WELL-POSEDNESS OF THE EQUATION
FOR THE THREE-FORM FIELD
IN ELEVEN-DIMENSIONAL SUPERGRAVITY

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ABSTRACT. We analyze a semi-linear gauge-invariant wave equation which arises in the theory of supergravity. We prove that the Cauchy problem is well-posed globally in time for the fixed-gauge version of the equation for small compactly supported smooth data. We employ the method of Klainerman vector fields along with a finer analysis of the nonlinearity to establish an integrable decay in the energy estimate.

1. INTRODUCTION

Let K be a compact 7-dimensional Riemannian manifold. Then the product $\mathbb{R}^{3+1} \times K$ becomes an 11-dimensional Lorentzian manifold. For a three-form u on $\mathbb{R}^{3+1} \times K$ we use the Hodge star $*$ and the de Rham differential d of the product to formulate the following Cauchy problem:[1]

$$\Box_{\mathbb{R}^{3+1}} u + \Delta_K u = - * (du \wedge du), \tag{1.1a}$$

$$u(0, \cdot) = u_0, u_t(0, \cdot) = u_1. \tag{1.1b}$$

In this article, we will prove the following statement.

Theorem 1.1. *There exist positive N, ϵ such that the Cauchy problem (1.1) is globally well-posed provided that initial data is localized in the ball of radius 1 in $\mathbb{R}^3$ for every point of K and obeys*

$$\|u_0\|_{H^N(\mathbb{R}^3 \times K)} + \|u_1\|_{H^{N-1}(\mathbb{R}^3 \times K)} \le \epsilon.$$

Moreover, in such a case the solution u satisfies the estimates

$$\sum_{|\alpha| \le N} \|\nabla_{t,x,y} \Gamma^\alpha u(t)\|_{L^2(\mathbb{R}^3 \times K)} \le C\epsilon(1+t)^{1/12},$$

$$\sum_{|\alpha| \le N-10} \|\nabla_{t,x,y} \Gamma^\alpha u(t)\|_{L^2(\mathbb{R}^3 \times K)} \le C\epsilon,$$

[1]Our sign convention for the d'Alambertian is

$$\Box_{\mathbb{R}^{3+1}} = -\frac{\partial^2}{\partial t^2} + \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2},$$

and the operator Δ_K is negative.

$$(1+t) \sum_{|\alpha| \leq N-20} \|\nabla_{t,x,y} \Gamma^\alpha \mathcal{P}_0 u(t)\|_{L^\infty(\mathbb{R}^3 \times K)}$$

$$+(1+t)^{3/2} \sum_{|\alpha| \leq N-20} \|\nabla_{t,x,y} \Gamma^\alpha \mathcal{P}_{>0} u(t)\|_{L^\infty(\mathbb{R}^3 \times K)} \leq C\epsilon,$$

where $\mathcal{P}_0, \mathcal{P}_{>0}$ are the spectral projections of the operator Δ_K defined in Section 3, Γ^α are compositions of a subset of Klainerman vector fields together with the operator $(-\Delta_K)^{1/2}$, which are defined in Section 4, $\nabla_{t,x,y}$ is the gradient in all the derivatives of $\mathbb{R}^{3+1} \times K$, and C is a constant that depends only on N and the geometry of K.

The theorem is true if the Cauchy data is supported in a larger ball but then the constant ϵ has to decrease as a negative power of the size of the support.

The equations have a connection to the theory of supergravity, which we explain in the next section. The mathematical aspects of the supergravity theory have recently drawn attention in the context of conformal geometry [3],[6], where the space-time was assumed to be a Riemannian manifold. The Lorentzian case was investigated earlier (see [1] and references therein).

Our methods are inspired by the work of Metcalfe, Sogge and Stewart [11] and Metcalfe and Stewart [12] who analyze the quasi-linear wave equation on $\mathbb{R}^{3+1} \times D$, where D is a bounded domain in $\mathbb{R}^n$ with various nonlinearities and boundary conditions. Their results do not cover the case under study but we employ some of their ideas in this work. See also [10] for the previous work on Klein-Gordon equations on $\mathbb{R}^{3+1} \times D$.

The article is organized as follows. In Section 2 we derive the equation from a gauge-invariant Lagrangian and explain how to fix the gauge. In Section 3 we recall some necessary facts from Riemannian geometry. In Section 4 we adapt the linear estimates for the wave equation on $\mathbb{R}^{3+1}$ to the product $\mathbb{R}^{3+1} \times K$. In Section 5 we perform a deeper analysis of the nonlinearity. In Section 6, we provide the proof of Theorem 1.1.

We will denote by k a constant which depends only on N and the geometry of K; this constant may change from line to line but for each inequality below there is an apriori computable constant such that the inequality holds. We will also write $A \lesssim B$ to mean $A \leq kB$.

2. Background

2.1. **Physical motivation.** The supergravity theory is a model of classical physics, which describes the low-energy, classical limit of superstring theory. The model describes the interaction of the field of gravity with other fields. In one of the simplest setups, one considers an 11-dimensional Lorentzian manifold as a space-time, with gravity field g and a field, whose strength is described by a closed differential four-form F. The Lagrangian is prescribed only locally by restricting attention to an open, topologically trivial subset U of the space-time. Then one solves the equation for a potential of F on U:

$$dA = F.$$

With this, the Lagrangian can be written as

$$\mathcal{L} = \int_U R dv + \int_U F \wedge *F + \int_U A \wedge F \wedge F,$$

where R, dv and $*$ are the (scalar) Ricci curvature, the volume form and the Hodge $*$ corresponding to g, respectively. The reader should consult [17] and textbooks [2, Section 3.3], [13, Section 16.1.1] for physical aspects of this theory.

2.2. **The Lagrangian and the equation.** In this paper we wish to concentrate on the global aspects of dynamics of the field F, leaving the more difficult question of interaction of F with gravity for further research. Therefore we will simplify our setup to consider the product manifold $\mathbb{R}^{3+1} \times K$ as the fixed space-time, where K is a 7-dimensional compact Riemannian manifold without a boundary (thus dropping the term $\int R\,dv$ from the Lagrangian). The metric on the product space will be the product of the Minkowski metric and the metric on K. We will also assume that the field strength F is not only closed but also exact, namely there exists a global three-form u, such that

$$du = F.$$

Then u will be the dynamical variable, for which we define a classical field theory Lagrangian

$$\mathcal{L}(u) = \int_{\mathbb{R}^{3+1} \times K} du \wedge *du + \int_{\mathbb{R}^{3+1} \times K} u \wedge du \wedge du.$$

The formal Euler-Lagrange equations are

$$(2.1) \qquad d * du = -du \wedge du.$$

We will take the Hodge-dual on both sides of the equation and use the notation $\delta = -*d*$ to arrive at the equation

$$(2.2) \qquad \delta du = *(du \wedge du).$$

Since our space-time is a product manifold, most of the operators which act on it can be decomposed in a natural way as operators acting on either $\mathbb{R}^{3+1}$ or K. We will denote by subscript $\parallel$ the operators acting on $\mathbb{R}^{3+1}$ and by subscript $\perp$ the operators acting on K. For instance, we will have

$$(2.3) \qquad d = d_{\mathbb{R}^{3+1} \times K} = d_{\mathbb{R}^{3+1}} \otimes \mathrm{id}_K + \mathrm{id}_{\mathbb{R}^{3+1}} \otimes d_K = d_{\parallel} \otimes \mathrm{id}_K + \mathrm{id}_{\mathbb{R}^{3+1}} \otimes d_{\perp}.$$

The tensor notation should be understood in terms of operations on differential forms $\Omega(\mathbb{R}^{3+1})$ and $\Omega(K)$. We will colloquially write equation (2.3) as

$$(2.4) \qquad d = d_{\parallel} + d_{\perp}.$$

2.3. **Hodge star and form Laplacian.** Let us recall a few simple facts regarding the Hodge star operator, which is an operator that takes differential n-forms to differential $(11 - n)$-forms. Let x^i, $i = 0, \ldots, 3$, be the coordinates on $\mathbb{R}^{3+1}$ and let x^i, $i = 4, \ldots, 10$, be a coordinate patch on K at a point where the metric tensor is the identity and its derivative vanishes, i.e. normal coordinates. Then the Hodge dual $*$ for an n-form $v = v_{i_0 i_2 \ldots i_{n-1}} dx^{i_0} dx^{i_1} \ldots dx^{i_{n-1}}$, $i_k = 0, \ldots, 10$, is defined as follows:

$$(2.5) \qquad *v = \sum_{i_0, i_1, \ldots, i_{10}=0}^{10} (-1)^{\alpha(\{i_0 \ldots i_{n-1}\})} \varepsilon^{i_0 \ldots i_{10}} v_{i_0 i_1 \ldots i_{n-1}} dx^{i_n} dx^{i_{n+1}} \ldots dx^{i_{10}},$$

where

$$\varepsilon^{i_0 \ldots i_{10}} = \begin{cases} 0, & i_k = i_l \text{ for some } k \neq l, \\ 1, & i_0 \ldots i_{10} \text{ is an even permutation}, \\ -1, & i_0 \ldots i_{10} \text{ is an odd permutation}, \end{cases}$$

and

$$\alpha(\{i_0 \dots i_n\}) = \begin{cases} 1, & 0 \in \{i_0 \dots i_n\}, \\ 0, & 0 \notin \{i_0 \dots i_n\}. \end{cases}$$

Thus the $*$ operator exchanges the components of the forms, multiplying those containing the time x^0 coordinate by -1. Next, we define δ which takes n-forms to $(n-1)$-forms by

$$\delta u = (-1)^{\deg u} * d(*u).$$

Lastly we define

$$\Box_{\mathbb{R}^{3+1} \times K} = -d\delta - \delta d.$$

We have the following facts about $*$ and δ:

- $** u = (-1)^{\deg u} u$.
- $u \wedge *v = g(u, v) d\text{vol}$, where g is the Lorentzian metric on $\mathbb{R}^{3+1} \times K$ and $d\text{vol}$ is the volume form.
- The operator $*$ is an isometry and in particular $d * u = 0$ if and only if $\delta u = 0$.
- δ is the Lorentzian adjoint of d in the sense that

$$\int_{\mathbb{R}^{3+1} \times K} g(\delta u, v) d\text{vol} = \int_{\mathbb{R}^{3+1} \times K} g(u, dv) d\text{vol}.$$

- In normal coordinates and with notational conventions of relativity we have

$$(\delta u)_{\alpha_1 \alpha_2 \dots} = -\partial^{\alpha_0} u_{\alpha_0 \alpha_1 \alpha_2 \dots},$$

$$(\Box_{\mathbb{R}^{3+1} \times K} u)_{\alpha_1 \alpha_2 \dots} = \partial^{\alpha_0} \partial_{\alpha_0} u_{\alpha_1 \alpha_2 \dots} + (f(R)u)_{\alpha_1 \alpha_2 \dots},$$

  where $f(R)$ is a linear, zeroth-order tensor that depends on the Riemann curvature tensor.
- $\Box_{\mathbb{R}^{3+1} \times K} * u = *\Box_{\mathbb{R}^{3+1} \times K} u$.
- We have

$$\Box_{\mathbb{R}^{3+1} \times K} du = d(\Box_{\mathbb{R}^{3+1} \times K} u), \quad \Box_{\mathbb{R}^{3+1} \times K} \delta u = \delta(\Box_{\mathbb{R}^{3+1} \times K} u).$$

  The first identity is a consequence of the fact that $d^2 = 0$. This fact also implies $\delta^2 = 0$ which leads to the second identity above.

2.4. Gauge fixing.
There is an obvious gauge freedom in equation (2.2) - if u is a solution of the equation, then for any two-form w, the three-form $u + dw$ is a solution as well. Therefore, we will fix the gauge by requiring that u satisfies

(2.6) $$d * u = 0,$$

which is equivalent to

$$\delta u = 0.$$

This choice of gauge is similar to the Lorenz gauge of the Maxwell equations, where for a one-form $\mathfrak{a}$ one requires $d * \mathfrak{a} = 0$. Since we work with 3-forms on a product manifold, the gauge is structurally more complicated. We will give a proof that equation (2.6) is a valid gauge choice in the end of this section but first we rewrite equation (2.2) using the gauge. We defined the Laplace(d'Alembert)-Beltrami operator on forms as $\Box_{\mathbb{R}^{3+1} \times K} u = -\delta d - d\delta$.[2] On a product manifold,

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[2]The operator Δ_K is negative. See footnote 1.

the operator decomposes into $\square_{\mathbb{R}^{3+1} \times K} = \square_{\mathbb{R}^{3+1}} + \Delta_K$, where Δ_K is the Laplace-Beltrami operator on $\Omega(K)$ (the space of differential forms). Thus we can rewrite the main equation (2.2) as

$$\square_{\mathbb{R}^{3+1} \times K} u = -\delta du - d\delta u = - * (du \wedge du).$$

Therefore, equation (2.2) with the gauge choice (2.6) becomes

$$(2.7) \qquad \square_{\mathbb{R}^{3+1}} u + \Delta_K u = - * (du \wedge du).$$

Let us now address the validity of the gauge choice.

Proposition 2.1. *Let $F \in \Omega^4(\mathbb{R}^{3+1} \times K)$. Suppose there exists a solution $A \in \Omega^3(\mathbb{R}^{3+1} \times K)$ to the equation $dA = F$. Then there exists a solution to the system*

$$d\tilde{A} = F, \quad d * \tilde{A} = 0.$$

Moreover, we can choose $\tilde{A}$ such that

$$\operatorname{supp} \tilde{A} \subseteq \{(t, x) \times K | (s, y, z) \in \operatorname{supp} A, |x - y| \leq |t - s|\}$$

for $t, s \in \mathbb{R}, x, y \in \mathbb{R}^3, z \in K$.

Proof. Let A be as above. Denote $\delta A = e$. The two-form e is the error we wish to eliminate by finding a two-form b such that $\delta(A + db) = 0$. We solve the equation

$$(2.8) \qquad -\square_{\mathbb{R}^{3+1} \times K} b = \delta db + d\delta b = -e$$

and define

$$\tilde{A} = A + db.$$

We have from equation (2.8)

$$(2.9) \qquad \delta \tilde{A} = -d\delta b.$$

We have

$$\square \delta b = \delta \square b = -\delta e = -\delta^2 A = 0.$$

Thus δb solves the homogeneous wave equation. We will prove that $\delta b = 0$ by choosing suitable Cauchy data for b at the hypersurface $t = 0$. Our goal is to make the Cauchy data for δb be zero. The Cauchy data that we prescribe for b in the normal coordinates are as follows:

$$b_{\alpha_1 \alpha_2} = 0, \quad \alpha_1, \alpha_2 = 0, \ldots, 10,$$

$$\frac{\partial}{\partial t} b_{0\alpha} = 0, \quad \alpha = 1, \ldots, 10,$$

$$(2.10) \qquad \frac{\partial}{\partial t} b_{\alpha_1 \alpha_2} = A_{0\alpha_1 \alpha_2} = -A_{\alpha_1 0 \alpha_2}, \quad \alpha_1, \alpha_2 = 0, \ldots, 10.$$

We check the Cauchy data for δb at $t = 0$:

$$(\delta b)_\alpha |_{t=0} = \frac{\partial}{\partial t} b_{0\alpha} - \sum_{\beta \neq 0} \frac{\partial}{\partial x_\beta} b_{\beta \alpha} = 0.$$

This is because the Cauchy data for $b_{0\alpha}$ is zero and the function $b_{\beta\alpha} = 0$ for $\alpha \neq 0$ at $t = 0$ and so are the spatial derivatives. To see that the time derivative of δb at $t = 0$ is zero, we employ the equation for b

$$\frac{\partial}{\partial t}(\delta b)_\alpha|_{t=0} = \frac{\partial^2 b_{0\alpha}}{\partial t^2} + \sum_{\mu \neq 0} \frac{\partial}{\partial t} \frac{\partial}{\partial x_\mu} b_{\mu\alpha}$$

$$= (\Delta_{\mathbb{R}^3} + \Delta_K) b_{0\alpha} - (\delta A)_{0\alpha} + \sum_\mu \frac{\partial}{\partial t} \frac{\partial}{\partial x_\mu} b_{\mu\nu}.$$

Observe that $(\Delta_{\mathbb{R}^3} + \Delta_K) b_{0\alpha} = 0$ since the function at $t = 0$ is zero and we take spatial derivatives and zero order terms which are linear in b to compute the Laplacian. The second and the third terms cancel since $A_{00\alpha} = 0$ by antisymmetry and thus

$$(\delta A)_{0\alpha} = -\sum_{\mu \neq 0} \frac{\partial}{\partial x_\mu} A_{\mu 0\alpha} = \frac{\partial}{\partial t} \sum_{\mu \neq 0} \frac{\partial}{\partial x_\mu} b_{\mu\alpha},$$

by equation (2.10). Therefore

$$(\delta b)_\alpha|_{t=0} = 0, \quad \frac{\partial}{\partial t}(\delta b)_\alpha|_{t=0} = 0$$

and thus δb obeys a homogeneous wave equation with zero Cauchy data, which makes it identically zero. Therefore

$$\delta \tilde{A} = \delta(A + db) = d\delta b = 0.$$

Observe that the support of the Cauchy data for b is contained in the support for A and thus the statement on the support follows from finite speed of propagation. $\quad\square$

We now wish to prove that the gauge condition $\delta u = 0$ persists for the equation (2.7). For that we need to discuss the initial conditions. Since the equation is of second order, the natural Cauchy data is $u|_{t=0}$ and $\frac{\partial}{\partial t} u|_{t=0}$. If we express (2.2) through the field strength $F = du$ we have

(2.11) $$\delta F = *(F \wedge F).$$

The natural initial condition for this first order equation is $F|_{t=0}$ but we first need to observe that there is a certain compatibility condition in (2.11), which is not of the evolution form. For that we recall the notion of the interior product of a form by a vector field. Let α be an n-form and let X be a vector field; then the interior product of α by X, denoted by $\alpha \lrcorner_X$, is an $(n-1)$-form defined by

$$\alpha \lrcorner_X (X_1, X_2, \ldots, X_{n-1}) = \alpha(X, X_1, X_2, \ldots, X_{n-1}).$$

We will be interested in interior products by $\frac{\partial}{\partial t} = \frac{\partial}{\partial x_0}$. Such a construction can be simply described as freezing the first index of the form α to be the zero (i.e. time) index. Thus in coordinates

$$(\alpha \lrcorner_{\frac{\partial}{\partial t}})_{a_1 a_2 \ldots} = \alpha_{0 a_1 a_2 \ldots}.$$

With this notation we prove the following observation.

Claim 2.2. *The form* $(\delta F) \lrcorner_{\frac{\partial}{\partial t}}$ *does not contain the time derivatives of* F.

Proof. We will give the proof in normal coordinates. We have

$$[(\delta F) \rfloor_{\frac{\partial}{\partial t}}]_{a_1 a_2} = (\delta F)_{0 a_1 a_2} = \frac{\partial}{\partial x_0} F_{0 0 a_1 a_2} - \sum_{i=1}^{10} \frac{\partial}{\partial x_i} F_{i 0 a_1 a_2}.$$

Only the first term contains the time derivative but $F_{0 0 a_1 a_2} = 0$ due to antisymmetry of F. $\qquad\square$

Thus applying $\rfloor_{\frac{\partial}{\partial t}}$ to (2.11) and restricting it to time $t = 0$ we see that both sides of the equality

$$(\delta F) \rfloor_{\frac{\partial}{\partial t}}|_{t=0} = [*(F \wedge F)] \rfloor_{\frac{\partial}{\partial t}}|_{t=0}$$

depend only on $F|_{t=0}$ and the spatial derivatives of $F|_{t=0}$ so they are functions of a gauge-invariant part of the Cauchy data and express a compatibility condition, which must hold in both gauge-invariant and gauge-fixed versions of the equations. We thus make the following definition.

Definition 2.3. The form $du|_{t=0}$ is compatible if

$$(\delta du) \rfloor_{\frac{\partial}{\partial t}}|_{t=0} = *(du \wedge du) \rfloor_{\frac{\partial}{\partial t}}|_{t=0}.$$

We now can prove that the gauge condition $\delta u = 0$ persists in the equation for the compatible Cauchy data.

Proposition 2.4. *Let u solve*

$$\Box_{\mathbb{R}^{3+1} \times K} u = - * (du \wedge du),$$

such that $\delta u|_{t=0} = 0$ and $du|_{t=0}$ is compatible. Then $\delta u = 0$ for all times.

Proof. We apply $\Box_{\mathbb{R}^{3+1} \times K}$ to δu to see that

$$\Box_{\mathbb{R}^{3+1} \times K}(\delta u) = \delta(\Box_{\mathbb{R}^{3+1} \times K}) u = \delta * (du \wedge du) = *d * *(du \wedge du)$$
$$= *d(du \wedge du) = 2 * (d^2 u \wedge du) = 0.$$

We check the Cauchy data: $\delta u|_{t=0} = 0$ by assumption. The term $\frac{\partial}{\partial t} \delta u|_{t=0}$ vanishes because of the equation and the compatibility condition. We prove this in normal coordinates. We have

$$[(d\delta u) \rfloor_{\frac{\partial}{\partial t}}]_{ab} = (d\delta u)_{0ab} = \frac{\partial}{\partial t}(\delta u)_{ab} - \frac{\partial}{\partial x_a}(\delta u)_{0b} + \frac{\partial}{\partial x_b}(\delta u)_{0a}.$$

If $a, b \neq 0$, then the last two terms above are spatial derivatives of δu which is zero when we compute at $t = 0$. Therefore, for $a, b \neq 0$

$$\frac{\partial}{\partial t}(\delta u)_{ab}|_{t=0} = [(d\delta u) \rfloor_{\frac{\partial}{\partial t}}]_{ab}|_{t=0} = [(-\Box u - \delta du) \rfloor_{\frac{\partial}{\partial t}}]_{ab}|_{t=0}$$
$$= \{[*(du \wedge du) - \delta du)] \rfloor_{\frac{\partial}{\partial t}}\}_{ab}|_{t=0} = 0,$$

where we applied the compatibility condition to conclude the last equality. Next we assume without loss of generality that $a = 0, b \neq 0$; then since $\delta^2 u = 0$ we have

$$\frac{\partial}{\partial t}(\delta u)_{0b} = \delta^2 u + \sum_{i \neq 0} \frac{\partial}{\partial x_i}(\delta u)_{ib} = \sum_{i \neq 0} \frac{\partial}{\partial x_i}(\delta u)_{ib}.$$

This vanishes because it is a sum of spatial derivatives of components of δu which vanish at $t = 0$. $\qquad\square$

Corollary 2.5. *Let u solve the equation*

$$\Box_{\mathbb{R}^{3+1} \times K} u = - * (du \wedge du)$$

with $\delta u|_{t=0} = 0$ and compatible $du|_{t=0}$. Then u solves

$$\delta du = *(du \wedge du).$$

3. Review of Hodge theory

The objects of our study are three-forms on $\mathbb{R}^{3+1} \times K$. The basic example of such a form would be $u_{\parallel} \wedge u_{\perp}$ where $u_{\perp}$ is a k-form (for $k = 0, 1, 2, 3$) on K and $u_{\parallel}$ is a $(3 - k)$-form on $\mathbb{R}^{3+1}$. The action of the Hodge-Laplacian of K is clearly $\Delta_K(u_{\parallel} \wedge u_{\perp}) = u_{\parallel} \wedge (\Delta_K u_{\perp})$ and it extends through density on all the forms on $\mathbb{R}^{3+1} \times K$. Moreover, if we use the eigenvectors of Δ_K, $\Delta_K e_{\lambda} = -\lambda^2 e_{\lambda}$, we can further decompose any form on $\mathbb{R}^{3+1} \times K$ as a series $u(x, y) = \sum_{\lambda} u_{\lambda}(x) e_{\lambda}(y)$, where x is a variable on $\mathbb{R}^{3+1}$ and y is the variable on K. Thus, we envision the equation being the system of equations on differential forms on $\mathbb{R}^{3+1}$ which are indexed by λ, in which case the equation will become

$$\Box u_{\lambda} - \lambda^2 u_{\lambda} = \sum_{\lambda', \lambda''} \mathcal{B}_{\lambda}^{\lambda', \lambda''}(u_{\lambda'}, u_{\lambda''}),$$

where the $\mathcal{B}$'s are bilinear differential operators. Thus we see that u_{λ} for $\lambda = 0$ evolve under a nonlinear wave equation, while u_{λ} for $\lambda \neq 0$ evolve under a nonlinear Klein-Gordon equation. This analysis follows the ideas of Metcalfe, Sogge and Stewart [11] and Metcalfe and Stewart [12], who analyze the wave equation on $\mathbb{R}^{n+1} \times D$, where D is a bounded domain in $\mathbb{R}^m$ with various boundary conditions. Their analysis splits the function to eigenfunctions of the Laplacian on D with appropriate boundary conditions.

In this section, we recall some properties of the eigenvectors of Δ_K which we need for the proof. The material is taken from textbooks, [5, section 2.1] and [16, Chapter 5, section 8]. For the rest of this section we will deal only with forms on K. We will continue to employ the subscript $\perp$ to maintain consistency. We begin with the following facts.

Proposition 3.1.   (1) *The operator Δ_K is a differential operator acting on the space of forms $\bigoplus_{i=0}^{7} \Omega^i(K)$ with the principal symbol $-g_{ij}\xi^i\xi^j Id$, where g is the Riemannian metric.*
  (2) *The operator Δ_K has a self-adjoint nonpositive-definite extension to the space of L^2-valued forms on K.*

Denote $\mathcal{P}_0 = \chi_{\{0\}}(-\Delta_K)$ and $\mathcal{P}_{>0} = \chi_{\{\lambda : \lambda > 0\}}(-\Delta_K)$. These are spectral projections on the zero- and nonzero-subspace of the spectrum of $-\Delta_K$, respectively.

3.1. **Hodge theory.** The range of $\mathcal{P}_0$, i.e all the forms ω that satisfy $\Delta_K \omega = 0$, are called the harmonic forms. We have the following simple fact.

Claim 3.2.

$$d_{\perp} \mathcal{P}_0 = 0.$$

Proof. Let $\delta_K = d_\perp^*$ be the $L^2(K)$ adjoint of $d_\perp$. We have the following characterization of Δ_K (see [5, Definition 2.1.2][3]):

$$-\Delta_K = \delta_K d_\perp + d_\perp \delta_K = d_\perp^* d_\perp + d_\perp d_\perp^*.$$

Therefore for $\omega = \mathcal{P}_0 \omega$, we have $\Delta_K \omega = 0$. Thus

$$0 = -\langle \Delta_K \omega, \omega \rangle_{L^2(K)} = \langle d_\perp^* d_\perp \omega + d_\perp d_\perp^* \omega, \omega \rangle_{L^2(K)}$$
$$= \|d_\perp \omega\|_{L^2(K)}^2 + \|d_\perp^* \omega\|_{L^2(K)}^2,$$

where $L^2(K)$ is the space of L^2-valued differential forms on K.   □

The full version of this claim can be found in [5, Proposition 2.1.5]. It is the basis of Hodge theory in algebraic topology. We will not require any of it in this paper but we will quote the following theorem for the sake of beauty.

Theorem 3.3. *Every nonempty de Rham cohomology class of K contains precisely one harmonic form.*

See [5, Theorem 2.2.1] for the proof. Thus existence and properties of harmonic forms are connected to the topological properties of the manifold. For instance the sphere $\mathbb{S}^7$ will have only two harmonic forms - the constant 0-form and the volume 7-form. The torus $\mathbb{T}^7$ will have $\binom{7}{n}$ linearly independent harmonic n-forms. Observe that both of these statements are independent of the choice of the Riemannian metric.

3.2. **Elliptic regularity for Δ_K.** We recall some basic regularity results for the form Laplacian. We have the following estimate.

Claim 3.4. *Let ω be a form on K. Then*

$$\|\omega\|_{H^2(K)} \leq C(\|\Delta_K \omega\|_{L^2(K)} + \|\omega\|_{L^2(K)}).$$

Proof. Combine the rudimentary elliptic estimate [16, Chapter 5, Theorem 1.3]

$$\|\omega\|_{H^2(K)} \leq C(\|\Delta_K \omega\|_{L^2(K)} + \|\omega\|_{H^1(K)}),$$

with some basic interpolation theory [16, Chapter 4, Proposition 3.1]

$$\|\omega\|_{H^1(K)} \leq C(\|\omega\|_{H^2(K)} \|\omega\|_{L^2(K)})^{1/2} \leq \varepsilon \|\omega\|_{H^2(K)} + C_\varepsilon \|\omega\|_{L^2(K)}.$$

□

Corollary 3.5.   (1) $\mathcal{P}_0 L^2(K)$ *is finite dimensional.*
    (2) *For every N there exists a constant C_N such that for every $\omega \in L^2(K)$*

$$\frac{1}{C_N} \|\mathcal{P}_0 \omega\|_{H^k(K)} \leq \|\mathcal{P}_0 \omega\|_{L^\infty(K)} \leq C_N \|\mathcal{P}_0 \omega\|_{H^k(K)}, k \leq N, x \in \mathbb{R}^3.$$

Corollary 3.6. *For every N, there are constants A_N such that for every $\omega \in H^n(K)$*

$$\frac{1}{A_N} \|\mathcal{P}_{>0} \omega\|_{H^n(K)} \leq \|(-\Delta_K)^{\frac{n}{2}} \omega\|_{L^2(K)} \leq A_N \|\mathcal{P}_{>0} \omega\|_{H^n(K)}, \quad \forall n \leq N.$$

See the discussion leading to [16, Equation (8.20)] for the proofs. The practical conclusion that we will draw from these two corollaries is that when measuring smoothness of the solution in K variables, we can ignore the question completely for $u = \mathcal{P}_0 u$ and use the $(-\Delta_K)^{1/2}$ operator for $u = \mathcal{P}_{>0} u$.

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[3]Our definition is the negative of [5].

4. Linear estimates

In this subsection, we would like to obtain decay estimates for the linear inhomogeneous equation. We will leverage this decay by employing the following subset of Klainerman vector fields:

$$(4.1) \qquad \tilde{\Gamma} = \{\partial_i, i = 0, \dots, 3\} \cap \{\Omega_{ij}, i, j = 0, \dots, 3\},$$

where

$$\Omega_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad 1 \leq i, j \leq 3,$$

$$\Omega_{0j} = x_0 \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_0}, \quad 1 \leq i \leq 3.$$

We augment $\tilde{\Gamma}$ with the operator $(-\Delta_K)^{1/2}$

$$\Gamma = \tilde{\Gamma} \cup \{(-\Delta_K)^{1/2}\} = \{\partial_i, i = 0, \dots, 3\} \cup \{\Omega_{ij}, i, j = 0, \dots, 3\} \cup \{(-\Delta_K)^{1/2}\}.$$

We will index the set Γ by $i = 1, \dots, 11$ and for a multi-index $I = (I_1, I_2, \dots, I_{|I|}) \in \{1, \dots, 11\}^{|I|}$ we define the composition

$$\Gamma^I = \Gamma_{I_1} \Gamma_{I_2} \dots \Gamma_{I_{|I|}}.$$

We will introduce some notation to simplify the presentation. We will denote for an integer N, an abstract vector-valued function:

$$\mathbf{\Gamma}^{(N)} f = (\Gamma^\alpha f)_{|\alpha| \leq N}.$$

Accordingly we will interpret the following notation:

$$|\mathbf{\Gamma}^{(N)} f| = \sum_{|\alpha| \leq N} |\Gamma^\alpha f|$$

and

$$\|\mathbf{\Gamma}^{(N)} f\|_p = \sum_{|\alpha| \leq N} \|\Gamma^\alpha f(t, x, y)\|_{L^p(\mathbb{R}^3 \times K)}.$$

We will also have a similar notation for the gradients

$$|\nabla \mathbf{\Gamma}^{(N)} f| = \sum_{|\alpha| \leq N} \sum_{i=0}^3 |\frac{\partial}{\partial x_i} \Gamma^\alpha f| + \sum_{|\alpha| \leq N} |(-\Delta_K)^{\frac{1}{2}} \Gamma^\alpha f|$$

and

$$\|\nabla \mathbf{\Gamma}^{(N)} f\|_p = \sum_{|\alpha| \leq N} \sum_{i=0}^3 \|\frac{\partial}{\partial x_i} \Gamma^\alpha f\|_{L^p(\mathbb{R}^3 \times K)}$$
$$+ \sum_{|\alpha| \leq N} \|(-\Delta_K)^{\frac{1}{2}} \Gamma^\alpha f\|_{L^p(\mathbb{R}^3 \times K)}.$$

All these norms will be taken at a certain time t, which we will drop from the notation when there is no ambiguity. We will fix coordinate patches on K, with the appropriate partition of unity. That will turn our objects into vector-valued functions on $\mathbb{R}^{3+1} \times \mathbb{R}^7$, so that we will apply the vector fields $\tilde{\Gamma}$ simply by applying them on every component of u.

4.1. **Linear estimates in $\mathbb{R}^{3+1}$.** We recall the following estimates in $\mathbb{R}^{3+1}$ which we seek to generalize to the product case $\mathbb{R}^{3+1} \times K$.

Proposition 4.1. *Let $w \in C^\infty(\mathbb{R}^{3+1})$ such that $\Box_{\mathbb{R}^{3+1}}w(t,x) = 0$ for $|x| > t - B$. Then for $t \geq 2B$ we have*

$$(1+t)|\nabla_{t,x}w(t,x)| \lesssim \|\nabla_{t,x}\mathbf{\Gamma}^{(2)}w(2B,\cdot)\|_{L^2(\mathbb{R}^3)}$$
$$+ \sum_k \sup_{\tau \in [2^{k-1},2^{k+1}]\cap[2B,t]} 2^k\|\mathbf{\Gamma}^{(2)}\Box w(\tau,\cdot)\|_{L^2(\mathbb{R}^3)}.$$

This proposition is proven in [12, Proposition 3.1]. Although [12] proves it with zero Cauchy data, the estimate with nonzero Cauchy data is proven in the same manner.

Proposition 4.2. *Let $w \in C^\infty(\mathbb{R}^{3+1})$ such that $(\Box_{\mathbb{R}^{3+1}} + 1)w(t,x) = 0$ for $|x| \geq t - B$. Then*

$$(1+t)^{3/2}\sup_x |w(t,x)| \lesssim \|\mathbf{\Gamma}^{(6)}w(2B,x)\|_{L^2(\mathbb{R}^3)}$$
$$+ \sum_k \sup_{\substack{\tau \in \\ [2^{k-1},2^{k+1}]\cap[2B,t]}} 2^k\|\mathbf{\Gamma}^{(5)}F(\tau,\cdot)\|_{L^2(\mathbb{R}^3)},$$

where $F = (\Box_{\mathbb{R}^{3+1}} + 1)w$.

This proposition is proved in [4, Proposition 7.3.6], refining the previous work of Klainerman [8].

4.2. **Linear estimates in $\mathbb{R}^{3+1} \times K$.** We turn to obtaining estimates for the equation

$$\Box u + \Delta_K u = F, \quad u(0) = u_0, \frac{\partial}{\partial t}u(0) = u_1.$$

Since the spectral projections $\mathcal{P}_A$ commute with this equation, we will split the equation into two equations

$$\Box \mathcal{P}_0 u = \mathcal{P}_0 F$$

and

$$\Box \mathcal{P}_{>0} u + \Delta_K \mathcal{P}_{>0} u = \mathcal{P}_{>0} F,$$

with the spectral projections applied to initial data as well. By elliptic regularity and the estimate for the wave equation (Proposition 4.1) we have the following estimate.

Proposition 4.3. *Let $\operatorname{supp}F(\cdot,\cdot,y) \subseteq \{(t,x) : |t - |x|| \leq 1\}$ for every $y \in K$. Then the solution of*

$$\Box \mathcal{P}_0 u = \mathcal{P}_0 F$$

obeys the estimate

$$(1+t)|\nabla_{t,x}(\mathcal{P}_0 u)(t,x,y)| \lesssim \|\nabla_{t,x}\mathbf{\Gamma}^{(2)}(\mathcal{P}_0 u)(0,\cdot,\cdot)\|_2$$
$$+ \sum_k \sup_{s \in [2^{k-1},2^{k+1}]\cap[0,t]} 2^k\|\mathbf{\Gamma}^{(2)}F(s,\cdot,\cdot)\|_2.$$

Proof. We wish to apply Proposition 4.1 with $B = 1$. For that we need to switch to a new coordinate $\tau = t + 2$ and then the proposition applies with one reservation: the vector fields in τ, x, y coordinates are different from the vector fields in t, x, y

coordinates but they are expressible in terms of sums of the old ones since $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t}$ and

$$\Omega_{0i}(\tau) = \tau \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial \tau} = (t+2) \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y} = \Omega_{0i}(t) + 2 \frac{\partial}{\partial t}.$$

Thus, Proposition 4.1 applies with possibly a different constant and (t, x, y) vector fields to show that for every $y \in K$

$$(1+t)|\nabla_{t,x}(\mathcal{P}_0 u)(t, x, y)| \lesssim \|\nabla_{t,x}\mathbf{\Gamma}^{(2)}(\mathcal{P}_0 u)(0, x, y)\|_{L^2(\mathbb{R}^3)}$$
$$+ \sum_k \sup_{s \in I_k} 2^k \|\mathbf{\Gamma}^{(2)}(\mathcal{P}_0 F)(s, \cdot, y)\|_{L^2(\mathbb{R}^3)},$$

where $I_k = [2^{k-1}, 2^{k+1}] \cap [0, t]$. Apply elliptic regularity (Corollary 3.5) to dominate $(\mathcal{P}_0 F)(s, \cdot, y)$ by its $L^2(K)$ norm. □

Theorem 4.4. *Let $u(t, x, y)$ solve*

$$(\Box_{\mathbb{R}^{3+1}} + \Delta_K)\mathcal{P}_{>0} u = \mathcal{P}_{>0} F.$$

Then

$$(1+t)^{3/2}|\mathcal{P}_{>0} u(t, x, y)| \lesssim \|\nabla \mathbf{\Gamma}^{(9)}\mathcal{P}_{>0} u(0, \cdot, \cdot)\|_2$$
$$+ \sum_k \sup_{s \in [2^{k-1}, 2^{k+1}] \cap [0, t]} 2^k \|\mathbf{\Gamma}^{(9)}\mathcal{P}_{>0} F(s, \cdot, \cdot)\|_2,$$

provided $\mathcal{P}_{>0} F(\cdot, \cdot, y)$ is supported in $\{(x, t) | |x| \leq 1 + |t|\}$ for every $y \in K$.

The proof of the theorem follows almost verbatim the proof of [4, Proposition 7.3.5] with the exception of the following modification of [4, Lemma 7.3.4].

Lemma 4.5. *Let K be a compact manifold and let $u \in \Omega(K)$ solve the equation*

$$-\frac{\partial^2}{\partial t^2}\mathcal{P}_{>0} u + \Delta_K \mathcal{P}_{>0} u = \mathcal{P}_{>0} F, \quad u(0) = u_0, \frac{\partial u}{\partial t}(0) = u_1.$$

Then

$$\|\mathcal{P}_{>0} u\|_{L^\infty(K)} \leq \|\Delta_K^2 \mathcal{P}_{>0} u_0\|_{L^2(K)} + \|(-\Delta_K)^{3/2} \frac{\partial}{\partial t}\mathcal{P}_{>0} u_0\|_{L^2(K)}$$
$$+ \int_0^t \|(-\Delta_K)^{3/2}\mathcal{P}_{>0} F(s, \cdot)\|_{L^2(K)} ds.$$

Proof of Lemma 4.5. We combine the energy estimate for the equation

$$(-\frac{\partial^2}{\partial t^2} + \Delta_K)(-\Delta_K)^{3/2} u = (-\Delta_K)^{3/2} F,$$

which is

$$\|\Delta_K^2 u(t)\|_{L^2(K)} \lesssim \|\Delta_K^2 u(0)\|_{L^2(K)} + \|\frac{\partial}{\partial t}(-\Delta_K)^{3/2} u(0)\|_{L^2(K)}$$
$$+ \int_0^t \|(-\Delta_K)^{3/2} F(s)\|_{L^2(K)} ds,$$

with the Sobolev embedding for a 7-dimensional manifold:

$$\|\mathcal{P}_{>0} u(t)\|_{L^\infty(K)} \lesssim \|\Delta_K^2 u(t)\|_{L^2(K)}.$$

□

We relegate the rest of the proof of Theorem 4.4 to the appendix. We combine the two decay estimates above with the possibility to apply Γ^α which are symmetries of the equation in the following statement.

Corollary 4.6. *Let M be an integer greater then 9. Let u solve*

$$(\Box_{\mathbb{R}^{3+1}} + \Delta_K)u = F,$$

with initial data supported in the ball of radius 1 for every $y \in K$ and

$$\mathrm{supp}F(\cdot, \cdot, y) \subseteq \{(t,x) : |x| \leq 1 + |t|\}$$

for every $y \in K$. Then the following estimate holds:

$$
\begin{aligned}
(4.2) \quad &(1+t)\|\nabla \mathbf{\Gamma}^{(M-10)} \mathcal{P}_0 u(t)\|_\infty \\
&+ (1+t)^{3/2}\|\nabla \mathbf{\Gamma}^{(M-10)} \mathcal{P}_{>0} u(t)\|_\infty \lesssim \|\nabla \mathbf{\Gamma}^{(M)} u(0)\|_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \sum_k \sup_{s \in I_k} 2^k \|\mathbf{\Gamma}^{(M)} F(s, \cdot)\|_2,
\end{aligned}
$$

where $I_k = [2^{k-1}, 2^{k+1}] \cap [0, t]$.

4.3. **Energy estimates.** We combine the energy estimates for the solution of $\Box_{\mathbb{R}^{3+1} \times K} u = F$ with the fact that the operators Γ are symmetries of the equation and use the notation introduced above.

Proposition 4.7. *Let u be the solution of $\Box_{\mathbb{R}^{3+1} \times K} u = F$. Then for any $M \geq 0$ we have*

$$\|\nabla \mathbf{\Gamma}^{(M)} u(t)\|_2 \leq \|\nabla \mathbf{\Gamma}^{(M)} u(0)\|_2 + \int_0^t \|\mathbf{\Gamma}^{(M)} F(s)\|_{L^2} ds.$$

5. ANALYSIS OF NONLINEARITY

In this section, we will treat the bilinear form $(u_1, u_2) \mapsto *(du_1 \wedge du_2)$. Recall from Subsection 2.3 the $*$ operator exchanges the components of the forms, multiplying those containing the time x^0 coordinate by -1. The $*$ operator loses the simple form when the metric on K is no longer the identity, but because of tensoriality, it will be multiplied by a function depending only on $x^4, \ldots, x^{11}$, which due to compactness will be bounded above and below. Therefore, when we take the $L^2(\mathbb{R}^3 \times K)$-norm at a certain time, we will consider $*v$ to be L^2 equivalent to v. Furthermore, we will be interested in the action of Γ operators on $*(du \wedge du)$. Clearly, the operators which act on $\mathbb{R}^{3+1}$ componentwise will commute with $*$. Equation (2.5) shows that the Laplacian Δ_K commutes with $*$ simply because $\Delta_K = \sum_{i=4}^{10} \frac{\partial^2}{\partial x_i^2}$ at that point and the relation is tensorial. Thus any function of Δ_K will commute with $*$ and we have

$$(-\Delta_K)^{1/2} * v = *(-\Delta_K)^{1/2} v.$$

From this discussion we conclude that

$$\|\Gamma^\alpha(*v(t))\|_{L^2(\mathbb{R}^3 \times K)} \cong \| * \Gamma^\alpha v\|_{L^2(\mathbb{R}^3 \times K)} \cong \|\Gamma^\alpha v\|_{L^2(\mathbb{R}^3 \times K)},$$

for any multi-index α, time t, with constants which depend only on the manifold K.

5.1. **The splitting of the nonlinearity.** Recall that the operator d splits into $d = d_\parallel + d_\perp$. Also any form u on $\mathbb{R}^{3+1} \times K$ can be written as $u = \mathcal{P}_0 u + \mathcal{P}_{>0} u$. Therefore, we can write

$$
\begin{aligned}
*(du \wedge du) &= * \left[(d_\parallel + d_\perp)(\mathcal{P}_0 u + \mathcal{P}_{>0} u) \wedge (d_\parallel + d_\perp)(\mathcal{P}_0 u + \mathcal{P}_{>0} u) \right] \\
&= * (d_\parallel \mathcal{P}_0 u \wedge d_\parallel \mathcal{P}_0 u) \\
&\quad + 2 * (d_\parallel \mathcal{P}_0 u \wedge d\mathcal{P}_{>0} u) + *(d\mathcal{P}_{>0} u \wedge d\mathcal{P}_{>0} u),
\end{aligned}
$$

where we used that $d_\perp \mathcal{P}_0 = 0$, which is the content of Claim 3.2. With this we proved the following splitting of the nonlinearity:

Claim 5.1. *Let u_1, u_2 be differential three-forms on $\mathbb{R}^{3+1} \times K$. Denote*

$$
B(\mathcal{P}_0 u_1, \mathcal{P}_0 u_2) = *(d_\parallel \mathcal{P}_0 u_1 \wedge d_\parallel \mathcal{P}_0 u_2),
$$

$$
C(\mathcal{P}_0 u_1, \mathcal{P}_{>0} u_2) = *(d_\parallel \mathcal{P}_0 u_1 \wedge d\mathcal{P}_{>0} u_2),
$$

$$
D(\mathcal{P}_{>0} u_1, \mathcal{P}_{>0} u_2) = *(d\mathcal{P}_{>0} u \wedge d\mathcal{P}_{>0} u).
$$

Then

$$
*(du \wedge du) = B(\mathcal{P}_0 u, \mathcal{P}_0 u) + 2C(\mathcal{P}_0 u, \mathcal{P}_{>0} u) + D(\mathcal{P}_{>0} u, \mathcal{P}_{>0} u).
$$

5.2. **The basic estimate.**

Proposition 5.2. *Let F be any of the forms B, C, D defined in Claim 5.1 or the total nonlinearity which is $B + 2C + D$. Let N be a positive integer. Then there exists a constant k such that for any $M \leq N$ we have*

$$
\begin{aligned}
\|\mathbf{\Gamma}^{(M)} F(v_1, v_2)\|_2 &\lesssim \|\nabla \mathbf{\Gamma}^{(\frac{M}{2})} v_1\|_{p_1} \|\nabla \mathbf{\Gamma}^{(M)} v_2\|_{q_1} \\
&\quad + \|\nabla \mathbf{\Gamma}^{(M)} v_1\|_{q_2} \|\nabla \mathbf{\Gamma}^{(\frac{M}{2})} v_2\|_{p_2},
\end{aligned}
$$

where $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{2}$, $2 \leq p_i, q_i \leq \infty$.

Proof. Choose a coordinate patch x^i, $i = 4, \ldots, 10$, for K. Then de Rham differentials $d, d_\perp, d_\parallel$ can be written as $a_i \frac{\partial}{\partial x_i}$ for a_i which depend only on $x^i, i \geq 4$. Thus we need to estimate an expression of the form

$$
I = \sum_{i,j=0}^{10} \Gamma^\alpha \left(a_i a_j' \frac{\partial}{\partial x_i} v_1 \frac{\partial}{\partial x_j} v_2 \right).
$$

Observe that all the operators in Γ besides $(-\Delta_K)^{1/2}$ are vector fields and thus obey Leibniz's rule. So assume first that in the composite operator Γ^α there are no $(-\Delta_K)^{1/2}$ operators. We treat the Hodge dual $*$ as a constant coefficient operator, which only permutes between different components. Next, we employ the Jacobi identity to write I as

$$
\sum_{\alpha' + \alpha'' = \alpha} C_{\alpha' \alpha''} \sum_{i,j=0}^{10} a_i a_j' \Gamma^{\alpha'} \left(\frac{\partial}{\partial x_i} v_1 \right) \Gamma^{\alpha''} \left(\frac{\partial}{\partial x_j} v_2 \right),
$$

where $C_{\alpha' \alpha''}$ are constants. Observe that Γ_i commutes with $\frac{\partial}{\partial x_i}$ for $i \geq 4$ and for $i, j \leq 4$ we have

$$
\left[\frac{\partial}{\partial x_k}, \Omega_{0j} \right] = \delta_{0k} \frac{\partial}{\partial x_j} + \delta_{jk} \frac{\partial}{\partial x_0}, \quad \left[\frac{\partial}{\partial x_k}, \Omega_{ij} \right] = \delta_{jk} \frac{\partial}{\partial x_k} - \delta_{ik} \frac{\partial}{\partial x_j}.
$$

Thus we commute $\frac{\partial}{\partial x_i}$ with Γ's to get

$$I = \sum_{|\beta'|+|\beta''|\leq M} \sum_{i,j=0}^{10} C_{\beta'\beta''ij}(\frac{\partial}{\partial x_i}\Gamma^{\beta'}v_1)(\frac{\partial}{\partial x_j}\Gamma^{\beta''}v_2),$$

for some constants $C_{\beta'\beta''ij}$. In the expression above only one of the $|\beta'|, |\beta''|$ can be larger then $M/2$. We split the sum accordingly:

$$|I| \lesssim \sum_{|\beta'|\leq \frac{M}{2},i} |\frac{\partial}{\partial x_i}\Gamma^{\beta'}v_1| \sum_{|\beta''|\leq M,j} |\frac{\partial}{\partial x_j}\Gamma^{\beta''}v_2|$$

$$+ \sum_{|\beta'|\leq M,i} |\frac{\partial}{\partial x_i}\Gamma^{\alpha'}v_1| \sum_{|\beta''|\leq \frac{M}{2},j} |\frac{\partial}{\partial x_j}\Gamma^{\beta''}v_2|.$$

We then obtain the required estimate by applying the L^2 norm to $|I|$ and using the appropriate Hölder inequalities.

In case Γ^α contains some m appearances of the operator $(-\Delta_K)^{1/2}$, we note that $(-\Delta_K)^{1/2}$ commutes with all the other Γ operators as the rest of Γ operate on $\mathbb{R}^{3+1}$ only. By elliptic regularity

$$\|I\|_2 = \|(-\Delta_K)^{\frac{m}{2}}\Gamma^{\alpha'}F\|_2 \lesssim \|\mathcal{P}_{>0}\Gamma^{\alpha'}F\|_{H^m(K)} \leq \|\Gamma^{\alpha'}F\|_{H^m(K)}.$$

Now the $H^m(K)$ norm obeys a Jacobi's "inequality", which is a primitive form of the Kato-Ponce estimates (see [7], [15, Chapter II, Prop. 1.1]) and thus the rest of the proof proceeds in a similar fashion. $\qquad\square$

5.3. **Null form.** Continuing with the notation of Claim 5.1, we need the observation that $B(\mathcal{P}_0 u_1, \mathcal{P}_0 u_2) = *(d_\| \mathcal{P}_0 u_1 \wedge d_\| \mathcal{P}_0 u_2)$ is a null form and the estimates that follow from it.

Proposition 5.3. *The bilinear form $B(\omega_1, \omega_2) = *(d_\|\omega_1 \wedge d_\|\omega_2)$ is a null-form.*

We will give two proofs of the proposition.

Fourier. It is enough to compute $B(\omega_1, \omega_2)$ for $\omega_i = A_i e^{ik_i x}$ for two parallel null-vectors, with A_i constant. If $B(\omega_1, \omega_2)$ vanishes in such a case, then B is a null-form. But

$$B(\omega_1, \omega_2) = *(k_1 \wedge A_1 \wedge k_2 \wedge A_2)e^{i(k_1+k_2)x}.$$

When two vectors in the wedge product are parallel, the wedge product vanishes. Thus

$$B(\omega_1, \omega_2) = 0.$$

$\qquad\square$

Coefficients. Let

$$A = A_{ijk}dx_i dx_j dx_k$$

and

$$B = B_{lmn}dx_l dx_m dx_n.$$

Then

$$d_\| A = \frac{\partial A_{ijk}}{\partial x_p} dx_p dx_i dx_j dx_k,$$

$$d_\| B = \frac{\partial B_{lmn}}{\partial x_s} dx_s dx_l dx_m dx_n.$$

Therefore

$$d_\| A \wedge d_\| B = \frac{\partial A_{ijk}}{\partial x_p}\frac{\partial B_{lmn}}{\partial x_s} dx_p dx_i dx_j dx_k dx_s dx_l dx_m dx_n$$

$$+ \frac{\partial A_{ijk}}{\partial x_s}\frac{\partial B_{lmn}}{\partial x_p} dx_s dx_i dx_j dx_k dx_p dx_l dx_m dx_n$$

$$= \left(\frac{\partial A_{ijk}}{\partial x_p}\frac{\partial B_{lmn}}{\partial x_s} - \frac{\partial A_{ijk}}{\partial x_s}\frac{\partial B_{lmn}}{\partial x_p}\right) dx_p dx_i dx_j dx_k dx_s dx_l dx_m dx_n.$$

Thus

$$*d_\| A \wedge d_\| B = \left(\frac{\partial A_{ijk}}{\partial x_p}\frac{\partial B_{lmn}}{\partial x_s} - \frac{\partial A_{ijk}}{\partial x_s}\frac{\partial B_{lmn}}{\partial x_p}\right)$$
$$* (dx_p dx_i dx_j dx_k dx_s dx_l dx_m dx_n).$$

Since

$$\frac{\partial A_{ijk}}{\partial x_p}\frac{\partial B_{lmn}}{\partial x_s} - \frac{\partial A_{ijk}}{\partial x_s}\frac{\partial B_{lmn}}{\partial x_p}$$

is a null-form, $*d_\| A \wedge d_\| B$ is a null-form. Observe that we needed to assume that only dx_p and dx_s are co-vectors on $\mathbb{R}^{3+1}$; all the other indices could have belonged to either K or $\mathbb{R}^{3+1}$. In order to see that the sign in front of the second term is negative, we count the transpositions needed to transform

$$\omega_1 = dx_s dx_i dx_j dx_k dx_p dx_l dx_m dx_n$$

to

$$\omega_2 = dx_p dx_i dx_j dx_k dx_s dx_l dx_m dx_n.$$

One needs four transpositions to bring dx_p to the front and then another three to bring dx_s behind $dx_i dx_j dx_k$; therefore there are seven transpositions in total and the sign is minus, $\omega_1 = -\omega_2$. $\qquad\square$

Denote

$$Q_{ij}(f,g) = \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}.$$

As the proof of Proposition 5.3 shows, the form $B(\omega_1,\omega_2) = *(d_\| \omega_1 \wedge d_\| \omega_2)$ is the sum of the forms Q_{ij} applied to different components. We have the following estimate.

Lemma 5.4.
$$|Q_{ij}(f,g)| \lesssim \frac{1}{1+t}\left(|\Gamma f||\nabla_{x,t}g| + |\nabla_{x,t}f||\Gamma g|\right).$$

Remark. The proof is taken from [9, Lemma 1.1]. We reproduce it here to stress that we have the estimate involving only vector fields Γ and not the full set of Klainerman vector fields, which includes the radial scaling field $x_0\frac{\partial}{\partial x_0} + \ldots + x_3\frac{\partial}{\partial x_3}$.

Proof. We have
$$\frac{\partial}{\partial x_i} = -\frac{x_i}{t}\frac{\partial}{\partial x_0} + \frac{1}{t}\Omega_{0i}, \quad i = 1,2,3.$$
Thus we have for $i,j \geq 1$
$$Q_{ij}(f,g) = \frac{1}{t}\left[-\frac{\partial f}{\partial x_0}\Omega_{ij}g + \left(\Omega_{0i}f\frac{\partial g}{\partial x_j} - \Omega_{0j}f\frac{\partial g}{\partial x_j}\right)\right].$$

For $i = 0$, we have

$$Q_{0j} = \frac{1}{t}\left(\frac{\partial f}{\partial x_0}\Omega_{0j}g - \Omega_{0j}f\frac{\partial g}{\partial x_0}\right).$$

$\square$

We wish to prove a variant of the basic estimate specialized to the null-form.

Proposition 5.5.

$$\|\mathbf{\Gamma}^{(M)}B(v,v)\|_2 \lesssim \frac{1}{1+t}\Big(\|\mathbf{\Gamma}^{(\frac{M}{2}+1)}v\|_{p_1}\|\nabla\mathbf{\Gamma}^{(M)}v\|_{q_1}$$
$$+ \|\mathbf{\Gamma}^{(M+1)}v\|_{q_2}\|\nabla\mathbf{\Gamma}^{(\frac{M}{2})}v\|_{p_2}\Big).$$

Proof. The bilinear form B is a linear combination of the forms Q_{ij}. The forms Q_{ij} are preserved under applications of the Γ operators (with the exception of $(-\Delta_K)^{1/2}$) because of the following formula which appears in [9, Lemma 1.2] and which can be obtained by direct calculation:

$$(5.1) \qquad \Omega_{\alpha\beta}Q_{\gamma\delta}(f,g) = Q_{\gamma\delta}(\Omega_{\alpha\beta}f,g) + Q_{\gamma\delta}(f,\Omega_{\alpha\beta}g) + \tilde{Q}(f,g).$$

Here

$$\tilde{Q}(f,g) = m_{\alpha\gamma}Q_{\beta\delta}(f,g) - m_{\beta\gamma}Q_{\alpha\delta}(f,g) + m_{\alpha\delta}Q_{\beta\gamma}(f,g) + m_{\beta\delta}Q_{\alpha\gamma}(f,g),$$

where $m_{\alpha\beta}$ are coefficients of the Minkowski metric. Thus we apply Lemma 5.4 and after grouping the multi-indices with order less than $\frac{M}{2}$ and applying the Hölder estimates we get the result as in Proposition 5.2. For $(-\Delta_K)^{1/2}$ we apply Kato-Ponce estimates. $\square$

6. PROOF OF THEOREM 1.1

To prove the theorem we seek to establish the following apriori bounds for the solutions of (1.1):

$$(6.1a) \qquad\qquad (1+t)^{-\delta}\|\nabla\mathbf{\Gamma}^{(N)}u(t)\|_2 \leq \epsilon,$$

$$(6.1b) \qquad\qquad \|\nabla\mathbf{\Gamma}^{(N-10)}u(t)\|_2 \leq \epsilon,$$

$$(6.1c) \qquad (1+t)\|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_0 u(t)\|_\infty + (1+t)^{3/2}\|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_{>0} u(t)\|_\infty \leq \epsilon,$$

where δ is smaller than $\frac{1}{12}$. Any of the global in time estimates above implies uniqueness, existence and well-posedness for the semilinear wave equations by employing the local theory which is explained in [14, Chapter 2] or [4, section 6.2].

We will prove the estimates by bootstrapping. Namely, we will prove that (6.1) imply

$$(6.2a) \qquad\qquad (1+t)^{-\delta}\|\nabla\mathbf{\Gamma}^{(N)}u(t)\|_2 \leq \frac{\epsilon}{2},$$

$$(6.2b) \qquad\qquad \|\nabla\mathbf{\Gamma}^{(N-10)}u(t)\|_2 \leq \frac{\epsilon}{2},$$

$$(6.2c) \qquad (1+t)\|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_0 u(t)\|_\infty + (1+t)^{3/2}\|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_{>0} u(t)\|_\infty \leq \frac{\epsilon}{2},$$

i.e. the right-hand side can be made $\frac{\epsilon}{2}$ instead of ϵ. Thus our goal is to establish the following statement.

Proposition 6.1. *Let N be an integer that satisfies*

$$\frac{N}{2} \leq N - 20.$$

Then there exists $\epsilon > 0$ small enough such that, for every $T > 0$, if a solution u to the Cauchy problem (1.1) *with initial data that satisfies*

$$\|\mathbf{\Gamma}^{(N)}u_0\|_2 + \|\mathbf{\Gamma}^{(N-1)}u_1\| \leq \frac{\epsilon}{4},\,^4$$

such that $u_0(\cdot, y), u_1(\cdot, y)$ are localized in a ball of radius 1 for every $y \in K$, and if the inequalities (6.1a), (6.1b), (6.1c) *hold for every $0 < t \leq T$, then the inequalities* (6.2a), (6.2b), (6.2c) *hold for every $0 < t \leq T$, where δ is a positive exponent which depends on ϵ and is smaller than $\frac{1}{12}$.*

Remark. There are two considerations that affect the smallness of ϵ. One is that we will see that we can replace ϵ in the right-hand side of (6.1) by $\frac{\epsilon}{4} + k\epsilon^2$, where k is an apriori computable constant that depends only on N and the geometry of the manifold K. Thus we will need to decrease ϵ to achieve the inequality

$$\frac{\epsilon}{4} + k\epsilon^2 \leq \frac{\epsilon}{2}.$$

The second consideration is that the exponent δ depends linearly on ϵ, $\delta = C\epsilon$ with C depending on N and the geometry of K. We have to decrease ϵ so that $\delta = C\epsilon \leq \frac{1}{12}$.

We prove the implication (6.2a) in the next lemma, while the implications (6.2b) and (6.2c) are proved in Lemma 6.5.

Lemma 6.2. *Under conditions of Proposition* 6.1, (6.1c) *implies* (6.2a).

Proof. We use the energy estimate for the equation $\Box\mathbf{\Gamma}^{(N)}u = \mathbf{\Gamma}^{(N)}(*du \wedge du)$.

$$(6.3) \qquad \|\nabla\mathbf{\Gamma}^{(N)}u(t)\|_2 \leq \|\nabla\mathbf{\Gamma}^{(N)}u(0)\|_2 + \int_0^t \|\mathbf{\Gamma}^{(N)} * du \wedge du(s)\|_2 ds.$$

We now use Proposition 5.2 to estimate the nonlinearity. We have

$$\|\mathbf{\Gamma}^{(N)} * du \wedge du(s)\|_2 \leq C\|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}u\|_\infty\|\nabla\mathbf{\Gamma}^{(N)}u\|_2.$$

Since $\frac{N}{2} \leq N - 20$ we employ the assumption (6.1c) in equation (6.3) to conclude

$$\|\nabla\mathbf{\Gamma}^{(N)}u(t)\|_2 \leq \frac{\epsilon}{4} + \int_0^t \frac{C\epsilon}{1+s}\|\nabla\mathbf{\Gamma}^{(N)}u(s)\|_2 ds.$$

After applying Gronwall's inequality we conclude

$$\|\nabla\mathbf{\Gamma}^{(N)}u(t)\|_2 \leq \frac{\epsilon}{4}(1+t)^{C\epsilon},$$

which is the required inequality. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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[4]Interpret high time derivatives of the initial data by using the equation.

We require the following interpolated intermediate result.

Claim 6.3. *Let $2 \le p \le \infty$. Then:*

(1) *Assumptions* (6.1a) *and* (6.1c) *imply*

$$(6.4) \quad (1+t)^{-\delta+1-\frac{2}{p}}\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_0 u\|_p + (1+t)^{-\delta+3/2(1-\frac{2}{p})}\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0}u\|_p \lesssim \epsilon.$$

(2) *Assumptions* (6.1b) *and* (6.1c) *imply*

$$(6.5) \quad (1+t)^{1-\frac{2}{p}}\|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_0 u\|_p + (1+t)^{3/2(1-\frac{2}{p})}\|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_{>0}u\|_p \lesssim \epsilon.$$

Proof. Obviously, the estimate for the sum with more derivatives is true for the sum with less derivatives and thus we will interpolate between equation (6.1b) and equation (6.1c) to get the second conclusion. To prove the first point, we use the following intermediate result:

$$(6.6) \quad (1+t)^{-\delta+1}\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_0 u\|_\infty + (1+t)^{-\delta+3/2}\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0}u\|_\infty \lesssim \epsilon.$$

By interpolation of equation (6.6) with equation (6.1a) we have

$$(1+t)^{-\delta+1-\frac{2}{p}}\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_0 u\|_p + (1+t)^{-\delta+3/2(1-\frac{2}{p})}\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0}u\|_p \lesssim \epsilon.$$

To establish (6.6) we use equation (4.2):

$$(1+t)\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_0 u\|_\infty$$
$$+(1+t)^{3/2}\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0}u\|_\infty \le \|\nabla\mathbf{\Gamma}^{(N)}u(0)\|_2$$
$$+\sum_n \sup_{\tau\in I_n} 2^n\|\mathbf{\Gamma}^{(N)}F(\tau))\|_2,$$

where $I_n = [2^{n-1}, 2^{n+1}] \cap [0,t]$ and $F = *(du \wedge du)$. We use Proposition 5.2 and combine it with the assumptions to get

$$\|\mathbf{\Gamma}^{(N)}F(\tau)\|_2 = \|\mathbf{\Gamma}^{(N)} * du \wedge du(\tau)\|_2 \lesssim C\|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}u\|_\infty\|\nabla\mathbf{\Gamma}^{(N)}u\|_2$$
$$\lesssim \epsilon^2(1+\tau)^{\delta-1}.$$

Therefore,

$$(1+t)\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_0 u\|_\infty$$
$$+(1+t)^{3/2}\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0}u\|_\infty \lesssim \|\mathbf{\Gamma}^{(N)}u(0)\|_2$$
$$+\sum_n \sup_{\tau\in[2^{n-1},2^{n+1}]\cap[0,t]} \epsilon 2^n \tau^{-1+\delta}$$
$$\le \frac{\epsilon}{4} + k\epsilon^2(1+t)^\delta,$$

which proves (6.6). □

To complete the proof of Proposition 6.1, we analyze the equation

$$\Box_{\mathbb{R}^{3+1}\times K}\mathbf{\Gamma}^{(N-10)}u = \mathbf{\Gamma}^{(N-10)} * du \wedge du.$$

We estimate the right-hand side of the equation in the following lemma.

Lemma 6.4. *Under assumptions* (6.1a), (6.1b), (6.1c) *we have*

$$\|\mathbf{\Gamma}^{(N-10)} * du \wedge du(t)\|_2 \lesssim \epsilon^2(1+t)^{-1-\frac{1}{3}+\delta} \le \epsilon^2(1+t)^{-1-\frac{1}{4}}.$$

Proof. Recall the splitting of the nonlinearity in Claim 5.1 and denote

$$(6.7a) \qquad B(t) = \|\mathbf{\Gamma}^{(N-10)}B(\mathcal{P}_0 u(t), \mathcal{P}_0 u(t))\|_2,$$

$$(6.7b) \qquad C(t) = \|\mathbf{\Gamma}^{(N-10)}C(\mathcal{P}_0 u(t), \mathcal{P}_{>0} u(t))\|_2,$$

$$(6.7c) \qquad D(t) = \|\mathbf{\Gamma}^{(N-10)}D(\mathcal{P}_{>0} u(t), \mathcal{P}_{>0} u(t))\|_2.$$

To prove the lemma, we will obtain the bound above for $B(t)$, $C(t)$, $D(t)$.

Estimate for $B(t)$: To estimate $B(t)$ we use Proposition 5.5 to get

$$(6.8) \qquad \|\mathbf{\Gamma}^{(N-10)}B(\mathcal{P}_0 u, \mathcal{P}_0 u)\|_2 \leq \frac{k}{1+t}\big(\|\mathbf{\Gamma}^{(\frac{N}{2}-4)}\mathcal{P}_0 u\|_6 \|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_0 u\|_3$$
$$+ \|\mathbf{\Gamma}^{(N-9)}\mathcal{P}_0 u\|_6 \|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}\mathcal{P}_0 u\|_3 \big).$$

We wish to use the homogeneous Sobolev embedding in $\mathbb{R}^3$. Thus, we have for every $y \in K$,

$$\|\mathbf{\Gamma}^{(M)}\mathcal{P}_0 u(t,\cdot,y)\|_{L^6(\mathbb{R}^3)} \leq \|\nabla\mathbf{\Gamma}^{(M)}\mathcal{P}_0 u(t,\cdot,y)\|_{L^2(\mathbb{R}^3)}.$$

Next we employ the compactness of K to see that the $L^6(K)$ norm of $\mathcal{P}_0 \mathbf{\Gamma}^{(M)} u$ is dominated by the $L^\infty(K)$ norm. At the same time, elliptic regularity shows that the $L^\infty(K)$ norm of $\mathcal{P}_0 \nabla\mathbf{\Gamma}^{(M)} u$ is dominated by its $L^2(K)$ norm. Therefore

$$(6.9) \qquad \|\mathbf{\Gamma}^{(\frac{N}{2}-4)}\mathcal{P}_0 u\|_6 \leq \|\mathbf{\Gamma}^{(N-20)}\mathcal{P}_0 u\|_6 \leq \|\nabla\mathbf{\Gamma}^{(N-20)} u\|_2 \lesssim \epsilon$$

by (6.1b) and the fact that $\frac{N}{2} \leq N - 20$. Also

$$(6.10) \qquad \|\mathbf{\Gamma}^{(N-9)}\mathcal{P}_0 u\|_6 \leq \|\nabla\mathbf{\Gamma}^{(N-9)} u\|_2 \lesssim \epsilon(1+t)^\delta$$

by (6.1a). Employing (6.4) again we have

$$(6.11) \qquad \|\nabla\mathbf{\Gamma}^{(N-10)} u\|_6 \lesssim \epsilon(1+t)^{\delta-\frac{1}{3}}.$$

Also equation (6.5) implies

$$(6.12) \qquad \|\nabla\mathbf{\Gamma}^{(\frac{N}{2}-5)}\mathcal{P}_0 u\|_3 \leq \|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_0 u\|_3 \lesssim \epsilon(1+t)^{-\frac{1}{3}}.$$

Thus, applying (6.9), (6.10), (6.11), (6.12) on (6.8) we have

$$(6.13) \qquad B(t) = \|\mathbf{\Gamma}^{(N-10)}B(\mathcal{P}_0 u, \mathcal{P}_0 u)\|_2 \leq k\epsilon^2 (1+t)^{\delta-\frac{1}{3}-1}.$$

Estimate for $C(t)$: For $C(t)$ we have the following estimate. We use Proposition 5.2 to see that

$$\|\mathbf{\Gamma}^{(N-10)}C(\mathcal{P}_0 u, \mathcal{P}_{>0} u)\|_2 \lesssim \|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_0 u\|_2 \|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}\mathcal{P}_{>0} u\|_\infty$$
$$+ \|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}\mathcal{P}_0 u\|_3 \|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0} u\|_6.$$

By equation (6.5)

$$\|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}\mathcal{P}_0 u\|_3 \leq \epsilon(1+t)^{-\frac{1}{3}}$$

and by (6.4)

$$\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0} u\|_6 \leq \epsilon(1+t)^{\delta-1}.$$

For the first summand, we will use the bootstrap assumptions (6.1b) and (6.1c). Thus we get

$$(6.14) \qquad C(t) = \|\mathbf{\Gamma}^{(N-10)}C(\mathcal{P}_0 u, \mathcal{P}_{>0} u)\|_2 \leq k\epsilon^2\big((1+t)^{\delta-\frac{4}{3}} + (1+t)^{-\frac{3}{2}}\big).$$

Estimate for $D(t)$: Lastly, we estimate $D(t)$. We have

$$\|\mathbf{\Gamma}^{(N-10)}D(\mathcal{P}_{>0}u, \mathcal{P}_{>0}u)\|_2 \le k(\|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}\mathcal{P}_{>0}u\|_\infty \|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0}u\|_2),$$

which according to the bootstrap assumptions satisfies

(6.15) $$D(t) = \|\mathbf{\Gamma}^{(N-10)}D(\mathcal{P}_{>0}u, \mathcal{P}_{>0}u)\|_2 \le k\epsilon^2 \frac{1}{(1+t)^{\frac{3}{2}}}.$$

We combine (6.13), (6.14), (6.15) to obtain the required estimate. $\qquad\square$

We are now ready to complete the proof of Proposition 6.1 in the following lemma.

Lemma 6.5. *Under the conditions of Proposition 6.1, the assumptions* (6.1a), (6.1b), (6.1c) *imply* (6.2b), (6.2c).

Proof. We analyze the equation

(6.16) $$\Box_{\mathbb{R}^{3+1}\times K}\mathbf{\Gamma}^{(N-10)}u = \mathbf{\Gamma}^{(N-10)}(*du \wedge du).$$

First apply the energy estimate to get

$$\|\nabla\mathbf{\Gamma}^{(N-10)}u(t)\|_2 \le \|\nabla\mathbf{\Gamma}^{(N-10)}u(0)\|_2 + \int_0^t \|\mathbf{\Gamma}^{(N-10)}(*du \wedge du)(s)\|_2 ds$$

$$\le \frac{\epsilon}{4} + k\epsilon^2 \int_0^t \frac{1}{(1+s)^{1+1/4}}ds \le \frac{\epsilon}{4} + k\epsilon^2 \le \frac{\epsilon}{2},$$

where we used Lemma 6.4 and the assumptions. This establishes (6.2b). To address (6.2c), apply Corollary 4.6 to equation (6.16) to get

$$(1+t)\|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_0 u(t)\|_\infty$$
$$+ (1+t)^{3/2}\|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_{>0}u(t)\|_\infty$$
$$\le \|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_0 u(0)\|_2$$
$$+ \sum_n \sup_{s \in I_n} 2^n \|\mathbf{\Gamma}^{(N-10)}(F(s))\|_2,$$

where $I_n = [2^{n-1}, 2^{n+1}] \cap [0, t]$ and $F = *du \wedge du$. Since

$$\|\mathbf{\Gamma}^{(N-10)}F(s)\|_2 \le \epsilon^2(1+s)^{-1-\frac{1}{4}}$$

by Lemma 6.4, we have

$$(1+t)\|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_0 u(t)\|_\infty$$
$$+ (1+t)^{3/2}\|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_{>0}u(t)\|_\infty \le \frac{\epsilon}{4} + k \sum_{n \le C\log t + 1} 2^n \epsilon^2 2^{-n(1+\frac{1}{4})}$$
$$\le \frac{\epsilon}{4} + k\epsilon^2,$$

which establishes (6.2c) and completes the proof. $\qquad\square$

Appendix A. Proof of Theorem 4.4

We will denote $\tau = t + 2$. We will again employ operators Γ in the τ-variable, which are different from operators in the t-variable but can be expressed as a linear combination with coefficients independent of u. See the proof of Proposition 4.3 for details regarding this substitution. We will use the following statement.

Proposition A.1. *Let g be supported in $\{(\tau, x); T/2 \le \tau \le T, |x| \le \tau\}$ and denote*

$$M(\rho) = \sup_{\tau^2 - |x|^2 = \rho^2} |g(\tau, x)|.$$

Then

$$T^2 \int M(\rho^2)^2 \rho d\rho \le C \sum_{|I| \le \frac{5}{2}} \int |\tilde{\Gamma}^I g(\tau, x)|^2 d\tau dx.$$

The proof of this statement is given in [4, Lemma 7.3.1]. Let u solve

$$\Box u + \Delta_K u = -\frac{\partial^2 u}{\partial \tau^2} + \sum_{i=1}^3 \frac{\partial^2 u}{\partial x_i^2} + \Delta_K u = F.$$

Introduce the hyperbolic polar coordinates $(\tau, x) = \rho \omega, \rho = (\tau^2 - |x|^2)^{\frac{1}{2}}, \omega \in \mathbb{S}^2$. Then the equation becomes

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{3}{\rho} \frac{\partial u}{\partial \rho} - \Delta_K u = \frac{1}{\rho^2} \Delta_H u + f,$$

where Δ_H is the Laplacian on the hyperbolic space. We have

$$\Delta_H = \sum_i (t \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t})^2 - \sum_{k,j} (x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j})^2.$$

Thus $v = \rho^{\frac{3}{2}} u$ obeys

$$\frac{\partial^2 v}{\partial \rho^2} - \Delta_K v = \rho^{3/2} (\rho^{-2} (\Delta_H u + 3\frac{u}{4}) + F).$$

We decompose the right-hand side dyadically in time. Let χ be a smooth function, supported on $[\frac{1}{2}, 2]$ s.t. $\sum_{k=-\infty}^{\infty} \chi(\frac{\tau}{2^k}) = 1$. Denote $f_k = \chi(\frac{\tau}{2^k})F$ and $u_k = \chi(\frac{\tau}{2^k})(\Delta_H u + n(n-2)\frac{u}{4})$. We apply Lemma 4.5. We have

$$(A.1) \qquad |v(\rho, \omega, y)| \le \sum_k \int_0^{\rho} \rho^{3/2} (\rho^{-2} \|u_k(\rho, \omega, \cdot)\|_{H^4(K)} + \|f_k(\rho, \omega, \cdot)\|_{H^4(K)}).$$

We wish to estimate $u(\overline{\tau}, \overline{x})$, thus we need to estimate the sum of the integrals above. Denote $M_k(\rho^2) = \sup_{\omega} \|f_k(\rho, \omega, \cdot)\|_{H^4(K)}$. Then we have

$$\int_0^{\overline{\rho}} \rho^{\frac{3}{2}} \|f_k\|_{H^4(K)} d\rho \le \int_0^{\overline{\rho}} M_k(\rho)^2 \rho d\rho \int_0^{\overline{\rho}} \rho^2 d\rho.$$

We have $2^{k-1} \le t \le 2^k$; therefore $\frac{\rho}{\rho} \le C\frac{\tau}{\overline{\tau}} \le C\frac{2^k}{\overline{\tau}}$ in the support of u. Thus

$$\int \rho^2 d\rho \le C \frac{\overline{\rho}^3 2^{3k}}{\overline{\tau}^3}.$$

According to Proposition A.1, we have

$$\int M_k(\rho^2)\rho d\rho \leq 2^{-2k}C\sum_{|I|\leq\frac{5}{2}}\int |\tilde{\Gamma}^I\|f_k(\tau,x,\cdot)\|_{H^4(K)}|^2 d\tau dx$$

$$\leq 2^{-2k}C\sum_{|I|\leq\frac{5}{2}}\int_{\tau\leq\overline{\tau}}\int_K |\tilde{\Gamma}^I\Delta_K^2 f_k(\tau,x,y)|^2 d\tau dx dy.$$

Therefore, we have

$$\int \rho^{3/2}\|f_k(\rho,\omega,\cdot)\|_{H^4(K)}$$

$$\leq \frac{2^{\frac{k}{2}}\rho^{3/2}}{\overline{\tau}^{3/2}}\sum_{|I|\leq\frac{5}{2}}\int_{2^{k-1}\leq\tau\leq 2^k}\int_K |\tilde{\Gamma}^I\Delta_K^2 f_k(\tau,x,y)|^2 d\tau dx dy$$

$$\leq \rho^{3/2}2^k\frac{1}{\overline{\tau}^{3/2}}\sum_{|I|\leq 7}\sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,\overline{\tau}]}\|\Gamma^I f(\tau,\cdot,\cdot)\|.$$

To deal with the first term on the right-hand side of equation (A.1), we repeat the argument in [4, Lemma 7.3.4] verbatim to get the estimate

$$\int \rho^{-\frac{1}{2}}\|u_k\|_{H^4} \leq \frac{\overline{\rho}^{3/2}}{\overline{\tau}^{3/2}}\sum_{|I|\leq\frac{9}{2}}\sup_{2B\leq t}\|\tilde{\Gamma}^I u(t,\cdot)\|_{H^4(K)}.$$

We use the energy estimate to bound this expression by the estimate (A.2) and the initial energy to get, after summation in k,

$$\overline{\rho}^{\frac{3}{2}}|u| = |v| \leq \frac{\overline{\rho}^{\frac{3}{2}}}{\overline{\tau}^{\frac{3}{2}}}\sum_{|I|\leq 6}(\sum_i \|\Gamma^I\Gamma_i u_0\| + \sum_k 2^k\sup_{2^{k-1}\leq\tau\leq 2^{k+1}}\|\Gamma^I f(\tau)\|).$$

Dividing by $\frac{\overline{\rho}^{\frac{3}{2}}}{\overline{\tau}^{\frac{3}{2}}} = \frac{\overline{\rho}^{\frac{3}{2}}}{(t+1)^{\frac{3}{2}}}$ completes the proof.

ACKNOWLEDGEMENTS

REFERENCES

[1] Yvonne Choquet-Bruhat, *The Cauchy problem in extended supergravity, N = 1, d = 11*, Comm. Math. Phys. **97** (1985), no. 4, 541–552. MR787117 (86i:83035)

[2] B. de Wit, *Supergravity*, Unity from duality: gravity, gauge theory and strings (Les Houches, 2001), NATO Adv. Study Inst., EDP Sci., Les Ulis, 2003, pp. 1–135, DOI 10.1007/3-540-36245-2_1. MR2010970 (2004i:83144)

[3] Robin Graham, *Dirichlet-to-Neumann map for Poincaré-Einstein metrics*, Oberwolfach Reports.

[4] Lars Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 26, Springer-Verlag, Berlin, 1997. MR1466700 (98e:35103)

[5] Jürgen Jost, *Riemannian geometry and geometric analysis*, 3rd ed., Universitext, Springer-Verlag, Berlin, 2002. MR1871261 (2002i:53001)

[6] Joshua M. Kantor, *Eleven dimensional supergravity on edge manifolds*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of Washington. MR2717775

[7] Tosio Kato and Gustavo Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), no. 7, 891–907, DOI 10.1002/cpa.3160410704. MR951744 (90f:35162)

[8] Sergiu Klainerman, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions*, Comm. Pure Appl. Math. **38** (1985), no. 5, 631–641, DOI 10.1002/cpa.3160380512. MR803252 (87e:35080)

[9] Sergiu Klainerman, *The null condition and global existence to nonlinear wave equations*, Mathematics, Part 1 (Santa Fe, N.M., 1984), Lectures in Appl. Math., vol. 23, Amer. Math. Soc., Providence, RI, 1986, pp. 293–326. MR837683 (87h:35217)

[10] Peter H. Lesky and Reinhard Racke, *Nonlinear wave equations in infinite waveguides*, Comm. Partial Differential Equations **28** (2003), no. 7-8, 1265–1301, DOI 10.1081/PDE-120024363. MR1998938 (2004f:35122)

[11] Jason Metcalfe, Christopher D. Sogge, and Ann Stewart, *Nonlinear hyperbolic equations in infinite homogeneous waveguides*, Comm. Partial Differential Equations **30** (2005), no. 4-6, 643–661, DOI 10.1081/PDE-200059267. MR2153511 (2006b:35230)

[12] Jason Metcalfe and Ann Stewart, *Almost global existence for quasilinear wave equations in waveguides with Neumann boundary conditions*, Trans. Amer. Math. Soc. **360** (2008), no. 1, 171–188, DOI 10.1090/S0002-9947-07-04290-0. MR2341999 (2008k:35325)

[13] Tomás Ortín, *Gravity and strings*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2004. MR2053633 (2005g:83001)

[14] Christopher D. Sogge, *Lectures on non-linear wave equations*, 2nd ed., International Press, Boston, MA, 2008. MR2455195 (2009i:35213)

[15] Michael E. Taylor, *Tools for PDE: Pseudodifferential operators, paradifferential operators, and layer potentials*, Mathematical Surveys and Monographs, vol. 81, American Mathematical Society, Providence, RI, 2000. MR1766415 (2001g:35004)

[16] Michael E. Taylor, *Partial differential equations I. Basic theory*, second ed., Applied Mathematical Sciences, vol. 115, Springer, New York, 2011. MR2744150 (2011m:35001)

[17] Edward Witten, *String theory dynamics in various dimensions*, Nuclear Phys. B **443** (1995), no. 1-2, 85–126, DOI 10.1016/0550-3213(95)00158-O. MR1334520 (96f:81104)

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