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THE RANGE OF LOCALIZATION OPERATORS AND LIFTING THEOREMS FOR MODULATION AND BARGMANN-FOCK SPACES

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Abstract. We study the range of time-frequency localization operators acting on modulation spaces and prove a lifting theorem. As an application we also characterize the range of Gabor multipliers, and, in the realm of complex analysis, we characterize the range of certain Toeplitz operators on weighted Bargmann-Fock spaces. The main tools are the construction of canonical isomorphisms between modulation spaces of Hilbert-type and a refined version of the spectral invariance of pseudodifferential operators. On the technical level we prove a new class of inequalities for weighted gamma functions.

1. Introduction

The precise description of the range of a linear operator is usually difficult, if not impossible, because this amounts to a characterization of which operator equations are solvable. In this paper we study the range of an important class of pseudodifferential operators, so-called time-frequency localization operators, and we prove an isomorphism theorem between modulation spaces with respect to different weights.

The guiding example to develop an intuition for our results is the class of multiplication operators. Let $m \geq 0$ be a weight function on $\mathbb{R}^d$ and define the weighted space $L^p_m(\mathbb{R}^d)$ by the norm $\|f\|_{L^p_m} = \left(\int_{\mathbb{R}^d} |f(x)|^p m(x)^p \, dx\right)^{1/p} = \|fm\|_{L^p}$. Let $\mathcal{M}_a$ be the multiplication operator defined by $\mathcal{M}_a f = af$. Then L^p_m is precisely the range of the multiplication operator $\mathcal{M}_{1/m}$.

We will prove a similar result for time-frequency localization operators between weighted modulation spaces. To set up terminology, let $\pi(z)g(t) = e^{2\pi i \xi \cdot t} g(t-x)$ denote the time-frequency shift by $z = (x,\xi) \in \mathbb{R}^{2d}$ acting on a function g on $\mathbb{R}^d$. The corresponding transform is the short-time Fourier transform of a function defined by

$$V_g f(z) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} \, e^{-2\pi i \xi \cdot t} \, dt = \langle f, \pi(z) g \rangle .$$

The standard function spaces of time-frequency analysis are the modulation spaces. The modulation space norms measure smoothness in the time-frequency space (phase space in the language of physics) by imposing a norm on the short-time

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Fourier transform of a function f. As a special case we mention the modulation spaces $M^p_m(\mathbb{R}^d)$ for $1 \le p \le \infty$ and a non-negative weight function m. Let

$$h(t) = 2^{d/4} e^{-\pi t^2} = 2^{d/4} e^{-\pi t \cdot t}, \qquad t \in \mathbb{R}^d$$

denote the (normalized) Gaussian. Then the modulation space $M^p_m(\mathbb{R}^d)$ is defined by the norm

$$\|f\|_{M^p_m} = \|V_h f\|_{L^p_m}.$$

The localization operator A^g_m with respect to the "window" g, usually some test function, and the symbol or multiplier m is defined formally by the integral

$$A^g_m f = \int_{\mathbb{R}^{2d}} m(z) V_g f(z) \pi(z) g \, dz.$$

Localization operators constitute an important class of pseudodifferential operators and occur under different names such as Toeplitz operators or anti-Wick operators. They were introduced by Berezin as a form of quantization [3], and are nowadays applied in mathematical signal processing for time-frequency masking of signals and for phase-space localization [12]. An equivalent form occurs in complex analysis as Toeplitz operators on Bargmann-Fock space [8, 4, 5]. In hard analysis they are used to approximate pseudodifferential operators, in some proofs of the sharp Gårding inequality and the Fefferman-Phong inequality [24, 25, 26, 30], and in PDE [1]. For the analysis of localization operators with time-frequency methods we refer to [10] and the references given there; for a more analytic point of view we recommend [32, 33].

A special case of our main result can be formulated as follows. By an isomorphism between two Banach spaces X and Y we understand a bounded and invertible operator from X onto Y.

Theorem 1.1. *Let m be a non-negative continuous symbol on $\mathbb{R}^{2d}$ satisfying $m(w+z) \le e^{a|w|^b} m(z)$ for $w, z \in \mathbb{R}^{2d}$, $0 \le b < 1$, and assume that m is radial in each time-frequency coordinate. Then for suitable test functions g the localization operator A^g_m is an isomorphism from the modulation space $M^p_\mu(\mathbb{R}^d)$ onto $M^p_{\mu/m}(\mathbb{R}^d)$ for all $1 \le p \le \infty$ and moderate weights μ.*

We see that the range of a localization operator exhibits the same behavior as the multiplication operators. For the precise formulation with all assumptions stated we refer to Section 4. The above isomorphism theorem can also be intepreted as a lifting theorem, quite in analogy with the lifting property of Besov spaces [35]. However, whereas Besov spaces $\dot{B}^{p,q}_s$ with different smoothness s are isomorphic via Fourier multipliers, the lifting operators between modulation spaces are precisely the localization operators with the weight m. This case is more subtle because localization operators are not closed under composition, quite in contrast to Fourier multipliers.

The isomorphism theorem for localization operators stated above is preceded by many contributions, which were already listed in [21]. In particular, in [11] it was shown that for moderate symbol functions with subexponential growth the localization operator is a Fredholm operator, i.e., it differs from an invertible operator only by a finite-rank operator. Here we show that such operators are isomorphisms between the corresponding modulation spaces, even under weaker conditions than needed for the Fredholm property. In the companion paper [21] we proved the isomorphism property for weight functions of *polynomial type*, i.e.,

$$c(1+|z|)^{-N} \le m(z) \le C((1+|z|)^N. \tag{1}$$

The contribution of Theorem 1.1 is the extension of the class of possible weights. In particular, the isomorphism property also holds for subexponential weight functions of the form $m(z) = e^{a|z|^b}$ or $m(z) = e^{a|z|/\log(e+|z|)}$ for $a>0, 0<b<1$, which are often considered in time-frequency analysis. We remark that this generality comes at the price of imposing the radial symmetry on the weight m. Therefore, our results are not applicable to all situations covered by [21], where no radial symmetry is required.

Although the extension to weights of ultra-rapid growth looks like a routine generalization, it is not. The proof of Theorem 1.1 for weights of polynomial type is based on a deep theorem of Bony and Chemin [7]. They construct a one-parameter group of isomorphisms from $L^2(\mathbb{R}^d)$ onto the modulation spaces $M^2_{m^t}(\mathbb{R}^d)$ for $t\in\mathbb{R}$. Unfortunately, the pseudodifferential calculus developed in [7] requires polynomial growth conditions, and, to our knowledge, an extention to symbols of faster growth is not available.

On a technical level, our contribution is the construction of canonical isomorphisms between $L^2(\mathbb{R}^d)$ and the modulation spaces $M^2_m(\mathbb{R}^d)$ of Hilbert type. In fact, such an isomorphism is given by an anti-Wick operator, i.e., by a time-frequency localization operator with Gaussian window $h(t) = 2^{d/4}e^{-\pi t^2}$.

Theorem 1.2. *Assume that m is a continuous moderate weight function of at most exponential growth and radial in each time-frequency coordinate. Then the localization operator A^h_m is an isomorphism from $L^2(\mathbb{R}^d)$ onto $M^2_{1/m}(\mathbb{R}^d)$ and from $M^2_m(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$.*

This theorem replaces the result of Bony and Chemin. Theorem 1.2 was proved by Shubin [27] for the special case $m(z) = (1+|z|^2)^{1/2}$. In fact, Shubin defined the "Shubin class" M^2_m as the range of the localization operator $A^h_{1/m}$. The Shubin class $A^h_{1/m}L^2$ was identified with the modulation space in [6]. The novelty is that Theorem 1.2 also covers symbols of ultra-rapid growth. Its proof requires new arguments and most of our efforts. We take a time-frequency approach rather than using classical methods from pseudodifferential calculus. In the course of its proof we will establish new inequalities for weighted gamma functions of the form

$$C^{-1} \le \int_0^\infty \theta(\sqrt{x/\pi})\frac{x^n}{n!}e^{-x}dx \int_0^\infty \frac{1}{\theta(\sqrt{x/\pi})}\frac{x^n}{n!}e^{-x}dx \le C \text{ for all } n\in\mathbb{N}\cup\{0\}.$$

The method of proof for Theorem 1.1 may be of interest in itself. Once the canonical isomorphisms are in place (Theorem 1.2), the proof of Theorem 1.1 proceeds as follows. It is easy to establish that A_m is an isomorphism from $M^2_{\sqrt{m}}$ to $M^2_{1/\sqrt{m}}$, so the composition $A^g_{1/m}A^g_m$ is an isomorphism on $M^2_{\sqrt{m}}$. Using the canonical isomorphisms of Theorem 1.2, one next shows that the operator $V = A^h_{\sqrt{m}}A^g_{1/m}A^g_m A^h_{1/\sqrt{m}}$ is an isomorphism on L^2 and that the (Weyl) symbol of this operator belongs to a generalized Sjöstrand class. After these technicalities we apply the machinery of spectral invariance of pseudodifferential operators from [18] to conclude that V is invertible on all modulation spaces M^p_μ (with μ compatible with the conditions on

m and the window g). Since V is a composition of three isomorphisms, we then deduce that A_m^g is an isomorphism from M_μ^p onto $M_{\mu/m}^p$.

As an application we prove (i) a new isomorphism theorem for so-called Gabor multipliers, which are a discrete version of time-frequency localization operators, and (ii) an isomorphism theorem for Toeplitz operators between weighted Bargmann-Fock spaces of entire functions. To formulate this result more explicitly, for a non-negative weight function μ and $1 \le p \le \infty$, let $\mathcal{F}_\mu^p(\mathbb{C}^d)$ be the space of entire functions of d complex variables defined by the norm

$$\|F\|_{\mathcal{F}_\mu^p}^p := \int_{\mathbb{C}^d} |F(z)|^p \mu(z)^p e^{-p\pi|z|^2/2} dz < \infty ,$$

and let P be the usual projection from $L^1_{\mathrm{loc}}(\mathbb{C}^d)$ to entire functions on $\mathbb{C}^d$. The Toeplitz operator with symbol m acting on a function F is defined to be $T_m F = P(mF)$. Then we show that *the Toeplitz operator T_m is an isomorphism from $\mathcal{F}_\mu^p(\mathbb{C}^d)$ onto $\mathcal{F}_{\mu/m}^p(\mathbb{C}^d)$ for every $1 \le p \le \infty$ and every moderate weight μ.*

The connection between Toeplitz operators on Bargmann-Fock space and localization operators is also used in [13].

The paper is organized as follows: In Section 2 we provide the precise definition of modulation spaces and localization operators, and we collect their basic properties. In particular, we investigate the Weyl symbol of the composition of two localization operators. In Section 3, which contains our main contribution, we construct the canonical isomorphisms and prove Theorem 1.2. In Section 4 we derive a refinement of the spectral invariance of pseudodifferential operators and prove the general isomorphism theorem, of which Theorem 1.1 is a special case. Finally in Section 5 we give applications to Gabor multipliers and Toeplitz operators.

2. Time-frequency analysis and localization operators

We first set up the vocabulary of time-frequency analysis. For the notation we follow the book [17]. For a point $z = (x, \xi) \in \mathbb{R}^{2d}$ in phase space the time-frequency shift of a function f is $\pi(z)f(t) = e^{2\pi i \xi \cdot t} f(t - x)$, $t \in \mathbb{R}^d$.

The short-time Fourier transform: Fix a non-zero function $g \in L^1_{loc}(\mathbb{R}^d)$ which is usually taken in a suitable space of Schwartz functions. Then the *short-time Fourier transform* of a function or distribution f on $\mathbb{R}^d$ is defined to be

$$V_g f(z) = \langle f, \pi(z) g \rangle \qquad z \in \mathbb{R}^{2d} , \tag{2}$$

provided the scalar product is well-defined for every $z \in \mathbb{R}^{2d}$. Here g is called a "window function". If $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$ for the conjugate parameter $p' = p/(p-1)$, then the short-time Fourier transform can be written in integral form as

$$V_g f(z) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)}\, e^{-2\pi i \xi \cdot t}\, dt.$$

In general, the bracket $\langle \cdot, \cdot \rangle$ extends the inner product on $L^2(\mathbb{R}^d)$ to any dual pairing between a distribution space and its space of test functions, for instance $g \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, but time-frequency analysis often needs larger distribution spaces.

Weight functions: We call a continuous, strictly positive weight function m on $\mathbb{R}^{2d}$ *moderate* if

$$\sup_{z\in\mathbb{R}^{2d}} \left(\frac{m(z+y)}{m(z)}, \frac{m(z-y)}{m(z)}\right) := v(y) < \infty \qquad \text{for all } y \in \mathbb{R}^{2d}.$$

The resulting function v is a submultiplicative weight function, i.e., v is even and satisfies $v(z_1+z_2) \le v(z_1)v(z_2)$ for all $z_1, z_2 \in \mathbb{R}^{2d}$, and then m satisfies

$$m(z_1+z_2) \le v(z_1)m(z_2) \qquad \text{for all } z_1, z_2 \in \mathbb{R}^{2d}. \tag{3}$$

Given a submultiplicative weight function v on $\mathbb{R}^{2d}$, any weight satisfying condition (3) is called v-moderate. For a fixed submultiplicative function v the set

$$\mathcal{M}_v := \{m \in L^\infty_{loc}(\mathbb{R}^{2d}) \,:\, 0 < m(z_1+z_2) \le v(z_1)m(z_2) \ \forall z_1, z_2 \in \mathbb{R}^{2d}\}$$

contains all v-moderate weights.

Several times we will use that every v-moderate weight $m \in \mathcal{M}_v$ satisfies the following bounds:

$$\frac{1}{v(z_1-z_2)} \le \frac{m(z_1)}{m(z_2)} \le v(z_1-z_2) \qquad \text{for all } z_1, z_2 \in \mathbb{R}^{2d}. \tag{4}$$

This follows from (3) by replacing z_1 with $z_1 - z_2$.

Modulation spaces $M^{p,q}_m$ for arbitrary weights: For the general definition of modulation spaces we choose the Gaussian function $h(t) = 2^{d/4}e^{-\pi t\cdot t}$ as the canonical window function. Then the short-time Fourier transform is defined for arbitrary elements in the Gelfand-Shilov space $(S^{1/2}_{1/2})'(\mathbb{R}^d)$ of generalized functions. The modulation space $M^{p,q}_m(\mathbb{R}^d)$, $1 \le p, q < \infty$ consists of all elements $f \in (S^{1/2}_{1/2})'(\mathbb{R}^d)$ such that the norm

$$\|f\|_{M^{p,q}_m} := \Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |V_h f(x,\xi)|^p m(x,\xi)^p\, dx\Big)^{q/p} d\xi\Big)^{1/q} = \|V_h f\|_{L^{p,q}_m} \tag{5}$$

is finite. If $p = \infty$ or $q = \infty$, we make the usual modification and replace the integral by the supremum norm $\|\cdot\|_{L^\infty}$. If $m = 1$, then we usually write $M^{p,q}$ instead of $M^{p,q}_m$. We also set $M^p_m = M^{p,p}_m$ and $M^p = M^{p,p}$.

The reader who does not like general distribution spaces may interprete $M^{p,q}_m(\mathbb{R}^d)$ as the completion of the finite linear combinations of time-frequency shifts $\mathcal{H}_0 = \operatorname{span}\{\pi(z)h : z \in \mathbb{R}^{2d}\}$ with respect to the $M^{p,q}_m$-norm for $1 \le p, q < \infty$ and as a weak*-relative closure, when $p = \infty$ or $q = \infty$. These issues arise only for extremely rapidly decaying weight functions. If $m \ge 1$ and $1 \le p, q \le 2$, then $M^{p,q}_m(\mathbb{R}^d)$ is in fact a subspace of $L^2(\mathbb{R}^d)$. If m is of polynomial type (cf. (1)), then $M^{p,q}_m(\mathbb{R}^d)$ is a subspace of tempered distributions. This is the case that is usually considered, although the theory of modulation spaces was developed from the beginning to include arbitrary moderate weight functions [14, 17, 19].

Norm equivalence: Definition (5) uses the Gauss function as the canonical window. The definition of modulation spaces, however, does not depend on the particular choice of the window. More precisely, if $g \in M^1_v$, $g \ne 0$, and $m \in \mathcal{M}_v$, then there exist constants $A, B > 0$ such that

$$A\,\|f\|_{M^{p,q}_m} \le \|V_g f\|_{L^{p,q}_m} \le B\|f\|_{M^{p,q}_m} = B\|V_h f\|_{L^{p,q}_m}. \tag{6}$$

We will usually write

$$\|V_g f\|_{L^{p,q}_m} \asymp \|f\|_{M^{p,q}_m}$$

for the equivalent norms.

Localization operators: Given a non-zero window function $g \in M^1_v(\mathbb{R}^d)$ and a symbol or multiplier m on $\mathbb{R}^{2d}$, the localization operator A^g_m is defined informally by

$$A^g_m f = \int_{\mathbb{R}^{2d}} m(z) V_g f(z) \pi(z) g \, dz \,, \tag{7}$$

provided the integral exists. A useful alternative definition of A^g_m is the weak definition

$$\langle A^g_m f, k\rangle_{L^2(\mathbb{R}^d)} = \langle m V_g f, V_g k\rangle_{L^2(\mathbb{R}^{2d})}. \tag{8}$$

While in general the symbol m may be a distribution in a modulation space of the form $M^\infty_{1/v}(\mathbb{R}^{2d})$ [10, 33], we will investigate only localization operators whose symbol is a moderate weight function.

Taking the short-time Fourier transform of (7), we find that

$$V_g(A^g_m f)(w) = \int_{\mathbb{R}^{2d}} m(z) V_g f(z) \langle \pi(z) g, \pi(w) g\rangle \, dz = \big((m V_g f) \natural V_g g\big)(w) \,,$$

with the usual twisted convolution $\natural$ defined by

$$(F \natural G)(w) = \int_{\mathbb{R}^{2d}} F(z) G(w - z) e^{2\pi i z_1 \cdot (z_2 - w_2)} \, dz.$$

Since

$$F \mapsto \int_{\mathbb{R}^{2d}} F(z) \langle \pi(z) g, \pi(\cdot) g\rangle \, dz$$

is the projection from arbitrary tempered distributions on $\mathbb{R}^{2d}$ onto functions of the form $V_g f$ for some distribution f, the localization operator can be seen as the composition of a multiplication operator and the projection onto the space of short-time Fourier transforms for the fixed window g. In this light, localization operators resemble the classical Toeplitz operators, which are multiplication operators followed by a projection onto analytic functions. Therefore they are sometimes called Toeplitz operators [21, 31, 33]. If the window g is chosen to be the Gaussian, then this formal similarity can be made more precise. See Proposition 5.5.

2.1. **Mapping properties of localization operators.** The mapping properties of localization operators on modulation spaces closely resemble the mapping properties of multiplication operators between weighted L^p-spaces. The boundedness of localization operators has been investigated on many levels of generality [10, 32, 36]. We will use the following boundedness result from [11, 33].

Lemma 2.1. *Let $m \in \mathcal{M}_v$ and $\mu \in \mathcal{M}_w$. Fix $g \in M^1_{vw}(\mathbb{R}^d)$. Then the localization operator $A^g_{1/m}$ is bounded from $M^{p,q}_\mu(\mathbb{R}^d)$ to $M^{p,q}_{\mu m}(\mathbb{R}^d)$.*

Remark. The condition on the window g is required to make sense of $V_g f$ for f in the domain space $M^{p,q}_\mu(\mathbb{R}^d)$ and of $V_g k$ for k in the dual $M^{p',q'}_{1/(\mu m)}(\mathbb{R}^d) = (M^{p,q}_{\mu m})'$ of the target space $M^{p,q}_{\mu m}$ for the full range of parameters $p, q \in [1, \infty]$. For fixed $p, q \in [1, \infty]$ weaker conditions may suffice, because the norm equivalence (6) still holds after relaxing the condition $g \in M^1_v$ into $g \in M^r_v$ for $r \le \min(p, p', q, q')$ [34].

On a special pair of modulation spaces, $A^g_{1/m}$ is even an isomorphism [21].

Lemma 2.2. *Let $g \in M^1_v$, $m \in \mathcal{M}_v$, and set $\theta = m^{1/2}$. Then A^g_m is an isomorphism from $M^2_\theta(\mathbb{R}^d)$ onto $M^2_{1/\theta}(\mathbb{R}^d)$.*

Likewise $A_{1/m}$ is an isomorphism from $M^2_{1/\theta}(\mathbb{R}^d)$ onto $M^2_\theta(\mathbb{R}^d)$. Consequently the composition $A^g_{1/m}A^g_m$ is an isomorphism on $M^2_\theta(\mathbb{R}^d)$, and $A^g_m A^g_{1/m}$ is an isomorphism on $M^2_{1/\theta}(\mathbb{R}^d)$.

Lemma 2.2 is based on the equivalence

$$\langle A^g_m f, f\rangle = \langle m, |V_g f|^2\rangle = \|V_g f \cdot \theta\|_2^2 \asymp \|f\|^2_{M^2_\theta}\,, \tag{9}$$

and is proved in detail in [21, Lemma 3.4].

2.2. **The symbol of $A^g_{1/m}A^g_m$.** The composition of localization operators is no longer a localization operator, but the product of two localization operators still has a well-behaved Weyl symbol. In the following we use the time-frequency calculus of pseudodifferential operators as developed in [18, 20]. Compared to the standard pseudodifferential operator calculus it is more restrictive because it is related to the constant Euclidean geometry on phase space. On the other hand, it is more general because it works for arbitrary moderate weight functions (excluding exponential growth).

Given a symbol $\sigma(x,\xi)$ on $\mathbb{R}^d \times \mathbb{R}^d \simeq \mathbb{R}^{2d}$, the corresponding pseudodifferential operator in the Weyl calculus $\mathrm{Op}(\sigma)$ is defined formally as

$$\mathrm{Op}(\sigma)f(x) = \iint_{\mathbb{R}^{2d}} \sigma\Big(\frac{x+y}{2},\xi\Big) e^{2\pi i (x-y)\cdot\xi} f(y)\, dyd\xi$$

with a suitable interpretation of the integral. If $\mathcal{C}$ is a class of symbols, we write $\mathrm{Op}(\mathcal{C}) = \{\mathrm{Op}(\sigma) : \sigma \in \mathcal{C}\}$ for the class of all pseudodifferential operators with symbols in $\mathcal{C}$. For the control of the symbol of composite operators we will use the following characterization of the generalized Sjöstrand class from [18]. For the formulation associate to a submultiplicative weight $v(x,\xi)$ on $\mathbb{R}^{2d}$ the rotated weight on $\mathbb{R}^{4d}$ defined by

$$\tilde v(x,\xi,\eta,y) = v(-y,\eta). \tag{10}$$

For radial weights, to which we will restrict later, the distinction between v and $\tilde v$ is unnecessary.

Theorem 2.3. *Fix a non-zero $g \in M^1_v$. An operator T possesses a Weyl symbol in $M^{\infty,1}_{\tilde v}$, $T \in \mathrm{Op}(M^{\infty,1}_{\tilde v})$, if and only if there exists a (semi-continuous) function $H \in L^1_v(\mathbb{R}^{2d})$ such that*

$$|\langle T\pi(z)g, \pi(y)g\rangle| \le H(y-z) \qquad \textit{for all } y,z \in \mathbb{R}^{2d}.$$

Remark. This theorem says the symbol class $M^{\infty,1}_{\tilde v}$ is *characterized* by the off-diagonal decay of its kernel with respect to time-frequency shifts. This kernel is in fact dominated by a convolution kernel. The composition of operators can then be studied with the help of convolution relations. Clearly this is significantly easier than the standard approaches that work with the Weyl symbol directly and the twisted product between Weyl symbols. See [22] for results in this direction.

Theorem 2.4. *Assume that $g \in M^1_{v^s}(\mathbb{R}^d)$, $T \in \mathrm{Op}(M^{\infty,1}_{\tilde v^s})$ for $s \ge 1/2$, and $\theta \in \mathcal{M}_{v^{1/2}}$. Then $A^g_\theta T A^g_{1/\theta} \in \mathrm{Op}(M^{\infty,1}_{\tilde v^{s-1/2}})$.*

Proof. We distinguish the window g of the localization operator A_m^g from the window h used in the expression of the kernel $\langle T\pi(z)h, \pi(y)h\rangle$. Choose h to be the Gaussian; then $h \in M_v^1$ for every submultiplicative weight v. Let us first write the kernel $\langle T\pi(z)h, \pi(y)h\rangle$ informally and later justify the convergence of the integrals. Recall that

$$T(A_{1/\theta}^g f) = T\Big(\int_{\mathbb{R}^{2d}} \theta(u)^{-1}\langle f, \pi(u)g\rangle \pi(u)g\, du\Big).$$

Then

$$\langle A_\theta^g T A_{1/\theta}^g \pi(z)h, \pi(y)h\rangle = \langle T A_{1/\theta}^g \pi(z)h, A_\theta^g \pi(y)h\rangle \tag{11}$$

$$= \iint_{\mathbb{R}^{4d}} \frac{1}{\theta(u)} \big\langle \pi(z)h, \pi(u)g\big\rangle \big\langle T\pi(u)g, \pi(u')g\big\rangle\, \theta(u') \big\langle \pi(y)h, \pi(u')g\big\rangle\, du du' .$$

Now set

$$G(z) = |\langle g, \pi(z)h\rangle| = |V_h g(z)| \quad \text{and} \quad G^*(z) = G(-z)$$

and let H be a dominating function in $L_v^1(\mathbb{R}^{2d})$ so that

$$|\langle T\pi(u)g, \pi(u')g\rangle| \le H(u'-u).$$

Since time-frequency shifts commute up to a phase factor, we have

$$|\langle \pi(z)h, \pi(u)g\rangle| = G(z-u).$$

Before substituting all estimates into (11), we recall that θ is $\sqrt{v}$-moderate by assumption, and so (4) says that

$$\frac{\theta(u')}{\theta(u)} \le v(u'-u)^{1/2} \qquad \text{for all } u, u' \in \mathbb{R}^{2d}.$$

Now by (11) we get

$$\begin{aligned}
&|\langle A_\theta^g T A_{1/\theta}^g \pi(z)h, \pi(y)h\rangle| \\
\le\ & \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \frac{\theta(u')}{\theta(u)} G(z-u) H(u'-u) G(y-u')\, du du' \\
\le\ & \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} G(z-u) v(u'-u)^{1/2} H(u'-u) G(y-u')\, du du' \\
=\ & \Big(G * (v^{1/2}H) * G^*\Big)(z-y).
\end{aligned}$$

Thus the kernel of $A_\theta^g T A_{1/\theta}^g$ is dominated by the function $G * (v^{1/2}H) * G^*$. By assumption $g \in M_{v^s}^1(\mathbb{R}^d)$ and thus $G \in L_{v^s}^1(\mathbb{R}^{2d})$, and $T \in \mathrm{Op}(M_{\tilde{v}^s}^{\infty,1})$ and thus $H \in L_{v^s}^1(\mathbb{R}^{2d})$. Then $v^{1/2}H \in L_{v^{s-1/2}}^1(\mathbb{R}^{2d})$. Consequently

$$G * (v^{1/2}H) * G^* \in L_{v^s}^1 * L_{v^{s-1/2}}^1 * L_{v^s}^1 \subseteq L_{v^{s-1/2}}^1. \tag{12}$$

The characterization of Theorem 2.3 now implies that $A_\theta^g T A_{1/\theta}^g \in \mathrm{Op}(M_{\tilde{v}^{s-1/2}}^{\infty,1})$. □

Corollary 2.5. *Assume that $g \in M_{v^2 w}^1(\mathbb{R}^d)$ and $m \in \mathcal{M}_v$ and w is an arbitrary submultiplicative weight. Then $A_{1/m}^g A_m^g \in \mathrm{Op}(M_{\tilde{v}\tilde{w}}^{\infty,1})$.*

Proof. In this case T is the identity operator and $\mathrm{Id} \in \mathrm{Op}(M^{\infty,1}_{v_0})$ for every submultiplicative weight $v_0(x,\xi,\eta,y) = v_0(-y,y)$. In particular, $\mathrm{Id} \in \mathrm{Op}(M^{\infty,1}_{\widetilde{v}^2})$. Now replace the weight θ in Theorem 2.4 by m and the condition $\theta \in \mathcal{M}_{v^{1/2}}$ by $m \in \mathcal{M}_v$ and modify the convolution inequality (12) in the proof of Theorem 2.4. □

3. Canonical isomorphisms between modulation spaces of Hilbert-type

In [21] we have used a deep result of Bony and Chemin [7] regarding the existence of isomorphisms between modulation spaces of Hilbert-type and then extended those isomorphisms to arbitrary modulation spaces. Unfortunately the result of Bony and Chemin is restricted to weights of polynomial-type and does not cover weights moderated by superfast growing functions, such as $v(z) = e^{a|z|^b}$ for $0 < b < 1$.

In this section we construct explicit isomorphisms between $L^2(\mathbb{R}^d)$ and the modulation spaces $M^2_\theta(\mathbb{R}^d)$ for a general class of weights. We will assume that the weights are radial in each time-frequency variable. Precisely, consider time-frequency variables

$$(x,\xi) \simeq z = x + i\xi \in \mathbb{C}^d \simeq \mathbb{R}^{2d}, \tag{13}$$

which we identify by

$$(x_1,\xi_1;x_2,\xi_2;\dots;x_d,\xi_d) = (z_1,z_2,\dots,z_d) \in \mathbb{C}^d \simeq \mathbb{R}^{2d}.$$

Then the weight function m should satisfy

$$m(z) = m_0(|z_1|,\dots,|z_d|) \qquad \text{for } z \in \mathbb{R}^{2d} \tag{14}$$

for some function m_0 on $\overline{\mathbb{R}}^d_+ = [0,\infty)^d$. Without loss of generality, we may also assume that m is continuous on $\mathbb{R}^{2d}$. (Recall that only weights of polynomial-type occur in the lifting results in [21]. On the other hand, no radial symmetry is needed in [21].)

For each multi-index $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}_0^d$ we denote the corresponding multivariate Hermite function by

$$h_\alpha(t) = \prod_{j=1}^d h_{\alpha_j}(t_j), \quad \text{where} \quad h_n(x) = \frac{2^{1/4}\pi^{n/2}}{n!^{1/2}} e^{\pi x^2} \frac{d^n}{dx^n}(e^{-2\pi x^2})$$

is the n-th Hermite function in one variable with the normalization $\|h_n\|_2 = 1$.

Then the collection of all Hermite functions $h_\alpha, \alpha \geq 0$, is an orthonormal basis of $L^2(\mathbb{R}^d)$. By identifying $\mathbb{R}^{2d}$ with $\mathbb{C}^d$ via (13), the short-time Fourier transform of h_α with respect to $h(t) = 2^{d/4}e^{-\pi t^2}$ is simply

$$V_h h_\alpha(\overline{z}) = e^{-\pi i x\cdot\xi}\Big(\frac{\pi^{|\alpha|}}{\alpha!}\Big)^{1/2} z^\alpha\, e^{-\pi|z|^2/2} = e^{-\pi i x\cdot\xi}\, e_\alpha(z)\, e^{-\pi|z|^2} \qquad \text{for } z \in \mathbb{C}^d. \tag{15}$$

Remark. We mention that a formal Hermite expansion $f = \sum_\alpha c_\alpha h_\alpha$ defines a distribution in the Gelfand-Shilov space $(S^{1/2}_{1/2})'$ if and only if the coefficients satisfy $|c_\alpha| = \mathcal{O}(e^{\epsilon|\alpha|})$ for every $\epsilon > 0$. The Hermite expansion then converges in the weak* topology. Here we have used the fact that the duality $(S^{1/2}_{1/2})' \times S^{1/2}_{1/2}$ extends the

L^2-form $\langle\cdot,\cdot\rangle$ on $S_{1/2}^{1/2}$, and likewise the duality of the modulation spaces $M^2_\theta(\mathbb{R}^d)\times M^2_{1/\theta}(\mathbb{R}^d)$. Consequently, the coefficient c_α of a Hermite expansion is uniquely determined by $c_\alpha=\langle f,h_\alpha\rangle$ for $f\in(S_{1/2}^{1/2})'$. See [23] for details.

In the following we take for granted the existence and convergence of Hermite expansions for functions and distributions in arbitrary modulation spaces. By $d\mu(z)=e^{-\pi|z|^2}\,dz$ we denote the Gaussian measure on $\mathbb{C}^d$.

Lemma 3.1. *Assume that $\theta(z)=\mathcal{O}(e^{a|z|})$ and that θ is radial in each coordinate.*

(a) *Then the monomials $z^\alpha,\alpha\ge 0$, are orthogonal in $L^2_\theta(\mathbb{C}^d,\mu)$.*

(b) *The finite linear combinations of the Hermite functions are dense in $M^2_\theta(\mathbb{R}^d)$.*

By using polar coordinates $z_j=r_je^{i\varphi_j}$, where $r_j\ge 0$ and $\varphi_j\in[0,2\pi)$, we get

$$z^\alpha=r^\alpha e^{i\alpha\cdot\varphi}\quad\text{and}\quad dz=r_1\cdots r_d\,d\varphi dr,\tag{16}$$

and the condition on θ in Lemma 3.1 can be recast as

$$\theta(z)=\theta_0(r),\tag{17}$$

for some appropriate function θ_0 on $[0,\infty)^d$, and $r=(r_1,\dots,r_d)$ and $\varphi=(\varphi_1,\dots,\varphi_d)$ as usual.

Proof. (a) This is well known and is proved in [12, 16]. In order to be self-contained, we recall the arguments. By writing the integral over $\mathbb{R}^{2d}$ in polar coordinates in each time-frequency pair, (16) and (17) give

$$\int_{\mathbb{R}^{2d}} z^\alpha\overline{z^\beta}\theta(z)^2\,e^{-\pi|z|^2}\,dz=\iint_{\mathbb{R}^d_+\times[0,2\pi)^d} e^{i(\alpha-\beta)\cdot\varphi}r^{\alpha+\beta}e^{-\pi r^2}\,\theta_0(r)^2r_1\cdots r_d\,d\varphi dr.$$

The integral over the angles φ_j is zero, unless $\alpha=\beta$, whence the orthogonality of the monomials.

(b) Density: Assume on the contrary that the closed subspace in $M^2_\theta(\mathbb{R}^d)$ spanned by the Hermite functions is a proper subspace of $M^2_\theta(\mathbb{R}^d)$. Then there exists a nonzero $f\in(M^2_\theta(\mathbb{R}^d))'=M^2_{1/\theta}(\mathbb{R}^d)$ such that $\langle f,h_\alpha\rangle=0$ for all Hermite functions $h_\alpha\in M^2_\theta(\mathbb{R}^d)$, $\alpha\in\mathbb{N}_0^d$. Consequently the Hermite expansion of $f=\sum_\alpha\langle f,h_\alpha\rangle h_\alpha=0$ in $(S_{1/2}^{1/2})'$, which contradicts the assumption that $f\neq 0$. □

Definition 1. The anti-Wick operator J_m is the localization operator A_m^h associated to the weight m and to the Gaussian window $h=h_0$. Specifically,

$$J_mf=\int_{\mathbb{R}^{2d}} m(z)\langle f,\pi(z)h\rangle\pi(z)h\,dz.\tag{18}$$

For $m=\theta^2$ we obtain

$$\langle J_mf,f\rangle=\langle mV_hf,V_hf\rangle_{\mathbb{R}^{2d}}=\|V_hf\,\theta\|_2^2=\|f\|^2_{M^2_\theta}\tag{19}$$

whenever f is in a suitable space of test functions.

Localization operators with respect to Gaussian windows have been used in several problems of analysis; see for instance [1, 24, 25, 27]. They have rather special properties. In view of the connection to the localization operators on the Bargmann-Fock space (see below), this is to be expected.

Theorem 3.2. *If θ is a continuous, moderate function and radial in each time-frequency coordinate, then each of the mappings*

$$J_\theta : \; M^2_\theta(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \qquad J_\theta : L^2(\mathbb{R}^d) \to M^2_{1/\theta}(\mathbb{R}^d),$$

$$J_{1/\theta} : M^2_{1/\theta}(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad J_{1/\theta} : L^2(\mathbb{R}^d) \to M^2_\theta(\mathbb{R}^d)$$

is an isomorphism.

The special case of the polynomial weight $\theta(z) = (1 + |z|^2)^{1/2}$ was already proved in [27] with completely different methods. The extension to weights of non-polynomial growth is non-trivial and requires a number of preliminary results. In these investigations we will play with different coefficients of the form

$$\tau_\alpha(\theta) := \langle J_\theta h_\alpha, h_\alpha \rangle,$$

or, more generally,

$$\tau_{\alpha,s}(\theta) := \tau_\alpha(\theta^s) = \langle J_{\theta^s} h_\alpha, h_\alpha \rangle = \int_{\mathbb{R}^{2d}} \theta(z)^s \frac{\pi^{|\alpha|}}{\alpha!} |z^\alpha|^2 e^{-\pi|z|^2}\, dz\,, \tag{20}$$

when θ is a weight function and $s \in \mathbb{R}$. We note that the $\tau_{\alpha,s}(\theta)$ are strictly positive, since θ is positive. If $\theta \equiv 1$, then $\tau_{\alpha,s}(\theta) = 1$, so we may consider the coefficients $\tau_{\alpha,s}(\theta)\alpha!$ as *weighted gamma functions.*

Proposition 3.3 (Characterization of M^2_θ with Hermite functions)**.** *Let θ be a moderate and radial function. Then*

$$\|f\|^2_{M^2_\theta} = \sum_{\alpha \geq 0} |\langle f, h_\alpha \rangle|^2 \tau_\alpha(\theta^2). \tag{21}$$

Proof. Let $f = \sum_{\alpha \geq 0} c_\alpha h_\alpha$ be a finite linear combination of Hermite functions. Since the short-time Fourier transform of f with respect to the Gaussian h is given by

$$V_h f(\overline{z}) = \sum c_\alpha V_h h_\alpha(\overline{z}) = e^{-\pi i x\cdot\xi} \sum_{\alpha \geq 0} c_\alpha e_\alpha(z) e^{-\pi|z|^2/2}$$

in view of (15), definition (18) gives

$$\begin{aligned} \|f\|^2_{M^2_\theta} &= \int_{\mathbb{R}^{2d}} |V_h f(z)|^2 \theta(z)^2\, dz \\ &= \sum_{\alpha,\beta \geq 0} c_\alpha \overline{c_\beta} \int_{\mathbb{R}^{2d}} e_\alpha(z) \overline{e_\beta(z)} \theta(z)^2\, e^{-\pi|z|^2}\, dz = \sum_{\alpha \geq 0} |c_\alpha|^2 \tau_\alpha(\theta^2). \end{aligned}$$

In the latter equalities it is essential that the weight θ is radial in each time-frequency coordinate so that the monomials e_α are orthogonal in $L^2_\theta(\mathbb{C}^d, \mu)$. □

In the next proposition, which is due to Daubechies [12], we represent the anti-Wick operator by a Hermite expansion.

Proposition 3.4. *Let θ be a moderate, continuous weight function on $\mathbb{R}^{2d}$ that is radial in each time-frequency coordinate.*

Then the Hermite function h_α is an eigenfunction of the localization operator J_θ with eigenvalue $\tau_\alpha(\theta)$ for $\alpha \in \mathbb{N}^d_0$, and J_θ possesses the eigenfunction expansion

$$J_\theta f = \sum_{\alpha \geq 0} \tau_\alpha(\theta) \langle f, h_\alpha \rangle h_\alpha \qquad \textit{for all } f \in L^2(\mathbb{R}^d). \tag{22}$$

Proof. By Lemma 3.1(a) we find that, for $\alpha \neq \beta$,

$$\begin{aligned}\langle J_\theta h_\beta, h_\alpha\rangle &= \int_{\mathbb{C}^d} \theta(z) V_h h_\beta(z)\, \overline{V_h h_\alpha(z)}\, dz \\ &= \int_{\mathbb{C}^d} \theta(z) e_\beta(z) \overline{e_\alpha(z)} e^{-\pi|z|^2}\, dz = 0.\end{aligned}$$

This implies that $J_\theta h_\alpha = c h_\alpha$, and therefore $c = c\langle h_\alpha, h_\alpha\rangle = \langle J_\theta h_\alpha, h_\alpha\rangle = \tau_\alpha(\theta)$.

For a (finite) linear combination $f = \sum_{\beta\geq 0} c_\beta h_\beta$, we obtain

$$\begin{aligned}J_\theta f = \sum_{\alpha\geq 0} \langle J_\theta f, h_\alpha\rangle h_\alpha &= \sum_{\alpha\geq 0}\sum_{\beta\geq 0} c_\beta \langle J_\theta h_\beta, h_\alpha\rangle h_\alpha \\ &= \sum_{\alpha\geq 0}\sum_{\beta\geq 0} \tau_\beta(\theta)\delta_{\alpha,\beta} c_\beta h_\alpha = \sum_{\alpha\geq 0} \tau_\alpha(\theta) c_\alpha h_\alpha.\end{aligned}$$

The proposition follows because the Hermite functions span $M^2_{1/\theta}(\mathbb{R}^d)$ and because the coefficients of a Hermite expansion are unique and given by $c_\alpha = \langle f, h_\alpha\rangle$. $\square$

Corollary 3.5. *If θ is moderate and radial in each coordinate, then $J_\theta : L^2(\mathbb{R}^d) \to M^2_{1/\theta}(\mathbb{R}^d)$ is one-to-one and possesses dense range in $M^2_{1/\theta}(\mathbb{R}^d)$.*

Proof. The coefficients in $J_\theta f = \sum_{\alpha\geq 0} \tau_\alpha(\theta)\langle f, h_\alpha\rangle h_\alpha$ are unique. If $J_\theta f = 0$, then $\tau_\alpha(\theta)\langle f, h_\alpha\rangle = 0$, and since $\tau_\alpha(\theta) > 0$ we obtain $\langle f, h_\alpha\rangle = 0$ and thus $f = 0$. Clearly the range of J_θ in $M^2_{1/\theta}(\mathbb{R}^d)$ contains the finite linear combinations of Hermite functions, and these are dense in $M^2_{1/\theta}(\mathbb{R}^d)$ by Lemma 3.1(b). $\square$

To show that J_θ maps $L^2(\mathbb{R}^d)$ onto $M^2_{1/\theta}(\mathbb{R}^d)$ is much more subtle. For this we need a new type of inequality valid for the weighted gamma functions in (20). By Proposition 3.4 the number $\tau_{\alpha,s}(\theta)$ is exactly the eigenvalue of the localization operator J_{θ^s} corresponding to the eigenfunction h_α.

Proposition 3.6. *If $\theta \in \mathcal{M}_w$ is continuous and radial in each time-frequency coordinate, then the mapping $s \mapsto \tau_{\alpha,s}(\theta)$ is "almost multiplicative". This means that for every $s,t \in \mathbb{R}$ there exists a constant $C = C(s,t)$ such that*

$$C^{-1} \leq \tau_{\alpha,s}(\theta)\tau_{\alpha,t}(\theta)\tau_{\alpha,-s-t}(\theta) \leq C \qquad \textit{for all multi-indices } \alpha. \tag{23}$$

Proof. The upper bound is easy. By Proposition 3.4 the Hermite function h_α is a common eigenfunction of $J_{\theta^s}, J_{\theta^t}$, and $J_{\theta^{-s-t}}$. Since the operator $J_{\theta^s} J_{\theta^t} J_{\theta^{-s-t}}$ is bounded on $L^2(\mathbb{R}^d)$ by repeated application of Lemma 2.1, we obtain that

$$\tau_{\alpha,s}(\theta)\tau_{\alpha,t}(\theta)\tau_{\alpha,-s-t}(\theta) = \|J_{\theta^s} J_{\theta^t} J_{\theta^{-s-t}} h_\alpha\|_{L^2} \leq C\|h_\alpha\|_{L^2} = C \tag{24}$$

for all $\alpha \geq 0$. The constant C is the operator norm of $J_{\theta^s} J_{\theta^t} J_{\theta^{-s-t}}$ on $L^2(\mathbb{R}^d)$.

For the lower bound we rewrite the definition of $\tau_{\alpha,s}(\theta)$ and make it more explicit by using polar coordinates $z_j = r_j e^{i\varphi_j}, r_j \geq 0, \varphi_j \in [0, 2\pi)$, in each variable. Then by assumption $\theta(z) = \theta_0(r)$ for some continuous moderate function θ_0 on $\mathbb{R}^d_+$, and we obtain

$$
\begin{aligned}
(25)\quad \tau_{\alpha,s}(\theta) &= \int_{\mathbb{R}^{2d}} \theta(z)^s \frac{\pi^{|\alpha|}}{\alpha!}|z^\alpha|^2 e^{-\pi|z|^2}\,dz \\
&= (2\pi)^d \int_{\mathbb{R}^d_+} \theta_0(r)\frac{\pi^{|\alpha|}}{\alpha!} r^{2\alpha} e^{-\pi|r|^2}\, r_1\cdots r_d\,dr \\
&= (2\pi)^d \int_0^\infty \dots \int_0^\infty \theta_0(r_1,\dots,r_d)\prod_{j=1}^d \frac{1}{\alpha_j!}\,(\pi r_j^2)^{\alpha_j} e^{-\pi r_j^2}\, r_1\cdots r_d\,dr_1\cdots dr_d \\
&= \int_0^\infty \dots \int_0^\infty \theta_0\big(\sqrt{u_1/\pi},\dots,\sqrt{u_d/\pi}\big)\prod_{j=1}^d \frac{u_j^{\alpha_j}}{\alpha_j!}\, e^{-u_j}\,du_1\cdots du_d.
\end{aligned}
$$

We first focus on a single factor in the integral. The function $f_n(x) = x^n e^{-x}/n!$ takes its maximum at $x = n$, and

$$f_n(n) = \frac{1}{n!}n^n e^{-n} = (2\pi n)^{-1/2}\big(1+\mathcal{O}(n^{-1})\big)$$

by Stirling's formula. Furthermore, f_n is almost constant on the interval $[n-\sqrt{n}/2, n+\sqrt{n}/2]$ of length $\sqrt{n}$. On this interval the minimum of f_n is taken at one of the endpoints $n \pm \sqrt{n}/2$, where the value is

$$f_n(n\pm\sqrt{n}/2) = \frac{1}{n!}(n\pm\sqrt{n}/2)^n e^{-(n\pm\sqrt{n}/2)} = \frac{1}{n!}\frac{n^n}{e^n}\big(1\pm\frac{1}{2\sqrt{n}}\big)^n e^{\mp\sqrt{n}/2}.$$

Since

$$\lim_{n\to\infty}\frac{(2\pi n)^{1/2}\left(\frac{n}{e}\right)^n}{n!} = 1 \quad\text{and}\quad \lim_{n\to\infty}\big(1\pm\frac{1}{2\sqrt{n}}\big)^n e^{\mp\sqrt{n}/2} = e^{-1/8}$$

by Stirling's formula and straightforward applications of Taylor's formula, we find that

$$(26)\qquad f_n(x) \geq \frac{c}{\sqrt{n}} \qquad \text{for } x\in[n-\sqrt{n}/2, n+\sqrt{n}/2] \text{ and all } n\geq 1.$$

For $n = 0$ we use the inequality $f_0(x) \geq e^{-1/2}$ for $x \in [0, 1/2]$.

Now consider the products of the f_n's occuring in the integral above. For $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}_0^d$ define the boxes

$$C_\alpha = \prod_{j=1}^d \big[\alpha_j - 2^{-1}\sqrt{\alpha_j}\,,\, \alpha_j + 2^{-1}\max(\sqrt{\alpha_j},1)\big] \subseteq \mathbb{R}^d\,,$$

with volume $\mathrm{vol}\,(C_\alpha) = \prod_{j=1}^d \sqrt{\max(2^{-1},\alpha_j)}$. Consequently, on the box C_α we have

$$(27)\qquad \prod_{j=1}^d \frac{u_j^{\alpha_j}}{\alpha_j!}\, e^{-u_j} \geq C_0 \prod_{j=1}^d \frac{1}{\sqrt{\max(2^{-1},\alpha_j)}} = C_0\big(\mathrm{vol}\,C_\alpha\big)^{-1}$$

for some constant $C_0 > 0$ which is independent of $\alpha \in \mathbb{N}_0^d$.

Next, to take into account the coordinate change in (25), we define the box

$$D_\alpha = \prod_{j=1}^{d} \Big[\frac{(\alpha_j - 2^{-1}\sqrt{\alpha_j})^{1/2}}{\sqrt{\pi}} \, , \, \frac{(\alpha_j + 2^{-1}\max(\sqrt{\alpha_j},1))^{1/2}}{\sqrt{\pi}} \Big] \subseteq \mathbb{R}^d \, .$$

Furthermore, the length of each edge of D_α is

$$\pi^{-1/2}\Big(\big(\alpha_j + \sqrt{\alpha_j}/2\big)^{1/2} - \big(\alpha_j - \sqrt{\alpha_j}/2\big)^{1/2} \Big) \le \pi^{-1/2}$$

when $\alpha_j \ge 1$, and likewise for $\alpha_j = 0$. Consequently,

$$\text{if } z_1, z_2 \in D_\alpha, \quad \text{then } z_1 - z_2 \subseteq [-\pi^{-1/2}, \pi^{-1/2}]^d. \tag{28}$$

After these preparations we start the lower estimate of $\tau_{\alpha,s}(\theta)$. Using (27) we obtain

$$\begin{aligned} \tau_{\alpha,s}(\theta) &= \int_0^\infty \cdots \int_0^\infty \theta_0\big(\sqrt{u_1/\pi}, \dots, \sqrt{u_d/\pi}\big)^s \prod_{j=1}^{d} \frac{u_j^{\alpha_j}}{\alpha_j!} \, e^{-u_j} \, du_1 \cdots du_d \\ &\ge C_0 \frac{1}{\mathrm{vol}\,(C_\alpha)} \int_{C_\alpha} \theta_0\big(\sqrt{u_1/\pi}, \dots, \sqrt{u_d/\pi}\big)^s \, du_1 \cdots du_d. \end{aligned}$$

Since θ is continuous, the mean value theorem asserts that there is a point $z = z(\alpha,s) = (z_1, z_2, \dots, z_d) \in C_\alpha$ such that

$$\tau_{\alpha,s}(\theta) \ge C_0 \, \theta_0\big(\sqrt{z_1/\pi}, \dots, \sqrt{z_d/\pi}\big)^s.$$

Note that the point with coordinates $\zeta = \zeta(\alpha,s) = \big(\sqrt{z_1/\pi}, \dots, \sqrt{z_d/\pi}\big)$ is in D_α; consequently,

$$\tau_{\alpha,s}(\theta) \ge C_0 \, \theta_0(\zeta(\alpha,s))^s \qquad \text{for } \zeta(\alpha,s) \in D_\alpha.$$

Finally

$$\tau_{\alpha,s}(\theta) \, \tau_{\alpha,t}(\theta) \, \tau_{\alpha,-s-t}(\theta) \ge C_0^3 \, \theta_0(\zeta)^s \, \theta_0(\zeta')^t \, \theta_0(\zeta'')^{-s-t} \tag{29}$$

for points $\zeta, \zeta', \zeta'' \in D_\alpha$. Since the weight θ is a w-moderate, θ_0 satisfies

$$\frac{\theta_0(z_1)}{\theta_0(z_2)} \ge \frac{1}{w(z_1 - z_2)}, \qquad z_1, z_2 \in \mathbb{R}^d.$$

Since $\zeta, \zeta', \zeta'' \in D_\alpha$, the differences $\zeta - \zeta''$ and $\zeta' - \zeta''$ are in the cube $[-\pi^{-1/2}, \pi^{-1/2}]^d$ as observed in (28). We conclude the non-trivial part of this estimate by

$$\begin{aligned} \tau_{\alpha,s}(\theta)\tau_{\alpha,t}(\theta)\tau_{\alpha,-s-t}(\theta) &\ge C_0^3 \frac{1}{w(\zeta - \zeta'')^s} \frac{1}{w(\zeta' - \zeta'')^t} \\ &\ge C_0^3 \Big(\max_{z \in [-\pi^{-1/2}, \pi^{-1/2}]^d} w(z) \Big)^{-s-t} = C. \end{aligned}$$

The proof is complete. □

The next result provides a sort of symbolic calculus for the anti-Wick operators J_{θ^s}. Although the mapping $s \to J_{\theta^s}$ is not a homomorphism from $\mathbb{R}$ to operators, it is multiplicative modulo bounded operators.

Theorem 3.7. *Let θ and μ be two moderate, continuous weight functions on $\mathbb{R}^{2d}$ that are radial in each time-frequency variable. For every $r, s \in \mathbb{R}$ there exists an operator $V_{s,t}$ that is invertible on every $M^2_\mu(\mathbb{R}^d)$ such that*

$$J_{\theta^s} J_{\theta^t} J_{\theta^{-s-t}} = V_{s,t}.$$

Proof. For $s, t \in \mathbb{R}$ fixed, set $\gamma(\alpha) = \tau_{\alpha,s}(\theta)\tau_{\alpha,t}(\theta)\tau_{\alpha,-s-t}(\theta)$ and

$$V_{s,t}f = \sum_{\alpha \geq 0} \gamma(\alpha)\langle f, h_\alpha\rangle h_\alpha.$$

Clearly, $V_{s,t} = J_{\theta^s}J_{\theta^t}J_{\theta^{-s-t}}$. Since $C^{-1} \leq \gamma(\alpha) \leq C$ for all $\alpha \geq 0$ by Proposition 3.6, Proposition 3.3 implies that $V_{s,t}$ is bounded on every modulation space M^2_μ. Likewise the formal inverse operator $V_{s,t}^{-1}f = \sum_{\alpha\geq 0} \gamma(\alpha)^{-1}\langle f, h_\alpha\rangle h_\alpha$ is bounded on $M^2_\mu(\mathbb{R}^d)$; consequently, $V_{s,t}$ is invertible on $M^2_\mu(\mathbb{R}^d)$. □

We can now finish the proof of Theorem 3.2.

Proof of Theorem 3.2. Choose $s = 1$ and $t = -1$; then $J_\theta J_{1/\theta} = V_{1,-1}$ is invertible on L^2. Similarly, the choice $s = -1, t = 1$ yields that $J_{1/\theta}J_\theta = V_{-1,1}$ is invertible on M^2_θ. The factorization $J_\theta J_{1/\theta} = V_{1,-1}$ implies that $J_{1/\theta}$ is one-to-one from L^2 to M^2_θ and that J_θ maps M^2_θ onto L^2. The factorization $J_{1/\theta}J_\theta = V_{-1,1}$ implies that J_θ is one-to-one from M^2_θ to L^2 and that $J_{1/\theta}$ maps L^2 onto M^2_θ.

We have proved that J_θ is an isomorphism from M^2_θ to L^2 and that $J_{1/\theta}$ is an isomorphism from L^2 to M^2_θ. The other isomorphisms are proved similarly. □

Remark. In dimension $d = 1$ the invertibility of $J_\theta J_{\theta^{-1}}$ follows from the equivalence $\tau_{n,1}(\theta)\tau_{n,-1}(\theta) \asymp 1$, which can be expressed as the following inequality for weighted gamma functions:

$$C^{-1} \leq \int_0^\infty \theta_0(\sqrt{x/\pi})\frac{x^n}{n!}e^{-x}dx \int_0^\infty \frac{1}{\theta_0(\sqrt{x/\pi})}\frac{x^n}{n!}e^{-x}dx \leq C \tag{30}$$

for all $n \geq 0$. Here θ_0 is the same as before. It is a curious and fascinating fact that this inequality implies that the localization operator $J_{1/\theta}$ is an isomorphism between $L^2(\mathbb{R})$ and $M^2_\theta(\mathbb{R})$.

4. The general isomorphism theorems

In Theorem 4.3 we will state the general isomorphism theorems. The strategy of the proof is similar to that of Theorem 3.2 in [21]. The main tools are the theorems about the spectral invariance of the generalized Sjöstrand classes [18] and the existence of a canonical isomorphism between $L^2(\mathbb{R}^d)$ and $M^2_\theta(\mathbb{R}^d)$ established in Theorem 3.2.

4.1. Variations on spectral invariance.

We first introduce the tools concerning the spectral invariance of pseudodifferential operators. Recall the following results from [18].

Theorem 4.1. *Let v be a submultiplicative weight on $\mathbb{R}^{2d}$ such that*

$$\lim_{n\to\infty} v(nz)^{1/n} = 1 \quad \textit{for all } z \in \mathbb{R}^{2d}, \tag{31}$$

and let $\tilde{v}$ be the same as in (10). *If $T \in \mathrm{Op}(M^{\infty,1}_{\tilde{v}})$ and T is invertible on $L^2(\mathbb{R}^d)$, then $T^{-1} \in \mathrm{Op}(M^{\infty,1}_{\tilde{v}})$.*

Consequently, T is invertible simultaneously on all modulation spaces $M^{p,q}_\mu(\mathbb{R}^d)$ for $1 \leq p, q \leq \infty$ and all $\mu \in \mathcal{M}_v$.

Condition (31) is usually called the Gelfand-Raikov-Shilov (GRS) condition.

We prove a more general form of spectral invariance. Since we have formulated all results about the anti-Wick operators J_θ for radial weights only, we will assume from now on that all weights are radial in each coordinate. In this case

$$\tilde{v}(x,\xi,\eta,y) = v(-\eta,y) = v(y,\eta),$$

and we do not need the somewhat ugly distinction between v and $\tilde{v}$.

Theorem 4.2. *Assume that v satisfies the GRS condition, $\theta^2 \in \mathcal{M}_v$, and that both v and θ are radial in each time-frequency coordinate.*

If $T \in \mathrm{Op}(M_v^{\infty,1})$ and T is invertible on $M_\theta^2(\mathbb{R}^d)$, then T is invertible on $L^2(\mathbb{R}^d)$.

As a consequence T is invertible on every modulation space $M_\mu^{p,q}(\mathbb{R}^d)$ for $1 \le p,q \le \infty$ and $\mu \in \mathcal{M}_v$.

Proof. Set $\widetilde{T} = J_\theta T J_{1/\theta}$. By Theorem 3.2, $J_{1/\theta}$ is an isomorphism from $L^2(\mathbb{R}^d)$ onto $M_\theta^2(\mathbb{R}^d)$ and J_θ is an isomorphism from $M_\theta^2(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$; therefore $\widetilde{T}$ is an isomorphism on $L^2(\mathbb{R}^d)$.

$$\begin{array}{ccc} M_\theta^2 & \xrightarrow{T} & M_\theta^2 \\ \uparrow J_{1/\theta} & & \downarrow J_\theta \\ L^2(\mathbb{R}^d) & \xrightarrow{\widetilde{T}} & L^2(\mathbb{R}^d) \end{array} \tag{32}$$

By Theorem 2.4 the operator $\widetilde{T}$ is in $\mathrm{Op}(M_{v^{1/2}}^{\infty,1})$. Since $\widetilde{T}$ is invertible on $L^2(\mathbb{R}^d)$, Theorem 4.1 on the spectral invariance of the symbol class $M_{v^{1/2}}^{\infty,1}$ implies that the inverse operator $\widetilde{T}$ also possesses a symbol in $M_{v^{1/2}}^{\infty,1}$, i.e., $\widetilde{T}^{-1} \in \mathrm{Op}(M_{v^{1/2}}^{\infty,1})$.

Now, since $\widetilde{T}^{-1} = J_{1/\theta}^{-1} T^{-1} J_\theta^{-1}$, we find that

$$T^{-1} = J_{1/\theta} \widetilde{T}^{-1} J_\theta.$$

Applying Theorem 2.4 once again, the symbol of T^{-1} must be in $M^{\infty,1}$. As a consequence, T^{-1} is bounded on $L^2(\mathbb{R}^d)$.

Since $T \in \mathrm{Op}(M_v^{\infty,1})$ and T is invertible on $L^2(\mathbb{R}^d)$, it follows that T is also invertible on $M_\mu^{p,q}(\mathbb{R}^d)$ for every weight $\mu \in \mathcal{M}_v$ and $1 \le p,q \le \infty$. □

4.2. **An isomorphism theorem for localization operators.** We now combine all steps and formulate and prove our main result, the isomorphism theorem for time-frequency localization operators with symbols of superfast growth.

Theorem 4.3. *Let $g \in M_{v^2 w}^1(\mathbb{R}^d)$, $\mu \in \mathcal{M}_w$ and $m \in \mathcal{M}_v$ be such that m is radial in each time-frequency coordinate and v satisfies* (31)*. Then the localization operator A_m^g is an isomorphism from $M_\mu^{p,q}(\mathbb{R}^d)$ onto $M_{\mu/m}^{p,q}(\mathbb{R}^d)$ for $1 \le p,q \le \infty$.*

Proof. Set $T = A_{1/m}^g A_m^g$. We have already established that

(1) $T = A_{1/m}^g A_m^g$ possesses a symbol in $M_{\tilde{v}\tilde{w}}^{\infty,1}$ by Corollary 2.5.
(2) T is invertible on $M_\theta^2(\mathbb{R}^d)$ by Lemma 2.2.

These are the assumptions of Theorem 4.2, and therefore T is invertible on $M_\mu^{p,q}(\mathbb{R}^d)$ for every $\mu \in \mathcal{M}_w \subseteq \mathcal{M}_{vw}$ and $1 \le p,q \le \infty$. The factorization $T = A_{1/m}^g A_m^g$ implies that A_m^g is one-to-one from $M_\mu^{p,q}(\mathbb{R}^d)$ to $M_{\mu/m}^{p,q}(\mathbb{R}^d)$ and that $A_{1/m}^g$ maps $M_{\mu/m}^{p,q}(\mathbb{R}^d)$ onto $M_\mu^{p,q}(\mathbb{R}^d)$.

Now we change the order of the factors and consider the operator $T' = A^g_m A^g_{1/m}$ from $M^{p,q}_{\mu/m}(\mathbb{R}^d)$ to $M^{p,q}_{\mu/m}(\mathbb{R}^d)$ and factoring through $M^{p,q}_{\mu}(\mathbb{R}^d)$. Again T' possesses a symbol in $M^{\infty,1}_{\tilde{v}\tilde{w}}$ and is invertible on $M^2_{1/\theta}(\mathbb{R}^d)$. With Theorem 4.2 we conclude that T' is invertible on all modulation spaces $M^{p,q}_{\mu/m}(\mathbb{R}^d)$. The factorization of $T' = A^g_m A^g_{1/m}$ now yields that $A^g_{1/m}$ is one-to-one from $M^{p,q}_{\mu/m}(\mathbb{R}^d)$ to $M^{p,q}_{\mu}(\mathbb{R}^d)$ and that A^g_m maps $M^{p,q}_{\mu}(\mathbb{R}^d)$ onto $M^{p,q}_{\mu/m}(\mathbb{R}^d)$.

As a consequence A^g_m is bijective from $M^{p,q}_{\mu}(\mathbb{R}^d)$ onto $M^{p,q}_{\mu/m}(\mathbb{R}^d)$ and $A^g_{1/m}$ is bijective from $M^{p,q}_{\mu/m}(\mathbb{R}^d)$ onto $M^{p,q}_{\mu}(\mathbb{R}^d)$. □

5. Consequences for Gabor multipliers and Toeplitz operators on the Bargmann-Fock space

5.1. **Gabor multipliers.** Gabor multipliers are time-frequency localization operators whose symbols are discrete measures. Their basic properties are the same as for the corresponding localization operators, but the discrete definition makes them more accessible for numerical computations.

Let $\Lambda = A\mathbb{Z}^{2d}$ for some $A \in \mathrm{GL}(2d, \mathbb{R})$ be a lattice in $\mathbb{R}^{2d}$, let g, γ be suitable window functions (as in Proposition 5.1) and m be a weight sequence defined on Λ. Then the Gabor multiplier $G^{g,\gamma,\Lambda}_m$ is defined to be

$$G^{g,\gamma,\Lambda}_m f = \sum_{\lambda\in\Lambda} m(\lambda)\langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma. \tag{33}$$

The boundedness of Gabor multipliers between modulation spaces is formulated and proved exactly as for time-frequency localization operators. See [15] for a detailed exposition of Gabor multipliers and [10] for general boundedness results that include distributional symbols.

Proposition 5.1. *Let $g, \gamma \in M^1_{vw}(\mathbb{R}^d)$, and let m be a continuous moderate weight function $m \in \mathcal{M}_v$. Then $G^{g,\gamma,\Lambda}_m$ is bounded from $M^{p,q}_{\mu}(\mathbb{R}^{2d})$ to $M^{p,q}_{\mu/m}(\mathbb{R}^d)$.*

In the following we will assume that the set $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is a Gabor frame for $L^2(\mathbb{R}^d)$. This means that the frame operator $S_{g,\Lambda} f = M^{g,g,\Lambda}_1 = \sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)g\rangle\pi(\lambda)g$ is invertible on $L^2(\mathbb{R}^d)$. From the rich theory of Gabor frames we quote only the following result: If $g \in M^1_v$ and $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, $1 \le p \le \infty$, and $m \in \mathcal{M}_v$, then

$$\|f\|_{M^p_m} \asymp \Big(\sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^p m(\lambda)^p\Big)^{1/p} \qquad \text{for all } f \in M^p_m(\mathbb{R}^d). \tag{34}$$

We refer to [17] for a detailed discussion, the proof, and for further references.

Our main result on Gabor multipliers is the isomorphism theorem.

Theorem 5.2. *Assume that $g \in M^1_{v^2w}(\mathbb{R}^d)$ and that $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. If $m \in \mathcal{M}_v$ is radial in each time-frequency coordinate, then $G^{g,g,\Lambda}_m$ is an isomorphism from $M^{p,q}_{\mu}(\mathbb{R}^{2d})$ onto $M^{p,q}_{\mu/m}(\mathbb{R}^d)$ for every $\mu \in \mathcal{M}_w$ and $1 \le p, q \le \infty$.*

The proof is similar to the proof of Theorem 4.3, and we only sketch the necessary modifications. In the following we fix the lattice Λ and choose $\gamma = g$ and drop the reference to these additional parameters by writing $G^{g,\gamma,\Lambda}_m$ as G^g_m in analogy to the localization operator A^g_m.

Proposition 5.3. *If $g \in M^1_v(\mathbb{R}^d)$ and $m \in \mathcal{M}_v$ and $\theta = m^{1/2}$, then G^g_m is an isomorphism from $M^2_\theta(\mathbb{R}^d)$ onto $M^2_{1/\theta}(\mathbb{R}^d)$.*

Proof. By Proposition 5.1 G^g_m is bounded from M^2_θ to $M^2_{\theta/m} = M^2_{1/\theta}$, and thus

$$\|G^g_m f\|_{M^2_{1/\theta}} \le C\|f\|_{M^2_\theta}.$$

Next we use the characterization of M^2_θ by Gabor frames and relate it to Gabor multipliers:

$$\langle G^g_m f, f\rangle = \sum_{\lambda\in\Lambda} m(\lambda)|\langle f, \pi(\lambda)g\rangle|^2 \asymp \|f\|^2_{M^2_\theta}. \tag{35}$$

This identity implies that G^g_m is one-to-one on M^2_θ and that $\|G^g_m f\|_{M_{1/\theta}}$ is an equivalent norm on M^2_θ. Since G^g_m is self-adjoint, it has dense range in $M^2_{1/\theta}$, whence G^g_m is onto as well. □

Using the weight $1/m$ instead of m and $1/\theta$ instead of θ, we see that $G^g_{1/m}$ is an isomorphism from $M^2_{1/\theta}(\mathbb{R}^d)$ onto $M^2_\theta(\mathbb{R}^d)$.

Proof of Theorem 5.2. We proceed as in the proof of Theorem 4.3. Define the operators $T = G^g_{1/m}G^g_m$ and $T' = G^g_m G^g_{1/m}$. By Proposition 5.1 T maps $M^2_\theta(\mathbb{R}^d)$ to $M^2_\theta(\mathbb{R}^d)$, and since T is a composition of two isomorphisms, T is an isomorphism on $M^2_\theta(\mathbb{R}^d)$. Likewise T' is an isomorphism on $M^2_{1/\theta}(\mathbb{R}^d)$.

As in Corollary 2.5 we verify that the symbol of T is in $M^{\infty,1}_{\tilde v\tilde w}(\mathbb{R}^{2d})$. Theorem 4.2 then asserts that T is invertible on $L^2(\mathbb{R}^d)$, and the general spectral invariance implies that T is an isomorphism on $M^{p,q}_\mu(\mathbb{R}^d)$. Likewise T' is an isomorphism on $M^{p,q}_{\mu/m}(\mathbb{R}^d)$. This means that each of the factors of T must be an isomorphism on the correct space. □

5.2. **Toeplitz operators.** The Bargmann-Fock space $\mathcal{F} = \mathcal{F}^2(\mathbb{C}^d)$ is the Hilbert space of all entire functions of d variables such that

$$\|F\|^2_{\mathcal{F}} = \int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2}\,dz < \infty. \tag{36}$$

Here dz is the Lebesgue measure on $\mathbb{C}^d = \mathbb{R}^{2d}$.

Related to the Bargmann-Fock space $\mathcal{F}^2$ are spaces of entire functions satisfying weighted integrability conditions. Let m be a moderate weight on $\mathbb{C}^d$ and $1 \le p, q \le \infty$. Then the space $\mathcal{F}^{p,q}_m(\mathbb{C}^d)$ is the Banach space of all entire functions of d complex variables such that

$$\|F\|^q_{\mathcal{F}^{p,q}_m} = \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |F(x+iy)|^p m(x+iy)^p e^{-p\pi|x+iy|^2/2}dx\Big)^{q/p} dy < \infty. \tag{37}$$

Let P be the orthogonal projection from $L^2(\mathbb{C}^d,\mu)$ with Gaussian measure $d\mu(z) = e^{-\pi|z|^2}\,dz$ onto $\mathcal{F}^2(\mathbb{C}^d)$. Then P is given by the formula

$$PF(w) = \int_{\mathbb{C}^d} F(z) e^{\pi\overline{z}w} e^{-\pi|z|^2}\,dz. \tag{38}$$

We remark that a function on $\mathbb{R}^{2d}$ with a certain growth is entire if and only if it satisfies $F = PF$.

The classical Toeplitz operator on Bargmann-Fock space with a symbol m is defined by $T_m F = P(mF)$ for $F \in \mathcal{F}^2(\mathbb{C}^d)$. Explicitly T_m is given by the formula

$$T_m F(w) = \int_{\mathbb{C}^d} m(z) F(z) e^{\pi \overline{z} w}\, e^{-\pi|z|^2}\, dz. \tag{39}$$

See [2, 3, 4, 5, 12, 16] for a sample of references.

In the following we assume that the symbol of a Toeplitz operator is continuous and radial in each coordinate, i.e., $m(z_1, \dots, z_d) = m_0(|z_1|, \dots, |z_d|)$ for some continuous function m_0 from $\mathbb{R}_+^d$ to $\mathbb{R}_+$.

Theorem 5.4. *Let $\mu \in \mathcal{M}_w$, and let $m \in \mathcal{M}_v$ be a continuous moderate weight function such that one of the following conditions is fulfilled:*

(i) *Either m is radial in each coordinate*
(ii) *or m is of polynomial type.*

Then the Toeplitz operator T_m is an isomorphism from $\mathcal{F}^{p,q}_\mu(\mathbb{C}^d)$ onto $\mathcal{F}^{p,q}_{\mu/m}(\mathbb{C}^d)$ for $1 \le p, q \le \infty$.

The formulation of Theorem 5.4 looks similar to the main theorem about time-frequency localization operators. In fact, after a suitable translation of concepts, it is a special case of Theorem 4.3.

To explain the connection, we recall the Bargmann transform that maps distributions on $\mathbb{R}^d$ to entire functions on $\mathbb{C}^d$:

$$\mathcal{B}f(z) = F(z) = 2^{d/4} e^{-\pi z^2/2} \int_{\mathbb{R}^d} f(t) e^{-\pi t^2} e^{2\pi t \cdot z} dt, \qquad z \in \mathbb{C}^d. \tag{40}$$

If f is a distribution, then we interpret the integral as the action of f on the function $e^{-\pi t^2} e^{2\pi t \cdot z}$.

The connection to time-frequency analysis comes from the fact that the Bargmann transform is just a short-time Fourier transform in disguise [17, Ch. 3]. As before, we use the normalized Gaussian $h(t) = 2^{d/4} e^{-\pi t^2}$ as a window. Identifying the pair $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ with the complex vector $z = x + i\xi \in \mathbb{C}^d$, the short-time Fourier transform of a function f on $\mathbb{R}^d$ with respect to h is

$$V_h f(\overline{z}) = e^{\pi i x \cdot \xi} \mathcal{B}f(z) e^{-\pi |z|^2/2}. \tag{41}$$

In particular, the Bargmann transform of the time-frequency shift $\pi(w)h$ is given by

$$\mathcal{B}(\pi(w)h)(z) = e^{-\pi i u \cdot \eta} e^{\pi \overline{w} z} e^{-\pi |w|^2/2}, \quad w, z \in \mathbb{C}^d, w = u + i\eta. \tag{42}$$

It is a basic fact that the Bargmann transform is a unitary mapping from $L^2(\mathbb{R})$ onto the Bargmann-Fock space $\mathcal{F}^2(\mathbb{C}^d)$. Furthermore, the Hermite functions $h_\alpha, \alpha \ge 0$, are mapped to the normalized monomials $e_\alpha(z) = \pi^{|\alpha|/2} (\alpha!)^{-1/2} z^\alpha$.

Let $m'(z) = m(\overline{z})$. Then (41) implies that the Bargmann transform $\mathcal{B}$ maps $M^{p,q}_m(\mathbb{R}^d)$ isometrically to $\mathcal{F}^{p,q}_{m'}(\mathbb{C}^d)$. By straightforward arguments it follows that the Bargmann transform maps the modulation space $M^{p,q}_m(\mathbb{R}^d)$ onto the Fock space $\mathcal{F}^{p,q}_{m'}(\mathbb{C}^d)$ [17, 28].

The connection between time-frequency localization operators and Toeplitz operators on the Bargmann-Fock space is given by the following statement.

Proposition 5.5. *Let m be a moderate weight function on $\mathbb{R}^{2d} \simeq \mathbb{C}^d$ and set $m'(z) = m(\overline{z})$. The Bargmann transform intertwines J_m and $T_{m'}$, i.e., for all $f \in L^2(\mathbb{R}^d)$ (or $f \in (S_{1/2}^{1/2})^*$) we have*

$$\mathcal{B}(J_m f) = T_{m'}(\mathcal{B}f). \tag{43}$$

Proof. This fact is well known [8, 12]. For completeness and consistency of notation, we provide the formal calculation.

We take the short-time Fourier transform of $J_m = A_m^h$ with respect to h. On the one hand we obtain that

$$\langle J_m f, \pi(\overline{w})h\rangle = e^{\pi i u\cdot\eta}\mathcal{B}(J_m f)(w)\, e^{-\pi|w|^2/2}\,, \tag{44}$$

and on the other hand, after substituting (41) and (42), we obtain

$$\begin{aligned}\langle J_m f, \pi(\overline{w})h\rangle &= \int_{\mathbb{R}^{2d}} m(\overline{z}) V_h f(\overline{z})\overline{V_h(\pi(\overline{w})h)(\overline{z})}\, dz \\ &= \int_{\mathbb{C}^d} m(\overline{z})\mathcal{B}f(z)\,\overline{\mathcal{B}(\pi(\overline{w})h)(\overline{z})} e^{-\pi|z|^2}\, dz \\ &= e^{\pi i u\cdot\eta}\int_{\mathbb{C}^d} m(\overline{z})\,\mathcal{B}f(z) e^{\pi\overline{z}w}\, e^{-\pi|z|^2}\, dz\; e^{-\pi|w|^2/2}.\end{aligned} \tag{45}$$

Comparing (44) and (45) we obtain

$$\mathcal{B}(J_m f)(w) = \int_{\mathbb{C}^d} m(\overline{z})\mathcal{B}f(z) e^{\pi\overline{z}w}\, e^{-\pi|z|^2}\, dz = T_{m'}\mathcal{B}f(w). \qquad \square$$

Proof of Theorem 5.4. First assume that (i) holds, i.e., m is radial in each variable. The Bargmann transform is an isomorphism between the modulation space $M_m^{p,q}(\mathbb{R}^d)$ and the Bargmann-Fock space $\mathcal{F}_{m'}^{p,q}(\mathbb{C}^d)$ for arbitrary moderate weight function m and $1 \le p, q \le \infty$. By Theorem 4.3 the anti-Wick operator J_m is an isomorphism from $M_\mu^{p,q}(\mathbb{R}^d)$ onto $M_{\mu/m}^{p,q}(\mathbb{R}^d)$. Since $T_{m'}$ is a composition of three isomorphisms (Proposition 5.5 and (46)), the Toeplitz operator $T_{m'}$ is an isomorphism from $\mathcal{F}_{\mu'}^{p,q}(\mathbb{C}^d)$ onto $\mathcal{F}_{\mu'/m'}^{p,q}(\mathbb{C}^d)$:

$$\begin{array}{ccc} \mathcal{F}_{\mu'}^{p,q} & \xrightarrow{T_{m'}} & \mathcal{F}_{\mu'/m'}^{p,q} \\ \uparrow \mathcal{B} & & \uparrow \mathcal{B} \\ M_\mu^{p,q} & \xrightarrow{J_m} & M_{\mu/m}^{p,q} \end{array}\,. \tag{46}$$

Finally replace m' and μ' by m and μ.

By using Theorem 3.2 in [21] instead of Theorem 4.3, the same arguments show that the result follows when (ii) is fulfilled. $\square$

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