TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 9, September 2014, Pages 4849–4871
S 0002-9947(2014)06261-2
Article electronically published on May 8, 2014

TAME DYNAMICS AND ROBUST TRANSITIVITY CHAIN-RECURRENCE CLASSES *VERSUS* HOMOCLINIC CLASSES

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ABSTRACT. One main task of smooth dynamical systems consists in finding a good decomposition into elementary pieces of the dynamics. This paper contributes to the study of chain-recurrence classes. It is known that C^1-generically, each chain-recurrence class containing a periodic orbit is equal to the homoclinic class of this orbit. Our result implies that in general this property is fragile.

We build a C^1-open set $\mathcal{U}$ of tame diffeomorphisms (their dynamics only splits into finitely many chain-recurrence classes) such that for any diffeomorphism in a C^∞-dense subset of $\mathcal{U}$, one of the chain-recurrence classes is not transitive (and has an isolated point). Moreover, these dynamics are obtained among partially hyperbolic systems with one-dimensional center.

1. Introduction

1.1. **Decomposition of the dynamics.** In the setting of hyperbolic diffeomorphisms, Smale's spectral decomposition theorem organizes the global dynamics by decomposing them into a finite number of homoclinic classes which play the role of elementary pieces. After the first examples of open sets of non-hyperbolic systems [AS, N] the research focused on non-hyperbolic dynamics trying to recover a decomposition in pieces in this new setting. Many such examples have been obtained, and we can distinguish two main different behaviors, tame and wild, which are defined as follows:

- The dynamics can be globally "indecomposable": there are open sets of *transitive* systems, i.e. having a half-orbit which is dense in the whole manifold; see [S, M, BD_1]. In the same spirit, other examples present a decomposition in finitely many disjoint, isolated, robustly transitive compact invariant sets [Ca, BV, BD_1].
- On the opposite side, the dynamics of generic systems in some C^r-open sets ($r \geq 1$) splits into infinitely many pieces; see [N, BD_2]. Necessarily in this case, some pieces of the decomposition are accumulated by other ones.

These two patterns (finite or infinite number of indecomposable pieces) are called *tame* and *wild* systems: in order to formalize these notions we have to explain what we mean by "pieces". Several definitions have been proposed:

- A natural way for a piece to be dynamically indecomposable is to be transitive. Among notable transitive sets are the *homoclinic classes* of hyperbolic

Received by the editors January 10, 2012 and, in revised form, December 10, 2012.
2010 *Mathematics Subject Classification.* Primary 37C20, 37D30, 37C29, 37G25.
The authors were partially supported by the ANR project *DynNonHyp* BLAN08-2 313375.

periodic orbits, that is, the closure of the transversal intersection of their invariant manifolds. These classes contain dense subsets of hyperbolic periodic orbits that are *homoclinically related*: the stable manifold of each of these orbits intersects transversally the unstable manifold of each other.

– Other natural candidates for pieces are the *maximal transitive sets* [BD$_2$]. They always exist but may fail to be disjoint, and there may be non-trivial dynamical behavior outside of those sets. By this we mean that the union of the maximal transitive sets may not cover the entire limit set.
– By weakening the notion of transitivity, Conley [Co] defined the very general notion of *chain-recurrence classes* for a homeomorphism f on a compact manifold M. The *chain-recurrent set* $\mathcal{R}(f)$ is the set of points x contained in closed ε-pseudo-orbits[1] for every $\varepsilon > 0$. This set splits into invariant compact subsets called *chain-recurrence classes*: two points $x, y \in \mathcal{R}(f)$ are in the same class if for every $\varepsilon > 0$ they are contained in the same closed ε-pseudo-orbit. The chain-recurrence classes cover the whole interesting dynamics, as this notion of recurrence is the weakest possible: the chain-recurrence set contains the limit and non-wandering sets. Moreover, the different chain-recurrence classes can be separated by filtrations of the dynamics (see [Co], [BDV, Chapter 10]).

Each of these notions of elementary piece has its advantages and drawbacks and in general they do not coincide. For C^1-generic diffeomorphisms, that is, for any f in a residual subset of $\mathrm{Diff}^1(M)$, one gets better properties. For instance, [BC] showed that any chain-recurrence class containing a hyperbolic periodic orbit O coincides with the homoclinic class $H(O)$ of O and is the unique maximal transitive set containing O.

In the case of hyperbolic diffeomorphisms (that is, diffeomorphisms whose chain-recurrent classes are hyperbolic), the elementary pieces indeed correspond to each of these notions: there are finitely many chain-recurrence classes which robustly are the homoclinic classes and are also the maximal transitive sets.

The different notions of elementary pieces of a hyperbolic periodic point do not coincide anymore for arbitrary diffeomorphisms. As we said before they coincide for C^1-generic diffeomorphisms. One can expect that they coincide on larger classes of systems, in particular for some close C^r-diffeomorphisms. More precisely one can ask:

Question 1. Consider a hyperbolic periodic orbit O_f of a diffeomorphism f. Its continuation exists in a C^1-neighborhood $\mathcal{U}_0$ of f. Under what conditions does there exist a dense open subset $\mathcal{U} \subset \mathcal{U}_0$ such that for any $g \in \mathcal{U}$ the chain-recurrence class containing O_g is transitive? coincides with the homoclinic class $H(O_g)$?

Note that, as hyperbolic periodic points have continuation, so does the homoclinic class of a periodic point (those may not vary continuously). On the other hand, chain-recurrence classes may have no well-defined continuation (see section 2.2, where these properties are discussed).

1.2. **Tame dynamics.** Question 1 can be asked for the important particular case where the continuation of the chain recurrence class is defined. A chain-recurrence class $\mathcal{E}$ of a diffeomorphism f is *robustly isolated* (for the C^1-topology) if there exists

[1] An ε-pseudo-orbit is a sequence $x_0, \dots, x_n$, $n \geq 1$, such that $n \geq 1$ and $\mathrm{dist}(f(x_k), x_{k+1}) \leq \varepsilon$, for all k.

a neighborhood $\mathcal{O}$ of $\mathcal{E}$ and a C^1-neighborhood $\mathcal{U}$ of f such that, for every $g \in \mathcal{U}$, the set $\mathcal{R}(g) \cap \mathcal{O}$ of chain-recurrent points in $\mathcal{O}$ coincides with a chain-recurrence class $\mathcal{E}_g$ of g called its *continuation*.

Following [BD$_3$] (see also [A] for an earlier definition using C^1-generic diffeomorphisms), one says that a diffeomorphism f is *tame* if each of its chain-recurrence classes is robustly isolated. In this case, the number of chain-recurrence classes is finite and constant on a C^1-neighborhood of f; in particular, tame diffeomorphisms form a C^1-open set of diffeomorphisms. A diffeomorphism is *wild* if it cannot be approached in the C^1 topology by tame ones: in other words, the set of wild diffeomorphisms is the complement of the closure of the set of tame diffeomorphisms, and therefore is C^1-open too. C^1-generic wild diffeomorphisms have infinitely many chain recurrence classes (see [A, BC]).

Morse-Smale and more generally Axiom A diffeomorphisms are tame, but there are also non-hyperbolic diffeomorphisms which are tame. Much more is known on the dynamics of C^1-generic tame systems than for wild ones; for instance, all their chain-recurrence classes are homoclinic classes [BC].

Until now, the unique known examples of tame dynamics have robustly transitive chain-recurrence classes, i.e. robustly isolated classes whose continuation remains transitive for any C^1-perturbation. For this reason it was natural to ask:

Question 2 ([BC, problem 1.3])**.** Is there a dense subset $\mathcal{D}$ in the space $\mathrm{Diff}^r(M)$ of C^r-diffeomorphisms such that every robustly isolated chain-recurrence class of any $f \in \mathcal{D}$ is robustly transitive?

The present paper gives a negative answer to this question.

Main theorem (first version). *Any compact manifold M, with* $\dim(M) \geq 3$, *admits a smooth C^∞-diffeomorphism f with a C^1-robustly isolated chain-recurrence class $\mathcal{E}_f$ satisfying for any $r \geq 1$ the following property: The diffeomorphisms g such that the continuation $\mathcal{E}_g$ of $\mathcal{E}_f$ is not transitive form a C^r-dense subset of a C^1-neighborhood of f in* $\mathrm{Diff}^r(M)$.

Remark 1. We mention that the diffeomorphism f can be constructed in *any* isotopy class of diffeomorphisms of M. This results from the fact that our construction is semilocal, in a ball and done by surgery at some sink. Every isotopy class of diffeomorphisms of a manifold M has some Axiom A representative admitting some periodic sink (see [F]). Therefore our construction builds a C^1-open set of tame diffeomorphisms having a (robustly isolated) chain-recurrence class which is not robustly transitive, in any isotopy class of diffeomorphims of compact manifolds of dimension ≥ 3 (see section 3.1 below).

Actually, for a class to be robustly isolated brings several restrictions on the dynamics. On surfaces, it is well known that such a class is far[2] from homoclinic tangencies [N, PS]. Also it was shown in [BDP] that it presents dominated splittings and volume hyperbolicity. Indeed the dominated splitting (more precisely, the fact that the stable/unstable splitting along the hyperbolic periodic orbits is dominated) is the structure which is an antagonist to the homoclinic tangencies [W, G]. Conversely, we expect (see [B, Conjecture 11]) that, C^1-generically, any chain-recurrent class exhibiting enough hyperbolicity is robustly isolated: this should be the case

[2]We say that f is *far from homoclinic tangencies* if it cannot be C^1-perturbed into a diffeomorphism admitting a hyperbolic periodic point with a homoclinic tangency.

for *partially hyperbolic* classes with a one-dimensional center bundle (see a precise definition in section 2.1).

The aim of this paper is to build an example of a robustly isolated chain-recurrent class as close as possible to being hyperbolic, in particular, C^1-far from homoclinic tangencies and partially hyperbolic with central dimension one, which nevertheless is not robustly transitive. Before stating more precisely our result, let us informally explain the main idea of our construction.

1.3. **Our main tool: *Cuspidal periodic points.***

Our construction comes from a very simple geometric property. Let us explain it for a diffeomorphism f on a 3-manifold. Consider a hyperbolic periodic point p of stable index 2 (it has two eigenvalues of modulus smaller than 1), and assume its orbit has three real eigenvalues of distinct moduli. For simplicity we may assume that p is fixed. The hypothesis on the eigenvalues means that there is a Df_p-invariant splitting $T_pM = E^{ss}_p \oplus E^c_p \oplus E^u_p$ such that the bundles E^{ss}_p and E^c_p are contracted by Df_p (though E^{ss}_p is more contracted than E^c_p) while E^u_p is expanded by Df_p. This gives rise to an invariant strong stable manifold $W^{ss}(p)$ tangent to E^{ss}_p contained in the stable manifold $W^s(p)$ which is tangent to $E^{ss}_p \oplus E^c_p$ (see [KH]).

Consider the chain-recurrence class $\mathcal{E}(p)$ of p. Assume that the strong stable manifold of p cuts $\mathcal{E}(p)$ only on the orbit of p. This is always a C^1-robust property: according to the Conley theory, chain-recurrence classes admit a basis of neighborhoods which are *filtrating sets* (the intersection of an attracting region with a repelling region); this implies in particular that the punctured strong stable manifold $W^{ss}(p) \setminus \{p\}$ admits a (compact) fundamental domain out of a filtrating neighborhood U of $\mathcal{E}(p)$. Small perturbations of f keep this fundamental domain out of U.

Now consider the intersection $W^s(p) \cap \mathcal{E}(p)$. As this intersection is a closed invariant set disjoint from the (punctured) strong stable manifold, it is contained, at least close to p, in a very thin region bounded by two weak stable curves of p which are both tangent to E^c_p at the point p. This region where the local stable manifold of p can intersect $\mathcal{E}(p)$ consists of two components which are cuspidal triangles that meet exactly at p: the component at the right of p and the one at the left of p. A priori, $\mathcal{E}(p)$ can meet both of these cuspidal regions (see the left part of Figure 1). We say that p is a *(stable) cuspidal point* if furthermore $\mathcal{E}(p)$ meets only one of these cuspidal regions (the right part of Figure 1).

This geometric behavior of $\mathcal{E}(p)$ on $W^s(p)$ extends along the unstable manifold of p: the class $\mathcal{E}(p)$ is contained in a cuspidal prism whose edge is the strong unstable manifold of p; see Figure 3.

Clearly, we can define an analogous notion for periodic points q of stable index 1: q is an *unstable cuspidal point* if it is a *stable cuspidal point* for f^{-1}. Thus, the class $\mathcal{E}(q)$ is contained in a cuspidal prism whose edge is the strong stable manifold of q.

What happens when the unstable manifold of a stable cuspidal point intersects the stable manifold of an unstable cuspidal point? The two cuspidal prisms are intersecting, and the position in which they intersect will be a key point for understanding how the dynamics bifurcates.

In this paper we present a global setting which ensures that each time the unstable manifold of p intersects the stable manifold of an unstable cuspidal point q at a

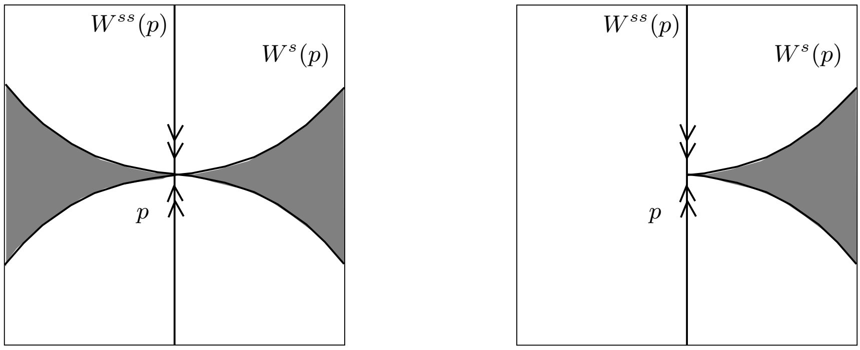

FIGURE 1. Cuspidal regions in the stable manifold of p.

point x, then x is the unique intersection point of the cuspidal prisms in a neighborhood of x: the cuspsidal prisms look like two blades hiting each other as in Figure 2 (notice that [BD$_5$] explores consequences of another geometric configuration of the prisms).

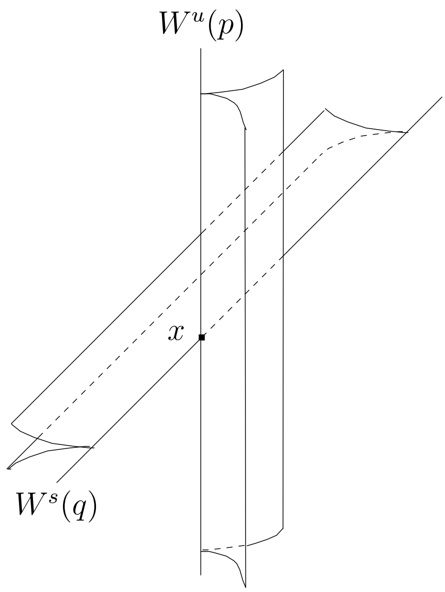

FIGURE 2. The geometry of the cuspidal prisms imposes that their intersection gives rise to an isolated point.

Let us describe this setting. We assume that:

- f is a diffeomorphism of a 3-manifold having two hyperbolic periodic points p and q, of stable dimension equal to 2 and 1 respectively, which belong robustly to the same chain recurrence class $\mathcal{E}_f = \mathcal{E}(p) = \mathcal{E}(q)$: for any diffeomorphism C^1-close to f, the continuations of p and q belong to the same class. This is obtained, as is often the case, by using large uniformly

hyperbolic sets called *blenders* for building *robust heterodimensional cycles* (see [BDV, Chapter 6]);
- in some filtrating neighborhood of $\mathcal{E}(p)$ f leaves invariant a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ into one-dimensional bundles;
- E^c is orientable and f preserves the orientation of E^c. Therefore, for each hyperbolic periodic point, the center direction has two sides, called right and left.
- p is a *right stable cuspidal point*: the cuspidal sector containing $W^s(p)\cap\mathcal{E}(p)$ is on the right of p;
- q is a *left unstable cuspidal point*: the cuspidal sector containing $W^u(q) \cap \mathcal{E}(q)$ is on the left of p.

In that setting, each time $W^u(p)$ intersects $W^s(q)$ at a point x, the point x is an isolated point of the chain recurrence class (the fact that the point belongs to the chain-recurrence class is immediate since it is forward and backward asymptotic to the class; what one must prove is the isolation by studying the geometry explained above). As p and q belong robustly to the same class, arbitrarily small perturbations produce heteroclinic intersections between $W^u(p)$ and $W^s(q)$, each time creating isolated points in the class which prevents the class from being transitive: Indeed, an infinite transitive set cannot have isolated points.

Let us summarize this phenomenon as a general criterion (see also Proposition 2 where this criterion is precisely restated):

Isolated point criterion. *Let $\mathcal{E}_f$ be a partially hyperbolic chain-recurrence class. Assume that its central bundle is one-dimensional and endowed with an invariant continuous orientation. Let p be a right stable cuspidal point in $\mathcal{E}_f$ and q a left unstable cuspidal point in $\mathcal{E}_f$. Then, any intersection point $x \in W^u(p) \cap W^s(q)$ is isolated in $\mathcal{E}_f$.*

1.4. Precise statement of our result.

Main theorem. *When $\dim(M) \geq 3$, there exist a C^1-open set $\mathcal{U} \subset \mathrm{Diff}^r(M)$, $1 \leq r \leq \infty$, a C^r-dense subset $\mathcal{D} \subset \mathcal{U}$ and an open set $U \subset M$ with the following properties:*

(I) Isolation*: For every $f \in \mathcal{U}$, the set $\mathcal{E}_f := U \cap \mathcal{R}(f)$ is a chain-recurrence class.*

(II) Non-robust transitivity*: For every $f \in \mathcal{D}$, the class $\mathcal{E}_f$ is not transitive.*

(III) Partial hyperbolicity*: For any $f \in \mathcal{U}$, the chain-recurrence class $\mathcal{E}_f$ is partially hyperbolic with a one-dimensional central bundle.*

More precisely:

(1) *For every $f \in \mathcal{U}$, there exists a hyperbolic periodic point $p \in \mathcal{E}_f$ whose homoclinic class H_f is inside $\mathcal{E}_f$ and coincides with the homoclinic class of any other hyperbolic periodic point of $\mathcal{E}_f$. Moreover, any two hyperbolic periodic points in $\mathcal{E}_f$ with the same stable dimension are homoclinically related.*

(2) *For any $f \in \mathcal{U}$ there exists a hyperbolic periodic point $q \in \mathcal{E}_f$ satisfying $\dim E^s_p = \dim E^s_q + 1$ and $\mathcal{E}_f$ is the disjoint union of H_f with $W^u(p)\cap W^s(q)$. Moreover, the points of $W^u(p) \cap W^s(q)$ are isolated in $\mathcal{E}_f$. In particular, if $W^u(p) \cap W^s(q) \neq \emptyset$, the class $\mathcal{E}_f$ is not transitive.*

(3) *The dense set $\mathcal{D}$ is given by $\mathcal{D} := \{f \in \mathcal{U} \ : \ W^u(p) \cap W^s(q) \neq \emptyset\}$ and is a countable union of one-codimensional submanifolds of $\mathcal{U}$.*

(4) *The chain-recurrent set of any $f \in \mathcal{U}$ is the union of $\mathcal{E}_f$ with a finite number of hyperbolic periodic points (which depend continuously on f).*

Remark 2. The isolated points $\mathcal{E}_f \setminus H_f$ are non-wandering for f. However, they do not belong to $\Omega(f|_{\Omega(f)})$ (since they are isolated in $\Omega(f)$ and non-periodic).

Our construction strongly uses the existence of strong stable and strong unstable manifolds outside of the chain-recurrence class $\mathcal{E}_f$. Therefore this class cannot be an *attractor*,[3] that is, a transitive set with a neighborhood U satisfying $\mathcal{E}_f = \bigcap_{n\geq 0} f^n(U)$, nor can it be an attractor for f^{-1}. The following question remains open.

Question 3. Is there a dense G_δ subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that every transitive attractor of any $f \in \mathcal{G}$ is robustly transitive?[4]

In section 2 we give a precise definition of cuspidal points and prove the isolated point criterium (see Proposition 2). Then in section 3, we construct an explicit example that satisfies the main theorem.

2. A mechanism for the existence of isolated points in a chain-recurrence class

In this section, we prove the isolated point criterium explained in the introduction. We first recall a few definitions regarding partially hyperbolic dynamics. We precisely define the notion of cuspidal points which plays a fundamental role in our construction. The mechanism is made explicit in section 2.4, and the criterium (Proposition 2) is finally proved in section 2.5.

2.1. **Preliminaries on invariant bundles.** Consider $f \in \mathrm{Diff}^1(M)$ preserving a set Λ, that is, $f(\Lambda) = \Lambda$.

A Df-invariant subbundle $E \subset T_\Lambda M$ is *uniformly contracted* (resp. *uniformly expanded*) if there exists $N > 0$ such that for every unit vector $v \in E$, we have

$$\|Df^N v\| < \frac{1}{2} \qquad (\text{resp. } > 2).$$

A Df-invariant splitting $T_\Lambda M = E^{ss} \oplus E^c \oplus E^{uu}$ is *partially hyperbolic* if E^{ss} is uniformly contracted, E^{uu} is uniformly expanded, both are non-trivial, and if there exists $N > 0$ such that for any $x \in \Lambda$ and any unit vectors $v_s \in E^{ss}_x, v_c \in E^c_x$ and $v_u \in E^{uu}_x$ we have

$$2\|Df^N v_s\| < \|Df^N v_c\| < \frac{1}{2}\|Df^N v_u\|.$$

E^{ss}, E^c and E^{uu} are called *strong stable*, *center*, and *strong unstable* bundles.

Remark 3. We will sometimes consider a Df-invariant continuous orientation of E^c. In a one-dimensional bundle, an orientation corresponds to a continuous unit vector field tangent. When Λ is the union of two different periodic orbits O_p, O_q and of a heteroclinic orbit $\{f^n(x)\} \subset W^u(O_p) \cap W^s(O_q)$, such an orientation exists if and only if above each orbit O_p, O_q, the tangent map Df preserves an orientation of the central bundle.

[3] In fact, it cannot be a quasi-attractor either, but we will not define this concept here; see for example [BC].

[4] Flavio Abdenur and the second author announced a positive answer in a particular case: Question 3 has a positive answer if f is transitive and preserves a global partially hyperbolic structure with a one-dimensional center bundle.

2.2. **Preliminaries on homoclinic classes and chain-recurrence classes.** Let $H_f(p)$ denote the homoclinic class of a hyperbolic periodic point p for a diffeomorphism f. As defined in the introduction, by this we mean the closure of the transverse intersections between the stable and unstable manifolds of points in the orbit of p.

We recall some of their main properties:

- They are compact and transitive invariant sets, and periodic points of the same index as p are dense in $H_f(p)$.
- If two hyperbolic periodic points p_1, p_2 with orbits O_{p_1} and O_{p_2} of the same stable dimension are *homoclinically related* (i.e. there are transverse intersections between $W^s(O_{p_1})$ and $W^u(O_{p_2})$ and between $W^s(O_{p_2})$ and $W^u(O_{p_1})$), then their homoclinic classes coincide.

An important property of a homoclinic class is that, as for hyperbolic periodic points, we can define their continuation. This means that if $H_f(p)$ is the homoclinic class of a periodic point p for a diffeomorphism f, then, in a neighborhood $\mathcal{U}_0$ where the continuation of p is well defined, we can define $H_g(p_g)$ to be the homoclinic class of p_g for $g \in \mathcal{U}_0$. By the continuous variation of stable and unstable manifolds in compact parts we know that the map $g \mapsto H_g(p_g)$ is *semicontinuous* in the sense that for $g_0 \in \mathcal{U}_0$ given an open set U of M such that $H_{g_0}(p_{g_0}) \cap U \neq \emptyset$, there exists $\mathcal{U}_1$ a neighborhood of g_0 in $\mathcal{U}_0$ such that if $g \in \mathcal{U}_1$, we have that $H_g(p_g) \cap U \neq \emptyset$.

We will also list some properties of chain-recurrence classes that we will use later and that are easy to prove:

- If $p, q \in \mathcal{E}$ a chain-recurrence class, then every point in $W^s(p) \cap W^u(q)$ also belongs to $\mathcal{E}$. In particular, the homoclinic class of a periodic point is contained in its chain-recurrence class.
- For any non-trivial chain-recurrence class $\mathcal{E}$ we have that if it possesses an isolated point, then the chain-recurrence class is not transitive.

In general, the continuation of a chain-recurrence class is not well defined, but as was explained in the introduction there is an important exception to this rule: If a chain-recurrence class is *robustly isolated* we know that it coincides robustly with the maximal invariant set of some neighborhood. In this case, it also varies semicontinuously but in the inverse sense that if $\mathcal{E}_f$ is disjoint from an open set U, then there is a neighborhood $\mathcal{U}$ of f for which if $g \in \mathcal{U}$ we will have that $\mathcal{E}_g \cap U = \emptyset$.

In both cases, using standard point-set topology arguments we know that for a residual subset of diffeomorphisms the variation is in fact continuous.

2.3. **Cuspidal periodic points.** Let p be a hyperbolic periodic point whose orbit is partially hyperbolic, with a stable one-dimensional central bundle: the stable bundle at p writes as $E^s_p = E^{ss}_p \oplus E^c_p$, and $\dim E^c_p = 1$. Then there exists a uniquely defined strong stable manifold $W^{ss}(p)$ tangent to E^{ss}_p that is invariant by the iterates f^τ that fixes p. It is contained in $W^s(p)$ and separates it into two closed manifolds whose boundary is $W^{ss}(p)$. We call them the *half stable manifolds*.

Let us consider an orientation of E^c_p. The unit vector at p defining the orientation goes inward on one of the half stable manifolds of p that we call the *right half stable manifold* $R^s(p)$. The other one is called the *left half stable manifold* $L^s(p)$. These half stable manifolds are invariant by an iterate f^τ which fixes p if and only if the orientation of E^c_p is preserved by Df^τ_p. When the central space is unstable, one similarly defines the right and left half unstable manifolds $R^u(p), L^u(p)$.

Definition 1. A hyperbolic periodic point p is *right stable cuspidal* if:

- its orbit is partially hyperbolic and the central bundle is one-dimensional and stable;
- the left half stable manifold of p intersects the chain-recurrence class of p only at p.

One symmetrically defines the *left stable cuspidal points* and can define *left/right unstable cuspidal points* in a similar way.

Let p be a right stable cuspidal point and assume that the chain-recurrence class $\mathcal{E}$ containing p is not reduced to the orbit O_p of p; then, there must be intersection points between $\mathcal{E}$ and $W^s(p)$ different from p. This implies that there must exist a Df-invariant orientation on the central bundle of O_p and that $\mathcal{E}$ intersects $W^s(p)$ in the right half stable manifold of p. A similar remark holds for any other type of cuspidal point.

The choice of the name cuspidal has to do with the geometry it imposes on $\mathcal{E} \cap W^s(p)$ in a neighborhood of p; see Figure 3. This notion appears with the same definition as in [BD$_5$] but in a different setting. It is stronger than the notion of *stable boundary points* defined in [CP].

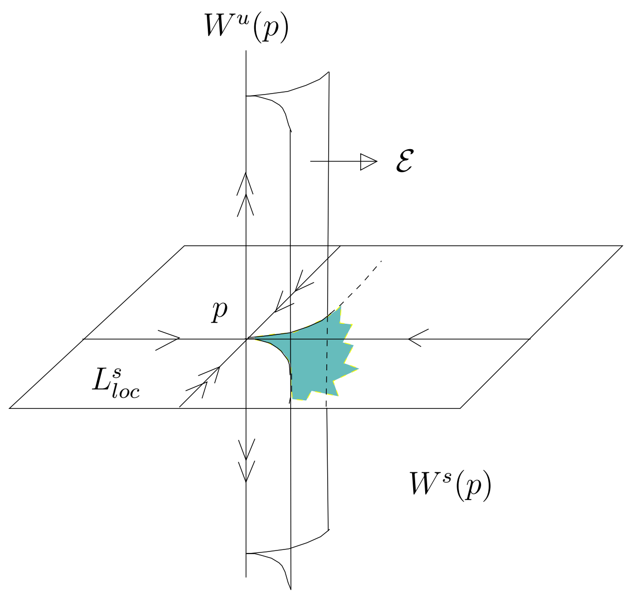

FIGURE 3. Geometry of a chain-recurrence class $\mathcal{E}$ near a stable cuspidal fixed point.

Lemma 1. *If p is a right stable cuspidal point, then the hyperbolic continuation p_g is still right stable cuspidal for every g that is C^1-close to f.*

Proof. Denote by τ the period of p. Let D^s be an open ball in $W^s(p)$ whose closure is contained in $W^s(p)$ and such that $f^\tau(\overline{D^s})$ is contained in D^s. Then $\Delta^s = \overline{D^s} \setminus f^\tau(D^s)$ contains a fundamental domain of $W^s(p) \setminus \{p\}$ for f^τ. In particular, $\Delta^{s,L} = \Delta^s \cap L^s(p)$ is a compact set which meets every orbit of $L^s(p) \setminus \{p\}$ and which is disjoint from $\mathcal{R}(f)$.

By the upper semicontinuity of the chain-recurrent set (see section 2.2), a small neighborhood V of $\Delta^{s,L}$ is disjoint from $\mathcal{R}(g)$ for any g close to f and meets every orbit of the continuation of $L^s(p) \setminus \{p\}$. $\square$

2.4. **Description of the mechanism.** Let x be a point in a chain-recurrence class $\mathcal{E}$. We introduce the following assumptions (see Figure 4):

(H1) $\mathcal{E}$ contains two periodic points p, q such that $\dim(E^s_p) = \dim(E^s_q) + 1$.

(H2) The point x belongs to $W^u(p) \cap W^s(q)$. The union Λ of the orbits of x, p, q has a partially hyperbolic decomposition with a one-dimensional central bundle.

(H3) There exists a Df-invariant continuous orientation of the central bundle over Λ for which:
  (i) the point p is a right stable cuspidal point;
  (ii) the point q is a left unstable cuspidal point.

Note that from Remark 3 and the fact that a central orientation is preserved for cuspidal points, a Df-invariant continuous orientation of the central bundle over Λ always exists. The following proposition implies the isolation point criterion stated in the introduction.

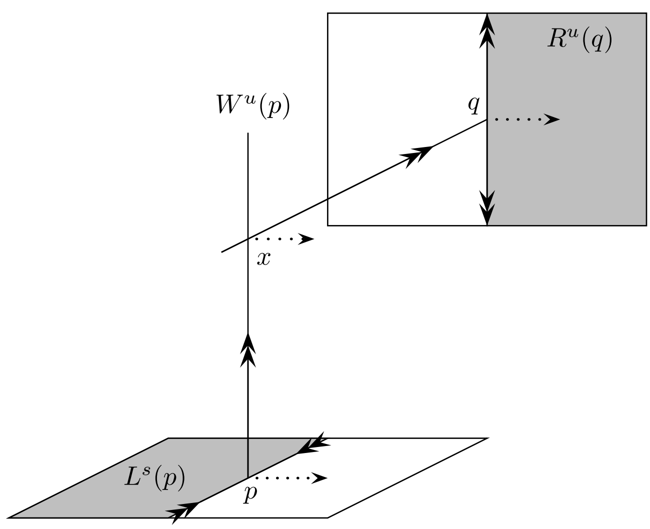

FIGURE 4. Hypotheses (H1)-(H3). The central orientation is given at each point by a unit vector tangent to the central bundle.

Proposition 2. *Under (H1)-(H3), the point x is isolated in the chain-recurrence class $\mathcal{E}$. Thus, $\mathcal{E}$ is not transitive and in particular not a homoclinic class.*

2.5. **Proof of Proposition 2.** We shall denote $d^u = \dim E^u_p = \dim E^u_q - 1$ and $d^s = \dim E^s_q = \dim E^s_p - 1$. We have that $\dim M = d^s + d^u + 1$.

This section consists of three parts. We first define a notion of coherence (with respect to the central orientation) for embeddings of the disk $[-1,1]^{d^u+1}$ centered at a point of $W^s(q)$ and that transversely intersect $W^s(q)$. Then Lemma 3 will give us that the right part of such small disks has an iterate close to a small neighborhood $V^{u,R}$ of a fundamental domain of the right-hand part of $W^u(q)$.

Finally we will deduce that all points in the right part of a small cube centered at x admit a positive iterate in $V^{u,R}$ and in particular do not intersect $\mathcal{E}$. The same result on the left will end the proof of Proposition 2 since it implies that inside the cube centered at x the only point of $\mathcal{E}$ is x.

Let q be the periodic point given by hypotheses (H2)-(H3). Any $z \in W^s(q)$ has a stable direction E^s_z of dimension d^s and a center stable direction E^{cs}_z of dimension d^s+1, both being uniquely defined. The first one is the tangent space $T_zW^s(q)$; the second one is the (d^s+1)-space made of the vectors $v \in T_zM \setminus \{0\}$ whose positive iterates stay away from E^{uu}_q.

If $E' \subset E$ are two vector subspaces of T_zM such that

$$E_z \oplus E^s_z = E'_z \oplus E^{cs}_z = T_zM$$

(in particular, $\dim E = d^u + 1$ and $\dim E' = d^u$), then $F = E^{cs}_z \cap E$ is a one-dimensional space whose forward iterates converge to the central bundle over the orbit of q (see Figure 5). As a consequence, there exists an orientation of F which converges by forward iterations to the orientation of the central bundle.

Recall that the orientation is seen as a unit vector in F. There is a connected component of $E \setminus E'$ that contains it. The closure of this component is called the *right half-plane* of $E \setminus E'$. The closure of the other component is the *left half-plane* of $E \setminus E'$.

Consider a C^1-embedding $\varphi\colon [-1,1]^{d^u+1} \to M$ such that $z := \varphi(0)$ belongs to $W^s(q)$.

Definition 2. The embedding φ is *coherent with the central orientation at* q if

- $E := D_0\varphi(\mathbb{R}^{d^u+1})$ and $E' := D_0\varphi(\{0\} \times \mathbb{R}^{d^u})$ are transverse to E^s_z, E^{cs}_z respectively;
- the half-plaque $\varphi([0,1] \times [-1,1]^{d^u})$ is tangent to the right half-plane of $E \setminus E'$.

Let $\tau \geq 1$ be the period of q and $\chi\colon [-1,1]^d \to M$ be coordinates such that

- $\chi(0) = q$;
- the image $D^u := \chi((-1,1) \times \{0\}^{d-d^u-1} \times (-1,1)^{d^u})$ is contained in $W^u(q)$;
- the image $D^{uu} := \chi(\{0\}^{d-d^u} \times (-1,1)^{d^u})$ is contained in $W^{uu}(q)$;
- the image $D^{u,R} := \chi([0,1) \times \{0\}^{d-d^u-1} \times (-1,1)^{d^u})$ is contained in $R^u(q)$;
- $f^{-\tau}(\overline{D^u})$ is contained in D^u.

We define $D^{u,\pm}$ as the components of $D^u \setminus D^{uu}$ with $+$ or $-$ corresponding to the chosen orientation.

Let $\Delta^u = \overline{D^{u,+}} \setminus f^{-\tau}(D^u)$. Then the right-hand part

$$\Delta^{u,R} = \Delta^u \cap R^u(q)$$

of Δ^u is a compact set that meets each orbit of $R^u(q) \setminus \{q\}$ (see Figure 5).

Lemma 3. *Let $\mathcal{A} = [-\varepsilon, \varepsilon]^{d_1}$ and $\{\varphi_a\}_{a \in \mathcal{A}}$ be a continuous family of C^1-embeddings that are coherent with the central orientation at q. Consider any $a_0 \in \mathcal{A}$ and any neighborhood $V^{u,R}$ of $\Delta^{u,R}_a$.*

Then, there exist $\delta > 0$ and some neighborhood A of a_0 in $\mathcal{A}$ such that, for any $a \in A$, any point $z \in \varphi_a([0,\delta] \times [-\delta,\delta]^{d^u})$ different from $\varphi_a(0)$ has a forward iterate in V^u.

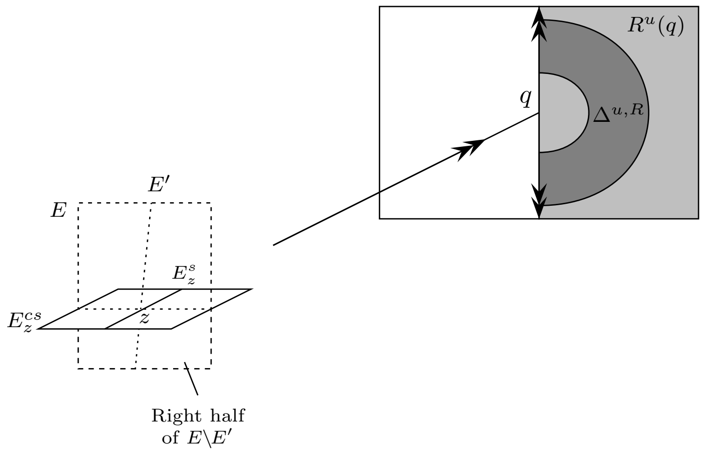

FIGURE 5. The positive iterates of the right half-plane of $E\setminus E'$ converge to the half-space tangent to $R^u(q)$ at q.

Proof. The graph transform argument (see for instance [KH, section 6.2]) gives the following generalization of the λ-lemma. See Figure 6.

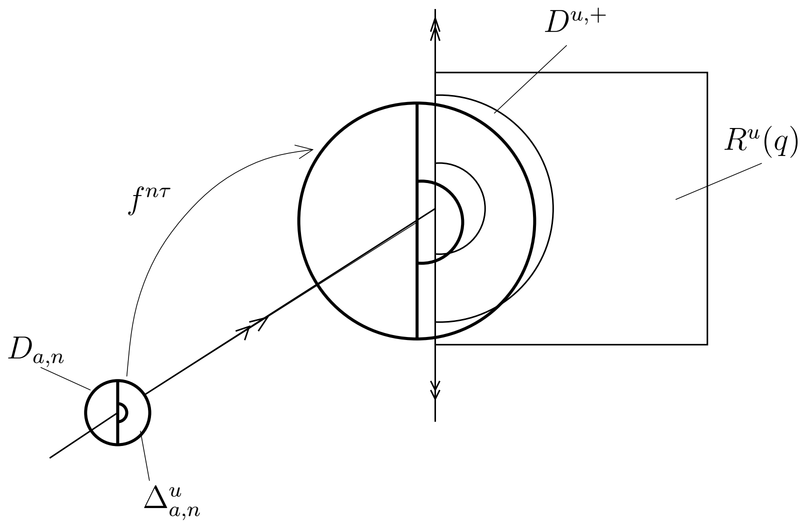

FIGURE 6. The λ-lemma type argument.

Claim. There exists $N \geq 0$ and, for all a in a neighborhood A of a_0, there exist some decreasing sequences of disks $(D_{a,n})$ of $[-1,1]^{d^u+1}$ and $(D'_{a,n})$ of $\{0\}\times[-1,1]^{d^u}$ which contain 0 and such that for any $n \geq N$ one has, in the coordinates of χ:

- $f^{n\tau} \circ \varphi_a(D_{a,n})$ is the graph of a function $F_n\colon D^u \to \mathbb{R}^{d-d^u-1}$ that is C^1-close to 0;
- $f^{n\tau} \circ \varphi_a(D'_{a,n})$ is the graph of a function $F'_n\colon D^{uu} \to \mathbb{R}^{d-d^u}$ that is C^1-close to 0.

In particular, for large n, for any $a \in A$, the image by $f^{n\tau}$ of

$$\Delta^u_{a,n} = \varphi_a(D_{a,n} \setminus D_{a,n+1})$$

is contained in a small neighborhood of Δ^u.

For small $\delta > 0$, for any $a \in A$, any point $y \in \varphi_a([-\delta,\delta] \times [-\delta,\delta]^{d^u})$ different from $\varphi_a(0)$ belongs to some $\Delta^u_{a,n}$, with $n \geq N$. Consequently:

(i) Any $y \in \varphi_a([-\delta,\delta] \times [-\delta,\delta]^{d^u}) \setminus \{\varphi_a(0)\}$ has a forward iterate $f^n(y)$ in a small neighborhood of Δ^u (and y belongs to $\varphi_a(D_{a,n})$).

Let us consider $a \in A$. The image by $f^{n\tau}$ of each component of $D_{a,n} \setminus D'_{a,n}$ is contained in a small neighborhood of one component of $D^u \setminus D^{uu}$. The angle between $f^{n\tau}(D'_{a,n})$ and the central direction at q is uniformly bounded from below. Since φ is coherent with the central orientation at q, one deduces that

(ii) $f^{n\tau} \circ \varphi_a\left(([0,1] \times [-1,1]^{d^u}) \cap D_{a,n}\right)$ is contained in a small neighborhood of $D^{u,R}$.

Putting properties (i)-(ii) together, one gets the announced property. □

Proof of Proposition 2. We have that d^s+1 (resp. d^u+1) is the stable dimension of p (resp. the unstable dimension of q) so that the dimension of M satisfies $d = d^s+d^u+1$. Consider a C^1-embedding $\varphi : [-1,1]^d \to M$ with $\varphi((0,0^{d^u},0^{d^s})) = x$ such that:

- $\varphi(\{0\} \times [-1,1]^{d^s} \times \{0\}^{d^u})$ is contained in $W^s(q)$;
- $\varphi(\{0\} \times \{0\}^{d^s} \times [-1,1]^{d^u})$ is contained in $W^u(p)$;
- $D_0\varphi.(1,0^{d^s},0^{d^u})$ is tangent to E^c_x and has positive orientation.

Note that, by construction, all the restrictions of φ to $[-1,1] \times \{a^s\} \times [-1,1]^{d^u}$ for $a^s \in \mathbb{R}^{d^s}$ close to 0 are embeddings coherent with the central orientation at q (cf. Definition 2).

Let $\Delta^{u,R} \subset R^u(q)\setminus\{q\}$ be as above. Since $\mathcal{E}$ is closed and q is left unstable cuspidal, there is a neighborhood $V^{u,R}$ of $\Delta^{u,R}$ in M that is disjoint from $\mathcal{E}$. Then Lemma 3 can be applied: the points in $\varphi([0,\delta] \times \{a^s\} \times [-\delta,\delta]^{d^u})$ distinct from $\varphi(0,a^s,0^{d^u})$ have an iterate in $V^{u,R}$, and hence do not belong to $\mathcal{E}$. This shows that

$$\mathcal{E} \cap \varphi\left([0,\delta] \times [-\delta,\delta]^{d-1}\right) \subset \varphi\left(\{0\} \times [-\delta,\delta]^{d^s} \times \{0\}^{d^u}\right).$$

Hypotheses (H1), (H2) and (H3) are symmetrical upon the exchange of p and q, replacement of f by f^{-1}, and reversal of central orientation. Thus, one argues the same way to get:

$$\mathcal{E} \cap \varphi\left([-\delta,0] \times [-\delta,\delta]^{d-1}\right) \subset \varphi\left(\{0\} \times \{0\}^{d^s} \times [-\delta,\delta]^{d^u}\right).$$

Both inclusions give that

$$\mathcal{E} \cap \varphi\left([-\delta,\delta]^d\right) = \{\varphi(0)\},$$

which says that $x = \varphi(0)$ is isolated in $\mathcal{E}$. □

3. Construction of examples

In this part we build a collection of diffeomorphisms satisfying the properties (I) and (II) stated in the theorem. The construction will be made only in dimension 3 for simplicity. The generalization to higher dimensions is straightforward.

3.1. **Construction of a diffeomorphism.** Let us consider an orientation-preserving C^∞ diffeomorphism h of the plane $\mathbb{R}^2$ and a closed subset $D = D^- \cup C \cup D^+$ such that (see Figure 7):

- D is an attracting region: $h(\overline{D}) \subset \mathrm{Int}(D)$, and every orbit of $\mathbb{R}^2$ meets D;
- $h(\overline{D^- \cup D^+}) \subset \mathrm{Int}(D^-)$;
- the forward orbit of any point in D^- converges towards a sink $S \in D^-$;
- C is the square $[0,5]^2$ whose maximal invariant set by h is a hyperbolic horseshoe;
- D^- and D^+ are topological disks.

On $C \cap h^{-1}(C)$ the map h is piecewise linear, it preserves and contracts by $1/5$ in the horizontal direction and it preserves and expands by 5 in the vertical direction (see Figure 7):

- The set $C \cap h(C)$ is the union of 4 disjoint vertical bands I_1, I_2, I_3, I_4 of width 1. We will assume that $I_1 \cup I_2 \subset (0, 2+\frac{1}{3}) \times [0,5]$ and $I_3 \cup I_4 \subset (2+\frac{2}{3}, 5) \times [0,5]$.
- The preimage $h^{-1}(C) \cap C$ is the union of 4 horizontal bands $h^{-1}(I_i)$. We will assume that $h^{-1}(I_1 \cup I_2) \subset [0,5] \times (0, 2+\frac{1}{3})$ and $h^{-1}(I_3 \cup I_4) \subset [0,5] \times (2+\frac{2}{3}, 5)$.

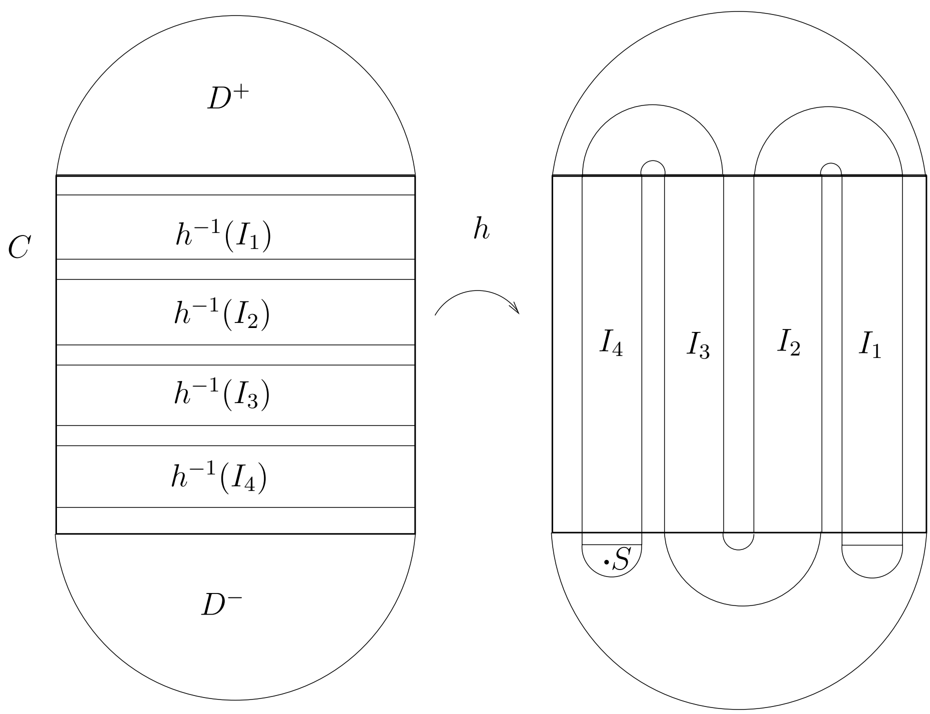

FIGURE 7. The map h.

We now define a C^∞ diffeomorphism F of $\mathbb{R}^3$ whose restriction to a neighborhood of $D \times [-1, 6]$ is a skew product of the form

$$F\colon (x,t) \mapsto (h(x), g_x(t)),$$

where the diffeomorphisms g_x are orientation-preserving and satisfy (see Figure 8):

(P1) g_x does not depend on x in the sets $h^{-1}(I_i)$ for every $i = 1, 2, 3, 4$.
(P2) For every $(x,t) \in D \times [-1, 6]$ one has $4/5 < g'_x(t) < 6/5$.
(P3) g_x has exactly two fixed points inside $[-1, 6]$, which are $\{0, 4\}$, $\{3, 4\}$, $\{1, 2\}$ and $\{1, 5\}$, when x belongs to $h^{-1}(I_i)$ for i respectively equal to $1, 2, 3$ and 4. All fixed points are hyperbolic; moreover,
  (i) $g'_x(t) < 1$ for $t \in [-1, 3 + 1/2]$ and $x \in h^{-1}(I_1) \cup h^{-1}(I_2)$.
  (ii) $g'_x(t) > 1$ for $t \in [1 + 1/2, 6]$ and $x \in h^{-1}(I_3) \cup h^{-1}(I_4)$.
(P4) For every $(x,t) \in (D^- \cup D^+) \times [-1, 6]$ one has $g_x(t) > t$.

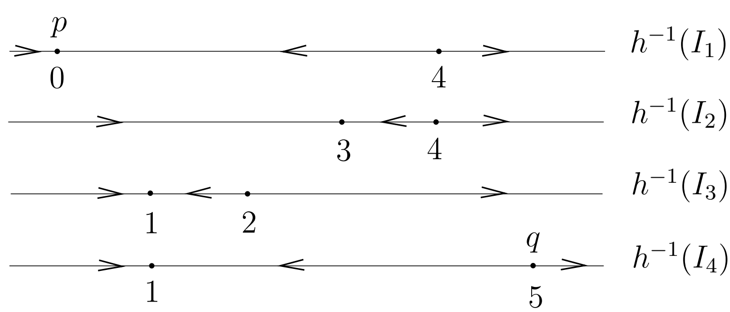

FIGURE 8. The map g_x above each rectangle $h^{-1}(I_i)$.

We assume furthermore that the following properties are satisfied:

(P5) $F(D \times [6, 8]) \subset \mathrm{Int}(D \times [6, 8])$;
(P6) there exists a sink which attracts the orbit of any point of $D \times [6, 8]$;
(P7) F coincides with a linear homothety outside a compact domain;
(P8) $D \times [-1, 8]$ is an attracting region (its closure is mapped into its interior), and any forward orbit of $\mathbb{R}^3$ meets $D \times [-1, 8]$.

One can build a diffeomorphism which coincides with the identity on a neighborhood of the boundary of $D_0 \times (-2, 9)$ and coincides with F in $D \times (-1, 8)$ (D_0 denotes a small neighborhood of D in $\mathbb{R}^2$).

On any three-dimensional manifold, one can consider an orientation-preserving Morse-Smale diffeomorphism and by surgery replace the dynamics on a neighborhood of a sink by the dynamics of F on $D_0 \times (-2, 9)$. We denote by f_0 the diffeomorphism obtained by this surgery.

More generally, any diffeomorphism of a three-dimensional manifold can be deformed by isotopy to be axiom A with a sink (see [F]). This implies that, on any three-dimensional manifold, every isotopy class of diffeomorphisms contains an element whose restriction to an invariant set is C^∞-conjugated to F.

3.2. **First robust properties.** We list some properties satisfied by f_0 which are also satisfied by any diffeomorphism f in a small C^1-neighborhood $\mathcal{U}_0$ of f_0.

Fixed points: By (P3), in each rectangle $\mathrm{Int}(I_i) \times (-1, 6)$ there exist two hyperbolic fixed points p_i, q_i. Their stable dimensions are respectively equal to 2 and 1. Since p_1 and q_4 will play special roles, we shall denote them as $p = p_1$ and $q = q_4$.

Isolation: The two open sets $V_0 = \mathrm{Int}(D)\times(-1, 8)$ and $V_1 = V_0\setminus(C\times[-1, 6])$ are isolating blocks, i.e. satisfy $f(\overline{V_0}) \subset V_0$ and $f(\overline{V_1}) \subset V_1$.

For V_0, the property follows immediately from the construction. Using that $h(\overline{D^- \cup D^+}) \subset \mathrm{Int}(D^-)$ and property (P3), one gets that the closure of the second set V_1 can be decomposed as the union of:
- $D^+ \times [-1, 6]$, which is mapped into $(D^- \times [-1, 6]) \cup (D \times [6, 8])$,
- $D^- \times [-1, 6]$, which is also mapped into $(D^- \times [-1, 6]) \cup (D \times [6, 8])$ and moreover has a forward iterate in $D \times [6, 8]$ by (P4),
- $D \times [6, 8]$, which is mapped into itself and whose limit set is a sink.

Hence, any chain-recurrence class which meets the rectangle $C \times [-1, 6]$ is inside the rectangle. The maximal invariant set of f in $C\times[-1, 6]$ will be denoted by $\mathcal{E}_f$. Notice that in principle $\mathcal{E}_f$ may not be a chain-recurrence class. We will prove below that this is indeed the case.

The set V_1 is contained in the basin of attraction of the sink inside $D \times [6, 9]$ (see (P6)); therefore any chain-recurrence class which meets V_1 is reduced to that sink.

Partial hyperbolicity (property (III) of the theorem) and (H2): On $C \times [-1, 6] \subset \mathbb{R}^3$, there exists some narrow cone fields Γ^s, Γ^{cs} around the coordinate direction $(1, 0, 0)$ and the plane $(x_1, 0, t)$ which are invariant by Df^{-1}. The vectors tangent to Γ^s are uniformly expanded by Df^{-1}. Similarly there exist some forward invariant cone fields Γ^u, Γ^{cu} close to the direction $(0, 1, 0)$ and the plane $(0, x_2, t)$.

In particular, $\mathcal{E}_f$ is partially hyperbolic. Moreover, the tangent map Df preserves the orientation of the central direction such that any positive unitary central vector is close to the vector $(0, 0, 1)$.

If there exists an intersection point $x \in W^u(p) \cap W^s(q)$ for f, then by the isolating property it must be contained in $\mathcal{E}_f$.[5]

Therefore it will be enough to have $W^u(p) \cap W^s(q) \neq \emptyset$ to satisfy (H2).

Central expansion: Property (P2) holds for f when one replaces the derivative $g'_x(t)$ by the tangent map $\|Df|_{E^c}(x, t)\|$ along the central bundle.

Property (H3): There is a central orientation as was noticed before, and for this orientation the point p is stable cuspidal and the point q is unstable cuspidal: the left half stable manifold of p and the right half unstable manifold of q are disjoint from $\mathcal{E}_f$. Since the chain-recurrence classes of p and q are contained in $\mathcal{E}_f$, this implies property (H3).

Let us explain how to prove this property: it is enough to discuss the case of the left half stable manifold of p and (arguing as in Lemma 1) to

[5] This does not need $\mathcal{E}_f$ to be a chain-recurrence class. Indeed, since $\mathcal{E}_f$ is the maximal invariant set in $V_0 \setminus V_1$ we have that $W^u(p) \subset \bigcap_{n\geq 0} f^n(\overline{V_0})$ and $W^s(q) \subset \bigcap_{n\leq 0} f^n((V^1)^c)$, so the intersection points must be in $\mathcal{E}_f$.

assume that $f = f_0$. From (P2) and (P3), we have:
- every point in $C \times [-1, 0)$ has a backward iterate outside $C \times [-1, 6]$;
- the same holds for every point in $(C \setminus I_1) \times \{0\}$;
- any point in $I_1 \times \{0\}$ has some backward image in $(C \setminus I_1) \times \{0\}$, unless it belongs to $W^u(p)$.

Combining these properties, one deduces that the connected component of $W^s(p) \cap (C \times [-1, 0])$ containing p intersects $\mathcal{E}_f$ only at p. Note that this is a left half stable manifold of p, giving the required property.[6]

Hyperbolic regions: By (P3) item (i), the maximal invariant set K_p in $Q_p := [0, 5] \times [0, 2 + \frac{1}{3}] \times [-1, 3 + \frac{1}{2}]$ is hyperbolic of stable dimension 2. The stable manifold of the points in K_p are nearly parallel to the plane $(x_1, 0, t)$, and the unstable ones are nearly parallel to the line $(0, x_2, 0)$. Since they are large, they give a local product structure in K_p, so K_p is compact and transitive (see [KH]). We can argue symmetrically on the maximal invariant set K_q in $Q_q := [0, 5] \times [2 + \frac{2}{3}, 5] \times [1 + \frac{1}{2}, 6]$, this time using (P3) item (ii).

Summarizing:
- K_p is a hyperbolic compact and transitive set of stable dimension 2 which contains p, p_2.
- K_q is a hyperbolic compact and transitive set of stable dimension 1 which contains q, q_3.

Tameness (property (4) of the theorem): Since f_0 has been obtained by surgery of a Morse-Smale diffeomorphism and f is a small perturbation of f_0, the chain-recurrent set of f in $M \setminus \mathcal{E}_f$ is a finite union of hyperbolic periodic orbits.

Any $x \in \mathcal{E}_f$ has a strong stable manifold $W^{ss}(x)$. Its *local* strong stable manifold $W^{ss}_{loc}(x)$ is the connected component containing x of the intersection $W^{ss}(x) \cap C \times [-1, 6]$. It is a curve bounded by $\{0, 5\} \times [0, 5] \times [-1, 6]$. Symmetrically, we define $W^{uu}(x)$ and $W^{uu}_{loc}(x)$.

3.3. **Central behaviors of the dynamics.** We analyze the local strong stable and strong unstable manifolds of points of $\mathcal{E}_f$ depending on their central position. For the next lemma see Figure 9.

Lemma 4. *There exists an open set $\mathcal{U}_1 \subset \mathcal{U}_0$ such that for every $f \in \mathcal{U}_1$ and $x \in \mathcal{E}_f$:*

(R1) *If $x \in R_1 := C \times [-1, 3 + \frac{1}{2}]$, then $W^{uu}_{loc}(x) \cap W^s(p) \neq \emptyset$.*
(R2) *If $x \in R_2 := C \times [1 + \frac{1}{2}, 6]$, then $W^{ss}_{loc}(x) \cap W^u(q) \neq \emptyset$.*
(R3) *If $x \in R_3 := C \times [\frac{1}{2}, 2 + \frac{1}{2}]$, then $W^{ss}_{loc}(x) \cap W^{uu}_{loc}(y) \neq \emptyset$ for some $y \in K_p$.*
(R4) *If $x \in R_4 := C \times [2 + \frac{1}{2}, 4 + \frac{1}{2}]$, then $W^{uu}_{loc}(x) \cap W^{ss}_{loc}(y) \neq \emptyset$ for some $y \in K_q$.*

Moreover, p_2 belongs to R_2 and q_3 belongs to R_1.

Proof. Properties (R1) and (R2) follow directly from the continuous variation of the stable and unstable manifolds; this may require reducing the neighborhood $\mathcal{U}_0$. Similarly $p_2 \in R_2$ and $q_3 \in R_1$ by continuity.

We prove (R3) with standard blender arguments (see [BD$_1$] and [BDV, chapter 6] for more details). The set K_p is a called *blender-horseshoe* in [BD$_4$, section 3.2]. We need some preliminary definitions.

[6] This argument does not work for the other periodic points of stable index two in $\mathcal{E}_f$: we rely strongly on the fact that the center coordinate of p is the smaller among periodic points.

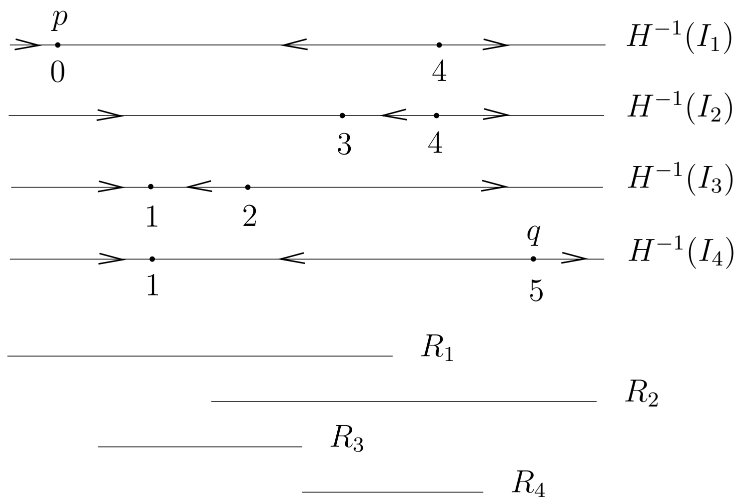

FIGURE 9. The maps g_x and the rectangles $R_1, \dots, R_4$

A *cs-strip* $\mathcal{S}$ is the image by a diffeomorphism $\phi : [-1,1]^2 \to Q_p = [0,5] \times [0, 2+\frac{1}{3}] \times [-1, 3+\frac{1}{2}]$ such that:

- The surface $\mathcal{S}$ is tangent to the center-stable cone field and meets $C \times [\frac{1}{2}, 2+\frac{1}{2}]$.
- The curves $\phi(t, [-1,1])$, $t \in [-1,1]$, are tangent to the strong stable cone field and cross Q_p, i.e. $\phi(t, \{-1,1\}) \subset \{0,5\} \times [0, 2+\frac{1}{3}] \times [-1, 3+\frac{1}{2}]$.
- $\mathcal{S}$ does not intersect $W^u_{loc}(p) \cup W^u_{loc}(p_2)$.

The *width* of $\mathcal{S}$ is the minimal length of the curves contained in $\mathcal{S}$, tangent to the center cone, and that joins $\phi(-1, [-1,1])$ and $\phi(1, [-1,1])$.

Condition (P2) (and (P3)(i)) is important to get the following (see [BDV, lemma 6.6] for more details):

Claim. There exists $\lambda > 1$ such that if $\mathcal{S}$ is a *cs*-strip of width ε, then, either $f^{-1}(\mathcal{S})$ intersects $W^u_{loc}(p) \cup W^u_{loc}(p_2)$ or it contains at least one *cs*-strip with width $\lambda\varepsilon$.

Proof. Using (P2), the set $f^{-1}(\mathcal{S}) \cap C \times [-1, 6]$ is the union of two strips crossing $C \times [-1, 6]$: the first has its two first coordinates near $h^{-1}(I_1)$, the second near $h^{-1}(I_2)$. Their width is larger than $\lambda\varepsilon$, where $\lambda > 1$ is a lower bound of the expansion of Df^{-1} in the central direction inside Q. We assume by contradiction that none of them intersect $W^u_{loc}(p) \cup W^u_{loc}(p_2)$ or $C \times [\frac{1}{2}, 2+\frac{1}{2}]$.

Since $\mathcal{S}$ intersects $C \times [\frac{1}{2}, 2+\frac{1}{2}]$, from conditions (P2) and (P3) the first strip intersects $C \times [\frac{1}{2}, 4]$. By our assumption it is thus contained in $C \times (2+\frac{1}{2}, 4]$. Using (P2) and (P3) again, this shows that $\mathcal{S}$ is contained in $C \times (2, 4]$. The same argument with the second strip shows that $\mathcal{S}$ is contained in $C \times [-1, 2)$, a contradiction. □

As the size of the strips that do not meet $W^u_{loc}(p) \cup W^u_{loc}(p_2)$ is bounded, we can repeat the procedure and get an intersection point between $W^u_{loc}(p) \cup W^u_{loc}(p_2)$ and a backward iterate of the $cs-$strip. In turn it gives a transverse intersection point z between the initial $cs-$strip and $W^u(p) \cup W^u(p_2)$. By construction, all the past iterates of z belong to Q_p. Hence z has a well-defined local strong unstable manifold. In particular, the intersection y between $W^{uu}_{loc}(z)$ and $W^{s}_{loc}(p)$ (which exists by (R1)) remains in Q_p both for future and past iterates, thus, it belongs to K_p.

For any point $x \in \mathcal{E}_f \cap R_3$, we can find an arbitrarily thin cs-strip containing its local strong stable manifold. We have proved that this $cs-$ strip intersects $W^{uu}_{loc}(y)$ for some $y \in K_p$. One can consider a sequence $\mathcal{S}_n$ of thinner strips converging to $W^{ss}_{loc}(x)$ and a sequence $y_n \in K_p$ such that $\mathcal{S}_n$ intersects $W^{uu}_{loc}(y_n)$. Since K_p is closed and the local strong unstable manifolds vary continuously, we get at the limit an intersection between $W^{ss}_{loc}(x)$ and $W^{uu}_{loc}(y')$ for some $y' \in K_p$ as desired.

This gives (R3). Property (R4) can be obtained similarly. □

We have controlled the local strong unstable manifold of points in $R_1 \cup R_4$ and the local strong stable manifold of points in $R_2 \cup R_3$. Since neither $R_1 \cup R_4$ nor $R_2 \cup R_3$ cover $C \times [-1, 6]$ completely, we shall also need the following result (see also Figure 9):

Lemma 5. *For every diffeomorphism in a small C^1-neighborhood $\mathcal{U}_2 \subset \mathcal{U}_0$ of f_0, if a point has its backward orbit contained in $C \times [-1, \frac{1}{2}]$, then it belongs to $W^u(p)$; symmetrically, if a point has its forward orbit contained in $C \times [4 + \frac{1}{2}, 6]$, then it is in $W^s(q)$.*

Proof. We argue as for property (H3) in section 3.2: the set of points whose past iterates stay in $C \times [-1, \frac{1}{2}]$ is the local strong unstable manifold of p (which coincides with its unstable manifold since p has stable index 2). □

3.4. **Properties (I) and (II) of the theorem.** We now check that (I) and (II) hold for the region $U = \mathrm{Int}(C \times [-1, 6])$ and the neighborhood $\mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2$.

Proposition 6. *For any $f \in \mathcal{U}$ and $x \in \mathcal{E}_f$, there are arbitrarily large $n_q, n_p \geq 0$, such that $W^{uu}_{loc}(f^{n_q}(x)) \cap W^{ss}(y_q) \neq \emptyset$ and $W^{ss}_{loc}(f^{-n_p}(x)) \cap W^{uu}(y_p) \neq \emptyset$ for some $y_q \in K_q$, $y_p \in K_p$.*

Proof. If $\{f^n(x), n \geq n_0\} \subset C \times [4 + \frac{1}{2}, 6]$ for some $n_0 \geq 0$, then $x \in W^s(q)$ by Lemma 5 and this concludes.

In the remaining case, there are arbitrarily large forward iterates in $R_1 \cup R_4$. Then one of the following applies:

- There are arbitrarily large iterates in R_1 so that $W^{uu}_{loc}(f^n(x))$ meets $W^s(p)$ by Lemma 4. Since the unstable manifold of p intersects the stable one of p_2, by the λ-lemma there exists $k \geq 0$ such that $f^k(W^{uu}_{loc}(f^n(x)))$ contains $W^{uu}_{loc}(x')$ for some $x' \in W^s(p_2) \cap R_4$ because $p_2 \in R_4$.
- There are arbitrarily large iterates of x intersecting R_4.

In any case, by Lemma 4, $f^k(W^{uu}_{loc}(f^n(x)))$ intersects $W^{ss}_{loc}(y'_q)$ for some $y'_q \in K_q$ showing that $W^{uu}_{loc}(f^n(x)) \cap W^{ss}(y_q) \neq \emptyset$ with $y_q = f^{-k}(y'_q)$ in K_q.

We have obtained the first property in all the cases. The second property is similar. □

The following corollary (together with the isolation property of section 3.2) implies that for every $f \in \mathcal{U}$, properties (I) and (H1) are verified.

Corollary 7. *For every $f \in \mathcal{U}$ the set $\mathcal{E}_f$ is a chain-recurrence class.*

Proof. Given $\varepsilon > 0$, there exists n_0 such that for every point z in $\mathcal{E}_f$ and $n > n_0$, the set $f^n(W^{uu}_{loc}(z) \cap B_\varepsilon(z))$ contains $W^{uu}_{loc}(f^n(z))$.

So, using Proposition 6 we find $n_q > n_0$ which allows us to create an ε-pseudo-orbit from x to K_q: We start from $x = x_0$ and we consider x_1 as the point $f^{-n_q+1}(y)$ where y is the intersection point between $W^{uu}_{loc}(f^{n_q}(x))$ and $W^{ss}_{loc}(y_q)$ with $y_q \in K_q$ given by Proposition 6; then, since K_q is invariant, we know that $f^n(x_1)$ converges to K_q with $n \to +\infty$. Since K_q is transitive we can in fact construct an ε-pseudo-orbit from x to q_3. Symmetrically, there exists an ε-pseudo-orbit from p to x.

By Lemma 4, the unstable manifold of q_3 intersects the stable manifold of p; hence there exists an ε-pseudo-orbit from q_3 to p.

We can now take the concatenation of these pseudo-orbits in the appropriate order to conclude. $\square$

Now, we show that (H2) holds for a C^r dense set $\mathcal{D}$ of $\mathcal{U}$. Since (H1) and (H3) are satisfied, Proposition 2 implies that property (II) of the theorem holds with the set $\mathcal{D} \subset \mathcal{U}$. In fact, as we noticed in section 3.2 it is enough to get the following.

Corollary 8. *For every $r \geq 1$, the set*

$$\mathcal{D} = \{f \in \mathcal{U},\ W^u(p) \cap W^s(q) \neq \emptyset\}$$

is dense in $\mathcal{U} \cap \mathrm{Diff}^r(M)$. It is a countable union of one-codimensional submanifolds.

In the C^1-topology, this result is a direct consequence of the C^1-connecting lemma (see [BDV, Appendix A]) together with Proposition 6; however, here we present another argument (without using the connecting lemma) which holds in any C^r-topology. This is possible thanks to the additional structure of our specific example.

Proof. Fix any $f \in \mathcal{U}$. By Proposition 6, there exists $x \in K_q$ such that $W^u(p)$ intersects $W^{ss}(x)$ at a point y (notice that $y \notin K_q \cup \{p\}$). Since K_q and p are disjoint and $y \in W^s(p)$ there exists U, a neighborhood of y such that:

- U is disjoint from the iterates of y, i.e. $\{f^n(y) \ :\ n \in \mathbb{Z}\} \cap U = \{y\}$;
- U is disjoint from $K_q \cup \{p\}$.

Given a C^r neighborhood $\mathcal{V}$ of the identity, there exists a neighborhood $V \subset U$ of y such that, for every $z \in V$, the set $\mathcal{V}$ contains a diffeomorphism φ_z which coincides with the identity in the complement of U and maps y at z.

Since K_q is locally maximal, there exists $\bar{x} \in K_q \cap W^s(q)$ near x. In particular, $W^{ss}_{loc}(\overline{x})$ intersects V in a point z whose backward orbit is disjoint from U.

For the diffeomorphism $\Phi = \varphi_z \circ f$ (which is C^r-close to f) the manifolds $W^s(q)$ and $W^u(p)$ intersect. Indeed, both f and Φ satisfy $f^{-1}(y) \in W^u(p)$ and $z \in W^{ss}_{loc}(\overline{x})$. Since $W^{ss}_{loc}(\overline{x}) \subset W^{ss}(q)$ and $\Phi(f^{-1}(y)) = z$ we get the density of $\mathcal{D}$ in $\mathcal{U} \cap \mathrm{Diff}^r(M)$.

For each integer $n \geq 1$, the manifolds $f^n(W^{uu}_{loc}(p))$ and $W^{ss}_{loc}(q)$ have disjoint boundary and intersect in at most finitely many points. One deduces that the set $\mathcal{D}_n$ of diffeomorphisms such that they intersect is a finite union of the one-codimensional submanifold of $\mathcal{U}$. The set $\mathcal{D}$ is the countable union of the $\mathcal{D}_n$. $\square$

3.5. **Other properties.** Here we show properties (1), (2) and (3) of the theorem.

Proposition 9. *For every $f \in \mathcal{U}$ and $x \in \mathcal{E}_f$ we have:*

- *If $x \notin W^s(q)$, there exists a large $n \geq 0$ such that $W^{uu}_{loc}(f^n(x)) \cap W^s(p) \neq \emptyset$.*
- *If $x \notin W^u(p)$, there exists a large $n \geq 0$ such that $W^{ss}_{loc}(f^{-n}(x)) \cap W^u(q) \neq \emptyset$.*

Moreover, in the first case x belongs to the homoclinic class of p and in the second it belongs to the homoclinic class of q.

Proof. By Lemma 5, any point $x \in \mathcal{E}_f \setminus W^s(q)$ has arbitrarily large iterates $f^n(x)$ in $R_1 \cup R_4$. Using Proposition 6 we get that proving that either $W^{uu}_{loc}(f^n(x)) \cap W^s(p) \neq \emptyset$ or x has an arbitrarily large iterate inside R_4. In the latter case, since p is homoclinically related to p_2 we also deduce that $W^{uu}_{loc}(f^n(x)) \cap W^s(p) \neq \emptyset$.

In particular, $W^s(p)$ intersects transversally $W^{uu}_{loc}(x)$ at points arbitrarily close to x. On the other hand, by Proposition 6 there exists a sequence z_n converging to x and points $y_n \in K_p$ such that $z_n \in W^u(y_n)$ for each n, proving that $W^{uu}_{loc}(z_n)$ intersects $W^u(p)$ transversally at a point close to x when n is large. By the λ-lemma, $W^{uu}_{loc}(y_n)$ is the C^1-limit of a sequence of discs contained in $W^u(p)$. This proves that $W^u(p)$ and $W^s(p)$ have a transverse intersection point close to x; hence x belongs to the homoclinic class of p.

The other properties are obtained analogously. □

The next corollary gives property (1) of the theorem.

Corollary 10. *For every $f \in \mathcal{U}$, the homoclinic class of any hyperbolic periodic point of $\mathcal{E}_f$ coincides with H_f. Moreover, the hyperbolic periodic points in $\mathcal{E}_f$ of the same stable dimension are homoclinically related.*

Proof. Let $z \in \mathcal{E}_f$ be a hyperbolic periodic point whose stable dimension is 2. By Proposition 6, we know that $W^{ss}(z)$ intersects $W^{uu}_{loc}(y)$ for some $y \in K_p$. This implies that $W^s(z)$ intersects $W^{uu}_{loc}(y)$, and since $W^{uu}_{loc}(y)$ is accumulated by $W^u(p)$ we get that $W^s(z)$ intersects $W^u(p)$. Now, by Proposition 9, $W^u(z)$ intersects $W^s(p)$. Moreover, the partial hyperbolicity implies that the intersections are transversal, proving that z and p are homoclinically related. By the properties of homoclinic classes mentioned in section 2.2, this implies that the homoclinic classes of p and z coincide.

One shows in the same way that any hyperbolic periodic point whose stable dimension is 1 is homoclinically related to q.

It remains to prove that the homoclinic classes of p and q coincide. The homoclinic class of q coincides with the homoclinic class of q_3, so, by definition, it contains a dense set of points of transverse intersection between $W^u(q_3)$ and $W^s(q_3)$. In particular, such points do not belong to $W^u(q)$ but hence belong to the homoclinic class of p by Proposition 9. This gives one inclusion. The other one is similar. □

Properties (2) and (3) of the theorem follow from Corollary 8 and the following.

Corollary 11. *For every $f \in \mathcal{U}$ we have $\mathcal{E}_f \setminus H_f = W^s(q) \cap W^u(p)$.*

Proof. By Corollary 10, a point $x \in \mathcal{E}_f \setminus H_f$ does not belong to the homoclinic class of q (nor to the homoclinic class of p by definition of H_f). Proposition 9 gives $\mathcal{E}_f \setminus H_f \subset W^s(q) \cap W^u(p)$.

Proposition 2 implies that the points of $W^s(q) \cap W^u(p)$ are isolated in $\mathcal{E}_f$. Since any point in a non-trivial homoclinic class is the limit of a sequence of distinct periodic points of the class (see section 2.2), we conclude that $W^s(q) \cap W^u(p)$ and H_f are disjoint. Since the set $W^s(q) \cap W^u(p)$ is contained in $\mathcal{E}_f$, we conclude the proof of the corollary. □

The proof of the theorem is now complete.

Acknowledgement

The authors thank the referee for the careful reading of the paper.

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