

SURFACES WITH PARALLEL MEAN CURVATURE IN $CP^n \times \mathbb{R}$ AND $CH^n \times \mathbb{R}$

DOREL FETCU AND HAROLD ROSENBERG

ABSTRACT. We consider surfaces with parallel mean curvature vector (pmc surfaces) in $CP^n \times \mathbb{R}$ and $CH^n \times \mathbb{R}$, and, more generally, in cosymplectic space forms. We introduce a holomorphic quadratic differential on such surfaces. This is then used in order to show that the anti-invariant pmc 2-spheres of a 5-dimensional non-flat cosymplectic space form of product type are actually the embedded rotational spheres $S_H^2 \subset \bar{M}^2 \times \mathbb{R}$ of Hsiang and Pedrosa, where \bar{M}^2 is a complete simply-connected surface with constant curvature. When the ambient space is a cosymplectic space form of product type and its dimension is greater than 5, we prove that an immersed non-minimal non-pseudo-umbilical anti-invariant 2-sphere lies in a product space $\bar{M}^4 \times \mathbb{R}$, where \bar{M}^4 is a space form. We also provide a reduction of codimension theorem for the pmc surfaces of a non-flat cosymplectic space form.

1. INTRODUCTION

Surfaces with constant mean curvature (cmc surfaces) in 3-dimensional ambient spaces have been intensively studied in the last six decades, and a very useful tool proved to be holomorphic quadratic forms defined on such surfaces.

In 1951, H. Hopf used for the first time a holomorphic quadratic form in order to show that any cmc surface in Euclidean 3-space, homeomorphic to a sphere, is actually a round sphere (see [20]). His result was then extended to cmc surfaces in 3-dimensional space forms by S.-S. Chern in [11]. When the codimension is greater than 1, a natural generalization of cmc surfaces are surfaces with parallel mean curvature vector (pmc surfaces). These surfaces have been studied since the early seventies. Among the first papers to treat this subject were [16] by D. Ferus, [9] by B.-Y. Chen and G. D. Ludden, [19] by D. A. Hoffman and [25] by S.-T. Yau. In this last paper it was proved that a pmc surface immersed in a space form either lies in a totally geodesic 3-dimensional space or it is a minimal surface of an umbilical hypersurface.

The next natural step was taken by U. Abresch and H. Rosenberg, who studied in [1, 2] cmc surfaces and obtained Hopf type results in product spaces of type $M^2(\rho) \times \mathbb{R}$, where $M^2(\rho)$ is a complete simply-connected surface with constant curvature ρ , as well as the homogeneous 3-manifolds $Nil(3)$, $\widetilde{PSL}(2, \mathbb{R})$ and Berger spheres. As for the study of pmc surfaces in product spaces of type $M^n(\rho) \times \mathbb{R}$,

Received by the editors November 22, 2010 and, in revised form, August 30, 2011.

2010 *Mathematics Subject Classification*. Primary 53A10, 53C42.

Key words and phrases. Surfaces with parallel mean curvature vector, cosymplectic space forms, quadratic forms.

The first author was supported by a Post-Doctoral Fellowship “Pós-Doutorado Sênior (PDS)” offered by FAPERJ, Brazil.

where $M^n(\rho)$ is a space form with constant sectional curvature ρ , the papers [4] and [5] by H. Alencar, M. do Carmo and R. Tribuzy are devoted to this subject. The principal tool they use is a holomorphic quadratic form, which in the 3-dimensional case is just the Abresch-Rosenberg differential, introduced in [1]. In [5] the authors proved, amongst others, a very nice reduction of the codimension theorem, showing that if Σ^2 is a pmc surface immersed in $M^n(\rho) \times \mathbb{R}$, then either

- Σ^2 is a minimal surface in a totally umbilical hypersurface of $M^n(\rho)$ or
- Σ^2 is a cmc surface in a 3-dimensional totally umbilical or in a totally geodesic submanifold of $M^n(\rho)$ or
- Σ^2 lies in $M^4(\rho) \times \mathbb{R}$.

In the recent paper [17] a similar result is proved for pmc surfaces immersed in a complex space form, i.e. a Kähler manifold with constant holomorphic sectional curvature. There it is shown that a non-minimal pmc surface immersed in a non-flat complex space form $N^n(\rho)$, where ρ is the (constant) holomorphic sectional curvature and $n \geq 3$, is a pseudo-umbilical totally real surface or it lies in a complex space form $N^r(\rho)$, with $r \leq 5$.

The products of a complex space form and a 1-dimensional manifold are the main examples of cosymplectic space forms, which are often seen as the odd-dimensional version of complex space forms. Therefore, working in such spaces seems to be the natural continuation of [17]. The other option for odd-dimensional ambient spaces with nice curvature properties is represented by the Sasakian space forms, amongst them being the odd-dimensional spheres and the generalized Heisenberg group. Although the present paper is devoted to the study of pmc surfaces in cosymplectic space forms, it is sure that interesting results could also be obtained by considering this second option.

The paper is organized as follows. In Section 2 we briefly recall some general facts about the cosymplectic space forms as they are presented in [3, 6, 7, 12]. In Section 3 we introduce a quadratic form Q defined on surfaces immersed in such a space and prove that its $(2, 0)$ -part is holomorphic when the mean curvature vector of the surface is parallel. In Section 4 we characterize the pmc surfaces of type $\Sigma^2 = \pi^{-1}(\gamma)$ in a product space $M^n(\rho) \times \mathbb{R}$, where $M^n(\rho)$ is a complex space form, $\pi : M^n(\rho) \times \mathbb{R} \rightarrow M^n(\rho)$ is the projection map and $\gamma : I \rightarrow M^n(\rho)$ is a Frenet curve of osculating order r in $M^n(\rho)$. We also prove that such surfaces with vanishing $(2, 0)$ -part of Q exist if and only if $\rho < 0$. The main result of Section 5 is a reduction theorem, which states that a non-minimal pmc surface Σ^2 in a non-flat cosymplectic space form $N^{2n+1}(\rho)$ is either pseudo-umbilical and then the characteristic vector field is orthogonal to Σ^2 and the surface is anti-invariant, or it is not pseudo-umbilical and lies in a totally geodesic invariant submanifold of $N^{2n+1}(\rho)$ with dimension less than or equal to 11. The last section is devoted to the study of anti-invariant pmc surfaces. We prove that any non-minimal anti-invariant pmc 2-sphere in $M^2(\rho) \times \mathbb{R}$ is an embedded rotationally invariant cmc sphere $S_H^2 \subset \bar{M}^2(\frac{\rho}{4}) \times \mathbb{R}$, where $\bar{M}^2(\frac{\rho}{4})$ is a complete simply-connected surface with constant curvature $\frac{\rho}{4}$, immersed as a totally-geodesic Lagrangian submanifold in the complex space form $M^2(\rho)$. When the dimension of the ambient space is greater than 5, we show that a non-minimal non-pseudo-umbilical anti-invariant 2-sphere immersed in $M^n(\rho) \times \mathbb{R}$ lies in a product space $\bar{M}^4(\frac{\rho}{4}) \times \mathbb{R}$, where $\bar{M}^4(\frac{\rho}{4})$ is a space form immersed as a totally geodesic totally real submanifold in $M^n(\rho)$.

2. PRELIMINARIES

Let $M^n(\rho)$ be a complex space form with the complex structure (J, \langle, \rangle_M) , consider the product manifold $N^{2n+1} = M^n(\rho) \times \mathbb{R}$ and define the following tensors on N^{2n+1} :

$$\varphi = J \circ d\pi, \quad \xi = \frac{\partial}{\partial t}, \quad \eta = dt \quad \text{and} \quad \langle, \rangle_N = \langle, \rangle_M + dt \otimes dt,$$

where $\pi : M^n(\rho) \times \mathbb{R} \rightarrow M^n(\rho)$ is the projection map and t is the standard coordinate function on the real axis. Then $(N^{2n+1}, \varphi, \xi, \eta, \langle, \rangle_N)$ is a cosymplectic space form with constant φ -sectional curvature equal to ρ (see [3, 7]). We shall explain what this means in the following.

An *almost contact metric structure* on an odd-dimensional manifold N^{2n+1} is given by $(\varphi, \xi, \eta, \langle, \rangle)$, where φ is a tensor field of type $(1, 1)$ on N , ξ is a vector field, η is its dual 1-form and \langle, \rangle is a Riemannian metric such that

$$\varphi^2 U = -U + \langle U, \xi \rangle \xi \quad \text{and} \quad \langle \varphi U, \varphi V \rangle = \langle U, V \rangle - \eta(U)\eta(V),$$

for all tangent vector fields U and V .

An almost contact metric structure $(\varphi, \xi, \eta, \langle, \rangle)$ is called *normal* if

$$\mathcal{N}_\varphi(U, V) + 2d\eta(U, V)\xi = 0,$$

where

$$\mathcal{N}_\varphi(U, V) = [\varphi U, \varphi V] - \varphi[\varphi U, V] - \varphi[U, \varphi V] + \varphi^2[U, V]$$

is the Nijenhuis tensor field of φ .

Definition 2.1. An almost contact metric manifold $(N, \varphi, \xi, \eta, \langle, \rangle)$ is a *cosymplectic manifold* if it is normal and both the 1-form η and the fundamental 2-form Ω , defined by $\Omega(U, V) = \langle U, \varphi V \rangle$, are closed.

Equivalently, an almost contact metric manifold is cosymplectic if and only if φ is parallel, i.e. $\nabla^N \varphi = 0$, where ∇^N is the Levi-Civita connection. This implies that the vector field ξ and the 1-form η are also parallel. We note that a cosymplectic manifold has a natural local product structure as a product of a Kähler manifold and a 1-dimensional manifold, but there exist compact cosymplectic manifolds which are not global products (see [6, 12]). We also recall that a submanifold M of a cosymplectic manifold is called *invariant* when $\varphi(TM) \subset TM$ and *anti-invariant* when $\varphi(TM) \subset NM$, where NM is the normal bundle of M .

Let $(N, \varphi, \xi, \eta, \langle, \rangle)$ be a cosymplectic manifold. The sectional curvature of a 2-plane generated by U and φU , where U is a unit vector orthogonal to ξ , is called a φ -*sectional curvature* determined by U . A cosymplectic manifold with constant φ -sectional curvature ρ is called a *cosymplectic space form* and is denoted by $N(\rho)$. The curvature tensor field of a cosymplectic space form $N(\rho)$ is given by

$$\begin{aligned} R^N(U, V)W &= \frac{\rho}{4} \{ \langle V, W \rangle U - \langle U, W \rangle V + \langle U, \varphi W \rangle \varphi V - \langle V, \varphi W \rangle \varphi U \\ &+ 2\langle U, \varphi V \rangle \varphi W + \eta(U)\eta(W)V - \eta(V)\eta(W)U \\ &+ \langle U, W \rangle \eta(V)\xi - \langle V, W \rangle \eta(U)\xi \}. \end{aligned} \tag{2.1}$$

3. A QUADRATIC FORM WITH HOLOMORPHIC $(2, 0)$ -PART

Although our main interest is to study immersed pmc surfaces in product spaces of type $M^n(\rho) \times \mathbb{R}$, where $M^n(\rho)$ is a complex space form, it is more convenient to treat the more general case where the surfaces are immersed in an arbitrary cosymplectic space form.

Let Σ^2 be an immersed surface in a cosymplectic space form $N^{2n+1}(\rho)$, endowed with the cosymplectic structure $(\varphi, \xi, \eta, \langle \cdot, \cdot \rangle)$ with constant φ -sectional curvature ρ . The second fundamental form σ of the surface is defined by the equation of Gauss

$$\nabla_X^N Y = \nabla_X Y + \sigma(X, Y),$$

while the shape operator A and the normal connection ∇^\perp are given by the equation of Weingarten

$$\nabla_X^N V = -A_V X + \nabla_X^\perp V,$$

for any vector fields X and Y tangent to the surface and any vector field V normal to Σ^2 , where ∇^N and ∇ are the Levi-Civita connections of $N^{2n+1}(\rho)$ and Σ^2 , respectively.

Definition 3.1. If the mean curvature vector \vec{H} of the surface Σ^2 is parallel in the normal bundle, i.e. $\nabla^\perp \vec{H} = 0$, then Σ^2 is called a *pmc surface*.

Theorem 3.2. *If Σ^2 is an immersed pmc surface in a cosymplectic space form $N^{2n+1}(\rho)$, then the $(2, 0)$ -part of the quadratic form Q , defined on Σ^2 by*

$$Q(X, Y) = 8|\vec{H}|^2 \langle \sigma(X, Y), \vec{H} \rangle - \rho |\vec{H}|^2 \eta(X) \eta(Y) + 3\rho \langle \varphi X, \vec{H} \rangle \langle \varphi Y, \vec{H} \rangle,$$

is holomorphic.

Proof. We first consider isothermal coordinates (u, v) on the surface. Then we have $ds^2 = \lambda^2(du^2 + dv^2)$, and let us define $z = u + iv$, $\hat{z} = u - iv$, $dz = \frac{1}{\sqrt{2}}(du + idv)$, $d\hat{z} = \frac{1}{\sqrt{2}}(du - idv)$ and

$$Z = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \hat{Z} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

We get $\langle Z, \hat{Z} \rangle = \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \rangle = \langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \rangle = \lambda^2$. We mention that this rather unusual notation for conjugation is used only for the reader's convenience.

Now, we compute

$$\hat{Z}(Q(Z, Z)) = \hat{Z}(8|\vec{H}|^2 \langle \sigma(Z, Z), \vec{H} \rangle - \rho |\vec{H}|^2 (\eta(Z))^2 + 3\rho \langle \varphi Z, \vec{H} \rangle^2).$$

We have

$$\begin{aligned} \hat{Z}(\langle \sigma(Z, Z), \vec{H} \rangle) &= \langle \nabla_{\hat{Z}}^N \sigma(Z, Z), \vec{H} \rangle + \langle \sigma(Z, Z), \nabla_{\hat{Z}}^N \vec{H} \rangle \\ &= \langle \nabla_{\hat{Z}}^\perp \sigma(Z, Z), \vec{H} \rangle + \langle \sigma(Z, Z), \nabla_{\hat{Z}}^\perp \vec{H} \rangle \\ &= \langle (\nabla_{\hat{Z}}^\perp \sigma)(Z, Z), \vec{H} \rangle + \langle \sigma(Z, Z), \nabla_{\hat{Z}}^\perp \vec{H} \rangle, \end{aligned}$$

since

$$(\nabla_{\hat{Z}}^\perp \sigma)(Z, Z) = \nabla_{\hat{Z}}^\perp \sigma(Z, Z) - 2\sigma(\nabla_{\hat{Z}} Z, Z) = \nabla_{\hat{Z}}^\perp \sigma(Z, Z)$$

and $\nabla_{\hat{Z}} Z = 0$, from the definition of the connection ∇ on the surface.

Next, using the Codazzi equation, we get

$$\begin{aligned}
 \widehat{Z}(\langle \sigma(Z, Z), \vec{H} \rangle) &= \langle (\nabla_{\widehat{Z}} \sigma)(\widehat{Z}, Z), \vec{H} \rangle + \langle (R^N(\widehat{Z}, Z)Z)^\perp, \vec{H} \rangle \\
 &\quad + \langle \sigma(Z, Z), \nabla_{\widehat{Z}}^\perp \vec{H} \rangle \\
 (3.1) \qquad &= \langle (\nabla_{\widehat{Z}} \sigma)(\widehat{Z}, Z), \vec{H} \rangle + \langle R^N(\widehat{Z}, Z)Z, \vec{H} \rangle \\
 &\quad + \langle \sigma(Z, Z), \nabla_{\widehat{Z}}^\perp \vec{H} \rangle.
 \end{aligned}$$

From formula (2.1), it follows that

$$(3.2) \quad \langle R^N(\widehat{Z}, Z)Z, \vec{H} \rangle = \frac{\rho}{4} \{ \langle Z, \widehat{Z} \rangle \eta(Z) \eta(\vec{H}) + 3 \langle \widehat{Z}, \varphi Z \rangle \langle \vec{H}, \varphi Z \rangle \}.$$

Working as in [5] (or in [17]), we can prove that

$$(3.3) \quad \langle (\nabla_{\widehat{Z}} \sigma)(\widehat{Z}, Z), \vec{H} \rangle = \langle \widehat{Z}, Z \rangle \langle \nabla_{\widehat{Z}}^\perp \vec{H}, \vec{H} \rangle.$$

Indeed, if we consider the unit vector fields e_1 and e_2 corresponding to $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$, respectively, then we get $Z = \frac{\lambda}{\sqrt{2}}(e_1 - ie_2)$ and

$$\sigma(\widehat{Z}, Z) = \frac{\lambda^2}{2} \sigma(e_1 - ie_2, e_1 + ie_2) = \frac{\lambda^2}{2} (\sigma(e_1, e_1) + \sigma(e_2, e_2)) = \langle \widehat{Z}, Z \rangle \vec{H}.$$

Since we also have $\nabla_Z Z = \frac{1}{\lambda^2} \langle \nabla_Z Z, \widehat{Z} \rangle Z$, it follows that

$$\begin{aligned}
 \langle (\nabla_{\widehat{Z}} \sigma)(\widehat{Z}, Z), \vec{H} \rangle &= \langle \nabla_{\widehat{Z}}^N \sigma(\widehat{Z}, Z), \vec{H} \rangle - \langle \sigma(\nabla_Z \widehat{Z}, Z), \vec{H} \rangle - \langle \sigma(\widehat{Z}, \nabla_Z Z), \vec{H} \rangle \\
 &= \langle \nabla_{\widehat{Z}}^N (\langle \widehat{Z}, Z \rangle \vec{H}), \vec{H} \rangle - \frac{1}{\lambda^2} \langle \nabla_Z Z, \widehat{Z} \rangle \langle \sigma(\widehat{Z}, Z), \vec{H} \rangle \\
 &= \langle \nabla_{\widehat{Z}}^N (\langle \widehat{Z}, Z \rangle \vec{H}), \vec{H} \rangle - \langle \nabla_Z Z, \widehat{Z} \rangle \langle \vec{H}, \vec{H} \rangle \\
 &= \langle \nabla_Z \widehat{Z}, Z \rangle \langle \vec{H}, \vec{H} \rangle + \langle \nabla_Z Z, \widehat{Z} \rangle \langle \vec{H}, \vec{H} \rangle \\
 &\quad + \langle \widehat{Z}, Z \rangle \langle \nabla_{\widehat{Z}}^\perp \vec{H}, \vec{H} \rangle - \langle \nabla_Z Z, \widehat{Z} \rangle \langle \vec{H}, \vec{H} \rangle \\
 &= \langle \widehat{Z}, Z \rangle \langle \nabla_{\widehat{Z}}^\perp \vec{H}, \vec{H} \rangle.
 \end{aligned}$$

Replacing (3.2) and (3.3) in (3.1), and using the fact that \vec{H} is parallel, it follows that

$$(3.4) \quad \widehat{Z}(\langle \sigma(Z, Z), \vec{H} \rangle) = \frac{\rho}{4} \{ \langle Z, \widehat{Z} \rangle \eta(Z) \eta(\vec{H}) + 3 \langle \widehat{Z}, \varphi Z \rangle \langle \vec{H}, \varphi Z \rangle \}.$$

As the characteristic vector field ξ is parallel, equation (3.3) also implies that

$$(3.5) \quad \widehat{Z}((\eta(Z))^2) = 2 \langle Z, \widehat{Z} \rangle \eta(Z) \eta(\vec{H}).$$

Finally, since $\nabla^N \varphi = 0$ and \vec{H} is parallel, using $\nabla_{\frac{\widehat{Z}}{2}}^N Z = \sigma(\widehat{Z}, Z) \vec{H}$ and $(\varphi Z)^\top = \frac{1}{\lambda^2} \langle \varphi Z, \widehat{Z} \rangle Z$ (that can be easily checked), one obtains

$$\begin{aligned}
 \widehat{Z}(\langle \varphi Z, \vec{H} \rangle^2) &= 2\langle \varphi Z, \vec{H} \rangle \{ \langle \nabla_{\frac{\widehat{Z}}{2}}^N \varphi Z, \vec{H} \rangle + \langle \varphi Z, \nabla_{\frac{\widehat{Z}}{2}}^N \vec{H} \rangle \} \\
 &= 2\langle \varphi Z, \vec{H} \rangle \{ \langle \varphi \nabla_{\frac{\widehat{Z}}{2}}^N Z, \vec{H} \rangle + \langle \varphi Z, \nabla_{\frac{\widehat{Z}}{2}}^N \vec{H} \rangle \} \\
 &= 2\langle \varphi Z, \vec{H} \rangle \{ \langle \widehat{Z}, Z \rangle \langle \varphi \vec{H}, \vec{H} \rangle - \langle (\varphi Z)^\top, A_{\vec{H}} \widehat{Z} \rangle \\
 (3.6) \quad &+ \langle (\varphi Z)^\perp, \nabla_{\frac{\widehat{Z}}{2}} \vec{H} \rangle \} \\
 &= -2\langle \varphi Z, \vec{H} \rangle \langle \sigma((\varphi Z)^\top, \widehat{Z}), \vec{H} \rangle \\
 &= -2\langle \varphi Z, \vec{H} \rangle \langle \varphi Z, \widehat{Z} \rangle |\vec{H}|^2.
 \end{aligned}$$

From (3.4), (3.5) and (3.6) we see that $\widehat{Z}(Q(Z, Z)) = 0$, which proves Theorem 3.2. \square

4. SOME EXAMPLES OF PMC SURFACES IN PRODUCT SPACES

Let $\gamma : I \subset \mathbb{R} \rightarrow M^n(\rho)$ be a curve parametrized by arc-length in a complex space form with complex dimension n and constant holomorphic sectional curvature ρ , i.e. $\mathbb{C}P^n(\rho)$, \mathbb{C}^n or $\mathbb{C}H^n(\rho)$ as $\rho > 0$, $\rho = 0$ or $\rho < 0$. The curve γ is called a *Frenet curve of osculating order* r , $1 \leq r \leq 2n$, if there exist r orthonormal vector fields $\{E_1 = \gamma', \dots, E_r\}$ along γ such that

$$(4.1) \quad \begin{cases} \nabla_{E_1}^M E_1 = \kappa_1 E_2, \\ \nabla_{E_1}^M E_i = -\kappa_{i-1} E_{i-1} + \kappa_i E_{i+1}, \quad \forall i = 2, \dots, r-1, \\ \nabla_{E_1}^M E_r = -\kappa_{r-1} E_{r-1}, \end{cases}$$

where $\{\kappa_1, \kappa_2, \kappa_3, \dots, \kappa_{r-1}\}$ are positive functions on I called the *curvatures* of γ and ∇^M denotes the Levi-Civita connection on $M^n(\rho)$.

A Frenet curve of osculating order r is called a *helix of order* r if $\kappa_i = \text{constant} > 0$ for $1 \leq i \leq r-1$. A helix of order 2 is called a *circle*, and a helix of order 3 is simply called a *helix*.

S. Maeda and Y. Ohnita defined in [23] the *complex torsions* of the curve γ by $\tau_{ij} = \langle E_i, J E_j \rangle$, $1 \leq i < j \leq r$, where (J, \langle, \rangle) is the complex structure on $M^n(\rho)$. A helix of order r is called a *holomorphic helix of order* r if all the complex torsions are constant. It is easy to see that a circle is always a holomorphic circle.

In order to find examples of pmc surfaces, we will focus our attention on the *vertical cylinders* $\Sigma^2 = \pi^{-1}(\gamma)$ in product spaces $M^n(\rho) \times \mathbb{R}$, where $\pi : M^n(\rho) \times \mathbb{R} \rightarrow M^n(\rho)$ is the projection map and $\gamma : I \rightarrow M^n(\rho)$ is a Frenet curve of osculating order r in $M^n(\rho)$. For any vector field X tangent to $M^n(\rho)$, we shall denote by X^H its horizontal lift to $M^n(\rho) \times \mathbb{R}$. As for the Riemannian metrics on $M^n(\rho)$ and $M^n(\rho) \times \mathbb{R}$, we will use the same notation \langle, \rangle .

Obviously, $\{E_1^H, \xi\}$ is a local orthonormal frame on Σ^2 and E_i^H , $1 < i \leq r$, are normal vector fields. Then the mean curvature vector \vec{H} is given by

$$\vec{H} = \frac{1}{2}(\sigma(E_1^H, E_1^H) + \sigma(\xi, \xi)) = \frac{1}{2}\kappa_1 E_2^H,$$

where $\kappa_1 = \kappa_1 \circ \pi$ and we used the first Frenet equation for γ and O'Neill's equation [24] in the case of cosymplectic space forms, i.e. $\nabla_{X^H}^N Y^H = (\nabla_X^M Y)^H$, for any vector fields X and Y tangent to $M^n(\rho)$ (see also [3]).

Next, from the second Frenet equation, we have

$$(4.2) \quad \nabla_{E_1^H}^N \vec{H} = \frac{1}{2}(\nabla_{E_1}^M(\kappa_1 E_2))^H = \frac{1}{2}(\kappa_1' E_2 - \kappa_1^2 E_1 + \kappa_1 \kappa_2 E_3)^H.$$

It is easy to verify that $\nabla_\xi E_1^H = \nabla_{E_1^H} \xi = 0$, where ∇ is the connection on the surface, and then we get that $[\xi, E_1^H] = 0$, which means $\nabla_\xi^N E_1^H = \nabla_{E_1^H}^N \xi = 0$. Now, since from (2.1) it follows that $R^N(\xi, E_1^H)E_1^H = 0$, we obtain

$$(4.3) \quad \nabla_\xi^N \vec{H} = \frac{1}{2} \nabla_\xi^N \nabla_{E_1^H}^N E_1^H = 0.$$

From (4.2) and (4.3) we see that \vec{H} is parallel if and only if either

- γ is a geodesic in $M^n(\rho)$ or
- γ is a circle in $M^n(\rho)$ with the curvature $\kappa_1 = 2|\vec{H}| = \text{constant} > 0$.

Obviously, in the first case, Σ^2 is a minimal surface. In the second case, the $(2, 0)$ -part of Q vanishes if and only if

$$16|\vec{H}|^4 + \rho|\vec{H}|^2 + 3\rho\langle\varphi E_1^H, \vec{H}\rangle^2 = 0,$$

that is equivalent to

$$4\kappa_1^2 + \rho(1 + 3\tau_{12}^2) = 0.$$

Now, we can conclude.

Proposition 4.1. *A vertical cylinder $\Sigma^2 = \pi^{-1}(\gamma)$ in $M^n(\rho) \times \mathbb{R}$ has non-zero parallel mean curvature vector, and the $(2, 0)$ -part of the quadratic form Q vanishes on Σ^2 if and only if $\rho < 0$ and the curve γ is a circle in $M^n(\rho)$ with curvature $\kappa = \frac{1}{2}\sqrt{-\rho(1 + 3\tau^2)}$, where τ is the complex torsion of γ .*

Remark 4.2. S. Maeda and T. Adachi proved in [22] that for any positive number κ and for any number τ , such that $|\tau| < 1$, there exists a circle with curvature κ and complex torsion τ in any complex space form. Therefore, for any $\rho < 0$, we know that circles γ , as in the previous proposition, do exist. Since $0 \leq \tau^2 \leq 1$ we get that $\frac{1}{2}\sqrt{-\rho} \leq \kappa \leq \sqrt{-\rho}$, which means that the mean curvature of a non-minimal pmc cylinder $\Sigma^2 = \pi^{-1}(\gamma)$, with vanishing $(2, 0)$ -part of Q , satisfies $\frac{\sqrt{-\rho}}{4} \leq |\vec{H}| \leq \frac{\sqrt{-\rho}}{2}$.

5. A REDUCTION THEOREM

Let Σ^2 be an immersed non-minimal pmc surface in a non-flat cosymplectic space form $N^{2n+1}(\rho)$, $n \geq 2$.

Lemma 5.1. *For any vector V normal to Σ^2 , which is also orthogonal to $\varphi T\Sigma^2$ and to $\varphi\vec{H}$, we have $[A_{\vec{H}}, A_V] = 0$, i.e. $A_{\vec{H}}$ commutes with A_V .*

Proof. The conclusion follows easily from the Ricci equation

$$\langle R^\perp(X, Y)\vec{H}, V \rangle = \langle [A_{\vec{H}}, A_V]X, Y \rangle + \langle R^N(X, Y)\vec{H}, V \rangle,$$

since

$$\begin{aligned} \langle R^N(X, Y)\vec{H}, V \rangle &= \frac{\rho}{4} \{ \langle X, \varphi\vec{H} \rangle \langle \varphi Y, V \rangle - \langle Y, \varphi\vec{H} \rangle \langle \varphi X, V \rangle + 2 \langle X, \varphi Y \rangle \langle \varphi\vec{H}, V \rangle \} \\ &= 0 \end{aligned}$$

and $R^\perp(X, Y)\vec{H} = 0$. □

Corollary 5.2. *Either \vec{H} is an umbilical direction or there exists a basis that diagonalizes $A_{\vec{H}}$ and A_V simultaneously, for all normal vectors V satisfying $V \perp \varphi(T\Sigma^2)$ and $V \perp \varphi\vec{H}$.*

5.1. \vec{H} is an umbilical direction. In the following, we shall assume that \vec{H} is an umbilical direction everywhere, which means that our surface is pseudo-umbilical, i.e. $A_{\vec{H}} = |\vec{H}|^2 I$. For such a surface, since \vec{H} is also parallel, we have

$$\begin{aligned} R^N(X, Y)\vec{H} &= \nabla_X \nabla_Y \vec{H} - \nabla_Y \nabla_X \vec{H} - \nabla_{[X, Y]}\vec{H} \\ &= -|\vec{H}|^2 (\nabla_X Y - \nabla_Y X - [X, Y]) = 0, \end{aligned}$$

for any tangent vector fields X and Y .

We shall prove that, in this case, $\xi \perp T\Sigma^2$ and $\varphi(T\Sigma^2) \subset N\Sigma^2$, where $N\Sigma^2$ is the normal bundle of the surface. First, we have

Lemma 5.3. *Under the hypothesis that \vec{H} is an umbilical direction everywhere, the following four relations are equivalent:*

- (i) $\xi \perp T\Sigma^2$;
- (ii) $\vec{H} \perp \xi$;
- (iii) $\varphi(T\Sigma^2) \subset N\Sigma^2$;
- (iv) $\varphi\vec{H} \perp T\Sigma^2$.

Proof. Let us again consider isothermal coordinates (u, v) on the surface, and the complex vector fields $Z = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right)$ and $\widehat{Z} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)$.

Since \vec{H} is umbilical, we have $\langle \sigma(Z, Z), \vec{H} \rangle = 0$ and, consequently, the $(2, 0)$ -part of Q is, in this case,

$$Q(Z, Z) = -\rho |\vec{H}|^2 (\eta(Z))^2 + 3\rho \langle \varphi Z, \vec{H} \rangle^2,$$

where Z and its conjugate \widehat{Z} are the complex vectors on Σ^2 , defined in Section 3.

Since $Q(Z, Z)$ is holomorphic and \vec{H} is umbilical and parallel, it follows that

$$\langle Z, \widehat{Z} \rangle \eta(Z) \eta(\vec{H}) + 3 \langle \varphi Z, \vec{H} \rangle \langle \varphi Z, \widehat{Z} \rangle = 0.$$

Now, it is easy to see that $\eta(Z) \eta(\vec{H}) = 0$ is equivalent to $\langle \varphi Z, \vec{H} \rangle \langle \varphi Z, \widehat{Z} \rangle = 0$, and then we only have to prove the equivalence between (i) and (ii) and between (iii) and (iv), respectively.

First, we prove (i) is equivalent to (ii). If $\eta(Z) = 0$, then $0 = \eta(\nabla_{\widehat{Z}}^N Z) = \langle Z, \widehat{Z} \rangle \eta(\vec{H})$, since $N^{2n+1}(\rho)$ is a cosymplectic space form and $\nabla_{\widehat{Z}}^N Z = \langle Z, \widehat{Z} \rangle \vec{H}$. Conversely, if $\eta(\vec{H}) = 0$, we have

$$\eta(\nabla_{\widehat{Z}}^N \vec{H}) = -\eta(A_{\vec{H}} Z) = -|\vec{H}|^2 \eta(Z) = 0.$$

Next, we prove (iii) is equivalent to (iv). Since $R^N(X, Y)\vec{H} = 0$, for any tangent vector fields X and Y , we have

$$0 = R^N(\widehat{Z}, Z)\vec{H} = \frac{\rho}{4}\{\langle\varphi\vec{H}, \widehat{Z}\rangle\varphi Z - \langle\varphi H, Z\rangle\varphi\widehat{Z} + \langle\varphi Z, \widehat{Z}\rangle\varphi\vec{H} \\ + \eta(\widehat{Z})\eta(\vec{H})Z - \eta(Z)\eta(\vec{H})\widehat{Z}\}.$$

Assume that relation (iii) holds, i.e. that $\langle\varphi Z, \widehat{Z}\rangle = 0$. As we have seen, this also implies $\eta(Z) = \eta(\widehat{Z}) = 0$ and $\eta(\vec{H}) = 0$. Then, by using the definition of the cosymplectic structure on $N^{2n+1}(\rho)$, we have

$$0 = \langle R^N(\widehat{Z}, Z)\vec{H}, \varphi Z \rangle = -\frac{\rho}{4}\langle Z, \widehat{Z} \rangle \langle \varphi\vec{H}, Z \rangle.$$

Conversely, if (iv) holds, i.e. if $\langle\varphi\vec{H}, Z\rangle = 0$, we have

$$0 = \langle \nabla_{\widehat{Z}}\varphi\vec{H}, Z \rangle = \langle \varphi\nabla_{\widehat{Z}}\vec{H}, Z \rangle + \langle \varphi\vec{H}, \nabla_{\widehat{Z}}Z \rangle = -\langle \varphi A_{\vec{H}}\widehat{Z}, Z \rangle + \langle Z, \widehat{Z} \rangle \langle \varphi\vec{H}, \vec{H} \rangle \\ = |\vec{H}|^2 \langle \varphi Z, \widehat{Z} \rangle,$$

which completes the proof of Lemma 5.3. \square

We choose an orthonormal frame $\{e_1, e_2\}$ on Σ^2 such that $e_1 \perp \xi$, i.e. $\eta(e_1) = 0$.

Lemma 5.4. *If the relations (i)-(iv) in Lemma 5.3 do not hold on Σ^2 , then we have*

- (1) $2|\vec{H}|^2\langle\varphi e_2, e_1\rangle = \langle\varphi\vec{H}, \sigma(e_1, e_2)\rangle$;
- (2) $\langle\varphi\vec{H}, \sigma(e_1, e_1)\rangle = \langle\varphi\vec{H}, \sigma(e_2, e_2)\rangle = 0$;
- (3) $\nabla_{e_2}e_2 = \nabla_{e_1}e_1 = 0$;
- (4) $\eta(\sigma(e_1, e_2)) = 0$ and $\langle\varphi e_1, \sigma(e_1, e_2)\rangle = 0$.

Proof. Let us assume that relations (i)-(iv) do not hold on our surface. Then, from $\langle R^N(e_1, e_2)\vec{H}, e_2 \rangle = 0$, we obtain

$$\langle\varphi e_2, e_1\rangle\langle\varphi\vec{H}, e_2\rangle = 0,$$

which means that $\langle\varphi\vec{H}, e_2\rangle = 0$, and then $R^N(e_1, e_2)\vec{H} = 0$ can be written as

$$(5.1) \quad 2\langle\varphi e_2, e_1\rangle\varphi\vec{H} + \langle\varphi\vec{H}, e_1\rangle\varphi e_2 - \eta(e_2)\eta(\vec{H})e_1 = 0.$$

We take the inner product of this equation with $\varphi\vec{H}$, e_1 and φe_2 , respectively, and obtain

$$(5.2) \quad \langle\varphi e_2, e_1\rangle\langle\varphi\vec{H}, \varphi\vec{H}\rangle = \eta(e_2)\eta(\vec{H})\langle\varphi\vec{H}, e_1\rangle,$$

$$(5.3) \quad 3\langle\varphi e_2, e_1\rangle\langle\varphi\vec{H}, e_1\rangle = \eta(e_2)\eta(\vec{H}),$$

and

$$(5.4) \quad 3\langle\varphi e_2, e_1\rangle\eta(e_2)\eta(\vec{H}) = \langle\varphi\vec{H}, e_1\rangle\langle\varphi e_2, \varphi e_2\rangle.$$

Since $\langle\varphi e_2, e_1\rangle \neq 0$ and $\langle\varphi\vec{H}, e_1\rangle \neq 0$, from the first two equations we get

$$(5.5) \quad \langle\varphi\vec{H}, \varphi\vec{H}\rangle = |\vec{H}|^2 - (\eta(\vec{H}))^2 = 3\langle\varphi\vec{H}, e_1\rangle^2,$$

and, from the last two,

$$(5.6) \quad \langle\varphi e_2, \varphi e_2\rangle = 1 - (\eta(e_2))^2 = 9\langle\varphi e_2, e_1\rangle^2.$$

Now, from equation (5.5), it follows that

$$2\langle\varphi\vec{H}, \varphi\nabla_{e_2}^N\vec{H}\rangle = 6\langle\varphi\vec{H}, e_1\rangle(\langle\varphi\nabla_{e_2}^N\vec{H}, e_1\rangle + \langle\varphi\vec{H}, \nabla_{e_2}^N e_1\rangle).$$

But we also know that $\nabla_{e_2}^N\vec{H} = -|\vec{H}|^2 e_2$ and, since $\langle\varphi\vec{H}, e_2\rangle = 0$, that $\langle\varphi\vec{H}, \nabla_{e_2}^N e_1\rangle = 0$. Replacing in the above equation and using equation (5.3) one obtains

$$2|\vec{H}|^2\langle\varphi e_2, e_1\rangle = \langle\varphi\vec{H}, \sigma(e_1, e_2)\rangle.$$

In the same manner, from equation (5.5), we obtain $\langle\varphi\vec{H}, \sigma(e_1, e_1)\rangle = 0$, and then $\langle\varphi\vec{H}, \sigma(e_2, e_2)\rangle = 0$.

As $\langle\varphi\vec{H}, e_2\rangle = 0$ we get $\langle\varphi\vec{H}, \nabla_{e_2}^N e_2\rangle = 0$, which implies

$$\langle\varphi\vec{H}, \nabla_{e_2}^N e_2\rangle = \langle\varphi\vec{H}, e_1\rangle\langle\nabla_{e_2}^N e_2, e_1\rangle = 0,$$

meaning that $\nabla_{e_2}^N e_2 = 0$. Since $e_1 \perp e_2$, we also have $\nabla_{e_2}^N e_1 = 0$.

Finally, $\eta(e_1) = 0$ and $\nabla^N \xi = 0$ imply $\eta(\nabla_{e_2}^N e_1) = 0$. Since $\nabla_{e_2}^N e_1 = 0$ it follows that $\eta(\sigma(e_1, e_2)) = 0$. Then the last identity in our lemma follows easily by taking the inner product of (5.1) with $\varphi\sigma(e_1, e_2)$. \square

Proposition 5.5. *Let Σ^2 be an immersed non-minimal pmc surface in a non-flat cosymplectic space form $N^{2n+1}(\rho)$, $n \geq 2$. If the mean curvature vector \vec{H} is an umbilical direction everywhere, then $\xi \perp T\Sigma^2$, $\varphi(T\Sigma^2) \subset N\Sigma^2$ and $n \geq 3$. Moreover, $\vec{H} \perp \xi$ and $\varphi\vec{H} \perp T\Sigma^2$.*

Proof. We shall assume that (i)–(iv) in Lemma 5.3 do not hold on Σ^2 and we will come to a contradiction. From the expression of the curvature tensor R^N it can be easily checked that R^N is parallel, i.e. $\nabla^N R^N = 0$. Therefore, we have $(\nabla_{e_1}^N R^N)(e_1, e_2, \vec{H}) = 0$ and then, since $R^N(X, Y)\vec{H} = 0$ for any tangent vectors X and Y , one obtains

$$|\vec{H}|^2 R^N(e_1, e_2, e_1) - R^N(\sigma(e_1, e_1), e_2, \vec{H}) - R^N(e_1, \sigma(e_1, e_2), \vec{H}) = 0.$$

By using (2.1), (5.1) and Lemma 5.4, the above equation becomes, after a straightforward computation,

$$\eta(\sigma(e_1, e_1))\eta(\vec{H})e_2 - \eta(e_2)\eta(\vec{H})\sigma(e_1, e_1) + \langle\varphi\vec{H}, e_1\rangle\varphi\sigma(e_1, e_2) - 5|\vec{H}|^2\langle\varphi e_2, e_1\rangle\varphi e_1 = 0,$$

and, by taking the inner product with e_2 , we obtain that

$$(5.7) \quad \eta(\sigma(e_1, e_1))\eta(\vec{H}) + 9|\vec{H}|^2\langle\varphi e_2, e_1\rangle^2 = 0.$$

Next, from equations (5.3), (5.5) and (5.6), it follows that

$$3|\vec{H}|^2\langle\varphi e_2, e_1\rangle^2 = (1 - 6\langle\varphi e_2, e_1\rangle^2)(\eta(\vec{H}))^2.$$

Hence, replacing in (5.7), we get $\eta(\sigma(e_1, e_1)) = 3\eta(\vec{H})(6\langle\varphi e_2, e_1\rangle^2 - 1)$ and then $\eta(\sigma(e_2, e_2)) = \eta(\vec{H})(5 - 18\langle\varphi e_2, e_1\rangle^2)$, which means that

$$(5.8) \quad \eta(\nabla_{e_2}^N e_2) = \eta(\vec{H})(5 - 18\langle\varphi e_2, e_1\rangle^2),$$

since $\nabla_{e_2}^N e_2 = 0$.

From equation (5.6), we obtain $2\eta(e_2)\eta(\nabla_{e_2}^N e_2) = -18\langle\varphi e_2, e_1\rangle e_2(\langle\varphi e_2, e_1\rangle)$, and then, from (5.8) and (5.3), we conclude

$$(5.9) \quad 3e_2(\langle\varphi e_2, e_1\rangle) = (18\langle\varphi e_2, e_1\rangle^2 - 5)\langle\varphi\vec{H}, e_1\rangle.$$

Finally, we differentiate equation (5.3), and, using equations (5.8) and (5.9), the fact that \vec{H} is both umbilical and parallel, and Lemma 5.4, we obtain

$$|\vec{H}|^2 + (5 - 18\langle\varphi e_2, e_1\rangle^2)(\eta(\vec{H}))^2 = 0.$$

But, from equation (5.6), we know that $9\langle\varphi e_2, e_1\rangle^2 < 1$. Therefore, the last equation is a contradiction.

Thus, we have that $\xi \perp T\Sigma^2$, $\varphi(T\Sigma^2) \subset N\Sigma^2$, $\vec{H} \perp \xi$ and $\varphi\vec{H} \perp T\Sigma^2$. Now, it is easy to see that if $\{e_1, e_2\}$ is an orthonormal frame on Σ^2 , then, at any point on the surface, the system $\{e_1, e_2, \varphi e_1, \varphi e_2, \vec{H}, \varphi\vec{H}, \xi\}$ is linearly independent, which means that $n \geq 3$. \square

Let $N^{2n+1}(\rho)$ be the product between a non-flat complex space form $M^n(\rho)$, with complex dimension n , and \mathbb{R} . If Σ^2 is an immersed surface in $N^{2n+1}(\rho)$ as in the previous proposition, it follows that Σ^2 is a totally real surface in $M^n(\rho)$. Moreover, since $N^{2n+1}(\rho)$ is a product space, we have $\nabla_{\frac{N}{Z}}^N Z = \nabla_{\frac{M}{Z}}^M Z$, $\nabla_{\frac{N}{Z}}^N Z = \nabla_{\frac{M}{Z}}^M Z$ and $\nabla_X^N \vec{H} = \nabla_X^M \vec{H}$, for any vector field X tangent to Σ^2 , where we have used the fact that $\vec{H} \perp \xi$. From these identities, we obtain that the surface is pseudo-umbilical and with parallel mean curvature vector in $M^n(\rho)$. Hence, we have

Corollary 5.6. *Let Σ^2 be an immersed non-minimal pmc surface in $M^n(\rho) \times \mathbb{R}$, $n \geq 2$, $\rho \neq 0$. If its mean curvature vector is an umbilical direction everywhere, then Σ^2 is a pseudo-umbilical non-minimal totally real pmc surface in $M^n(\rho)$, and $n \geq 3$.*

Remark 5.7. If the mean curvature vector of the surface Σ^2 is umbilical everywhere, then the $(2, 0)$ -part of the quadratic form Q defined on Σ^2 vanishes.

5.2. \vec{H} is not an umbilical direction. The next step is to study the case when the mean curvature vector of the surface is nowhere umbilical. We shall prove that such a surface lies in a totally geodesic submanifold of $N^{2n+1}(\rho)$, with dimension less than or equal to 11.

Proposition 5.8. *Assume that \vec{H} is nowhere an umbilical direction. Then there exists a parallel subbundle of the normal bundle that contains the image of the second fundamental form σ and has dimension less than or equal to 9.*

Proof. We consider a subbundle L of the normal bundle, defined by

$$L = \text{span}\{\text{Im } \sigma \cup (\varphi(\text{Im } \sigma))^\perp \cup (\varphi(T\Sigma^2))^\perp \cup \xi^\perp\},$$

where $(\varphi(T\Sigma^2))^\perp = \{(\varphi X)^\perp : X \text{ tangent to } \Sigma^2\}$, $(\varphi(\text{Im } \sigma))^\perp = \{(\varphi\sigma(X, Y))^\perp : X, Y \text{ tangent to } \Sigma^2\}$ and ξ^\perp is the normal component of ξ along the surface. We will show that L is parallel.

First, we have to prove that if V is orthogonal to L , then $\nabla_{e_i}^\perp V$ is orthogonal to $\varphi(T\Sigma^2)$ and to $\varphi\vec{H}$, where $\{e_1, e_2\}$ is a frame satisfying

$$\langle\sigma(e_1, e_2), V\rangle = \langle\sigma(e_1, e_2), \vec{H}\rangle = 0.$$

Indeed, one has

$$\begin{aligned}
\langle (\varphi \vec{H})^\perp, \nabla_{e_i}^\perp V \rangle &= \langle (\varphi \vec{H})^\perp, \nabla_{e_i}^N V \rangle = -\langle \nabla_{e_i}^N (\varphi \vec{H})^\perp, V \rangle \\
&= -\langle \nabla_{e_i}^N \varphi \vec{H}, V \rangle + \langle \nabla_{e_i}^N (\varphi \vec{H})^\top, V \rangle \\
&= \langle \varphi A_{\vec{H}} e_i, V \rangle + \langle \sigma(e_i, (\varphi \vec{H})^\top), V \rangle \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\langle (\varphi e_j)^\perp, \nabla_{e_i}^\perp V \rangle &= -\langle \nabla_{e_i}^N (\varphi e_j)^\perp, V \rangle \\
&= -\langle \nabla_{e_i}^N \varphi e_j, V \rangle + \langle \nabla_{e_i}^N (\varphi e_j)^\top, V \rangle \\
&= -\langle \varphi \nabla_{e_i} e_j, V \rangle - \langle \varphi \sigma(e_i, e_j), V \rangle + \langle \sigma(e_i, (\varphi e_j)^\top), V \rangle \\
&= 0.
\end{aligned}$$

Next, we shall prove that if a normal vector V is orthogonal to L , then so is $\nabla^\perp V$, i.e.

$$\begin{aligned}
\langle \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle &= 0, \quad \langle \varphi \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle = 0, \\
\langle \varphi e_i, \nabla_{e_k}^\perp V \rangle &= 0, \quad \langle \xi^\perp, \nabla_{e_k}^\perp V \rangle = 0,
\end{aligned}$$

for all i, j, k . We only have to prove the first two identities and the last one, since the third one has been obtained above.

Let us denote $A_{ijk} = \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle$. As σ is symmetric, we have $A_{ijk} = A_{jik}$, and also $A_{ijk} = -\langle \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle$, since V is orthogonal to L . We get

$$\begin{aligned}
\langle (\nabla_{e_k}^\perp \sigma)(e_i, e_j), V \rangle &= \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle - \langle \sigma(\nabla_{e_k} e_i, e_j), V \rangle - \langle \sigma(e_i, \nabla_{e_k} e_j), V \rangle \\
&= \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle,
\end{aligned}$$

and, from the Codazzi equation, again using $V \perp L$,

$$\begin{aligned}
\langle (\nabla_{e_k}^\perp \sigma)(e_i, e_j), V \rangle &= \langle (\nabla_{e_i}^\perp \sigma)(e_k, e_j) + (R^N(e_k, e_i)e_j)^\perp, V \rangle \\
&= \langle (\nabla_{e_j}^\perp \sigma)(e_k, e_i) + (R^N(e_k, e_j)e_i)^\perp, V \rangle \\
&= \langle (\nabla_{e_i}^\perp \sigma)(e_k, e_j), V \rangle = \langle (\nabla_{e_j}^\perp \sigma)(e_k, e_i), V \rangle.
\end{aligned}$$

We have just proved that $A_{ijk} = A_{kji} = A_{ikj}$.

Next, since $\nabla_{e_k}^\perp V$ is orthogonal to $\varphi(T\Sigma^2)$ and to $\varphi \vec{H}$, it follows that the frame field $\{e_1, e_2\}$ diagonalizes $A_{\nabla_{e_k}^\perp V}$ as well, and we get

$$A_{ijk} = -\langle \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle = -\langle e_i, A_{\nabla_{e_k}^\perp V} e_j \rangle = 0$$

for any $i \neq j$. Hence, we have obtained that if two indices are different from each other, then $A_{ijk} = 0$.

Next, we have

$$\begin{aligned} A_{iii} &= -\langle \sigma(e_i, e_i), \nabla_{e_i}^\perp V \rangle = -\langle 2\vec{H}, \nabla_{e_i}^\perp V \rangle + \langle \sigma(e_j, e_j), \nabla_{e_i}^\perp V \rangle \\ &= \langle 2\nabla_{e_i}^\perp \vec{H}, V \rangle - A_{jji} = 0, \end{aligned}$$

and, therefore, the first identity is proved.

In order to obtain the second one, we observe first that if V is orthogonal to L , then φV is also normal and orthogonal to L . It follows that

$$\begin{aligned} \langle (\varphi\sigma(e_i, e_j))^\perp, \nabla_{e_k}^\perp V \rangle &= -\langle \nabla_{e_k}^N (\varphi\sigma(e_i, e_j))^\perp, V \rangle \\ &= -\langle \nabla_{e_k}^N \varphi\sigma(e_i, e_j), V \rangle + \langle \nabla_{e_k}^N (\varphi\sigma(e_i, e_j))^\top, V \rangle \\ &= \langle \varphi A_{\sigma(e_i, e_j)} e_k, V \rangle - \langle \varphi \nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle \\ &\quad + \langle \sigma(e_k, (\varphi\sigma(e_i, e_j))^\top), V \rangle \\ &= \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), \varphi V \rangle = -\langle \sigma(e_i, e_j), \nabla_{e_k}^\perp \varphi V \rangle \\ &= 0. \end{aligned}$$

Finally, we get

$$\begin{aligned} \langle \xi^\perp, \nabla_{e_k}^\perp V \rangle &= \langle \xi^\perp, \nabla_{e_k}^N V \rangle = -\langle \nabla_{e_k}^N \xi^\perp, V \rangle \\ &= -\langle \nabla_{e_k}^N \xi, V \rangle + \langle \nabla_{e_k}^N \xi^\top, V \rangle = \langle \sigma(e_k, \xi^\top), V \rangle \\ &= 0, \end{aligned}$$

which completes the proof. \square

Since $\varphi(L \oplus T\Sigma^2) \subset L \oplus T\Sigma^2$ and $\xi \in L \oplus T\Sigma^2$ along the surface, it follows that $R^N(X, Y)Z \in L \oplus T\Sigma^2$ for any $X, Y, Z \in L \oplus T\Sigma^2$. Therefore, by using a result of J. H. Eschenburg and R. Tribuzy (Theorem 2 in [14]) and the result of H. Endo in [13], we get

Proposition 5.9. *Let Σ^2 be an immersed non-minimal pmc surface in a non-flat cosymplectic space form $N^{2n+1}(\rho)$, $n \geq 2$. If its mean curvature vector is nowhere an umbilical direction, then the surface lies in a cosymplectic space form $N^r(\rho)$, where $r \leq 11$.*

If we consider the cosymplectic space form $N^{2n+1}(\rho)$ to be the product between a complex space form $M^n(\rho)$ and \mathbb{R} and again use the fact that $\varphi(L \oplus T\Sigma^2) \subset L \oplus T\Sigma^2$ and $\xi \in L \oplus T\Sigma^2$, then we have the following.

Corollary 5.10. *Let Σ^2 be an immersed non-minimal pmc surface in $M^n(\rho) \times \mathbb{R}$, $n \geq 2$, $\rho \neq 0$. If its mean curvature vector is nowhere an umbilical direction, then the surface lies in $M^r(\rho) \times \mathbb{R}$, where $r \leq 5$.*

Remark 5.11. Since the map $p \in \Sigma^2 \rightarrow (A_{\vec{H}} - \mu I)(p)$, where μ is a constant, is analytic, it follows that if \vec{H} is an umbilical direction, then this either holds on Σ^2 or only for a closed set without interior points. In this second case \vec{H} is not an umbilical direction in an open dense set, and then Proposition 5.8 holds on this set.

By continuity it holds on Σ^2 . Consequently, only the two above studied cases can occur.

Summarizing, we can state

Theorem 5.12. *Let Σ^2 be an immersed non-minimal pmc surface in a non-flat cosymplectic space form $N^{2n+1}(\rho)$, $n \geq 2$. Then, one of the following holds:*

- (1) Σ^2 is pseudo-umbilical and then $\xi \perp T\Sigma^2$, $\varphi(T\Sigma^2) \subset N\Sigma^2$, $\vec{H} \perp \xi$, $\varphi\vec{H} \perp T\Sigma^2$ and $n \geq 3$ or
- (2) Σ^2 is not pseudo-umbilical and lies in a cosymplectic space form $N^r(\rho)$, where $r \leq 11$.

Corollary 5.13. *Let Σ^2 be an immersed non-minimal pmc surface in $N^{2n+1}(\rho) = M^n(\rho) \times \mathbb{R}$, where $M^n(\rho)$ is a non-flat complex space form, with complex dimension $n \geq 2$. Then one of the following holds:*

- (1) Σ^2 is pseudo-umbilical in $N^{2n+1}(\rho)$ and then it is a pseudo-umbilical non-minimal totally real pmc surface in $M^n(\rho)$ and $n \geq 3$ or
- (2) Σ^2 is not pseudo-umbilical in $N^{2n+1}(\rho)$ and then it lies in $M^r(\rho) \times \mathbb{R}$, where $r \leq 5$.

6. ANTI-INVARIANT PMC SURFACES

Let Σ^2 be an immersed non-minimal anti-invariant pmc surface in a non-flat cosymplectic space form $N^{2n+1}(\rho)$ and define a new quadratic form Q' on Σ^2 by

$$Q'(X, Y) = 8\langle \sigma(X, Y), \vec{H} \rangle - \rho\eta(X)\eta(Y).$$

In the same way as in Section 3 it can be proved that the $(2, 0)$ -part of Q' is holomorphic. Indeed, if (u, v) are isothermal coordinates on the surface and we again consider the complex vector fields $Z = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$ and $\widehat{Z} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$, then, from equation (3.4), we get

$$\widehat{Z}(\langle \sigma(Z, Z), \vec{H} \rangle) = \frac{\rho}{4}\{\langle Z, \widehat{Z} \rangle \eta(Z)\eta(\vec{H}) + 3\langle \widehat{Z}, \varphi Z \rangle \langle \vec{H}, \varphi Z \rangle\} = \frac{\rho}{4}\langle Z, \widehat{Z} \rangle \eta(Z)\eta(\vec{H}),$$

since the surface is anti-invariant. Now, from (3.5), it follows that $\widehat{Z}(Q'(Z, Z)) = 0$.

In the following, we shall assume that the $(2, 0)$ -parts of Q and Q' vanish on the surface, i.e. the following equations hold on Σ^2 :

$$(6.1) \quad \begin{cases} 8|\vec{H}|^2\langle \sigma(e_1, e_1) - \sigma(e_2, e_2), \vec{H} \rangle - \rho|\vec{H}|^2((\eta(e_1))^2 - (\eta(e_2))^2) \\ + 3\rho(\langle \varphi e_1, \vec{H} \rangle^2 - \langle \varphi e_2, \vec{H} \rangle^2) = 0, \\ 8|\vec{H}|^2\langle \sigma(e_1, e_2), \vec{H} \rangle - \rho|\vec{H}|^2\eta(e_1)\eta(e_2) + 3\rho\langle \varphi e_1, \vec{H} \rangle \langle \varphi e_2, \vec{H} \rangle = 0 \end{cases}$$

and

$$(6.2) \quad \begin{cases} 8\langle \sigma(e_1, e_1) - \sigma(e_2, e_2), \vec{H} \rangle - \rho((\eta(e_1))^2 - (\eta(e_2))^2) = 0 \\ 8\langle \sigma(e_1, e_2), \vec{H} \rangle - \rho\eta(e_1)\eta(e_2) = 0, \end{cases}$$

where $\{e_1, e_2\}$ is an orthonormal frame on the surface.

From (6.2) it results that ξ is orthogonal to the surface at a point p if and only if \vec{H} is an umbilical direction at p . Therefore, using Remark 5.11, we obtain that either ξ is orthogonal to the surface at every point or this holds only in a closed set without interior points. From Theorem 5.12 we know that the first case is possible only for $n \geq 3$.

Next, if ξ_p is tangent to the surface at any point p in an open, connected subset of Σ^2 , it follows that the Gaussian curvature K of Σ^2 vanishes on this set, since ξ is parallel. Therefore, K vanishes on the entire surface, and this cannot occur for 2-spheres. However, we studied this case in Section 4, where $N^{2n+1}(\rho)$ is the product between a non-flat complex space form and \mathbb{R} . In general, for a surface in an arbitrary cosymplectic space form $N^{2n+1}(\rho)$, we can choose an orthonormal frame $\{e_1, \xi\}$ on the surface and easily see that $\sigma(\xi, \xi) = 0$, $\sigma(e_1, \xi) = 0$ and $\sigma(e_1, e_1) = 2\vec{H}$. Moreover, from (6.1) and (6.2), we have that $\vec{H} \perp \varphi e_1$ and $\rho = -16|\vec{H}|^2$.

Remark 6.1. We shall now use an argument in [5] in order to show that either ξ is tangent to Σ^2 everywhere or this holds only in a closed set without interior points. Let $f : \Sigma^2 \rightarrow \mathcal{L}(N\Sigma^2, \mathbb{R})$ be the map that takes any point $p \in \Sigma^2$ to the linear function f_p on $N_p\Sigma^2$, given by $f_p(X_p) = \eta_p(X_p)$, for any normal vector X_p at p . Obviously, ξ is tangent to the surface at $p \in \Sigma^2$ if and only if f_p vanishes identically. By analyticity, either f is identically zero on the surface or the set of its zeroes is closed and without interior points.

Now, in order to treat the case where ξ has non-vanishing tangent and normal components in an open dense set $T \subset \Sigma^2$, we shall split our study into two cases, as $n = 2$ or $n \geq 3$. We will work in the open dense set T , and all results obtained below that hold on this set actually hold on Σ^2 , by continuity.

Case I: $n = 2$. Let us assume that our surface Σ^2 is immersed in $N^5(\rho)$, and consider the orthonormal basis $\{e_1, e_2\}$ in $T_p\Sigma^2$ for any $p \in T$, where $e_2 = \frac{\xi^\top}{|\xi^\top|}$ is the unit vector in the direction of the projection of ξ on the tangent space. Then, since $\eta(e_1) = 0$ and, from (6.1) and (6.2), we have $\varphi e_1 \perp \vec{H}$ and $\varphi e_2 \perp \vec{H}$, it follows that $\{e_1, e_2, e_3 = \varphi e_1, e_4 = \frac{\varphi e_2}{|\varphi e_2|}, e_5 = \frac{\vec{H}}{|\vec{H}|}\}$ is an orthonormal basis in T_pN^5 . Observe that, at any point $p \in T$, the characteristic vector field ξ can be written as

$$(6.3) \quad \xi = \mu e_2 + \nu e_5,$$

where $\mu = \eta(e_2)$ and $\nu = \eta(e_5) = \frac{\eta(\vec{H})}{|\vec{H}|}$ is called the *angle function*.

Next, from the second equation of (6.2), we get that $\{e_1, e_2\}$ diagonalizes $A_{\vec{H}}$. Moreover, using the Ricci equation, one obtains that $\{e_1, e_2\}$ also diagonalizes $A_{\varphi e_1}$ and $A_{\varphi e_2}$, since $\langle R^N(e_1, e_2)\vec{H}, \varphi e_1 \rangle = \langle R^N(e_1, e_2)\vec{H}, \varphi e_2 \rangle = 0$.

Finally, the first equation of (6.2) leads to

$$(6.4) \quad A_{e_5} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} |\vec{H}|(1 - \frac{\rho}{16|\vec{H}|^2}\mu^2) & 0 \\ 0 & |\vec{H}|(1 + \frac{\rho}{16|\vec{H}|^2}\mu^2) \end{pmatrix}.$$

Lemma 6.2. *If ξ has non-vanishing tangent and normal components on an open dense set of Σ^2 , then, with the above notation, we have*

- (1) $e_1(\mu) = e_1(\nu) = 0$;
- (2) $e_2(\mu) = \lambda_2\nu$ and $e_2(\nu) = -\lambda_2\mu$;
- (3) $\nabla_{e_1}e_1 = -\lambda_1\frac{\nu}{\mu}e_2$ and $\nabla_{e_2}e_2 = 0$;
- (4) $\sigma(e_i, e_i) = \frac{\lambda_i}{|\vec{H}|}\vec{H}$, $i \in \{1, 2\}$.

Proof. Using the fact that ξ is parallel, (6.3) and (6.4) imply that

$$\begin{aligned} 0 &= \nabla_{e_1}^N \xi = \nabla_{e_1}^N (\mu e_2 + \nu e_5) \\ &= e_1(\mu)e_2 + \mu \nabla_{e_1} e_2 - \nu A_{e_5} e_1 + \mu \sigma(e_1, e_2) + e_1(\nu)e_5 \\ &= e_1(\mu)e_2 + \mu \nabla_{e_1} e_2 - \lambda_1 \nu e_1 + e_1(\nu)e_5. \end{aligned}$$

The tangent and the normal part on the right hand side vanish, and then, since $\nabla_{e_1} e_2 \perp e_2$, it follows that $e_1(\mu) = e_1(\nu) = 0$ and $\nabla_{e_1} e_2 = \lambda_1 \frac{\nu}{\mu} e_1$. Since $\langle \nabla_{e_1} e_2, e_1 \rangle + \langle \nabla_{e_1} e_1, e_2 \rangle = 0$ and $\nabla_{e_1} e_1 \perp e_1$, the last identity is equivalent to $\nabla_{e_1} e_1 = -\lambda_1 \frac{\nu}{\mu} e_2$.

In the same way, we get

$$\begin{aligned} 0 &= \nabla_{e_2}^N \xi = \nabla_{e_2}^N (\mu e_2 + \nu e_5) \\ &= e_2(\mu)e_2 + \mu \nabla_{e_2} e_2 - \nu A_{e_5} e_2 + \mu \sigma(e_2, e_2) + e_2(\nu)e_5 \\ &= e_2(\mu)e_2 + \mu \nabla_{e_2} e_2 - \lambda_2 \nu e_2 + e_2(\nu)e_5 + \mu \sigma(e_2, e_2) \end{aligned}$$

and then $\nabla_{e_2} e_2 = 0$, $e_2(\mu) = \lambda_2 \nu$ and, since $\mu^2 + \nu^2 = 1$, $e_2(\nu) = -\lambda_2 \mu$. We also obtain that $\sigma(e_2, e_2) = -\frac{e_2(\nu)}{\mu} e_5 = \frac{\lambda_2}{|\vec{H}|} \vec{H}$ and $\sigma(e_1, e_1) = 2\vec{H} - \sigma(e_2, e_2) = \frac{\lambda_1}{|\vec{H}|} \vec{H}$. \square

Remark 6.3. A direct consequence of the previous lemma is that $A_{\varphi e_1}$ and $A_{\varphi e_2}$ vanish, and then the only non-zero component of A is A_{e_5} .

Now, assume that the characteristic vector field ξ is either tangent to the surface or it has a non-vanishing tangent and normal components in an open dense set $T \subset \Sigma^2$, and consider the subbundle of the normal bundle $L = \text{Im } \sigma$. On the one hand, it is easy to see that L is parallel, $\dim L = 1$, and $\varphi X \perp Y$ for any $X, Y \in T\Sigma^2 \oplus L$. Also $T\Sigma^2 \oplus L$ is invariant by R^N , since $\xi \in T\Sigma^2 \oplus L$ along the surface.

On the other hand, any non-minimal cmc surface immersed in an anti-invariant totally geodesic 3-dimensional submanifold of $N^5(\rho)$ is an immersed non-minimal anti-invariant pmc surface in $N^5(\rho)$. Moreover, if we assume that the $(2, 0)$ -part of Q' vanishes on such a surface, it follows also that the $(2, 0)$ -part of Q vanishes.

Therefore, using Theorem 2 in [14], we get

Theorem 6.4. *A surface Σ^2 can be immersed as a non-minimal anti-invariant pmc surface in a non-flat cosymplectic space form $N^5(\rho)$, with vanishing $(2, 0)$ -parts of the quadratic forms Q and Q' , if and only if Σ^2 is an immersed non-minimal cmc surface in a 3-dimensional totally geodesic anti-invariant submanifold of $N^5(\rho)$ such that the $(2, 0)$ -part of Q' vanishes.*

The 3-dimensional totally geodesic anti-invariant submanifolds of $M^2(\rho) \times \mathbb{R}$, where M^2 is a non-flat complex space form, are $\bar{M}^2 \times \mathbb{R}$, where \bar{M}^2 is a totally geodesic Lagrangian submanifold of $M^2(\rho)$. B.-Y. Chen and K. Ogiue proved in [10] (Proposition 3.2) that a totally geodesic totally real submanifold \bar{M}^m of a non-flat complex space form $M^n(\rho)$ is necessarily a space form with constant curvature $\frac{\rho}{4}$. Moreover, it is known that $\mathbb{S}^2(\frac{\rho}{4})$ and $\mathbb{H}^2(\frac{\rho}{4})$ can be isometrically immersed as

totally geodesic Lagrangian submanifolds in $\mathbb{C}P^2(\rho)$ and $\mathbb{C}H^2(\rho)$, respectively (see [8]).

Hence, an immersed non-minimal anti-invariant pmc surface in $M^2(\rho) \times \mathbb{R}$ on which the $(2, 0)$ -parts of Q and Q' vanish is a non-minimal cmc surface in $\bar{M}^2(\frac{\rho}{4}) \times \mathbb{R}$ with vanishing $(2, 0)$ -part of Q' , which in this case is just the Abresch-Rosenberg differential introduced in [1], where $\bar{M}^2(\frac{\rho}{4})$ is a complete simply-connected surface with constant curvature $\frac{\rho}{4}$. U. Abresch and H. Rosenberg proved that there are four classes of such surfaces, the first three of them, namely the cmc spheres $S_H^2 \subset \bar{M}^2(\frac{\rho}{4}) \times \mathbb{R}$ of Hsiang and Pedrosa, their non-compact cousins D_H^2 and the surfaces of catenoidal type C_H^2 , being embedded and rotationally invariant, and the fourth one being comprised of parabolic surfaces P_H^2 (see [1] and [2] for a detailed description of all these surfaces).

Corollary 6.5. *Any immersed non-minimal anti-invariant pmc surface in $M^2(\rho) \times \mathbb{R}$ with vanishing $(2, 0)$ -parts of the quadratic forms Q and Q' is one of the surfaces S_H^2 , D_H^2 , C_H^2 and P_H^2 in the product space $\bar{M}^2(\frac{\rho}{4}) \times \mathbb{R}$.*

Therefore, we have

Theorem 6.6. *Any immersed non-minimal anti-invariant pmc 2-sphere in a non-flat cosymplectic space form $M^2(\rho) \times \mathbb{R}$ is one of the embedded rotationally invariant cmc spheres $S_H^2 \subset \bar{M}^2(\frac{\rho}{4}) \times \mathbb{R}$.*

Remark 6.7. A surface Σ^2 immersed in a cosymplectic space form is called a *slant surface* if for all vectors X tangent to Σ^2 and orthogonal to ξ , the angle θ between φX and $T_p \Sigma^2$ is constant, i.e. θ does not depend on X or on the point p on the surface. Obviously, the invariant and anti-invariant surfaces are slant surfaces. A slant surface which is neither invariant nor anti-invariant is called a proper slant surface. If Σ^2 is a proper slant surface, then ξ is orthogonal to the surface (see [21]). It follows that, if Σ^2 is an immersed proper slant surface in $M^2(\rho) \times \mathbb{R}$, then it lies in $M^2(\rho)$. On the other hand, there are no non-minimal pmc 2-spheres in a non-flat complex space form $M^2(\rho)$ (see [18]). Therefore, $S_H^2 \subset \bar{M}^2(\frac{\rho}{4}) \times \mathbb{R}$ are the only non-minimal slant pmc 2-spheres in $M^2(\rho) \times \mathbb{R}$.

In the following, we shall see that Lemma 6.2 allows us to make some remarks about the admissible range of the angle function ν .

Proposition 6.8. *Let Σ^2 be a complete non-minimal cmc surface in $\bar{M}^2(\frac{\rho}{4}) \times \mathbb{R}$ with vanishing Abresch-Rosenberg differential and non-negative Gaussian curvature. Then we have that:*

- (1) $\nu^2 \geq \frac{1}{5}$ cannot occur on an open dense subset of Σ^2 ;
- (2) if $\nu^2 \leq \frac{1}{3}$ on an open dense subset of Σ^2 , then ν vanishes identically and the surface is a vertical cylinder over a circle in $\mathbb{H}^2(-4|\vec{H}|^2)$, with curvature $\kappa = 2|\vec{H}|$ and complex torsion equal to 0.

Proof. From Lemma 6.2, it follows, after a straightforward computation, that

$$(6.5) \quad \Delta \nu^2 = 2\lambda_2^2(1 - 3\nu^2)$$

and

$$(6.6) \quad \Delta |A|^2 = \frac{\rho^2}{32|\vec{H}|^2} \lambda_2^2 \mu^2 (5\nu^2 - 1).$$

If $\nu^2 \geq \frac{1}{5}$ on an open dense subset of Σ^2 , then, from (6.6), it follows that $|A|^2$ is a subharmonic function on a parabolic space, and, since $|A|^2$ is bounded by (6.4), we get that either $\lambda_2^2 = 0$ or $\mu^2 = 0$ or $\nu^2 = \frac{1}{5}$. J. M. Espinar and H. Rosenberg proved in [15] that if the angle function ν is constant and the Gaussian curvature is non-negative, then $\nu^2 = 0$ or $\nu^2 = 1$, the second case being possible only when the surface is minimal. Therefore, since we also know that μ^2 cannot vanish on an open dense subset of Σ^2 , one obtains that $\lambda_2^2 = |\vec{H}|^2(1 + \frac{\rho}{16|\vec{H}|^2}\mu^2)^2 = 0$ and then that μ and ν are constant, which means that $\nu^2 = 0$ and $\mu^2 = 1$. But this is a contradiction, since we assumed that $\nu^2 \geq \frac{1}{5}$.

If $\nu^2 \leq \frac{1}{3}$ on an open dense subset of Σ^2 , then, from (6.5), in the same way as above, we obtain that $\nu^2 = 0$, $K = 0$ and $\rho = -16|\vec{H}|^2$. In this case, Σ^2 is a vertical cylinder over a circle in $\mathbb{H}^2(-4|\vec{H}|^2)$, with curvature $\kappa = 2|\vec{H}|$ and complex torsion equal to 0 (see also [15]). \square

Case II: $n \geq 3$. We note first that, according to Theorem 5.12, the surface cannot be pseudo-umbilical, since we have assumed that the tangent part of ξ does not vanish in an open dense set.

Theorem 6.9. *A non-minimal non-pseudo-umbilical anti-invariant pmc surface immersed in a non-flat cosymplectic space form $N^{2n+1}(\rho)$, $n \geq 3$, with vanishing $(2,0)$ -parts of Q and Q' , lies in a totally geodesic anti-invariant submanifold of $N^{2n+1}(\rho)$, with dimension less than or equal to 5.*

Proof. Let us again consider the orthonormal basis $\{e_1, e_2\}$ in $T_p\Sigma^2$, $p \in T$, where e_2 is the unit vector in the direction of the projection of ξ on the tangent space. From (6.1) and (6.2), we can see that $\{e_1, e_2\}$ diagonalizes $A_{\vec{H}}$ in this case, too. Since the surface is anti-invariant, from the Ricci equation, we get $[A_{\vec{H}}, A_V] = 0$, for any normal vector V and, therefore, $\{e_1, e_2\}$ diagonalizes A_V , for any normal vector V . We define the subbundle $L = \text{span}\{\text{Im } \sigma \cup \xi^\perp\}$ in the normal bundle, and, in the same way as in Lemma 5.8, we can prove that, for any normal vector V , orthogonal to L , we have $\langle \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle = 0$ and $\langle \xi^\perp, \nabla_{e_k}^\perp V \rangle = 0$, $i, j, k = \{1, 2\}$, which means that L is parallel. It is also easy to see that $T\Sigma^2 \oplus L$ is invariant by R^N . We shall prove that $\varphi X \perp Y$, for any $X, Y \in T\Sigma^2 \oplus L$. Since the surface is anti-invariant, we have $\varphi e_1 \perp e_2$ and, moreover, $\langle \varphi e_1, \xi^\perp \rangle = \langle \varphi e_1, \xi - \xi^\top \rangle = 0$. Next, we obtain

$$\langle \varphi e_1, \sigma(e_2, e_2) \rangle = \langle \varphi e_1, \nabla_{e_2}^N e_2 \rangle = -\langle \varphi \nabla_{e_2}^N e_1, e_2 \rangle = -\langle \varphi \nabla_{e_2} e_1, e_2 \rangle = 0,$$

again using the fact that Σ^2 is anti-invariant and $\sigma(e_1, e_2) = 0$. From the equations (6.1) and (6.2) it follows that $\varphi e_i \perp \vec{H}$, $i = \{1, 2\}$, and then

$$\langle \varphi e_1, \sigma(e_1, e_1) \rangle = \langle \varphi e_1, 2\vec{H} - \sigma(e_2, e_2) \rangle = 0.$$

Since $T\Sigma^2 \oplus L = \text{span}\{e_1, e_2, \sigma(e_1, e_1), \sigma(e_2, e_2), \xi^\perp\}$, we have just proved that φe_1 is orthogonal to $T\Sigma^2 \oplus L$. In the same way we get that φe_2 and $\varphi \xi^\perp = -\varphi \xi^\top = -|\xi^\top| \varphi e_2$ are orthogonal to $T\Sigma^2 \oplus L$. Finally, since $\varphi \vec{H} \perp e_i$, $i = \{1, 2\}$, it results

that $\varphi\vec{H}$ is normal and one gets

$$\begin{aligned} \langle \varphi\sigma(e_1, e_1), \sigma(e_2, e_2) \rangle &= \langle \varphi\sigma(e_1, e_1), 2\vec{H} - \sigma(e_1, e_1) \rangle = 2\langle \varphi\sigma(e_1, e_1), \vec{H} \rangle \\ &= -2\langle \nabla_{e_1}^N e_1, \varphi\vec{H} \rangle = 2\langle e_1, \varphi\nabla_{e_1}^N \vec{H} \rangle = -2\langle e_1, \varphi A_{\vec{H}} e_1 \rangle \\ &= 0, \end{aligned}$$

which means $\varphi\sigma(e_i, e_i) \perp T\Sigma^2 \oplus L$.

Hence, $T\Sigma^2 \oplus L$ is parallel, invariant by R^N , anti-invariant by φ and its dimension is less than or equal to 5. Now, again using Theorem 2 in [14], we come to the conclusion. \square

If $N^{2n+1}(\rho)$ is of product type, we again use Proposition 3.2 in [10] in order to obtain

Corollary 6.10. *A non-minimal non-pseudo-umbilical anti-invariant pmc surface immersed in $M^n(\rho) \times \mathbb{R}$, $n \geq 3$, $\rho \neq 0$, with vanishing $(2, 0)$ -parts of Q and Q' , lies in a product space $\bar{M}^4(\frac{\rho}{4}) \times \mathbb{R}$, where $\bar{M}^4(\frac{\rho}{4})$ is a space form immersed as a totally geodesic totally real submanifold in the complex space form $M^n(\rho)$.*

Remark 6.11. The non-minimal non-pseudo-umbilical pmc 2-spheres immersed in $\bar{M}^4(\frac{\rho}{4}) \times \mathbb{R}$ were characterized by H. Alencar, M. do Carmo and R. Tribuzy in [5, Theorem 2(4)]. In the same paper, they also described the non-minimal non-pseudo-umbilical complete pmc surfaces with non-negative Gaussian curvature with vanishing $(2, 0)$ -part of Q' (see [5, Theorem 3(4)]).

Remark 6.12. As we have seen, a proper slant surface Σ^2 immersed in $M^n(\rho) \times \mathbb{R}$, $\rho \neq 0$, lies in $M^n(\rho)$. Moreover, as an immersed surface in this space, it has constant Kähler angle. In [17] it is proved that there are no non-minimal non-pseudo-umbilical pmc 2-spheres with constant Kähler angle in a non-flat complex space form. Therefore, there are no non-minimal non-pseudo-umbilical proper slant pmc 2-spheres in $M^n(\rho) \times \mathbb{R}$.

REFERENCES

- [1] U. Abresch and H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$* , Acta Math. 193(2004), 141–174. MR2134864 (2006h:53003)
- [2] U. Abresch and H. Rosenberg, *Generalized Hopf differentials*, Mat. Contemp. 28(2005), 1–28. MR2195187 (2006h:53004)
- [3] P. Alegre, D. E. Blair and A. Carriazo, *Generalized Sasakian space forms*, Israel J. Math. 141(2004), 157–183. MR2063031 (2005f:53057)
- [4] H. Alencar, M. do Carmo and R. Tribuzy, *A theorem of Hopf and the Cauchy-Riemann inequality*, Comm. Anal. Geom. 15(2007), 283–298. MR2344324 (2009c:53071)
- [5] H. Alencar, M. do Carmo and R. Tribuzy, *A Hopf Theorem for ambient spaces of dimensions higher than three*, J. Differential Geometry 84(2010), 1–17. MR2629507 (2011e:53086)
- [6] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser Boston, Progress in Mathematics, 203, 2002. MR1874240 (2002m:53120)
- [7] D. E. Blair and S. I. Goldberg, *Topology of almost contact manifolds*, J. Differential Geometry 1(1967), 347–354. MR0226539 (37:2128)
- [8] I. Castro, C. R. Monteleagre and F. Urbano, *Minimal Lagrangian submanifolds in the complex hyperbolic space*, Illinois J. Math. 46(2002), 695–721. MR1951236 (2004e:53089)
- [9] B.-Y. Chen and G. D. Ludden, *Surfaces with mean curvature vector parallel in the normal bundle*, Nagoya Math. J. 47(1972), 161–167. MR0331231 (48:9565)
- [10] B.-Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Am. Math. Soc. 193(1974), 257–266. MR0346708 (49:11433)

- [11] S.-S. Chern, *On surfaces of constant mean curvature in a three-dimensional space of constant curvature*, Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math. 1007, Springer, Berlin, 1983, 104–108. MR730266 (86b:53058)
- [12] D. Chinea, M. de León and J. C. Marrero, *Topology of cosymplectic manifolds*, J. Math. Pures Appl. 72(1993), 567–591. MR1249410 (95c:53036)
- [13] H. Endo, *A note on invariant submanifolds in an almost cosymplectic manifold*, Tensor (N.S.) 43(1986), 75–78. MR878962 (88b:53042)
- [14] J. H. Eschenburg and R. Tribuzy, *Existence and uniqueness of maps into affine homogeneous spaces*, Rend. Sem. Mat. Univ. Padova 89(1993), 11–18. MR1229038 (94f:53092)
- [15] J. M. Espinar and H. Rosenberg, *Complete constant mean curvature surfaces in homogeneous spaces*, Comment. Math. Helv. 86(2011), 659–674.
- [16] D. Ferus, *The torsion form of submanifolds in E^N* , Math. Ann. 193(1971), 114–120. MR0287493 (44:4697)
- [17] D. Fetcu, *Surfaces with parallel mean curvature vector in complex space forms*, J. Differential Geom. 91(2012), 215–232. MR2971287
- [18] S. Hirakawa, *Constant Gaussian curvature surfaces with parallel mean curvature vector in two-dimensional complex space forms*, Geom. Dedicata 118(2006), 229–244. MR2239458 (2007c:53073)
- [19] D. A. Hoffman, *Surfaces in constant curvature manifolds with parallel mean curvature vector field*, Bull. Amer. Math. Soc. 78(1972), 247–250. MR0298601 (45:7653)
- [20] H. Hopf, *Differential Geometry in the Large*, Lecture Notes in Math. 1000, Springer-Verlag, 1983. MR707850 (85b:53001)
- [21] A. Lotta, *Slant submanifolds in contact geometry*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 39(1996), 183–198.
- [22] S. Maeda and T. Adachi, *Holomorphic helices in a complex space form*, Proc. Amer. Math. Soc. 125(1997), 1197–1202. MR1353391 (97f:53114)
- [23] S. Maeda and Y. Ohnita, *Helical geodesic immersions into complex space forms*, Geom. Dedicata 30(1989), 93–114. MR995941 (90b:53067)
- [24] B. O’Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Pure and Applied Mathematics 103, Academic Press, New York, 1983. MR719023 (85f:53002)
- [25] S.-T. Yau, *Submanifolds with constant mean curvature*, Amer. J. Math. 96(1974), 346–366. MR0370443 (51:6670)

DEPARTMENT OF MATHEMATICS AND INFORMATICS, “GH. ASACHI” TECHNICAL UNIVERSITY OF IASI, Bd. CAROL I NO. 11, 700506 IASI, ROMANIA

E-mail address: `dfetcu@math.tuiasi.ro`

INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA, ESTRADA DONA CASTORINA, 110, 22460-320 RIO DE JANEIRO, BRASIL

E-mail address: `rosen@impa.br`