} \\ &\leq C T^{\alpha/p} \left\| b_{h(\xi,\Omega)} \right\|_{X_{p,q}^T} \\ &\leq C T^{\alpha/p} (\|\xi\|_{W^{1,p}(J_T)} + \|\Omega\|_{W^{1,p}(J_T)}), \end{aligned}$$

for a suitable choice of R and $0 \le \alpha < \frac{1}{2q}$. In addition, if $\beta = \min\{\frac{1}{2p}, \frac{1}{3}(1-\frac{1}{q})\}$ and $r := \frac{1}{1/p-\beta}$, then $1+\frac{1}{q}+\beta < 2$ and $r \ge 2p$, so by (4.16) and Proposition 4.3,

$$\begin{aligned}
\|\mathcal{J}\mathcal{E}^{(\mathcal{U}_h(\xi,\Omega))}\|_p &\le C\,\|\mathcal{U}_h(\xi,\Omega)\|_{L^p(J_T;W^{1+1/q+\varepsilon/2,q}(\mathcal{D}))} \\
&\le CT^\beta\,\|\mathcal{U}_h(\xi,\Omega)\|_{L^r(J_T;W^{1+1/q+\varepsilon/2,q}(\mathcal{D}))} \\
&\le CT^\beta\,\|\mathcal{U}_h(\xi,\Omega)\|_{H^{\alpha,p}(J_T;H^{2-2\alpha,q}(\mathcal{D}))} \\
&\le CT^\beta\,\|\mathcal{U}_h(\xi,\Omega)\|_{X^T_{p,q}} \\
&\le CT^\beta(\|\xi\|_{W^{1,p}(J_T)} + \|\Omega\|_{W^{1,p}(J_T)}).
\end{aligned}$$

Let $\mathcal{D}_R$ and $\mathcal{P}(\xi,\Omega)$ be such that $\mathcal{P}(\xi,\Omega) \in L^q(\mathcal{D}_R)$, $\int_{\mathcal{D}_R}\mathcal{P}(\xi,\Omega) = 0$ and $c := \frac{1}{q}+\beta < 1$. Then by interpolation, Poincaré's inequality, and Lemma 4.2,

$$\begin{aligned}
\|\mathcal{J}(\mathcal{P}_h(\xi,\Omega)\mathrm{Id})\|_{L^p(J_T)} &\le C\,\|\mathcal{P}_h(\xi,\Omega)\|_{L^p(J_T;W^{1/q+\varepsilon,q}(\mathcal{D}_R))} \\
&\le C\,\|\mathcal{P}_h(\xi,\Omega)\|^c_{L^p(J_T;L^q(\mathcal{D}_R))}\,\|\mathcal{P}_h(\xi,\Omega)\|^{1-c}_{L^p(J_T;W^{1,q}(\mathcal{D}_R))} \\
&\le CT^{c\alpha/p}\,\|b_h\|^c_{p,q}\,\|\mathcal{P}_h(\xi,\Omega)\|^{1-c}_{L^p(J_T;\widehat{W}^{1,q}(\mathcal{D}))} \\
&\le CT^{c\alpha/p}(\|\xi\|_{W^{1,p}(J_T)} + \|\Omega\|_{W^{1,p}(J_T)}).
\end{aligned}$$

In conclusion,

$$\begin{aligned}
\|\mathcal{R}(\xi,\Omega)\|_{W^{1,p}(0,T)} &\le C\,\|\mathcal{J}\mathbf{T}(\mathcal{U}_h(\xi,\Omega),\mathcal{P}_h(\xi,\Omega))\|_p \\
&\le C(T^{1/3p} + T^{c\alpha/p})(\|\xi\|_{W^{1,p}(J_T)} + \|\Omega\|_{W^{1,p}(J_T)})
\end{aligned}$$

and $\mathcal{R}(\xi,\Omega)(0) = 0$ by definition, so that

$$L := \|\mathcal{R}\|_{\mathcal{L}(W^{1,p}_0(J_T),W^{1,p}_0(J_T))} \quad < \quad 1$$

for small T. Moreover,

$$\begin{aligned}
\|f^*\|_{W^{1,p}(0,T)} &\le C\left\|\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \mathcal{J}\mathbf{T}(u^*,p^*)\right\|_p \\
&\le C(\|f_0\|_{p,q} + \|f_1\|_p + \|f_2\|_p + \|u_0\|_{B^{2-2/p}_{q,p}(\mathcal{D})} + |\xi_0| + |\Omega_0|),
\end{aligned}$$

by (4.12). □

It follows from Lemma 4.4 that there is a unique $(\hat\xi,\hat\Omega) \in W^{1,p}_0(J_T;\mathbb{R}^6)$ which satisfies (4.18). Furthermore, we obtain

$$\begin{aligned}
&\|\hat\xi\|_{W^{1,p}(J_T)} + \|\hat\Omega\|_{W^{1,p}(J_T)} \\
&\quad \le (1-L)C(\|f_0\|_{p,q} + \|f_1\|_p + \|f_2\|_p + \|u_0\|_{B^{2-2/p}_{q,p}(\mathcal{D})} + |\xi_0| + |\Omega_0|).
\end{aligned}$$

Plugging $\hat\xi,\hat\Omega$ into (4.15) and (4.14) yields a solution

$$\begin{aligned}
u &= \hat u + \nabla v_{\hat\xi+\xi_0,\hat\Omega+\Omega_0} + u^* \in X^T_{p,q}, \\
p &= \hat p - \partial_t v_{\hat\xi+\xi_0,\hat\Omega+\Omega_0} + p^* \in Y^T_{p,q}, \\
\xi &= \hat\xi + \xi_0 \in W^{1,p}(J_T), \\
\Omega &= \hat\Omega + \Omega_0 \in W^{1,p}(J_T)
\end{aligned}$$

of (4.1) fulfilling the estimate

$$\begin{aligned}&\|u\|_{X^T_{p,q}} + \|p\|_{Y^T_{p,q}} + \|\xi\|_{W^{1,p}(J_T)} + \|\Omega\|_{W^{1,p}(J_T)}\\ &\quad\le C(\|f_0\|_{p,q} + \|f_1\|_p + \|f_2\|_p + \|u_0\|_{B^{2-2/p}_{q,p}(\mathcal{D})} + |\xi_0| + |\Omega_0|).\end{aligned}$$

Since $T > 0$ does not depend on the initial data, the process of finding a solution for (4.17) can be iterated to prove Theorem 4.1 for arbitrary time intervals.

5. Fixed point argument

In this section, we rewrite the non-linear transformed system (3.12) as a fixed point problem in u, p, ξ, and Ω. We first deal with the gravitation term and enforce homogeneous initial conditions. By Theorem 4.1, there is a unique solution $u^*, p^*, \xi^*, \Omega^*$ to

$$\left\{\begin{array}{rcll} u^*_t - \Delta u^* + \nabla(p^* - \mathbf{g}\cdot y) &=& 0, & \text{in } J_T\times\mathcal{D},\\ \operatorname{div} u^* &=& 0, & \text{in } J_T\times\mathcal{D},\\ u^*(t,y) - \Omega^*(t)\times y - \xi^*(t) &=& 0, & \text{on } J_T\times\Gamma,\\ u^*(0) &=& v_0, & \text{in } \mathcal{D},\\ \mathrm{m}(\xi^*)' + \int_\Gamma \mathbf{T}(u^*,p^*)N\,\mathrm{d}\sigma &=& \mathrm{m}\mathbf{g}, & t\in J_T,\\ I(\Omega^*)' + \int_\Gamma y\times\mathbf{T}(u^*,p^*)N\,\mathrm{d}\sigma &=& 0, & t\in J_T),\\ \xi^*(0) = \eta_0 & \text{and} & \Omega^*(0) = \omega_0, & \end{array}\right. \tag{5.1}$$

which satisfies

$$\begin{aligned}&\|u^*\|_{X^T_{p,q}} + \|p^* - \mathbf{g}\cdot y\|_{Y^T_{p,q}} + \|\xi^*\|_{W^{1,p}(J_T)} + \|\Omega^*\|_{W^{1,p}(J_T)}\\ &\quad\le C(|\mathrm{m}\mathbf{g}| + \|v_0\|_{B^{2-2/p}_{q,p}(\mathcal{D})} + |\eta_0| + |\omega_0|).\end{aligned}$$

Let $\hat{u} = u - u^*$, $\hat{p} = p - p^*$, $\hat{\xi} = \xi - \xi^*$, and $\hat{\Omega} = \Omega - \Omega^*$. Then the full transformed problem (3.12) can be reformulated equivalently by

$$\left\{\begin{array}{rcll} \hat{u}_t - \Delta\hat{u} + \nabla\hat{p} &=& F_0(\hat{u},\hat{p},\hat{\xi},\hat{\Omega}), & \text{in } J_T\times\mathcal{D},\\ \operatorname{div}\hat{u} &=& 0, & \text{in } J_T\times\mathcal{D},\\ \hat{u}(t,y) - \hat{\Omega}(t)\times y - \hat{\xi}(t) &=& 0, & \text{on } J_T\times\Gamma,\\ \hat{u}(0) &=& 0, & \text{in } \mathcal{D},\\ \mathrm{m}(\hat{\xi})' + \int_\Gamma \mathbf{T}(\hat{u},\hat{p})N\,\mathrm{d}\sigma &=& F_1(\hat{u},\hat{p},\hat{\xi},\hat{\Omega}), & t\in J_T,\\ I(\hat{\Omega})' + \int_\Gamma y\times\mathbf{T}(\hat{u},\hat{p})N\,\mathrm{d}\sigma &=& F_2(\hat{u},\hat{p},\hat{\Omega}), & t\in J_T,\\ \hat{\xi}(0) = 0 & \text{and} & \hat{\Omega}(0) = 0, & \end{array}\right. \tag{5.2}$$

where

$$\begin{aligned} F_0(\hat{u},\hat{p},\hat{\xi},\hat{\Omega}) &:= \mathbf{G} - \mathbf{g} - \mathcal{N}(\hat{u}+u^*) + \mathcal{H}(\hat{u}+u^*,\hat{p}+p^*,\hat{\xi}+\xi^*,\hat{\Omega}+\Omega^*) - \Delta u^*,\\ \mathcal{H}(u,p,\xi,\Omega) &:= -\mathcal{M}u + (\mathcal{L}-\Delta)u + (\mathcal{G}-\nabla)p,\\ F_1(\hat{u},\hat{p},\hat{\xi},\hat{\Omega}) &:= \mathrm{m}(\mathbf{G}(\cdot,0) - \mathbf{g}) - \mathrm{m}(\hat{\xi}+\xi^*)\times(\hat{\Omega}+\Omega^*) + \int_\Gamma(\mathbf{T}-\mathcal{T})(\hat{u},\hat{p})N\,\mathrm{d}\sigma,\\ F_2(\hat{u},\hat{p},\hat{\Omega}) &:= -(\hat{\Omega}+\Omega^*)\times I(\hat{\Omega}+\Omega^*) + \int_\Gamma y\times(\mathbf{T}-\mathcal{T})(\hat{u},\hat{p})N\,\mathrm{d}\sigma. \end{aligned}$$

Note that in this definition, the operators $\mathbf{G}, \mathcal{T}, \mathcal{H}$, etc., defined at the end of Section 3, depend on the coordinate transform and therefore on $\tilde{\xi}, \xi^*, \tilde{\Omega}$ and Ω^*. We solve

(5.2) in the set $\mathcal{K}_R^T$ given by

$$\mathcal{K}_R^T := \{(u,p,\xi,\Omega) \in X_{p,q,0}^T \times Y_{p,q}^T \times W_0^{1,p}(J_T;\mathbb{R}^6) : \\ \|u\|_{X_{p,q}^T} + \|p\|_{Y_{p,q}^T} + \|(\xi,\Omega)\|_{W^{1,p}(J_T)} \le R\},$$

where p, q satisfy the assumptions in Theorem 2.1.

Let

$$\phi_R^T : \begin{pmatrix} \tilde{u} \\ \tilde{p} \\ \tilde{\xi} \\ \tilde{\Omega} \end{pmatrix} \mapsto \begin{pmatrix} F_0(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega}) \\ F_1(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega}) \\ F_2(\tilde{u},\tilde{p},\tilde{\Omega}) \end{pmatrix} \mapsto \begin{pmatrix} u \\ p \\ \xi \\ \Omega \end{pmatrix}$$

be the function which maps $(\tilde{u},\tilde{p},\tilde{p},\tilde{\Omega}) \in \mathcal{K}_R^T$ to the solution (u,p,ξ,Ω) of the linear problem (4.1) with right-hand sides F_0, F_1, F_2 and initial values $\xi_0 = \Omega_0 = 0, u_0 = 0$. For suffiently small $R, T > 0$, we show that the image of ϕ_R^T is contained in $\mathcal{K}_R^T$ and that it is contractive. The fixed point of ϕ_R^T then satisfies (5.2).

6. Estimates on the non-linear terms

In the first part of this section we show estimates on the transforms X and Y, uniformly in the body velocities ξ and Ω. Then we can estimate the transformed operators $\mathcal{G}, \mathcal{L}, \mathcal{M}, \dots$ in subsection 6.2.

6.1. Estimates on the coordinate transforms.

For $T > 0$ we retrace the construction of X and Y from given $\xi, \Omega \in W^{1,p}(J_T)$ in five small steps.

(1) Determine $Q \in W^{2,p}(J_T;\mathbb{R}^{3\times 3})$ by solving the system of ordinary differential equations

$$\dot{Q}^T = MQ^T,\ Q^T(0) = \mathrm{Id}, \tag{6.1}$$

where the matrix-valued function M satisfies $M(t)x = \Omega(t) \times x$ for all $t \in J_T$ and $x \in \mathbb{R}^3$.

(2) Calculate the original body velocities $\eta = Q^T\xi$ and $\omega = Q^T\Omega$.

(3) Define $b = \chi(\cdot - x_c)v_{\mathcal{B}} - B_K(\nabla\chi(\cdot - x_c)v_{\mathcal{B}})$ as in (3.7), where

$$v_{\mathcal{B}}(t,x) = \omega(t) \times (x - x_c(t)) + \eta(t)$$

and $x_c(t) = \int_0^t \eta(s)\,\mathrm{d}s$ are known from step (2).

(4) Solve the equations (3.5) with right-hand side b from step (3) to get X.

(5) Define $b^{(Y)} := J_X^{-1} b(\cdot, X)$ as in (3.9) and solve (3.8) to get the inverse Y of X.

From this procedure, we get the following estimates, which will be essential in the next subsection.

Proposition 6.1. *Let $T > 0$, $\xi_1, \xi_2, \Omega_1, \Omega_2 \in W^{1,p}(J_T)$. Then for $i \in \{1,2\}$ it holds that $X_i, Y_i \in C^1(J_T; C^\infty(\mathbb{R}^n))$,*

$$\begin{aligned} \|\partial^\alpha X_i\|_{\infty,\infty} + \|\partial^\alpha Y_i\|_{\infty,\infty} &\le C \quad \textit{and} \\ \|\partial^\beta (X_1 - X_2)\|_{\infty,\infty} + \|\partial^\beta (Y_1 - Y_2)\|_{\infty,\infty} &\le CT(\|\xi_1 - \xi_2\|_\infty + \|\Omega_1 - \Omega_2\|_\infty) \end{aligned}$$

for all multi-indices α, β satisfying $1 \le |\alpha| \le 3$ and $0 \le |\beta| \le 3$. Moreover, the constants do not depend on ξ_i or Ω_i, but only on their norms $K_i := \|\xi_i\|_\infty + \|\Omega_i\|_\infty$.

The following lemmas are devoted to the proof of this proposition. Roughly speaking, they correspond to steps (2)–(5). It is always assumed that $\xi_1, \xi_2, \Omega_1, \Omega_2 \in W^{1,p}(J_T)$ are given and that the index $i \in \{1,2\}$ on a function means that it is associated to ξ_i, Ω_i by the definitions in steps (1)–(5).

Note that the generic constants which appear in the estimates may depend on the K_i and on T. When ξ_i, Ω_i are chosen from $\mathcal{K}_R^T$ for fixed $T, R > 0$, there is a uniform upper bound on K_i for all possible ξ_i, Ω_i so that we may ignore this dependence.

Lemma 6.1. *The vector-valued functions η_i, ω_i associated to ξ_i, Ω_i in step* (2) *satisfy*

$$\begin{aligned} \|\eta_1 - \eta_2\|_\infty &\leq C(\|\xi_1 - \xi_2\|_\infty + \|\Omega_1 - \Omega_2\|_\infty) \quad \textit{and} \\ \|\omega_1 - \omega_2\|_\infty &\leq C\,\|\Omega_1 - \Omega_2\|_\infty\,. \end{aligned}$$

Proof. First consider $Q_i \in W^{2,p}(J_T; \mathbb{R}^{3\times 3})$. From (6.1), we obtain the estimate

$$|Q_i(t)| \leq C + \int_0^t |M_i(s)|\,|Q_i(s)|\,\mathrm{d}s,$$

so that by Gronwall's Inequality,

$$|Q_i(t)| \leq C + \int_0^t C\,|M_i(s)|\,e^{\int_0^s |M_i(\tau)|\,\mathrm{d}\tau}\,\mathrm{d}s.$$

This yields

$$\|Q_i\|_\infty + \|Q_i^T\|_\infty \leq C(1 + TK_i e^{TK_i}).$$

Similarly, we use

$$\dot{Q}_1 - \dot{Q}_2 = M_1 Q_1 - M_2 Q_2, \qquad (Q_1 - Q_2)(0) = 0$$

to get the estimate

$$\|Q_1 - Q_2\|_\infty \leq CT\,\|M_1 - M_2\|_\infty \leq CT\,\|\Omega_1 - \Omega_2\|_\infty\,. \tag{6.2}$$

Now

$$\|\eta_1 - \eta_2\|_\infty \leq \|Q_1(\xi_1 - \xi_2) + (Q_1 - Q_2)\xi_2\|_\infty \leq C(\|\xi_1 - \xi_2\|_\infty + \|\Omega_1 - \Omega_2\|_\infty)$$

and the estimate

$$\|\omega_1 - \omega_2\|_\infty \leq C\,\|\Omega_1 - \Omega_2\|_\infty$$

follows analogously. □

Lemma 6.2. *The maps b_i associated to η_i, ω_i by step* (3) *satisfy*

$$\begin{aligned} \|\partial^\beta b_i\|_{\infty,\infty} &\leq C \quad \textit{and} \\ \|\partial^\beta (b_1 - b_2)\|_{\infty,\infty} &\leq C(\|\omega_1 - \omega_2\|_\infty + \|\eta_1 - \eta_2\|_\infty) \end{aligned}$$

for all multi-indices β with $0 \leq |\beta| \leq 3$.

Proof. First note that by Proposition 4.2 and Sobolev's embedding theorem,

$$\|B_K g\|_{C^3(K)} \leq C\|g\|_{C^3(K)}, \qquad \text{for all } g \in W_0^{4,p}(K), p > 4. \tag{6.3}$$

Calculating the derivatives of b_i in terms of the derivatives of χ shows that

$$\begin{aligned} \left\|\partial^\beta b_i\right\|_{\infty,\infty} &\leq C\,\|\chi\|_{C^{|\alpha|+1}(\overline{B_2})}\,(\|\omega_i\|_\infty + \|\eta_i\|_\infty) \\ &\leq \|Q_i\|_\infty\,\|\xi_i\|_\infty + \|Q_i\|_\infty\,\|\Omega_i\|_\infty \leq C_{K_i} \end{aligned} \tag{6.4}$$

and also

$$\left\|\partial^\beta(b_1-b_2)\right\|_{\infty,\infty} \quad \leq \quad C\left\|\chi\right\|_{C^{|\alpha|+1}(\overline{B_2})}\left(\left\|\omega_1-\omega_2\right\|_\infty+\left\|\eta_1-\eta_2\right\|_\infty\right).$$

□

We may now estimate the transforms X_i, Y_i with respect to b_i and $b_i^{(Y)}$.

Lemma 6.3. *The coordinate transforms X_i and Y_i associated to $b_i, b_i^{(Y)}$ by steps* (3) *and* (5)*, respectively, satisfy*

$$\left\|\partial^\alpha X_i\right\|_{\infty,\infty} \quad \leq \quad C$$

for multi-indices α with $1 \leq |\alpha| \leq 3$ and

$$\begin{aligned}\left\|\partial^\beta(X_1-X_2)\right\|_{\infty,\infty} &\leq CT\left\|b_1-b_2\right\|_{L^\infty(J_T;C^{|\beta|}(\mathbb{R}^3))},\\ \left\|\partial^\beta(Y_1-Y_2)\right\|_{\infty,\infty} &\leq CT\|b_1^{(Y)}-b_2^{(Y)}\|_{L^\infty(J_T;C^{|\beta|}(\mathbb{R}^3))}\end{aligned}$$

for multi-indices β with $0 \leq |\beta| \leq 3$.

Proof. First we show the second estimate for $|\alpha| = 0$. Our starting point is the differential equations

$$\begin{cases}\partial_t(X_1-X_2)(t,y) = b_1(t,X_1(t,y))-b_2(t,X_2(t,y)), & t\in J_T, y\in\mathbb{R}^3,\\ (X_1-X_2)(0,y) = 0, & y\in\mathbb{R}^3.\end{cases}$$

By integration in time and Lemma 6.2,

$$\begin{aligned}&|(X_1-X_2)(t,y)|\\ &\quad\leq \int_0^t |b_1(s,X_1(s,y))-b_2(s,X_2(s,y))|\,\mathrm{d}s\\ &\leq \int_0^t \|\nabla b_1(s)\|_{L^\infty(\mathbb{R}^3)}\,|(X_1-X_2)(s,y)|+\|(b_1-b_2)(s)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s\end{aligned}$$

for all $(t,y)\in J_T\times\mathbb{R}^3$. This yields

$$\left\|X_1-X_2\right\|_{\infty,\infty} \leq CT\left\|b_1-b_2\right\|_{\infty,\infty}$$

by Gronwall's Inequality. Now let $(z_i)_{kj}(t,y) := \partial_j(X_i)_k(t,y)$, $j,k\in\{1,2,3\}$ be the partial derivative of the kth component of X_i, and let J_{b_i} be the Jacobian of b_i. By differentiating (3.5) with respect to the spatial variables, we get

$$\begin{cases}\partial_t(z_i)_{kj}(t,y) = J_{b_i}(t,X_i(t,y))(z_i)_{kj}(t,y), & t\in J_T, y\in\mathbb{R}^3,\\ (z_i)_{kj}(0,y) = e_j, & y\in\mathbb{R}^3.\end{cases}$$

So Gronwall's Inequality and Lemma 6.2 imply

$$\left\|(z_i)_{kj}\right\|_{\infty,\infty} \leq e^{T\|J_{b_i}\|_{\infty,\infty}} \leq C.$$

For two different transforms, the relation

$$\begin{cases}\partial_t((z_1)_{kj}-(z_2)_{kj})(t,y) = J_{b_1}(t,X_1(t,y))(z_1)_{kj}(t,y)\\ \qquad\qquad -J_{b_2}(t,X_2(t,y))(z_2)_{kj}(t,y), & t\in J_T, y\in\mathbb{R}^3,\\ ((z_1)_{kj}-(z_2)_{kj})(0,y) = 0, & y\in\mathbb{R}^3,\end{cases}$$

holds true. Again, we integrate and use Gronwall's Inequality as well as the estimates obtained above to show

$$\left\|(z_1)_{kj}-(z_2)_{kj}\right\|_{\infty,\infty} \leq CT\left\|\nabla(b_1-b_2)\right\|_{\infty,\infty}+CT^2\left\|b_1-b_2\right\|_{\infty,\infty}. \tag{6.5}$$

The second and third derivatives can be done in a similar way and clearly, the same arguments yield the estimates for Y. □

Since the $b_i^{(Y)}$ are defined implicitly through the transforms X_i, it remains to show that this does not considerably worsen its dependence on the ξ_i and Ω_i.

Lemma 6.4. *Let b_i be given by η_i, ω_i as in step* (3) *and $b_i^{(Y)}$ correspondingly as in step* (5)*. Then*

$$\|\partial^\beta(b_1^{(Y)} - b_2^{(Y)})\|_{L^\infty(J_T;C(\mathbb{R}^3))} \le C \left\|\partial^\beta(b_1 - b_2)\right\|_{L^\infty(J_T;C^1(\mathbb{R}^3))}$$

for all multi-indices β with $0 \le |\beta| \le 3$.

Proof. A simple calculation shows that for all $(t, y) \in [0, T) \times \mathbb{R}^3$,

$$\begin{aligned}\|b_1^{(Y)} - b_2^{(Y)}\|_{\infty,\infty} \quad \le \quad & \|J_{X_1}^{-1} - J_{X_2}^{-1}\|_{\infty,\infty} \|b_1\|_{\infty,\infty} \\ & + \|J_{X_2}^{-1}\|_{\infty,\infty} \|\nabla b_1\|_{\infty,\infty} \|X_1 - X_2\|_{\infty,\infty} \\ & + \|J_{X_2}^{-1}\|_{\infty,\infty} \|b_1 - b_2\|_{\infty,\infty},\end{aligned}$$

where

$$\|J_{X_1}^{-1} - J_{X_2}^{-1}\|_{\infty,\infty} \quad \le \quad \|J_{X_1}^{-1}\|_{\infty,\infty}\|J_{X_2} - J_{X_1}\|_{\infty,\infty}\|J_{X_2}^{-1}\|_{\infty,\infty}.$$

By Lemma 6.3, we get the desired estimate if $\|J_{X_i}^{-1}\|_{\infty,\infty} \le C$, uniformly for fixed K_i. This follows from the uniform estimate $\|J_{X_i}\|_{\infty,\infty} \le C_{K_i}$ and the fact that the X_i are volume-preserving. In a similar way, the derivatives of the $b_i^{(Y)}$ can be treated. □

Putting the above lemmas together now yields Proposition 6.1. The first estimate on $\partial^\alpha X_i$ was already proved in Lemma 6.3. The boundedness of the $\partial^\alpha Y_i$ follows from the arguments used for $J_{X_i}^{-1}$ in the proof of Lemma 6.4. Furthermore, the differences satisfy

$$\begin{aligned}\left\|\partial^\beta(X_1 - X_2)\right\|_{\infty,\infty} \quad &\le \quad CT \|b_1 - b_2\|_{L^\infty(J_T;C^{|\beta|}(\mathbb{R}^3))} \\ &\le \quad CT(\|\eta_1 - \eta_2\|_\infty + \|\omega_1 - \omega_2\|_\infty) \\ &\le \quad CT(\|\xi_1 - \xi_2\|_\infty + \|\Omega_1 - \Omega_2\|_\infty)\end{aligned}$$

by Lemmas 6.3, 6.2 and 6.1. Adding Lemma 6.4 similarly yields

$$\begin{aligned}\left\|\partial^\beta(Y_1 - Y_2)\right\|_{\infty,\infty} \quad &\le \quad CT\|b_1^{(Y)} - b_2^{(Y)}\|_{L^\infty(J_T;C^{|\beta|}(\mathbb{R}^3))} \\ &\le \quad CT \|b_1 - b_2\|_{L^\infty(J_T;C^{|\beta|+1}(\mathbb{R}^3))} \\ &\le \quad CT(\|\xi_1 - \xi_2\|_\infty + \|\Omega_1 - \Omega_2\|_\infty).\end{aligned}$$

6.2. **Estimates on F_0, F_1 and F_2.** We fix $T_0, R_0 > 0$. In the following, $C > 0$ denotes a generic constant which does not depend on T, R for $0 < T \le T_0$, $0 < R \le R_0$. We then set

$$C_0 := \|u^*\|_{X_{p,q}^{T_0}} + \|p^* - \mathbf{g} \cdot y\|_{Y_{p,q}^{T_0}} + \|(\xi^*, \Omega^*)\|_{W^{1,p}(J_{T_0})}.$$

We always assume $(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega}), (\tilde{u}_1, \tilde{p}_1, \tilde{\xi}_1, \tilde{\Omega}_1), (\tilde{u}_2, \tilde{p}_2, \tilde{\xi}_2, \tilde{\Omega}_2) \in \mathcal{K}_R^T$ and that $u^*, p^*, \xi^*, \Omega^*$ are given by (5.1) and set $u = \tilde{u} + u^*, p = \tilde{p} + p^*, \xi = \tilde{\xi} + \xi^*, \Omega = \tilde{\Omega} + \Omega^*$ and $u_1 = \tilde{u}_1 + u^*, p_1 = \tilde{p}_1 + p^*$, etc., throughout this section. We often use the corresponding estimate

$$\|(u, p, \xi, \Omega)\|_{X_{p,q}^T \times Y_{p,q}^T \times W^{1,p}(J_T)} \quad \le \quad R + C_0. \tag{6.6}$$

Given ξ, ξ_1, ξ_2 and $\Omega, \Omega_1, \Omega_2$ from $\tilde{\xi}, \tilde{\xi}_1, \tilde{\xi}_2, \xi^*$ and $\tilde{\Omega}, \tilde{\Omega}_1, \tilde{\Omega}_2, \Omega^*$, we calculate the corresponding coordinate transforms X, X_1, X_2 and Y, Y_1, Y_2 in the way described in steps (1)–(5) in the last subsection. They determine the covariant and the contravariant metric tensors g^{ij}, g_{ij} and the Christoffel symbol Γ^i_{jk} of the transformed problem (3.12).

Lemma 6.5. *Let $T > 0$. The coefficients given by the coordinate transforms X, X_1, X_2 and Y, Y_1, Y_2 satisfy*

$$\|\partial^\alpha g^{ij}\|_{\infty,\infty} + \|\partial^\alpha g_{ij}\|_{\infty,\infty} + \|\partial^\alpha \Gamma^i_{jk}\|_{\infty,\infty} \quad \leq \quad C$$

and

$$\|\partial^\alpha((g_1)^{ij} - (g_2)^{ij})\|_{\infty,\infty} + \|\partial^\alpha((g_1)_{ij} - (g_2)_{ij})\|_{\infty,\infty} + \|\partial^\alpha((\Gamma_1)^i_{jk} - (\Gamma_2)^i_{jk})\|_{\infty,\infty} \\ \leq CT \left\|(\tilde{\xi}_1 - \tilde{\xi}_2, \tilde{\Omega}_1 - \tilde{\Omega}_2)\right\|_\infty$$

for all multi-indices α with $0 \leq |\alpha| \leq 1$.

Proof. The estimates are a direct consequence of Proposition 6.1. Since $g_{ij}(t,y) = \sum_{l=1}^3 (\partial_l X_i)(\partial_l X_j)(t,y)$,

$$\|g_{ij}\|_{\infty,\infty} + \|\partial_k g_{ij}\|_{\infty,\infty} \leq C \sup_l \|\partial_l X\|^2_{\infty,\infty} + \|\partial_k \partial_l X\|_{\infty,\infty} \|\partial_l X\|_{\infty,\infty} \leq C. \tag{6.7}$$

In Proposition 6.1, the constant C depends on $\|(\xi,\Omega)\|_\infty$, but here we can additionally use

$$\begin{aligned} \|(\xi,\Omega)\|_\infty &\leq \|(\tilde{\xi},\tilde{\Omega})\|_\infty + \|(\xi^*,\Omega^*)\|_{L^\infty(J_{T_0})} \\ &\leq C_1 \|(\tilde{\xi},\tilde{\Omega})\|_{W^{1,p}(J_T)} + C_2 \|(\xi^*,\Omega^*)\|_{W^{1,p}(J_{T_0})} \\ &\leq C(R + C_0), \end{aligned}$$

where the embedding constant C_1 can be chosen independently of T since $\tilde{\xi}, \tilde{\Omega} \in W^{1,p}_0(J_T)$. Furthermore, the differences of two coefficients given by two different transforms satisfy

$$\begin{aligned} \|(g_1)_{ij} - (g_2)_{ij}\|_{\infty,\infty} &\leq C \sup_l (\|\partial_l X_1\|_{\infty,\infty} + \|\partial_l X_2\|_{\infty,\infty}) \|\partial_l (X_1 - X_2)\|_{\infty,\infty} \\ &\leq CT \|(\tilde{\xi}_1 - \tilde{\xi}_2, \tilde{\Omega}_1 - \tilde{\Omega}_2)\|_\infty. \end{aligned}$$

The remaining estimates follow analogously. □

With these estimates on the coefficients, we can control the transformed differential operators appearing in F_0.

Lemma 6.6. *Let $T_0 \geq T > 0$ and $\frac{3}{2q} + \frac{1}{p} \leq \frac{3}{2}$. Furthermore, let $s = 3p, s' = 3p/2$, $r = 3q, r' = 3q/2$ and*

$$C_*(T) := \|\nabla u^*\|_{s',r'} + \|u^*\|_{s,r}.$$

Then

$$\begin{aligned}
\|\mathbf{G}-\mathbf{g}\|_{p,q} &\leq CT\,\|(\xi,\Omega)\|_{W^{1,p}(J_T)}\,,\\
\|\mathcal{M}u\|_{p,q} &\leq C(T^{1/2}+T^{1/p})\,\|u\|_{X^T_{p,q}}\,,\\
\|(\mathcal{L}-\Delta)u\|_{p,q} &\leq C(T+T^{1/2}+T^{1/p})\,\|u\|_{X^T_{p,q}}\,,\\
\|(\mathcal{G}-\nabla)p\|_{p,q} &\leq CT\,\|p\|_{Y^T_{p,q}}\,,\\
\|\mathcal{N}(u)\|_{p,q} &\leq C[(R+C_*(T))^2+T^{1/2}].
\end{aligned}$$

Moreover, we obtain

$$\|\mathcal{J}((\mathbf{T}-\mathcal{T})(\tilde{u},\tilde{p}))\|_p \leq CT\|\tilde{u}\|_{X^T_{p,q}}.$$

Proof. These estimates follow from Lemma 6.5 and the embedding properties of $X^T_{p,q}$.

If $p\geq 2$, let $k'=\infty$ and fix $\frac{1}{k'}=\frac{1}{p}-\frac{1}{2}$ otherwise. Proposition 4.3 yields the embeddings

$$X^T_{p,q}\hookrightarrow L^s(J_T;L^r(\mathcal{D})),\quad X^T_{p,q}\hookrightarrow L^{s'}(J_T;W^{1,r'}(\mathcal{D}))\quad \text{and}$$

$$X^T_{p,q}\hookrightarrow L^{k'}(J_T;W^{1,q}(\mathcal{D})),$$

where the embedding constants do not depend on T provided $\tilde{u}\in X^T_{p,q,0}$ and $u^*\in X^{T_0}_{p,q}$.

There are two operators of highest order, $\mathcal{L}$ and $\mathcal{G}$, which differ from Δ and ∇ smoothly in time. The identity transform $X(t,y)=y$ for all $t>0, y\in\mathbb{R}^3$ corresponds to the body velocities $\xi=\Omega=0$, so

$$\begin{aligned}
\|g^{jk}-\delta_{jk}\|_{\infty,\infty} &\leq CT\,\|(\xi,\Omega)\|_{W^{1,p}(J_T)} \leq CT(R+C_0) \qquad \text{and}\\
\|\partial_j Y_k-\delta_{jk}\|_{\infty,\infty} &\leq CT\,\|(\xi,\Omega)\|_{W^{1,p}(J_T)} \leq CT(R+C_0)
\end{aligned}$$

by Lemma 6.5 and (6.6). It follows that

$$\|\mathbf{G}-\mathbf{g}\|_{p,q} \leq C\sup_{j,k}\|\partial_j Y_k-\delta_{jk}\|_{\infty,\infty}\,|\mathbf{g}| \leq CT\,\|(\xi,\Omega)\|_{W^{1,p}(J_T)} \tag{6.8}$$

and that

$$\begin{aligned}
\|(\mathcal{L}-\Delta)u\|_{p,q} &\leq C\sup_{i,j,k,l}\big[\|g^{jk}-\delta_{jk}\|_{\infty,\infty}\|\Delta u\|_{p,q}\\
&\quad+\big(\|\partial_j g^{jk}\|_{\infty,\infty}+\|g^{ki}\|_{\infty,\infty}\|\Gamma^i_{jk}\|_{\infty,\infty}\big)\|\nabla u\|_{p,q}\\
&\quad+\big(\|\partial_k g^{kl}\|_{\infty,\infty}\|\Gamma^i_{kl}\|_{\infty,\infty}+\|g^{kl}\|_{\infty,\infty}\|\partial_k\Gamma^i_{kl}\|_{\infty,\infty}\\
&\quad+\|g^{kl}\|_{\infty,\infty}\|\Gamma^m_{jl}\|_{\infty,\infty}\|\Gamma^i_{km}\|_{\infty,\infty}\big)\|u\|_{p,q}\big]\\
&\leq CT(R+C_0)\,\|u\|_{X^T_{p,q}}+CT^{1/p-1/k'}\,\|u\|_{L^{k'}(J_T;W^{1,q}(\mathcal{D}))}\\
&\quad+CT^{1/p}\,\|u\|_{\infty,q}\\
&\leq C(T+T^{1/2}+T^{1/p})(R+C_0)\,\|u\|_{X^T_{p,q}}
\end{aligned}$$

as well as

$$\|(\mathcal{G}-\nabla)p\|_{p,q} \leq C\sup_{j,k}\|g^{jk}-\delta_{jk}\|_{\infty,\infty}\,\|\nabla p\|_{p,q} \leq CT\,\|p\|_{Y^T_{p,q}}\,.$$

The additional terms arising from the material derivative satisfy

$$\begin{aligned}\|\mathcal{M}u\|_{p,q} &\leq C\|b^{(Y)}\|_{\infty,\infty}\|\nabla u\|_{p,q} + \sup_{i,j,k}(\|\Gamma^i_{jk}\|_{\infty,\infty}\|b^{(Y)}\|_{\infty,\infty} \\ &\quad + \|\partial_k Y_i(\cdot,X)\|_{\infty,\infty}\|J_b\|_{\infty,\infty})\|u\|_{p,q} \\ &\leq C(T^{1/2}+T^{1/p})\|u\|_{L^{k'}(J_T;W^{1,q}(\mathcal{D}))} + CT^{1/p}\|u\|_{\infty,q} \\ &\leq C(T^{1/2}+T^{1/p})\|u\|_{X^T_{p,q}}\end{aligned}$$

by Proposition 6.1 and Lemmas 6.4, 6.2, and 6.1. For the transformed convection term, we use the fact that $X^T_{p,q} \hookrightarrow L^l(J_T;L^{2q}(\mathcal{D}))$ for $\frac{1}{l} := \frac{1}{p} + \frac{3}{4q} - 1$. Thus,

$$\begin{aligned}\|\mathcal{N}(u)\|_{p,q} &\leq \|(u\cdot\nabla)u)\|_{p,q} + \sup_{i,j,k}\|\Gamma^i_{jk}\|_{\infty,\infty}\|u\|^2_{2p,2q} \\ &\leq \|u\|_{r,s}\|\nabla u\|_{r',s'} + CT^{2/2p-2/l}\|u\|^2_{L^l(J_T;L^{2q}(\mathcal{D}))} \\ &\leq C[(R+C_*(T))^2 + T^{1/2}].\end{aligned}$$

Concerning the non-linear term in the rigid body equations, we can set $M_2 = 0$ in (6.2) to see the estimate

$$\|Q - \mathrm{Id}_{\mathbb{R}^3}\|_\infty \leq CT\|\Omega\|_\infty.$$

By (4.16), it follows that

$$\|\mathcal{J}(\mathcal{E}^{(\tilde{u})} - Q^T\mathcal{E}^{(Q\tilde{u})}Q)\|_p \leq CT\|\tilde{u}\|_{X^T_{p,q}},$$

and clearly, $\tilde{p}\mathrm{Id} = Q^T\tilde{p}Q$. □

We now have the main estimates at hand to prove the following lemma.

Lemma 6.7. *Given $T, R > 0$ sufficiently small, the function ϕ^T_R maps $\mathcal{K}^T_R$ into itself.*

Proof. Clearly by Lemma 6.6,

$$\begin{aligned}\|F_1(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega})\|_p &\leq \|\mathbf{G}(\cdot,0) - \mathbf{g}\|_p + \|\xi\times\Omega\|_p + \|\mathcal{J}((\mathbf{T}-\mathcal{T})(\hat{u},\hat{p}))\|_p \\ &\leq \|Q^T - \mathrm{Id}_{\mathbb{R}^3}\|_p|\mathbf{g}| + \|\xi\|_\infty\|\Omega\|_p + CTR \\ &\leq CT^{1/p}\end{aligned}$$

and

$$\|F_2(\tilde{u},\tilde{p},\tilde{\Omega})\|_p \leq \|\Omega\times I\Omega\|_p + \|\mathcal{J}((\mathbf{T}-\mathcal{T})(\tilde{u},\tilde{p}))\|_p \leq CT^{1/p}.$$

Furthermore, by Lemma 6.6 we also get

$$\begin{aligned}\|F_0(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega})\|_{p,q} &\leq \|\mathbf{G}-\mathbf{g}\|_{p,q} + \|\mathcal{H}(u,p)\|_{p,q} + \|\mathcal{N}(u)\|_{p,q} \\ &\leq C\big[T^{1/2} + T^{1/p} + (R+C_*(T))^2\big].\end{aligned}$$

Note here that $C_*(T) \to 0$ as $T \to 0$ by the definition in Lemma 6.6. Thus by Theorem 4.1,

$$\begin{aligned}\|\phi^T_R(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega})\|_{\mathcal{K}^T_R} &\leq C\big(\|F_0(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega})\|_{p,q} \\ &\quad + \|F_1(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega})\|_p + \|F_2(\tilde{u},\tilde{p},\tilde{\Omega})\|_p\big) \\ &\leq C(T^{1/2} + T^{1/p} + (R+C_*(T))^2) \\ &\leq R,\end{aligned}$$

if T and R are chosen sufficiently small. □

In a similar way, we show

Lemma 6.8. *The map ϕ_R^T is contractive.*

Proof. The proof is mainly a matter of writing out the estimates on F_0. We show that

$$\begin{aligned} &\|F_0(\tilde{u}_1,\tilde{p}_1,\tilde{\xi}_1,\tilde{\Omega}_1) - F_0(\tilde{u}_2,\tilde{p}_2,\tilde{\xi}_2,\tilde{\Omega}_2)\|_{p,q} \\ \leq\ & L_{R,T}\|(\tilde{u}_1-\tilde{u}_2,\tilde{p}_1-\tilde{p}_2,\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_{\mathcal{K}_R^T}, \end{aligned} \tag{6.9}$$

where $L_{R,T}$ can be made arbitrarily small for $T, R \to 0$. First, we rewrite the term as

$$\begin{aligned} &F_0(\tilde{u}_1,\tilde{p}_1,\tilde{\xi}_1,\tilde{\Omega}_1) - F_0(\tilde{u}_2,\tilde{p}_2,\tilde{\xi}_2,\tilde{\Omega}_2) \\ &\quad = \mathcal{H}(\tilde{u}_1-\tilde{u}_2,\tilde{p}_1-\tilde{p}_2,\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2) + \mathbf{G}_1 - \mathbf{G}_2 \\ &\qquad + (\mathcal{M}_1-\mathcal{M}_2)u_2 + (\mathcal{L}_1-\mathcal{L}_2)u_2 \\ &\qquad + (\mathcal{G}_1-\mathcal{G}_2)p_2 + \mathcal{N}_1(u_1) - \mathcal{N}_2(u_2), \end{aligned}$$

where $\mathcal{H}$ is defined as in (5.2) and its estimates are already known from Lemma 6.6. Moreover, it follows from Proposition 6.1, Lemma 6.5, and (6.6) that

$$\begin{aligned} \|\mathbf{G}_1-\mathbf{G}_2\|_{p,q} &\leq C\,\|J_{Y_1}(\cdot,X_1) - J_{Y_2}(\cdot,X_2)\|_{p,q} \\ &\leq CT\|(\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_{W^{1,p}(J_T)}, \\ \|(\mathcal{M}_1-\mathcal{M}_2)u_2\|_{p,q} &\leq CT(R+C_0)\|(\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_{W^{1,p}(J_T)}, \\ \|(\mathcal{G}_1-\mathcal{G}_2)p_2\|_{p,q} &\leq CT\|(\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_{W^{1,p}(J_T)}, \end{aligned}$$

as well as

$$\|(\mathcal{L}_1-\mathcal{L}_2)u_2\|_{p,q} \leq CT\|(\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_{W^{1,p}(J_T)}. \tag{6.10}$$

Moreover,

$$\begin{aligned} &\|\mathcal{N}_1(u_1)-\mathcal{N}_2(u_2)\|_{p,q} \\ &\quad\leq \|(u_1\cdot\nabla)u_1-(u_2\cdot\nabla)u_2\|_{p,q} \\ &\qquad + \sup_i \|\sum_{j,k}\big((\Gamma_1)^i_{jk}(u_1)_j(u_1)_k - (\Gamma_2)^i_{jk}(u_2)_j(u_2)_k\big)\|_{p,q} \\ &\quad\leq C\sup_{i,j,k}(\|u_1\|_{s,r}\,\|\nabla(\tilde{u}_1-\tilde{u}_2)\|_{s',r'} + \|\nabla u_2\|_{s',r'}\,\|\tilde{u}_1-\tilde{u}_2\|_{s,r} \\ &\qquad + \|(\Gamma_1)^i_{jk}\|_{\infty,\infty}(\|u_1\|_{2p,2q}+\|u_2\|_{2p,2q})\,\|\tilde{u}_1-\tilde{u}_2\|_{2p,2q} \\ &\qquad + \|(\Gamma_1)^i_{jk}-(\Gamma_2)^i_{jk}\|_{\infty,\infty}\,\|u_2\|^2_{2p,2q}) \\ &\quad\leq C(R+C_*(T))(\|\tilde{u}_1-\tilde{u}_2\|_{X^T_{p,q}} + \|(\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_{W^{1,p}(J_T)}), \end{aligned}$$

where s, s', r and r' are defined as in Lemma 6.7. In conclusion, (6.9) holds true if we set $L_{R,T} := C(R+C_*(T)+T+T^{1/2}+T^{1/p})$ and use that $C_*(T)\to 0$ as $T\to 0$. Concerning the functions F_1, F_2, we can show that

$$\begin{aligned} D\mathcal{J}_p &:= \|\mathcal{J}((\mathbf{T}-\mathcal{T}_1)(\tilde{u}_1,\tilde{p}_1) - (\mathbf{T}-\mathcal{T}_2)(\tilde{u}_2,\tilde{p}_2))\|_p \\ &\leq \|\mathcal{J}((\mathbf{T}-\mathcal{T}_1)(\tilde{u}_1-\tilde{u}_2,\tilde{p}_1-\tilde{p}_2))\|_p + \|\mathcal{J}((\mathcal{T}_1-\mathcal{T}_2)(\tilde{u}_2,\tilde{p}_2))\|_p \\ &\leq CT(\|\tilde{u}_1-\tilde{u}_2\|_{X^T_{p,q}} + \|\tilde{\Omega}_1-\tilde{\Omega}_2\|_{W^{1,p}(J_T)}), \end{aligned}$$

by Lemma 6.6 and using (4.16). In conclusion,

$$\begin{aligned}
&\|F_1(\tilde{u}_1,\tilde{p}_1,\tilde{\xi}_1,\tilde{\Omega}_1)-F_1(\tilde{u}_2,\tilde{p}_2,\tilde{\xi}_2,\tilde{\Omega}_2)\|_p+\|F_2(\tilde{u}_1,\tilde{p}_1,\tilde{\Omega}_1)-F_2(\tilde{u}_2,\tilde{p}_2,\tilde{\Omega}_2)\|_p\\
&\quad\le \|\mathbf{G}_1(\cdot,0)-\mathbf{G}_2(\cdot,0)\|_p+\|(\tilde{\xi}_1+\xi^*)\times(\tilde{\Omega}_1-\tilde{\Omega}_2)\|_p\\
&\qquad+\|(\tilde{\xi}_1-\tilde{\xi}_2)\times(\tilde{\Omega}_2+\Omega^*)\|_p\\
&\qquad+\|(\tilde{\Omega}_1+\Omega^*)\times I(\tilde{\Omega}_1-\tilde{\Omega}_2)\|_p+\|(\tilde{\Omega}_1-\tilde{\Omega}_2)\times I(\tilde{\Omega}_2+\Omega^*)\|_p+D\mathcal{J}_p\\
&\quad\le C\,\|Q_1-Q_2\|_p+C(R+C_0)\|(\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_p\\
&\qquad+CT(\|\tilde{u}_1-\tilde{u}_2\|_{X^T_{p,q}}+\|\tilde{\Omega}_1-\tilde{\Omega}_2\|_{W^{1,p}(J_T)})\\
&\quad\le CT\|(\tilde{u}_1-\tilde{u}_2,0,\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_{\mathcal{K}^T_R}.
\end{aligned}$$

By Theorem 4.1, it follows from these estimates that ϕ^T_R is strongly contractive for sufficiently small T,R, i.e.

$$\begin{aligned}
&\|\phi^T_R(\tilde{u}_1,\tilde{p}_1,\tilde{\xi}_1,\tilde{\Omega}_1)-\phi^T_R(\tilde{u}_2,\tilde{p}_2,\tilde{\xi}_2,\tilde{\Omega}_2)\|_{\mathcal{K}^T_R}\\
&\quad\le C\big(\|F_0(\tilde{u}_1,\tilde{p}_1,\tilde{\xi}_1,\tilde{\Omega}_1)-F_0(\tilde{u}_2,\tilde{p}_2,\tilde{\xi}_2,\tilde{\Omega}_2)\|_{p,q}\\
&\qquad+\|F_1(\tilde{u}_1,\tilde{p}_1,\tilde{\xi}_1,\tilde{\Omega}_1)-F_1(\tilde{u}_2,\tilde{p}_2,\tilde{\xi}_2,\tilde{\Omega}_2)\|_p\\
&\qquad+\|F_2(\tilde{u}_1,\tilde{p}_1,\tilde{\Omega}_1)-F_2(\tilde{u}_2,\tilde{p}_2,\tilde{\Omega}_2)\|_p\big)\\
&\quad\le CL_{R,T}\|(\tilde{u}_1-\tilde{u}_2,\tilde{p}_1-\tilde{p}_2,\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_{\mathcal{K}^T_R}.
\end{aligned}$$

□

7. Proofs of Theorems 2.1 and 2.4

The contraction mapping theorem yields a unique fixed point of ϕ^T_R which is a strong solution $(\hat{u},\hat{p},\hat{\xi},\hat{\Omega})$ of problem (5.2). The solutions of the original problem (2.1) can be found by adding the reference solution $(u^*,p^*+\mathbf{g}\cdot y,\xi^*,\Omega^*)$ and performing the corresponding backward change of coordinates and variables given in (3.10), which preserves regularity. Moreover, the solution (v,q,η,ω) to the original problem must be unique as a consequence of the uniqueness of the fixed point. Note that the coordinate transform Y which appears in the pressure term $q=q_0+g\cdot Y$ depends on the choice of the cut-off function χ in the transform, the complete term q, however, does not.

In order to use the above method in the case of a bounded domain and to prove Theorem 2.4, we modify the cut-off function in the change of coordinates given by b such that it respects the outer boundary. Since we cannot deal with possible contact of the body and the boundary $\partial\mathcal{O}$, we assume that the body starts from a position with some distance

$$\operatorname{dist}(\mathcal{B}(0),\partial\mathcal{O})>d \qquad \text{for some } d>0.$$

Moreover, since the body moves with a continuous velocity, we restrict the solution to a time which guarantees that a small distance, e.g. $\frac{d}{2}$, remains. We define $\overline{\chi}\in C^\infty(\mathbb{R}^3;[0,1])$ by

$$\overline{\chi}(x):=\begin{cases}1 & \text{if } \operatorname{dist}(x,\partial\mathcal{O})\ge d,\\ 0 & \text{if } \operatorname{dist}(x,\partial\mathcal{O})\le \frac{d}{2},\end{cases}$$

and a new vector field $\overline{b}:[0,T]\times\mathbb{R}^3\to\mathbb{R}^3$ by

$$\overline{b}(t,x):=\overline{\chi}(x)[m(t)(x-x_C(t))+\eta(t)]-B_{\mathcal{O}}(\nabla\overline{\chi}(\cdot)[m(t)(\cdot-x_C(t))+\eta(t)])(x).$$

The new change of coordinates is also a volume-preserving diffeomorphism of $\mathcal{O}$ which is continuously differentiable in time and has continuous mixed partial derivatives. The results from subsection 4.1 still hold true if $\partial\mathcal{O}$ is of class $C^{2,1}$, so the estimates from Sections 4 and 6 can be reproduced by the same techniques to prove Theorem 2.4. There is an additional restriction on the time interval $(0,T)$ of existence of this solution given by

$$\operatorname{dist}(\mathcal{B}(t),\partial\mathcal{O}) > \frac{d}{2} \quad \text{for all } t\in(0,T).$$

Note that both Theorems 2.1 and 2.4 still hold true if we impose external forces other than $\mathbf{g}$ which fit into the L^q-setting, i.e. $f\in L^p(J_T;L^q(\mathcal{D}(\cdot)))$ in (1.1) and $\mathbf{F},\mathbf{M}\in L^p(J_T)$ in (1.2).

8. The generalized Newtonian case

We now consider the fluid-rigid body interaction problem in generalized Newtonian fluids, in order to prove Theorem 2.2. We recall from Section 2 that this means that the stress tensor is of type (G), i.e. it is given by

$$\mathbf{T}^\mu(v,q) := \mu(|\mathcal{E}^{(v)}|_2^2)\mathcal{E}^{(v)} - q\mathrm{Id},$$

where the viscosity function $\mu\in C^{1,1}(\mathbb{R}_+;\mathbb{R})$ satisfies the assumptions

$$\mu(s)>0 \quad\text{and}\quad \mu(s)+2s\mu'(s)>0 \quad \text{for all } s\geq 0, \tag{8.1}$$

and

$$\mathcal{E}^{(v)} := \frac{1}{2}(\nabla v+(\nabla v)^T), \qquad \varepsilon_{ij}^{(v)} = \frac{1}{2}(\partial_i v_j+\partial_j v_i).$$

The system of equations governing the motion of both fluid and rigid body is described by the balance equations for mass and momentum of the fluid as well as the momentum and angular momentum of the rigid body,

$$\left\{\begin{array}{rcll} v_t - \operatorname{div}\mathbf{T}^\mu(v,q) + (v\cdot\nabla)v &=& \mathbf{g}, & \text{in } Q_{\mathcal{D}},\\ \operatorname{div} v &=& 0, & \text{in } Q_{\mathcal{D}},\\ v &=& v_{\mathcal{B}}, & \text{on } Q_\Gamma,\\ v(0) &=& v_0, & \text{in } \mathcal{D}(0),\\ \mathrm{m}\eta' + \int_{\Gamma(t)}\mathbf{T}^\mu(v,q)n\,\mathrm{d}\sigma &=& \mathrm{m}\mathbf{g}, & \text{in } \mathbb{R}_+,\\ (J\omega)' + \int_{\Gamma(t)}(x-x_c)\times\mathbf{T}^\mu(v,q)n\,\mathrm{d}\sigma &=& 0, & \text{in } \mathbb{R}_+,\\ \eta(0) &=& \eta_0, & \\ \omega(0) &=& \omega_0. & \end{array}\right. \tag{8.2}$$

The new aspect in the non-Newtonian situation is the fact that we need to replace the Laplacian in the fluid equations by a quasi-linear operator arising from the term $\operatorname{div}\mathbf{T}^\mu(v,q)$. More precisely, we write $\operatorname{div}\mu(|\mathcal{E}^{(v)}|_2^2)\mathcal{E}^{(v)}$ as

$$\begin{aligned}(A(v)v)_i := (\operatorname{div}\mu(|\mathcal{E}^{(v)}|_2^2)\mathcal{E}^{(v)})_i &= \sum_{j=1}^3 \mu(|\mathcal{E}^{(v)}|_2^2)\partial_j\varepsilon_{ij}^{(v)} + \partial_j(\mu(|\mathcal{E}^{(v)}|_2^2))\varepsilon_{ij}^{(v)}\\ &= \sum_{j=1}^3 \big[\mu(|\mathcal{E}^{(v)}|_2^2)(\partial_j^2 v_i + \partial_i\partial_j v_j)\\ &\qquad + 2\mu'(|\mathcal{E}^{(v)}|_2^2)(\sum_{k,l=1}^3 \varepsilon_{kl}^{(v)}\partial_j\varepsilon_{kl}^{(v)}\varepsilon_{ij}^{(v)})\big].\end{aligned}$$

Using $\operatorname{div} v = 0$, this simplifies to

$$\begin{aligned}(A(v)v)_i &= \mu(|\mathcal{E}^{(v)}|_2^2)\Delta v_i + 2\mu'(|\mathcal{E}^{(v)}|_2^2)\sum_{j,k,l=1}^{3}\varepsilon_{ij}^{(v)}\varepsilon_{kl}^{(v)}\partial_j\partial_l v_k \\ &= \mu(|\mathcal{E}^{(v)}|_2^2)\Delta v_i + \sum_{j,k,l=1}^{3}\alpha_{ij}^{kl}(v)\partial_j\varepsilon_{kl}^{(v)},\end{aligned}$$

where

$$\alpha_{ij}^{kl}(v)(t,x) := 2\mu'(|\mathcal{E}^{(v)}(t,x)|_2^2)\varepsilon_{ij}^{(v)}(t,x)\varepsilon_{kl}^{(v)}(t,x), \qquad (t,x)\in Q_{\mathcal{D}}.$$

Consider the corresponding linearized second-order operator $A(w)$, given by

$$(A(w)v)_i := \mu(|\mathcal{E}^{(w)}|_2^2)\Delta v_i + \sum_{j,k,l=1}^{3}\alpha_{ij}^{kl}(w)\partial_j\varepsilon_{kl}^{(v)}$$

for some fixed w. It is shown by Bothe and Prüss [3], that the operator $A(w)$ admits maximal regularity and that the corresponding quasi-linear equations can be solved by a fixed point argument. This result will help prove Theorem 2.2 in a similar way as Theorem 2.1.

9. The transformed generalized Stokes operator

In this section, we transfer the set of equations (8.2) defined on an unknown moving domain to a fixed domain by applying the coordinate transforms X, Y defined in (3.5) and (3.8). To this end, for $T > 0$ and $(t,y)\in[0,T)\times\mathbb{R}^3$ set

$$\begin{aligned}
u(t,y) &:= J_Y(t,X(t,y))v(t,X(t,y)),\\
p(t,y) &:= q(t,X(t,y)),\\
\xi(t) &:= Q^T(t)\eta(t),\\
\Omega(t) &:= Q^T(t)\omega(t),\\
\mathbf{G}(t,y) &:= J_Y(t,X(t,y))\mathbf{g},\\
\mathcal{T}^\mu(u(t,y),p(t,y)) &:= Q^T(t)\mathbf{T}^\mu(Q(t)u(t,y),p(t,y))Q(t),\\
I &:= Q^T(t)J(t)Q(t),\\
N &:= Q^T(t)n(t),\\
\mathcal{D} := \mathcal{D}(0), \Gamma := \Gamma(0) &\quad\text{and}\quad \mathcal{B} := \mathcal{B}(0).
\end{aligned}$$

The terms v_t, $(v\cdot\nabla)v$ and ∇q in equation (8.2) are then transformed into $u_t + \mathcal{M}u$, $\mathcal{N}(u)$ and $\mathcal{G}p$ similarly to the Newtonian setting; see Section 3. In order to transform

$A(v)v$, we set

$$
\begin{aligned}
2\varepsilon_{ij}^{(v)}(t,x) &= (\partial_i v_j)(t,x) + (\partial_j v_i)(t,x) \\
&= \sum_{k,l=1}^{3} \big[(\partial_i Y_k)(t,X(t,y))(\partial_k\partial_l X_j)(t,y) \\
&\quad + (\partial_j Y_k)(t,X(t,y))(\partial_k\partial_l X_i)(t,y)\big] u_l(t,y) \\
&\quad + \big[(\partial_i Y_k)(t,X(t,y))(\partial_l X_j)(t,y) \\
&\quad + (\partial_j Y_k)(t,X(t,y))(\partial_l X_i)(t,y)\big](\partial_k u_l)(t,y) \\
&=: 2\sum_{k,l=1}^{3} \tilde{e}_{kl}^{ij} u_l(t,y) + \tilde{d}_{kl}^{ij}\partial_k u_l(t,y) \\
&=: 2\tilde{\varepsilon}_{ij}^{(u)}(t,y)
\end{aligned} \tag{9.1}
$$

as the transformed symmetric part of the gradient of v and use the notation $\tilde{\mathcal{E}}^{(u)} := (\tilde{\varepsilon}_{ij}^{(u)})_{ij}$. The transformed quasi-linear fluid operator now reads as

$$
(\mathcal{A}(w)u)_i = \mu(|\tilde{\mathcal{E}}^{(w)}|_2^2)(\mathcal{L}u)_i + \sum_{j,k,l,m=1}^{3} a_{ij}^{klm}(w)\partial_m \tilde{\varepsilon}_{kl}^{(u)},
$$

where

$$
a_{ij}^{klm}(w)(t,y) := 2\mu'(|\tilde{\mathcal{E}}^{(w)}|_2^2)(\partial_j Y_m)(t,X(t,y))\tilde{\varepsilon}_{ij}^{(w)}(t,y)\tilde{\varepsilon}_{kl}^{(w)}(t,y),
$$

and where $\mathcal{L}$ is the transformed Laplacian as defined in (3.13).

The transformed set of the full system of equations reads

$$
\begin{cases}
u_t - \mathcal{A}(u)u + \mathcal{G}p = \mathbf{G} - \mathcal{N}(u) - \mathcal{M}u & \text{in } J_T \times \mathcal{D}, \\
\operatorname{div} u = 0 & \text{in } J_T \times \mathcal{D}, \\
u(t,y) = \Omega(t) \times y + \xi(t) & t \in J_T, y \in \mathcal{D}, \\
u(0) = v_0 & \text{in } \mathcal{D}, \\
\mathrm{m}\xi' + \int_\Gamma \mathcal{T}^\mu(u,p)N\,\mathrm{d}\sigma = \mathrm{m}\mathbf{G}(\cdot,0) - \mathrm{m}(\Omega \times \xi) & t \in J_T, \\
I\Omega' + \int_\Gamma y \times \mathcal{T}^\mu(u,p)N\,\mathrm{d}\sigma = -\Omega \times (I\Omega) & t \in J_T, \\
\xi(0) = \eta_0, & \\
\Omega(0) = \omega_0, &
\end{cases} \tag{9.2}
$$

and it is equivalent to (8.2).

10. Maximal regularity of the linearized system

In this section, we consider the linearization of (9.2) and show that it satisfies the property of maximal regularity provided $p > 5$. Surely, many arguments from the Newtonian case may be carried over to the present situation; however, new estimates are necessary with respect to the operators $\mathcal{A}$ and $\mathcal{T}^\mu$.

In order to formulate the maximal regularity result, we recall the defintions of the following function spaces:

$$
\begin{aligned}
X_{p,p}^T(\mathcal{D}) &:= L^p(J_T; W^{2,p}(\mathcal{D})) \cap W^{1,p}(J_T; L^p(\mathcal{D})), \\
X_{p,p,0}^T(\mathcal{D}) &:= \{u \in X_{p,p}^T : u(0) = 0\}, \\
Y_{p,p}^Y(\mathcal{D}) &:= L^p(J_T : \widehat{W}^{1,p}(\mathcal{D})), \\
W_0^{1,p}(J_T) &:= \{\omega \in W^{1,p}(J_T) : \omega(0) = 0\}.
\end{aligned}
$$

We linearize $\mathcal{A}$ by the operator A_*, given by

$$A_* u := A(u^*)u,$$

which fixes the coefficients in the non-transformed operator A to a reference solution $u^* \in X^T_{p,p}(\mathcal{D})$ of the problem

$$\begin{cases} u^*_t - \Delta u^* + \nabla(p^* - \mathbf{g}\cdot y) &= 0, & \text{in } J_T \times \mathcal{D}, \\ \operatorname{div} u^* &= 0, & \text{in } J_T \times \mathcal{D}, \\ u^*(t,y) - \xi^*(t) - \Omega^*(t) \times y &= 0, & t \in J_T, y \in \Gamma, \\ u^*(0) &= v_0, & \text{in } \mathcal{D}, \\ \mathrm{m}(\xi^*)' + \int_\Gamma \mathbf{T}(u^*,p^*)N \,\mathrm{d}\sigma &= \mathrm{m}\mathbf{g}, & \text{in } J_T, \\ I(\Omega^*)' + \int_\Gamma y \times (\mathbf{T}(u^*,p^*)N) \,\mathrm{d}\sigma &= 0, & \text{in } J_T, \\ \xi^*(0) &= \eta_0, \\ \Omega^*(0) &= \omega_0. \end{cases} \tag{10.1}$$

The existence of a solution $u^*, p^*, \xi^*, \Omega^*$ to (10.1) follows from Theorem 4.1. The stress tensor $\mathcal{T}^\mu$ near the interface Γ is linearized by the Newtonian tensor

$$\mathbf{T}(u,p) := 2\mu_0 \mathcal{E}^{(u)} - \mathrm{Id}p,$$

where $\mu_0 := 1$.

We then define $\hat{u} := u - u^*, \hat{p} := p - (p^* - \mathbf{g}\cdot y), \hat{\xi} := \xi - \xi^*, \hat{\Omega} := \Omega - \Omega^*$ and add $A_*\hat{u}$ to the first line of (9.2). Setting

$$\mathcal{Q}(u^*, \hat{u}) := A_*\hat{u} - \mathcal{A}(u^* + \hat{u})(u^* + \hat{u})$$

and

$$\begin{aligned} G_0(\hat{u},\hat{p},\hat{\xi},\hat{\Omega}) &:= \mathbf{G} - \mathbf{g} - \mathcal{I}(\hat{u} + u^*, \hat{p} + p^*, \hat{\xi} + \xi^*, \hat{\Omega} + \Omega^*) - \Delta u^* - \mathcal{Q}(u^*, \hat{u}), \\ \mathcal{I}(u,p,\xi,\Omega) &:= \mathcal{M}u + (\mathcal{G} - \nabla)p - \mathcal{N}(u) \\ G_1(\hat{u},\hat{p},\hat{\xi},\hat{\Omega}) &:= \int_\Gamma (\mathbf{T} - \mathcal{T}^\mu)(\hat{u},\hat{p})N \,\mathrm{d}\sigma + \int_\Gamma (\mathbf{T} - \mathcal{T}^\mu)(u^*,p^*)N \,\mathrm{d}\sigma \\ &\quad + \mathrm{m}(\mathbf{G}(\cdot,0) - \mathbf{g}) - (\Omega^* + \hat{\Omega}) \times (\xi^* + \hat{\xi}) \\ G_2(\hat{u},\hat{p},\hat{\xi},\hat{\Omega}) &:= \int_\Gamma y \times (\mathbf{T} - \mathcal{T}^\mu)(\hat{u},\hat{p})N \,\mathrm{d}\sigma + \int_\Gamma y \times (\mathbf{T} - \mathcal{T}^\mu)(u^*,p^*)N \,\mathrm{d}\sigma \\ &\quad - (\Omega^* + \hat{\Omega}) \times I(\Omega^* + \hat{\Omega}), \end{aligned} \tag{10.2}$$

cf. (5.2), we rewrite (9.2) into the equivalent system

$$\begin{cases} \hat{u}_t - A_*\hat{u} + \nabla\hat{p} &= G_0(\hat{u},\hat{p},\hat{\xi},\hat{\Omega}), & \text{in } J_T \times \mathcal{D}, \\ \operatorname{div}\hat{u} &= 0, & \text{in } J_T \times \mathcal{D}, \\ \hat{u}(t,y) &= \hat{\Omega}(t) \times y + \hat{\xi}(t), & t \in J_T, y \in \Gamma, \\ \hat{u}(0) &= 0, & \text{in } \mathcal{D}, \\ \int_\Gamma \mathbf{T}(\hat{u},\hat{p})N \,\mathrm{d}\sigma + \mathrm{m}\hat{\xi}' &= G_1(\hat{u},\hat{p},\hat{\xi},\hat{\Omega}), & \text{in } J_T, \\ \int_\Gamma y \times \mathbf{T}(\hat{u},\hat{p})N \,\mathrm{d}\sigma + I\hat{\Omega}' &= G_2(\hat{u},\hat{p},\hat{\xi},\hat{\Omega}), & \text{in } J_T, \\ \hat{\xi}(0) &= 0, \\ \hat{\Omega}(0) &= 0. \end{cases} \tag{10.3}$$

Fixing G_0, G_1, G_2 yields the linearization of (9.2) that we would like to investigate in this section.

The main result of this section is the following theorem.

Theorem 10.1. *Let $\mathcal{D}$ be an exterior domain of class $C^{2,1}$, $T_0 > 0$, and $p > 5$. Assume that $g_0 \in L^p(J_{T_0}; L^p(\mathcal{D})), g_1, g_2 \in L^p(J_{T_0})$. Then the problem*

$$\left\{\begin{array}{rcll} u_t - A_* u + \nabla p &=& g_0, & \text{in } J_{T_0} \times \mathcal{D}, \\ \operatorname{div} u &=& 0, & \text{in } J_{T_0} \times \mathcal{D}, \\ u(t,y) - \xi(t) - \Omega(t) \times y &=& 0, & t \in J_{T_0}, y \in \Gamma, \\ \int_\Gamma \mathbf{T}(u,p) N \,\mathrm{d}\sigma + \mathrm{m}\xi' &=& g_1, & \text{in } J_{T_0}, \\ \int_\Gamma y \times (\mathbf{T}(u,p)N) \,\mathrm{d}\sigma + I\Omega' &=& g_2, & \text{in } J_{T_0}, \end{array}\right. \tag{10.4}$$

with initial conditions

$$u(0) = 0, \xi(0) = \Omega(0) = 0,$$

admits a unique solution

$$u \in X^{T_0}_{p,p,0}, \quad p \in Y^{T_0}_{p,p}, \quad (\xi, \Omega) \in W^{1,p}_0(J_{T_0}; \mathbb{R}^6)$$

satisfying the estimate

$$\|u\|_{X^{T_0}_{p,p}} + \|p\|_{Y^{T_0}_{p,p}} + \|(\xi,\Omega)\|_{W^{1,p}(J_{T_0})} \le C(\|g_0\|_{p,p} + \|(g_1,g_2)\|_{L^p(J_{T_0})}), \tag{10.5}$$

where the constant C does not depend on g_0, g_1, or g_2.

As in the Newtonian case, it will again be important to split the tangential and the normal component of the inhomogeneous boundary data at the interface to prove Theorem 10.1. This allows us to use known existence results and estimates on the generalized Stokes equations in order to solve the coupled problem.

In the following subsection, we cite these results and prove the corresponding pressure estimate. In subsection 10.2, we give a reformulation of (10.4) in the unknowns $\hat{\xi}, \hat{\Omega}$ only, analogously to equation (4.17).

10.1. **Estimates for the fluid equations.** The following result is a special case of a theorem due to Bothe and Prüss, [3, Theorem 4.1 and Section 3], which yields maximal regularity estimates for the linearized fluid part in our problem.

Theorem 10.2. *Let $T > 0$ and A_* as above. Suppose that $p > 5$, $f \in L^p(J_T; L^p(\mathcal{D}))$ and $u_0 \in W^{2-2/p}_p(\mathcal{D}), \operatorname{div} u_0 = 0, u_0|_\Gamma = h(0)$, where*

$$h \in W^{T,\Gamma}_p := W^{1-1/2p}_p(J_T; L^p(\Gamma)) \cap L^p(J_T; W^{2-1/p}_p(\Gamma))$$

and $h \cdot N|_\Gamma = 0$. Then the problem

$$\left\{\begin{array}{rcll} u_t - A_* u + \nabla p &=& f, & \text{in } J_T \times \mathcal{D}, \\ \operatorname{div} u &=& 0, & \text{in } J_T \times \mathcal{D}, \\ u &=& h, & \text{on } J_T \times \Gamma, \\ u(0) &=& u_0, & \text{in } \mathcal{D}, \end{array}\right. \tag{10.6}$$

has a unique strong solution

$$(u,p) \in X^T_{p,p}(\mathcal{D}) \times Y^T_{p,p}(\mathcal{D}),$$

which satisfies

$$\|u\|_{X^T_{p,p}} + \|p\|_{Y^T_{p,p}} \quad \le \quad C(\|f\|_{p,p} + \|h\|_{W^{T,\Gamma}_p} + \|u_0\|_{W^{2-2/p,p}(\mathcal{D})}). \tag{10.7}$$

As a consequence, we may refer to the solution operators corresponding to (10.6) by using the notation

$$u = \mathcal{U}_{A_*}(f,h,u_0) \quad \text{and} \quad p = \mathcal{P}_{A_*}(f,h,u_0) \tag{10.8}$$

for suitable data. Here, the condition $p > 5$ provides the embedding

$$X^T_{p,p} \hookrightarrow C(J_T; W^{2-2/p,p}(\mathcal{D})) \hookrightarrow C([0,T]; C^1(\overline{\mathcal{D}})). \tag{10.9}$$

We also need the following estimate on the pressure term, which corresponds to Lemma 4.2 in the Newtonian case.

Lemma 10.3. *Let $u := \mathcal{U}_{A_*}(f,0,0) \in X^T_{p,p}$ and $p := \mathcal{P}_{A_*}(f,0,0) \in Y^T_{p,p}$ be solutions of*

$$\left\{ \begin{array}{rcll} u_t - A_* u + \nabla p &=& f & \text{in } J_T \times \mathcal{D}, \\ \operatorname{div} u &=& 0 & \text{in } J_T \times \mathcal{D}, \\ u &=& 0 & \text{on } J_T \times \Gamma, \\ u(0) &=& 0 & \text{in } \mathcal{D}, \end{array} \right. \tag{10.10}$$

where $f \in L^p(J_T; L^p_\sigma(\Omega))$ and $p > 5$. Choose $R > 0$, $\mathcal{D}_R = \mathcal{D} \cap B_R$ and $p_R = \mathcal{P}_{A_*}(f,0,0)$ in $L^p(J_T; \widehat{W}^{1,p}(\mathcal{D}))$ such that $p_R \in L^p(J_T; L^p(\mathcal{D}_R))$ and $\int_{\mathcal{D}_R} p_R = 0$. Then

$$\|p_R\|_{L^p(J_T;L^p(\mathcal{D}_R))} \quad \leq \quad CT^{\alpha/p} \|\mathcal{U}_{A_*}(f,0,0)\|_{X^T_{p,p}} \tag{10.11}$$

for $\alpha = \frac{1}{2} - \frac{1}{2p} - \frac{\varepsilon}{2}$ and some $0 < \varepsilon < 1 - \frac{1}{p}$.

Proof. Since u, p_R strongly solve (10.10) and $f \in L^q_\sigma(\mathcal{D})$, it follows that

$$\nabla p_R(t,x) = ((\mathrm{Id} - P_{\mathcal{D},p})(A_* u)(x))(t) \quad \text{for a.a. } t \in J_T. \tag{10.12}$$

Solving the following Neumann problem, we construct suitable test functions for p_R. Assume that $\psi \in L^{p'}(J_T; L^{p'}(\mathcal{D}_R))$ such that $\int_{\mathcal{D}_R} \psi(t) = 0$ for a.a. $t \in J_T$ and extend ψ to $\mathcal{D} \backslash \mathcal{D}_R$ by 0. Then by [41, Prop. 5.6], there is a solution ϕ_ψ of

$$\left\{ \begin{array}{rcll} \Delta \phi_\psi &=& \psi & \text{in } \mathcal{D}, \\ \frac{\partial \phi_\psi}{\partial N} &=& 0 & \text{on } \Gamma, \end{array} \right.$$

which satisfies the estimate

$$\|\nabla \phi_\psi\|_{L^{p'}(J_T;W^{1,p'}(\mathcal{D}))} \quad \leq \quad C\|\psi\|_{L^{p'}(J_T;L^{p'}(\mathcal{D}_R))}. \tag{10.13}$$

Integrating by parts and using $\int_\Gamma p_R(\nabla \phi_\psi \cdot n) = 0$ as well as (10.12), it follows that

$$\begin{aligned} &\int_0^T \int_{\mathcal{D}_R} p_R \psi \\ &\quad = \int_0^T \int_{\mathcal{D}} p_R \Delta \phi_\psi = -\int_0^T \int_{\mathcal{D}} \nabla p_R(t) \cdot \nabla \phi_\psi \\ &= -\int_{\mathcal{D}} (\mathrm{Id} - P_{\mathcal{D},p}) A_* u \cdot \nabla \phi_\psi \\ &= -\sum_{i=1}^3 \int_0^T \int_{\mathcal{D}} \big[\mu(|\mathcal{E}^{(u^*)}|_2^2)\Delta u_i + \sum_{j,k,l=1}^3 \alpha^{kl}_{ij}(\partial_j \partial_k u_l)\big] \big[(\mathrm{Id} - P_{\mathcal{D},p})\nabla \phi_\psi\big]_i \\ &= -\sum_{i=1}^3 \big[\int_0^T \int_{\mathcal{D}} \mu(|\mathcal{E}^{(u^*(t))}|_2^2)\Delta u_i(\partial_i \phi_\psi) + \sum_{j,k,l=1}^3 \int_0^T \int_{\mathcal{D}} \partial_j \partial_k u_l(\alpha^{kl}_{ij})\partial_i \phi_\psi\big]. \end{aligned}$$

We use integration by parts a second time to write

$$\begin{aligned}\sum_{i=1}^{3}\int_0^T\int_{\mathcal{D}}\mu(|\mathcal{E}^{(u^*)}|_2^2)\Delta u_i(\partial_i\phi_\psi) &= \sum_{i=1}^{3}\Big[-\int_0^T\int_{\mathcal{D}}\nabla u_i\cdot\nabla\big(\mu(|\mathcal{E}^{(u^*)}|_2^2)\partial_i\phi_\psi\big)\\ &\quad+\int_0^T\int_\Gamma\mu(|\mathcal{E}^{(u^*)}|_2^2)\partial_i\phi_\psi(\nabla u_i\cdot N)\Big]\\ &=: I+I_\Gamma,\end{aligned}$$

as well as

$$\begin{aligned}\sum_{i,j,k,l=1}^{3}\int_0^T\int_{\mathcal{D}}(\partial_j\partial_k u_l)\alpha_{ij}^{kl}(\partial_i\phi_\psi) &= \sum_{i,j,k,l=1}^{3}\Big[-\int_0^T\int_{\mathcal{D}}(\partial_k u_l)\partial_j(\alpha_{ij}^{kl}\partial_i\phi_\psi)\\ &\quad+\int_0^T\int_\Gamma(\partial_k u_l)(\alpha_{ij}^{kl}\partial_i\phi_\psi N_j)\Big]\\ &=: II+II_\Gamma.\end{aligned}$$

Now let $r:=\frac{p}{p-2}$. By Hölder's inequality, we get

$$\begin{aligned}|I| &\le C\|\nabla u\|_{p,p}\|\nabla^2\phi_\psi\|_{p',p'}\|\mu(|\mathcal{E}^{(u^*)}|_2^2)\|_{\infty,\infty}\\ &\quad+C\|\nabla u\|_{\infty,p}\|\nabla\phi_\psi\|_{p',r}\|\mu'(|\mathcal{E}^{(u^*)}|_2^2)\|_{\infty,\infty}\|\mathcal{E}^{(u^*)}\|_{\infty,\infty}\|\nabla\mathcal{E}^{(u^*)}\|_{p,p}\end{aligned}$$

and

$$\begin{aligned}|II| &\le C\sup_{i,j,k,l}\|\nabla u\|_{p,p}\|\nabla^2\phi_\psi\|_{p',p'}\|\alpha_{ij}^{kl}\|_{\infty,\infty}\\ &\quad+C\|\nabla u\|_{\infty,p}\|\nabla\phi_\psi\|_{p',r}(\|\mu''(|\mathcal{E}^{(u^*)}|_2^2)\|_{\infty,\infty}+\|\mu'(|\mathcal{E}^{(u^*)}|_2^2)\|_{\infty,\infty})\\ &\quad(\|\mathcal{E}^{(u^*)}\|_{\infty,\infty}+\|\mathcal{E}^{(u^*)}\|_{\infty,\infty}^3)\|\nabla\mathcal{E}^{(u^*)}\|_{p,p}.\end{aligned}$$

Let $0<\varepsilon<\frac{1}{2}-\frac{1}{p}$ be small. It follows that $\|\nabla\phi_\psi\|_{p',r}\le\|\nabla\phi_\psi\|_{L^{p'}(J_T;W^{1,p'}(\mathcal{D}))}$ and that $\theta:=1-\frac{2}{p}-2\varepsilon>0$. By Proposition 4.4,

$$X_{p,p}^T\hookrightarrow H^{1/2,p}(J_T;W^{1,p}(\mathcal{D}))$$

and thus by interpolation,

$$\begin{aligned}\|\nabla u\|_{\infty,p} &\le \|u\|_{H^{1/p+\varepsilon,p}(J_T;W^{1,p}(\mathcal{D}))}\le\|u\|_{H^{1/2,p}(J_T;W^{1,p}(\mathcal{D}))}^{1-\theta}\|u\|_{L^p(J_T;W^{1,p}(\mathcal{D}))}^{\theta}\\ &\le CT^{\theta/p}\|u\|_{L^\infty(J_T;W^{1,p}(\mathcal{D}))}^{\theta}\|u\|_{X_{p,p}^T}^{1-\theta}\le CT^{\theta/p}\|u\|_{X_{p,p}^T}.\end{aligned}$$

By (10.13), this implies that

$$|I|+|II|\le CT^{\theta/p}(T^{1/p}+1)\|u\|_{X_{p,p}^T}\|\psi\|_{L^{p'}(J_T;L^{p'}(\mathcal{D}_R))}.$$

Furthermore, now let $0<\hat{\varepsilon}<1-\frac{1}{p}$ be small and $\alpha:=\frac{1}{2}-\frac{1}{2p}-\frac{\hat{\varepsilon}}{2}>0$. From the embedding (10.9), from the boundedness of the trace operator $\gamma:H^{1/p+\hat{\varepsilon},p}(\mathcal{D})\to$

$L^p(\Gamma)$ for $1 < p < \infty$ and by interpolation, we get

$$\begin{aligned}
|I_\Gamma| &\leq \|\nabla u\|_{L^p(J_T;L^p(\Gamma))}\|\mu(|\mathcal{E}^{(u^*)}|_2^2)\|_{L^\infty(J_T;L^\infty(\Gamma))}\|\nabla\phi_\psi\|_{L^{p'}(J_T;L^{p'}(\Gamma))}\\
&\leq C\|u\|_{L^p(J_T;H^{1+1/p+\hat{\varepsilon},p}(\mathcal{D}))}\|\mu(|\mathcal{E}^{(u^*)}|_2^2)\|_{C(J_T;C(\overline{\mathcal{D}}))}\|\nabla\phi_\psi\|_{L^{p'}(J_T;W^{1,p'}(\mathcal{D}))}\\
&\leq C\|u\|^{1-\alpha}_{L^p(J_T;W^{2,p}(\mathcal{D}))}\|u\|^{\alpha}_{L^p(0,T;L^p(\mathcal{D}))}\|\psi\|_{L^{p'}(J_T;L^{p'}(\mathcal{D}_R))}\\
&\leq CT^{\alpha/p}\|u\|^{1-\alpha}_{X^T_{p,p}}\|u\|^{\alpha}_{L^\infty(0,T;L^p(\mathcal{D}))}\|\psi\|_{L^{p'}(J_T;L^{p'}(\mathcal{D}_R))}\\
&\leq CT^{\alpha/p}\|u\|_{X^T_{p,p}}\|\psi\|_{L^{p'}(J_T;L^{p'}(\mathcal{D}_R))}.
\end{aligned}$$

The estimate

$$|II_\Gamma| \leq CT^{\alpha/p}\|u\|_{X^T_{p,p}}\|\psi\|_{L^{p'}(J_T;L^{p'}(\mathcal{D}_R))}$$

follows in a similar way. From the estimates above and by (10.7), it follows that

$$\begin{aligned}
\|p_R\|_{L^p(J_T;L^p(\mathcal{D}_R))} &\leq C(T^{(\theta+1)/p}+T^{\theta/p}+T^{\alpha/p})\|u\|_{X^T_{p,p}}\\
&\leq C(T^{(\theta+1)/p}+T^{\theta/p}+T^{\alpha/p})\|f\|_{p,p},
\end{aligned}$$

where the constant C may be chosen independently of T for $T \leq T_0 \in \mathbb{R}_+$. Note that $\theta > \alpha$ if ε and $\hat{\varepsilon}$ are chosen suitably, so that the estimate (10.11) is proved. □

10.2. **Proof of Theorem 10.1.** The estimates from the last subsection allow us to repeat the reformulation of the linearized problem and the subsequent estimates from subsection 4.2 in a similar way. First we correct for the normal component of the boundary using the function $v_{\xi,\Omega}$ defined in (4.14). Recall that for almost all $t \in J_T$, $0 < T < T_0$, it satisfies the Neumann problem

$$\begin{cases}
\Delta v_{\xi,\Omega}(t) = 0, & \text{in } \mathcal{D},\\
\frac{\partial v_{\xi,\Omega}(t)}{\partial N}(y) = (\Omega(t)\times y+\xi(t))\cdot N(y), & y\in\Gamma,
\end{cases}$$

and the estimate

$$\|\nabla v_{\xi,\Omega}\|_{W^{1,p}(J_T;W^{2,p}(\mathcal{D}))}+\|\partial_t v_{\xi,\Omega}\|_{Y^T_{p,p}} \leq C\,\|(\xi,\Omega)\|_{W^{1,p}(J_T)}.$$

We use the momentum matrix $\mathbb{I}$, the added mass $\mathbb{M}$ and the operator $\mathcal{J}$ from subsection 4.2. This yields the equation

$$\begin{pmatrix}\hat{\xi}\\ \hat{\Omega}\end{pmatrix} = \mathcal{R}_{A_*}\begin{pmatrix}\hat{\xi}\\ \hat{\Omega}\end{pmatrix}+g^* \tag{10.14}$$

as an equivalent reformulation of (10.4), where

$$\mathcal{R}_{A_*}: W^{1,p}_0(J_T;\mathbb{R}^6)\to W^{1,p}_0(J_T;\mathbb{R}^6)$$

is given by

$$\mathcal{R}_{A_*}(\hat{\xi},\hat{\Omega})(t) := \int_0^t (\mathbb{I}+\mathbb{M})^{-1}\mathcal{J}\left[\mathbf{T}(\mathcal{U}_{h,A_*}(\hat{\xi},\hat{\Omega}),\mathcal{P}_{h,A_*}(\hat{\xi},\hat{\Omega}))\right](s)\,\mathrm{d}s$$

and

$$g^*(t) := \int_0^t (\mathbb{I}+\mathbb{M})^{-1}\Big[\begin{pmatrix}g_1\\ g_2\end{pmatrix}+\mathcal{J}\mathbf{T}(\mathcal{U}_{A_*}(g_0,0,0),\mathcal{P}_{A_*}(g_0,0,0))\Big]\,\mathrm{d}s.$$

We only have to modify the argument for the Newtonian case by replacing $\mathcal{U}_h(\hat{\xi},\hat{\Omega}),\mathcal{P}_h(\hat{\xi},\hat{\Omega})$ with $\mathcal{U}_{h,A_*}(\hat{\xi},\hat{\Omega}) := \mathcal{U}_{A_*}(0,\hat{\xi}+\hat{\Omega}\times y-\nabla v_{\hat{\xi},\hat{\Omega}}|_\Gamma,0)$ and $\mathcal{P}_{h,A_*}(\hat{\xi},\hat{\Omega})$

$:= \mathcal{P}_{A_*}(0, \hat{\xi} + \hat{\Omega} \times y - \nabla v_{\hat{\xi},\hat{\Omega}}|_\Gamma, 0)$. From (10.7) and the properties of $v_{\hat{\xi},\hat{\Omega}}$, it follows that

$$\begin{aligned}&\|\mathcal{U}_{h,A_*}(\hat{\xi},\hat{\Omega})\|_{X^T_{p,p}} + \|\mathcal{P}_{h,A_*}(\hat{\xi},\hat{\Omega})\|_{Y^T_{p,p}}\\ &\quad\le C\|\hat{\xi} + \hat{\Omega} \times \cdot - \nabla v_{\hat{\xi},\hat{\Omega}}\|_{W^{T,\Gamma}_p} \le C\|(\hat{\xi},\hat{\Omega})\|_{W^{1,p}(J_T)}.\end{aligned}$$

As in equation (4.6), $\mathcal{P}_{h,A_*}(\hat{\xi},\hat{\Omega})$ is the solution of a generalized Stokes problem of the form (10.10), by rewriting the inhomogeneous boundary value as a solenoidal exterior force to the problem. Thanks to Lemma 10.3, we thus obtain

$$\|\mathcal{P}_{h,A_*}(\hat{\xi},\hat{\Omega})\|_{L^p(J_T;L^q(\mathcal{D}_R))} \le CT^{\alpha/p}\|\mathcal{U}_{h,A_*}(\hat{\xi},\hat{\Omega})\|_{X^T_{p,q}} \le CT^{\alpha/p}\|(\hat{\xi},\hat{\Omega})\|_{W^{1,p}(J_T)}$$

for $\alpha = \frac{1}{2} - \frac{1}{2p} - \frac{\varepsilon}{2}$, $0 < \varepsilon < 1 - \frac{1}{p}$. From the estimates on the operator $\mathcal{J}$, we see that

$$\|\mathcal{R}_{A_*}(\hat{\xi},\hat{\Omega})\|_{W^{1,p}(0,T)} \le C(T^{1/3p} + T^{c\alpha/p})\|(\hat{\xi},\hat{\Omega})\|_{W^{1,p}(J_T)}$$

and

$$\begin{aligned}\|g^*\|_{W^{1,p}(0,T)} &\le C\left\|\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \mathcal{J}\mathbf{T}(\mathcal{U}_{A_*}(g_0,0,0), \mathcal{P}_{A_*}(g_0,0,0))\right\|_p\\ &\le C(\|g_0\|_{p,p} + \|g_1\|_p + \|g_2\|_p).\end{aligned}$$

For sufficiently small $T > 0$, we get $\|\mathcal{R}_{A_*}\|_{L(W^{1,p}_0(J_T))} < 1$, which yields a solution $(\hat{\xi},\hat{\Omega})$ of (10.14). The fluid velocity $\hat{u}$ and pressure $\hat{p}$ are given by

$$\hat{u} = \mathcal{U}_{h,A_*}(\hat{\xi},\hat{\Omega}) + \mathcal{U}_{A_*}(g_0,0,0) + \nabla v_{\hat{\xi},\hat{\Omega}}$$

and

$$\hat{p} = \mathcal{P}_{h,A_*}(\hat{\xi},\hat{\Omega}) + \mathcal{P}_{A_*}(g_0,0,0) - \partial_t v_{\hat{\xi},\hat{\Omega}}.$$

As in the Newtonian case, we extend this solution from J_T to J_{T_0} by iteration. The proof of Theorem 10.1 is complete.

11. The fixed point argument

Theorem 10.1 now allows us to solve (10.3) via a contraction mapping argument. Indeed, we choose

$$\begin{aligned}\mathcal{K}^T_R := \{(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega}) \in X^T_{p,p,0} \times Y^T_{p,p} \times W^{1,p}_0(J_T;\mathbb{R}^6)\\ : \|\tilde{u}\|_{X^T_{p,p}} + \|\tilde{p}\|_{Y^T_{p,p}} + \|(\tilde{\xi},\tilde{\Omega})\|_{W^{1,p}(J_T)} \le R\}\end{aligned}$$

as the underlying set in the natural function spaces for $p > 5$. Let G_0, G_1 and G_2 be the right-hand sides of (10.3) as in (10.2). Let

$$\psi^T_R : \begin{pmatrix} \tilde{u} \\ \tilde{p} \\ \tilde{\xi} \\ \tilde{\Omega} \end{pmatrix} \mapsto \begin{pmatrix} G_0(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega}) \\ G_1(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega}) \\ G_2(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega}) \end{pmatrix} \mapsto \begin{pmatrix} u \\ p \\ \xi \\ \Omega \end{pmatrix},$$

be the function which first maps $(\tilde{u},\tilde{p},\tilde{\xi},\tilde{\Omega}) \in \mathcal{K}^T_R$ to G_0, G_1, G_2 and then to the solution of the linear problem with fixed right-hand sides, using Theorem 10.1. For sufficiently small $R, T > 0$, we show that the Banach fixed point theorem can be applied to ψ^T_R.

For the remainder of the section, we assume $(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega}) \in \mathcal{K}_R^T$ and that $u^*, p^*, \xi^*, \Omega^*$ are given by (10.1). We set

$$C_0 := \|u^*\|_{X_{p,p}^{T_0}} + \|p^* - g \cdot y\|_{Y_{p,p}^{T_0}} + \|\xi^*\|_{W^{1,p}(J_{T_0})} + \|\Omega^*\|_{W^{1,p}(J_{T_0})},$$

and let $K_*(T) := \|u^*\|_{X_{p,p}^T}$, so that $K_*(T) \to 0$ as $T \to 0$. Note that the embedding constants in (10.9) are uniform in T on the subspace $X_{p,p,0}^T$ by Proposition 4.3. Since $\mu \in C^{1,1}(\mathbb{R}_+)$, we may define

$$m_\mu := m_{\mu,R,C_0} := \sup_{\tilde{u} \in X_{p,p,0}^T, \|\tilde{u}\|_{X_{p,p}^T} \le R} \|\mu(|\mathcal{E}^{(\tilde{u}+u^*)}|_2^2 + |\tilde{\mathcal{E}}^{(\tilde{u}+u^*)}|_2^2)\|_{C^{1,1}(\mathbb{R}_+)}.$$

We will show estimates of the type

$$\|\psi_R^T(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega})\|_{\mathcal{K}_R^T} \le L(T,R)\|(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega})\|_{\mathcal{K}_R^T},$$

where $L(T,R) \to 0$ as $T, R \to 0$. Thanks to maximal regularity of the linear problem, they follow directly from the estimates

$$\|G_0\|_{p,p} + \|G_1\|_p + \|G_2\|_p \le CL(T,R)\|(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega})\|_{\mathcal{K}_R^T}.$$

In order to show the latter estimate for G_0, it suffices to deal with the remaining term

$$\mathcal{Q}(u^*, \tilde{u}) = A_*\tilde{u} - \mathcal{A}(\tilde{u} + u^*)(\tilde{u} + u^*),$$

which did not appear in subsection 6.2. We write

$$\begin{aligned} &A_*\tilde{u} - \mathcal{A}(\tilde{u} + u^*)(\tilde{u} + u^*) \\ &\quad = (A_* - A(\tilde{u} + u^*))(\tilde{u} + u^*) - A_*u^* + [A(u^* + \tilde{u}) - \mathcal{A}(\tilde{u} + u^*)](\tilde{u} + u^*) \\ &\quad =: \mathcal{Q}_I + \mathcal{Q}_{II} + \mathcal{Q}_{III}, \end{aligned}$$

so that $\mathcal{Q}_I$ refers to the difference between the frozen and the quasi-linear generalized fluid operator and $\mathcal{Q}_{III}$ is given by the difference between the original and the transformed generalized fluid operator. First we let $w := \tilde{u} + u^*$ for simplicity and keep in mind that

$$\|w\|_{X_{p,p}^T} \le R + K_*(T) \quad \text{and} \quad \|w\|_{\infty,\infty} + \|\mathcal{E}^{(w)}\|_{\infty,\infty} \le C(R + C_0),$$

due to the embedding (10.9). In the following, we deal with $\mathcal{Q}_I$ as in [3, p. 417]. By definition,

$$\begin{aligned} &((A_* - A(w))w)_i \\ &\quad = (\mu(|\mathcal{E}^{(u^*)}|_2^2) - \mu(|\mathcal{E}^{(w)}|_2^2))\Delta w_i + \sum_{j,k,l=1}^{3} [\alpha_{ij}^{kl}(u^*) - \alpha_{ij}^{kl}(w)]\partial_j\partial_k w_l. \end{aligned}$$

We calculate

$$\begin{aligned} &\|\mu'(|\mathcal{E}^{(u^*)}|_2^2) - \mu'(|\mathcal{E}^{(u^*+\tilde{u})}|_2^2)\|_{\infty,\infty} \\ &\quad \le m_\mu \sum_{i,j=1}^{3} \|(\varepsilon_{ij}^{(u^*)})^2 - (\varepsilon_{ij}^{(u^*)} + \varepsilon_{ij}^{(u^*+\tilde{u})})^2\|_{\infty,\infty} \\ &\quad \le C \sup_{i,j} \|\varepsilon_{ij}^{(\tilde{u})}(\varepsilon_{ij}^{(\tilde{u})} - 2\varepsilon_{ij}^{(u^*)})\|_{\infty,\infty} \le CR(R + C_0), \end{aligned}$$

so that, by definition,

$$\begin{aligned}
&\|\alpha_{ij}^{kl}(u^*) - \alpha_{ij}^{kl}(u^*+\tilde{u})\|_{\infty,\infty} \\
&\quad \le \|4\left(\mu'(|\mathcal{E}^{(u^*)}|_2^2) - \mu'(|\mathcal{E}^{(u^*+\tilde{u})}|_2^2)\right)\varepsilon_{ij}^{(u^*)}\varepsilon_{kl}^{(u^*)}\|_{\infty,\infty} \\
&\qquad + \|\mu'(|\mathcal{E}^{(u^*+\tilde{u})}|_2^2)(\varepsilon_{ij}^{(u^*)}\varepsilon_{kl}^{(u^*)} - \varepsilon_{ij}^{(u^*+\tilde{u})}\varepsilon_{kl}^{(u^*+\tilde{u})})\|_{\infty,\infty} \\
&\quad \le CR(R+C_0) + Cm_\mu \sup_{i,j,k,l} \|\varepsilon_{ij}^{(u^*)}\varepsilon_{kl}^{(\tilde{u})} + \varepsilon_{ij}^{(\tilde{u})}\varepsilon_{kl}^{(u^*+\tilde{u})}\|_{\infty,\infty} \\
&\quad \le CR(R+C_0).
\end{aligned}$$

This shows that

$$\|\mathcal{Q}_I\|_{p,p} \quad \le \quad CR(R+C_0)(\|\Delta w\|_{p,p} + \|D^2 w\|_{p,p}) \le CR(R+C_0)(R+K_*(T)).$$

Moreover, it follows immediately from the definition that $\|\mathcal{Q}_{II}\|_{p,p} \le CK_*(T)$. Concerning the last term $\mathcal{Q}_{III}$, we define

$$\begin{aligned}
[\mathcal{Q}_{III}]_i &= [(A(w) - \mathcal{A}(w))w]_i \\
&= \left(\mu(|\tilde{\mathcal{E}}^{(w)}|_2^2)(\Delta - \mathcal{L})w_i\right) + \left(\mu(|\mathcal{E}^{(w)}|_2^2) - \mu(|\tilde{\mathcal{E}}^{(w)}|_2^2)\right)\Delta w_i \\
&\quad + \left(\sum_{j,k,l=1}^{3} \alpha_{ij}^{kl}(w)\partial_j\partial_k w_l - \sum_{j,k,l,m=1}^{3} a_{ij}^{klm}(w)\partial_m \tilde{\varepsilon}_{kl}^{(w)}\right) \\
&=: (i) + (ii) + (iii).
\end{aligned}$$

From Lemma 6.6, it follows that

$$\|(i)\|_{p,p} \le Cm_\mu T(R+K_*(T)). \tag{11.1}$$

For brevity, we omit the argument w in $\varepsilon^{(w)}, \tilde{\varepsilon}^{(w)}, \alpha_{ij}^{kl}(w), \dots$ in the following estimates. The second term (ii) satisfies

$$\|(ii)\|_{p,p} \quad \le \quad Cm_\mu(R+K_*(T)) \sum_{i,j=1}^{3} (\|\varepsilon_{ij} + \tilde{\varepsilon}_{ij}\|_{\infty,\infty}\|\varepsilon_{ij} - \tilde{\varepsilon}_{ij}\|_{\infty,\infty}),$$

so we have to look at

$$\begin{aligned}
\varepsilon_{ij} - \tilde{\varepsilon}_{ij} = \sum_{k,l=1}^{3} \big(&-[(\partial_i Y_k)(\partial_k\partial_l X_j) + (\partial_j Y_k)(\partial_k\partial_l X_i)]w_l \\
&+ [\delta_{ik}\delta_{lj} - (\partial_i Y_k)(\partial_l X_j)]\partial_k w_l + [\delta_{jk}\delta_{li} - (\partial_j Y_k)(\partial_l X_i)]\partial_k w_l\big).
\end{aligned}$$

By Proposition 6.1,

$$\|\delta_{ik}\delta_{lj} - (\partial_i Y_k)(\partial_l X_j)\| \quad \le \quad CT(R+C_0), \tag{11.2}$$

and similarly

$$\|\partial_k\partial_l X_i\|_{\infty,\infty} \quad \le \quad CT(R+C_0), \tag{11.3}$$

for all $i,j,k,l \in \{1,2,3\}$. Adding the corresponding estimates on Y yields

$$\begin{aligned}
\|\varepsilon_{ij} - \tilde{\varepsilon}_{ij}\|_{\infty,\infty} \quad &\le \quad C \sup_{i,j,k,l} (\|(\partial_i Y_k)(\partial_k\partial_l X_j) + (\partial_j Y_k)(\partial_k\partial_l X_i)\|_{\infty,\infty}\|w_l\|_{\infty,\infty})^2 \\
&\qquad + C\|\delta_{jk}\delta_{li} - (\partial_j Y_k)(\partial_l X_i)\|_{\infty,\infty}\|\partial_k w_l\|_{\infty,\infty}) \\
&\le \quad CT(R+C_0)^2,
\end{aligned} \tag{11.4}$$

and therefore

$$\|(ii)\|_{p,p} \leq Cm_\mu T(K_*(T)+C_0)(R+C_0)^2. \tag{11.5}$$

In order to do (iii) in a similar way, we must first calculate

$$\begin{aligned}
\partial_m \tilde{\varepsilon}_{kl} \quad = \quad & \frac{1}{2}\Bigg(\sum_{n,o=1}^{3} \big[(\partial_m\partial_k Y_n)(\partial_n\partial_o X_l) + (\partial_k Y_n)(\partial_m\partial_n\partial_o X_l) \\
& + (\partial_m\partial_l Y_n)(\partial_n\partial_o X_k) + (\partial_l Y_n)(\partial_m\partial_n\partial_o X_k)\big] w_o \\
& + \sum_{n,o=1}^{3} \big[(\partial_k Y_n)(\partial_n\partial_o X_l) + (\partial_l Y_n)(\partial_n\partial_o X_k)\big]\partial_m w_o \\
& + \sum_{n,o=1}^{3} \big[(\partial_m\partial_k Y_n)(\partial_o X_l) + (\partial_k Y_n)(\partial_m\partial_o X_l) \\
& + (\partial_m\partial_l Y_n)(\partial_o X_k) + (\partial_l Y_n)(\partial_m\partial_o X_k)\big]\partial_n w_o \\
& + \sum_{n,o=1}^{3} \big[(\partial_k Y_n)(\partial_o X_l) + (\partial_l Y_n)(\partial_o X_k)\big]\partial_m\partial_n w_o \Bigg) \\
=: \quad & \sum_{n,o=1}^{3} (d^{klmno} w_o + c^{klno}\partial_m w_o + \tilde{c}^{klmno}\partial_n w_o + b^{klno}\partial_m\partial_n w_o).
\end{aligned} \tag{11.6}$$

Thus, we get

$$\begin{aligned}
(iii) \quad = \quad & \left(\sum_{j,k,l=1}^{3} \alpha_{ij}^{kl}\partial_j\partial_k w_l - \sum_{j,k,l,m,n,o=1}^{3} a_{ij}^{klm} b^{klno}\partial_m\partial_n w_o \right) \\
& - \left(\sum_{j,k,l,m,n,o=1}^{3} a_{ij}^{klm}(c^{klno}\partial_m w_o + \tilde{c}^{klmno}\partial_n w_o) \right) \\
& - \left(\sum_{j,k,l,m,n,o=1}^{3} a_{ij}^{klm} d^{klmno} w_o \right) \\
=: \quad & (iii)_b + (iii)_c + (iii)_d.
\end{aligned}$$

It is easy to see the estimates for $(iii)_c$ and $(iii)_d$, because (11.3) can be applied to $c^{klno}, \tilde{c}^{klmno}, d^{klmno}$, and by definition

$$\|a_{ij}^{klm}\|_{\infty,\infty} \leq C\|\mu'(|\tilde{\mathcal{E}}|_2^2)\|_{\infty,\infty}\|\nabla w\|_{\infty,\infty}^2 \leq Cm_\mu(C_0+R)^2,$$

so that

$$\|(iii)_c\|_{\infty,\infty} + \|(iii)_d\|_{\infty,\infty} \leq Cm_\mu T(C_0+R)^4. \tag{11.7}$$

We rewrite the remaining part $(iii)_b$ by using the symmetry $a_{ij}^{klm} = a_{ij}^{klm}$,

$$\begin{aligned}(iii)_b &= \sum_{j,k,l,m,n,o=1}^{3} [\alpha_{ij}^{kl}\delta_{jm}\delta_{kn}\delta_{ol} - a_{ij}^{klm}(\partial_k Y_n)(\partial_o X_l)]\partial_m\partial_n w_o \\ &= \sum_{j,k,l,m,n,o=1}^{3} \Big[2\mu'(|\mathcal{E}|_2^2)\varepsilon_{ij}\varepsilon_{kl}\delta_{jm}\delta_{kn}\delta_{ol} \\ &\qquad - 2\mu'(|\tilde{\mathcal{E}}|_2^2)\tilde{\varepsilon}_{ij}\tilde{\varepsilon}_{kl}(\partial_j Y_m)(\partial_k Y_n)(\partial_o X_l)\Big]\partial_m\partial_n w_o.\end{aligned}$$

It is now clear from the structure of this term that by using the estimates (11.4) and (11.2), it follows that

$$\|(iii)_b\|_{\infty,\infty} \le C m_\mu T(C_0 + R). \tag{11.8}$$

In conclusion, from (11.1), (11.5), (11.7), and (11.8), we obtain

$$\|\mathcal{Q}_{III}\|_{p,p} \le CT(C_0 + R).$$

Putting together the estimates on $\mathcal{Q}_I$, $\mathcal{Q}_{II}$ and $\mathcal{Q}_{III}$ yields

$$\|\mathcal{Q}(u^*, \tilde{u})\|_{p,p} \le C(R^2 + K_*(T) + T). \tag{11.9}$$

Lemma 11.1. *For sufficiently small $T > 0$ and $R > 0$, the image $\mathrm{Im}(\psi_R^T)$ of ψ_R^T is contained in $\mathcal{K}_R^T$ and ψ_R^T is contractive.*

Proof. Recall the estimates for the terms F_0, F_1, F_2 given in Lemmas 6.7 and 6.8. By definition and by (11.9),

$$\begin{aligned}\|G_0(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega})\|_{p,p} &\le \|F_0(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega})\|_{p,p} + K_*(T) + \|\mathcal{Q}(\tilde{u}, u^*)\|_{p,p} \\ &\le C(T^{1/2} + T^{1/p} + R^2 + K_*(T)).\end{aligned}$$

Moreover, by definition,

$$\begin{aligned}\|G_1(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega})\|_p &\le \|F_1(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega})\|_p \\ &\quad + \left\|\int_\Gamma (\mathcal{T} - \mathcal{T}^\mu)(\tilde{u}, \tilde{p}) N \,\mathrm{d}\sigma\right\|_p + \left\|\int_\Gamma (\mathbf{T} - \mathcal{T}^\mu)(u^*, p^*) N \,\mathrm{d}\sigma\right\|_p \\ &\le CT^{1/p} + C\|\mathcal{J}(Q^T(\mu_0 - \mu(|\mathcal{E}^{(\tilde{u})}|_2^2))\mathcal{E}^{(Q\tilde{u})}Q)\|_p \\ &\quad + \|\mathcal{J}(\mu_0\mathcal{E}^{(u^*)} - Q^T\mu(|\mathcal{E}^{(u^*)}|_2^2)\mathcal{E}^{(Qu^*)}Q)\|_p \\ &\le CT^{1/p} + C\|\mathcal{E}^{(Q(u^*+\tilde{u}))}\|_{L^p(J_T;C(\overline{D}))} \le CT^{1/p}\end{aligned}$$

and similarly,

$$\begin{aligned}\|G_2(\tilde{u}, \tilde{p}, \tilde{\Omega})\|_p &\le \|F_2(\tilde{p}, \tilde{\xi}, \tilde{\Omega})\|_p + \|\mathcal{J}((\mathcal{T} - \mathcal{T}^\mu)(\tilde{u}, \tilde{p}))\|_p \\ &\quad + \|\mathcal{J}((\mathbf{T} - \mathcal{T}^\mu)(u^*, p^*))\|_p \le CT^{1/p}.\end{aligned}$$

Thus by Theorem 10.1,

$$\|\psi_R^T(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega})\|_{\mathcal{K}_R^T} \le C(\|G_0(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega})\|_{p,p} + \|(G_1(\tilde{u}, \tilde{p}, \tilde{\xi}, \tilde{\Omega}), G_2(\tilde{u}, \tilde{p}, \tilde{\Omega}))\|_p) \le R$$

for sufficiently small R, T. Now let $(\tilde{u}_1, \tilde{p}_1, \tilde{\xi}_1, \tilde{\Omega}_1), (\tilde{u}_2, \tilde{p}_2, \tilde{\xi}_2, \tilde{\Omega}_2) \in \mathcal{K}_R^T$. In the following, we again use the notation from Section 6 and put an index 1 or 2 on a function or an operator to indicate that it is constructed using either the change of coordinates X_1 corresponding to $\tilde{\xi}_1, \tilde{\Omega}_1$ or using X_2 corresponding to $\tilde{\xi}_2, \tilde{\Omega}_2$, respectively. Note that this does not apply to the functions F_0, F_1, F_2, G_0, G_1 and

G_2. Adding $\pm A(u^*+\tilde{u}_1)(\tilde{u}_1-\tilde{u}_2)$ and $\pm\mathcal{A}_1(u^*+\tilde{u}_1)\tilde{u}_2$ to $\mathcal{Q}_1(u^*,\tilde{u}_1)-\mathcal{Q}_2(u^*,\tilde{u}_2)$, we write

$$\begin{aligned}\|\mathcal{Q}_1(u^*,\tilde{u}_1)-\mathcal{Q}_2(u^*,\tilde{u}_2)\|_{p,p} \;\le\;& \|[A_*-A(u^*+\tilde{u}_1)](\tilde{u}_1-\tilde{u}_2)\|_{p,p}\\ &+\|[A(u^*+\tilde{u}_1)-\mathcal{A}_1(u^*+\tilde{u}_1)](\tilde{u}_1-\tilde{u}_2)\|_{p,p}\\ &+\|[\mathcal{A}_2(u^*+\tilde{u}_2)-\mathcal{A}_1(u^*+\tilde{u}_1)](u^*+\tilde{u}_2)\|_{p,p}\\ =:\;& q_I+q_{II}+q_{III}.\end{aligned}$$

From the estimates for $\mathcal{Q}_I$ and $\mathcal{Q}_{III}$ we directly see that

$$\|q_I\|_{p,p}+\|q_{II}\|_{p,p}\le C(R+T)\|\tilde{u}_1-\tilde{u}_2\|_{X^T_{p,p}}. \tag{11.10}$$

Now let $w_1:=\tilde{u}_1+u^*$ and $w_2:=\tilde{u}_2+u^*$. We split the remaining term q_{III} as follows,

$$\begin{aligned}\Big([\mathcal{A}_2(w_2)-\mathcal{A}_1(w_1)](w_2)\Big)_i \;=\;& \Big([\mu(|\tilde{\mathcal{E}}_2^{(w_2)}|_2^2)-\mu(|\tilde{\mathcal{E}}_1^{(w_1)}|_2^2)](\mathcal{L}_2w_2)\Big)_i\\ &+\Big(\mu(|\tilde{\mathcal{E}}_1^{(w_1)}|_2^2)(\mathcal{L}_2-\mathcal{L}_1)(w_2)\Big)_i\\ &+\sum_{j,k,l,m=1}^{3}((a_2)_{ij}^{klm}(w_2)-(a_1)_{ij}^{klm}(w_1))\partial_m(\tilde{\varepsilon}_2)_{kl}^{(w_2)}\\ &+\sum_{j,k,l,m=1}^{3}(a_1)_{ij}^{klm}(w_1)\partial_m((\tilde{\varepsilon}_2)_{kl}^{(w_2)}-(\tilde{\varepsilon}_1)_{kl}^{(w_1)})\\ =:\;& A_I+A_{II}+A_{III}+A_{IV}.\end{aligned}$$

From the definition in (9.1), Proposition 6.1, and the embedding (10.9), we obtain

$$\begin{aligned}\|(\tilde{\varepsilon}_1)_{ij}^{(w_1)}-(\tilde{\varepsilon}_2)_{ij}^{(w_2)}\|_{\infty,\infty}\le C\sup_{k,l}\Big[&\|(\tilde{e}_1)_{kl}^{ij}-(\tilde{e}_2)_{kl}^{ij}\|_{\infty,\infty}\|w_1\|_{\infty,\infty}\\ &+\|(\tilde{e}_2)_{kl}^{ij}\|_{\infty,\infty}\|\tilde{u}_1-\tilde{u}_2\|_{\infty,\infty}\\ &+\|(\tilde{d}_1)_{kl}^{ij}-(\tilde{d}_2)_{kl}^{ij}\|_{\infty,\infty}\|\nabla w_1\|_{\infty,\infty}\\ &+\|(\tilde{d}_2)_{kl}^{ij}\|_{\infty,\infty}\|\nabla(\tilde{u}_1-\tilde{u}_2)\|_{\infty,\infty}\Big]\\ \le C(&\|(\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_{W^{1,p}(J_T)}\\ &+\|\tilde{u}_1-\tilde{u}_2\|_{X^T_{p,p}}).\end{aligned}$$

Hence, it follows that

$$\begin{aligned}&\|\mu'(|\tilde{\mathcal{E}}_2^{(w_2)}|_2^2)-\mu'(|\tilde{\mathcal{E}}_1^{(w_1)}|_2^2)\|_{\infty,\infty}\\ &\quad\le Cm_\mu\sup_{i,j}\|(\tilde{\varepsilon}_1)_{ij}^{(w_1)}+(\tilde{\varepsilon}_2)_{ij}^{(w_2)}\|_{\infty,\infty}\|(\tilde{\varepsilon}_1)_{ij}^{(w_1)}-(\tilde{\varepsilon}_2)_{ij}^{(w_2)}\|_{\infty,\infty}\\ &\quad\le C(\|(\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\Omega}_1-\tilde{\Omega}_2)\|_{W^{1,p}(J_T)}+\|\tilde{u}_1-\tilde{u}_2\|_{X^T_{p,p}}),\end{aligned}$$

and therefore

$$\begin{aligned}
&\|(a_1)_{ij}^{klm}(w_1) - (a_2)_{ij}^{klm}(w_2)\|_{\infty,\infty} \\
&\quad \le \|2[\mu'(|\tilde{\mathcal{E}}_2^{(w_2)}|_2^2) - \mu'(|\tilde{\mathcal{E}}_1^{(w_1)}|_2^2)]\partial_j(Y_2)_m(\tilde{\varepsilon}_2)_{ij}^{(w_2)}(\tilde{\varepsilon}_2)_{kl}^{(w_2)}\|_{\infty,\infty} \\
&\qquad + \|2\mu'(|\tilde{\mathcal{E}}_1^{(w_1)}|_2^2)[\partial_j(Y_2)_m - \partial_j(Y_1)_m](\tilde{\varepsilon}_2)_{ij}^{(w_2)}(\tilde{\varepsilon}_2)_{kl}^{(w_2)}\|_{\infty,\infty} \\
&\qquad + \|2\mu'(|\tilde{\mathcal{E}}_1^{(w_1)}|_2^2)\partial_j(Y_1)_m[(\tilde{\varepsilon}_2)_{ij}^{(w_2)} - (\tilde{\varepsilon}_1)_{ij}^{(w_1)}](\tilde{\varepsilon}_2)_{kl}^{(w_2)}\|_{\infty,\infty} \\
&\qquad + \|2\mu'(|\tilde{\mathcal{E}}_1^{(w_1)}|_2^2)\partial_j(Y_1)_m(\tilde{\varepsilon}_1)_{ij}^{(w_2)}[(\tilde{\varepsilon}_2)_{kl}^{(w_2)} - (\tilde{\varepsilon}_1)_{kl}^{(w_1)}]\|_{\infty,\infty} \\
&\quad \le C(\|(\tilde{\xi}_1 - \tilde{\xi}_2, \tilde{\Omega}_1 - \tilde{\Omega}_2)\|_{W^{1,p}(J_T)} + \|\tilde{u}_1 - \tilde{u}_2\|_{X_{p,p}^T}).
\end{aligned}$$

Similarly, we get

$$\|\mathcal{L}_2 w_2\|_{p,p} + \sup_{k,l,m} \|\partial_m(\tilde{\varepsilon}_2)_{kl}^{(w_2)}\|_{p,p} \le C\|w_2\|_{X_{p,p}^T} \le C(R + K_*(T)).$$

As a direct consequence, we obtain

$$\|A_I\|_{p,p} + \|A_{III}\|_{p,p} \le C(R + K_*(T))(\|(\tilde{\xi}_1 - \tilde{\xi}_2, \tilde{\Omega}_1 - \tilde{\Omega}_2)\|_{W^{1,p}(J_T)} + \|\tilde{u}_1 - \tilde{u}_2\|_{X_{p,p}^T}).$$

Moreover, the estimate

$$\|A_{II}\|_{p,p} \le CT\|(\tilde{\xi}_1 - \tilde{\xi}_2, \tilde{\Omega}_1 - \tilde{\Omega}_2)\|_{W^{1,p}(J_T)}$$

follows directly from (6.10).

In a similar way, we treat the term A_{IV}. The coefficients in (11.6) satisfy

$$\begin{aligned}
\sup_{k,l,n,o} \|(b_1 - b_2)^{klno}\|_{\infty,\infty} \quad &\le \quad \sup_{k,l,n,o}\Big[\|\partial_k(Y_1 - Y_2)_n\|_{\infty,\infty}\|\partial_o(X_1)_l\|_{\infty,\infty} \\
&\qquad + \|\partial_k(Y_2)_n\|_{\infty,\infty}\|\partial_o(X_1 - X_2)_l\|_{\infty,\infty}\Big] \\
&\le \quad CT\|(\tilde{\xi}_1 - \tilde{\xi}_2, \tilde{\Omega}_1 - \tilde{\Omega}_2)\|_\infty,
\end{aligned}$$

and similarly

$$\begin{aligned}
&\sup_{k,l,m,n,o}\Big[\|(c_1 - c_2)^{klno}\|_{\infty,\infty} + \|(\tilde{c}_1 - \tilde{c}_2)^{klmno}\|_{\infty,\infty} + \|(d_1 - d_2)^{klmno}\|_{\infty,\infty}\Big] \\
&\quad \le CT\|(\tilde{\xi}_1 - \tilde{\xi}_2, \tilde{\Omega}_1 - \tilde{\Omega}_2)\|_\infty
\end{aligned}$$

by Proposition 6.1. We finally obtain

$$\|q_{III}\|_{p,p} \quad \le \quad C(K_*(T) + R + T)(\|\tilde{u}_1 - \tilde{u}_2\|_{X_{p,p}^T} + \|(\tilde{\xi}_1 - \tilde{\xi}_2, \tilde{\Omega}_1 - \tilde{\Omega}_2)\|_{W^{1,p}(J_T)}),$$

and therefore

$$\begin{aligned}
&\|\mathcal{Q}_1(u^*, \tilde{u}_1) - \mathcal{Q}_2(u^*, \tilde{u}_2)\|_{p,p} \\
&\quad \le \|q_I\|_{p,p} + \|q_{II}\|_{p,p} + \|q_{III}\|_{p,p} \\
&\quad \le C(K_*(T) + R + T)(\|\tilde{u}_1 - \tilde{u}_2\|_{X_{p,p}^T} + \|(\tilde{\xi}_1 - \tilde{\xi}_2, \tilde{\Omega}_1 - \tilde{\Omega}_2)\|_{\infty,\infty}).
\end{aligned}$$

From Lemma 6.6 and the estimates in the proof of Lemma 6.8 and the estimate above, we get

$$\begin{aligned}
&\|G_0(\tilde{u}_1, \tilde{p}_1, \tilde{\xi}_1, \tilde{\Omega}_1) - G_0(\tilde{u}_2, \tilde{p}_2, \tilde{\xi}_2, \tilde{\Omega}_2)\|_{p,p} \\
&\quad \le L_{R,T}(\|\tilde{u}_1 - \tilde{u}_2\|_{X_{p,p}^T} + \|\tilde{p}_1 - \tilde{p}_2\|_{Y_{p,p}^T} + \|(\tilde{\xi}_1 - \tilde{\xi}_2, \tilde{\Omega}_1 - \tilde{\Omega}_2)\|_{\infty,\infty}),
\end{aligned}$$

where $L_{R,T} = C(K_*(T) + R + T + T^{1/p})$.

Next we consider the right-hand sides of the rigid body equations, G_1 and G_2. In particular, it remains to show that

$$\begin{aligned}\mathcal{J}_*^\mu &:= \|\mathcal{J}((\mathcal{T}_1-\mathcal{T}_1^\mu)(\tilde u_1,\tilde p_1)-(\mathcal{T}_2-\mathcal{T}_2^\mu)(\tilde u_2,\tilde p_2))+(\mathcal{T}_1^\mu-\mathcal{T}_2^\mu)(u^*,p^*-\mathbf{g}\cdot y)\|_{p,p}\\ &\le CT\big(\|\tilde u_1-\tilde u_2\|_{X^T_{p,q}}+\|\tilde\Omega_1-\tilde\Omega_2\|_{W^{1,p}(J_T)}\big),\end{aligned}$$

where the $\mathcal{T}_i$ are the transformed Newtonian stress tensors from Section 3. Note that the pressure terms $\tilde p_1$ and $\tilde p_2$ disappear since $\mathcal{T}$ and $\mathcal{T}^\mu$ only differ in the viscosity μ. The estimate now follows as in Lemmas 6.7 and 6.8. Lemma 6.8 also implies that

$$\begin{aligned}&\|G_1(\tilde u_1,\tilde p_1,\tilde\xi_1,\tilde\Omega_1)-G_1(\tilde u_2,\tilde p_2,\tilde\xi_2,\tilde\Omega_2)\|_p\\ &\qquad+\|G_2(\tilde u_1,\tilde p_1,\tilde\Omega_1)-G_2(\tilde u_2,\tilde p_2,\tilde\Omega_2)\|_p\\ &\le\|F_1(\tilde u_1,\tilde p_1,\tilde\xi_1,\tilde\Omega_1)-F_1(\tilde u_2,\tilde p_2,\tilde\xi_2,\tilde\Omega_2)\|_p\\ &\qquad+\|F_2(\tilde u_1,\tilde p_1,\tilde\Omega_1)-F_2(\tilde u_2,\tilde p_2,\tilde\Omega_2)\|_p+\mathcal{J}_*^\mu\\ &\le CT\|(\tilde u_1-\tilde u_2,\tilde p_1-\tilde p_2,\tilde\xi_1-\tilde\xi_2,\tilde\Omega_1-\tilde\Omega_2)\|_{\mathcal{K}^T_R},\end{aligned}$$

so that in conclusion,

$$\begin{aligned}&\|\psi_R^T(\tilde u_1,\tilde p_1,\tilde\xi_1,\tilde\Omega_1)-\psi_R^T(\tilde u_2,\tilde p_2,\tilde\xi_2,\tilde\Omega_2)\|_{\mathcal{K}^T_R}\\ &\quad\le C\big(\|G_0(\tilde u_1,\tilde p_1,\tilde\xi_1,\tilde\Omega_1)-G_0(\tilde u_2,\tilde p_2,\tilde\xi_2,\tilde\Omega_2)\|_{p,p}\\ &\qquad+\|G_1(\tilde u_1,\tilde p_1,\tilde\xi_1,\tilde\Omega_1)-G_1(\tilde u_2,\tilde p_2,\tilde\xi_2,\tilde\Omega_2)\|_p\\ &\qquad+\|G_2(\tilde u_1,\tilde p_1,\tilde\Omega_1)-G_2(\tilde u_2,\tilde p_2,\tilde\Omega_2)\|_p\big)\\ &\quad\le C(R+K_*(T)+T^{1/p}+T^{1/2})\|(\tilde u_1-\tilde u_2,\tilde p_1-\tilde p_2,\tilde\xi_1-\tilde\xi_2,\tilde\Omega_1-\tilde\Omega_2)\|_{\mathcal{K}^T_R}.\end{aligned}$$

And thus for sufficiently small R,T, the map ψ_R^T is a contraction. □

12. Proof of Theorem 2.2

From the Banach fixed point theorem applied to ψ_R^T, it follows that there is a unique strong solution $(\hat u,\hat p,\hat\xi,\hat\Omega)$ to problem (10.3). The solution to the original problem (8.2) can be found by adding the reference solution $(u^*,p^*+g\cdot y,\xi^*,\Omega^*)$ and performing the corresponding backward coordinate transform, exactly as in the proof of Theorem 2.1 in Section 7.

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