

**CORRECTIONS AND NOTES TO
“VALUE GROUPS, RESIDUE FIELDS AND BAD PLACES
OF RATIONAL FUNCTION FIELDS”**

ANNA BLASZCZOK AND FRANZ-VIKTOR KUHLMANN

ABSTRACT. We correct mistakes in a 2004 paper by the second author and report on recent new developments which settle cases left open in that paper.

1. INTRODUCTION

The main theorem of [K1] (Theorem 1.6) describes all extensions of a valuation v of a base field K to a rational function field

$$F = K(x_1, \dots, x_n)$$

of transcendence degree $n \geq 1$. There was an omission in the last part, B4), of the theorem; we will state a corrected version of the full theorem in Section 2. The case B4), where the desired value group and residue field extensions $vF|vK$ and $Fv|Kv$ satisfy that

$$(1.1) \quad vF|vK \text{ and } Fv|Kv \text{ are finite,}$$

was the only one that was not completely understood. In particular, a full converse of the assertion of the theorem in this case is not known. This is due to deep open problems in the theory of immediate extensions of valued fields with residue fields of positive characteristic.

In the paragraph following Theorem 1.7 in [K1] (which presents a partial converse of Theorem 1.6), the second author wrote that a full converse can be given if Kv has characteristic 0 or (K, v) is a Kaplansky field. This claim, not proven in [K1], is correct, and we will show an even stronger result in Theorem 2.4 below. But he also claimed that the reason was that for such valued fields, if (\tilde{K}, v) admits an immediate extension of transcendence degree n , then so does (K, v) . This statement is false, as we will show in Section 2.

Working from the other direction on closing the gap, we will show in Theorem 2.3 that the conditions of B4) can be slightly relaxed. To facilitate a quicker assessment, the proofs of Theorems 2.3 and 2.4 are shifted to Section 5. Throughout Sections 2 and 5, we take $F|K$ as above and assume that v is nontrivial on K (since otherwise there is no extension of v to F satisfying conditions (1.1)).

Received by the editors January 30, 2014 and, in revised form, February 24, 2014.

2010 *Mathematics Subject Classification*. Primary 12J10; Secondary 12J15, 16W60.

Part of this paper was prepared while the first author was a guest of the Department of Mathematics and Statistics of the University of Saskatchewan. She gratefully acknowledges their hospitality and support.

The second author was partially supported by a Canadian NSERC grant. He also gratefully acknowledges the hospitality of the Institute of Mathematics of the University of Silesia in Katowice, Poland.

In Section 5 of [K1] the second author defined homogeneous sequences as a tool for the computation of the value group and residue field of a simple extension $(K(x), v)$ of (K, v) . Due to a last-minute change in the definition, the crucial Lemma 5.1 of [K1] became false. In Section 3 of this paper we give the correct definition and show that with the help of it, all results of Section 5 of [K1] can be proved. For the convenience of the reader, we will include all results of that section in the present paper, but will omit those proofs that do not need to be changed.

Finally, we include a list of corrections for some typing errors in [K1] in Section 4 of the present paper.

We take over the notions and notation from [K1].

2. EXTENSIONS OF VALUATIONS TO RATIONAL FUNCTION FIELDS

Here is a version of Theorem 1.6 of [K1] with a small correction which we will discuss afterwards:

Theorem 2.1. *Let (K, v) be any valued field, n, ρ, τ nonnegative integers, $n \geq 1$, $\Gamma \neq \{0\}$ an ordered abelian group extension of vK such that Γ/vK is of rational rank ρ , and $k|Kv$ a field extension of transcendence degree τ .*

Part A. *Suppose that $n > \rho + \tau$ and that*

A1) Γ/vK and $k|Kv$ are countably generated,

A2) Γ/vK or $k|Kv$ is infinite.

Then there is an extension of v to the rational function field $K(x_1, \dots, x_n)$ in n variables such that

$$(2.1) \quad vK(x_1, \dots, x_n) = \Gamma \quad \text{and} \quad K(x_1, \dots, x_n)v = k.$$

Part B. *Suppose that $n \geq \rho + \tau$ and that*

B1) Γ/vK and $k|Kv$ are finitely generated,

B2) *if v is trivial on K , $n = \rho + \tau$ and $\rho = 1$, then k is a simple algebraic extension of a rational function field in τ variables over Kv (or of Kv itself if $\tau = 0$), or a rational function field in one variable over a finitely generated field extension of Kv of transcendence degree $\tau - 1$,*

B3) *if $n = \tau$, then k is a rational function field in one variable over a finitely generated field extension of Kv of transcendence degree $\tau - 1$,*

B4) *if $\rho = 0 = \tau$, then there is an immediate separable-algebraic extension $(L|K, v)$ which either is separable-algebraic such that the extension $(L^h|K^h, v)$ of their respective henselizations is infinite, or is of transcendence degree at least n .*

Then again there is an extension of v to $K(x_1, \dots, x_n)$ such that (2.1) holds.

In the original version of assumption B4), it was stated that the existence of an infinite immediate separable-algebraic extension $(L|K, v)$ is sufficient for the assertion of Theorem 1.6 to hold. But it has to be required in addition that also the extension $(L^h|K^h, v)$ of their respective henselizations is infinite. With this additional assumption the proof of the theorem as presented in [K1] remains unchanged. Note that the henselization is always a separable-algebraic extension, but the assumption that it is infinite is not enough for our theorem.

As an example, take the Laurent series field $k((t))$ with its t -adic valuation. This is a maximal field, that is, it does not admit any proper immediate extensions. Take a transcendence basis T of $k((t))|k(t)$. Then $k((t))$ is the henselization of $K := k(t)(T)$, an infinite separable-algebraic and immediate extension of K . But K does not admit any extension of the valuation to F such that $vF = vK$ and

$Fv = Kv$, since then the henselization F^h would be a proper immediate extension of $K^h = k((t))$.

It suffices to assume the existence of an infinite immediate separable-algebraic extension $(L|K^h, v)$ because then L , being an algebraic extension of a henselian field, is henselian itself, $(L|K, v)$ is also an immediate separable-algebraic extension, and $L^h = L$ is an infinite extension of K^h . In order to analyze the situation further, we cite the following theorem, which is a special case of Theorem 1.1 in the paper [BK1]. This paper is a significantly extended version of the paper [KU5] cited in [K1].

Theorem 2.2. *Take an algebraic extension $(L|K, v)$ and assume that the extension $(L^h|K^h, v)$ of their henselizations contains an infinite separable-algebraic subextension. Then each maximal immediate extension of (L, v) has infinite transcendence degree over L .*

If $(L|K^h, v)$ is an infinite immediate separable-algebraic extension, then by this theorem, (L, v) admits an immediate extension $(M|L, v)$ of infinite transcendence degree. Since $(L|K, v)$ is immediate too, it follows that also $(M|K, v)$ is immediate of infinite transcendence degree. This shows that the above assumption actually implies the other assumption stated in case B4) of Theorem 1.6 of [K1]: the existence of an immediate extension of transcendence degree n .

The main problem with case (1.1) is that we do not know a full converse of the corresponding assertion in Theorem 1.6 of [K1]. In fact, Theorem 1.7 of [K1] states only this much:

Let $n \geq 1$, and let v be a valuation on the rational function field $F = K(x_1, \dots, x_n)$. Set $\rho = \text{rr } vF/vK$ and $\tau = \text{trdeg } Fv|Kv$. Then $n \geq \rho + \tau$, vF/vK is countable, and $Fv|Kv$ is countably generated.

If $n = \rho + \tau$, then vF/vK is finitely generated and $Fv|Kv$ is a finitely generated field extension. Assertions B2) and B3) of Theorem 2.1 hold for $k = Fv$, and if $\rho = 0 = \tau$, then there is an immediate extension of (\tilde{K}, v) of transcendence degree n (for any extension of v from K to \tilde{K}).

Hence if an extension of v from K to F with properties (1.1) exists, then the algebraic closure \tilde{K} of K admits an immediate extension of transcendence degree n , namely, $(\tilde{K}.F|\tilde{K}, v)$. But we would like to know something about K , not only about its algebraic closure. Is it true that the existence of an extension with properties (1.1) always implies the existence of an immediate extension of transcendence degree n ? The following generalization of the assertion of Theorem 2.1 for the case (1.1), which we will prove in Section 5, casts some doubt on this. However, we do not know of any example where the condition of the theorem is met but an immediate extension of transcendence degree n does not exist.

Theorem 2.3. *Take a finite ordered abelian group extension Γ of vK and a finite extension k of Kv . Assume that the henselization K^h admits an infinite separable-algebraic extension $(L|K^h, v)$ with $vL \subseteq \Gamma$ and $Lv \subseteq k$. Then there is an extension of v from K to F such that $vF = \Gamma$ and $Fv = k$.*

Theorem 2.2 can also be used to show that the statement from the paragraph following Theorem 1.7 in [K1] which we indicated in the introduction is false: even if (\tilde{K}, v) admits an immediate extension of transcendence degree n , the same is not necessarily true for (K, v) . Indeed, take any maximal valued field (K, v) that is not separable-algebraically closed or real closed. Certainly, there is an abundance of

valued fields with residue characteristic 0 and of Kaplansky fields that satisfy the stated conditions. Then (K, v) is henselian and its separable-algebraic closure is an infinite extension. Hence by Theorem 2.2, each maximal immediate extension of (\tilde{K}, v) has infinite transcendence degree over \tilde{K} , while (K, v) itself does not have any proper immediate extensions.

While the question about a full converse is still open, the following theorem settles the problem for a large class of valued fields which includes valued fields of residue characteristic 0, Kaplansky fields, and tame fields. Note that the definition of the **implicit constant field**, which in [K1] was given for valued rational function fields in one variable, can be generalized without problems; we take $\text{IC}(F|K, v)$ to be the relative algebraic closure of K in a fixed henselization of (F, v) .

Theorem 2.4. *Take p to be the characteristic exponent of Kv . Assume that vK is p -divisible and Kv is perfect. Further, take an ordered abelian group extension Γ of vK such that Γ/vK is a torsion group, and an algebraic extension k of Kv , both countably generated. Then there is an extension of v from K to F with $vF = \Gamma$ and $Fv = k$ if and only if at least one of the two extensions $\Gamma|vK$ and $k|Kv$ is infinite or (K, v) admits an immediate extension of transcendence degree n . In this case, $\text{IC}(F|K, v)$ is a separable-algebraic extension (L, v) of (K, v) with $vL = \Gamma$ and $Lv = k$, and it is infinite over the henselization of K if $\Gamma|vK$ or $k|Kv$ is infinite.*

For the proof, which we will give in Section 5, we will use the following powerful theorem from [B] (see also [BK2]):

Theorem 2.5. *Take a valued field (K, v) of positive residue characteristic p , with p -divisible value group and perfect residue field.*

- 1) *If (K, v) admits a maximal immediate extension of finite transcendence degree, then the maximal immediate extension of K is unique up to valuation preserving isomorphism.*
- 2) *If (K, v) admits a finite separable-algebraic extension (K', v) such that the valuation v extends in a unique way from K to K' and $(K'|K, v)$ has nontrivial defect, then every maximal immediate extension of (K, v) is of infinite transcendence degree over K .*

3. HOMOGENEOUS SEQUENCES

In Section 5.1 of [K1], the notion of “homogeneous approximation” was introduced. But the definition was incorrect, with the consequence that Lemma 5.1 of [K1] does not hold for this definition. The correct definition is as follows.

Let (K, v) be any valued field and a, b elements in some valued field extension (L, v) of (K, v) . We will say that a is **strongly homogeneous over (K, v)** if $a \in K^{\text{sep}} \setminus K$, the extension of v from K to $K(a)$ is unique, and

$$va = \text{kras}(a, K) := \max\{v(\tau a - \sigma a) \mid \sigma, \tau \in \text{Gal } K \text{ and } \tau a \neq \sigma a\} \in v\tilde{K}.$$

We call $a \in L$ a **homogeneous approximation of b over K** if there is some $d \in K$ such that $a - d$ is strongly homogeneous over K and $v(b - a) > v(b - d) \geq vb$. It then follows that $va = vb$ and $v(a - d) = v(b - d)$. With this definition, Lemma 5.1 of [K1] holds:

Lemma 3.1. *If $a \in L$ is a homogeneous approximation of b , then a lies in the henselization of $K(b)$ w.r.t. every extension of the valuation v from $K(a, b)$ to $\tilde{K}(b)$.*

Proof. From Lemma 2.21 of [K1] we obtain that $a - d$ and hence also a lies in the henselization of $K(b - d) = K(b)$ w.r.t. every extension of the valuation v from $K(a, b)$ to $\widetilde{K(b)}$. \square

Lemmas 5.2 and 5.3 of [K1] remain unchanged:

Lemma 3.2. *Let (K', v) be any henselian extension field of (K, v) such that $a \notin K'$. If a is homogeneous over (K, v) , then it is also homogeneous over (K', v) , and $\text{kras}(a, K) = \text{kras}(a, K')$. If a is strongly homogeneous over (K, v) , then it is also strongly homogeneous over (K', v) .*

Lemma 3.3. *Suppose that $a \in \widetilde{K}$ and that there is some extension of v from K to $K(a)$ such that if e is the least positive integer for which $eva \in vK$, then*

- a) e is not divisible by $\text{char } Kv$,
- b) there exists some $c \in K$ such that $vca^e = 0$, $ca^e v$ is separable-algebraic over Kv , and the degree of ca^e over K is equal to the degree f of $ca^e v$ over Kv .

Then $[K(a) : K] = ef$ and if $a \notin K$, then a is strongly homogeneous over (K, v) .

Lemma 5.4 of [K1] should read:

Lemma 3.4. *Assume that b is an element in some algebraically closed valued field extension (L, v) of (K, v) . Suppose that there is some $e \in \mathbb{N}$ not divisible by $\text{char } Kv$, and some $c \in K$ such that $vcb^e = 0$ and $cb^e v$ is separable-algebraic over Kv . If the smallest possible $e \in \mathbb{N}$ is bigger than 1 or if $cb^e v \notin Kv$, then we can find $a \in L$, strongly homogeneous over K and such that $v(b - a) > vb$. In particular, a is a homogeneous approximation of b over K .*

Proof. Take a monic polynomial g over K with v -integral coefficients whose reduction modulo v is the minimal polynomial of $cb^e v$ over Kv . Then let $a_0 \in \widetilde{K}$ be the root of g whose residue is $cb^e v$. The degree of a_0 over K is the same as that of $cb^e v$ over Kv . We have that $v(\frac{a_0}{cb^e} - 1) > 0$. So there exists $a_1 \in \widetilde{K}$ with residue 1 and such that $a_1^e = \frac{a_0}{cb^e}$. Then for $a := a_1 b$, we find that $v(a - b) = vb + v(a_1 - 1) > vb$ and $ca^e = a_0$. It follows that $va = vb$ and $ca^e v = cb^e v$. By the foregoing lemma, this shows that a is strongly homogeneous over K . \square

The definition of homogeneous sequences and Lemma 5.5 of [K1] remain unchanged:

Let $(K(x)|K, v)$ be any extension of valued fields. We fix an extension of v to $\widetilde{K(x)}$. Let S be an initial segment of \mathbb{N} , that is, $S = \mathbb{N}$ or $S = \{1, \dots, n\}$ for some $n \in \mathbb{N}$ or $S = \emptyset$. A sequence

$$\mathfrak{S} := (a_i)_{i \in S}$$

of elements in \widetilde{K} will be called a **homogeneous sequence for x** if the following conditions are satisfied for all $i \in S$ (where we set $a_0 := 0$):

(HS) $a_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1}$ over $K(a_0, \dots, a_{i-1})$.

Recall that, by the definition of “strongly homogeneous”, $a_i \notin K(a_0, \dots, a_{i-1})^h$. We call S the **support** of the sequence \mathfrak{S} . We set

$$K_{\mathfrak{S}} := K(a_i \mid i \in S).$$

If \mathfrak{S} is the empty sequence, then $K_{\mathfrak{S}} = K$.

Lemma 3.5. *If $i, j \in S$ with $1 \leq i < j$, then*

$$(3.1) \quad v(x - a_j) > v(x - a_i) = v(a_{i+1} - a_i).$$

If $S = \mathbb{N}$, then $(a_i)_{i \in \mathbb{N}}$ is a pseudo Cauchy sequence in $K_{\mathfrak{S}}$ with pseudo limit x .

Lemma 5.6 of [K1] now reads:

Lemma 3.6. *Take $x, x' \in L$.*

1) *If $a \in L$ is a homogeneous approximation of x over K and if $v(x - x') \geq v(x - a)$, then a is also a homogeneous approximation of x' over K .*

2) *Assume that $(a_i)_{i \in S}$ is a homogeneous sequence for x over K . If $v(x - x') > v(x - a_k)$ for all $k \in S$, then $(a_i)_{i \in S}$ is also a homogeneous sequence for x' over K .*

In particular, for each $k \in S$ such that $k > 1$, $(a_i)_{i < k}$ is a homogeneous sequence for a_k over K .

Proof. Only the proof of part 1) changes:

Suppose that a is a homogeneous approximation of x over K , with $v(x - a) > v(x - d) \geq vx$ and $a - d$ strongly homogeneous over K . If in addition $v(x - x') \geq v(x - a) > v(x - d)$, then $v(x' - d) = \min\{v(x - x'), v(x - d)\} = v(x - d)$ and $v(x' - a) \geq \min\{v(x - x'), v(x - a)\} \geq v(x - a) > v(x - d) = v(x' - d)$. Furthermore, $v(x - x') > v(x - d) \geq vx$, hence $vx = vx'$ and $v(x' - d) \geq vx'$. This yields the first assertion. \square

The remainder of Section 5.2 of [K1] remains unchanged, but for the convenience of the reader, we repeat the results here without proof:

Lemma 3.7. *Assume that $(a_i)_{i \in S}$ is a homogeneous sequence for x over K . Then*

$$(3.2) \quad K_{\mathfrak{S}} \subset K(x)^h.$$

For every $n \in S$, $a_1, \dots, a_n \in K(a_n)^h$. If $S = \{1, \dots, n\}$, then

$$(3.3) \quad K_{\mathfrak{S}}^h = K(a_n)^h.$$

Proposition 3.8. *Assume that $\mathfrak{S} = (a_i)_{i \in S}$ is a homogeneous sequence for x over K with support $S = \mathbb{N}$. Then $(a_i)_{i \in \mathbb{N}}$ is a pseudo Cauchy sequence of transcendental type in $(K_{\mathfrak{S}}, v)$ with pseudo limit x , and $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is immediate and pure.*

This proposition leads to the following definition. A homogeneous sequence \mathfrak{S} for x over K will be called a **(weakly) pure homogeneous sequence** if $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is (weakly) pure in x . Hence if $S = \mathbb{N}$, then \mathfrak{S} is always a pure homogeneous sequence. The empty sequence is a (weakly) pure homogeneous sequence for x over K if and only if already $(K(x)|K, v)$ is (weakly) pure in x .

Theorem 3.9. *Suppose that \mathfrak{S} is a (weakly) pure homogeneous sequence for x over K . Then*

$$K_{\mathfrak{S}}^h = \text{IC}(K(x)|K, v).$$

Further, $K_{\mathfrak{S}}v$ is the relative algebraic closure of Kv in $K(x)v$, and the torsion subgroup of $vK(x)/vK_{\mathfrak{S}}$ is finite. If \mathfrak{S} is pure, then $vK_{\mathfrak{S}}$ is the relative divisible closure of vK in $vK(x)$.

In Section 5.3 of [K1], Proposition 5.10 remains unchanged:

Proposition 3.10. *Suppose that (K, v) is henselian. If a is homogeneous over (K, v) , then $(K(a)|K, v)$ is a tame extension. If \mathfrak{S} is a homogeneous sequence over (K, v) , then $K_{\mathfrak{S}}$ is a tame extension of K .*

It can also be shown that if $va = \text{kras}(a, K)$ and a is separable over K , then a satisfies the conditions of Lemma 3.3. In [K1], the separability condition was forgotten.

Because of the change in the definition, the proof of Proposition 5.11 of [K1] changes significantly. Here is the proposition with its new proof:

Proposition 3.11. *An element $b \in \tilde{K}$ belongs to a tame extension of the henselian field (K, v) if and only if there is a finite homogeneous sequence a_1, \dots, a_k for b over (K, v) such that $b \in K(a_k)$.*

Proof. Suppose that such a sequence exists. By Proposition 5.10 of [K1], $K_{\mathfrak{S}}$ is a tame extension of K . Since $b \in K(a_k) \subseteq K_{\mathfrak{S}}$, it contains b .

For the converse, let b be an element in some tame extension of (K, v) . Since $K(b)|K$ is finite, also the extensions $vK(b)|vK$ and $K(b)v|Kv$ are finite. Take $\eta_i \in K(b)$ with $\eta_1 = 1$ such that $v\eta_i$, $1 \leq i \leq \ell$, belong to distinct cosets modulo vK . Further, take $\vartheta_j \in \mathcal{O}_{K(b)}$ with $\vartheta_1 = 1$ such that $\vartheta_j v$, $1 \leq j \leq m$, are Kv -linearly independent. Then by Lemma 2.8 of [K1], the elements $\eta_i \vartheta_j$, $1 \leq i \leq \ell$, $1 \leq j \leq m$, are K -linearly independent. Since $(K(b)|K, v)$ is tame, we have that $[K(b) : K] = \ell m$, so these elements form a basis of $K(b)|K$. Now we write

$$b = \sum_{i,j} c_{ij} \eta_i \vartheta_j$$

with $c_{ij} \in K$. Again by Lemma 2.8 of [K1],

$$vb = v \sum_{i,j} c_{ij} \eta_i \vartheta_j = \min_{i,j} v c_{ij} \eta_i \vartheta_j = \min_{i,j} (v c_{ij} + v \eta_i).$$

If $c_{11} \eta_1 \vartheta_1 = c_{11} \in K$ happens to be the unique summand of minimal value, then we set $d = c_{1,1}$; otherwise, we set $d = 0$.

Choose i_0 such that $v(b-d)$ is in the coset of $v\eta_{i_0}$. If $vc_{i_1 j_1} \eta_{i_1} = vc_{i_2 j_2} \eta_{i_2}$, then $i_1 = i_2$ since the values $v\eta_i$ are in distinct cosets modulo vK . So we can list the summands of minimal value as $c_{i_0 j_r} \eta_{i_0} \vartheta_{j_r}$, $1 \leq r \leq t$, for some $t \leq m$. We obtain that

$$(3.4) \quad v \left(b - d - \sum_{r=1}^t c_{i_0 j_r} \eta_{i_0} \vartheta_{j_r} \right) > v(b-d).$$

Take e to be the least positive integer such that $ev(b-d) \in vK$. Choose $c \in K$ such that $vc(b-d)^e = 0$. Since Kv is perfect, $c(b-d)^e v$ is separable-algebraic over Kv . If $i_0 > 1$, then $v(b-d) \notin vK$. Hence $e > 1$ and since (K, v) is a tame field, e is not divisible by $\text{char } Kv$. If $i_0 = 1$, then $e = 1$ and $\eta_{i_0} = 1$, and in view of (3.4),

$$c(b-d)v = \sum_{r=1}^t (cc_{i_0 j_r} v) \cdot \vartheta_{j_r} v.$$

This is not in Kv since by our choice of d , some $j_k > 1$ must appear in the sum and the residues $\vartheta_{j_r} v$, $1 \leq r \leq t$, are linearly independent over Kv .

We conclude that $b-d$ satisfies the assumptions of Lemma 3.4. Hence there is an element $a \in \tilde{K}$, strongly homogeneous over K and such that $v(b-d-a) > v(b-d) \geq vb$. We set $a_1 := a + d$ to obtain that a_1 is a homogeneous approximation of b over K . By the foregoing proposition, $K(a_1)$ is a tame extension of K and therefore, by the general facts we have noted following the definition of tame extensions in [K1], $K(a_1, b-a_1)$ is a tame extension of $K(a_1)$.

We repeat the above construction, replacing b by $b - a_1$. By induction, we build a homogeneous sequence for b over K . It cannot be infinite since b is algebraic over K (cf. Proposition 3.8). Hence it stops with some element a_k . Our construction shows that this can only happen if $b \in K(a_1, \dots, a_k)$, which by Lemma 3.7 is equal to $K(a_k)$. \square

Finally, some small corrections are needed in the proof of Proposition 5.12 of [K1]:

Proposition 3.12. *Assume that (K, v) is a henselian field. Then (K, v) is a tame field if and only if for every element x in any extension (L, v) of (K, v) there exists a weakly pure homogeneous sequence for x over K , provided that x is transcendental over K .*

Proof. First, let us assume that (K, v) is a tame field and that x is an element in some extension (L, v) of (K, v) , transcendental over K . We set $a_0 = 0$. We assume that $k \geq 0$ and that a_i for $i \leq k$ are already constructed. Like K , also the finite extension $K_k := K(a_0, \dots, a_k)$ is a tame field. Therefore, if x is the pseudo limit of a pseudo Cauchy sequence in K_k , then this pseudo Cauchy sequence must be of transcendental type, and $K_k(x)|K_k$ is pure and hence weakly pure in x .

If $K_k(x)|K_k$ is weakly pure in x , then we take a_k to be the last element of \mathfrak{S} if $k > 0$, and \mathfrak{S} to be empty if $k = 0$.

Assume that this is not the case. Then x cannot be the pseudo limit of a pseudo Cauchy sequence without pseudo limit in K_k . So the set $v(x - a_k - K_k)$ must have a maximum, say $x - a_k - d$ with $d \in K_k$. Since we assume that $K_k(x)|K_k$ is not weakly pure in x , there exist $e \in \mathbb{N}$ and $c \in K_k$ such that $vc(x - a_k - d)^e = 0$ and $c(x - a_k - d)^e v$ is algebraic over $K_k v$. Since K_k is a tame field, its residue field is perfect, so $c(x - a_k - d)^e v$ is separable-algebraic over $K_k v$. Also, if $\text{char } K v = p > 0$, then vK_k is p -divisible and therefore, e can be chosen to be prime to p . Since $v(x - a_k - d)$ is maximal in $v(x - a_k - K_k)$, we must have that $e > 1$ or $c(x - a_k - d)^e v \notin K_k v$.

Now Lemma 3.4 shows that there exists $a \in \tilde{K}$, strongly homogeneous over K_k and such that $v(x - a_k - d - a) > v(x - a_k - d)$. So $a + d$ is a homogeneous approximation of $x - a_k$ over K_k , and we set $a_{k+1} := a_k + a + d$. This completes our induction step. If our construction stops at some k , then $K_k(x)|K_k$ is weakly pure in x and we have obtained a weakly pure homogeneous sequence. If the construction does not stop, then $S = \mathbb{N}$ and the obtained sequence is pure homogeneous.

For the converse, assume that (K, v) is not a tame field. We choose an element $b \in \tilde{K}$ such that $K(b)|K$ is not a tame extension. On $K(b, x)$ we take the valuation $v_{b, \gamma}$ with γ an element in some ordered abelian group extension such that $\gamma > vK$. Choose any extension of v to $\tilde{K}(x)$. Since vK is cofinal in $v\tilde{K}$, we have that $\gamma > v\tilde{K}$. Since $b \in \tilde{K}$, we find $\gamma \in v\tilde{K}(x)$. Hence, $(\tilde{K}(x)|\tilde{K}, v)$ is value-transcendental.

Now suppose that there exists a weakly pure homogeneous sequence \mathfrak{S} for x over K . By Lemma 3.3 of [K1], also $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is value-transcendental. Since $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is also weakly pure, it follows that there must be some $c \in K_{\mathfrak{S}}$ such that $x - c$ is a value-transcendental element (all other cases in the definition of “weakly pure” lead to immediate or residue-transcendental extensions). But if $c \neq b$, then $v(b - c) \in v\tilde{K}$ and thus $v(c - b) < \gamma$. This implies

$$v(x - c) = \min\{v(x - b), v(b - c)\} = v(b - c) \in v\tilde{K},$$

a contradiction. This shows that $b = c \in K_{\mathfrak{S}}$. On the other hand, $K_{\mathfrak{S}}$ is a tame extension of K by Proposition 3.10 and cannot contain b . This contradiction shows that there cannot exist a weakly pure homogeneous sequence for x over K . \square

4. OTHER CORRECTIONS FOR THE PAPER [K1]

- In the paragraph before Example 3.9, “relatively closed subfield” should be: “relatively algebraically closed subfield”.
- The sentence after the first display in Lemma 3.13 should read: “Then $K(a) \subseteq \text{IC}(K(x)|K, v)$ ”.
- In the third line of Theorem 6.1, “ Γ ” should be replaced by “ G ”.
- In the proof of Theorem 6.1, “for the proof of assertion a) it now suffices...” (line after the fifth display) should be replaced by: “for the proof of assertions a) and b) it now suffices...”.

5. PROOFS OF THEOREMS 2.3 AND 2.4

Proof of Theorem 2.3. Since Γ/vL is a finite group, $k|Lv$ is a finite extension, and v is nontrivial on L , from Theorem 2.14 of [K1] it follows that there is a separable-algebraic extension $(L(a), v)$ of (L, v) such that $vL(a) = \Gamma$ and $L(a)v = k$. Then $L(a)$ is an infinite separable-algebraic extension of K^h . Therefore, without loss of generality we can assume that $vL = \Gamma$ and $Lv = k$.

Since $L|K^h$ is an infinite separable-algebraic extension, from Theorem 2.2 it follows that (L, v) admits an immediate extension (M, v) of infinite transcendence degree. Take elements $x_1, \dots, x_{n-1}, y \in M$ algebraically independent over L and set

$$E := K(x_1, \dots, x_{n-1}) \subseteq M.$$

As $(L(x_1, \dots, x_{n-1})|L, v)$ is an immediate extension, we obtain that

$$vE \subseteq vL(x_1, \dots, x_{n-1}) = \Gamma \quad \text{and} \quad Ev \subseteq L(x_1, \dots, x_{n-1})v = k.$$

Since $L|K^h$ and $K^h|K$ are separable-algebraic extensions, also $L|K$ is separable-algebraic. Furthermore vL/vK is a finite group and the extension $Lv|Kv$ is finite, hence there is a finite subextension $L'|K$ of $L|K$ such that $vL' = vL$ and $L'v = Lv$. Moreover, by the Theorem of Primitive Element, we can choose $L'|K$ to be a simple extension $K(b)|K$ for some $b \in L$. Then $E(b) \subseteq L(x_1, \dots, x_{n-1})$, hence $vE(b) = vL = \Gamma$ and $E(b)v = Lv = k$.

Multiplying y by an element in K^\times of large enough value if necessary, we can assume that

$$vy > \text{kras}(b, E) \in \widetilde{vE} = \widetilde{vK}.$$

Since

$$E(b) \subseteq E(y, b) \subseteq L(x_1, \dots, x_{n-1}, y) \subseteq M,$$

the extensions $(L(x_1, \dots, x_{n-1}, y)|E(b), v)$ and $(E(y, b)|E(b), v)$ are immediate. Take a transcendental element x_n in some field extension of E and define by $y \mapsto x_n - b$ an isomorphism of $E(y, b)$ onto $E(x_n, b)$. This isomorphism induces a valuation w on $E(x_n, b)$, which is an extension of the valuation v of $E(b)$ with $w(x_n - b) = vy$. Hence, $w(x_n - b) > \text{kras}(b, E)$ and from Lemma 3.13 of [K1] we deduce that

$$\begin{aligned} vL &= vE(b) \subseteq wE(x_n) = vE(y + b) \subseteq vL(x_1, \dots, x_{n-1}, y) = vL, \\ vL &= E(b)v \subseteq E(x_n)w = E(y + b)v \subseteq L(x_1, \dots, x_{n-1}, y)v = Lv, \end{aligned}$$

since $(L(x_1, \dots, x_{n-1}, y)|L, v)$ is an immediate extension. Thus equality holds everywhere and w is an extension of v from K to $K(x_1, \dots, x_n)$ such that

$$wK(x_1, \dots, x_n) = wE(x_n) = vL = \Gamma \quad \text{and} \quad K(x_1, \dots, x_n)w = E(x_n)w = Lw = k.$$

□

For the proof of Theorem 2.4 we will need the following lemma from [B].

Lemma 5.1. *Assume that vF/vK is a torsion group and $Fv|Kv$ is an algebraic extension. Fix an extension of v to \tilde{F} and set $L := \text{IC}(F|K, v)$. If the order of each element of vF/vK is prime to the characteristic exponent of Kv and $Fv|Kv$ is separable, then $vL = vF$ and $Lv = Fv$, and the extension $(L(x_1, \dots, x_n)|L, v)$ is immediate.*

Proof of Theorem 2.4. Assume that at least one of the two countably generated extensions $\Gamma|vK$ and $k|Kv$ is infinite or (K, v) admits an immediate extension of transcendence degree n . Then parts A2) and B4) of Theorem 1.6 of [K1] state that in both cases the valuation v admits an extension to F such that $vF = \Gamma$ and $Fv = k$.

Assume now that there is an extension of v to F with $vF = \Gamma$ and $Fv = k$. Fix an extension of this valuation to \tilde{F} and denote it again by v . Take K^h and F^h to be the henselizations of K and F with respect to this extension. Set $L := \text{IC}(F|K, v)$. Then L is a separable-algebraic extension of K which contains K^h . As vK is p -divisible and Kv is perfect by assumption, the order of each element of Γ/vK is prime to p and $k|Kv$ is a separable-algebraic extension. Hence, Lemma 5.1 yields that $(L(x_1, \dots, x_n)|L, v)$ is an immediate extension with $vL = \Gamma$ and $Lv = k$. Moreover, if $\Gamma/vK = vL/vK^h$ or $k|Kv = Lv|K^hv$ is infinite, then by the fundamental inequality, also the extension $L|K^h$ is infinite.

Suppose that the extensions $\Gamma|vK$ and $k|Kv$ are finite. Assume first that $L|K^h$ is an infinite extension. Take a finite subextension $E|K^h$ of degree bigger than $(\Gamma : vK)[k : Kv]$. Then

$$[E : K^h] > (\Gamma : vK)[k : Kv] = (vL : vK^h)[Lv : K^hv] \geq (vE : vK^h)[Ev : K^hv],$$

and thus the extension $(E|K^h, v)$ has a nontrivial defect. In this case, or if $L|K^h$ is itself a finite defect extension, we have that $p > 1$ and part 2) of Theorem 2.5 yields that every maximal immediate extension of (K^h, v) is of infinite transcendence degree. Thus the same holds for (K, v) and in particular, (K, v) admits an immediate extension of transcendence degree n .

It remains to consider the case of $(L|K^h, v)$ finite and defectless. As the extension $(L(x_1, \dots, x_n)|L, v)$ is immediate, it is contained in a maximal immediate extension $(M|L, v)$. If there is a maximal immediate extension of finite transcendence degree over L , then by part 1) of Theorem 2.5 it is isomorphic to M over L . This shows that every maximal immediate extension is of transcendence degree at least n over L .

Take a maximal immediate extension (M, w) of (K^h, v) . Take the unique extension of the valuation w of M to $M.L$ and call it again w . Since K^h is henselian, the restriction of w to L coincides with v . As $L|K^h$ is a finite defectless extension of henselian fields, by Lemma 2.5 of [K2] it is linearly disjoint from $M|K^h$ and the extension $(M.L|L, w)$ is immediate. Since a finite extension of a maximal field is again maximal, the field $(M.L, w)$ is a maximal immediate extension of (L, v) .

As we have already shown, $\text{trdeg} M.L|L \geq n$. Hence also $\text{trdeg} M|K^h \geq n$. Since (M, v) is also a maximal immediate extension of (K, v) , we deduce that K admits an immediate extension of transcendence degree n . \square

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF SILESIA, BANKOWA 14, 40-007 KATOWICE, POLAND
E-mail address: anna.blaszcok@us.edu.pl

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SASKATCHEWAN, 106 WIGGINS ROAD, SASKATOON, SASKATCHEWAN, CANADA S7N 5E6
E-mail address: fvk@math.usask.ca