TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 6, June 2015, Pages 4207–4233
S 0002-9947(2014)06200-4
Article electronically published on December 5, 2014

ON REGULAR G-GRADINGS

ELI ALJADEFF AND OFIR DAVID

Abstract. Let A be an associative algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero and let G be a finite abelian group. Regev and Seeman introduced the notion of a regular G-grading on A, namely a grading $A = \bigoplus_{g\in G} A_g$ that satisfies the following two conditions: (1) for every integer $n \geq 1$ and every n-tuple $(g_1, g_2, \ldots, g_n) \in G^n$, there are elements, $a_i \in A_{g_i}$, $i = 1, \ldots, n$, such that $\prod_1^n a_i \neq 0$; (2) for every $g, h \in G$ and for every $a_g \in A_g, b_h \in A_h$, we have $a_g b_h = \theta_{g,h} b_h a_g$ for some nonzero scalar $\theta_{g,h}$. Then later, Bahturin and Regev conjectured that if the grading on A is regular and minimal, then the order of the group G is an invariant of the algebra. In this article we prove the conjecture by showing that $ord(G)$ coincides with an invariant of A which appears in PI theory, namely $exp(A)$ (the exponent of A). Moreover, we extend the whole theory to (finite) nonabelian groups and show that the above result holds also in that case.

1. Introduction and statement of the main results

Group gradings on associative algebras (as well as on Lie and Jordan algebras) have been an active area of research in the last 15 years or so. In this article we will consider group gradings on associative algebras over an algebraically closed field $\mathbb{F}$ of characteristic zero. The fact that a given algebra admits additional structures, namely graded by a group G, provides additional information which may be used in the study of the algebra itself , e.g. in the study of group rings, twisted group rings and crossed product algebras in Brauer theory (indeed "gradings" on central simple algebras is an indispensable tool in Brauer theory, as it provides the isomorphism of $Br(k)$ with the second cohomology group $H^2(G_k, \bar{k}^\times)$. Here k is any field, G_k is the absolute Galois group of k and $\bar{k}^\times$ denotes the group of units of the separable closure of k). In addition, and more relevant to the purpose of this article, G-gradings play an important role in the theory of polynomial identities. Indeed, if A is a PI-algebra which is G-graded, then one may consider the T-ideal of G-graded identities (see subsection 2.1), denoted by $Id_G(A)$, and it turns out in general that it is easier to describe G-graded identities than the ordinary ones for the simple reason that the former ones are required to vanish on G-graded evaluations rather than on arbitrary evaluations. Nevertheless, two algebras A and B which are G-graded PI-equivalent are PI-equivalent as well, that is, $Id_G(A) = Id_G(B) \Rightarrow Id(A) = Id(B)$.

Recall that a G-grading on an algebra A is a vector space decomposition

$$A \cong \bigoplus_{g\in G} A_g$$

Received by the editors March 6, 2013 and, in revised form, March 7, 2013 and May 29, 2013.
2010 *Mathematics Subject Classification.* Primary 16R99, 16W50.
The first author was partially supported by the Israel Science Foundation (grant No. 1283/08 and grant No. 1017/12) and by the Glasberg-Klein Research Fund.

such that $A_g A_h \subseteq A_{gh}$ for every $g, h \in G$.

A particular type of G-gradings which is of interest was introduced by Regev and Seeman in [12], namely regular G-gradings where G is a finite abelian group. Let us recall the definition.

Definition 1 (Regular grading)**.** Let A be an associative algebra over a field $\mathbb{F}$ and let G be a finite abelian group. Suppose A is G-graded. We say that the G-grading on A is *regular* if there is a commutation function $\theta : G \times G \to \mathbb{F}^\times$ such that

(1) For every integer $n \geq 1$ and every n-tuple $(g_1, g_2, \ldots, g_n) \in G^n$, there are elements $a_i \in A_{g_i}$, $i = 1, \ldots, n$, such that $\prod_1^n a_i \neq 0$.
(2) For every $g, h \in G$ and for every $a_g \in A_g, b_h \in A_h$, we have $a_g b_h = \theta_{g,h} b_h a_g$.

Remark 2. One of our main tasks in this article is to extend the definition above to groups which are not necessarily abelian and prove the main results in that general context. For clarity we will continue with the exposition of the abelian case, and towards the end of the introduction we will discuss extensions to the nonabelian setting. It seems to us that the extension to the nonabelian case is rather natural in view of the abelian case. In those cases below where the statement in the general case is identical to the abelian case, we will make a note indicating it. As for the proofs (Sections 2 and 3), in case the result holds for arbitrary groups, we present the general setting only, possibly with some remarks concerning the abelian case.

In the first section of this article we present examples of regular G-gradings on finite and infinite dimensional algebras. We also explain how one can compose algebras with regular gradings. One of the most important examples of a regular grading on an algebra is the well known $\mathbb{Z}/2\mathbb{Z}$-grading on the infinite dimensional Grassmann algebra E.

Example 3. Let E be the Grassmann algebra, defined as the free algebra $\mathbb{F}\langle e_i \mid i \in \mathbb{N}\rangle$ with noncommuting variables, modulo the relations $e_i^2 = 0$ and $e_i e_j = -e_j e_i$ for $i \neq j$. We set E_0 to be the span of the monomials with even number of variables, and E_1 the span of monomials with odd number of variables. It is easy to see that $E = E_0 \oplus E_1$, and this is actually a regular $\mathbb{Z}/2\mathbb{Z}$-grading with commutation function $\tau_{0,0} = \tau_{0,1} = \tau_{1,0} = 1$ and $\tau_{1,1} = -1$.

Remark 4. It is sometimes more convenient to use the multiplicative group $C_2 = \{\pm 1\}$ for the grading on the Grassmann algebra. Hence we also write $E = E_1 \oplus E_{-1}$.

The Grassmann algebra with its $\mathbb{Z}/2\mathbb{Z}$-grading has remarkable properties which are fundamental in the theory of PI-algebras. Indeed, if $B = B_0 \oplus B_1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, we let $E\widehat{\otimes}B$ be the Grassmann $\mathbb{Z}/2\mathbb{Z}$-envelope of B. Recall that the algebra $E\widehat{\otimes}B$ is $\mathbb{Z}/2\mathbb{Z}$-graded as well and its grading is determined by $(E\widehat{\otimes}B)_0 = E_0 \otimes B_0$ and $\left(E\widehat{\otimes}A\right)_1 = E_1 \otimes B_1$.

A key property of the "envelope operation" is the following equality of $\mathbb{Z}/2\mathbb{Z}$-graded T-ideals of identities (and hence also of the corresponding ungraded T-ideals of identities):

$$Id_{\mathbb{Z}/2\mathbb{Z}}(E\widehat{\otimes}(E\widehat{\otimes}B)) = Id_{\mathbb{Z}/2\mathbb{Z}}(B). \tag{1.1}$$

We refer to the "envelope operation" as being *involutive*. It is well known that applying this operation, one can extend the solution of the Specht problem and proof of "representability" from affine to nonaffine PI-algebras (see [9], [2]).

Interestingly, the property satisfied by the Grassmann algebra we just mentioned follows from the fact that the $\mathbb{Z}/2\mathbb{Z}$-grading on E is regular, and indeed in Theorem 6 we show that a similar property holds for arbitrary regular graded algebras. In order to state the result precisely we introduce the notion of G-envelope of two algebras A and B where G is a finite abelian group.

Definition 5 (G-envelope)**.** Let A, B be two G-graded algebras. We denote by $A\widehat{\otimes}B$ the G-graded algebra defined by $\left(A\widehat{\otimes}B\right)_g = A_g \otimes B_g$.

The following result generalizes (1.1). The proof is presented in Section 2.

Theorem 6. *Let A be a regularly G-graded algebra with commutation function θ, and let B, C be two G-graded algebras.*

(1) *If $\theta \equiv 1$, then $Id_G(A\widehat{\otimes}B) = Id_G(B)$.*
(2) *Let $\tilde{A} = \widehat{\otimes}^{|G|-1}A$ be the envelope of $|G|-1$ copies of A; then $\tilde{A}$ is regularly G-graded and $Id_G(\tilde{A}\widehat{\otimes}(A\widehat{\otimes}B)) = Id_G(B)$.*
(3) *$Id_G(B) = Id_G(C)$ if and only if $Id_G(A\widehat{\otimes}B) = Id_G(A\widehat{\otimes}C)$.*

Our main goal in this article is to investigate the general structure of (minimal) regular gradings on associative algebras over an algebraically closed field of characteristic zero and in particular to give a positive answer to Conjecture 2.5 posed by Bahturin and Regev in [4].

It is easy to see that a given algebra A may admit regular gradings with nonisomorphic groups and even with groups of distinct orders. Therefore, in order to put some restrictions on the possible regular gradings on an algebra A, Bahturin and Regev introduced the notion of regular gradings which are *minimal*. A regular G-grading on an algebra A with commutation function θ is said to be *minimal* if for any $e \neq g \in G$ there is $g' \in G$ such that $\theta(g, g') \neq 1$.

Given a regular G-grading on an algebra A with commutation function θ, one may construct a minimal regular grading with a homomorphic image $\bar{G}$ of G. To see this, let

$$H = \{h \in G \mid \theta(h, g) = 1 : \text{for all } g \in G\}.$$

One checks easily that θ is a skew symmetric bicharacter and hence H is a subgroup of G. Consequently, the commutation function θ on G induces a commutation function $\tilde{\theta}$ on $\bar{G} = G/H$. Moreover, the induced regular $\bar{G}$-grading on A is minimal.

In this article we consider the problem of uniqueness of a minimal regular G-grading on an algebra A (assuming it exists). It is not difficult to show that an algebra A may admit nonisomorphic minimal regular G-gradings. Furthermore, an algebra A may admit minimal regular gradings with nonisomorphic abelian groups. However, it follows from our results (as conjectured by Bahturin and Regev) that the order of the group is uniquely determined. In fact, the order of any group which provides a minimal regular grading on an algebra A coincides with a numerical invariant of A which arises in PI-theory, namely the PI-exponent of the algebra A (denoted by $exp(A)$). In order to state the result precisely we need some terminology, which we recall now.

Given a regular G-grading on an $\mathbb{F}$-algebra A we consider the corresponding commutation matrix M^A defined by $\left(M^A\right)_{g,h} = \theta(g, h)$, $g, h \in G$ (see [4]). The commutation matrix encodes properties of θ. For instance, a regular grading is minimal if and only if there is only one row of ones in M^A (resp. with columns).

Next we recall the definition of $\exp(A)$. For any positive integer n we consider the $n!$-dimensional $\mathbb{F}$-space P_n, spanned by all monomials of degree n on n different variables $\{x_1, \dots, x_n\}$, and let

$$c_n(A) = \dim_F(P_n/(P_n \cap Id(A))).$$

This is the n-th coefficient of the codimension sequence of the algebra A. It was shown by Giambruno and Zaicev (see [5], [6]) that the limit

$$\lim_{n\to\infty} c_n(A)^{1/n}$$

exists and is a nonnegative integer. The limit is denoted by $\exp(A)$.

We can now state the main result of the paper in case the gradings on A are given by abelian groups.

Theorem 7. *Let A be an algebra over an algebraic closed field $\mathbb{F}$ of characteristic zero and suppose it admits minimal regular gradings by finite abelian groups G and H.*

Then

(1) $|G| = |H|$ *and this invariant is equal to* $exp(A)$. *In particular the algebra A is PI.*
(2) *The commutation matrices M_G^A and M_H^A are conjugate. In particular $tr(M_G^A) = tr(M_H^A)$ and* $\det(M_G^A) = \det(M_H^A)$.
(3) *In fact,* $\det(M_G^A) = \pm\,|G|^{|G|/2}$.

Remark 8. Some of the results stated in Theorem 7 were conjectured in [4]. Specifically, as mentioned above, Bahturin and Regev conjectured that the order of a group which provides a minimal regular grading on an algebra A is uniquely determined. Moreover, they conjectured that if M_G^A and M_H^A are the commutation matrices of two minimal regular gradings on A with groups G and H respectively, then $\det(M_G^A) = \det(M_H^A) \neq 0$.

1.1. **Not necessarily abelian groups.** Suppose now that G is an arbitrary finite group and let A be a G-graded algebra. As above, A is an associative algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero. We denote by A_g the corresponding $g \in G$-homogeneous component.

Definition 9. We say that the G-grading on A is *regular* if the following two conditions hold.

(1) (commutation) For any n-tuple $(g_1, \dots, g_n) \in G^n$ and for any permutation $\sigma \in Sym(n)$ such that the products $g_1 \cdots g_n$ and $g_{\sigma(1)} \cdots g_{\sigma(n)}$ yield the same element of G, there is a nonzero scalar $\theta_{((g_1,\dots,g_n),\sigma)} \in F^\times$ such that for any n-tuple $(a_1, \dots, a_n) \in A^n$ with $a_i \in A_{g_i}$, the following equality holds:
$$a_1 \cdots a_n = \theta_{((g_1,\dots,g_n),\sigma)} a_{\sigma(1)} \cdots a_{\sigma(n)}.$$
(2) (regularity) For any n-tuple $(g_1, \dots, g_n) \in G^n$, there exists an n-tuple $(a_1, \dots, a_n) \in A^n$ with $a_i \in A_{g_i}$ such that $a_1 \cdots a_n \neq 0$.

Remark 10. In the special case where the elements $g, g' \in G$ commute we write $\theta_{g,g'}$ instead of $\theta_{((g,g'),(12))}$. In particular we will often use the notation $\theta_{g,g}$. Note that if G is abelian, then $\theta_{((g_1,\dots,g_n),\sigma)}$ is determined by θ_{g_i,g_j}, $1 \le i, j \le n$.

Typical examples of regularly graded algebras (G arbitrary) are the well known group algebras $\mathbb{F}G$, and more generally, any twisted group algebra $\mathbb{F}^\alpha G$ where α is a 2-cocycle on G with values in $\mathbb{F}^\times$. Indeed, this follows easily from the fact that each homogeneous component is 1-dimensional and every nonzero homogeneous element is invertible (for a definition of twisted group algebra, see the paragraph after Lemma 31).

Additional examples can be obtained as follows.

(1) If A is a regularly G-graded algebra, then $E \otimes A$ has a natural regular $\mathbb{Z}/2\mathbb{Z} \times G$-grading where E is the infinite dimensional Grassmann algebra.

(2) Let A be a regularly G-graded algebra and suppose the group G contains a subgroup H of index 2. Then we may view A as a $\mathbb{Z}/2\mathbb{Z} \cong G/H$-graded algebra and we let $A = A_0 \oplus A_1$ be the corresponding decomposition. Let $E(A) = (E_0 \otimes A_0) \oplus (E_1 \otimes A_1)$ be the Grassmann envelope of A and consider the following G-grading on it. For any $g \in H$, we put $E(A)_g = E_0 \otimes A_g$, whereas if $g \notin H$ we put $E(A)_g = E_1 \otimes A_g$. We claim the grading is regular. Indeed, let $(g_1, \dots, g_n) \in G^n$ and let $\sigma \in Sym(n)$ be a permutation such that $g_1 \cdots g_n = g_{\sigma(1)} \cdots g_{\sigma(n)}$. Then for elements $z_1, \dots, z_n$ where $z_i \in E(A)_{g_i}$, we have

$$z_1 \cdots z_n = \tau((g_1H, \dots, g_nH), \sigma)\theta((g_1, \dots, g_n), \sigma) z_{\sigma(1)} \cdots z_{\sigma(n)}$$

where τ is the commutation function of the infinite Grassmann algebra. For future reference we denote the commutation function which corresponds to the regular G-grading on $E(A)$ by $\tau\theta$.

Following the discussion in the abelian case we define now nondegenerate gradings for arbitrary finite groups as the counterpart of minimal gradings. Let A be an associative algebra and suppose it has a regular G-grading with commutation function θ. We say that the grading is nondegenerate if for every $g \neq e$ in G, there is an element $g' \in C_G(g)$ (the centralizer of g in G) such that $\theta_{(g,g')} \neq 1$.

Remark 11. It turns out (see Lemma 37) that $g \mapsto \theta_{g,g}$ is a homomorphism from G to $\{\pm 1\}$ and therefore its kernel $H = \{g \in G : \theta_{g,g} = 1\}$ is a subgroup of G (of index ≤ 2). In case $H = G$, there is a cohomology class $[\alpha] \in H^2(G, \mathbb{F}^\times)$ such that $\mathbb{F}^\alpha G$ has commutation function θ. Then, the nondegeneracy of the G-grading on A corresponds to α being a nondegenerate 2-cocycle. Groups G which admit nondegenerate 2-cocycles are called "central type". It is a rather difficult problem to classify central type groups. It is known, using the classification of finite simple groups(!), that any central type group must be solvable. It seems to be an interesting problem to classify finite groups which admit nondegenerate commutation functions (modulo the classification of central type groups).

Our main results in the general case are extensions of the results appearing in Theorem 7.

Theorem 12. *Theorem* 7(1) *holds for arbitrary nondegenerate regular gradings (i.e. G is not necessarily abelian).*

Remark 13. In case G is abelian, the commutation function θ is a skew symmetric bicharacter (see Definition 24). In this case, it is a well known theorem of Scheunert [13] that θ arises from a 2-cocycle (as mentioned in the previous remark). The notion of a bicharacter was considerably generalized to cocommutative Hopf algebras

(and hence in particular to group algebras) (see [3]). Furthermore, whenever the bicharacter is skew symmetric, the theorem of Scheunert can be extended to that case. However, it should be noted that already for group algebras $\mathbb{F}G$ where G is a nonabelian group, the linear extension of $\beta(g,h) = f(g,h)/f(h,g)$ for a 2-cocycle $f : G \times G \to \mathbb{F}^\times$ is not in general a skew symmetric bicharacter on $\mathbb{F}G$.

Our results should be viewed or interpreted so as to overcome this problem by considering commutation functions which satisfy certain natural necessary conditions in case they arise from 2-cocycles on G. Then Lemma 32 provides a generalization of Scheunert's theorem to that context: namely every such commutation function on G indeed arises from a 2-cocycle on G.

1.1.1. *Commutation matrix.* As for the commutation matrix and its characteristic values, we need to fix some notation. Suppose A has a regular G-grading and let θ^A be the corresponding commutation function. Suppose first that $\theta^A_{g,g} = 1$ for every $g \in G$. In that case we know that the commutation function corresponds to an element $[\alpha] \in H^2(G, \mathbb{F}^\times)$. With this data we consider the corresponding twisted group algebra $B = \mathbb{F}^\alpha G$ which is regularly G-graded with commutation function θ^A. It is well known that the algebra $\mathbb{F}^\alpha G$ is spanned over $\mathbb{F}$ by a set of invertible homogeneous elements $\{U_g\}_{g\in G}$ that satisfy $U_g U_h = \alpha(g,h)U_{gh}$ for every $g, h \in G$.

Let us construct the corresponding commutation matrix. For every pair $(g,h) \in G^2$ we consider the element $U_g U_h U_g^{-1} U_h^{-1} \in \mathbb{F}^\alpha G$. The matrix M^A_G is determined by $(M^A_G)_{(g,h)} = U_g U_h U_g^{-1} U_h^{-1}$ for every $g, h \in G$, and we note that this element does not depend on the choice of the basis $\{U_g : g \in G\}$.

Next we consider the general case. Let $\psi : G \to \mathbb{F}^\times$ be the map determined by $\psi(g) = \theta_{g,g}$. The function ψ will be shown to be a homomorphism with its image contained in $\{\pm 1\}$, and we set $H = \ker(\psi) = \{g \in G : \theta_{g,g} = 1\}$. Applying the construction above we may define a G-grading on $E(A)$ where the $\mathbb{Z}/2\mathbb{Z}$-grading on A is defined by $A = A_H \oplus A_{G\setminus H}$. The commutation function $\tau \cdot \theta^A$ of $E(A)$ satisfies $(\tau\theta^A)_{g,g} = 1$ for every $g \in G$. As in the previous case the function $\tau\theta^A$ corresponds to a cohomology class $[\alpha] \in H^2(G, F^\times)$ where α is a representing 2-cocycle. We let $B = \mathbb{F}^\alpha G$ be the corresponding twisted group algebra with commutation function $\theta^B = \tau\theta^A$, and for every $g, h \in G$ we consider the element $U_g U_h U_g^{-1} U_h^{-1} \in \mathbb{F}^\alpha G$. The commutation matrix is defined by

$$(M_G)_{g,h} = \tau_{(\psi(g),\psi(h))} U_g U_h U_g^{-1} U_h^{-1}.$$

We will usually write $\tau_{g,h}$ instead of $\tau_{\psi(g),\psi(h)}$. Note that if $\psi \equiv 1$, then $H = G$ and $\tau\mid_G \equiv 1$, so we have that $\theta^B = \theta^A$ as in the first case.

Theorem 14. *Let A be an associative algebra over an algebraically closed field of characteristic zero. Suppose A admits a nondegenerate regular G-grading and let θ be the corresponding commutation function. Let M_G be the commutation matrix constructed above. Then $M_G^2 = |G| \cdot Id$.*

As a consequence we extend Theorem 7(2),(3) for arbitrary nondegenerate regular gradings (see subsection 3.1, Corollary 47). In case $\theta_{g,g} = 1$ for every $g \in G$, we have that the elements $U_g U_h U_g^{-1} U_h^{-1} \in \mathbb{F}^\alpha G \cong M_r(\mathbb{F})$ and so the commutation matrix may be viewed as a matrix in $M_{r^3}(\mathbb{F})$. In that case we obtain the following corollary.

Corollary 15. $\det(M_G) = \pm r^{(r^3)}$, *where* $|G| = r^2$.

2. Preliminaries, examples and some basic results

In the first part of this section we recall some general facts and terminology on G-graded PI-theory which will be used in the proofs of the main results (we refer the reader to [2] for a detailed account on this topic). In the second part of this section we present some additional examples of regular gradings on finite and infinite dimensional algebras. Finally, we present properties of regular gradings and prove Theorem 6.

2.1. **Graded polynomial identities.** Let W be a G-graded PI-algebra over $\mathbb{F}$ and $I = Id_G(W)$ be the ideal of G-graded identities of W. These are polynomials in $\mathbb{F}\langle X_G\rangle$, the free G-graded algebra over $\mathbb{F}$ generated by X_G, that vanish upon any admissible evaluation on W. Here $X_G = \bigcup_{g\in G} X_g$ and X_g is a set of countably many variables of degree g. An evaluation is admissible if the variables from X_g are replaced only by elements of W_g. It is known that I is a G-graded T-ideal, i.e. closed under G-graded endomorphisms of $\mathbb{F}\langle X_G\rangle$.

We recall from [2] that the T-ideal $I = Id_G(W)$ is generated by multilinear polynomials and so it does not change when passing to $\bar{\mathbb{F}}$, the algebraic closure of $\mathbb{F}$, in the sense that the ideal of identities of $W_{\bar{\mathbb{F}}}$ over $\bar{\mathbb{F}}$ is the span (over $\bar{\mathbb{F}}$) of the T-ideal of identities of W over $\mathbb{F}$.

2.2. **Additional examples of regular gradings.** We present here some more examples (in addition to the ones presented in the introduction). The following example corresponds to the grading determined by the symbol algebra $(1,1)_n$.

Example 16. Let $M_n(\mathbb{F})$ be the matrix algebra over the field $\mathbb{F}$, and let $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. For ζ a primitive n-th root of 1 we define

$$X = diag(1,\zeta,\dots,\zeta^{n-1}) = \begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & \zeta & 0 & & \vdots \\ & 0 & \zeta^2 & \ddots & \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & \zeta^{n-1} \end{bmatrix},$$

$$Y = E_{n,1} + \sum_{1}^{n-1} E_{i,i+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Note that $\zeta XY = YX$. Furthermore, the set $\left\{X^iY^j \mid 0 \le i,j \le n-1\right\}$ is a basis of $M_n(\mathbb{F})$, and so we can define a G-grading on $M_n(\mathbb{F})$ by $(M_n(\mathbb{F}))_{(i,j)} = \mathbb{F}X^iY^j$. Let us check that the G-grading is regular. For any two basis elements we have that

$$\begin{aligned} (X^{i_1}Y^{j_1})(X^{i_2}Y^{j_2}) &= \zeta^{i_2j_1}X^{i_1}X^{i_2}Y^{j_1}Y^{j_2} = \zeta^{i_2j_1}X^{i_2}X^{i_1}Y^{j_2}Y^{j_1} \\ &= \zeta^{i_2j_1-i_1j_2}\left(X^{i_2}Y^{j_2}\right)\left(X^{i_1}Y^{j_1}\right) \\ &\Rightarrow \theta_{(i_1,j_1)(i_2,j_2)} = \zeta^{i_2j_1-i_1j_2}, \end{aligned}$$

and hence the second condition in the definition of a regular grading is satisfied. The first condition in the definition follows at once from the fact that the elements X and Y are invertible. Finally we note that since ζ is a primitive n-th root of unity, the regular grading is in fact minimal.

Example 17. For any $n \in \mathbb{N}$ and $c \in \mathbb{F}^\times$, we can define a regular $\mathbb{Z}/n\mathbb{Z}$-grading on $A = \mathbb{F}[x]/\langle x^n - c\rangle$ by setting $A_k = \mathbb{F} \cdot x^k$. Clearly, the commutation function here is given by $\theta_{h,g} = 1$ for all $g, h \in \mathbb{Z}/n\mathbb{Z}$.

Example 18. For any algebra A we have the trivial $G = \{e\}$-grading by setting $A_e = A$. In this case the grading is regular if and only if A is abelian and nonnilpotent.

Example 19. We present an algebra with a *nondegenerate* regular G-grading where G is isomorphic to the dihedral group of order 8.

Consider the presentation $\langle x, y : x^4 = y^2 = e, yxy^{-1} = x^3\rangle$ of the group G. It is well known that there is a (unique) nonsplit extension

$$\alpha_G : 1 \longrightarrow \{\pm 1\} \longrightarrow Q_{16} \xrightarrow{\pi} G \longrightarrow 1$$

where $Q_{16} = \langle u, v : u^8 = v^4 = e, \ , u^4 = v^2, \ vuv^{-1} = u^3\rangle$ is isomorphic to the quaternion group of order 16. The map π is determined by $\pi(u) = x$ and $\pi(v) = y$. Note that the extension is nonsplit on any nontrivial subgroup of G; that is, if $\{e\} \neq H \leq G$, then the restricted extension

$$\alpha_H : 1 \longrightarrow \{\pm 1\} \longrightarrow \pi^{-1}(H) \xrightarrow{\pi} H \longrightarrow 1$$

is nonsplit. Consider the twisted group algebra $\mathbb{F}^{\alpha_G}G$ where the values of the cocycle are viewed in $\mathbb{F}^\times$. Clearly, $\mathbb{F}^{\alpha_G}G$ is $G/K = C_2$-graded where K is the Klein 4-group $K = \{e, x^2, y, x^2y\}$, and so we can consider the corresponding Grassmann envelope $E(\mathbb{F}^{\alpha_G}G)$. We show that $E(\mathbb{F}^{\alpha_G}G)$ is regularly G-graded and moreover that the grading is nondegenerate. Clearly the natural G-grading on the twisted group algebra $\mathbb{F}^{\alpha_G}G$ is regular, and hence the corresponding G-grading on $E(\mathbb{F}^{\alpha_G}G)$ is also regular. Let θ be the corresponding commutation function. To see that the G-grading on $E(\mathbb{F}^{\alpha_G}G)$ is nondegenerate, note that since the cocycle α_H is nontrivial on every subgroup $H \neq \{e\}$ of G, the group $\pi^{-1}(K)$ is isomorphic to the quaternion group of order 8, and hence the twisted group subalgebra $\mathbb{F}^{\alpha_K}K$ of $\mathbb{F}^{\alpha_G}G$ is isomorphic to $M_2(\mathbb{F})$. This shows that the nondegeneracy condition (see Definition 21) is satisfied by any nontrivial element of K. For elements g in $G \setminus K$ we have that $\theta_{g,g} = -1$ and we are done. We will return to this example at the end of the paper.

2.2.1. *The commutation function θ and the commutation matrix.* We now turn to study some properties of the commutation function θ. We start with some notation.

Let G be a group and $\bar{g} = (g_1, \ldots, g_n) \in G^n$.

- Denote by $Sym(\bar{g})$ the set $\{\sigma \in Sym(n) \mid g_1 \cdots g_n = g_{\sigma(1)} \cdots g_{\sigma(n)}\}$.
- For any $\sigma \in Sym(n)$ we write $\bar{g}^\sigma = (g_{\sigma(1)}, \ldots, g_{\sigma(n)})$.

The conditions in the following lemma correspond to the properties of T-ideal: namely (1) closed to multiplication, (2) closed to substitution and (3) closed to addition.

Lemma 20. *Let G be a group and A a regularly G-graded algebra with commutation function θ. Then θ satisfies the following conditions.*

(1) *Let $\bar{g} = (g_1, \dots, g_n) \in G^n$, $1 \le i \le j \le n$ and $\sigma \in Sym(\bar{g})$ such that $\sigma(k) = k$ for all $k < i$ or $k > j$. Denote by $\tilde{\sigma} \in Sym(j - i + 1)$ the restriction of σ to $\{i, i+1, \dots, j\}$; then $\theta_{(\bar{g},\sigma)} = \theta_{((g_i, g_{i+1}, \dots, g_j), \tilde{\sigma})}$.*

(2) *Let $\bar{h} = (h_1, \dots, h_k) \in G^k$. Let $\bar{g}_i = (g_{i,1}, \dots, g_{i,n_i}) \in G^{n_i}$ such that $\prod_{j=1}^{n_i} g_{i,j} = h_i$ and set $\bar{g} = (\bar{g}_1, \dots, \bar{g}_k) \in G^{\sum_1^k n_i}$. For each $\sigma \in Sym(\bar{h})$ let $\tilde{\sigma} \in Sym(n_1 + \cdots + n_k)$ be the permutation that moves the blocks of size $n_1, \dots, n_k$ according to the permutation σ. Then $\theta_{(\bar{h},\sigma)} = \theta_{(\bar{g},\tilde{\sigma})}$.*

(3) *For every $g_1, \dots, g_n \in G$ and $\sigma, \tau \in Sym(n)$ such that $g_1 \cdots g_n = g_{\sigma(1)} \cdots g_{\sigma(n)} = g_{\sigma\tau(1)} \cdots g_{\sigma\tau(n)}$ we have*

$$\theta_{((g_1,\dots,g_n),\sigma)}\theta_{\left((g_{\sigma(1)},\dots,g_{\sigma(n)}),\tau\right)} = \theta_{(g_1,\dots,g_n,\sigma\tau)}.$$

Proof. (1) This is an immediate consequence of the associativity of the product in A.

(2) The result follows from the fact that $A_{g_{i,1}} \cdots A_{g_{i,n_i}} \subseteq A_{g_{i,1} \cdots g_{i,n_i}} = A_{h_i}$.

(3) Let $A = \bigoplus_{g \in G} A_g$ and choose some $a_i \in A_{g_i}$ such that $\prod a_i \neq 0$. Then

$$\begin{aligned} a_1 \cdots a_n &= \theta_{((g_1,\dots,g_n),\sigma)} a_{\sigma(1)} \cdots a_{\sigma(n)} \\ &= \theta_{((g_1,\dots,g_n),\sigma)}\theta_{\left((g_{\sigma(1)},\dots,g_{\sigma(n)}),\tau\right)} a_{\sigma\tau(1)} \cdots a_{\sigma\tau(n)}, \\ a_1 \cdots a_n &= \theta_{(g_1,\dots,g_n,\sigma\tau)} a_{\sigma\tau(1)} \cdots a_{\sigma\tau(n)}. \end{aligned}$$

Finally, since $a_1 \cdots a_n \neq 0$, we have $\theta_{((g_1,\dots,g_n),\sigma)}\theta_{\left((g_{\sigma(1)},\dots,g_{\sigma(n)}),\tau\right)} = \theta_{(g_1,\dots,g_n,\sigma\tau)}$ as desired. □

Next, we define G-commutation functions. We remind the reader that if $g, h \in G$ commute, we denote by $\theta_{g,h}$ the scalar $\theta_{((g,h),(1,2))}$.

Definition 21. Let θ be a function from the pairs $\bar{g} = (g_1, \dots, g_n) \in G^n$, $\sigma \in Sym(\bar{g})$ with values in $\mathbb{F}^\times$. We say that θ is a *G-commutation function* if it satisfies conditions (1), (2), (3) from the last lemma. The function θ is said to be nondegenerate if for any $e \neq g \in G$ there is some $h \in C_G(g)$ such that $\theta_{g,h} \neq 1$.

In Lemma 37 below we show that each G-commutation function is in fact the commutation function of some regularly G-graded algebra. By the definition, we get that a regular grading is nondegenerate if and only if the commutation function is nondegenerate.

Lemma 22. *Let θ be a G-commutation function. Then the following hold.*

(1) *For every $\bar{g} \in G^n$ we have $\theta_{(\bar{g},id)}\theta_{(\bar{g},id)} = \theta_{(\bar{g},id)}$ and so $\theta_{(\bar{g},id)} = 1$.*

(2) *For every commuting pair $g, h \in G$, we have $\theta_{g,h} = \theta_{h,g}^{-1}$.*

(3) *For any fixed $g \in G$, the functions $h \mapsto \theta_{g,h}$ and $h \mapsto \theta_{h,g}$ are characters on $C_G(g)$.*

Proof. (1) Part (1) follows from part (3) of the last lemma, where $\sigma = \tau = id \in Sym(n)$.

Let $\sigma = (1, 2) \in Sym(2)$.

(2) If g, h commute, then $\theta_{g,h}\theta_{h,g} = \theta_{((g,h),\sigma)}\theta_{((h,g),\sigma)} = \theta_{((g,h),\sigma^2)} = \theta_{((g,h),id)} = 1$.

(3) By the conditions in Lemma 20 we get that if $g \in G$ and $h_1, h_2 \in C_G(g)$, then

$$\begin{aligned}\theta_{g,h_1h_2} &= \theta_{((g,h_1h_2),\sigma)} = \theta_{((g,h_1,h_2),(1,3,2))} \\ &= \theta_{((g,h_1,h_2),(1,2))}\theta_{((h_1,g,h_2),(2,3))} = \theta_{g,h_1}\theta_{g,h_2}.\end{aligned}$$

Similarly we have that $\theta_{h_1h_2,g} = \theta_{h_1,g}\theta_{h_2,g}$. □

Remark 23. Notice in particular that $\theta_{e,g} = \theta_{g,e} = 1$ for all $g \in G$. Using conditions (1) and (2) in Lemma 20 we see that if $\sigma \in Sym(n)$ is a permutation which moves rigidly in $\bar{g} = (g_1, \ldots, g_n) \in G^n$, a block $(g_i, g_{i+1}, \ldots, g_j)$ with $g_i \cdot g_{i+1} \cdots g_j = e$, then $\theta_{(\bar{g},\sigma)} = 1$.

If G is abelian, then the commutation function $\theta_{(\bar{g},\sigma)}$ is defined by its values on pairs $\theta_{g,h}$. In that case we get that $C_G(g) = G$ for all $g \in G$ and the conditions in Lemma 20 follow from those in the last lemma. We recall the definition of such functions.

Definition 24 (Bicharacter)**.** Let $\eta : G \times G \to \mathbb{F}^\times$ be a map where G is a group and $\mathbb{F}^\times$ is the group of units of the field $\mathbb{F}$. We say that the map η is a *bicharacter of* G if for any $g_0, h_0 \in G$ the maps $h \mapsto \eta(g_0, h)$ and $g \mapsto \eta(g, h_0)$ are characters (i.e group homomorphisms $G \to \mathbb{F}^\times$). A bicharacter of G is called *skew-symmetric* if $\eta(g, h) = \eta(h, g)^{-1}$ for any $h, g \in G$. A bicharacter is said to be *nondegenerate* if for any $e \neq g \in G$ there is an element $h \in G$ such that $\theta(g, h) \neq 1$.

Remark 25. In general, if θ is a commutation function on a finite group G, then for any commuting elements $g, h \in G$ we have $ord(\ \theta(g,h)\ ) \mid\ gcd(\ ord(g),\ ord(h)\ )$, so $\theta(g, h)$ is contained in the group of roots of unity of order $|G|$ in $\mathbb{F}^\times$. In fact, as it will be shown below, this holds for any $\theta_{(\bar{g},\sigma)}$. Also, we have that $\theta(g, g) = \theta(g, g)^{-1}$ so $\theta(g, g) \in \{\pm 1\}$ for every $g \in G$.

We present now two lemmas which summarize properties of the commutation function and the "G-envelope operation". The proof of the first lemma follows directly from the definitions and is left to the reader.

Lemma 26. *Suppose that A, B are G, H-regularly graded algebras with commutation functions θ and η respectively. Then the following hold.*

(1) *$A \otimes B$ is a regularly $G \times H$-graded algebra with $(A \otimes B)_{(g,h)} = A_g \otimes B_h$ and $(\theta \otimes \eta)_{((g_1,h_1),\ldots,(g_n,h_n),\sigma)} = \theta_{((g_1,\ldots,g_n),\sigma)}\theta_{((h_1,\ldots,h_n),\sigma)}$ for all $\sigma \in Sym(\bar{g}) \cap Sym(\bar{h})$.*
(2) *If $G = H$ and $\theta = \eta$, then the algebra $A \oplus B$ is regularly G-graded where $(A \oplus B)_g = A_g \oplus B_g$. Furthermore, the corresponding commutation function is θ. In particular $\bigoplus_1^n A$ is regularly G-graded for any $n \in \mathbb{N}$.*
(3) *If $N \leq G$ is a subgroup, then $A_N = \bigoplus_{g \in N} A_g$ is a regularly N-graded algebra with commutation function $\theta\mid_N$, the restriction to tuples in N.*
(4) *If $G = H$, then $A \widehat{\otimes} B$ is a regularly G-graded algebra with commutation function $(\theta \widehat{\otimes} \eta)_{(\bar{g},\sigma)} = \theta_{(\bar{g},\sigma)}\eta_{(\bar{g},\sigma)}$.*

If the groups G, H are abelian, then the commutation matrix which corresponds to the cases considered in the lemma are calculated as follows: (1) $M^{A \otimes B} = M^A \otimes M^B$, (2) $M^{A \oplus B} = M^A$, (3) M^{A_N} *is the restriction of* M^A *to the group* N, *and* (4) $M^{A \widehat{\otimes} B}$ *is the Schur product (entry wise multiplication) of* M^A *and* M^B.

In the nonabelian case we have a similar connection between the commutation matrices, though the rings over which the matrices are defined may differ. More details are presented in the end of subsection 3.1.

Suppose A is regularly G-graded and let θ be the corresponding commutation function. Given a multilinear polynomial

$$f(x_{g_1,1},\dots,x_{g_n,n}) = \sum_{\sigma\in Sym(\bar{g})} \lambda_\sigma \prod x_{g_{\sigma(i)},\sigma(i)},$$

we denote by f^θ the polynomial

$$f^\theta(x_{g_1,1},\dots,x_{g_n,n}) = \sum_{\sigma\in Sym(\bar{g})} \lambda_\sigma \theta(\bar{g},\sigma)^{-1} \prod_i x_{g_{\sigma(i)},\sigma(i)}.$$

Lemma 27. *Let A be a regularly G-graded algebra with commutation function θ and let B be any G-graded algebra. Let $f(x_{g_1,1},\dots,x_{g_n,n}) = \sum_{\sigma\in Sym(\bar{g})} \lambda_\sigma \prod x_{g_{\sigma(i)},\sigma(i)}$. Then $f^\theta \in Id_G(B)$ if and only if $f \in Id_G(A\widehat{\otimes}B)$.*

Proof. By multilinearity of f we only check that f vanishes on a spanning set. For any $a_i \in A_{g_i}$ and $b_i \in B_{g_i}$ we get that

$$\begin{aligned} f(a_1\otimes b_1,\dots,a_n\otimes b_n) &= \sum_{\sigma\in Sym(\bar{g})} \lambda_\sigma \prod (a_{\sigma(i)}\otimes b_{\sigma(i)}) \\ &= \sum_{\sigma\in Sym(\bar{g})} \lambda_\sigma \prod_i a_{\sigma(i)} \otimes \prod_i b_{\sigma(i)} \\ &= \prod_i a_i \otimes \sum_{\sigma\in Sym(\bar{g})} \lambda_\sigma \theta(\bar{g},\sigma)^{-1} \prod_i b_{\sigma(i)} \\ &= \prod a_i \otimes f^\theta(b_1,\dots,b_n). \end{aligned}$$

If $f^\theta \in Id_G(B)$, then the last term is zero so $f \in Id_G(A\widehat{\otimes}B)$. On the other hand, if $f \in Id_G(A\widehat{\otimes}B)$, then the first term is always zero. Since the grading on A is regular, we can find a_i such that $\prod a_i \neq 0$, so $\prod a_i \otimes f^\theta(b_1,\dots,b_n) = 0$ if and only if $f^\theta(b_1,\dots,b_n) = 0$ and we get that $f \in Id_G(B)$. □

Before we proceed with the proof of Theorem 6 we recall that for any G-graded algebra over a field of characteristic zero $\mathbb{F}$, the T-ideal of G-graded identities is generated by multilinear polynomials which are strongly homogeneous, namely polynomials of the form

$$f(x_{g_1,1},\dots,x_{g_n,n}) = \sum_{\sigma\in Sym(\bar{g})} \lambda_\sigma \prod x_{g_{\sigma(i)},\sigma(i)}.$$

Proof of Theorem 6. (1) This is immediate since $f^\theta = f$ when $\theta \equiv 1$.

(2) $\tilde{A}\widehat{\otimes}A$ is the product of $|G|$ copies of A. This is a regularly G-graded algebra with commutation function $\theta^{|G|} \equiv 1$. We now use the associativity of the envelope operation and part (1) to conclude that $Id_G(\tilde{A}\widehat{\otimes}(A\widehat{\otimes}B)) = Id_G(B)$.

(3) This follows immediately from the previous lemma. □

3. Main theorem

Our main objective in this section is to prove Theorem 7. The first step is to translate the definition of "regular grading" into the language of graded polynomial identities.

Lemma 28. *Let A be an algebra over $\mathbb{F}$, G a finite group and $A = \bigoplus_{g\in G} A_g$ a G-grading on A. Then the grading is regular if and only if the following conditions hold.*

(1) *There are no monomials with distinct indeterminates in $Id_G(A)$.*
(2) *There is a function θ from pairs $(\bar{g}, \sigma)$, where $\bar{g} \in G^n$ and $\sigma \in Sym(\bar{g})$, such that $x_{g_1,1} \cdots x_{g_n,n} - \theta_{(\bar{g},\sigma)} x_{g_{\sigma(1)},\sigma(1)} \cdots x_{g_{\sigma(n)},\sigma(n)} \in Id_G(A)$ (binomial identity).*

Proof. The proof is clear. Indeed, condition (1) (resp. (2)) of the lemma is equivalent to the first (resp. second) condition in the definition of a regular grading. □

As mentioned above, the conditions in Lemma 20 correspond to the properties of the T-ideal $Id_G(A)$, where $(1), (2), (3)$ correspond to closure under multiplication, closure under endomorphisms and closure under addition respectively. Here is the precise statement.

Proposition 29. *Let θ be a commutation function on a finite group G. Let $\mathbb{F}\langle X_G\rangle$ be the graded free algebra over $\mathbb{F}$ on the set X_G, where $X_G = \{x_{g,i} : g \in G, i \in \mathbb{N}\}$ is a set of noncommuting variables. For $\bar{g} \in G^n$, $\sigma \in Sym(\bar{g})$ and $\bar{i} \in \mathbb{N}^n$ we write $s(\bar{g}, \sigma, \bar{i}) = x_{g_1,i_1} \cdots x_{g_n,i_n} - \theta(\bar{g}, \sigma) x_{g_{\sigma(1)},i_{\sigma(1)}} \cdots x_{g_{\sigma(n)},i_{\sigma(n)}}$. Finally, we let I be the $\mathbb{F}$-subspace spanned by*

$$S = \{s(\bar{g}, \sigma, \bar{i}) \mid \bar{g} = (g_1, \ldots, g_n) \in G^n,\ \sigma \in Sym(\bar{g}),\ i_1, \ldots, i_n \in \mathbb{N}\}.$$

Then the following hold.

(1) *The vector space I is a T-ideal.*
(2) *The G-grading on ${}^{\mathbb{F}\langle X_G\rangle}/_{I}$ is regular with commutation function θ.*

In particular, any G-commutation function is a commutation function of some regular algebra.

Proof. The proof is based on translating the conditions of Lemma 20 into the language of T-ideals. We give here only an outline of the proof and leave the details to the reader.

(1) By definition I is closed under addition. To see I is closed under the multiplication of arbitrary polynomials, it is sufficient to show that it is closed under multiplication by $x_{g,j}$ for any $g \in G$ and $j \in \mathbb{N}$, which is exactly condition (1) in Lemma 20.

    Next we show the ideal I is closed under endomorphisms. Notice that if $s \in S$ is multilinear and $\varphi \in End(\mathbb{F}\langle X_G\rangle)$, one can decompose $\varphi = \varphi_1 \circ \varphi_2$ such that φ_2 sends each x_{g_j,i_j} to a sum of multilinear monomials, all disjoint from each other, and φ_1 sends each $x_{g,j}$ to some $x_{g',j'}$. It now follows from condition (2) in Lemma 20 that $\varphi_2(s) = \sum s_l$ for some $s_l \in S$ multilinear and that $\varphi_1(S) \subseteq S$. This completes the proof.
(2) The algebra ${}^{\mathbb{F}\langle X_G\rangle}/_{I}$ has a natural G-grading, and by its definition it satisfies condition (2) in the definition of a regular grading. Therefore, we only need

to show that it has no monomial identities with distinct indeterminates. Condition (3) in Lemma 20 translates into the equation

$$s(\bar{g},\sigma,\bar{i})+\theta(\bar{g},\sigma)s(\bar{g}^{\sigma},\tau,\bar{i}^{\sigma})=s(\bar{g},\sigma\tau,\bar{i}),\qquad \bar{i}^{\sigma}=(i_{\sigma(1)},\dots,i_{\sigma(n)})$$

for any $\bar{g}\in G^n$, $\bar{i}\in\mathbb{N}^n$, $\sigma\in Sym(\bar{g})$ and $\tau\in Sym(\bar{g}^{\sigma})$. For each $\bar{g}=(g_1,\dots,g_n)\in G^n$ and $\bar{i}\in\mathbb{N}^n$, we define

$$\begin{aligned} S(\bar{g},\bar{i}) &= \{s(\bar{g}^{\sigma},\tau,\bar{i}^{\sigma})\ \mid\ \sigma\in Sym(\bar{g}),\ \tau\in Sym(\bar{g}^{\alpha})\}\\ V(\bar{g},\bar{i}) &= span\,\{S(\bar{g},\bar{i})\}=span\,\{s(\bar{g},\sigma,\bar{i})\ \mid\ e\neq\sigma\in Sym(\bar{g})\}\,. \end{aligned}$$

It is easy to see that if I contains a monomial $x_{g_1,i_1}\cdots x_{g_n,i_n}$ with distinct indeterminates, then it must be in $V=V((g_1,\dots,g_n),(i_1,\dots,i_n))$. The term $\prod x_{g_{\sigma(j)},i_{\sigma(j)}}$ for $e\neq\sigma\in Sym(\bar{g})$ appears only in $s(\bar{g},\sigma,\bar{i})$, so we see that V does not contain monomials and we are done.

□

Definition 30. Let θ be a G-commutation function. The algebra $\mathbb{F}\langle X_G\rangle/I$ defined in the previous proposition is called the *θ-relatively free algebra.*

Let A be a G-graded algebra. Let $\pi:G\to\bar{G}$ be a surjective homomorphism and let $A=\bigoplus_{\bar{g}\in\bar{G}}A_{\bar{g}}$ be the induced grading on A by $\bar{G}$ (that is, $A_{\bar{g}}=\bigoplus_{\pi(g)=\bar{g}}A_g$). Clearly, for a multilinear polynomial f we have $f(x_{\bar{g}_1,1},\dots,x_{\bar{g}_n,n})\in Id_{\bar{G}}(A)$ if and only if $f(x_{g_1,1},\dots,x_{g_n,n})\in Id_G(A)$ for every $g_i\in G$ with $\pi(g_i)=\bar{g}_i$ and so, in the particular case where $\pi:G\to\{e\}$, we obtain the aforementioned fact that algebras which are G-graded PI-equivalent, are also PI-equivalent. This simple but important fact will enable us to replace the algebra A by a more tractable G-graded algebra B (satisfying the same G-graded identities as A) from which it will be easier to deduce the invariance of the order of the group which provides a nondegenerate regular grading on A.

For the rest of this section, unless stated otherwise, we assume that $\mathbb{F}$ is algebraically closed and $char(\mathbb{F})=0$.

Lemma 31. *Let A,B be two regularly G-graded algebras with commutation functions θ_A and θ_B respectively. If $\theta_A=\theta_B$, then $Id_G(A)=Id_G(B)$. In particular, $Id(A)=Id(B)$.*

Proof. Clearly, it is sufficient to consider multilinear polynomials. Let $\bar{g}=(g_1,\dots,g_n)\in G^n$. Applying binomial G-graded identities of A (see Lemma 28), a polynomial $f(x_{g_1,1},\dots,x_{g_n,n})=\sum_{\sigma\in Sym(\bar{g})}\lambda_\sigma\prod_i x_{g_{\sigma(i)},\sigma(i)}$ is a G-graded identity of A if and only if $f'(x_{g_1,1},\dots,x_{g_n,n})=\left(\sum_{\sigma\in Sym(\bar{g})}\lambda_\sigma\theta^{-1}_{(\bar{g},\sigma)}\right)\prod_i x_{g_i,i}$ is a G-graded identity as well. But since there are no monomials with distinct variables in $Id_G(A)$, $f'\in Id_G(A)$ if and only if $\sum_{\sigma\in Sym(\bar{g})}\lambda_\sigma\theta^{-1}_{(\bar{g},\sigma)}=0$. Thus, the statement

$$f(x_{g_1,1},\dots,x_{g_n,n})\in Id_G(A)$$

is equivalent to a condition on the commutation function θ_A and the result follows.

□

As mentioned above, we wish to replace any regularly G-graded algebra A with commutation function θ_A by a better understood regularly G-graded algebra B with commutation function $\theta_B=\theta_A$.

We first deal with the case where $\theta_{g,g}=1$ for all $g\in G$ (we remind the reader that in general $\theta_{g,g}=\pm1$ for all $g\in G$). Here the algebra B will be isomorphic

to a suitable twisted group algebra $B = \mathbb{F}^\alpha G$, where α is a 2-cocycle on G with values in $\mathbb{F}^\times$. Recall that $B = \mathbb{F}^\alpha G$ is isomorphic to the group algebra $\mathbb{F}G$ as an $\mathbb{F}$-vector space and if $\{U_g : g \in G\}$ is an $\mathbb{F}$-basis of $\mathbb{F}^\alpha G$, then the multiplication is defined by the rule $U_g U_h = \alpha(g,h)U_{gh}$ for every $g, h \in G$. It is well known that up to a G-graded isomorphism, the twisted group algebra $\mathbb{F}^\alpha G$ depends only on the cohomology class of $\bar{\alpha} \in H^2(G, \mathbb{F}^\times)$ and not on the representative α. In order to construct the 2-cocycle $\alpha = \alpha_\theta$, we show that the commutation function $\theta = \theta_A$ (with $\theta_{g,g} = 1$, for all $g \in G$) determines uniquely an element in $Hom(M(G), \mathbb{F}^\times)$, where $M(G)$ denotes the Schur multiplier of the group G. Then applying the Universal Coefficient Theorem, we obtain an element in $H^2(G, \mathbb{F}^\times)$ which by abuse of notation we denote again by θ.

Here is the precise statement and its proof.

Lemma 32. *Let θ be a G-commutation function such that $\theta_{g,g} = 1$ for all $g \in G$. Then there is a 2-cocycle $\alpha \in Z^2(G, \mathbb{F}^\times)$ such that the commutation function of $B = \mathbb{F}^\alpha G$ is θ.*

Proof. The next construction follows the one in [1, Prop. 1].

Recall that from the Universal Coefficient Theorem we get that for any group G we have an exact sequence

$$1 \longrightarrow Ext^1(G_{ab}, \mathbb{F}^\times) \longrightarrow H^2(G, \mathbb{F}^\times) \xrightarrow{\pi} Hom(M(G), \mathbb{F}^\times) \longrightarrow 1\ ,$$

where $M(G)$ is the Schur multiplier of G. Note that since $\mathbb{F}$ is assumed to be algebraically closed, we have that $Ext^1(G_{ab}, \mathbb{F}^\times) = 0$, and hence in that case, the map π is an isomorphism. Thus, our task is to find a suitable element in $Hom(M(G), \mathbb{F}^\times)$ and then show that its inverse image in $H^2(G, \mathbb{F}^\times)$ satisfies the required property.

To start with, we fix a presentation of $M(G)$ via the Hopf formula: Let F be the free group $F = \langle y_g \mid g \in G\rangle$ and define $\varphi : F \to G$ by $\varphi(y_g) = g$. Setting $R = \ker(\varphi)$ we have the exact sequence

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\varphi} G \longrightarrow 1\ .$$

The Schur multiplier is then isomorphic to $R\cap[F,F]/[F,R]$.

Next we show how an element $\alpha \in Z^2(G, \mathbb{F}^\times)$ determines a map $\pi([\alpha])$ on $R\cap[F,F]/[F,R]$. Let $\bar{g} = (g_1, \ldots, g_n) \in G^n$ and $\sigma \in Sym(\bar{g})$. Then $y_{g_1} \cdots y_{g_n} y_{g_{\sigma(n)}}^{-1} \cdots y_{g_{\sigma(1)}}^{-1} \in R \cap [F,F]$, and hence by the Hopf formula it determines an element in $M(G)$. On the other hand, from [1] we know that any element in $M(G)$ has a presentation $y_{g_1} \cdots y_{g_n} y_{g_{\sigma(n)}}^{-1} \cdots y_{g_{\sigma(1)}}^{-1} [F,R]$ for some $\bar{g} \in G^n$ and $\sigma \in Sym(\bar{g})$, and moreover the map

$$\pi\left([\alpha]\right)\left(y_{g_1} \cdots y_{g_n} y_{g_{\sigma(n)}}^{-1} \cdots y_{g_{\sigma(1)}}^{-1} [R,F]\right) = \frac{\alpha(g_1, \ldots, g_n)}{\alpha(g_{\sigma(1)}, \ldots, g_{\sigma(n)})}$$

is a well defined homomorphism.

Our next step is to show that $\psi(y_{g_1} \cdots y_{g_n} y_{g_{\sigma(n)}}^{-1} \cdots y_{g_{\sigma(1)}}^{-1} [R,F]) = \theta(\bar{g}, \sigma)$ is a well defined homomorphism. This will complete the proof of the lemma.

Let $A = \mathbb{F}\langle X_G\rangle/I$ be the θ-relatively free algebra defined in Proposition 29. Then A is regularly G-graded with commutation function θ. If we can find elements $a_g \in A_g$, $g \in G$, which are invertible, then we can define a group homomorphism $\tilde{\psi} : F \to A^\times$ induced by the map $y_g \mapsto a_g$. Notice that the image of any commutator in $[R,F]$ is mapped to 1 (because R is mapped to A_e which is in the center) while

$y_{g_1}\cdots y_{g_n}y^{-1}_{g_{\sigma(n)}}\cdots y^{-1}_{g_{\sigma(1)}}$ is mapped to $\theta(\bar{g},\sigma)1$. The induced map $\varphi : M(G)\to\mathbb{F}^\times$ is the required map. In general A might not have such invertible elements, so we need to construct new elements.

Let $S=\left\{x_{g,1}x_{g^{-1},2}\right\}_{g\in G}\subseteq\mathbb{F}\langle X_G\rangle$. Note that it is sufficient to show that the elements in S represent nonzero divisors in A since in that case, the localized algebra $A'=AS^{-1}$ will still be regularly G-graded with commutation function θ, and in addition each $x_{g,1}$ will be invertible (notice that $x_{g^{-1},2}x_{g,1}=\theta(g^{-1},g)x_{g,1}x_{g^{-1},2}$ and $\theta(g^{-1},g)=\theta(g,g)^{-1}=1$, so $x_{g,1}$ is right and left invertible).

Suppose that there is some $0\neq f\in A$ such that $x_{g,1}x_{g^{-1},2}\cdot f\equiv 0$. We can assume that f is homogeneous (i.e. its monomials have the same G-homogeneous degree), and by standard methods (since the field is infinite) we can assume that every variable $x_{h,i}$ appears with the same total degree in each monomial of $x_{g,1}x_{g^{-1},2}\cdot f$ and therefore this is true also in f. Finally, using the binomial identities we can assume that f is a monomial. Now, by assumption $x_{g,1}x_{g^{-1},2}\cdot f$ cannot be a monomial with different variables, so we need to show there is no general monomial identities (i.e. with possibly repeated variables).

Let $a_1x_{g,i}a_2x_{g,i}a_3\cdots a_nx_{g,i}a_{n+1}\in Id_G(A)$ be a monomial identity where $x_{g,i}$ does not appear in the monomials a_i. Using linearization we get that the polynomial

$$f=\sum_{\sigma\in S_n}a_1z_{g,\sigma(1)}a_2z_{g,\sigma(2)}a_3\cdots a_nz_{g,\sigma(n)}a_{n+1}$$

is an identity as well. We claim that the monomials in f are equal modulo identities of A. In order to see this suppose that a,b,c are monomials and denote by h the degree of a. Let g be some element in G. Then

$$\begin{aligned}\left(y_{(hg)^{-1}}ax_{g,1}\right)bx_{g,2}c&\equiv b\left(y_{(hg)^{-1}}ax_{g,1}\right)x_{g,2}c\\&\equiv b\left(y_{(hg)^{-1}}ax_{g,2}\right)x_{g,1}c\equiv\left(y_{(hg)^{-1}}ax_{g,2}\right)bx_{g,1}c\end{aligned}$$

where the middle equation is true since $\theta_{g,g}=1$, and the first and third equalities are true because monomials of degree e are in the center. We therefore have $ax_{g,1}bx_{g,2}c\equiv ax_{g,2}bx_{g,1}c$. Applying this equivalence we have that

$$\sum_{\sigma\in S_n}a_1y_{g,\sigma(1)}a_2y_{g,\sigma(2)}a_3\cdots a_ny_{g,\sigma(n)}a_{n+1}\equiv n!a_1y_{g,1}a_2y_{g,2}a_3\cdots a_ny_{g,n}a_{n+1}.$$

Finally, we see that if we repeat this process for every pair $g\in G,\ i\in\mathbb{N}$ such that $x_{g,i}$ has total degree greater than 1 in our monomial identity, we obtain a monomial identity with distinct variables, a contradiction. □

Suppose that A has a nondegenerate regular grading with commutation function θ such that $\theta(g,g)=1$ for all $g\in G$. Let $B=\mathbb{F}^\alpha G$ as constructed in the last lemma. Clearly, the twisted group algebra B is regularly G-graded and the commutation function is θ. Invoking Lemma 31 we have the following corollary.

Corollary 33. *$Id_G(B)=Id_G(A)$. Consequently $Id(B)=Id(A)$. In particular $exp(B)=exp(A)$.*

Our goal is to extract the cardinality of G from $Id(B)$.

By Maschke's theorem, we know that any twisted group algebra $B=\mathbb{F}^\alpha G$ is a direct sum of matrix algebras. We wish to show that the commutation function θ is nondegenerate if and only if B is simple or equivalently $\dim(Z(B))=1$. It is easily seen that the center $Z(B)$ is spanned by elements of the form $\sum_{\sigma\in G}\lambda_\sigma U_{\sigma g\sigma^{-1}}$ where

$g \in G$ and $\lambda_\sigma \in \mathbb{F}$. We call a conjugacy class that contributes a nonzero central element a *ray class*. The determination of the ray classes and their corresponding central elements is well known (for example see [10], Section 2). The next lemma gives the condition for a conjugacy classes to be a ray class, and Lemma 38 will generalize this idea to the $\mathbb{Z}/2\mathbb{Z}$-simple case.

Lemma 34. *Let $g \in G$ and choose some set of left coset representatives $\{t_i\}_1^k$ of $C_G(g)$ in G. For any 2-cocycle $\alpha \in Z^2(G, \mathbb{F}^\times)$ the following conditions are equivalent:*

(1) *For every $h \in C_G(g)$ we have $U_g U_h = U_h U_g$ in $\mathbb{F}^\alpha G$.*
(2) *The element $a = \sum_{i=1}^k U_{t_i} U_g U_{t_i}^{-1}$ is central in $\mathbb{F}^\alpha G$.*

In addition, if there are $\lambda_i \in \mathbb{F}, i = 1, \dots, k$, not all zero, such that $b = \sum_{i=1}^k \lambda_i U_{t_i} U_g U_{t_i}^{-1}$ is central in $\mathbb{F}^\alpha G$, then $\lambda_i = \lambda_1$ for all i. In particular we get that $a = \frac{1}{\lambda_1} b$ is central in $\mathbb{F}^\alpha G$.

Proof. Suppose first that (1) holds. Let $w \in G$. Then for every $i \in \{1, \dots, k\}$, there are $\tau(i) \in \{1, \dots, k\}$, $h_i \in C_G(g)$ and $c_i \in \mathbb{F}^\times$ such that $U_w U_{t_i} = c_i U_{t_{\tau(i)}} U_{h_i}$. Note that $\tau = \tau_w$ is a permutation of $\{1, \dots, k\}$. Then we have that

$$\begin{aligned} U_w a U_w^{-1} &= \sum (U_w U_{t_i}) U_g (U_w U_{t_i})^{-1} = \sum \left(c_i U_{t_{\tau(i)}} U_{h_i}\right) U_g \left(c_i U_{t_{\tau(i)}} U_{h_i}\right)^{-1} \\ &= \sum U_{t_{\tau(i)}} \left(U_{h_i} U_g U_{h_i}^{-1}\right) U_{t_{\tau(i)}}^{-1} = \sum U_{t_{\tau(i)}} U_g U_{t_{\tau(i)}}^{-1} = a, \end{aligned}$$

and so $U_w a = a U_w$. Since the set $\{U_w : w \in G\}$ spans $\mathbb{F}^\alpha G$, we get that a is central.

On the other hand, if a is central and $h \in C_G(g)$, then there is some $c \in \mathbb{F}^\times$ such that $U_h U_g U_h^{-1} = c U_g$. Assume that $t_1 \in C_G(g)$, so there is $c' \in \mathbb{F}^\times$ such that $U_{t_1} U_g U_{t_1}^{-1} = c' U_g$. Thus we have

$$\begin{aligned} a &= \sum_{i=1}^k U_{t_i} U_g U_{t_i}^{-1} = c' U_g + \sum_{i=2}^k U_{t_i} U_g U_{t_i}^{-1}, \\ a &= U_h a U_h^{-1} = c' U_h U_g U_h^{-1} + \sum_{i=2}^k (U_h U_{t_i}) U_g (U_h U_{t_i})^{-1} \\ &= c \cdot c' U_g + \sum_{i=2}^k U_{ht_i} U_g U_{ht_i}^{-1}, \end{aligned}$$

so we must have that $c = 1$ and we get that (2) $\Rightarrow$ (1).

Assume that $b = \sum \lambda_i U_{t_i} U_g U_{t_i}^{-1}$ is central for some $\lambda_i \in \mathbb{F}$ not all zero. For any $j \in \{1, \dots, k\}$ we get that

$$b = U_{t_j}^{-1} b U_{t_j} = \lambda_j U_g + \sum_{i \neq j} \lambda_i \left(U_{t_j}^{-1} U_{t_i}\right) U_g \left(U_{t_i}^{-1} U_{t_j}\right),$$

so we must have $\lambda_j = \lambda_1$. In particular $\lambda_1 \neq 0$, so $b = \lambda_1 a$. □

By the last lemma, each ray class contributes only one central element up to a scalar multiplication, which we call a *ray element*. In addition, ray elements from different ray classes are linearly independent. Thus we get that $\dim(Z(B))$ is the number of ray classes.

Lemma 35. *Let $B = \mathbb{F}^\alpha G$, where G is a finite group of order n. Then B is simple if and only if α is nondegenerate. Furthermore, in that case we have $B \cong M_{\sqrt{n}}(\mathbb{F})$.*

Proof. As we remarked before the last lemma, B is simple if and only if $\dim(Z(B)) = 1$. Since $Z(B)$ is spanned by ray elements, then B is simple if and only if there is only one ray class (which is $\{e\}$). By the previous lemma this holds if and only if the cocycle α is nondegenerate.

Finally, we note that if B is simple, then $B \cong M_k(\mathbb{F})$ and $k^2 = \dim(B) = |G| = n$. $\square$

We can now complete the proof of Theorem 7 in case the commutation function satisfies $\theta_{g,g} = 1$ for every $g \in G$. Indeed, in [11] Regev showed that $\exp(M_k(\mathbb{F})) = \dim(M_k(\mathbb{F})) = k^2$, and since the exponent of an algebra depends only on its ideal of identities, we have from Lemma 31 that if A has a regular G-grading such that $\theta_{g,g} = 1$ for all $g \in G$, then $|G|$ is an invariant of A (as an algebra and independent of the grading).

Corollary 36. *Let A be an algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero. If G is a finite group such that A has a nondegenerate regular G-grading with $\theta_{g,g} = 1$ for all $g \in G$, then $|G| = \exp(A)$.*

We move on to the general case where $\theta_{g,g}$ can be -1. Let $H = \{g \in G \mid \theta_{g,g} = 1\}$. We are to show that H is a subgroup of G of index 1 or 2. Then, if the index is 1, we are in the previous case where $\theta_{g,g} = 1$ for all $g \in G$, whereas in the second case, we will find a twisted group algebra for the group G such that its Grassmann envelope will be PI-equivalent to A.

Let $E = E_1 \oplus E_{-1}$ be the infinite dimensional Grassmann algebra over the field $\mathbb{F}$, where E_1 and E_{-1} are the even and odd components of E. As noted above, this grading on E is a regular C_2-grading with commutation function $\tau : C_2 \times C_2 \to \mathbb{F}^\times$ determined by $\tau(1,1) = \tau(1,-1) = \tau(-1,1) = 1$ and $\tau(-1,-1) = -1$.

Lemma 37. *Let θ be a G-commutation function. Then there is a 2-cocycle $\alpha \in Z^2(G, \mathbb{F}^\times)$ and a subgroup $H \leq G$ such that for $B = \mathbb{F}^\alpha G$, $B_1 = \bigoplus_{h \in H} B_h$, $B_{-1} = \bigoplus_{g \notin H} B_g$, the Grassmann envelope $\tilde{B} = (B_1 \otimes E_1) \oplus (B_{-1} \otimes E_{-1})$ is a regularly G-graded algebra with commutation function θ.*

Proof. Let $\psi : G \to \{\pm 1\}$ be the map $\psi(g) = \theta_{g,g}$. We claim that ψ is a homomorphism. To see this, let A be the θ-relatively free algebra (see Proposition 29). For $h, g \in G$, let $\theta_{gh,gh}$ be the (unique) scalar such that $x_{g,1} y_{h,1} x_{g,2} y_{h,2} = \theta_{gh,gh} x_{g,2} y_{h,2} x_{g,1} y_{h,1}$ in A (we use $y_{h,*}$ instead of $x_{h,*}$ for clarity). From Remark 23 we see that monomials of total degree e are central, and hence we have

$$\begin{aligned}
\left(w_{g^{-1}} x_{g,1}\right) y_{h,1} x_{g,2} \left(y_{h,2} z_{h^{-1}}\right) &= y_{h,1} \left(y_{h,2} z_{h^{-1}}\right) \left(w_{g^{-1}} x_{g,1}\right) x_{g,2} \\
&= \theta_{h,h} \theta_{g,g} \left(y_{h,2} y_{h,1}\right) z_{h^{-1}} w_{g^{-1}} \left(x_{g,2} x_{g,1}\right) \\
&= \theta_{h,h} \theta_{g,g} y_{h,2} \left(y_{h,1} z_{h^{-1}}\right) \left(w_{g^{-1}} x_{g,2}\right) x_{g,1} \\
&= \theta_{h,h} \theta_{g,g} \left(w_{g^{-1}} x_{g,2}\right) y_{h,2} x_{g,1} \left(y_{h,1} z_{h^{-1}}\right), \\
w_{g^{-1}} \left(x_{g,1} y_{h,1}\right) \left(x_{g,2} y_{h,2}\right) z_{h^{-1}} &= \theta_{gh,gh} w_{g^{-1}} \left(x_{g,2} y_{h,2}\right) \left(x_{g,1} y_{h,1}\right) z_{h^{-1}}.
\end{aligned}$$

It follows that $\theta_{gh,gh} = \theta_{g,g} \theta_{h,h}$, and hence letting $H = \ker(\psi)$ we have either $H = G$ or $[G : H] = 2$. The case where $H = G$ is the case considered above, so we can assume that $H \neq G$. In this case, roughly speaking, we apply first the Grassmann envelope operation to "turn the -1's (in the image of $\theta_{g,g}$) into $+1$'s", then use the previous case to find some $B = \mathbb{F}^\alpha G$, and finally apply the Grassmann envelope operation once again in order to return to the original identities.

To start with, we consider the group $\tilde{G} = \{(g, \psi(g)) \mid g \in G\} \leq G \times \mathbb{Z}/2\mathbb{Z}$ which is clearly isomorphic to G. Then we define on G a new commutation function η by $\eta_{((g_1,\dots,g_n),\sigma)} = \theta_{((g_1,\dots,g_n),\sigma)}\tau_{((\psi(g_1),\dots,\psi(g_n)),\sigma)}$ where τ is the commutation function of the Grassmann algebra. The function η satisfies $\eta_{g,g} = \theta_{g,g}\tau_{\psi(g),\psi(g)} = 1$ by the definition of ψ. We may apply now the case considered above (that is, when $\theta_{g,g} = 1$, for all $g \in G$) and obtain a suitable twisted group algebra $B = \mathbb{F}^\alpha G$, $\alpha \in Z^2(G, \mathbb{F}^\times)$, with commutation function η. We now apply the Grassmann envelope operation once again. Let $\tilde{B}$ be the $\tilde{G}$ graded algebra $\tilde{B} = (B \otimes E)_{\tilde{G}} = (B_H \otimes E_0) \oplus (B_{G\setminus H} \otimes E_1)$, which is the Grassmann envelope of $B = B_H \oplus B_{G\setminus H}$. Then $\tilde{B}$ is a regularly $\tilde{G} \cong G$-graded algebra (since it is a subalgebra of the regularly $G \times C_2$-graded algebra $B \otimes E$). We claim the commutation function $\tilde{\eta}$ of $\tilde{B}$ equals θ. Indeed,

$$\tilde{\eta}_{((g_1,\dots,g_n),\sigma)} = \eta_{((g_1,\dots,g_n),\sigma)}\tau_{((\psi(g_1),\dots,\psi(g_n)),\sigma)} = \theta_{((g_1,\dots,g_n),\sigma)},$$

and so $\tilde{B}$ is the required envelope. □

Let us pause for a moment and summarize what we have so far. By the previous lemma we have constructed an algebra $\tilde{B}$ which has a regular G-grading whose commutation function coincides with a given commutation function θ, and hence, if θ is the commutation function of a regularly G-graded algebra A, we have in fact constructed a regularly G-graded algebra $\tilde{B}$ with the same commutation function. It follows from Lemma 31 that $Id_G(A) = Id_G(\tilde{B})$, $Id(A) = Id(\tilde{B})$ and hence $\exp(A) = \exp(\tilde{B})$. The main point for constructing the algebra $\tilde{B}$ is that in case the grading is nondegenerate, it enables us to show that $ord(G) = \exp(\tilde{B})$. For this we need to further analyze the algebra $\tilde{B}$ (constructed in Lemma 37).

Note that the algebras B and $\tilde{B}$ in Lemma 37 satisfy $\theta_{g_1,g_2} = \tau_{\psi(g_1),\psi(g_2)}\tilde{\theta}_{g_1,g_2}$. In particular if $h \in H$ and $g \in C_G(h)$, then $\theta_{g,h} = \tilde{\theta}_{g,h}$. Since the grading on $\tilde{B}$ is nondegenerate, then for every $h \in H$ there is some $g \in C_G(h)$ with $\theta_{h,g} \neq 1$, and from what we just said, this is also true for B.

Lemma 38. *Let G be a finite group and H a subgroup of index 2. Let $B = \mathbb{F}^\alpha G$ be a twisted group algebra such that for every $e \neq h \in H$ there is some $g \in C_G(h)$ such that $U_h U_g \neq U_g U_h$. Then the induced $\mathbb{Z}_2 \cong G/H$ grading on B is $\mathbb{Z}_2$-simple.*

Proof. Suppose first that the twisted group algebra $\mathbb{F}^\beta H$, where $\beta = \alpha\mid_H$, is simple. Let $0 \neq I$ be a $\mathbb{Z}_2$-graded ideal of B and denote $I_0 = I \cap B_H$ and $I_1 = I \cap B_{G\setminus H}$, so $I = I_0 \oplus I_1$. Observe that since I_0 is an ideal of $\mathbb{F}^\beta H$, it is either 0 or $\mathbb{F}^\beta H$. On the other hand, taking any U_g where $g \notin H$, we have $U_g \cdot I_0 \subseteq I_1$, and since U_g is invertible in $\mathbb{F}^\alpha G$ we have equality. It follows that $I_0 = \mathbb{F}^\beta G$, for otherwise $I = 0$. We now have

$$\dim(I) = \dim(I_0) + \dim(I_1) = 2\dim(I_0) = 2\,|H| = |G| = \dim(B),$$

so we see that $I = B$. This proves that B is $\mathbb{Z}_2$-simple in that case.

If B is simple, then it must also be $\mathbb{Z}_2$-simple, so assume that neither B nor $\mathbb{F}^\beta H$ are simple or equivalently both α and β are degenerate 2-cocycles. This means that there is $h_0 \in H$ such that $U_{h_0}U_h = U_h U_{h_0}$ for all $h \in C_H(h_0)$, and similarly there is $g_0 \in G$ such that $U_g U_{g_0} = U_{g_0} U_g$ for all $g \in C_G(g_0)$. Note that by the assumption on B we must have $g_0 \notin H$. Let $\{t_i\}, \{s_i\}$ be left coset representatives of $C_H(h_0)$ and $C_G(g_0)$ respectively. By Lemma 34 we have $a = \sum U_{t_i} U_{h_0} U_{t_i}^{-1} \in Z(\mathbb{F}^\beta H)$ and

$b = \sum U_{s_i} U_{g_0} U_{s_i}^{-1} \in Z(\mathbb{F}^\alpha G)$. If $s_i \notin H$ then $s_i g_0 \in H$ is a representative of the same left coset of $C_G(g_0)$ as s_i, so we may assume that $s_i \in H$ for all i.

By the assumption on B, there is some $g_1 \in C_G(h_0)$ such that $U_{g_1} U_{h_0} U_{g_1}^{-1} = cU_{h_0}$ with $c \neq 1$, and in particular $g_1 \notin H$ by the choice of h_0. It is easily seen that $\{g_1 t_i g_1^{-1}\}$ is again a set of left coset representatives of $C_H(h_0)$ in H (H is normal in G and $g_1 \in C_G(h_0)$). We now have that

$$U_{g_1} a U_{g_1}^{-1} = \sum U_{g_1 t_i g_1^{-1}} c U_{h_0} U_{g_1 t_i g_1}^{-1} = ca.$$

Let $h \in H$ be such that $hg_1 = g_0$. Then

$$\begin{aligned} ab = ba &= \sum U_{s_i} U_{g_0} a U_{s_i}^{-1} = \alpha(h, g_1)^{-1} \sum U_{s_i} U_h U_{g_1} a U_{s_i}^{-1} \\ &= \alpha(h, g_1)^{-1} ca \sum U_{s_i} U_h U_{g_1} U_{s_i}^{-1} = ca \sum U_{s_i} U_{g_0} U_{s_i}^{-1} = cab, \end{aligned}$$

and we get a contradiction. Thus, we must have that either $\mathbb{F}^\alpha G$ or $\mathbb{F}^\beta H$ are simple. In both cases the algebra $\mathbb{F}^\alpha G$ is $\mathbb{Z}_2 \cong {}^G/_H$-simple and the lemma is proved. □

Lemma 39. *Let G be a finite group and H a subgroup of index 2. Let $B = \mathbb{F}^\alpha G$ and let $\tilde{B} = (E_1 \otimes B_H) \oplus \left(E_{-1} \otimes B_{G \setminus H}\right)$ be the Grassmann envelope of B. We denote by θ and $\tilde{\theta}$ the commutation functions of B and $\tilde{B}$ respectively. If the regular G-grading on $\tilde{B}$ is nondegenerate, then B is a $\mathbb{Z}_2$-simple algebra.*

Proof. By nondegeneracy of the grading, we have for any $e \neq h \in H$ an element $g \in C_G(h)$ such that $\theta_{g,h} \neq 1$ in $\tilde{B}$. But the Grassmann envelope operation does not change this property, so it holds for the G-graded algebra B. Now use the previous lemma. □

The fact that the algebra $B = \mathbb{F}^\alpha G$ is finite dimensional over $\mathbb{F}$ ($\mathbb{F}$ is algebraically closed of characteristic zero) and $\mathbb{Z}_2$-simple almost determines the structure of B.

Corollary 40. *The algebra $B = \mathbb{F}^\alpha G$ in the last lemma is $\mathbb{Z}_2$-isomorphic to one of the following algebras.*

(1) *$B = M_n(\mathbb{F})$ with the grading $B_1 = B$ and $B_{-1} = 0$.*
(2) *$B = M_n(\mathbb{F})$ with the grading*

$$\begin{aligned} B_1 &= M^1_{(n,m)} = \left\{ \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \mid D_1 \in \mathbb{F}^{m \times m} \; D_2 \in \mathbb{F}^{(n-m)\times(n-m)} \right\}, \\ B_{-1} &= M^{-1}_{(n,m)} = \left\{ \begin{pmatrix} 0 & D_1 \\ D_2 & 0 \end{pmatrix} \mid D_1 \in \mathbb{F}^{m \times (n-m)} \; D_2 \in \mathbb{F}^{(n-m)\times m} \right\}. \end{aligned}$$

(3) *$B = M_n({}^{\mathbb{F}[t]}/_{t^2=1})$ with the grading $B_1 = M_n(\mathbb{F})$ and $B_{-1} = t \cdot M_n(\mathbb{F})$.*

Proof. This is well known. See for instance Lemma 6 in [8]. □

In our case, the algebra B satisfies an additional condition, namely $\dim(B_1) = \dim(B_2) = |H|$, so if B is of the second type above we must have $n = 2m$.

We can now complete the proof of part 1 of Theorem 7.

Corollary 41. *Let A be an algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. For every finite group G, if A has a nondegenerate regularly G-graded structure, then $|G| = \exp(A)$.*

Proof. We know that there is a simple $\mathbb{Z}_2$-graded algebra $B = \mathbb{F}^\alpha G$ (where B is one of the three types mentioned in the corollary above) such that the algebra $\tilde{B} = E(B)$ satisfies $Id_G(\tilde{B}) = Id_G(A)$. In [7] Giambruno and Zaicev computed the exponent of the Grassmann envelope of any $\mathbb{Z}_2$-simple algebras and showed that $\exp(\tilde{B}) = \exp(E(B)) = \dim(B) = |G|$. Because $Id_G(\tilde{B}) = Id_G(A)$ we get that $\exp(A) = \exp(\tilde{B}) = |G|$. □

We close this section with some additional corollaries of Lemmas 37 and 39.

Let us denote the Grassmann envelope of the algebra B in Corollary 40 (types (2) and (3) respectively) as follows:

$$\begin{aligned} M_{2m,m}(E) &= \left[E_1 \otimes M^1_{(2m,m)}(\mathbb{F})\right] \oplus \left[E_{-1} \otimes M^{-1}_{(2m,m)}(\mathbb{F})\right], \\ M_n(E) &= [E_1 \otimes M_n(\mathbb{F})] \oplus [E_{-1} \otimes t \cdot M_n(\mathbb{F})]; \quad t^2 = 1. \end{aligned}$$

Corollary 42. *Suppose that A has a nondegenerate regular G-grading for some finite group G. Then one of the following holds.*

(1) $Id(A) = Id(M_n(\mathbb{F}))$ *for some* $n \in \mathbb{N}$ *and then* $\exp(A) = n^2$.
(2) $Id(A) = Id(M_{2m,m}(E))$ *for some* $m \in \mathbb{N}$ *and then* $\exp(A) = (2m)^2$.
(3) $Id(A) = Id(M_n(E))$ *for some* $n \in \mathbb{N}$ *and then* $\exp(A) = 2n^2$.

It is well known that the families considered in the corollary above are mutually exclusive. Furthermore, different integers n or m yield algebras which are PI-nonequivalent. Indeed, algebras within the same type are PI-nonequivalent as their exponent is different. Next, any algebra of type 1 satisfies a Capelli polynomial, whereas any algebra of type 2 or 3 does not. Finally, the exponent of any algebra of type 2 is an exact square, whereas this is not the case for any algebra of type 3. Thus if we let $\mathfrak{U} = \{M_n(\mathbb{F}) : n \in \mathbb{N}\} \cup \{M_{2m,m}(E) : m \in \mathbb{N}\} \cup \{M_n(E) : n \in \mathbb{N}\}$ we have

Corollary 43. *Suppose that A has a nondegenerate regular G-grading for some finite group G. Then there is a unique algebra $C \in \mathfrak{U}$ such that A and C are PI-equivalent.*

From the results above we can now derive easily a consequence on the commutation matrix M^A for a regularly G-graded algebra A with commutation function θ.

The complete proof of Theorem 7 (parts 2 and 3) is presented in the next section.

From the definition of the commutation matrix (see subsection 1.1.1) we see that $M^A_{g,g} = \theta_{g,g} U_e$. Recall that for nondegenerate regularly G-graded algebras of type 1 we have $\theta_{g,g} = 1$ for all $g \in G$, whereas for type 2 and 3 half of the entries on the diagonal of M^A are U_e and half are $-U_e$. This clearly implies the following corollary.

Corollary 44. *Let A be an $\mathbb{F}$ algebra with a nondegenerate regular G grading and commutation matrix M^A. Then $tr(M^A)$ is an invariant of A and either $tr(M^A) = 0$ or $tr(M^A) = \exp(A)U_e = |G|\, U_e$.*

3.1. **The commutation matrix.** It is easy to exhibit algebras with nonisomorphic nondegenerate regular G-gradings for some group G as well as examples of algebras with minimal regular gradings with nonisomorphic groups. For instance the algebra $M_4(\mathbb{F})$ admits (precisely) two nonisomorphic minimal gradings with the group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \langle g, h \rangle$. These gradings are determined by bicharacters θ_1 and θ_2, where

$\theta_1(g,h) = \zeta_4$ and $\theta_2(g,h) = \zeta_4^3$. Here ζ_4 denotes a primitive 4-th root of unity. On the other hand, the algebra $M_2(\mathbb{F})$ admits a (unique) nondegenerate grading with the Klein 4-group, and hence the algebra $M_4(\mathbb{F}) \cong M_2(\mathbb{F}) \otimes M_2(\mathbb{F})$ admits a nondegenerate regular grading with the group $(\mathbb{Z}/2\mathbb{Z})^4$.

We therefore see that in general the entries of commutation matrices which correspond to different nondegenerate regular gradings on an algebra A may be distinct. However, the last corollary shows that the trace of the commutation matrices remains invariant.

Our goal in this section is to extend Corollary 44 and show that any two such matrices corresponding to nondegenerate gradings are conjugate (Theorem 7).

We will follow the notation from subsection 1.1.1. In particular we have $B = \mathbb{F}^\alpha G$, $H = \ker(g \mapsto \theta_{g,g} = \psi(g))$ (a subgroup of G of index ≤ 2) and A is PI-equivalent to the Grassmann envelope of B with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading $B = B_H \oplus B_{G\setminus H}$.

Before we consider nondegenerate gradings, let us analyze briefly the degenerate case. If G is abelian, the commutation matrix is given by $M^A_{g,h} = \theta(g,h)U_e$. Hence, since the grading is not minimal, there exists $g \neq e$ such that $\theta(g,h) = \theta(h,g) = 1$ for all $h \in G$ and so M^A is not invertible. The next proposition shows that this is true in the nonabelian case as well.

Proposition 45. *Let A be a regularly G-graded algebra with a degenerate grading. Then M^A is not invertible.*

Proof. Let $B = \mathbb{F}^\alpha G$ be the twisted group algebra which corresponds to the G-graded algebra A and denote by θ^B and θ^A the corresponding commutation functions. We note that for commuting elements $g_1, g_2 \in G$ we have $\theta^B_{g_1,g_2} = -\theta^A_{g_1,g_2}$ if $\theta^A_{g_1,g_1} = \theta^A_{g_2,g_2} = -1$ and $\theta^B_{g_1,g_2} = \theta^A_{g_1,g_2}$ otherwise.

Since the field $\mathbb{F}$ is algebraically closed of characteristic zero, B is a direct product of matrix algebras over $\mathbb{F}$. Fix a representation $\rho : B \to M_n(\mathbb{F})$. The grading on A is degenerate, so there is some $e \neq h \in G$ such that $\theta^A_{h,g} = 1$ for all $g \in C_G(h)$ and in particular $\theta^A_{h,h} = 1$. We thus have $\theta^B_{h,g} = 1$ for all $g \in C_G(h)$. As a consequence, applying Lemma 34, the element $z = \sum U_{t_i} U_h U_{t_i}^{-1}$, where $\{t_i\}$ are left coset representatives of $C_G(h)$, is central in B. Note that since H is normal, we get that z is in $\mathbb{F}^\alpha H$.

Let $v \in \prod_{g\in G} B$ (a vector of size $ord(G)$ with entries in B) where $v_g = \lambda_g U_g$ for some $\lambda_g \in \mathbb{F}$, and consider

$$\left(M^A v\right)_g = \sum_{\omega\in G} M^A_{g,\omega} v_\omega = \sum_{\omega\in G} \tau_{g,\omega} U_g U_\omega U_g^{-1} U_\omega^{-1} \lambda_\omega U_\omega = U_g \left[\sum_{\omega\in G} \tau_{g,\omega}\lambda_\omega U_\omega\right] U_g^{-1}.$$

Clearly, we may choose the λ_h's such that $\sum_{h\in G} \tau_{g,h}\lambda_h U_h = \lambda_1 U_e + \lambda_2 z$ for all $g \in G$. This element is central, so we have $\left(M^A v\right)_g = \lambda_1 U_e + \lambda_2 z$. But the center of $M_n(\mathbb{F})$ is $\mathbb{F}\cdot I$, so there is some $c \in \mathbb{F}$ such that $\rho(z) = c \cdot I$ and hence $\rho(\left(M^A v\right)_g) = (\lambda_1 + c\lambda_2)\cdot I$. We see that if we choose λ_1, λ_2 not both zero such that $\lambda_1 + c\lambda_2 = 0$, we have that $\rho(M^A v) = 0$ for some $v \neq 0$. Moreover, we note that the nonzero entries of v are invertible in B.

Let ρ_i be the distinct representations of B and let $e_i \in B$ be such that $\rho_i(e_j) = \delta_{i,j}\cdot I$. For each i let v^i be a vector corresponding to ρ_i as constructed above and let $v = \sum v^i e_i \in \prod_{g\in G} B$. Then $\rho_i(v_g) = \rho_i(\sum v^i_g e_j) = \rho_i(v^i_g)$. Furthermore, taking

$g \in G$ such that $v_g^i \neq 0$, we know that v_g^i is invertible and so $\rho_i(v_g^i) \neq 0$. This implies that $v \neq 0$. On the other hand, we get for each i,

$$\rho_i(M^A v) = \sum_j \rho_i(M^A v^j)\rho_i(e_j) = \rho_i(M^A v^i) = 0$$

and so $M^A v = 0$. We conclude that M^A is not invertible from the left, and similar computations show that it is not invertible from the right. □

Now we consider the case where the grading is nondegenerate.

Proposition 46. *Let A be a nondegenerate regularly G-graded algebra; then $M^A \cdot M^A = |G|\, Id \cdot U_e$.*

Proof. Recall that for any fixed $g \in G$, the function $\theta^A_{(\cdot,g)} : C_G(g) \to \mathbb{F}^\times$ is a character, and since the grading is nondegenerate, this character is nontrivial for $g \neq e$.

For fixed $a, c \in G$, set $N = C_G(a^{-1}c)$ and choose a set of left coset representatives $\{t_i\}_{i=1}^{[G:N]} \subset G$ of N in G. Then

$$\begin{aligned}
M^2_{a,c} &= \sum_{b\in G} M_{a,b}M_{b,c} = \sum_{b\in G} \tau_{a,b}\tau_{b,c}U_aU_bU_a^{-1}U_b^{-1}U_bU_cU_b^{-1}U_c^{-1} \\
&= \sum_{b\in G} \tau_{ac^{-1},b}U_aU_b\left(U_a^{-1}U_c\right)U_b^{-1}U_c^{-1} \\
&= \sum_i U_aU_{t_i}\tau_{ac^{-1},t_i}\left[\sum_{h\in N}\tau_{ac^{-1},h}U_h\left(U_a^{-1}U_c\right)U_h^{-1}\right]U_{t_i}^{-1}U_c^{-1} \\
&= \sum_i U_aU_{t_i}\tau_{ac^{-1},t_i}\left[\sum_{h\in N}\tau_{ac^{-1},h}\theta^B_{h,a^{-1}c}\right]\left(U_a^{-1}U_c\right)U_{t_i}^{-1}U_c^{-1}.
\end{aligned}$$

Notice that since $\psi : G \to \{\pm 1\}$, we have $\tau_{ac^{-1},h} = \tau_{h,ca^{-1}} = \tau_{h,a^{-1}c}$, where $\tau_{g,h} = \tau_{\psi(g),\psi(h)}$ is the commutation function of the Grassmann algebra with the $\mathbb{Z}_2$-grading. In addition, the character $\theta^A(\cdot, a^{-1}c) : N \to \mathbb{F}^\times$ is trivial if and only if $a = c$, and so we get that

$$\sum_{h\in N}\tau_{ac^{-1},h}\theta^B_{h,a^{-1}c} = \sum_{h\in N}\tau_{h,a^{-1}c}\theta^B_{h,a^{-1}c} = \sum_{h\in N}\theta^A_{h,a^{-1}c} = \begin{cases} 0, & a \neq c, \\ |N|, & a = c, \end{cases}$$
$$\Rightarrow M^2 = |G|\, Id \cdot U_e.$$

This completes the proof of the proposition. □

In the next discussion we use the notation of abelian groups, namely $M^A \in M_{|G|}(\mathbb{F})$ with $M^A_{g,h} = \theta(g,h)$. This can be generalized to the nonabelian (i.e. not necessarily abelian) in the following way. We may view $B = \mathbb{F}^\alpha G$ as a direct sum of matrices, and then also $M_{|G|}(\mathbb{F}^\alpha G)$ is isomorphic to a direct sum of matrices. Alternatively, we may factor through a representation $\rho : B \to M_t(\mathbb{F})$ of B and then extend it to $\rho : M_{|G|}(\mathbb{F}^\alpha G) \to M_{|G|t}(\mathbb{F})$. In any case, the matrix M^A can be viewed as a matrix in $M_k(\mathbb{F})$ for some k large enough.

It follows from the last proposition that the commutation matrix M^A satisfies the polynomial $p(x) = x^2 - n = (x - \sqrt{n})(x + \sqrt{n})$ where $n = |G| \neq 0$. Hence, the corresponding minimal polynomial is either $(x - \sqrt{n})$,$(x + \sqrt{n})$ or $p(x)$. In each

case the minimal polynomial has only simple roots and hence the matrix M^A is diagonalizable.

Let α^+ and α^- denote the multiplicities of the eigenvalues $\sqrt{n}$ and $-\sqrt{n}$ respectively. Then we have $\alpha^+ + \alpha^- = n$ and $\alpha^+ - \alpha^- = \frac{tr(M^A)}{\sqrt{n}}$.

In our case, M^A has only 1's on the diagonal (the first type of regular algebras), or half 1's and half -1's (the second and third type of regular algebras). Moreover, by Corollary 44 we know that this depends only on the algebra A and not on the grading.

Thus, for algebras of the first type (in Corollary 42) we have that $n = \exp(A) = |G|$ is a square and $tr(M^A) = n$. In that case, the equalities above take the form

$$\alpha^+ + \alpha^- = n = m^2,$$
$$\alpha^+ - \alpha^- = \frac{tr(M^A)}{\sqrt{n}} = \frac{n}{\sqrt{n}} = m,$$

and hence

$$\alpha^+ = \binom{m+1}{2}, \quad \alpha^- = \binom{m}{2}.$$

For algebras of the second or third type (in Corollary 42) we have $n = \exp(A)$, which is either $2m^2$ or $(2m)^2$ for some m, and $tr(M^A) = 0$. Then, here, the corresponding equalities are

$$\alpha^+ + \alpha^- = n,$$
$$\alpha^+ - \alpha^- = 0,$$

and hence

$$\alpha^+ = \alpha^- = \frac{n}{2}.$$

Corollary 47. *Suppose the algebra A admits nondegenerate regular gradings with groups G and H and let M_G^A and M_H^A be the corresponding commutation matrices. Then the following hold.*

(1) *The matrices M_G^A and M_H^A are conjugate.*
(2) *The characteristic and minimal polynomial of M_G^A (in fact we may write "of M^A") are in $\mathbb{Z}[x]$.*

Proof. (1) By the proposition above we have $\left(M_G^A\right)^2 = \left(M_H^A\right)^2 = \exp(A)I$ and the trace is an invariant of A. Furthermore, the matrices M_G^A and M_H^A are both diagonalizable and have the same eigenvalues (with multiplicities). In particular, M_G^A and M_H^A are conjugate.

(2) If $n = \exp(A)$, then the eigenvalues of M^A are $\pm\sqrt{n}$. In particular, if n is a square, then the minimal and characteristic polynomials of M^A are in $\mathbb{Z}[x]$.

In case $n = 2m^2$ we have $\alpha^+ = \alpha^- = \frac{n}{2} = m^2$, and so the characteristic polynomial is

$$\begin{aligned}\prod_1^{m^2}(x - m\sqrt{2})\prod_1^{m^2}(x + m\sqrt{2}) &= \prod_1^{m^2}\left[(x - m\sqrt{2})\left(x + m\sqrt{2}\right)\right] \\ &= \prod_1^{m^2}\left(x^2 - 2m^2\right) = \left(x^2 - n\right)^{m^2} \in \mathbb{Z}[x],\end{aligned}$$

and the minimal polynomial is $(x - \sqrt{n})(x + \sqrt{n}) = x^2 - n$. □

Finally, an easy computation of the free coefficient of the characteristic polynomial in each one of the cases considered above yields that $\det(M_G^A) = \pm |G|^{|G|/2}$. This proves part 3 of Theorem 7, and hence the entire theorem is now proved.

As promised, we now compute the commutation matrix for the G-regular algebras constructed in Lemma 26, where G is an arbitrary finite group.

In case (2), the algebras A, B and $A \oplus B$ have the same commutation function θ. Thus, the cocycles corresponding to $A, B, A \oplus B$ are isomorphic (up to a coboundary), and therefore the corresponding twisted group algebras of A, B and $A \oplus B$ are isomorphic. With this identification of twisted group algebras we get that the commutation matrices of A, B and $A \oplus B$ are the same.

In case (3) we consider $A_N = \bigoplus_{g \in N} A_g$ for some subgroup N of G. Let α_N be the restriction of α to $N \times N$. Then α_N is the cocycle corresponding to the algebra A_N and there is a natural graded embedding $\mathbb{F}^{\alpha_N} N \hookrightarrow \mathbb{F}^{\alpha} G$. Let M' be the restriction of M^A to the coordinates in $N \times N$. Then the entries of M' are in $\mathbb{F}^{\alpha_N} N$, and this submatrix is actually M^{A_N}.

In cases (1) and (4) we have algebras A, B with commutation functions θ^A, θ^B and cocycles α, β which are defined on groups G and H respectively. In case the groups G and H are abelian, the matrix $M^{A \otimes B}$ is just $M^A \otimes M^B$. For the general case, let $\alpha \otimes \beta \in Z^2(G \times H, \mathbb{F}^\times)$ be the cocycle defined by

$$(\alpha \otimes \beta)((g_1, h_1), (g_2, h_2)) = \alpha(g_1, g_2)\beta(h_1, h_2).$$

Clearly, $\alpha \otimes \beta$ represents the regular algebra $A \otimes B$ (with commutation function $\theta^A \otimes \theta^B$). Furthermore, since $\mathbb{F}^\alpha G \otimes \mathbb{F}^\beta H \cong \mathbb{F}^{\alpha \otimes \beta}(G \times H)$, we can extend this product to a "matrix tensor product". In other words, if $\varphi : \mathbb{F}^\alpha G \otimes \mathbb{F}^\beta H \to \mathbb{F}^{\alpha \otimes \beta}(G \times H)$ is an isomorphism, then $M^{A \otimes B}$ is determined by $M^{A \otimes B}_{(g_1,h_1),(g_2,h_2)} = \varphi(M^A_{g_1,g_2} \otimes M^B_{h_1,h_2})$.

In case (4), a similar computation shows that there is an isomorphism $\psi : \mathbb{F}^\alpha G \widehat{\otimes} \mathbb{F}^\beta G \to \mathbb{F}^{\alpha \cdot \beta} G$ and then $M^{A \widehat{\otimes} B}$ (which is defined over $\mathbb{F}^{\alpha\beta} G$) is determined by $M^{A \widehat{\otimes} B}_{g,h} = \psi(M^A_{g,h} \otimes M^B_{g,h})$. Finally, we note that there is a natural embedding $G \cong \tilde{G} = \{(g, g) \mid g \in G\} \leq G \times G$. Hence we may view $A \widehat{\otimes} B$ as $(A \otimes B)_{\tilde{G}}$, and with this identification $M^{A \widehat{\otimes} B}$ is the restriction of $M^{A \otimes B}$ to $\tilde{G}$.

3.2. **Nondegenerate skew-symmetric bicharacters.** If the group G is abelian, then any G-commutation function θ is defined by the skew-symmetric bicharacter $\theta(g, h) = \theta_{g,h}$ for every $g, h \in G$ (the commutation of two elements).

Our goal in this section is to present a classification of the the pairs (G, ϕ) where G is a (finite) abelian group and ϕ is a nondegenerate skew-symmetric bicharacter defined on G. In fact, this classification is known and can be found in [14]. Nevertheless, for the reader's convenience and completeness of the article, we recall the main results here.

Definition 48. Let θ_1, θ_2 be two bicharacters on G_1, G_2 respectively. We say that θ_1, θ_2 are isomorphic and write $\theta_1 \cong \theta_2$ if there is an isomorphism $\varphi : G_1 \to G_2$ such that $\theta_1(g, h) = \theta_2(\varphi(g), \varphi(h))$.

Definition 49. Let $\theta : G \times G \to \mathbb{F}^\times$ be a skew-symmetric bicharacter. We say that θ is reducible if there are groups $\{e\} \neq H_i$ and bicharacters θ_i on H_i, for $i = 1, 2$, such that $G \cong H_1 \times H_2$ and $\theta \cong \theta_1 \otimes \theta_2$.

In what follows, we present three types of regularly graded algebras. It turns out that the bicharacters which correspond to some special cases of these gradings are irreducible and generate all possible skew-symmetric bicharacters.

Example 50. The standard $\mathbb{Z}/2\mathbb{Z}$-grading on the Grassmann algebra (Example 3).

Example 51. The $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$-grading on $M_n(\mathbb{F})$ defined in Example 16.

Example 52. Let ζ be a primitive root of unity of order $2n$, and define

$$X_\zeta = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \zeta^2 & 0 & & 0 \\ \vdots & 0 & \zeta^4 & \ddots & \vdots \\ 0 & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \zeta^{2n-2} \end{pmatrix},$$

$$Y = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & & 1 & 0 \\ 0 & & & & 0 & 1 \\ 1 & 0 & \cdots & & 0 & 0 \end{pmatrix} \in M_n(\mathbb{F}),$$

$$U = \begin{pmatrix} X_\zeta & 0 \\ 0 & \zeta X_\zeta \end{pmatrix}, \; V = \begin{pmatrix} 0 & I \\ Y & 0 \end{pmatrix} \in M_{2n}(\mathbb{F}).$$

Then

$$\begin{aligned} UV &= \begin{pmatrix} X_\zeta & 0 \\ 0 & \zeta X_\zeta \end{pmatrix} \begin{pmatrix} 0 & I \\ Y & 0 \end{pmatrix} = \begin{pmatrix} 0 & X_\zeta \\ \zeta X_\zeta Y & 0 \end{pmatrix}, \\ VU &= \begin{pmatrix} 0 & I \\ Y & 0 \end{pmatrix} \begin{pmatrix} X_\zeta & 0 \\ 0 & \zeta X_\zeta \end{pmatrix} = \begin{pmatrix} 0 & \zeta X_\zeta \\ Y X_\zeta & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \zeta X_\zeta \\ \zeta^2 X_\zeta Y & 0 \end{pmatrix} = \zeta UV. \end{aligned}$$

Define a $\mathbb{Z}_{2n} \times \mathbb{Z}_{2n}$-grading on $M_{2n,n}(E)$ by $M_{2n,n}(E)_{(k,l)} = U^k V^l \otimes E_{(-1)^l}$, where $E = E_1 \oplus E_{-1}$ is the usual grading on the Grassmann algebra. This induces a minimal regular grading on $M_{2n,n}(E)$ with commutation function θ determined by

$$\theta\left[(1,0),(1,0)\right] = 1 \quad \theta\left[(1,0),(0,1)\right] = \zeta^{-1} \quad \theta\left[(0,1),(0,1)\right] = -1.$$

We consider the bicharacters which correspond to (some special cases of) the gradings just described.

(1) $(\{\pm 1\}, \tau)$: where $\tau(1,1) = \tau(1,-1) = \tau(-1,1) = 1,\ \tau(-1,-1) = -1$.
(2) $(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}, \eta_{p^m})$: where $a = (1,0),\ b = (0,1)$ and $\eta_{p^m}(a,a) = \eta_{p^m}(b,b) = 1,\ \eta_{p^m}(a,b) = \zeta$ for some primitive p^m root of unity ζ.
(3) $(\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}, \epsilon_{2^m})$: where $a = (1,0),\ b = (0,1)$ and $\epsilon_{2^m}(a,a) = 1,\ \epsilon_{2^m}(b,b) = -1,\ \epsilon_{2^m}(a,b) = \zeta$ for some primitive 2^m root of unity ζ.

Definition 53. A bicharacter is called *basic* if it is isomorphic to one of bicharacters (1)-(3).

Remark 54. Let a and b be the elements which appear in the definition of the second or third basic bicharacter (of order p^m and with primitive root of unity ζ). Note that for any prime to p integer k, we have $G = \langle ka \rangle \times \langle b \rangle$, $\theta(ka, ka) = 1$ and $\theta(ka, b) = \zeta^k$, so for different choices of primitive p^m-roots of unity we get isomorphic bicharacters. In particular, for each group G, if θ_1, θ_2 are two basic bicharacters of the same type on G, then they are isomorphic.

The next 3 results were proved in [14] (see Lemma 6, Lemma 7 and Theorem 1).

Lemma 55. *The basic characters $\tau, \eta_{p^m}, \epsilon_{2^n}$, where $m, n \geq 1$, are nonisomorphic. Furthermore, the set of nondegenerate irreducible bicharacters on nontrivial groups coincides with the set $\{\tau, \eta_{p^m}, \epsilon_{2^n} : m \geq 1, n \geq 2\}$ (that is, the set of basic bicharacters except ϵ_2).*

We can now write each nondegenerate skew-symmetric bicharacter θ as a product of basic bicharacters. In general, this presentation is not unique. Nevertheless, using the isomorphisms below, there exists a canonical presentation for any θ.

(1) $\epsilon_{2^n} \otimes \epsilon_{2^m} \cong \epsilon_{2^n} \otimes \eta_{2^m}$ for all $1 \leq n \leq m \in \mathbb{N}$.
(2) $\epsilon_{2^n} \otimes \tau \cong \eta_{2^n} \otimes \tau$ for all $n \in \mathbb{N}$.
(3) $\tau \otimes \tau \cong \epsilon_2$.

Theorem 56. *Let G be a finite abelian p-group and θ a nondegenerate skew-symmetric bicharacter on G. Then there is a unique canonical presentation $G = \prod_1^n G_i$, $\theta \cong \bigotimes \theta \mid_{G_i}$, such that for each i, the bicharacter $(G_i, \theta \mid_{G_i})$ is a basic bicharacter, where at most one basic bicharacter is of type 1 or type 3.*

Note, in particular, that the three types of bicharacters in the last theorem correspond to the three types of algebras in Corollary 40. We can now determine the abelian groups G which admit a nondegenerate regular grading or equivalently a nondegenerate skew-symmetric bicharacter. To this end, let θ be a nondegenerate skew-symmetric bicharacter on an abelian group G. If the canonical decomposition of θ has no factor isomorphic to τ, then the group G is isomorphic to $N \times N$ (i.e. central type, abelian). On the other hand, if one of the components in the canonical decomposition is isomorphic to τ, then we have $G = H \times \mathbb{Z}/2\mathbb{Z}$, where

(1) H is a group of central type determined by $H = \{h \in G \mid \theta(h, h) = 1\}$.
(2) $\theta \mid_{\mathbb{Z}/2\mathbb{Z}} \cong \tau$.

In the general case, if H is of central type, then clearly $H \times \mathbb{Z}/2\mathbb{Z}$ admits a nondegenerate commutation function. However, the following example shows that one may have nondegenerate commutation functions on groups which are not of this kind. For instance, in Example 19, we considered the 2-cocycle $\alpha \in Z^2(D_8, \mathbb{F}^\times)$ induced by the extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow Q_{16} \longrightarrow D_8 \longrightarrow 1 ,$$

where we view the group $\{\pm 1\}$ as a subgroup of $\mathbb{F}^\times$. One can check easily that the (natural) D_8-grading on the Grassmann envelope $A = E \widehat{\otimes} \mathbb{F}^\alpha D_8$ is regular and nondegenerate.

References

[1] Eli Aljadeff, Darrell Haile, and Michael Natapov, *Graded identities of matrix algebras and the universal graded algebra*, Trans. Amer. Math. Soc. **362** (2010), no. 6, 3125–3147, DOI 10.1090/S0002-9947-10-04811-7. MR2592949 (2011b:16079)

[2] Eli Aljadeff and Alexei Kanel-Belov, *Representability and Specht problem for G-graded algebras*, Adv. Math. **225** (2010), no. 5, 2391–2428, DOI 10.1016/j.aim.2010.04.025. MR2680170 (2012c:16064)

[3] Yuri Bahturin, Davida Fischman, and Susan Montgomery, *Bicharacters, twistings, and Scheunert's theorem for Hopf algebras*, J. Algebra **236** (2001), no. 1, 246–276, DOI 10.1006/jabr.2000.8520. MR1808354 (2001k:16067)

[4] Yuri Bahturin and Amitai Regev, *Graded tensor products*, J. Pure Appl. Algebra **213** (2009), no. 9, 1643–1650, DOI 10.1016/j.jpaa.2008.12.010. MR2518166 (2010d:16027)

[5] A. Giambruno and M. Zaicev, *On codimension growth of finitely generated associative algebras*, Adv. Math. **140** (1998), no. 2, 145–155, DOI 10.1006/aima.1998.1766. MR1658530 (99k:16049)

[6] A. Giambruno and M. Zaicev, *Exponential codimension growth of PI algebras: an exact estimate*, Adv. Math. **142** (1999), no. 2, 221–243, DOI 10.1006/aima.1998.1790. MR1680198 (2000a:16048)

[7] A. Giambruno and M. Zaicev, *Codimension growth and minimal superalgebras*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 5091–5117 (electronic), DOI 10.1090/S0002-9947-03-03360-9. MR1997596 (2004h:16025)

[8] A. R. Kemer, *Finite basability of identities of associative algebras* (Russian), Algebra i Logika **26** (1987), no. 5, 597–641, 650. MR985840 (90b:08008)

[9] A. R. Kemer, *Ideals of identities of associative algebras*, Translations of Mathematical Monographs, vol. 87, American Mathematical Society, Providence, RI, 1991. Translated from the Russian by C. W. Kohls. MR1108620 (92f:16031)

[10] E. Nauwelaerts and F. Van Oystaeyen, *The Brauer splitting theorem and projective representations of finite groups over rings*, J. Algebra **112** (1988), no. 1, 49–57, DOI 10.1016/0021-8693(88)90131-7. MR921963 (89e:20028)

[11] Amitai Regev, *Codimensions and trace codimensions of matrices are asymptotically equal*, Israel J. Math. **47** (1984), no. 2-3, 246–250, DOI 10.1007/BF02760520. MR738172 (85j:16024)

[12] Amitai Regev and Tamar Seeman, *Z_2-graded tensor products of p.i. algebras*, J. Algebra **291** (2005), no. 1, 274–296, DOI 10.1016/j.jalgebra.2005.01.049. MR2158524 (2006e:16043)

[13] M. Scheunert, *Generalized Lie algebras*, J. Math. Phys. **20** (1979), no. 4, 712–720, DOI 10.1063/1.524113. MR529734 (80f:17007)

[14] A. A. Zolotykh, *Commutation factors and varieties of associative algebras* (Russian, with English and Russian summaries), Fundam. Prikl. Mat. **3** (1997), no. 2, 453–468. MR1793455

Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel

E-mail address: `aljadeff@tx.technion.ac.il`

Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel

E-mail address: `ofirdav@tx.technion.ac.il`