TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 10, October 2014, Pages 5541–5566
S 0002-9947(2014)06105-9
Article electronically published on February 10, 2014

# EXISTENCE AND CONVERGENCE TO A PROPAGATING TERRACE IN ONE-DIMENSIONAL REACTION-DIFFUSION EQUATIONS

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Abstract. We consider one-dimensional reaction-diffusion equations for a large class of spatially periodic nonlinearities – including multi-stable ones – and study the asymptotic behavior of solutions with Heaviside type initial data. Our analysis reveals some new dynamics where the profile of the propagation is not characterized by a single front, but by a layer of several fronts which we call a terrace. Existence and convergence to such a terrace is proven by using an intersection number argument, without much relying on standard linear analysis. Hence, on top of the peculiar phenomenon of propagation that our work highlights, several corollaries will follow on the existence and convergence to pulsating traveling fronts even for highly degenerate nonlinearities that have not been treated before.

## 1. Introduction

In this work we consider a Cauchy problem for the following reaction-diffusion equation in one space dimension:

$$(E) \qquad \partial_t u(t,x) = \partial_{xx} u(t,x) + f(x, u(t,x)), \quad \forall (t,x) \in [0,+\infty) \times \mathbb{R},$$

supplemented with the initial condition

$$(1.1) \qquad u(0,x) = u_0(x) \geq 0, \quad \forall x \in \mathbb{R}.$$

Here the function $f \in C^1(\mathbb{R}^2;\mathbb{R})$ satisfies the periodicity condition

$$(1.2) \qquad f(x+L,u) \equiv f(x,u) \ \text{ and } f(x,0) \equiv 0,$$

for some $L > 0$. We will assume, throughout this paper, that there exists a positive and $L$-periodic stationary solution $p(x)$ of $(E)$:

$$(P) \qquad \begin{cases} p''(x) + f(x,p(x)) = 0, & \forall x \in \mathbb{R}, \\ p(x) > 0, \quad p(x+L) \equiv p(x). \end{cases}$$

The function $p$ is also a stationary solution of the following auxiliary equation, the $L$-periodic counterpart of $(E)$:

$$(E_{per}) \qquad \begin{cases} \partial_t u(t,x) = \partial_{xx} u(t,x) + f(x, u(t,x)), & \forall (t,x) \in [0,+\infty) \times \mathbb{R}, \\ u(t,\cdot) \ L\text{-periodic for any } t \in \mathbb{R}. \end{cases}$$

Received by the editors March 12, 2012 and, in revised form, February 4, 2013.
2010 *Mathematics Subject Classification.* Primary 35K55, 35C07, 35B08, 35B40.
*Key words and phrases.* Multi-stable reaction-diffusion equation, periodic environment, long time behavior, propagating terrace, zero-number argument.

It is obvious that any solution of $(E_{per})$ is also a solution of $(E)$. Equation $(E_{per})$ will later play an important auxiliary role in the analysis of $(E)$.

Our aim is to investigate the profile of solutions of $(E)$ connecting the two stationary states 0 and $p$. In particular, we will study the long time behavior of solutions with Heaviside type initial data. Roughly speaking, our main results state that the solution will converge to what we call a "propagating terrace", the meaning of which will be specified later. This result, in particular, implies the existence of pulsating traveling waves (or a set of traveling waves) under rather mild assumptions.

We now state our main assumptions. The first one is concerned with the attractiveness of $p$ with respect to at least one compactly supported initial data. Our main theorems (Theorems 1.10 and 1.11) will only need this assumption:

**Assumption 1.1.** *There exists a solution $u$ of $(E)$-(1.1) with compactly supported initial data $0 \leq u_0(x) < p(x)$ that converges locally uniformly to $p$ as $t \to +\infty$.*

This assumption covers a wide variety of nonlinearities that include not only such standard ones as monostable, bistable or combustion nonlinearities, but also much more general and complex ones. For instance, it even allows an infinite number of stationary solutions between 0 and $p$. In this paper, we will show that this rather weak condition is in fact sufficient for deriving our main results on the convergence to a "propagating terrace".

The next assumption guarantees that our propagating terrace consists of a single (pulsating) traveling wave. Thus, under this additional assumption, our main results imply the existence of a pulsating traveling wave, as well as the convergence of solutions to this traveling wave (Theorem 1.12):

**Assumption 1.2.** *There exists no $L$-periodic stationary solution $q$ with $0 < q(x) < p(x)$ that is both isolated from below and stable from below with respect to $(E_{per})$.*

Let us clarify the notions introduced in this assumption. A stationary solution $q$ of $(E_{per})$ is said to be **isolated from below** (resp. **above**) if there exists no sequence of other stationary solutions converging to $q$ from below (resp. above). A stationary solution $q$ is said to be **stable from below** (resp. **above**) with respect to equation $(E_{per})$ if it is stable in the $L^\infty$ topology under nonpositive (resp. nonnegative) perturbations. Otherwise, $q$ is called **unstable from below** (resp. **above**). It is known that if $q$ is isolated from below, then it is stable from below if and only if there exists a solution $u < q$ converging to $q$ as $t \to +\infty$, and unstable from below if and only if there exists an ancient solution (that is, a solution defined for all sufficiently negative $t$) $u < q$ converging to $q$ as $t \to -\infty$ (see Theorem 8 in [19]).

Note that this additional assumption holds for a large class of standard nonlinearities including the following:

*Case* 1.3 (Monostable nonlinearity). There exists no $L$-periodic stationary solution $q$ satisfying $0 < q(x) < p(x)$ for all $x \in \mathbb{R}$. Furthermore, 0 is unstable from above.

*Case* 1.4 (Bistable nonlinearity). The stationary solution 0 is stable from above with respect to $(E_{per})$, and $p$ is stable from below with respect to $(E_{per})$. Furthermore, all other stationary solutions between 0 and $p$ are unstable.

*Case* 1.5 (Combustion nonlinearity). There exists a family of $L$-periodic stationary solutions $(q_\lambda)_{\lambda \in [0,1]}$ that forms a continuum in $L^\infty(\mathbb{R})$ and satisfies $0 = q_0 < q_1 < p$.

Furthermore, there exists no stationary solution $q$ satisfying $q_1(x) < q(x) < p(x)$ for all $x \in \mathbb{R}$.

A classical example of the bistable nonlinearity is the Allen-Cahn nonlinearity $u(1-u)(u-a(x))$, where $0 < a(x) < 1$, $a(x+L) \equiv a(x)$. An important subclass of the monostable nonlinearity is the KPP type nonlinearity, in which 0 is assumed to be linearly unstable and $f$ is sublinear with respect to $u$, a typical example being $(R(x)-u)u$, with $R(x+L) \equiv R(x) > 0$.

KPP type equations have been widely studied, even in the periodic setting, by numerous authors including [2, 3, 12, 14, 24]. While most of those studies rely heavily on the linear instability of 0, our approach in the present paper largely avoids the need for linear analysis, allowing our results to be applicable even to strongly degenerate situations that have not been treated before.

We now introduce some notions which will play a fundamental role in this paper. We begin with the following:

**Definition 1.6.** Let $u_1$, $u_2$ be two entire solutions of $(E)$. We say that $u_1$ is **steeper than** $u_2$ if for any $t_1$, $t_2$ and $x_1$ in $\mathbb{R}$ such that $u_1(t_1, x_1) = u_2(t_2, x_1)$, we have either

$$u_1(\cdot + t_1, \cdot) \equiv u_2(\cdot + t_2, \cdot) \quad \text{or} \quad \partial_x u_1(t_1, x_1) < \partial_x u_2(t_2, x_1).$$

Here, by an "entire solution" we mean a solution that is defined for all $t \in \mathbb{R}$. The above property implies that the graph of the solution $u_1$ (at any chosen time moment $t_1$) and that of the solution $u_2$ (at any chosen time moment $t_2$) can intersect at most once unless they are identical, and that if they intersect at a single point, then $u_1 - u_2$ is positive on the left-hand side of the intersection point, while negative on the right-hand side. Note that, according to this definition, if the ranges of $u_1$ and $u_2$ are disjoint, then $u_1$ and $u_2$ are steeper than each other, since their graphs never intersect.

**Definition 1.7** (Pulsating traveling wave)**.** Given two distinct periodic stationary states $p_1$ and $p_2$, by a **pulsating traveling wave solution** (or **pulsating traveling front**) of $(E)$ connecting $p_1$ to $p_2$, we mean any entire solution $u$ satisfying, for some $T > 0$,

$$u(t, x - L) = u(t + T, x),$$

for any $x \in \mathbb{R}$ and $t \in \mathbb{R}$, along with the asymptotics

$$u(-\infty, \cdot) = p_1(\cdot) \quad \text{and} \quad u(+\infty, \cdot) = p_2(\cdot),$$

where the convergence is understood to hold locally uniformly in the space variable. The ratio $c := \frac{L}{T} > 0$ is called the **average speed** (or simply the **speed**) of this pulsating traveling wave.

*Remark* 1.8. One can easily check that, for any $c > 0$, $u(t, x)$ is a pulsating traveling wave connecting $p_1$ to $p_2$ with speed $c$ if and only if it can be written in the form $u(t, x) = U(x - ct, x)$, where $U(z, x)$ satisfies

$$U(\cdot, x + L) \equiv U(\cdot, x),$$
$$U(+\infty, \cdot) = p_1(\cdot) \quad \text{and} \quad U(-\infty, \cdot) = p_2(\cdot),$$

along with the following equation that is equivalent to $(E)$:

$$(\partial_x + \partial_z)^2 U + cU_z + f(x, U) = 0, \quad \forall (z, x) \in \mathbb{R}^2.$$

Let us recall some known results on traveling waves from the literature. In the case of spatially homogeneous problems, existence of traveling waves is well studied (see for instance [6] for a review of the area). More precisely, in the KPP case, there exists a continuum of admissible speeds $[c^*, +\infty)$, while in the bistable or combustion cases, the admissible speed is unique. Stability and convergence to those traveling waves are also studied extensively. Among other things, in the one-dimensional KPP case (or, more generally, the monostable case), Uchiyama [23], Bramson [7] and Lau [15] proved that solutions of the Cauchy problem with compactly supported initial data converge to the traveling front with minimal speed as $t \to \infty$. In this case, the solution does not converge to the traveling wave with an asymptotic phase, but a phase drift of order $\ln t$ occurs [7]. Similar results hold for multi-dimensional problems as long as the parameters of the equation are invariant in the direction of propagation [17].

In the case of spatially periodic problems, the state of research is slightly behind, for obvious technical difficulties. Nonetheless, in the KPP case, the existence of a continuum of admissible speeds is well established, as in the case of spatially homogeneous problems. It is also known that there is a close relation between the speed of a traveling wave $u(x,t)$ and its decay rate as $x \to +\infty$, at least under some assumptions on the linearized problem around 0; the smaller the speed $c$, the faster the decay, hence the steeper the front profile. Convergence to those traveling waves was studied in [2, 3, 14] in a periodic framework. More precisely, it has been shown that if the initial data has the same exponential decay as a given traveling wave as $x \to +\infty$, then the solution of the Cauchy problem converges to this traveling wave as $t \to +\infty$. However, the case of very fast decaying initial data (for instance, a Heaviside or a compactly supported function) has been left open up to now in the periodic framework (see the notes at the end of the paper for recent developments).

1.1. **The notion of a terrace.** Let us now come back to the main theme of the present paper — a propagating terrace. As we mentioned earlier, a traveling wave is a special case of a propagating terrace, but the latter is a more suitable notion for describing typical frontal behaviors in equations of multi-stable nature. The aim of the present paper is to study properties of propagating terraces in a spatially periodic setting, thereby generalizing (and improving) some of the aforementioned results on pulsating traveling waves.

**Definition 1.9.** A **propagating terrace** connecting 0 to $p$ is a pair of finite sequences $(p_k)_{0\le k\le N}$ and $(U_k)_{1\le k\le N}$ such that:

- Each $p_k$ is an $L$-periodic stationary solution of $(E)$ satisfying
$$p = p_0 > p_1 > ... > p_N = 0.$$
- For each $1 \le k \le N$, $U_k$ is a pulsating traveling wave solution of $(E)$ connecting $p_k$ to $p_{k-1}$.
- The speed $c_k$ of each $U_k$ satisfies $0 < c_1 \le c_2 \le \cdots \le c_N$.

Furthermore, a propagating terrace $T = ((p_k)_{0\le k\le N}, (U_k)_{1\le k\le N})$ connecting 0 to $p$ is said to be **minimal** if it also satisfies the following:

- For any propagating terrace $T' = ((q_k)_{0\le k\le N'}, (V_k)_{1\le k\le N'})$ connecting 0 to $p$, one has that
$$\{p_k \mid 0 \le k \le N\} \subset \{q_k \mid 0 \le k \le N'\}.$$

- For each $1 \leq k \leq N$, the traveling wave $U_k$ is steeper than any other traveling wave connecting $p_k$ to $p_{k-1}$.

Roughly speaking, a propagating terrace can be pictured as a layer of several traveling fronts going at various speeds, the lower the faster (Figure 1).

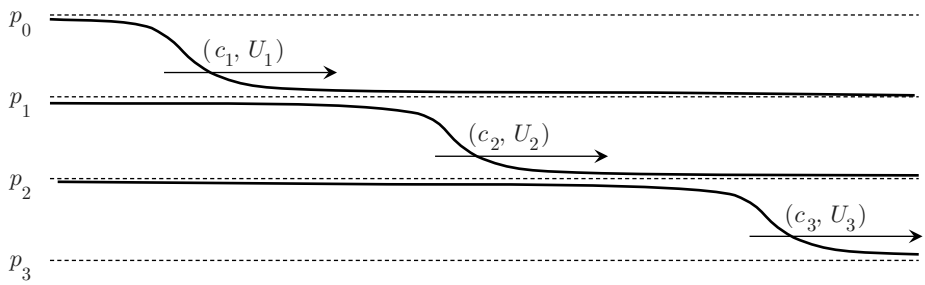


FIGURE 1. A three-step terrace

The aim of the present paper is to show that the solution of $(E)$ with Heaviside type initial data will converge to a minimal propagating terrace, as illustrated in Figure 2.

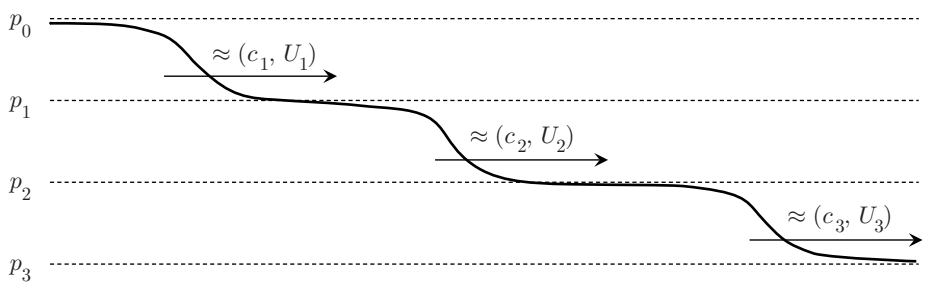


FIGURE 2. Terrace-shaped profile of propagation

In some standard problems such as the KPP and the bistable equations, the terrace actually consists of a single front (that is, $N = 1$), which means that the solution will eventually look like a single traveling wave; see Theorem 1.12. However, in more general equations, one cannot expect such simple dynamics, and this is where the notion of a terrace plays a fundamental role.

The existence of a multi-step terrace has been known in the spatially homogeneous case (where $f = f(u)$). Let us give a simple example. Consider $f$ as in Figure 3 (left), which is KPP on $[0, \theta_1]$, and bistable on $[\theta_1, 1]$. The speed of the upper part of the solution is bounded from above by the speed, say $c$, of the traveling wave for the bistable nonlinearity $f_{|[\theta_1,1]}$. On the other hand, the lower part of the solution is pushed from behind by a spreading front for the KPP nonlinearity $f_{|[0,\theta_1]}$, whose speed is known to approach $c_* := 2\sqrt{f'(0)}$. Therefore, if $c_* > c$, the upper and lower parts of the solution necessarily move at two distinct speeds.

Another example was exhibited by Fife and McLeod in [10], where they considered a specific case of a nondegenerate tristable nonlinearity, that is, when $f_{|[0,\theta]}$

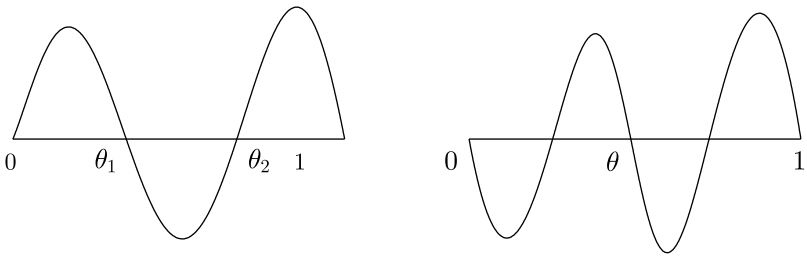


FIGURE 3. Examples of $f(u)$ that produce a multi-step terrace

and $f_{|[\theta,1]}$ are both bistable for some $\theta \in (0,1)$, and that $f'(0)$, $f'(\theta)$ and $f'(1)$ are all strictly negative; see Figure 3 (right). They showed that if the speed of the upper bistable part is smaller than the speed of the lower bistable part, then there does not exist any single front connecting 0 to 1. Furthermore, some solutions of the Cauchy problem, in particular for Heaviside type initial data, converge to a combination of those two fronts. This may be seen as an early study of a propagating terrace for some very specific examples.

Although the method in [10] was expected to hold for homogeneous nonlinearities composed of a finite number of bistable parts, it relied strongly on the particular shape of $f$, and on the nondegeneracy of the equilibria. This means that they needed some important a priori knowledge on the shape of the nonlinearity, which we do not need in the present paper. More importantly, what makes our work different from those early observations is that we are not simply giving examples of propagating terraces but are establishing the ubiquity of such terraces for large classes of reaction nonlinearities, thus showing that the notion of a propagating terrace is fundamental for studying the dynamics of fronts in general reaction-diffusion equations.

1.2. **Main results.** We consider solutions of $(E)$-(1.1) whose initial data are given in the form

$$u_0(x) = p(x)H(a-x), \tag{1.3}$$

where $a \in \mathbb{R}$ is any constant, and $H$ denotes the Heaviside function, which is defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Hereafter, for each $a \in \mathbb{R}$, we denote by $\widehat{u}(t,x;a)$ such solutions. We will prove that $\widehat{u}$ converges in some sense to a minimal propagating terrace as $t \to \infty$.

**Theorem 1.10** (Existence of a minimal terrace)**.** *Let Assumption* 1.1 *hold. Then there exists a propagating terrace* $((p_k)_{0\leq k\leq N}, (U_k)_{1\leq k\leq N})$ *that is minimal in the sense of Definition* 1.9*. Such a minimal propagating terrace is unique, in the sense that any minimal propagating terrace shares the same* $(p_k)_k$ *and that* $U_k$ *is unique up to time shift for each* $k$*. Moreover, it satisfies:*

*(i) For any* $0 \leq k < N$*, the* $L$*-periodic stationary solution* $p_k$ *is isolated and stable from below with respect to* $(E_{per})$*.*
*(ii) All the* $p_k$ *and* $U_k$ *are steeper than any other entire solution of* $(E)$*.*

The existence of a minimal terrace as stated in the above theorem gives various types of useful information about the qualitative properties of the equation. For example, statement $(ii)$ implies, in particular, that there exists no traveling wave that intersects any of the $p_k$. Note that, in the spatially homogeneous case (namely, $f = f(u)$), the stability of $p_k$ in statement $(i)$ implies that each $p_k$ is a constant; hence the terrace consists of flat steps.

We now state our convergence result:

**Theorem 1.11** (Convergence to a minimal terrace)**.** *Let Assumption* 1.1 *hold. Then for any $a \in \mathbb{R}$, the solution $\widehat{u}(t,x;a)$ converges as $t \to +\infty$ to the minimal propagating terrace $((p_k)_{0\le k\le N}, (U_k)_{1\le k\le N})$ in the following sense:*

*(i) There exist functions $(m_k(t))_{1\le k\le N}$ with $m_k(t) = o(t)$ as $t \to +\infty$ such that*

$$\begin{aligned} &\widehat{u}(t, x + c_k(t - m_k(t)); a) - U_k(t - m_k(t), x + c_k(t - m_k(t))) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to 0 \quad \text{as} \quad t \to +\infty, \end{aligned} \tag{1.4}$$

*locally uniformly on $\mathbb{R}$, $c_k$ being the speed of $U_k$.*

*(ii) For any $\delta > 0$, there exists $C > 0$ and $T > 0$ such that, for any $0 \le k \le N$, $t \ge T$ and $x \in I_{k,C}(t)$:*

$$\|\widehat{u}(t,\cdot;a) - p_k(\cdot)\|_{L^\infty(I_k(t))} \le \delta,$$

*where*

$$\begin{aligned} &I_{k,C}(t) = [c_k(t - m_k(t)) + C, c_{k+1}(t - m_{k+1}(t)) - C] \text{ for } 1 \le k \le N-1, \\ &I_{0,C}(t) = (-\infty, c_1(t - m_1(t)) - C] \quad \text{and } I_{N,C}(t) = [c_N(t - m_N(t)) + C, +\infty). \end{aligned}$$

Roughly speaking, statements $(i)$ and $(ii)$ of the above theorem describe, respectively, the ascending part and the stationary part of the terrace, the latter being flat if $f = f(u)$, as mentioned above. It should be noted that, under Assumption 1.1, there may exist an infinite number of isolated stationary solutions between 0 and $p$, but our Theorem 1.10 states that only a finite number of layers appear in the limiting terrace. The solution seems to ignore excessive complexity of such nonlinearities.

The proofs of Theorems 1.10 and 1.11 are, in a sense, one and the same. In fact, by first showing that the steepness of the Heaviside type initial data (1.3) implies that the limiting profile of the solution is steeper than any other entire solution, we will then use this fact to prove the convergence of the solution to a minimal terrace without assuming the existence of a terrace. Hence it automatically implies the existence of a minimal terrace.

In the special case where Assumption 1.2 also holds, the above two theorems reduce to the following result on pulsating traveling waves. Thus, our paper gives a new and original proof — based mainly on the zero number argument — for the existence of pulsating traveling waves.

**Theorem 1.12** (Monostable/bistable/combustion cases)**.** *Let Assumptions* 1.1 *and* 1.2 *hold. Then there exists a pulsating traveling wave $U^*(t,x)$ connecting* 0 *to $p$ with speed $c > 0$ that is steeper than any other entire solution between* 0 *and $p$. Furthermore, for any $a \in \mathbb{R}$, there exists a function $m(t)$ with $m(t) = o(t)$ as $t \to +\infty$ such that*

$$\|\widehat{u}(t,\cdot;a) - U^*(t - m(t),\cdot)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad t \to +\infty. \tag{1.5}$$

Let us make some comments on Theorem 1.12, which is a special case of the previous two theorems. As regards the existence part, there have been earlier studies of the existence of pulsating traveling waves for monostable (possibly degenerate) and combustion cases [4], as well as for some periodic bistable case [9]. In contrast to those earlier results, which are derived by various different methods depending on the type of nonlinearity, our theorem relies on a new, unified and rather straightforward proof, thus avoiding having to deal directly with the particular features and difficulties of each case.

Convergence to generalized transition waves from fast decaying initial data has been proven in the ignition case, even for nonperiodic environments [20, 25]. The bistable case may be treated by similar methods as the above papers. However, as far as the KPP or more general monostable cases are concerned, such a type of convergence result was not known before, even for Heaviside type initial data such as ours, aside from the classical homogeneous setting. Indeed, even in the standard KPP case with spatially periodic coefficients, almost all the convergence results in the literature are concerned with solutions whose initial data have roughly the same decay rate as one of the traveling waves near $x = +\infty$. Note that, though our results cover a large class of equations, they are concerned only with a specific type of initial data (1.3). However, by analogy with the homogeneous case (see for instance the proofs in [23]), we expect this to lead us to similar convergence results for more general initial data, such as compactly supported ones. This is a topic of particular relevance from an applied point of view (see the notes at the end of this paper for more recent developments).

**Plan of the paper.** Our paper is organized as follows. In Section 2, we will present some preliminaries. As our proofs largely rely on the so-called intersection number (or the zero number) argument, we will first give its precise definition and basic properties. We will then use this method to prove our fundamental lemma (Lemma 2.8), which roughly states the following:

> **Fundamental lemma:** *Any function that appears in the $\omega$-limit set of $\widehat{u}(\cdot,\cdot;a)$ is steeper than any other entire solution, where $\widehat{u}(\cdot,\cdot;a)$ is the solution of $(E)$ for the initial data* (1.3).

One immediate consequence of the above fundamental lemma is that any two elements of the $\omega$-limit set are steeper than each other. This means that they are either identical (up to time shift) or strictly ordered, that is, one is above or below the other. This observation is important both for establishing convergence results and for the construction of a multi-step terrace.

Here we note that the definition of the $\omega$-limit set in this paper is slightly different from the standard one, in that we consider arbitrary spatial translations while taking the limit as $t_k \to \infty$; see Definition 2.6. The reason for adopting this slightly nonstandard definition is that, since each step of the terrace moves at a different speed, we cannot capture the asymptotic profile of the solution in a single frame. A multi-speed observation is unavoidable in the case of a multi-step terrace. In the last part of Section 2, we will also give a simple lemma on the inner and outer spreading speeds of $\widehat{u}$, which will be used repeatedly in later sections.

In Section 3, we will use our fundamental lemma to prove the convergence of solutions with Heaviside type initial data to a unique limit around any level set. The entire solution thereby constructed possesses, in some sense, some qualitative

properties of traveling fronts, such as monotonicity in time. The same result could in fact be shown with a similar argument in a more general setting without the periodicity assumption.

We will then show in Section 4 that this limit is a pulsating traveling wave connecting some pair of $L$-periodic stationary solutions $p_- < p_+$ that lie between 0 and $p$. Once again the above-mentioned fundamental lemma plays a key role in deriving this result. This leads to the construction of a multi-step minimal terrace inductively, as described in Theorems 1.10 and 1.11. In the special case where Assumption 1.2 holds, we have $p_- = 0$, $p_+ = p$, thus the terrace is a single traveling wave. As the existence of any traveling wave is not a priori assumed, this leads to both the existence and the convergence results in Theorem 1.12.

## 2. Preliminaries

2.1. **Zero number.** Our proof of the main results relies strongly on a zero-number argument. The application of this argument — or the "Sturmian principle" — to the convergence proof in semilinear parabolic equations first appeared in [18]. But what makes the present paper different from earlier work is that we employ the zero-number argument to prove not only the convergence but also the existence of the target objects, namely the terrace and pulsating traveling waves.

In this paper, besides the standard zero-number $Z[\,\cdot\,]$, we introduce a related notion $SGN[\,\cdot\,]$, which turns out to be exceedingly useful in establishing our fundamental lemma.

**Definition 2.1.** For any real-valued function $w$ on $\mathbb{R}$, we define:

- $Z\,[w(\cdot)]$ is the number of sign changes of $w$, namely the supremum over all $k \in \mathbb{N}^*$ such that there exist real numbers $x_1 < x_2 < ... < x_{k+1}$ with

$$w(x_i).w(x_{i+1}) < 0 \quad \text{for all} \quad i = 1, 2, ..., k.$$

  When such a $k$ does not exist, that is, $w$ does not change sign, we set $Z[w] = 0$ if $w \not\equiv 0$, and $Z[w] = -1$ if $w \equiv 0$.
- $SGN\,[w(\cdot)]$, which is defined when $Z\,[w(\cdot)] < \infty$, is the word consisting of $+$ and $-$ that describes the signs of $w(x_1), \ldots, w(x_{k+1})$, where $x_1 < \cdots < x_{k+1}$ is the sequence that appears in the definition of $Z[w]$ with maximal $k$.

  When such a $k$ does not exist, we set $SGN[w] = sgn(w(x))$ if $w \not\equiv 0$ and $w(x) \neq 0$, where $sgn$ denotes the classical sign function, and $SGN\,[0] = [\,]$, the empty word.

If $w$ is a smooth function having only simple zeros on $\mathbb{R}$, then $Z\,[w]$ coincides with the number of zeros of $w$. For example,

$$Z[x^2 - 1] = 2, \quad SGN[x^2 - 1] = [+ - +].$$

By definition, the length of the word $SGN[w]$ is equal to $Z[w] + 1$.

If $A, B$ are two words consisting of $+$ and $-$, we write $A \rhd B$ (or, equivalently, $B \lhd A$) if $B$ is a subword of $A$. For example,

$$[+-] \rhd B \quad \text{for } B = [+-],\ [+],\ [-],\ [\,] \qquad \text{but not} \quad [+-] \rhd [-+].$$

Let us recall some properties of $Z$ and $SGN$:

**Lemma 2.2.** *Let $w(t,x) \not\equiv 0$ be a bounded solution of a parabolic equation of the form*

$$\partial_t w = \partial_{xx} w + c(t,x) w \quad \text{on a domain } (t_1,t_2) \times \mathbb{R}, \tag{2.6}$$

*where $c$ is bounded. Then, for each $t \in (t_1,t_2)$, the zeros of $w(t,\cdot)$ do not accumulate in $\mathbb{R}$. Furthermore,*

(*i*) *$Z[w(t,\cdot)]$ and $SGN[w(t,\cdot)]$ are nonincreasing in $t$, that is, for any $t' > t$,*

$$Z[w(t,\cdot)] \geq Z[w(t',\cdot)], \qquad SGN[w(t,\cdot)] \rhd SGN[w(t',\cdot)];$$

*here the assertion remains true even for $t = t_1$ if $w$ can be extended to a continuous function on $[t_1,t_2) \times \mathbb{R}$.*

(*ii*) *If $w(t',x') = \partial_x w(t',x') = 0$ for some $t' \in (t_1,t_2)$ and $x' \in \mathbb{R}$, then*

$$Z[w(t,\cdot)] - 2 \geq Z[w(s,\cdot)] \geq 0 \quad \text{for any } t \in (t_1,t') \text{ and } s \in (t',t_2)$$

*whenever $Z[w(t,\cdot)] < \infty$.*

The second inequality of statement (*ii*) above implies that, for any $t \in (t_1,t_2)$, the function $w(t,x)$ does not vanish entirely on $\mathbb{R}$ unless $w \equiv 0$ on $(t_1,t_2) \times \mathbb{R}$. Statement (*ii*) is due to [1], where this result is proved by using similarity variables and expansion by Hermitian polynomials. Though [1] deals only with equations on bounded intervals, the result can easily be extended to $\mathbb{R}$ by applying the maximum principle near $x = \pm\infty$; see [8].

The statement (*i*) for $Z[w]$ follows from (*ii*), at least when the domain is a bounded interval, but it can be shown more directly by a combination of the maximum principle and a topological argument similar to the Jordan curve theorem (which is a more standard way to prove this statement). In fact, this direct proof proves the assertion for $SGN[w]$, from which the assertion for $Z[w]$ follows automatically; see, for example, [18] for a similar argument.

One can also check that $Z$ is semi-continuous with respect to the pointwise convergence, that is:

**Lemma 2.3.** *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of functions converging to $w$ pointwise on $\mathbb{R}$. Then*

$$Z[w] \leq \liminf_{n\to\infty} Z[w_n],$$

$$SGN[w] \lhd \liminf_{n\to\infty} SGN[w_n].$$

Combining the above two lemmas, we obtain the following lemma:

**Lemma 2.4.** *Let $u_1$ and $u_2$ be solutions of $(E)$ such that the initial data $u_1(0,x)$ is a piecewise continuous bounded function on $\mathbb{R}$, while $u_2(0,x)$ is bounded and continuous on $\mathbb{R}$. Assume also that $u_1(0,x) - u_2(0,x)$ changes sign at most finitely many times on $\mathbb{R}$. Then*

(*i*) *For any $0 \leq t < t' < \infty$,*

$$\begin{aligned} Z[u_1(t,\cdot) - u_2(t,\cdot)] &\geq Z[u_1(t',\cdot) - u_2(t',\cdot)], \\ SGN[u_1(t,\cdot) - u_2(t,\cdot)] &\rhd SGN[u_1(t',\cdot) - u_2(t',\cdot)]. \end{aligned} \tag{2.7}$$

(*ii*) *If, for some* $t' > 0$, *the graph of* $u_1(x,t')$ *and that of* $u_2(x,t')$ *are tangential at some point in* $\mathbb{R}$, *and if* $u_1 \not\equiv u_2$, *then for any* $t, s$ *with* $0 \le t < t' < s$,

$$Z\left[u_1(t,\cdot) - u_2(t,\cdot)\right] - 2 \ge Z\left[u_1(s,\cdot) - u_2(s,\cdot)\right] \ge 0.$$

*The same conclusion holds if* $u_1, u_2$ *are entire solutions of* $(E)$, *in which case* $t' \in \mathbb{R}$ *is arbitrary and* $-\infty < t < t' < s < \infty$.

*Proof.* The function $w := u_1 - u_2$ satisfies an equation of the form (2.6) on $(0,\infty)\times\mathbb{R}$ with $c(x,t) := (f(x,u_1) - f(x,u_2))/(u_1 - u_2)$ being bounded. Thus the conclusion of the lemma follows from Lemma 2.2 except for (2.7) with $t = 0$. Moreover, statement (2.7) with $t = 0$ also follows from Lemma 2.2 if $u_1(0,x)$ and $u_2(0,x)$ are both continuous. In the general case where $u_1(0,x)$ is only piecewise continuous, we approximate $u_1$ by a sequence of solutions of $(E)$, say $u_{1,n}$, whose initial data $u_{1,n}(0,x)$ are continuous and satisfy

(a) $\sup_n \|u_{1,n}(0,\cdot)\|_{L^\infty(\mathbb{R})} < \infty$ and $u_{1,n}(0,x) \to u_1(0,x)$ pointwise on $\mathbb{R}$;
(b) $SGN[u_{1,n}(0,\cdot) - u_2(0,\cdot)] = SGN[u_1(0,\cdot) - u_2(0,\cdot)]$ for $n = 1,2,3,\ldots$.

Then we have, for each $t' > 0$,

$$SGN\left[u_1(0,\cdot) - u_2(0,\cdot)\right] \rhd SGN\left[u_{1,n}(t',\cdot) - u_2(t',\cdot)\right].$$

Letting $n \to \infty$ and applying Lemma 2.3, we obtain the desired conclusion. □

The following corollary will later be used repeatedly:

**Corollary 2.5.** *Let* $v_1$, $v_2$ *be two entire solutions of* $(E)$, *and assume that*

$$SGN\left[v_1(t_1,\cdot) - v_2(t_2,\cdot)\right] \lhd [+\ -] \quad \textit{for any} \ \ t_1, t_2 \in \mathbb{R}.$$

*Then* $v_1$ *is steeper than* $v_2$ *in the sense of Definition* 1.6.

*Proof.* Fix $t_1, t_2 \in \mathbb{R}$ arbitrarily. From the assumption we see that

$$Z\left[v_1(t+t_1,\cdot) - v_2(t+t_2,\cdot)\right] \le 1 \quad \text{for all} \ \ t \in \mathbb{R}.$$

If $v_1(\cdot + t_1,\cdot) \not\equiv v_2(\cdot + t_2,\cdot)$, then by Lemma 2.4 (ii), the function $v_1(t_1,x) - v_2(t_2,x)$ has at most one zero on $\mathbb{R}$, and that this zero is simple. Let $x_1$ be such a zero; that is, $v_1(t_1,x_1) = v_2(t_2,x_1)$. Then the simplicity of this zero and the sign property $SGN\left[v_1 - v_2\right] \lhd [+\ -]$ imply that $\partial_x v_1(t_1,x_1) < \partial_x v_2(t_2,x_1)$. This proves that $v_1$ is steeper than $v_2$. □

2.2. **Fundamental lemma on the $\omega$-limit set of $\widehat{u}(\cdot,\cdot;a)$.** The following definition of the $\omega$-limit set of a solution $u$ is slightly different from the standard one, as we add arbitrary spatial translations while taking the long-time limit. The reason for adopting this definition is that, since each step of the terrace moves at a different speed, we need multi-speed observations in order to fully capture the asymptotic profile of the solution.

**Definition 2.6.** Let $u(t,x)$ be any bounded solution of the Cauchy problem $(E)$-(1.1). We call $v(t,x)$ an **$\omega$-limit orbit** of $u$ if there exist two sequences $t_j \to +\infty$ and $k_j \in \mathbb{Z}$ such that

$$u(t+t_j, x+k_jL) \to v(t,x) \ \ \text{as } j \to +\infty \ \ \text{locally uniformly on } \mathbb{R}^2.$$

*Remark* 2.7. By parabolic estimates, the above convergence takes place in $C^2$ in $x$ and $C^1$ in $t$. Hence one can easily check that any $\omega$-limit orbit of $u$ is an entire solution of $(E)$. Moreover, if $v(t,x)$ is an $\omega$-limit orbit of $u$, then so is $v(t+\tau, x+kL)$ for any $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Let us now state a fundamental lemma that will be used repeatedly throughout our paper:

**Lemma 2.8.** *Let $a \in \mathbb{R}$ and let $v_1$ be any $\omega$-limit orbit of $\widehat{u}(t,x;a)$. Then $v_1$ is steeper than any entire solution of $(E)$ in the sense of Definition* 1.6*, provided that this entire solution lies between* 0 *and $p$.*

*Proof.* Fix $a \in \mathbb{R}$ and let the sequences $t_j \to +\infty$ and $k_j \in \mathbb{Z}$ be such that $\widehat{u}(t + t_j, x + k_j L; a) \to v_1(t,x)$ locally uniformly as $j \to +\infty$. By standard parabolic estimates, the convergence in fact holds in $C^1_{loc}(\mathbb{R}^2)$.

Let $v$ be any entire solution lying between 0 and $p$. Since $0 \le v(t,x) \le p(x)$, we have $\widehat{u}(0,x;a) \ge v(t,x)$ for $x < a$ and $\widehat{u}(0,x;a) \le v(t,x)$ for $x > a$. Consequently, for any $j \in \mathbb{N}$ and $\tau \in \mathbb{R}$,

$$Z\left[\widehat{u}(0,\cdot;a) - v(\tau - t_j,\cdot)\right] = 1 \quad \text{and} \quad SGN\left[\widehat{u}(0,\cdot;a) - v(\tau - t_j,\cdot)\right] = [+\,-].$$

It follows from Lemma 2.4 that, for all $j \in \mathbb{N}$ and $t \ge -t_j$,

$$Z\left[\widehat{u}(t + t_j,\cdot;a) - v(t+\tau,\cdot)\right] \le 1,$$

$$SGN\left[\widehat{u}(t + t_j,\cdot;a_j) - v(t+\tau,\cdot)\right] \lhd [+\,-].$$

Passing to the limit as $j \to +\infty$, we get

$$SGN\left[v_1(t,\cdot) - v(t+\tau,\cdot)\right] \lhd [+\,-] \qquad \text{for any} \quad t,\tau \in \mathbb{R}.$$

Hence, by Corollary 2.5, $v_1$ is steeper than $v$ in the sense of Definition 1.6. □

2.3. **Spreading of the solution with positive speed.** Before going to the proof of our main results, we investigate some spreading property of solutions of $(E)$ that will be used repeatedly later. Recall that $\widehat{u}(t,x;a)$ is the solution of $(E)$ with initial data $u_0(x) = p(x)H(a-x)$.

**Lemma 2.9.** *Let Assumption* 1.1 *be satisfied. Then there exist constants $0 < c_* < c^* < +\infty$ that do not depend on $a$, such that*

(i) *for each $c > c^*$, one has $\lim_{t\to\infty} \sup_{x \ge ct} \widehat{u}(t,x;a) = 0$;*

(ii) *for each $c \in (0, c_*)$ one has*

$$\lim_{t\to\infty} \sup_{x \le ct} |\widehat{u}(t,x;a) - p(x)| = 0.$$

This lemma follows from a rather standard comparison argument. Results of this type may be found in earlier papers, but for the sake of completeness, we give a proof here.

*Proof.* Note that from the $C^1$-regularity and the periodicity of $f$, there exists $K > 0$ such that for any $x \in \mathbb{R}$ and $0 \le u \le \sup_{x\in\mathbb{R}} p(x)$, we have that $f(x,u) \le Ku$. Then the function

$$\overline{u}(t,x) := \min\left\{p(x), e^{-\sqrt{K}(x-a-2\sqrt{K}t)}\|p\|_\infty\right\}$$

is clearly a supersolution of $(E)$. Since $\overline{u}(0,x) = p(x) = \widehat{u}(0,x;a)$ for all $x \le a$, while $\overline{u}(0,x) \ge 0 = \widehat{u}(0,x;a)$ for all $x > a$, it follows from the comparison principle that for all $t \ge 0$ and $x \in \mathbb{R}$,

$$\widehat{u}(t,x;a) \le \overline{u}(t,x).$$

Therefore, statement $(i)$ of Lemma 2.9 clearly holds for $c^* = 2\sqrt{K}$.

Let us now find a positive lower bound for the spreading speed. Let $u_0$ be the compactly supported function given in Assumption 1.1. This means that the solution $\underline{u}$ of the Cauchy problem $(E)$-(1.1) with initial data $0 \le u_0 < p$ converges locally uniformly to $p$ as $t \to +\infty$. Thanks to the periodicity of $(E)$, one can assume without loss of generality that $supp(u_0) \subset [-\infty, a]$.

Since $\underline{u} \to p$ as $t \to \infty$, there exists $T > 0$ such that

$$\underline{u}(T, x) \ge \max\{u_0(x), u_0(x - L)\} \quad \text{for any } x \in \mathbb{R}\ .$$

By the comparison principle, it follows that

$$\begin{aligned} \underline{u}(2T, x) &\ge \max\{\underline{u}(T, x), \underline{u}(T, x - L)\} \\ &\ge \max\{u_0(x), u_0(x - L), u_0(x - 2L)\}\,. \end{aligned}$$

By induction, we obtain that for all $k \in \mathbb{N}$,

$$\underline{u}(kT, x) \ge \max\{u_0(x - jL) \mid j \in \mathbb{N},\ 0 \le j \le k\}\,.$$

Since $\widehat{u}(0, x; a) \ge \max\{u_0(x + jL) \mid j \in \mathbb{N}\}$, one gets by applying the comparison principle that for all $t > 0$ and $x \in \mathbb{R}$,

$$\widehat{u}(t, x; a) \ge \max\{\underline{u}(t, x + jL) \mid j \in \mathbb{N}\}\,.$$

Hence, for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\begin{aligned} \widehat{u}(\tau + kT, x; a) &\ge \max\{\underline{u}(\tau, x - jL) \mid j \in \mathbb{Z},\ j \le k\} \\ &\longrightarrow p(x) \end{aligned}$$

where the convergence holds, thanks to Assumption 1.1, as $\tau \to +\infty$ uniformly with respect to $k \in \mathbb{N}$ and $x \in (-\infty, kL]$.

Let us now define $c_* = L/T > 0$ and choose any $c$ with $0 < c < c_*$. Denote by $\lceil y \rceil$ the ceiling function of $y$, that is, the least integer not smaller than $y$. Then for any $t \ge 0$, let

$$\tau(t) := t - \left\lceil \frac{ct}{L} \right\rceil T.$$

As $c < c_* = L/T$, one can easily check that $\tau(t) \to +\infty$ as $t \to +\infty$. Thus,

$$\sup_{x \le \left\lceil \frac{ct}{L} \right\rceil L} \left| \widehat{u}\left(\tau(t) + \left\lceil \frac{ct}{L} \right\rceil T, x; a\right) - p(x) \right| \longrightarrow 0 \ \text{ as } t \to +\infty,$$

and, since $ct \le \left\lceil \frac{ct}{L} \right\rceil L$ and $t = \tau(t) + \left\lceil \frac{ct}{L} \right\rceil T$ for all $t \ge 0$,

$$\sup_{x \le ct} |\widehat{u}(t, x; a) - p(x)| \longrightarrow 0 \ \text{ as } t \to +\infty,$$

which concludes the proof of Lemma 2.9. □

## 3. Convergence of the solutions with shifted initial data

In this section, we aim to prove an important lemma on the convergence of the solutions of the Cauchy problem with some shifted Heaviside type initial data around a given level set.

**Lemma 3.1.** *Let Assumption* 1.1 *be satisfied. Let* $x_0 \in \mathbb{R}$ *be given. For any* $0 < \alpha < p(x_0)$ *and* $a < x_0$*, let us define*

$$\tau(x_0, \alpha, a) := \min\{t > 0 \mid \widehat{u}(t, x_0; a) = \alpha\}\,. \tag{3.8}$$

*Then the following limit exists for the topology of $C^1_{loc}(\mathbb{R}^2)$:*

$$\lim_{k\to+\infty} \widehat{u}(t+\tau(x_0,\alpha,a-kL),x;a-kL) := w_\infty(t,x;\alpha). \tag{3.9}$$

*The function $w_\infty(t,x;\alpha)$ is an entire solution of $(E)$ that is steeper than any other entire solution. Furthermore, the following alternative holds true: either it is a stationary solution, or $\partial_t w_\infty(t,x;\alpha) > 0$ for all $(t,x) \in \mathbb{R}^2$. The former assertion is impossible for each $\alpha$ close enough to $p(x_0)$.*

This lemma states that if we look at some well chosen level set for shifted initial data, the profile of the solution locally converges to a monotonically increasing entire solution $w_\infty$ of $(E)$, which connects two stationary solutions. In the sequel we will show that $w_\infty$ is a pulsating traveling wave.

One could prove, by the same steepness argument as we use here, that the limit $\lim_{a\to-\infty} \widehat{u}(t+\tau(x_0,\alpha,a),x;a)$ also exists and satisfies the same properties. This result then even holds in a nonperiodic heterogeneous framework. In a parallel work, Nadin uses it to exhibit "critical traveling waves", namely steepest entire solutions [22]. However, it is in general not known what features those particular solutions do share with classical fronts, or whether the convergence holds for a given Heaviside type initial data. Therefore, we choose to restrict ourselves to the more standard periodic setting where we can derive many such properties from the steepness.

The proof of this result is split into three parts. Before we begin the proof of this lemma, let us make the following remark, explaining the choice of such shifted initial data.

*Remark* 3.2. Notice that one has

$$\widehat{u}(t,x;a+L) = \widehat{u}(t,x-L;a), \quad \forall (t,x,a) \in [0,\infty)\times\mathbb{R}\times\mathbb{R}.$$

This implies that for any $0<\alpha<p(x_0)$ and $k\in\mathbb{N}$,

$$\tau(x_0,\alpha,a-kL) = \tau(x_0+L,\alpha,a-(k-1)L).$$

Now since the initial data $p(x)H(a-x)$ is increasing with respect to $a$, the comparison principle provides that for each given $(t,x)\in\mathbb{R}^2$, the maps $a\mapsto \widehat{u}(t,x;a)$ and $k\mapsto \tau(x_0,\alpha,a-kL)$ are nondecreasing.

Let $x_0\in\mathbb{R}$ and $a<x_0$ be given. Then the function $w_\infty$ defined in Lemma 3.1 rewrites as

$$\begin{aligned} w_\infty(t,x;\alpha) &= \lim_{k\to+\infty} \widehat{u}\left(t+\tau(x_0,\alpha,a-kL),x;a-kL\right) \\ &= \lim_{k\to+\infty} \widehat{u}\left(t+\tau(x_0,\alpha,a-kL),x+kL;a\right). \end{aligned}$$

As we will see below that $\tau(x_0,\alpha,a-kL)\to+\infty$ as $k\to+\infty$, the above computations explain the choice of the shifts of the initial data in order to study the large time behavior of the solution of the Cauchy problem.

3.1. **Existence of $\tau(x_0,\alpha,a)$ and $w_\infty$.** Let us first recall that Assumption 1.1 holds true. Let $x_0\in\mathbb{R}$ and $0<\alpha<p(x_0)$ be given. Let us first note that for any $a<x_0$, the following quantity exists and is finite:

$$\tau(x_0,\alpha,a) := \min\left\{t>0 \mid \widehat{u}(t,x_0;a)=\alpha\right\} < +\infty.$$

This follows from the fact that $\widehat{u}(0,x_0;a)=0<\alpha$ and that, for instance using part $(ii)$ of Lemma 2.9, $\lim_{t\to+\infty}\widehat{u}(t,x_0;a)=p(x_0)>\alpha$. Moreover, it follows

either from part $(i)$ of Lemma 2.9, or more directly from the fact that $\widehat{u}(t,x;a)$ converges locally uniformly in time and space to 0 as $a \to -\infty$, that

$$\tau(x_0,\alpha,a) \to +\infty \text{ as } a \to -\infty.$$

We now aim to prove that the following limit exists for all $(t,x) \in \mathbb{R}^2$:

$$w_\infty(t,x;\alpha) = \lim_{k\to+\infty} \widehat{u}(t+\tau(x_0,\alpha,a-kL),x;a-kL).$$

To do so, let us first notice that from parabolic estimates, the family of functions $\{\widehat{u}(t+\tau(x_0,\alpha,a),x;a)\}_{a<x_0}$ is uniformly bounded along with their derivatives. Therefore it is relatively compact for the topology of $C^1_{loc}(\mathbb{R}^2)$ with respect to $(t,x)$.

Let $(k_j)_{j\in\mathbb{N}}$ be a given sequence of integers such that $k_j \to +\infty$ as $j \to +\infty$, and such that the following limit holds true:

$$\widehat{u}(t+\tau(x_0,\alpha,a-k_jL),x;a-k_jL) \to w_\infty(t,x),$$

as $k \to +\infty$, wherein $w_\infty$ is some function and where the convergence holds in $C^1_{loc}(\mathbb{R}^2)$. One can check from Remark 3.2 that $w_\infty$ is an $\omega$-limit orbit of $\widehat{u}(t,x;a)$, and it therefore follows from Lemma 2.8 that it is steeper than any other entire solution in the sense of Definition 1.6.

Recalling Definition 1.6, there is a unique entire solution $w_\infty$ of $(E)$ that is steeper than any other entire solution and such that $w_\infty(0,x_0) = \alpha$. It follows that $w_\infty$ does not depend on the choice of the sequence $\{k_j\}$. Finally the relative compactness of the family of functions $\{\widehat{u}(t+\tau(x_0,\alpha,a),x;a)\}_{a<x_0}$ completes the proof of the existence of

$$w_\infty(t,x;\alpha) = \lim_{k\to+\infty} \widehat{u}(t+\tau(x_0,\alpha,a-kL),x;a-kL),$$

together with the convergence for the topology of $C^1_{loc}(\mathbb{R}^2)$.

3.2. **Monotonicity in time of $w_\infty$.** In order to complete the proof of Lemma 3.1, it remains to prove the alternative part. To do so, we will show that function $w_\infty$ is nondecreasing with respect to time. We will more precisely prove that for any $t \in \mathbb{R}$, $\partial_t w_\infty(t,\cdot;\alpha)$ does not change sign. To prove this statement, we will argue by contradiction by assuming that for some given $t_1 \in \mathbb{R}$, there exist $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ such that

$$\partial_t w_\infty(t_1,x_1;\alpha) > 0 \text{ and } \partial_t w_\infty(t_1,x_2;\alpha) < 0. \tag{3.10}$$

It is then clear that for any $\tau$ small enough, one has

$$Z\left[w_\infty(t_1+\tau,\cdot;\alpha) - w_\infty(t_1,\cdot;\alpha)\right] \geq 1.$$

Besides, recall that $w_\infty$ is an $\omega$-limit orbit of $\widehat{u}$, and so is $w_\infty(\cdot+\tau,\cdot)$ for any $\tau \in \mathbb{R}$. Therefore, they are steeper than each other, and it immediately follows that $w_\infty(t_1+\tau,\cdot;\alpha) \equiv w_\infty(t_1,\cdot;\alpha)$ for each $\tau$ small enough. Hence,

$$\partial_t w_\infty(t_1,\cdot;\alpha) \equiv 0,$$

a contradiction together with (3.10). This implies that for any $t \in \mathbb{R}$, one has

$$SGN\left[\partial_t w_\infty(t,\cdot;\alpha)\right] = [\,] \text{ or } [+] \text{ or } [-]. \tag{3.11}$$

Let us denote $\Phi := \partial_t w_\infty$. It is an entire solution of the linear parabolic equation

$$\partial_t \Phi = \partial_{xx}\Phi + \partial_u f(x,w_\infty)\Phi.$$

We infer from (3.11) and the strong maximum principle that either $\partial_t w_\infty < 0$, $\partial_t w_\infty > 0$ or $\partial_t w_\infty \equiv 0$. Next due to the definition of $\tau(x_0, \alpha, a)$, one has $\partial_t w_\infty(0, x_0) \geq 0$. This completes the proof of the alternative part of Lemma 3.1.

To conclude the proof of Lemma 3.1, let us show that when $\alpha$ is chosen close enough to $p(x_0)$, then $w_\infty$ cannot be a stationary solution of $(E)$. To show that, let us first notice that due to Assumption 1.1, the stationary solution $p$ is isolated with respect to the other stationary solutions. Therefore, one can choose $\alpha$ close enough to $p(x_0)$ so that there is no stationary solution $q$ with $q(x_0) = \alpha$. Then due to Assumption 1.1, $w_\infty$ is not a stationary solution and it converges to $p$ as $t \to +\infty$. This completes the proof of the lemma.

## 4. Convergence to a propagating terrace

The aim of this section is to prove the convergence of the solutions to a propagating terrace. Since only small differences will arise depending on whether Assumption 1.2 holds or not, in this section we give a common proof for both Theorem 1.12 and Theorem 1.10. We will explicitly write whenever we use Assumption 1.2.

In this section, we will first show that the functions $w_\infty(t, x; \alpha)$, constructed in the previous sections, are either traveling waves or stationary solutions. Using some well chosen values of $\alpha$, we will then be able to construct, by using iterative arguments, the minimal propagating terrace describing the long time behavior of the solution $\widehat{u}$ of $(E)$ with a Heaviside type initial data, as stated in Theorem 1.10. Lastly, we will prove that it satisfies all the required statements.

### 4.1. Convergence to a pulsating traveling wave for some level sets.

Recalling the definition of (3.8), let us define the sequence

$$\tau_k := \begin{cases} \tau(x_0, \alpha, a - kL) - \tau(x_0, \alpha, a - (k-1)L) & \text{if } k \geq 1, \\ \tau(x_0, \alpha, a) & \text{if } k = 0 \end{cases}$$

so that for all $k \in \mathbb{N}$,

$$\tau(x_0, \alpha, a - kL) = \sum_{i=0}^{k} \tau_i.$$

Then the following result holds true:

**Lemma 4.1.** *For any $\alpha \in (0, p(x_0))$, the entire solution $w_\infty$ provided by Lemma 3.1 is either a positive periodic stationary solution or a pulsating traveling wave.*

*Proof.* The proof of this result relies on some properties of the sequence $\{\tau_k\}$. It is split into two parts. Let us first assume that there exists some subsequence $(\tau_{k_j})_{j \in \mathbb{N}}$ converging to some $T > 0$. Then we obtain that

$$\begin{aligned} w_\infty(t + T, x; \alpha) &= \lim_{k \to +\infty} \widehat{u}\left(t + \tau_k + \tau(x_0, \alpha, a - (k-1)L), x; a - (k-1)L\right) \\ &= \lim_{k \to +\infty} \widehat{u}\left(t + \tau(x_0, \alpha, a - kL), x - L; a - kL\right) \\ &= w_\infty(t, x - L; \alpha). \end{aligned}$$

Moreover, Lemma 3.1 provides that $\partial_t w_\infty \geq 0$, and therefore it converges as $t \to \pm\infty$ to two periodic stationary solutions $p_\pm$ (the periodicity follows from the above equality). If the two functions $p_+$ and $p_-$ are distinct, then $w_\infty$ is a pulsating traveling wave. If they are identically equal, then $w_\infty$ is a periodic stationary

solution. Furthermore, it is positive in view of $w_\infty(0, x_0; \alpha) = \alpha > 0$ and the strong maximum principle.

Let us now consider the case when no subsequence of $(\tau_k)_k$ converges to some positive constant and let us show that $w_\infty$ is stationary. Due to Remark 3.2, it is clear that for all $k \in \mathbb{N}$, $\tau_k \geq 0$. On the other hand, it follows from the spreading speed property provided by Lemma 2.9 that

$$\frac{L}{c^*} \leq \liminf_{k\to\infty} \frac{\tau(x_0, \alpha, a - kL)}{k} \leq \limsup_{k\to\infty} \frac{\tau(x_0, \alpha, a - kL)}{k} \leq \frac{L}{c_*}. \tag{4.12}$$

Therefore, since no subsequence of $(\tau_k)_k$ converges to some positive constant, (4.12) implies that one can find two subsequences converging respectively to 0 and $+\infty$.

By considering a subsequence converging to 0, the same computations as above with $T = 0$ lead us to

$$w_\infty(t, x; \alpha) \quad = \quad w_\infty(t, x - L; \alpha), \tag{4.13}$$

and the function $w_\infty$ is $L$-periodic with respect to the space variable for all time.

In order to show that $w_\infty$ is stationary, let us argue by contradiction by assuming that $w_\infty$ is not stationary. Then using Lemma 3.1, one has that

$$\partial_t w_\infty(0, x_0 + L; \alpha) > 0.$$

Thanks to the $C^1_{loc}$ convergence of $\widehat{u}(\cdot + \tau(x_0, \alpha, a - kL), \cdot; a - kL)$ to $w_\infty$ as $k \to +\infty$, there exists some $\delta > 0$ such that for any $0 \leq t \leq \delta$ and $k \in \mathbb{N}$ large enough, one has

$$\partial_t \widehat{u}(t + \tau(x_0, \alpha, a - kL), x_0 + L; a - kL) \geq \frac{\partial_t w_\infty(0, x_0 + L; \alpha)}{2} > 0.$$

On the other hand, using (4.13), for any $\epsilon > 0$, the following holds true for any $k$ large enough:

$$\widehat{u}(\tau(x_0, \alpha, a - kL), x_0 + L; a - kL) \geq \alpha - \epsilon.$$

Then one gets

$$\widehat{u}(\delta + \tau(x_0, \alpha, a - kL), x_0 + L; a - kL) \geq \alpha - \epsilon + \frac{\partial_t w_\infty(0, x_0 + L; \alpha)}{2}\, \delta.$$

By choosing $\epsilon$ small enough, we conclude that $\widehat{u}(\delta + \tau(x_0, \alpha, a - kL); x_0 + L; a - kL) > \alpha$; thus

$$\delta > \tau(x_0 + L, \alpha, a - kL) - \tau(x_0, \alpha, a - kL) = \tau(x_0, \alpha, a - (k+1)L) - \tau(x_0, \alpha, a - kL),$$

for any $k$ large enough. In particular, one gets that the sequence $\{\tau_k\}$ is bounded, which contradicts the existence of a subsequence going to $+\infty$. This completes the proof of the result. □

In this subsection, we have proven that the limit $w_\infty$ is either a periodic positive stationary solution or a pulsating traveling wave. In the monostable case, there is no periodic positive stationary solution between 0 and $p$, so that $w_\infty$ is always a pulsating traveling wave connecting 0 to $p$, which already gives part $(i)$ of Theorem 1.12. Together with Assumption 1.1, it is clear that $p$ is isolated, so that one can choose $\alpha$ close enough to $p(x_0)$, so that function $w_\infty$ is a pulsating traveling wave connecting some periodic stationary solution $p_1$ to $p$.

*Remark* 4.2. Note that it follows from the above proof as well as the uniqueness of the speed, that in the case where $w_\infty$ is a pulsating traveling wave, then the whole sequence $\tau_k$ converges to $\frac{L}{c}$ where $c$ is the speed of $w_\infty$. This will be used later in the paper.

4.2. **Construction of the terrace of traveling fronts.** We now aim to construct a terrace composed of pulsating fronts. We will proceed by iteration to construct such a terrace. Let us first notice that, as mentioned above, by choosing $\alpha$ close enough to $p(x_0)$, one can find a wave $U_1(t,x) = w_\infty(t,x;\alpha)$ connecting some periodic stationary solution $p_1 < p$ to $p$. This gives us the first step of our iterative argument which is related to the following claim:

**Lemma 4.3.** *Assume that for some $0 < \alpha_k < p(x_0)$, the function*

$$U_k(t,x) := w_\infty(t,x;\alpha_k)$$

*is a pulsating traveling wave connecting $p_k > 0$ to $p_{k-1} > p_k$.*

*Then $p_k$ is isolated from below and stable from below with respect to $(E_{per})$. Furthermore, there exists some $\alpha_{k+1} < p_k(x_0)$ such that*

$$U_{k+1}(t,x) := w_\infty(t,x;\alpha_{k+1})$$

*is a pulsating traveling wave connecting some stationary periodic solution $p_{k+1} < p_k$ to $p_k$.*

*Remark* 4.4. Note that the iteration will clearly end if one obtains $p_k \equiv 0$ at some step $k$.

In order to prove the above lemma, we begin with some preliminary claims:

*Claim* 4.5. The stationary solution $p_k$ is steeper than any other entire solution and, moreover,

$$w_\infty(t,x;p_k(x_0)) \equiv p_k(x).$$

*Proof.* Let $v$ be an entire solution of $(E)$ such that $0 < v < p$ and that $v(t_1,x_1) = p_k(x_1)$ for some $(t_1,x_1) \in \mathbb{R}^2$. As before, we know from Lemma 3.1 that $w_\infty(\cdot,\cdot;\alpha_k)$ is steeper than any other entire solution between 0 and $p$; thus for any $t'$ and $t$ in $\mathbb{R}$,

$$Z\left[w_\infty(t',\cdot;\alpha_k) - v(t,\cdot)\right] \le 1,$$

$$SGN\left[w_\infty(t',\cdot;\alpha_k) - v(t,\cdot)\right] \lhd [+\,-].$$

Passing to the limit as $t' \to -\infty$, one gets, for all $t \in \mathbb{R}$,

$$Z\left[p_k(\cdot) - v(t,\cdot)\right] \le 1,$$

$$SGN\left[p_k(\cdot) - v(t,\cdot)\right] \lhd [+\,-].$$

This implies, by Corollary 2.5, that $p_k$ is steeper than $v$. Since $v$ is arbitrary, in particular, $p_k$ is steeper than $w_\infty(\cdot,\cdot;p_k(x_0))$. On the other hand, we also know by Lemma 3.1 that $w_\infty(\cdot,\cdot;p_k(x_0))$ is steeper than $p_k$. Thus these two functions are steeper than each other. Furthermore, neither lies strictly above or below the other since $p_k(x_0) = w_\infty(0,x_0;p_k(x_0))$. Thus we conclude that $w_\infty(t,x;p_k(x_0)) \equiv p_k(x)$, which completes the proof of Claim 4.5. $\square$

*Claim* 4.6. Let $v \equiv v(t,x)$ be a given function satisfying $0 < v(t,x) < p(t,x)$ and let $x(t)$ be a nondecreasing function moving with average speed $0 < c < c_*$ (where $c_*$ is the minimal speed of spreading of $\widehat{u}$ provided by Lemma 2.9). Assume that

$$v(t,x(t)) = p_k(x(t)), \quad \forall t \in \mathbb{R},$$

$$v(t,x) \text{ is a super-solution of } (E) \text{ on } D := \{(t,x) |\, x \geq x(t)\}.$$

Then there exists a sequence $t_j \to +\infty$ such that for any $x \geq 0$,

$$\liminf_{j\to+\infty} v(t_j, x(t_j)+x) - p_k(x(t_j)+x) \geq 0.$$

*Proof.* Let us look at the intersection of $\widehat{u}(t,\cdot;a)$ and $v_j(t,x) := v(t,x-jL)$ for any $j \in \mathbb{N}$. Note that $v_j$ is a super-solution for $(E)$ on the domain

$$D_j := \{(t,x) \mid (t,x-jL) \in D\},$$

and, moreover, that $\widehat{u}(0,\cdot;a) = 0 \leq v_j(0,\cdot)$ on the half-space $x \geq x(0)+jL$ for each $j \in \mathbb{N}$ large enough.

Since $x(t)$ moves with the average speed $c$ smaller than the minimal spreading speed $c_*$ of $\widehat{u}$, one has that for any $j \in \mathbb{N}$,

$$\widehat{u}(t,x(t)+jL;a) \to p(x(t)+jL) > v_j(t,x(t)+jL) = v(t,x(t)),$$

as $t \to +\infty$. Thus, for any $j$ large enough, there exists some minimal time $t_j$ such that

$$\widehat{u}(t_j, x(t_j)+jL;a) = v_j(t_j, x(t_j)+jL) = p_k(x(t_j)).$$

Since $v_j$ is a super-solution on the domain $D_j$, one can check that

$$\begin{aligned} \widehat{u}(t_j,x;a) &\leq v_j(t_j,x) \text{ for any } x \geq x(t_j)+jL \\ &\leq v(t_j, x-jL) \text{ for any } x \geq x(t_j)+jL. \end{aligned}$$

One can easily check that $t_j \to +\infty$ as $j \to +\infty$. Therefore, by standard parabolic estimates and possibly up to a subsequence, one may assume that $x(t_j) \to x_\infty$ in $\mathbb{R}_{/L\mathbb{Z}}$ and that $\widehat{u}(t+t_j, x(t_j)-x_\infty+jL+x;a)$ converges as $j \to +\infty$ to some $\omega$-limit $v_\infty$ that satisfies

$$v_\infty(0,x_\infty) = p_k(x_\infty).$$

Since $v_\infty$ is steeper than $p_k$, and conversely from Claim 4.5, it follows that $v_\infty \equiv p_k$. Therefore, we get that

$$\liminf_{j\to+\infty} v(t_j, x(t_j)+x) - p_k(x(t_j)+x) \geq 0,$$

for any $x \geq 0$, and the result follows. □

We are now able to prove Lemma 4.3.

*Proof of Lemma* 4.3. We will split the proof of this lemma into several parts.

*Step* 1: *$p_k$ is isolated from below.* Assume by contradiction that there exists some sequence $\{q_j\}_j$ of periodic stationary solutions such that $q_j \to p_k$ as $j \to +\infty$ and $q_j < p_k$ for any $j \in \mathbb{N}$. Using standard elliptic estimates, one can easily show that the convergence holds uniformly in $C^1(\mathbb{R})$.

Let us introduce the following principal eigenvalue problem:

$$\left\{ \begin{array}{rcl} -\partial_{xx}\phi_\lambda + 2\lambda\partial_x\phi_\lambda - \frac{\partial f}{\partial u}(x,p_k(x))\phi_\lambda & = & \mu(\lambda)\phi_\lambda \text{ in } \mathbb{R}, \\ \phi_\lambda > 0 \text{ and } L\text{-periodic}. & & \end{array} \right. \tag{4.14}$$

Note that the principal eigenvalue $\mu(\lambda)$, which is associated to the linearized problem around $p_k$, satisfies:

$(i)$ $\mu(\lambda) - \mu(0) = O(\lambda^2)$ on a neighborhood of 0;
$(ii)$ $\mu(0) = 0$.

The first statement $(i)$ follows from the following formula taken from [16] (see Proposition 7.1), which is adapted from Nadin in [21]:

$$\mu(\lambda) = \min_{\substack{\eta \in H^1_{per} \\ \eta > 0}} \frac{1}{\int_0^L \eta^2 dx} \left( \mathcal{F}(p_k, \eta) + \lambda^2 \left( \int_0^L \eta^2 dx - \frac{L^2}{\int_0^L \eta^{-2} dx} \right) \right), \tag{4.15}$$

where $\mathcal{F}(p_k, \eta)$ is the functional defined by

$$\mathcal{F}(p_k, \eta) = \int_0^L \left( \eta'^2 - \frac{\partial f}{\partial u}(x, p_k(x)) \right) dx.$$

Next $(ii)$ follows from the fact that $p_k$ is an accumulation point of periodic stationary solutions.

Let us now construct a super-solution of $(E)$. Consider the function $v$ defined by

$$v(t,x) := \min \left\{ p_k(x), e^{-\lambda(x-ct)}\phi_\lambda + q_j(x) \right\},$$

wherein $0 < c < c_*$ ($c_*$ being the minimal speed of spreading of $\widehat{u}$, provided by Lemma 2.9), and $\lambda > 0$ while $\phi_\lambda$ is a solution of (4.14).

It is clear that there exists some increasing map $t \mapsto x(t)$ moving with the average speed $c$ and such that

$$v(t, x(t)) = p_k(x(t)), \quad \forall t \in \mathbb{R},$$

$$v(t,x) < p_k(x), \forall t \in \mathbb{R} \text{ and } \forall x > x(t).$$

Let us now define

$$D := \{(t,x) \mid x \geq x(t)\},$$

and compute on this set the following quantity:

$$\begin{aligned}
&\partial_t v - \partial_{xx} v - f(x,v) \\
&= e^{-\lambda(x-ct)} \left( (c\lambda - \lambda^2)\phi_\lambda + 2\lambda \partial_x \phi_\lambda - \partial_{xx}\phi_\lambda \right) - \partial_{xx} q_j \\
&\quad - f\left( x, q_j + e^{-\lambda(x-ct)}\phi_\lambda \right) \\
&= e^{-\lambda(x-ct)} \left( (c\lambda - \lambda^2)\phi_\lambda + 2\lambda \partial_x \phi_\lambda - \partial_{xx}\phi_\lambda \right) \\
&\quad - \frac{\partial f}{\partial u}(x, q_j) e^{-\lambda(x-ct)}\phi_\lambda + o\left( \min\left\{ p_k - q_j, \phi_\lambda e^{-\lambda(x-ct)} \right\} \right) \\
&= e^{-\lambda(x-ct)} \left( (c\lambda - \lambda^2)\phi_\lambda + 2\lambda \partial_x \phi_\lambda - \partial_{xx}\phi_\lambda \right) \\
&\quad - \frac{\partial f}{\partial u}(x, p_*) e^{-\lambda(x-ct)}\phi_\lambda + o\left( \min\left\{ p_k - q_j, \phi_\lambda e^{-\lambda(x-ct)} \right\} \right) \\
&= e^{-\lambda(x-ct)} (c\lambda - \lambda^2 + \mu(\lambda))\phi_\lambda + o\left( \min\left\{ p_k - q_j, \phi_\lambda e^{-\lambda(x-ct)} \right\} \right) \\
&> 0,
\end{aligned}$$

where the last inequality holds for any $j$ large enough and any $\lambda$ small enough, on the domain $D$.

Next Claim 4.6 applies and provides the existence of a sequence $t_j \to +\infty$ such that

$$\liminf_{j\to+\infty} v(t_j, x(t_j)+x) - p_k(x(t_j)+x) \geq 0, \tag{4.16}$$

as $j \to +\infty$ and for any $x \geq 0$. On the other hand, from the definition of $v$, there exists some constant $A > 0$ such that for any $t \in \mathbb{R}$,

$$v(t, x(t)+A; a) < p_k(x(t)+A).$$

This contradicts (4.16) and we conclude that $p_k$ is isolated from below with respect to $(E_{per})$.

*Step* 2: *Stability from below.* To prove this statement we will argue by contradiction and we assume that $p_k$ is unstable from below with respect to $(E_{per})$. Let us distinguish two cases.

Assume first that $p_k$ is linearly unstable, that is, $\mu(0) < 0$, where $\mu$ is defined as in (4.14). Then, proceeding as in the well known monostable case (see for instance [5]), one can find a stationary super-solution of the form $v(x) := p_k(x) - \kappa\psi_R(x)$, with $\kappa > 0$ small enough and wherein $\psi_R$ is a principal eigenfunction of the following problem:

$$\begin{cases} -\partial_{xx}\psi_R - \frac{\partial f}{\partial u}(x, p_k(x))\psi_R = \mu_R\psi_R \ \text{ in } (-R, R), \\ \psi_R > 0 \text{ and } \psi_R(\pm R) = 0. \end{cases} \tag{4.17}$$

One can check, using the regularity of $f$ and the fact that $\mu_R \to \mu(0) < 0$ as $R \to +\infty$, that $v$ is a super-solution of $(E)$. As before, one can then apply Claim 4.6 to reach a contradiction in this case.

Assume now that $p_k$ is not linearly unstable, namely $\mu(0) = 0$. Since $p_k$ is unstable from below with respect to $(E_{per})$, we also know that there exists some entire solution $U(t, x)$, decreasing in time and periodic with respect to the space variable, that belongs to the unstable set of $p_k$ in the downward direction (we refer to [19]), that is, such that

$$U(t, x) < p_k(x) \ \forall (t, x) \in \mathbb{R}^2,$$

$$U(t, x) \to p_k(x) \ \text{ as } t \to -\infty.$$

Let $\{t_j\}_{j\in\mathbb{N}}$ be a sequence converging to $-\infty$ as $j \to +\infty$, and such that the $L$-periodic function

$$r_j(x) := U(t_j, x)$$

satisfies

$$\partial_{xx} r_j + f(x, r_j(x)) < 0.$$

Similarly as above, we construct a super-solution crossing $p_k$ and use Claim 4.6 to reach a contradiction. Let us consider the function

$$w(t, x) := r_j(x) + e^{-\lambda(x-ct)}\phi_\lambda(x),$$

wherein $0 < c < c_*$ and $c_*$ is the minimal speed of spreading of $\widehat{u}$, $\lambda > 0$, while $\phi_\lambda$ is a solution of (4.14). As before, one can check that function $w$ satisfies the hypotheses of Claim 4.6 and stays away from below to $p_k$ as $t \to +\infty$. We again reach a contradiction.

*Step* 3: *Convergence to a pulsating traveling wave for* $\alpha < p_k(x_0)$. This last step is rather easier. We already know that for any $0 < \alpha < p_k(x_0)$, we have that $w_\infty(t, x; \alpha)$ is either a pulsating traveling wave or a positive and periodic stationary solution. But we have just shown that $p_k$ is isolated from below, therefore similarly as we have done before in the case $p_k = p$ to begin our iteration, for any $\alpha$ close enough to $p_k(x_0)$, $w_\infty(t, x; \alpha)$ is a pulsating traveling wave connecting some stationary solution $p_{k+1}$ to $p_k$. □

To conclude the proof of the existence of a propagating terrace, it remains to show that the sequence is finite (or equivalently, that $p_k \equiv 0$ at some step $k$).

Let us argue by contradiction and assume that it is not. Since the sequence $\{p_k\}_k$ is monotonically decreasing, it converges uniformly to some $p_\infty \geq 0$, a periodic stationary solution of $(E)$. As in the proof of Lemma 4.3, we use a super-solution crossing some $p_k$ to get a contradiction.

We introduce the following principal eigenvalue problem:

$$\left\{ \begin{array}{rcl} -\partial_{xx}\psi_\lambda + 2\lambda\partial_x\psi_\lambda - \frac{\partial f}{\partial u}(x, p_\infty(x))\psi_\lambda & = & \nu(\lambda)\psi_\lambda \ \text{ in } \mathbb{R}, \\ \psi_\lambda > 0 \text{ and } L\text{-periodic.} & & \end{array} \right. \tag{4.18}$$

The above defined eigenproblem is the same as (4.14), with $p_k$ replaced by $p_\infty$. As before, one has that $\nu(\lambda)$ satisfies:

(*i*) $\nu(\lambda) \geq \nu(0)$ for any $\lambda$, and $\nu(\lambda) - \nu(0) = O(\lambda^2)$ on a neighborhood of 0;
(*ii*) $\nu(0) = 0$.

Let us now introduce the function

$$z(t, x) := e^{-\lambda(x-ct)}\psi_\lambda + p_\infty(x),$$

wherein $0 < c < c_*$, $\lambda > 0$ and $\psi_\lambda$ is a solution of (4.18). Using the same computations as in the proof of Lemma 4.3, one gets that

$$\partial_t z - \partial_{xx} z - f(x, z) > 0,$$

for all $\lambda$ small enough, on some domain of the form $\{(t, x) |\ x \geq x(t)\}$, wherein $x(t)$ moves with the average speed $c$ and satisfies for some $k$ large enough and any $t \in \mathbb{R}$:

$$z(t, x(t)) = p_k(x(t)) \text{ and } z(t, x) \leq p_k(x) \text{ for } x \geq x(t).$$

As $z(t, x(t) + x) \to p_\infty(x(t) + x) < p_k(x(t) + x)$ as $x \to +\infty$ uniformly with respect to $t \in \mathbb{R}$, one can proceed as before to reach a contradiction together with Claim 4.6.

We conclude that the iterative process stops in a finite number $N$ of steps. This allows us to construct a propagating terrace, which is called $T^*$.

*Remark* 4.7. In fact, we have not yet proven that the sequence $(c_k)_k$ of the speeds of the pulsating traveling waves $U_k$ is nondecreasing. However, this directly follows from Lemma 4.3. Indeed Lemma 4.3 also states that we have a decreasing sequence $(\alpha_k)_k$ such that

$$U_k(t, x) = w_\infty(t, x; \alpha_k).$$

Then, from Remark 4.2, the speed $c_k$ of $U_k$ can be obtained as

$$c_k = \lim_{j \to +\infty} \frac{jL}{\tau(x_0, \alpha_k, a - jL)},$$

and since $\alpha \mapsto \tau(x_0, \alpha, b)$ is increasing for any $x_0$ and $b$ (see Remark 3.2), one obtains that the sequence $c_k$ is nondecreasing.

To conclude the proof of Theorem 1.10, it remains to check that the propagating terrace $T^*$ satisfies all the required statements. This is in fact straightforward from all the above. Part $(i)$ indeed immediately follows from the construction of the terrace and Lemma 4.3. Part $(ii)$ follows from Claim 4.5 and the construction of the $U_k$ as some $\omega$-limit orbits of $\widehat{u}$.

Moreover, one can easily check that $T^*$ is minimal. Indeed, let us argue by contradiction by assuming that it is not. Then, one can easily check that there exist some $k$ and some traveling wave $V$ crossing $p_k$. This contradicts the fact that it is steeper than any other entire solution.

Lastly, let us check that any other minimal propagating terrace $T$ is equal to $T^*$. Let $T = ((q_k)_k, (V_k)_k)$ be a given other minimal propagating terrace. Then it immediately follows from the definition that the two sequences $(q_k)_k$ and $(p_k)_k$ are identically equal. Then, for any $k$, $V_k$ and $U_k$ are steeper than each other (from Definition 1.9 and part $(ii)$ of Theorem 1.10) and intersect, hence they are identically equal up to some time shift.

This ends the proof of Theorem 1.10.

4.3. **Locally or uniform convergence to the waves.** In this section, we prove the convergence part of Theorem 1.12 as well as Theorem 1.11.

Let us first show the locally uniform convergence to the pulsating traveling waves $(U_k)_{1\leq k\leq N}$ along the moving frames with speed $c_k$ and some sublinear drifts. Let us fix some $1 \leq k \leq N$. For any large enough $t$, let us define $j(t) \in \mathbb{N}$ such that

$$j(t)\frac{L}{c_k} \leq t < (j(t)+1)\,\frac{L}{c_k},$$

and let us introduce

$$t_{j(t)} := \sum_{i=0}^{i=j(t)} \tau_i,$$

wherein $\tau_j$ is defined as in Section 4.1, with $\alpha = \alpha_k$ chosen so that $U_k(\cdot,\cdot) = w_\infty(\cdot,\cdot;\alpha_k)$. Let us now consider $m_k(t)$, the piecewized affine function, defined by

$$m_k(t) = t_{j(t)} - t \quad \text{if} \quad t = j(t)\frac{L}{c_k}.$$

Recall that the sequence $\{\frac{1}{j}\sum_{i=0}^{i=j}\tau_i\}_j$ converges to $\frac{L}{c_k}$, so that $m_k(t) = o(j(t)) = o(t)$ as $t \to +\infty$.

Furthermore, since

$$U_k(t,x) = w_\infty(t,x;\alpha_k) = \lim_{j\to+\infty} \widehat{u}(t+\tau_0+...+\tau_j, x+jL; a),$$

where the above convergence is understood to hold locally uniformly with respect to $(t,x) \in \mathbb{R}^2$, and since $t + m_k(t) - t_{j(t)} \sim (t - j(t)\frac{L}{c_k})$ and $x + c_k t - j(t)L$ stay bounded, one can check that

$$\widehat{u}(t+m_k(t), x+c_k t; a) - U_k\left(t - j(t)\frac{L}{c_k}, x - j(t)L + c_k t\right) \to 0 \quad \text{as} \quad t \to +\infty.$$

Thus, we obtain

$$\widehat{u}(t, x+c_k(t-m_k(t)); a) - U_k\left(t-m_k(t), x+c_k(t-m_k(t))\right) \to 0 \quad \text{as} \quad t \to +\infty,$$

wherein both of the above convergences hold locally uniformly with respect to $x \in \mathbb{R}$. This completes, in the general case, the convergence result (1.4) stated in Theorem 1.11.

It now remains to consider what happens "outside" of the moving frames with speed $(c_k)_{1\leq k\leq N}$. This will follow from the following monotonicity property:

*Claim* 4.8. For all $(t,x) \in [0,+\infty) \times \mathbb{R}$, one has

$$\widehat{u}(t,x;a) \geq \widehat{u}(t,x+L;a).$$

*Proof.* This claim directly follows from Remark 3.2. □

Let us first look on the left of the terrace, that is, when $x+c_1(t-m_1(t)) \to -\infty$. In that case, we will use the fact that

$$\lim_{t\to+\infty} U_1(t,x) \equiv p(x). \tag{4.19}$$

Let $\delta > 0$ be a given small enough number. From the asymptotics of $U_1$, there exists $x_\delta$ such that for all $t$,

$$p(x) - \frac{\delta}{2} \leq U_1(t-m_1(t), x+c_1(t-m_1(t))) \leq p(x) \text{ for all } x \leq -x_\delta + L.$$

Next there exists some large time $T$ such that for all $x \in [-x_\delta, x_\delta]$,

$$|\widehat{u}(t,x+c_1(t-m_1(t));a) - U_1(t-m_1(t), x+c_1(t-m_1(t)))| \leq \frac{\delta}{2}.$$

Then, using Claim 4.8, one gets for all $t \geq T$,

$$p(x) - \delta \leq \widehat{u}(t,x+c_1(t-m_1(t));a) \leq p(x) \text{ for all } x \leq -x_\delta + L.$$

One can proceed similarly to get that for any $\delta > 0$, there exist $C$ and $T$ such that for any $x \geq C$ and $t \geq T$,

$$|\widehat{u}(t,x+c_N(t-m_N(t));a)| \leq \delta.$$

Note that under Assumption 1.2, since $N = 1$, the uniform convergence (1.5) immediately follows from the above computations.

Lastly, let $1 \leq k < N$ be a given integer. Then one has

$$\lim_{t\to-\infty} U_k(t,x) \equiv p_k(x) \text{ and } \lim_{t\to+\infty} U_{k+1}(t,x) \equiv p_k(x).$$

As above, one can use Claim 4.8 to show that there exist some constants $C > 0$ and $T > 0$ such that for each large time $t \geq T$,

$$p_k(x) + \delta \geq \widehat{u}(t,x+c_k(t-m_k(t));a) \text{ for all } x \geq C,$$

$$p_k(x) - \delta \leq \widehat{u}(t,x+c_{k+1}(t-m_{k+1}(t));a) \leq p(x) \text{ for all } x \leq -C.$$

Note that because we are looking at a finite number of $k$, the constants $C$ and $T$ can be chosen to depend only on $\delta$ by taking the largest ones. This ends the proof of Theorem 1.11.

## Notes

Since the submission of this work, some recent developments have appeared in the study of the convergence to the pulsating traveling wave with minimal speed of the KPP equation, from fast decaying but not necessarily Heaviside type initial data. In [13], by locating precisely the profile of the solution, the convergence is proven for any compactly supported initial data. In a sequel of this paper [11], one of the authors will also prove a similar result using the zero number argument.

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