

VARIATIONS IN NONCOMMUTATIVE POTENTIAL THEORY: FINITE-ENERGY STATES, POTENTIALS AND MULTIPLIERS

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Dedicated to Gabriel Mokobodzki

ABSTRACT. In this work we undertake an extension of various aspects of the potential theory of Dirichlet forms to noncommutative C^* -algebras with trace. In particular we introduce finite-energy states, potentials and multipliers of Dirichlet spaces. We prove several results among which are the celebrated Deny's embedding theorem, Deny's inequality, the fact that the carré du champ of bounded potentials are finite-energy functionals and the fact that bounded eigenvectors are multipliers. Deny's embedding theorem and Deny's inequality are also crucial to prove that the algebra of finite-energy multipliers is a form core and that it is dense in A provided the resolvent has the Feller property.

Examples include Dirichlet spaces on group C^* -algebras associated to negative definite functions, Dirichlet forms arising in free probability, Dirichlet forms on algebras associated to aperiodic tilings, Dirichlet forms of Markovian semigroups on locally compact spaces, in particular on post critically finite self-similar fractals, and Bochner and Hodge-de Rham Laplacians on Riemannian manifolds.

1. INTRODUCTION AND DESCRIPTION OF RESULTS

In the present work we develop further the potential theory of Dirichlet forms on C^* -algebras with trace. We introduce and investigate *finite-energy states, potentials and multipliers*, objects naturally associated to Dirichlet spaces and which are meant to encode or reveal the geometric nature of the latter.

In a companion work the results obtained here will be crucial to construct, on C^* -algebras endowed with a Dirichlet form, the building blocks of a metric differential geometry (Dirac operators and spectral triples), pseudo-differential calculus and topological invariants (summable Fredholm modules in K-homology) in the framework of the noncommutative geometry developed by A. Connes [Co].

Classical potential theory, studying harmonic functions on Euclidean spaces \mathbb{R}^n , finite-energy measures and their potentials, was based on the properties of the kernel $|x - y|^{-1}$, the so-called Green function, to be understood as the integral kernel of the inverse of the Laplace operator (see [Bre], [Ca], [Do]).

In the late fifties, A. Beurling and J. Deny outlined, in two seminal papers [BeDe1], [BeDe2], the way to develop a kernel-free potential theory on locally compact Hausdorff spaces X . There, the central role was no longer played by the Green

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function, but rather by a quadratic form that possesses the fundamental *Markovian contraction property*

$$(1.1) \quad \mathcal{E}[f \wedge 1] \leq \mathcal{E}[f],$$

generalizing the Dirichlet integral of Euclidean spaces

$$\mathcal{E}_{\mathbb{R}^n}[f] = \int_{\mathbb{R}^n} |\nabla f|^2 dm.$$

The second fundamental property these quadratic forms are required to have is *lower semicontinuity* on the algebra $C_0(X)$. Lower semicontinuity is reminiscent of the fact that Dirichlet forms may represent energy functionals of physical systems (distributions of electric charges or quantum spinless particles in the ground state representation, for example). On the other hand this property allows us, by a result of G. Mokobodzki [Moko], to interpret the quadratic form as a lower semicontinuous form on the Hilbert spaces $L^2(X, m)$, with respect to a wide family of Borel measures m on X . In this context they give rise to a positive self-adjoint operator L ,

$$\mathcal{E}[f] = \|L^{1/2}f\|_{L^2(X, m)}^2,$$

which in turns generates a Markovian semigroup e^{-tL} on $L^2(X, m)$.

Semigroups in this class are precisely the symmetric, strongly continuous, contractive, positivity preserving semigroups on $L^2(X, m)$ which extend to weakly*-continuous, contractive, positivity preserving semigroups on $L^\infty(X, m)$, symmetric with respect to the measure m .

L^2 -theory is particularly interesting because, as noticed by A. Beurling and J. Deny, there exists a one-to-one correspondence between Dirichlet forms and Markovian symmetric semigroups on $L^2(X, m)$.

The third requirement a Dirichlet form \mathcal{E} on $L^2(X, m)$ has to satisfy is called *regularity*, and concerns the existence of a form core which is also a dense subalgebra of $C_0(X)$. This allows us to develop a rich theory of finite-energy measures and their potentials and, in particular, the construction of a Choquet capacity on the space X . Sets having vanishing capacity can be considered to be negligible from the point of view of potential theory. M. Fukushima made crucial use of them to construct the essentially unique Hunt's Markov stochastic processes (\mathbb{E}_x, ω_t) on X , associated to the regular Dirichlet form, by the formula

$$(e^{-tL}f)(x) = \mathbb{E}_x(f(\omega_t)) \quad x \in X, \quad t \in [0, +\infty)$$

(see [F1], [F2], [FOT]).

The idea to generalize the notion of Markovian semigroups to C^* -algebras A more generally than the commutative ones arose in quantum field theory when L. Gross [G1], [G2] approached the problem of the existence and uniqueness of the ground state of an assembly of $\frac{1}{2}$ spin particles, in terms of certain hypercontractivity properties of the Markovian semigroup on the Clifford C^* -algebra of an infinite dimensional (one-particle) Hilbert space, generated by the Hamiltonian operator.

Later, S. Albeverio and R. Hoegh-Krohn [AHK] introduced Dirichlet forms on C^* -algebras with trace (A, τ) as closed, quadratic forms on the G.N.S. Hilbert space $L^2(A, \tau)$, satisfying a suitable contraction property generalizing (1.1) and having a form core which is a dense subalgebra of A . They also generalized the Beurling-Deny correspondence between Dirichlet forms and Markovian semigroups on $L^2(A, \tau)$. This theory was subsequently developed by J.-L. Sauvageot [S2],

E. B. Davies and M. Lindsay [DL]. Applications were found in Riemannian geometry by E. B. Davies and O. Rothaus [DR1], [DR2] to spectral bounds for the Bochner Laplacian and in noncommutative geometry by J.-L. Sauvageot [S3], [S4] to the transverse heat semigroup on the C*-algebra of a Riemannian foliation.

In [CS1] it is shown that one can associate a canonical differential calculus to any regular Dirichlet form. This calculus allows us to represent it as

$$\mathcal{E}[a] = \|\partial a\|_{\mathcal{H}}^2$$

in terms of an essentially unique closable derivation ∂ on A taking its values in a Hilbert A -bimodule \mathcal{H} . The derivation thus appears as a differential square root of the generator:

$$L = \partial^* \circ \partial.$$

This differential calculus allowed a potential theoretic characterization of Riemannian manifolds having a positive curvature operator as those for which the semigroup generated by the Dirac Laplacian on the Clifford C*-algebra is Markovian [CS3].

Among the other applications of Dirichlet forms and their differential calculus on a C*-algebra with trace, we mention the use made by D. Voiculescu [V1], [V2] and Ph. Biane [Bi] in free probability theory to define and investigate free entropy and the recent appearance in K-theory of Banach algebras [V3] and in K-homology of fractals [CGIS1], [CGIS2].

A suitable Dirichlet form and a related derivation, playing the rôle of a noncommutative Sobolev space and gradient operator, play an important role in the J. Bellissard approach to the quantum Hall effect, in particular to determine the range of application of the Chern-Kubo formula (see [B], [BES]).

Derivations and their associated Markovian semigroups have been used by J. Peterson to approach L^2 -rigidity in von Neumann algebras [Pe1], [Pe2] in order to characterize von Neumann algebras having the property T (a generalization of the Kazhdan property T for groups); and by Y. Dabrowski to prove the non- Γ property of von Neumann algebras generated by noncommuting self-adjoint generators under finite nonmicrostates free Fisher information, still in the framework of D. Voiculescu free entropy theory [Da]. Markov semigroups and Dirichlet forms appear in connection with Lévy processes on compact quantum groups [CFK].

The paper is organized as follows. In Section 2 we recall the basic definitions and properties of Dirichlet forms \mathcal{E} on C*-algebras with traces, their Dirichlet spaces \mathcal{F} , Markovian semigroups and resolvents. In Section 3 we introduce finite-energy functionals and potentials associated to Dirichlet spaces. We prove a correspondence between these two classes of objects, the positivity of potentials and a version of a “noncommutative maximum principle”. As an important tool, we introduce the fine C*-algebra \mathcal{C} , intermediate between the C*-algebra A and the von Neumann algebra \mathcal{M} , to which finite-energy functionals automatically extend. The section also contains a detailed discussion of a class of examples on reduced C*-algebras $C_{red}^*(\Gamma)$ of discrete groups Γ , associated to negative definite functions on them. In Section 4 we provide a version, in our noncommutative framework, of Deny’s embedding theorem by which the Dirichlet space \mathcal{F} can be continuously embedded in the G.N.S. space $L^2(A, \omega)$ of any finite-energy state ω whose potential is bounded. We also prove a version of Deny’s inequality

$$\omega\left(b^* \frac{1}{G(\omega)} b\right) \leq \|b\|_{\mathcal{F}}^2, \quad b \in \mathcal{F},$$

for finite energy ω with potential $G(\omega)$. In Section 5, making use of the canonical differential calculus associated to Dirichlet spaces, we recall the definition of *energy functionals or carré du champ* $\Gamma[a]$ associated to a Dirichlet space and we show that the carré du champ $\Gamma[G]$ of a bounded potential G is a finite-energy functional. In the last Section 6, we introduce *multipliers* of a Dirichlet space and we illustrate this notion discussing examples on group C^* -algebras and on noncommutative tori. We then prove, making use of Deny's embedding theorem, that bounded potentials g , whose carré du champ $\Gamma[g]$ have a bounded potential $G(\Gamma[g])$, are multipliers. This shows a certain abundance of multipliers and, in particular, that the algebra of multipliers is dense in the Dirichlet space. Finally we prove that bounded eigenvectors of the generators associated to the Dirichlet form, on any L^p -space, are necessarily multipliers. The notion of multiplier and the results concerning them are new even in the classical commutative case. In particular, we discuss examples involving ultracontractive Markov semigroups on locally compact spaces and Dirichlet forms associated to harmonic structures on post critically finite self-similar fractals.

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2. DIRICHLET FORMS ON C^* -ALGEBRAS

In this section we summarize the main definitions and some fundamental results of the theory of noncommutative Dirichlet forms on C^* -algebras with trace, for which one may refer to [AHK], [C2], [CS1], [DL].

2.1. C^* -algebras, traces and their standard forms. Let us denote by (A, τ) a separable C^* -algebra A and a densely defined, faithful, semifinite, lower semicontinuous, positive trace τ on it.

We denote by $L^2(A, \tau)$ the Hilbert space of the Gelfand–Naimark–Segal (G.N.S.) representation π_τ associated to τ , and by \mathcal{M} or $L^\infty(A, \tau)$ the von Neumann algebra $\pi_\tau(A)''$ in $\mathcal{B}(L^2(A, \tau))$ generated by A through the G.N.S. representation.

When unnecessary, we shall not distinguish between τ and its canonical normal extension on \mathcal{M} ; between elements of A and their representation in \mathcal{M} as bounded operators in $L^2(A, \tau)$; nor between elements a of A or \mathcal{M} which are square integrable, in the sense that $\tau(a^*a) < +\infty$, and their canonical image in $L^2(A, \tau)$.

Then $\|a\|$ stands for the uniform norm of a in A or in \mathcal{M} , $\|\xi\|_2$ or $\|\xi\|_{L^2(A, \tau)}$ for the norm of ξ in $L^2(A, \tau)$ and $1_{\mathcal{M}}$ for the unit of \mathcal{M} .

As usual A_+ , \mathcal{M}_+ or $L^\infty_+(A, \tau)$ and $L^2_+(A, \tau)$ will denote the positive part of A , \mathcal{M} and $L^2(A, \tau)$ respectively.

Recall that $(\mathcal{M}, L^2(A, \tau), L^2_+(A, \tau))$ is a *standard form* of the von Neumann algebra \mathcal{M} (see [Ara]). In particular $L^2_+(A, \tau)$ is a self-polar, closed convex cone in $L^2(A, \tau)$, inducing an antilinear isometry (the modular conjugation) J on $L^2(A, \tau)$ which is an extension of the involution $a \mapsto a^*$ of \mathcal{M} . The subspace of J -invariant elements (called *real*) will be denoted by $L^2_h(A, \tau)$ (cf. [Dix]). Any element $\xi \in L^2(A, \tau)$ can be written uniquely as $\xi = \xi_r + i\xi_i$ for real elements $\xi_r, \xi_i \in L^2_h(A, \tau)$, and any real element $\xi \in L^2_h(A, \tau)$ can be written uniquely as $\xi = \xi_+ - \xi_-$ for orthogonal positive elements $\xi_\pm \in L^2_+(A, \tau)$, called the positive and negative parts.

Recall that ξ_+ is the Hilbert projection of $\xi \in L^2_h(A, \tau)$ onto the closed convex set $L^2_+(A, \tau)$. For a real element $\xi \in L^2_h(A, \tau)$, the positive element $|\xi| := \xi_+ + \xi_- \in L^2_+(A, \tau)$ will be called the modulus of ξ .

Whenever $\xi \in L^2_h(A, \tau)$ is real, the symbol $\xi \wedge 1$ will denote its Hilbert projection onto the closed and convex subset C of $L^2_h(A, \tau)$ obtained as the L^2 -closure of $\{a \in A \cap L^2(A, \tau) : a \leq 1_{\mathcal{M}}\}$. It can also be obtained from ξ through an obvious functional calculus.

2.2. C*-Dirichlet forms, Dirichlet spaces and Dirichlet algebras. Let $M_n(\mathbb{C})$ be, for $n \geq 1$, the C*-algebra of $n \times n$ matrices with complex entries and tr_n its normalized trace. For every $n \geq 1$, we will indicate by τ_n the trace $\tau \otimes tr_n$ of the C*-algebra $M_n(A) = A \otimes M_n(\mathbb{C})$ of $n \times n$ with entries in A .

The main object of our investigation is the class of C*-Dirichlet forms on $L^2(A, \tau)$ whose definition we recall here (cf. [AHK], [DL], [C1], [CS1]).

Definition 2.1 (C*-Dirichlet forms). A closed, densely defined, nonnegative quadratic form $(\mathcal{E}, \mathcal{F})$ on $L^2(A, \tau)$ is said to be:

i) *real* if

$$(2.1) \quad J(\xi) \in \mathcal{F}, \quad \mathcal{E}[J(\xi)] = \mathcal{E}[\xi], \quad \xi \in \mathcal{F},$$

ii) a *Dirichlet form* if it is real and *Markovian* in the sense that

$$(2.2) \quad \xi \wedge 1 \in \mathcal{F}, \quad \mathcal{E}[\xi \wedge 1] \leq \mathcal{E}[\xi], \quad \xi \in \mathcal{F} \cap L^2_h(A, \tau),$$

iii) a *completely Dirichlet form* if the canonical extension $(\mathcal{E}_n, \mathcal{F}_n)$ to $L^2(M_n(A), \tau_n)$

$$(2.3) \quad \mathcal{E}_n[[\xi_{i,j}]^n_{i,j=1}] := \sum_{i,j=1}^n \mathcal{E}[\xi_{i,j}], \quad [\xi_{i,j}]^n_{i,j=1} \in \mathcal{F}_n := M_n(\mathcal{F}),$$

is a Dirichlet form for all $n \geq 1$,

iv) a C*-Dirichlet form if it is a completely Dirichlet form which is *regular* in the sense that the subspace $\mathcal{B} := A \cap \mathcal{F}$ is dense in the C*-algebra A and is a *form core* for $(\mathcal{E}, \mathcal{F})$.

Notice that, in general, the property

$$|\xi| \in \mathcal{F}, \quad \mathcal{E}[|\xi|] \leq \mathcal{E}[\xi], \quad \xi \in \mathcal{F} \cap L^2_h(A, \tau),$$

is a consequence of the property (2.2) and that it is actually equivalent to it when τ is finite, the cyclic and separating vector ξ_τ representing τ belongs to \mathcal{F} and $\mathcal{E}[\xi_\tau] = 0$ (see [C1]).

Remark 2.2. Even if in this paper we formulate the results in the setting of the G.N.S. standard form of (A, τ) , they can be equivalently stated and proved in a general standard form of (A, τ) (see [C1]). This may be an important advantage when considering specific examples where an *ad hoc* standard form can be more manageable than the G.N.S. one.

*To simplify notation, in the rest of the paper
"Dirichlet form" will always mean C*-Dirichlet form.*

We will denote by $(L, D(L))$ the densely defined, self-adjoint, nonnegative operator on $L^2(A, \tau)$ associated with the closed quadratic form $(\mathcal{E}, \mathcal{F})$,

$$(2.4) \quad \mathcal{E}[\xi] = \|L^{1/2}\xi\|^2, \quad \xi \in \mathcal{F} = \mathcal{D}(L^{1/2}).$$

This operator is the generator of the strongly continuous, contractive semigroup $\{e^{-tL} : t \geq 0\}$ on the Hilbert space $L^2(A, \tau)$. This semigroup is Markovian in the sense that it is positivity preserving and extends to a weakly*-continuous semigroup of contractions on the von Neumann algebra \mathcal{M} . By duality and interpolation this semigroup extends also as a strongly continuous, positivity preserving, contractive semigroup on the noncommutative L^p -space $L^p(A, \tau)$ for each $p \in [1, +\infty]$.

As practice, several aspects of potential theory are more easily managed working with the resolvent family $\{(I + \varepsilon L)^{-1} : \varepsilon \geq 0\}$ than using the semigroup itself. In particular, we will make use of the following obvious properties.

Lemma 2.3. *For $\varepsilon > 0$, the resolvent $(I + \varepsilon L)^{-1}$ is a symmetric contraction in $L^2(A, \tau)$ which operates as a σ -weakly continuous, completely positive, contraction of the von Neumann algebra \mathcal{M} . This family converges strongly to the identity on \mathcal{F} as ε tends to zero.*

Definition 2.4 (Dirichlet spaces, Dirichlet algebras and their fine C*-algebras).

Dirichlet space. The domain \mathcal{F} of the Dirichlet form will be called the *Dirichlet space* when considered as a Hilbert space endowed with its *graph norm*

$$(2.5) \quad \|\xi\|_{\mathcal{F}} := (\mathcal{E}[\xi] + \|\xi\|_{L^2(A, \tau)}^2)^{1/2}, \quad \xi \in \mathcal{F},$$

and the scalar product

$$(2.6) \quad \langle \xi, \eta \rangle_{\mathcal{F}} := \mathcal{E}(\xi, \eta) + (\xi, \eta)_{L^2(A, \tau)}, \quad \xi, \eta \in \mathcal{F}.$$

Dirichlet algebra. The subspace $\mathcal{B} := \mathcal{F} \cap A$ is automatically an involutive, subalgebra of A called the *Dirichlet algebra* (see [DL], [CFK]). By the regularity assumption, it is dense in the Dirichlet space \mathcal{F} as well as in the C*-algebra A , with respect to their own topologies.

The subspace $\tilde{\mathcal{B}} := \mathcal{F} \cap \mathcal{M}$ is an involutive subalgebra of \mathcal{M} called the *extended Dirichlet algebra*. It is dense in the Dirichlet space \mathcal{F} as well as in the von Neumann algebra \mathcal{M} with respect to its σ -weak topology.

Fine C-algebra.* In our approach to potential theory on noncommutative C*-algebras, a distinguished role will be played by the *fine C*-algebra* $\mathcal{C} \supseteq A$, closure of the extended Dirichlet algebra $\tilde{\mathcal{B}}$ in the norm topology of the von Neumann algebra \mathcal{M} . In particular, we will make use of the fact that the Dirichlet form $(\mathcal{E}, \mathcal{F})$, originally assumed to be regular with respect to the C*-algebra A , is still regular with respect to the larger fine C*-algebra \mathcal{C} (see Section 5 below).

Notice that the resolvents map the Hilbert algebra into the fine C*-algebra

$$(I + \varepsilon L)^{-1}(L^2(A, \tau) \cap \mathcal{M}) \subset \mathcal{C}, \quad \varepsilon > 0.$$

In particular, they also leave globally invariant the fine C*-algebra

$$(I + \varepsilon L)^{-1}(\mathcal{C}) \subset \mathcal{C}, \quad \varepsilon > 0.$$

We conclude this section with four examples of Dirichlet space. In the first one we recall the classical Beurling-Deny theory on locally compact spaces X , where the C*-algebra A is the commutative algebra $C_0(X)$ of continuous functions vanishing at infinity, endowed with its uniform norm. In the second one we deal with the

Dirichlet form of the Bochner Laplacian of a Riemannian manifold V and with the Dirichlet form of the Hodge-de Rham Laplacian of a Riemannian manifold with nonnegative curvature. Both these Dirichlet forms are considered on the noncommutative Clifford C^* -algebra of V . The third one deals with typical situations in harmonic analysis where the (reduced) group C^* -algebra $C_{red}^*(\Gamma)$ of a discrete group Γ is most of the time noncommutative. The fourth one illustrates the standard Dirichlet form on noncommutative tori.

Example 2.5 (Dirichlet spaces on commutative C^* -algebras). By a fundamental result of I. M. Gelfand (see [Dix]), commutative C^* -algebras are of type $C_0(X)$ for a suitable locally compact, Hausdorff space X . In this case, positive maps are automatically completely positive so that positive or Markovian semigroups are automatically completely positive or Markovian and all Dirichlet forms are automatically completely Dirichlet forms. In the commutative case our framework thus coincides with that introduced by A. Beurling and J. Deny [BeDe2] to develop potential theories on locally compact, metrizable Hausdorff spaces.

The model Dirichlet form on the Euclidean space \mathbb{R}^n or, more generally, on any Riemannian manifold V endowed with its Riemannian measure m , is the Dirichlet integral

$$\mathcal{E}[f] = \int_V |\nabla f|^2 dm, \quad f \in L^2(V, m).$$

In this case the trace on $C_0(M)$ is given by the integral with respect to the measure m and the Dirichlet space is the Sobolev space $H^1(V) \subset L^2(V, m)$.

Much of the potential theory of Dirichlet forms on locally compact spaces relies on a notion of smallness for subsets of X called polarity. This can be expressed in terms of a suitable Choquet capacity associated to the Dirichlet form (see [FOT]). In the present, possibly noncommutative, setting, the fine C^* -algebra $\mathcal{C} \subseteq \mathcal{M}$ will play the role of the Choquet capacity (see Lemma 5.7 below).

Example 2.6 (Dirichlet spaces on the Clifford algebra of Riemannian manifolds). On a Riemannian manifold (V, g) consider the associated Clifford bundle $\mathcal{C}\ell(V)$, whose fiber $\mathcal{C}\ell_x(V)$ at $x \in V$ is the (complexification of the) Clifford algebra of the tangent space $T_x V$. By Clifford multiplication, $\mathcal{C}\ell_x(V)$ is canonically a (finite dimensional, noncommutative) C^* -algebra with trace τ_x , isomorphic to a finite sum of full matrix algebras. Sections of $\mathcal{C}\ell(V)$ vanishing at infinity thus form, in a natural way, a C^* -algebra $C_0^*(V)$, called the *Clifford algebra* of V . Its center reduces to the commutative C^* -subalgebra $C_0(V)$ of continuous functions vanishing at infinity on V . The Riemannian measure m on (V, g) gives rise to a trace $\tau = \int_V \tau_x m(dx)$ on $C_0^*(V)$ whose associated G.N.S. space $L^2(C_0^*(V), \tau)$ coincides with the Hilbert space $L^2(\mathcal{C}\ell(V))$ of square integrable sections of the Clifford bundle (see e.g. [LM]). The Levi-Civita connection of (V, g) , through its covariant derivative ∇ , gives rise to a Dirichlet form on $C_0^*(V)$

$$\mathcal{E}_B[\sigma] := \int_V |\nabla \sigma|^2,$$

whose associated self-adjoint operator is the Bochner Laplacian $\Delta_B = \nabla^* \nabla$ of V (see [LM], [DR1],[DR2]). The Dirac operator D on $\mathcal{C}\ell(V)$ gives rise to a nonnegative, closed quadratic form

$$\mathcal{E}_D[\sigma] := \|D\sigma\|^2,$$

whose associated self-adjoint operator is the so-called Dirac Laplacian $\Delta_D = D^2$. Under the canonical isomorphism of Hilbert spaces between $L^2(\mathcal{C}\ell(V))$ and the Hilbert space $L^2(\Lambda^*V)$ of square integrable sections of the exterior bundle of V , the Dirac Laplacian translates into the Hodge-de Rham Laplacian $\Delta_D \simeq (d+d^*)^2 = d^*d+dd^*$, where d is the exterior derivative and d^* is its adjoint. Manifolds in which \mathcal{E}_D is a Dirichlet form, i.e. the heat semigroup $e^{-t\Delta_D}$ is a Markovian semigroup on the Clifford algebra $C_0^*(V)$, may be characterized geometrically as those in which the curvature operator of V is nonnegative (see [CS3]).

Example 2.7 (Dirichlet spaces on group C^* -algebras). Let Γ be a discrete group, with unit $e \in G$, whose elements will be denoted by s, t, \dots . Denote by λ_Γ its left regular representation on $l^2(\Gamma)$ acting by

$$(\lambda_\Gamma(s)a)(t) := a(s^{-1}t), \quad s, t \in \Gamma, \quad a \in l^2(\Gamma),$$

and by $C_{red}^*(\Gamma)$ its reduced C^* -algebra in $\mathcal{B}(l^2(\Gamma))$ generated by $\{\lambda_\Gamma(s) \in \mathcal{B}(l^2(\Gamma)) : s \in \Gamma\}$ (see [Dix]). More explicitly, for $a, b \in c_c(\Gamma) \subseteq C_{red}^*(\Gamma)$ their product is defined by convolution

$$(a * b)(s) := \sum_{t \in \Gamma} a(t)b(st^{-1}), \quad s \in \Gamma,$$

while involution is defined by

$$(a^*)(s) := \overline{a(s^{-1})}, \quad s \in \Gamma.$$

The left regular representation of Γ extends to a $*$ -representation of the reduced C^* -algebra and will be denoted by the same symbol. The functional $C_{red}^*(\Gamma) \supseteq c_c(\Gamma) \ni a \mapsto a(e) \in \mathbb{C}$ extends to a trace state τ on $C_{red}^*(\Gamma)$ and the associated G.N.S. representation coincides with the left regular representation above. In particular the G.N.S. Hilbert space $L^2(C_{red}^*(\Gamma), \tau)$ can be identified with $l^2(\Gamma)$ and its positive cone with the cone of positive definite, square integrable functions.

Any positive, conditionally negative definite function $\ell : \Gamma \rightarrow [0, +\infty)$ (see for example [CCJJV]) gives rise to a regular Dirichlet form

$$\mathcal{E}_\ell[a] = \sum_{s \in \Gamma} |a(s)|^2 \ell(s),$$

with domain the space \mathcal{F}_ℓ of those a in $l^2(\Gamma)$ for which the series converges (see [CS1], [C2]).

Examples of the above framework arise on \mathbb{Z}^n , where as the negative definite function one can choose the Euclidean length $\ell(k) := |k|$ or its square $\ell(k) := |k|^2$, and on free groups \mathbb{F}_n with $n \in \{1, 2, \dots\}$ generators where the most important negative definite functions are the length functions associated to systems of generators (see [Haa1]).

Example 2.8 (Dirichlet forms on noncommutative tori). Noncommutative tori are a family of C^* -algebras which represent a sort of gymnasium for noncommutative geometry [Co]. They are defined, for any fixed irrational $\theta \in [0, 1]$, as the universal C^* -algebras A_θ generated by two unitaries U and V , satisfying the relation

$$VU = e^{2i\pi\theta}UV.$$

The functional $\tau : A_\theta \rightarrow \mathbb{C}$ given by

$$\tau(U^n V^m) = \delta_{n,0} \delta_{m,0}, \quad n, m \in \mathbb{Z},$$

is a tracial state and the *heat semigroup* $\{T_t : t \geq 0\}$ on A_θ is defined by

$$T_t(U^n V^m) = e^{-t(n^2+m^2)} U^n V^m, \quad n, m \in \mathbb{Z}.$$

It is τ -symmetric and the associated Dirichlet form is the closure of the quadratic form given by

$$\mathcal{E} \left[\sum_{n,m \in \mathbb{Z}} \alpha_{n,m} U^n V^m \right] = \sum_{n,m \in \mathbb{Z}} (n^2 + m^2) |\alpha_{n,m}|^2$$

defined on the algebra $\{\sum_{n,m \in \mathbb{Z}} \alpha_{n,m} U^n V^m \in A_\theta : [\alpha_{n,m}]_{n,m \in \mathbb{Z}} \in c_c(\mathbb{Z}^2)\}$.

3. FINITE-ENERGY FUNCTIONALS AND POTENTIALS

In this section we introduce the first two objects of our investigation: the class of finite-energy functionals and the class of potentials of a Dirichlet space. In the classical case they were introduced by A. Beurling and J. Deny in their work on Dirichlet forms on locally compact spaces [BeDe2].

Definition 3.1 (Finite-energy functionals and potentials). Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on the separable C^* -algebra (A, τ) endowed with a densely defined, faithful, semifinite, lower semicontinuous, positive trace.

- A positive functional $\omega \in A_+^*$ will be said to be a *finite-energy functional* if

$$(3.1) \quad \omega(b) \leq c_\omega \|b\|_{\mathcal{F}}, \quad b \in \mathcal{B}_+ := \mathcal{B} \cap L_+^2(A, \tau),$$

for some $c_\omega \geq 0$.

- An element $\xi \in \mathcal{F}$ will be called a *potential* if

$$(3.2) \quad \langle \xi, b \rangle_{\mathcal{F}} \geq 0, \quad b \in \mathcal{B}_+.$$

- Let $\omega \in A_+^*$ be a finite-energy functional. By the contraction property of the Dirichlet and (3.1), for a self-adjoint $b \in \tilde{\mathcal{B}}$ we have

$$|\omega(b)| \leq \omega(|b|) \leq c_\omega \cdot \mathcal{E}[|b|] \leq c_\omega \cdot \mathcal{E}[b].$$

Consequently, as the Dirichlet algebra \mathcal{B} is a form core, there exists a unique element $\xi \in \mathcal{F}$ determined by the relation

$$(3.3) \quad \omega(b) = \langle \xi, b \rangle_{\mathcal{F}} = \mathcal{E}(\xi, b) + (\xi, b)_2, \quad b \in \mathcal{B}.$$

The element ξ will be called the *potential of ω* and will be denoted by $G(\omega)$.

Thus, finite-energy functionals and their potentials satisfy the relation

$$(3.4) \quad \omega(b) = \langle G(\omega), b \rangle_{\mathcal{F}}, \quad b \in \mathcal{B}.$$

Moreover, by the formula above, any finite-energy functional can then be extended to the whole Dirichlet space \mathcal{F} , the quantity

$$(3.5) \quad \mathcal{E}[\omega] := \mathcal{E}[G(\omega)] = \omega(G(\omega))$$

is called the *energy content* of ω and one has $|\omega(b)| \leq \sqrt{\mathcal{E}[\omega]} \|b\|_{\mathcal{F}}$ for all $b \in \mathcal{F}$.

The set \mathcal{P}_+ of potentials is, by definition, the polar cone of the positive cone $\mathcal{F}_+ := \mathcal{F} \cap L_+^2(A, \tau)$ in the Dirichlet space:

$$\mathcal{P}_+ := \mathcal{F}_+^\circ = \{\xi \in \mathcal{F} : \langle \xi, \eta \rangle_{\mathcal{F}} \geq 0 \text{ for all } \eta \in \mathcal{F}_+\}.$$

We will prove in Proposition 3.7 below that potentials are necessarily positive elements of $L_+^2(A, \tau)$ so that $\mathcal{P}_+ \subseteq \mathcal{F}_+$ and then $\mathcal{P}_+ \subseteq \mathcal{P}_+^\circ$.

Example 3.2 (Finite-energy normal functionals). Let $h \in L^2_+(A, \tau) \cap L^1(A, \tau)$ and consider the normal positive functional $\omega_h \in \mathcal{M}_{*+}$ defined by

$$\omega_h(b) := \tau(hb), \quad b \in \mathcal{M}.$$

Since $h \in L^2(A, \tau)$, then $\xi := (I + L)^{-1}h \in \mathcal{F}$ is such that

$$\langle \xi, b \rangle_{\mathcal{F}} = (L^{1/2}\xi, L^{1/2}b) + (\xi, b) = \tau(hb), \quad b \in \mathcal{B},$$

the vector $\xi \in \mathcal{F}$ is a potential, the normal positive linear form ω_h is a finite-energy functional, its potential coincides with ξ

$$G(\omega_h) = (I + L)^{-1}h$$

and its energy content is given by $\mathcal{E}[\omega_h] = \omega_h((I + L)^{-1}h) = \tau(h(I + L)^{-1}h)$.

Example 3.3 (Finite-energy functionals and potentials on group C^* -algebras). Let us consider the Dirichlet form on a group algebra $C^*_{red}(\Gamma)$ of a discrete group Γ associated to a negative definite function $\ell : \Gamma \rightarrow [0, +\infty)$, as in Example 2.7,

$$\mathcal{E}_\ell[a] = \sum_{s \in \Gamma} \ell(s)|a(s)|^2, \quad a \in l^2(\Gamma).$$

In this case ω is a finite-energy state on $C^*_{red}(\Gamma)$ if and only if

$$\sum_{s \in \Gamma} \frac{|\varphi_\omega(s)|^2}{1 + \ell(s)} < +\infty$$

and its potential $G(\omega)$ is given by

$$G(\omega)(s) = \frac{\varphi_\omega(s)}{(1 + \ell(s))}, \quad s \in \Gamma,$$

where $\varphi_\omega : \Gamma \rightarrow \mathbb{C}$ is the normalized, positive definite function associated to the state ω and defined as $\varphi_\omega(s) := \omega(\delta_s)$ for all $s \in \Gamma$. In particular the energy content of ω is equal to

$$\mathcal{E}_\ell[\omega] = \mathcal{E}_\ell[G(\omega)] = \sum_{s \in \Gamma} \frac{|\varphi_\omega(s)|^2}{1 + \ell(s)}.$$

In other words, since states ω on $C^*_{red}(\Gamma)$ are characterized by the fact that for the associated function φ_ω (see [Dix]), potentials $\xi \in \mathcal{P}_+$ associated to the Dirichlet form \mathcal{E}_ℓ have the form

$$\xi(s) = \frac{\varphi_\xi(s)}{1 + \ell(s)}, \quad s \in \Gamma,$$

for some positive definite function $\varphi_\xi : \Gamma \rightarrow \mathbb{C}$. Notice that, since ℓ is a negative definite function, the function $(1 + \ell)^{-1}$ is positive definite so that the potential ξ is a positive definite element of $L^2(G)$. It will be shown later in this section that positivity of potentials is a general fact valid in all Dirichlet spaces.

On groups having the Kazhdan property T, all negative definite functions are bounded so that the cone of potential associated to any such negative definite function ℓ simply coincides with the cone of square integrable, positive definite functions. Richer classes of examples can be found on groups having the Haagerup property, where there exist proper, negative definite functions (see for example [CCJJV]).

Suppose that Γ has polynomial growth (i.e. by a theorem of M. Gromov, it has a nilpotent subgroup of finite index) so that, with respect to a system of generators $S \subset \Gamma$, the associated length function ℓ_S , assumed to be negative definite, has

spherical growth $\sigma_S : \mathbb{N} \rightarrow \mathbb{N}$ behaving as $\sigma_S(k) \sim k^{d-1}$ for some $d > 1$. If Γ is nilpotent, by a theorem of J. Dixmier, the exponent d coincides with the homogeneous dimension $d(\Gamma)$, defined in terms of the relative indexes of its lower central series (see [CCJJV]). Then

$$\|(1 + \ell)^{-1}\|_{\ell^q(\Gamma)}^q = \sum_{s \in \Gamma} (1 + \ell(s))^{-q} = \sum_{k \in \mathbb{N}} (1 + k)^{-q} \sigma_S(k) < +\infty$$

for all $q > d$. If $\omega \in A_+^*$ is a (pure) state whose cyclic (irreducible) representation is $l^p(\Gamma)$ -integrable for some $2 \leq p < \frac{2d}{d-1}$, by definition this means that $\varphi \in l^p(\Gamma)$, then, by the Hölder inequality, it is a finite-energy state with respect to the Dirichlet form \mathcal{E}_ℓ

$$\mathcal{E}_\ell[\omega] = \mathcal{E}_\ell[G(\omega)] = \sum_{s \in \Gamma} \frac{|\varphi_\omega(s)|^2}{1 + \ell(s)} \leq \|\varphi_\omega\|_{l^p(\Gamma)} \cdot \|(1 + \ell)^{-1}\|_{\ell^q(\Gamma)}^q < +\infty.$$

For a specific example one may consider the Heisenberg group which is nilpotent with homogeneous dimension $d(\Gamma) = 4$.

As ℓ is a negative definite function, so is its square root $\sqrt{\ell}$. Hence $(1 + \sqrt{\ell})^{-1}$ is a positive definite, normalized function and there exists a state $\omega_\ell \in A_+^*$ such that $\varphi_{\omega_\ell}(s) = (1 + \sqrt{\ell}(s))^{-1}$ for all $s \in \Gamma$. Since

$$(1 + \sqrt{x})^2 \leq 2(1 + x) \leq 2(1 + \sqrt{x})^2, \quad x > 0,$$

a functional $\omega \in A_+^*$ is a finite-energy state if and only if

$$\sum_{s \in \Gamma} \frac{|\varphi_\omega(s)|^2}{(1 + \sqrt{\ell}(s))^2} = \sum_{s \in \Gamma} |\varphi_{\omega_\ell}(s) \cdot \varphi_\omega(s)|^2 < +\infty.$$

Notice that $\varphi_{\omega_\ell} \cdot \varphi_\omega$ is a coefficient of a cyclic subrepresentation of the tensor product $\pi_{\omega_\ell} \otimes \pi_\omega$ of the cyclic representations $(\pi_\ell, \mathcal{H}_\ell, \xi_\ell)$ and $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ associated to the states ω_ℓ and ω . Hence if ω is a finite-energy state, the representation $\pi_{\omega_\ell} \otimes \pi_\omega$ is not disjoint from the left regular representation λ_Γ .

Moreover, since a state ω has finite energy with respect to the Dirichlet form generated by a negative definite function ℓ if and only if it is a finite-energy state with respect to the Dirichlet forms associated to each negative type function $\lambda^{-2}\ell$ for all $\lambda > 0$, we have that the family of normalized, positive definite functions $\{\varphi_\lambda := \varphi_{\omega_{\lambda^{-2}\ell}} \cdot \varphi_\omega : \lambda > 0\}$, explicitly given by

$$\varphi_\lambda(s) = \frac{\lambda}{\lambda + \sqrt{\ell}(s)} \cdot \varphi_\omega(s), \quad s \in \Gamma,$$

generates a family of cyclic representations $\{\pi_\lambda : \lambda > 0\}$, contained in the left regular representation λ_Γ which interpolate between the left regular representation λ_Γ and the cyclic representation π_ω associated to the finite-energy state ω . In fact

$$\lim_{\lambda \rightarrow 0^+} \varphi_\lambda = \delta_e, \quad \lim_{\lambda \rightarrow +\infty} \varphi_\lambda = \varphi_\omega$$

pointwise.

Now we prove that finite-energy functionals extend to positive functionals on the fine C*-algebras \mathcal{C} . For this we need the following approximation result.

Lemma 3.4. *Let $b \in \tilde{\mathcal{B}}$ such that $b^* = b$. Then there exists a sequence of self-adjoint elements $\{b_n\}_{n \in \mathbb{N}} \subset \mathcal{B}$ such that $\|b_n - b\|_{\mathcal{F}} \rightarrow 0$, $\|b_n\| \leq \|b\|$ and $b_n \rightarrow b$ σ -weakly in \mathcal{M} . If $b \geq 0$, one can choose $b_n \geq 0$ for all n .*

Proof. By the regularity of $(\mathcal{E}, \mathcal{F})$, the Dirichlet algebra \mathcal{B} is a form core, so that there exists a sequence $\{b_n\}_{n \in \mathbb{N}} \subset \mathcal{B}$ which converges to b in \mathcal{F} . By reality (2.1) of \mathcal{E} , the sequence b_n^* converges also to b^* , so that one can suppose $b_n = b_n^*$ for all n .

Set $K := \|b\|$ and, for each n , let e_n be the spectral projection of b_n corresponding to the interval $(-\infty, K]$. Set $b'_n = b_n \wedge K = e_n b_n + K(I - e_n)$. One has $\|b'_n\|_{L^2(A, \tau)} \leq \|b_n\|_{L^2(A, \tau)}$ (since $b_n'^2 \leq b_n^2$) and, by the Markovian property (2.2) of the Dirichlet form, $\mathcal{E}[b'_n] \leq \mathcal{E}[b_n]$. Hence, the sequence b'_n is bounded in \mathcal{F} . Replacing it by a subsequence, one can suppose that it has a weak limit γ in \mathcal{F} , with $\gamma \leq b$.

As $b'_n \rightarrow \gamma$ weakly in $L^2(A, \tau)$, we have

$$(3.6) \quad \tau(\gamma^2) \leq \liminf \tau(b_n'^2) \leq \lim \tau(b_n^2) = \tau(b^2)$$

and, weakly in $L^2(A, \tau)$,

$$(3.7) \quad (b_n - KI)(I - e_n) = b_n - b_n \wedge K \rightarrow b - \gamma.$$

As $K^2\tau(I - e_n) \leq \tau(b_n^2(I - e_n)) \leq \tau(b_n^2) \rightarrow \tau(b^2)$, one can suppose that $I - e_n$ has a weak limit p in $L^2(A, \tau)$, which is also a σ -weak limit in \mathcal{M} . Hence, $b_n(1 - e_n)$ converges weakly to bp in $L^2(A, \tau)$ and (3.7) provides

$$(b - KI)p = b - \gamma.$$

As b_n commutes with e_n , b will commute with p , so that, in this equality, the left hand side is a negative operator while the right hand side is a positive operator. This proves $\gamma = b$ and, by (3.6), that $b'_n \rightarrow b$ strongly in $L^2(A, \tau)$. As the sequence b'_n is bounded in \mathcal{F} , it converges to b weakly in \mathcal{F} . As moreover $\mathcal{E}[b'_n] \leq \mathcal{E}[b_n]$ which converges to $\mathcal{E}[b]$, this must be a strong limit in \mathcal{F} .

Similarly, $b''_n = b'_n \vee (-K) = -(-b'_n \wedge K)$ converges to b in \mathcal{F} . It is a bounded sequence in \mathcal{M} , with norm less than $K = \|b\|$. As its only possible σ -weak limit is b , it converges to b σ -weakly in \mathcal{M} .

Note that, if $b \geq 0$, one can replace $b''_n = b'_n \vee (-K)$ by $b''_n = b'_n \vee 0$, so that $b''_n \geq 0$ for all n . □

Proposition 3.5. *If $\omega \in A_+^*$ is a finite-energy functional, then the linear map $\tilde{\omega} : \tilde{\mathcal{B}} \rightarrow \mathbb{C}$*

$$(3.8) \quad \tilde{\omega}(b) := \langle G(\omega), b \rangle_{\mathcal{F}}$$

extends to the C^ -algebra \mathcal{C} as a positive linear functional with norm equal to $\|\omega\|_{A^*}$.*

Proof. Note first that $G(\omega)^* = G(\omega)$ since, by symmetry of \mathcal{E} , one has, for $b \in \mathcal{B}$:

$$\langle G(\omega)^*, b \rangle_{\mathcal{F}} = \langle b^*, G(\omega) \rangle_{\mathcal{F}} = \overline{\omega(b^*)} = \omega(b) = \langle G(\omega), b \rangle_{\mathcal{F}}.$$

The same computation proves that $\tilde{\omega}$ is hermitian: $\tilde{\omega}(b^*) = \overline{\tilde{\omega}(b)}$ for $b \in \tilde{\mathcal{B}}$.

Let $b = b^* \in \tilde{\mathcal{B}}$ and let b_n be a sequence in \mathcal{B} provided by Lemma 3.4. Since any finite-energy functional is continuous with respect to the topology of \mathcal{F} and since $\|b_n\| \leq \|b\|$ one has

$$|\tilde{\omega}(b)| = \lim |\omega(b_n)| \leq \|\omega\|_{A^*} \limsup \|b_n\|_A \leq \|\omega\|_{A^*} \|b\|_{\mathcal{M}}.$$

By definition, $\tilde{\mathcal{B}}$ is dense in \mathcal{C} so that $\tilde{\omega}$ extends by continuity to \mathcal{C} . To prove positivity, recall that, again by Lemma 3.4, if $b \geq 0$ we may assume the approximating sequence to be positive so that $\tilde{\omega}(b) = \lim \omega(b_n) \geq 0$. □

The next proposition contains approximation and positivity results needed in the forthcoming section. They will also be used below to prove that potentials of finite-energy functionals are positive.

Proposition 3.6. *Let $\omega \in A_+^*$ be a finite-energy functional, $\tilde{\omega} \in C_+^*$ its canonical extension to the fine algebra \mathcal{C} and $\varepsilon > 0$. Then*

- i) $\tilde{\omega} \circ (I + \varepsilon L)^{-1}|_A$ is a positive finite-energy functional on A ;
- ii) one has $G(\tilde{\omega} \circ (I + \varepsilon L)^{-1}|_A) = (I + \varepsilon L)^{-1}G(\omega)$;
- iii) one has $(I + L)(I + \varepsilon L)^{-1}G(\omega) \in L^1(A, \tau) \cap L_+^2(A, \tau)$.

Proof. As $(I + \varepsilon L)^{-1}$ is a positivity preserving, norm contraction on \mathcal{M} , the functional $\tilde{\omega} \circ (I + \varepsilon L)^{-1}$ is positive on \mathcal{C} and so it is its restriction to A , thus proving the statement in i).

As $(I + \varepsilon L)^{-1}(b) \in \mathcal{D}(L)$ for $b \in \mathcal{B}$, the identities

$$\begin{aligned}
 \tilde{\omega}((I + \varepsilon L)^{-1}(b)) &= \langle G(\omega), (I + \varepsilon L)^{-1}(b) \rangle_{\mathcal{F}} \\
 &= (G(\omega), L(I + \varepsilon L)^{-1}(b))_2 + (G(\omega), (I + \varepsilon L)^{-1}(b))_2 \\
 (3.9) \qquad &= ((I + L)(I + \varepsilon L)^{-1}G(\omega), b)_2 \\
 &= \langle (I + \varepsilon L)^{-1}G(\omega), b \rangle_{\mathcal{F}}
 \end{aligned}$$

allow us to conclude that $\tilde{\omega} \circ (I + \varepsilon L)^{-1}|_A$ has finite energy, its potential is given by $G(\tilde{\omega} \circ (I + \varepsilon L)^{-1}|_A) = (I + \varepsilon L)^{-1}G(\omega)$ and $(I + L)(I + \varepsilon L)^{-1}G(\omega)$ is a positive element in $L^2(A, \tau)$.

The second line in equation (3.9) tells us that the element

$$h := (I + L)(I + \varepsilon L)^{-1}G(\omega) \in L_+^2(A, \tau)$$

satisfies

$$|\tau(hb)| = |(h, b)_2| = |\tilde{\omega}((I + \varepsilon L)^{-1}b)| \leq \|\tilde{\omega}\|_{C^*} \|b\|_A, \quad b \in \mathcal{B},$$

which suffices to imply $h \in L^1(A, \tau)$, and thus proving assertion iii). □

Proposition 3.7. *The cone of potentials is contained in the standard cone: $\mathcal{P}_+ \subset L_+^2(A, \tau)$.*

Proof. Let us consider a potential $G \in \mathcal{P}_+$. By the positivity preserving property of the resolvents, we have that $(I + L)^{-1}b \in \mathcal{F}_+ := \mathcal{F} \cap L_+^2(A, \tau)$ for any $b \in L_*^2(A, \tau)$ and then

$$(G, b)_2 = (G, (I + L)(I + L)^{-1}b)_2 = \langle G, (I + L)^{-1}b \rangle_{\mathcal{F}} \geq 0.$$

□

Here we prove a useful property shared by potentials which will be needed later on in this work.

Lemma 3.8. *If $G \in \mathcal{P}_+$ is a potential, then $\frac{1}{\sqrt{G + \delta}}$ is a multiplier of the fine C^* -algebra \mathcal{C} , for all $\delta > 0$.*

Proof. The function

$$f : [0, +\infty) \rightarrow \mathbb{R}, \quad f(t) := \frac{1}{\sqrt{t + \delta}} - \frac{1}{\delta}$$

vanishes at 0, and is bounded and differentiable with bounded derivative. Hence by [CS1], Lemma 7.2, we have $f(G) \in \tilde{\mathcal{B}} \subset \mathcal{C}$. Adding the constant operator $\frac{1}{\delta}$ we get a multiplier of \mathcal{C} . \square

Lemma 3.9. *For $\xi, \eta \in \mathcal{F}$ we have*

$$(3.10) \quad \frac{d}{dt} \left\langle e^{-t(1+L)}\xi, \eta \right\rangle_{L^2(A, \tau)} = - \left\langle e^{-t(1+L)}\xi, \eta \right\rangle_{\mathcal{F}}, \quad t \geq 0.$$

Proof. For $\xi \in \text{Dom}_{L^2}(L)$ the identity is obvious. Writing it in integral form

$$\left\langle e^{-t(1+L)}\xi, \eta \right\rangle_{L^2(A, \tau)} = \langle \xi, \eta \rangle_{L^2(A, \tau)} - \int_0^t \left\langle e^{-s(1+L)}\xi, \eta \right\rangle_{\mathcal{F}} ds,$$

it extends easily to $\xi, \eta \in \mathcal{F}$. \square

Lemma 3.10. *For any potential $G \in \mathcal{P}_+$ one has*

$$e^{-t(1+L)}G \leq G \text{ in } L^2(A, \tau), \quad t \geq 0,$$

and

$$\frac{1}{1 + \varepsilon L}G \leq \frac{1}{1 - \varepsilon}G \text{ in } L^2(A, \tau), \quad 0 < \varepsilon < 1.$$

Viceversa, any one of the above two properties for a nonnegative G implies that G is a potential.

Proof. Applying (3.10), for $b \in \mathcal{F}_+$ one has

$$\frac{d}{dt} \left\langle e^{-t(1+L)}G, b \right\rangle_{L^2(A, \tau)} = - \left\langle e^{-t(1+L)}G, b \right\rangle_{\mathcal{F}} \leq 0$$

and then $e^{-t(1+L)}G \leq G$. Integrating this inequality between 0 and $+\infty$ with respect to the probability measure $me^{-tm} dt$ for $m > 0$, one gets

$$\frac{m}{m + 1 + L}G \leq G,$$

and the second result choosing m such that $(m + 1)\varepsilon = 1$. The converses of the above two results are easily obtained by deriving the inequalities, weakly in \mathcal{F} , in $t = 0$ and $\varepsilon = 0$, respectively. \square

We conclude this section with a “noncommutative maximum principle” in Dirichlet spaces (for other versions see [C3], [CS2], [S4]). We will need it in the proof of Proposition 4.2 below.

Proposition 3.11. *Let ω and ω' in A_+^* be such that $\omega' \leq \omega$ and ω has finite energy. Then ω' has finite energy, the potential of ω' is dominated by the potential of ω*

$$G(\omega') \leq G(\omega),$$

meaning that $G(\omega) - G(\omega') \in \mathcal{P}_+$, and the energy content of ω' is not greater than the energy content of ω

$$\mathcal{E}[\omega'] \leq \mathcal{E}[\omega].$$

Proof. If $b \in \mathcal{B}$ is positive one has $\omega'(b) \leq \omega(b) \leq c_\omega \|b\|_{\mathcal{F}}$ for some $c_\omega > 0$. Decomposing a generic $b \in \mathcal{B}$ as a linear combination of four positive elements in \mathcal{B} one gets $|\omega'(b)| \leq 4c_\omega \|b\|_{\mathcal{F}}$ so that ω' is a finite-energy functional.

Notice that, for the same reason, $\omega - \omega'$ is a finite-energy functional on A whose potential is given by $G(\omega - \omega') = G(\omega) - G(\omega') \in \mathcal{P}_+$. We conclude the proof by the estimate

$$\mathcal{E}[\omega'] = \omega'(G(\omega')) \leq \omega(G(\omega')) \leq \omega(G(\omega)) = \mathcal{E}[\omega].$$

□

4. DENY'S EMBEDDING AND DENY'S INEQUALITY

This section is devoted, in the present setting of Dirichlet spaces over noncommutative C^* -algebras with traces, to prove a theorem obtained by J. Deny [Den] in the classical framework.

What Deny proved is that, if μ is a finite-energy measure on the locally compact space X , having a bounded potential, then the Dirichlet space \mathcal{F} is continuously imbedded in the space $L^2(X, \mu)$. In other words, the Dirichlet form, initially considered as a closed form on $L^2(X, m)$ with respect to a fixed positive measure m , results to be closable on all the spaces $L^2(\mu, X)$ with respect to finite-energy measures having bounded potentials. The probabilistic counterpart of this property is the “change of speed measure” or “random time change” of the stochastic Hunt processes X associated to the Dirichlet form and to the different reference measures. A detailed discussion about this can be found in [FOT].

We will prove below that if $\omega \in A_+^*$ is a finite-energy functional with respect to a Dirichlet form $(\mathcal{E}, \mathcal{F})$, based on the Hilbert space $L^2(A, \tau)$ of a trace τ on A , having a bounded potential $G(\omega) \in \mathcal{M}$, then the Dirichlet space \mathcal{F} is embedded in the G.N.S. space $L^2(A, \omega)$ with embedding norm less than $\sqrt{\|G(\omega)\|_{\mathcal{M}}}$.

One of the problems to circumvent in the proof of the result is that, in general, the functional ω need not be a trace and consequently the extension of bounded maps on the von Neumann algebra \mathcal{M} to bounded maps on the Hilbert space $L^2(A, \omega)$ cannot rely on their G.N.S.-symmetry but rather on their K.M.S.-symmetry with respect to ω (as introduced in [C1], [C2]). Note that, in general, finite-energy functionals need not be absolutely continuous with respect to the trace τ and, as a matter of fact, in current examples most of them are singular with respect to τ .

In the following we will denote by $\Omega \in L_+^2(A, \omega)$ the cyclic vector representing the functional $\omega \in A_+^*$:

$$\omega(b) = (\Omega, b\Omega)_{L^2(A, \omega)}, \quad b \in A.$$

We also prove below Deny's inequality in the noncommutative framework.

Theorem 4.1 (Deny's embedding theorem). *Let $\omega \in A_+^*$ be a finite-energy functional. If its potential $G(\omega) \in \mathcal{F}$ is bounded, hence belongs to the extended Dirichlet algebra $\mathcal{F} \cap \mathcal{M} = \tilde{\mathcal{B}}$, then one has*

$$(4.1) \quad \omega(b^*b) \leq \|G(\omega)\|_{\mathcal{M}} \|b\|_{\mathcal{F}}^2, \quad b \in \mathcal{B}.$$

Hence, there exists a continuous imbedding $T : \mathcal{F} \rightarrow L^2(A, \omega)$, with norm less than $\|G(\omega)\|_{\mathcal{M}}^{1/2}$, such that $Tb = b\Omega$ for $b \in \mathcal{B}$.

Before proving the theorem in its full generality, we investigate the special case where \mathcal{E} is bounded and ω is the restriction of a faithful normal functional on \mathcal{M} . The general case will be deduced from this special one with the help of Proposition 3.6.

Proposition 4.2. *Let \mathcal{E} be a bounded Dirichlet form on $L^2(A, \tau)$ and $\omega \in \mathcal{M}_{*+}$ be faithful with finite energy. If its potential is bounded $G(\omega) \in \mathcal{F}_+ \cap \mathcal{M}$, then one has*

$$(4.2) \quad \omega(b^*b) \leq \|G(\omega)\|_{\mathcal{M}} \|b\|_{\mathcal{F}}^2, \quad b \in \tilde{\mathcal{B}}.$$

Proof. The proof proceeds in several steps.

Step 1 (Construction of a completely positive kernel). Notice first that, by assumption, there exists $h \in L^1_+(A, \tau)$ such that $\omega(x) = \tau(hx)$ for $x \in \mathcal{M}$. In this case one may realize the G.N.S. representation of ω in the Hilbert space $L^2(A, \tau)$ setting $\Omega := h^{1/2} \in L^2_+(A, \tau)$,

$$\omega(b) = (\Omega, b\Omega)_2, \quad b \in \mathcal{M}.$$

One checks easily that $G(\omega) = (I+L)^{-1}h \in L^2(A, \tau) \cap \mathcal{M}$ and that it is nonsingular: in fact, if $p \in \mathcal{M}$ is the support projection of $G(\omega)$ in \mathcal{M} , one has

$$0 = \tau(G(\omega)(1_{\mathcal{M}} - p)) = \omega((I + L)^{-1}(1_{\mathcal{M}} - p)),$$

hence $(I + L)^{-1}(1_{\mathcal{M}} - p) = 0$ by faithfulness of ω , so that $p = 1_{\mathcal{M}}$.

For $x \in \mathcal{M}$, denote by $\rho_x \in \mathcal{M}_*$ the σ -weakly continuous linear form on \mathcal{M} defined by

$$\rho_x(y) = (Jx^*\Omega, y\Omega)_2, \quad y \in \mathcal{M}.$$

By the properties of the standard forms of von Neumann algebras (see [Ara]), if $x \in \mathcal{M}_+$, then

$$\rho_x(y) = (Jx^*\Omega, y\Omega)_2 \geq 0, \quad y \in \mathcal{M}_+,$$

so that $\rho_x \in \mathcal{M}_{*+}$. The map $\mathcal{M} \ni x \rightarrow \rho_x \in \mathcal{M}_*$ is antilinear, $\sigma(\mathcal{M}, \mathcal{M}_*)$ - $\sigma(\mathcal{M}_*, \mathcal{M})$ continuous and satisfies

$$0 \leq \rho_x \leq \|x\| \cdot \omega, \quad x \in \mathcal{M}_+.$$

Notice that, since, by assumption, \mathcal{E} is bounded, we have $\mathcal{F} = L^2(A, \tau)$ and $\tilde{\mathcal{B}} = L^2(A, \tau) \cap \mathcal{M}$. Applying Proposition 3.11, we get that ρ_x has finite energy and

$$G(\rho_x) \leq \|x\|G(\omega), \quad x \in \mathcal{M}_+.$$

Since, by assumption, the potential of ω is bounded, $G(\omega) \in \tilde{\mathcal{B}} = L^2(A, \tau) \cap \mathcal{M}$, we have a well-defined σ -weakly continuous, positive linear map $V : \mathcal{M} \rightarrow \mathcal{M}$ characterized by

$$V(x) := G(\rho_x), \quad x \in \mathcal{M}_+,$$

and satisfying $V(x) \in \tilde{\mathcal{B}} = L^2(A, \tau) \cap \mathcal{M}$ as well as

$$(4.3) \quad (Jx^*\Omega, b\Omega)_2 = \rho_x(b) = \langle V(x), b \rangle_{\mathcal{F}}, \quad x \in \mathcal{M}, \quad b \in \tilde{\mathcal{B}}.$$

We now proceed to check that $V : \mathcal{M} \rightarrow \mathcal{M}$ is a completely positive map. We first check that V is completely positive when considered as a map $V : \mathcal{M} \rightarrow \mathcal{F}$ between the ordered Banach spaces \mathcal{M} and \mathcal{F} : for $b_1, \dots, b_n \in \tilde{\mathcal{B}}$, $c_1, \dots, c_n \in \mathcal{M}$, we compute

$$\begin{aligned} \sum_{i,j} \langle V(c_i^*c_j), b_i^*b_j \rangle_{\mathcal{F}} &= (Jc_j^*c_i\Omega, b_i^*b_j\Omega)_2 \\ &= \sum_{i,j} (b_iJc_i\Omega, b_jJc_j\Omega)_2 \\ &= \left\| \sum_i b_iJc_i\Omega \right\|_2^2 \geq 0. \end{aligned}$$

This means that, not only is $V(x) \in \widetilde{\mathcal{B}}$ a potential for any $x \in \mathcal{M}_+$, so that it is positive in \mathcal{M} , because of Proposition 3.7, but also the matrix $[V(c_i^*c_j)]_{i,j=1}^n \in \mathbb{M}_n(\widetilde{\mathcal{B}})$ is positive in $\mathbb{M}_n(\mathcal{M})$ just applying again Proposition 3.7 to the matrix amplification \mathcal{E}_n of the complete Dirichlet form \mathcal{E} to $L^2(\mathbb{M}_n(A), \tau_n)$ described in Definition 2.1 iii).

Notice that $V(1_{\mathcal{M}}) = G(\omega)$ so that the endomorphism $V : \mathcal{M} \rightarrow \mathcal{M}$ has norm not greater than $\|G(\omega)\|_{\mathcal{M}}$. More precisely, for $x = x^* \in \mathcal{M}$ one has

$$V(x_+) - V(x) = V(x_-) = G(\rho_{x_-}) \geq 0,$$

hence $V(x) \leq V(x_+) \leq \|x_+\| G(\omega)$ and, for the sake of symmetry,

$$(4.4) \quad -\|x_-\| G(\omega) \leq V(x) \leq \|x_+\| G(\omega), \quad x = x^* \in \mathcal{M}.$$

Step 2 (Reduction of V and ω). Let us consider now the normal, positive functional $\omega' := \rho_{G(\omega)} \in \mathcal{M}_{*+} = L^1_+(A, \tau)$. By the properties of standard forms of von Neumann algebras (see [Ara]), there exists $\Omega' \in L^2_+(A, \tau)$ such that

$$(4.5) \quad \omega'(x) = (\Omega', x\Omega')_2, \quad x \in \mathcal{M}.$$

Moreover, $\|x\Omega'\|_2^2 = (x\Omega, JG(\omega)Jx\Omega)_2 \leq \|G(\omega)\|_{\mathcal{M}} \|x\Omega\|_2^2$. Consequently, there exists $\beta' \in \mathcal{M}'$ (the von Neumann algebra commutant of \mathcal{M} in $\mathcal{B}(L^2(A, \tau))$) such that $\Omega' = \beta' \Omega$ characterized by

$$\beta'(x\Omega) := x\Omega', \quad x \in \mathcal{M}.$$

Notice that, as Ω and Ω' belong to the self-polar cone $L^2_+(A, \tau)$ of a standard form, one has $J\Omega = \Omega$ and $J\Omega' = \Omega'$. Setting $\beta = J\beta'J \in J\mathcal{M}'J = \mathcal{M}$, one has $\beta\Omega = \Omega'$.

Notice also that, as ω and ω' are faithful states (by assumption for ω , and by nonsingularity of $G(\omega)$ for ω'), so that the vectors Ω and Ω' are cyclic and separating. Hence β and β' act in $L^2(A, \tau)$ as one-to-one operators with dense range.

Then, for $x, y \in \mathcal{M}$ one has

$$\begin{aligned} (y\Omega, \beta'^* \beta' x\Omega)_2 &= (y\beta'\Omega, x\beta'\Omega)_2 \\ &= (y\Omega', x\Omega')_2 \\ &= \omega'(y^*x) = (JG(\omega)J\Omega, y^*x\Omega)_2 \\ &= (y\Omega, JG(\omega)Jx\Omega)_2 \end{aligned}$$

so that $\beta'^*\beta' = JG(\omega)J$ and, finally,

$$(4.6) \quad \beta^*\beta = G(\omega).$$

As V is completely positive and $V(1_{\mathcal{M}}) = G(\omega) = \beta^*\beta$, with β having initial and final support equal to $1_{\mathcal{M}}$, there will exist a σ -weakly continuous completely positive endomorphism $W : \mathcal{M} \rightarrow \mathcal{M}$ such that

$$(4.7) \quad V(x) = \beta^*W(x)\beta, \quad x \in \mathcal{M}.$$

Moreover, $W(1_{\mathcal{M}}) = 1_{\mathcal{M}}$, so that W is a noncommutative Markov kernel and, in particular, a contraction of \mathcal{M} .

Step 3 (ω' -KMS-symmetry and $L^2(A, \omega')$ -contractivity of W). By the properties of standard forms (see [Ara]), we have, for $x, y \in \mathcal{M}$, the identities

$$\begin{aligned} (Jy\Omega', W(x)\Omega')_2 &= (JyJ\beta\Omega, W(x)\beta\Omega)_2 \\ &= (Jy\Omega, \beta^*W(x)\beta\Omega)_2 \\ &= (Jy\Omega, V(x)\Omega)_2 \\ &= \langle V(y^*), V(x) \rangle_{\mathcal{F}} \\ &= \langle V(x^*), V(y) \rangle_{\mathcal{F}} \\ &= (Jx\Omega', W(y)\Omega')_2 \\ &= (JW(y)\Omega', x\Omega')_2. \end{aligned}$$

This reveals that W is ω' -KMS-symmetric so that it extends to a bounded map on $L^2(A, \omega')$ by [C2], Proposition 2.24. As it is a contraction of \mathcal{M} , it will also be a contraction in $L^2(A, \omega')$. Alternatively, we can check the boundedness of the extension to $L^2(A, \omega')$ invoking the 2-positivity of W :

$$\begin{aligned} \|W(x)\Omega'\|_2 &= (\Omega, W(x)^*W(x)\Omega')_2 \\ (4.8) \quad &\leq (\Omega', W(x^*x)\Omega') \\ &= (JW(1_{\mathcal{M}})\Omega', x^*x\Omega') \\ &= (\Omega', x^*x\Omega')_2 = \|x\Omega'\|_2^2, \quad x \in \mathcal{M}. \end{aligned}$$

Consider now $x \in \mathcal{M}$ and compute

$$\begin{aligned} \langle V(x), V(x) \rangle_{\mathcal{F}} &= (Jx\Omega, V(x)\Omega)_{L^2} \\ &= (Jx\Omega', W(x)\Omega')_2 \\ &\leq \|x\Omega'\|_{L^2(A, \tau)}^2 \end{aligned}$$

so that

$$(4.9) \quad \|V(x)\|_{\mathcal{F}} \leq \|x\Omega'\|_2, \quad x \in \mathcal{M}. \quad \square$$

For x and y in \mathcal{M} , with y such that $\beta^*y\beta \in L^2(A, \tau)$, one computes

$$\begin{aligned} |\langle Jy\Omega', x\Omega' \rangle_2| &= |\langle J\beta^*y\beta\Omega, x\Omega \rangle_2| \\ &= |\langle \beta^*y\beta, V(x) \rangle_{\mathcal{F}}| \\ &\leq \|\beta^*y\beta\|_{\mathcal{F}} \|V(x)\|_{\mathcal{F}} \\ &\leq \|\beta^*y\beta\|_{\mathcal{F}} \|x\Omega'\|_2 \text{ by (4.9),} \end{aligned}$$

which provides $\|y\Omega'\|_2 \leq \|\beta^*y\beta\|_{\mathcal{F}}$ for all $y \in \mathcal{M}$ and then

$$(4.10) \quad \|y\beta\Omega\|_2 \leq \|\beta^*y\beta\|_{\mathcal{F}}, \quad y \in \mathcal{M}.$$

As we are assuming that the Dirichlet form is bounded, the $\|\cdot\|_{\mathcal{F}}$ norm is equivalent to the $L^2(A, \tau)$ norm. Moreover, since the functional ω is assumed to be faithful, the potential $G(\omega)$ has been proved to be nonsingular and, since $\beta^*\beta = G(\omega)$, $\beta \in \mathcal{M}$ is nonsingular too. Hence (4.10) extends as

$$(4.11) \quad \|x\Omega\|_2 \leq \|\beta^*x\|_{\mathcal{F}}, \quad x \in \mathcal{F} = L^2(A, \tau).$$

Considering the polar decomposition, there exists a unitary $u \in \mathcal{M}$ such that $\beta^* = G(\omega)^{1/2}u^*$ which implies

$$(4.12) \quad \|x\Omega\|_2 \leq \|G(\omega)^{1/2}u^*x\|_{\mathcal{F}}, \quad x \in \mathcal{F} = L^2(A, \tau),$$

or

$$(4.13) \quad \|ux\Omega\|_2 \leq \|G(\omega)^{1/2}x\|_{\mathcal{F}}, \quad x \in \mathcal{F} = L^2(A, \tau),$$

and finally

$$(4.14) \quad \|x\Omega\|_2 \leq \|G(\omega)^{1/2}x\|_{\mathcal{F}}, \quad x \in \mathcal{F} = L^2(A, \tau),$$

which provides the result:

$$\frac{1}{\|G(\omega)\|_{\mathcal{M}}} \|x\Omega\|_2^2 \leq \omega(x^*G(\omega)^{-1}x) \leq \|x\|_{\mathcal{F}}^2, \quad x \in \mathcal{F} = L^2(A, \tau).$$

Proof of the theorem. For $\varepsilon > 0$, the operator

$$L_\varepsilon = L(I + \varepsilon L)^{-1} = \frac{1}{\varepsilon}(I - (I + \varepsilon L)^{-1})$$

acts as a bounded positive operator in $L^2(A, \tau)$, but also (as it is of the form *constant* \times (*identity* $-$ *completely positive contraction*)) it acts on \mathcal{M} as the generator of a semigroup of symmetric completely positive contractions. This means that

$$(4.15) \quad \mathcal{E}_\varepsilon : L^2(A, \tau) \rightarrow [0, +\infty), \quad \mathcal{E}_\varepsilon[\xi] = \langle \xi, L_\varepsilon \xi \rangle_2, \quad \xi \in L^2(A, \tau),$$

is a bounded symmetric Dirichlet form on $L^2(A, \tau)$.

The associated Dirichlet space, denoted by \mathcal{F}_ε , is the vector space $L^2(A, \tau)$, equipped with the scalar product

$$\langle \eta, \xi \rangle_{\mathcal{F}_\varepsilon} = \langle \eta, (I + L_\varepsilon) \xi \rangle_2, \quad \xi, \eta \in L^2(A, \tau).$$

Notice that

$$(4.16) \quad \|\xi\|_{\mathcal{F}} = \lim_{\varepsilon \downarrow 0} \|\xi\|_{\mathcal{F}_\varepsilon} \quad \forall \xi \in \mathcal{F}.$$

Consider now the positive linear form $\tilde{\omega} \circ (I + \varepsilon L)^{-1}$ on \mathcal{C} , with $\tilde{\omega}$ provided by Proposition 3.5. It is well defined since $(I + \varepsilon L)^{-1}$ acts as a positive contraction on $L^2(A, \tau)$, hence as a positive contraction of \mathcal{F} (since it commutes with L), but also as a σ -weakly continuous completely positive contraction of \mathcal{M} , so that it maps $\tilde{\mathcal{B}}$ into $\tilde{\mathcal{B}}$ and \mathcal{C} into itself. One has, for $b \in \tilde{\mathcal{B}}$,

$$\begin{aligned} \tilde{\omega}((I + \varepsilon L)^{-1}(b)) &= \langle G(\omega), (I + \varepsilon L)^{-1}b \rangle_{\mathcal{F}} \\ &= \langle (I + \varepsilon L)^{-1}G(\omega), b \rangle_{\mathcal{F}} \\ &= \tau(h_\varepsilon b) \end{aligned}$$

with $h_\varepsilon = (I + L)(I + \varepsilon L)^{-1}G(\omega)$ well defined in $L^2(A, \tau)$, since $(I + L)(I + \varepsilon L)^{-1}$ is bounded. One has $\tau(h_\varepsilon b) \geq 0$ whenever $b \geq 0$, and $|\tau(h_\varepsilon b)| \leq \|\tilde{\omega}\|_{\mathcal{C}^*} \|b\|_{\mathcal{M}}$ for any $b \in \tilde{\mathcal{B}}$, so that $h_\varepsilon \in L^1(A, \tau)_+$ and that $\tilde{\omega} \circ (I + \varepsilon L)^{-1}$ extends as a normal positive linear form on \mathcal{M} .

The functional $\tilde{\omega} \circ (I + \varepsilon L)^{-1}$ has finite energy with respect to the Dirichlet form \mathcal{E}_ε , and the corresponding potential is

$$\begin{aligned} G_\varepsilon(\tilde{\omega} \circ (I + \varepsilon L)^{-1}) &= (I + L_\varepsilon)^{-1}h_\varepsilon \\ &= (I + L_\varepsilon)^{-1}(I + L)(I + \varepsilon L)^{-1}G(\omega) \\ &= \frac{1}{1 + \varepsilon}G(\omega) + \frac{\varepsilon}{1 + \varepsilon}(1 + (1 + \varepsilon)L)^{-1}G(\omega) \end{aligned}$$

so that this potential is bounded, with

$$(4.17) \quad \|G_\varepsilon(\tilde{\omega} \circ (I + \varepsilon L)^{-1})\|_{\mathcal{M}} \leq \|G(\omega)\|_{\mathcal{M}} \quad \forall \varepsilon > 0.$$

As A is separable, there will exist $h_0 \in L^2(A, \tau) \cap L^1(A, \tau) \cap \mathcal{M}_+$ which acts as a nonsingular operator on $L^2(A, \tau)$. Let $\omega_0 \in \mathcal{M}_{*+}$ be the corresponding normal positive linear functional on \mathcal{M} defined by $\omega_0(x) = \tau(h_0x)$ for $x \in \mathcal{M}$. Since ω_0 is, by construction, faithful and has finite energy with respect to \mathcal{E}_ε , the corresponding potential $G_\varepsilon(\omega_0) = (I + L_\varepsilon)^{-1}h_0$ is thus bounded, with

$$(4.18) \quad \|G_\varepsilon(\omega_0)\|_{\mathcal{M}} \leq \|h_0\|_{\mathcal{M}} \quad \forall \varepsilon > 0.$$

Applying now Proposition 4.2 to the Dirichlet form \mathcal{E}_ε and to the faithful, normal, positive linear functional $\tilde{\omega} \circ (I + \varepsilon L)^{-1} + \varepsilon\omega_0 \in \mathcal{M}_*$, we get, for all $b \in \mathcal{B}$,

$$(4.19) \quad \begin{aligned} \tilde{\omega}((I + \varepsilon L)^{-1}(b^*b)) + \varepsilon\omega_0(b^*b) &\leq \|G_\varepsilon(\tilde{\omega} \circ (I + \varepsilon L)^{-1}) + \varepsilon G_\varepsilon(\omega_0)\|_{\mathcal{M}} \|b\|_{\mathcal{F}_\varepsilon}^2 \\ &\leq (\|G(\omega)\|_{\mathcal{M}} + \varepsilon\|h_0\|_{\mathcal{M}}) \|b\|_{\mathcal{F}_\varepsilon}^2. \end{aligned}$$

As $\varepsilon \rightarrow 0$, $\|b\|_{\mathcal{F}_\varepsilon}^2$ tends to $\|b\|_{\mathcal{F}}^2$ (cf. (4.16)). The convergence in the left hand side is a bit more delicate, since ω does not necessarily extend as a linear form on \mathcal{M} . Nevertheless, for $b \in \mathcal{B}$, we have

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \tilde{\omega}((I + \varepsilon L)^{-1}(b^*b)) &= \lim_{\varepsilon \downarrow 0} \langle G(\omega), (I + \varepsilon L)^{-1}(b^*b) \rangle_{\mathcal{F}} \\ &= \langle G(\omega), b^*b \rangle = \omega(b^*b) \end{aligned}$$

since $(I + \varepsilon L)^{-1}\xi \rightarrow \xi$ in \mathcal{F} as $\varepsilon \downarrow 0$, for any $\xi \in \mathcal{F}$. Letting $\varepsilon \downarrow 0$ in (4.19), we get

$$\omega(b^*b) \leq \|G(\omega)\|_{\mathcal{M}} \|b\|_{\mathcal{F}}^2 \quad \forall b \in \mathcal{B}$$

and the theorem is proved. □

Remark 4.3. According to Lemma 3.8, for $b \in \tilde{\mathcal{B}}$, the operator $b^* \frac{1}{G(\omega)+\delta} b$ lies in the fine algebra \mathcal{C} . Passing to the increasing limit as $\delta \rightarrow 0$, one gets $b^* \frac{1}{G(\omega)} b$ as a nonnegative operator affiliated to the enveloping von Neumann algebra \mathcal{C}^{**} (cf. [Haa2]).

Consequently, for all $\omega \in \mathcal{C}_+^*$, the quantity $\omega(b^* \frac{1}{G(\omega)} b)$ is well defined in the extended half line $[0, +\infty]$. In particular, if $\omega \in A_+^*$ is a finite-energy functional, it extends as $\tilde{\omega}$ in \mathcal{C}_+^* and the quantity $\tilde{\omega}(b^* \frac{1}{G(\omega)} b)$ is well defined in the extended half line $[0, +\infty]$. The following Deny’s inequality provides a universal bound for this quantity.

Theorem 4.4 (Deny’s inequality). *For any finite-energy functional $\omega \in A_+^*$ the following inequality holds true:*

$$(4.20) \quad \tilde{\omega}\left(b^* \frac{1}{G(\omega)} b\right) \leq \|b\|_{\mathcal{F}}^2, \quad b \in \tilde{\mathcal{B}}.$$

If the potential is bounded, the inequality is saturated by the choice $b = G(\omega)$.

Proof. The proof goes through the discussion of several particular cases.

First particular case: *the Dirichlet form \mathcal{E} is bounded, the finite-energy functional $\omega \in A_+^*$ is faithful and its potential $G(\omega) \in \mathcal{P}_+$ is bounded too.* In this case the inequality 4.20 is just (4.13) or (4.14) at the end of the proof of Proposition 4.2.

Second particular case: *the Dirichlet form \mathcal{E} is bounded, the potential $G(\omega) \in \mathcal{P}_+$ of the finite-energy functional $\omega \in A_+^*$ is bounded (but ω is not necessarily faithful).* Choose a nonsingular $h_0 \in L_+^1(A, \tau) \cap \mathcal{M} \subset L^2(A, \tau)$ and consider the functional $\omega_0(\cdot) := \tau(h_0\cdot)$. Then ω_0 is faithful, it has finite energy (since h_0 lies in $L^2(A, \tau)$)

and it has bounded potential $G(\omega_0) = (I + L)^{-1}h_0$ (see Example 3.2). The first particular case applies to $\omega + \varepsilon\omega_0$ so that

$$(\omega + \varepsilon\omega_0)\left(b^* \frac{1}{G(\omega) + \varepsilon G(\omega_0) + \delta} b\right) \leq \|b\|_{\mathcal{F}}^2, \quad \varepsilon, \delta > 0, b \in \tilde{\mathcal{B}}.$$

Passing to the limit first as $\varepsilon \rightarrow 0$ and then as $\delta \rightarrow 0$ provides the result in this case.

Third particular case: *the Dirichlet form \mathcal{E} is bounded (but neither the finite-energy functional $\omega \in A_+^*$ is assumed to be faithful nor its potential $G(\omega) \in \mathcal{P}_+$ is assumed to be bounded)*. As \mathcal{E} is bounded, the generator L is a bounded operator on $L^2(A, \tau)$ so that $\omega(\cdot) = \tau(h \cdot)$ where $h \in L_+^1(A, \tau) \cap L^2(A, \tau)$ and $h = (I + L)G(\omega)$ for $G(\omega) \in \mathcal{P}_+ \subset L^2(A, \tau)$. For any fixed $M > 0$, consider $h_M := h \wedge M \in L_+^1(A, \tau) \cap \mathcal{M}$ and the corresponding finite-energy functional $\omega_M(\cdot) := \tau(h_M \cdot)$. One has $G(\omega_M) = (I + L)^{-1}h_M \leq (I + L)^{-1}h = G(\omega)$. According to the second particular case

$$\omega_M\left(b^* \frac{1}{G(\omega) + \delta} b\right) \leq \omega_M\left(b^* \frac{1}{G(\omega_M) + \delta} b\right) \leq \|b\|_{\mathcal{F}}^2, \quad \delta > 0, b \in \tilde{\mathcal{B}}.$$

Passing to the limit first $M \rightarrow +\infty$ and then $\delta \searrow 0$ one gets the result in case.

General case: *\mathcal{E} is any Dirichlet form and $\omega \in A_+^*$ is any finite-energy functional*. For any $\varepsilon > 0$, define the functional $\omega_\varepsilon = \omega \circ \frac{1}{1 + \varepsilon L}$ and the bounded Dirichlet form \mathcal{E}_ε with generator $\frac{L}{1 + \varepsilon L}$. By Lemma 3.10, ω_ε has finite energy with respect to \mathcal{E} , and a fortiori with respect to \mathcal{E}_ε .

Let us identify for $b \in \tilde{\mathcal{B}}$,

$$\begin{aligned} \omega\left(\frac{1}{1 + \varepsilon L} b\right) &= \left\langle G(\omega), \frac{1}{1 + \varepsilon L} b \right\rangle_{\mathcal{F}} \\ &= \left\langle \frac{1}{1 + \varepsilon L} G(\omega), b \right\rangle_{\mathcal{F}} \\ &= \langle G_\varepsilon(\omega_\varepsilon), b \rangle_{\mathcal{F}_\varepsilon} \\ &= \left\langle \frac{1}{1 + \varepsilon L} \frac{1 + (1 + \varepsilon)L}{1 + L} G_\varepsilon(\omega_\varepsilon), b \right\rangle_{\mathcal{F}} \end{aligned}$$

so that we get, applying Lemma 3.10,

$$\begin{aligned} G_\varepsilon(\omega_\varepsilon) &= \frac{1 + L}{1 + (1 + \varepsilon)L} G(\omega) \\ &= \frac{1}{1 + \varepsilon} G(\omega) + \frac{\varepsilon}{1 + \varepsilon} \frac{1}{1 + (1 + \varepsilon)L} G(\omega) \\ &\leq \frac{1}{1 + \varepsilon} G(\omega) + \frac{\varepsilon}{1 + \varepsilon} \frac{1}{1 - \varepsilon} G(\omega) = \frac{1}{1 - \varepsilon^2} G(\omega). \end{aligned}$$

Now the previous particular case allows us to write, for any $\delta > 0$:

$$\begin{aligned} (1 - \varepsilon^2)\omega_\varepsilon\left(b^* \frac{1}{G(\omega) + \delta} b\right) &\leq \omega_\varepsilon\left(b^* \frac{1}{G_\varepsilon(\omega_\varepsilon)} b\right) \leq \|b\|_{\mathcal{F}_\varepsilon}^2, \\ \omega\left(b^* \frac{1}{G(\omega) + \delta} b\right) &\leq \|b\|_{\mathcal{F}}^2. \end{aligned}$$

Passing to the limit first as $\varepsilon \rightarrow 0$ and then as $\delta \rightarrow 0$ provides the result. □

As a corollary of the generalized Deny’s embedding theorem, we get the following bound which will be used below in Proposition 5.8 and Proposition 6.3.

Corollary 4.5. *Let us consider a bounded potential $G \in \mathcal{P}_+ \cap \mathcal{M} = \mathcal{P}_+ \cap \tilde{\mathcal{B}}$. Then one has*

$$(4.21) \quad \langle G, b^*b \rangle_{\mathcal{F}} \leq \|G\|_{\mathcal{M}} \|b\|_{\mathcal{F}}^2 \quad \forall b \in \tilde{\mathcal{B}}.$$

Proof. When $G = G(\omega)$, with $\omega \in A_+^*$ having finite energy, this is exactly Theorem 4.1. Now, fix $\varepsilon > 0$ and consider $G_\varepsilon = (I + \varepsilon L)^{-1}G$, $h_\varepsilon = (I + L)G_\varepsilon$. By Proposition 3.6 we have $h_\varepsilon \in L^2(A, \tau)_+$.

For $\delta > 0$, let p_δ be the spectral projection of h_ε corresponding to the interval $[\delta, +\infty[$. Then, $p_\delta h_\varepsilon \in L^1(A, \tau)_+$ and the corresponding linear form $b \rightarrow \tau(p_\delta h_\varepsilon b)$ has a potential $G_{\varepsilon, \delta}$ equal to

$$G_{\varepsilon, \delta} = (I + L)^{-1}(p_\delta h_\varepsilon) \leq (I + L)^{-1}h_\varepsilon = G_\varepsilon.$$

Theorem 4.1 applied to this linear form provides

$$(4.22) \quad \langle G_{\varepsilon, \delta}, b^*b \rangle_{\mathcal{F}} \leq \|G_\varepsilon\|_{\mathcal{M}} \|b\|_{\mathcal{F}}^2 \quad \forall b \in \tilde{\mathcal{B}}$$

since $\|G_{\varepsilon, \delta}\|_{\mathcal{M}} \leq \|G_\varepsilon\|_{\mathcal{M}} \leq \|G\|_{\mathcal{M}}$. The convergence in \mathcal{F} , $\lim_{\delta \rightarrow 0} G_{\varepsilon, \delta} = G_\varepsilon$, is obvious and we already noticed that $G_\varepsilon \rightarrow G$ in \mathcal{F} as $\varepsilon \rightarrow 0$. Hence the result. \square

5. ENERGY FUNCTIONALS OR “CARRÉ DU CHAMP” OF DIRICHLET SPACES

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ on the space $L^2(A, \tau)$ of a faithful, semifinite, lower semicontinuous, positive trace τ on a C^* -algebra A gives rise to a family of positive functionals $\{\Gamma[a] \in A_+^* : a \in \mathcal{F}\}$, called *carré du champ*, from which the quadratic form can be recovered:

$$\mathcal{E}[a] = \langle \Gamma[a], 1_{A^{**}} \rangle.$$

In the noncommutative setting they were introduced in [CS1] to analyze the structure of Dirichlet forms on possibly noncommutative C^* -algebras. In the commutative case, where $A = C_0(X)$, they were defined by Y. Le Jan [LJ] as *energy measures*. This appellation being justified by the fact that in applications the positive measure $\Gamma[a]$ may represent the energy distributions over X of the finite-energy configuration $a \in \mathcal{F}$.

Since in the case of the Dirichlet integral on a Riemannian manifold V with measure m one has $d\Gamma[f] = |\nabla f|^2 \cdot dm$, they are often called “*carré du champ*” (even if in general the functional $\Gamma[a]$ is not absolutely continuous with respect to the reference trace τ).

In this section we show that the carré du champ $\Gamma[G]$ of bounded potentials $G \in \mathcal{P}_+ \cap \mathcal{M}$ form a natural class of finite-energy functionals, intimately associated to a Dirichlet space.

5.1. Carré du champ and differential calculus of a Dirichlet space.

Definition 5.1 (Carré du champ [CS1]). The *carré du champ* $\Gamma[a] \in A_+^*$ of $a \in \mathcal{B}$ is the functional on A defined by

$$(5.1) \quad \langle \Gamma[a], b \rangle := \frac{1}{2}(\mathcal{E}(a, ab) + \mathcal{E}(ab^*, a) - \mathcal{E}(b^*, a^*a)), \quad b \in \mathcal{B}.$$

It can be shown (see [CS1]) that $\Gamma[a]$ extends as a bounded positive functional on A whose norm is $\mathcal{E}[a]$.

In order to extend the definition to all elements $a \in \mathcal{F}$ of the Dirichlet space and to give a short proof of the main result of this section, we briefly recall the main properties of the differential calculus associated to a regular Dirichlet form (see [CS1], [C2]), in terms of which an alternative and more manageable description of $\Gamma[a]$ can be given.

Any regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(A, \tau)$ can be described as

$$\mathcal{E}[a] = \|\partial a\|_{\mathcal{H}}^2, \quad a \in \mathcal{F},$$

by a map $\partial : \mathcal{F} \rightarrow \mathcal{H}$ which is closed on $L^2(A, \tau)$, taking its values in a Hilbert A - A -bimodule \mathcal{H} and which is a derivation on the Dirichlet algebra $\mathcal{B} \subseteq \mathcal{F}$, in the sense that satisfies the Leibniz rule

$$\partial(ab) = (\partial a) \cdot b + a \cdot (\partial b), \quad a, b \in \mathcal{B}$$

(the dots denote the left and right actions of elements of \mathcal{B} on vectors in \mathcal{H}). Moreover, on the bimodule there exists a symmetry $\mathcal{J} : \mathcal{H} \rightarrow \mathcal{H}$, i.e. an antiunitary involution which intertwines the left and right actions of A

$$\mathcal{J}(a\xi b) = b^*(\mathcal{J}\xi)a^*, \quad a, b \in A, \quad \xi \in \mathcal{H},$$

such that

$$\partial(a^*) = \mathcal{J}(\partial a), \quad a \in A.$$

Summarizing, one describes the self-adjoint, nonnegative operator L on $L^2(A, \tau)$ whose quadratic form is the Dirichlet form $(\mathcal{E}, \mathcal{F})$ as the *divergence of a derivation*: $L = \partial^* \circ \partial$ or, in other words, one can refer to the derivation as the *differential square root of the generator L* . The derivation representing a regular Dirichlet form is essentially unique (see [CS1], Theorem 8.3, for details).

Example 5.2. Derivation associated to negative definite functions on group C^* -algebras. In Example 2.7 we considered the Dirichlet form \mathcal{E}_ℓ on the reduced group C^* -algebra $C_{red}^*(\Gamma)$ of discrete group Γ , associated to a negative definite function $\ell : \Gamma \rightarrow [0, +\infty)$. To describe the derivation it gives rise to, recall that there exists a 1-cocycle (π, \mathcal{K}, c) , where $\pi : G \rightarrow \mathcal{K}$ is an orthogonal representation of G in a suitable real Hilbert space \mathcal{K} and $c : \Gamma \rightarrow \mathcal{K}$ is a function satisfying

$$c(st) = c(s) + \pi(s)c(t), \quad s, t \in \Gamma,$$

such that $\ell(s) = \|c(s)\|_{\mathcal{K}}^2$ for all $s \in \Gamma$. Denote by $\mathcal{K}_{\mathbb{C}}$ the complexification of the real Hilbert space \mathcal{K} and by $\mathcal{K}_{\mathbb{C}} \ni \xi \mapsto \bar{\xi} \in \mathcal{K}_{\mathbb{C}}$ its canonical conjugation. The tensor product of complex Hilbert spaces $\mathcal{K}_{\mathbb{C}} \otimes l^2(\Gamma)$ is a $C_{red}^*(\Gamma)$ -bimodule under the commuting actions $\pi_l := \pi \otimes \lambda$ and $\pi_r := id \otimes \rho$ constructed by the left and right regular representations λ, ρ of $C_{red}^*(\Gamma)$ in $l^2(\Gamma)$. This bimodule structure turns out to be symmetric with respect to the antilinear involution given by

$$\mathcal{J}(\xi \otimes a) := \bar{\xi} \otimes J(a), \quad \xi \otimes a \in \mathcal{K}_{\mathbb{C}} \otimes l^2(\Gamma),$$

where $J(a)(s) = \overline{a(s^{-1})}$, $s \in \Gamma$, is just the involution associated to the standard cone of positive definite functions in $l^2(\Gamma)$. The map $\partial : D(\partial) \rightarrow \mathcal{K}_{\mathbb{C}} \otimes l^2(\Gamma) \simeq l^2(\Gamma, \mathcal{K}_{\mathbb{C}})$ defined by

$$\text{dom}(\partial) := c_c(\Gamma), \quad \partial(a)(t) := c(t)f(t), \quad a \in c_c(\Gamma), \quad t \in \Gamma,$$

is a closable derivation such that

$$\mathcal{E}[a] = \|\partial a\|_{\mathcal{K}_{\mathbb{C}} \otimes l^2(\Gamma)}^2, \quad a \in D(\partial) \subseteq \mathcal{F}_\ell.$$

See [CS1], [C2] for the details.

Example 5.3. Derivation on noncommutative tori. The derivation associated to the Dirichlet form introduced in Example 2.8 and given by

$$\mathcal{E} \left[\sum_{n,m \in \mathbb{Z}} \alpha_{n,m} U^n V^m \right] = \sum_{n,m \in \mathbb{Z}} (n^2 + m^2) |\alpha_{n,m}|^2$$

on the noncommutative torus A_θ is the direct sum

$$\partial(a) = \partial_1(a) \oplus \partial_2(a)$$

of the following derivations ∂_1 and ∂_2 :

$$\partial_1(U^n V^m) = inU^n V^m, \quad \partial_2(U^n V^m) = imU^n V^m, \quad n, m \in \mathbb{Z}.$$

The A_θ -bimodule \mathcal{H} associated with \mathcal{E} is a sub-bimodule of the direct sum $L^2(A, \tau) \oplus L^2(A, \tau)$ of two copies of the standard A_θ -bimodule.

Example 5.4. A Dirichlet form occurring in the NCG approach to the quantum Hall effect. A specific example of a Dirichlet form and associated differential calculus, arising in mathematical physics, has been considered by J. Bellissard [B], [BES] to model the quantum Hall effect. An algebra can be constructed starting from a continuous action $\alpha : \mathbb{R}^2 \times \Omega \rightarrow \Omega$ of the additive group \mathbb{R}^2 on a compact Hausdorff space Ω and a number $B \in \mathbb{R}$ representing the intensity of a constant magnetic field. The space $C_c(\mathbb{R}^2 \times \Omega)$ of continuous functions with compact support may be considered as a $*$ -algebra under product and involution given by

$$\begin{aligned} (a * b)(x, \omega) &:= \int_{\mathbb{R}^2} dy a(y, \omega) b(x - y, \alpha(-y, \omega)) e^{-i \frac{qB}{2\hbar} \cdot x \wedge y}, \quad a, b \in C_c(\mathbb{R}^2 \times \Omega), \\ b^*(\omega, x) &:= \overline{b(-x, \alpha(-x, \omega))}, \quad (x, \omega) \in \mathbb{R}^2 \times \Omega. \end{aligned}$$

For fixed $\omega \in \Omega$, one considers the representation π_ω of $C_c(\mathbb{R}^2 \times \Omega)$ on the Hilbert space $L^2(\mathbb{R}^2)$ given by

$$\pi_\omega(a)\psi(x) := \int_{\mathbb{R}^2} dy a(y - x, \alpha(-x, \omega)) e^{-i \frac{qB}{2\hbar} \cdot x \wedge y} \psi(y), \quad a \in C_c(\mathbb{R}^2 \times \Omega), \quad x \in \mathbb{R}^2.$$

A C^* -norm can then be defined on $C_c(\mathbb{R}^2 \times \Omega)$ by

$$\|a\| := \sup_{\omega \in \Omega} \|\pi_\omega(a)\|_{B(\mathcal{H})}, \quad a \in C_c(\mathbb{R}^2 \times \Omega),$$

so that its completion $C^*(\alpha, \mathbb{R}^2, \Omega, B)$ is a C^* -algebra. For any α -invariant probability measure P on Ω , there exists a faithful, semifinite, lower semicontinuous trace on $C^*(\alpha, \mathbb{R}^2, \Omega, B)$, characterized by

$$\tau(a) = \int_{\Omega} P(d\omega) a(0, \omega), \quad a \in C_c(\mathbb{R}^2 \times \Omega).$$

The G.N.S. space $L^2(C^*(\alpha, \mathbb{R}^2, \Omega, B), \tau)$ coincides, at the Hilbert space level, with the ordinary Lebesgue space $L^2(\mathbb{R}^2 \times \Omega)$. The above C^* -algebra contains elements representing physical observables of the system in which one electron is moving in a metal strip under the influence of a periodic or aperiodic potential created by the distribution of atomic ions, occupying the fixed position of a lattice, and a magnetic field normal to the strip. The compact space Ω is the limit set of all translates, by the action α , of the ionic lattice. Its compactness represents

the macroscopic homogeneity of the system. If $P_{(-\infty, E]}$ represents the spectral projection of a Hamiltonian operator H , affiliated to the C^* -algebra, corresponding to energies below the level E , then $\tau(P_{(-\infty, E]})$ represents the integrated density of states, i.e. the total number of states corresponding to particles having energy below E . Alternatively, combining the spectral projection-valued measure of H with the trace above (extending it to the von Neumann algebra) one gets the density of states of the system, a measure on the spectrum of H representing the distribution of states with respect to energy. From it other mean values of physical observables can be derived such as conductivity. A Dirichlet form is given by

$$\mathcal{E}[a] := \int_{\Omega} P(d\omega) \int_{\mathbb{R}^2} dx |x|^2 \cdot |a(x, \omega)|^2, \quad a \in L^2(\mathbb{R}^2 \times \Omega),$$

and $C_c(\mathbb{R}^2 \times \Omega)$ is a form core. Moreover the associated derivation is given by

$$\partial a(x, \omega) := ixa(x, \omega) \in \mathbb{C} \oplus \mathbb{C}, \quad (x, \omega) \in \mathbb{R}^2 \times \Omega,$$

taking values in the $C^*(\alpha, \mathbb{R}^2, \Omega, B)$ -bimodule $L^2(\mathbb{R}^2 \times \Omega) \oplus L^2(\mathbb{R}^2 \times \Omega)$. The finiteness of $\mathcal{E}[P_{(-\infty, E]}]$ represents a sort of localization and it is a sufficient condition to have finite Hall conductivity proportional to the Connes-Chern character of the element of the K-theory of $C^*(\alpha, \mathbb{R}^2, \Omega, B)$ represented by the projection $P_{(-\infty, E]}$.

Example 5.5. Curvature and derivations on Riemannian manifolds. From the discussion in Example 2.6, it is clear that the derivation associated to the quadratic form \mathcal{E}_B of the Bochner Laplacian Δ_B is just the covariant derivative of the Levi-Civita connection of the Clifford bundle. For manifolds with nonnegative curvature operator, the derivation associated to the quadratic form \mathcal{E}_D of the Dirac Laplacian D^2 is not a local operator in general, as it may contain also approximately bounded parts due to the nonvanishing of the curvature (see [CS3]).

Example 5.6. A Dirichlet form arising in free probability. In the free probability theory, developed by D. V. Voiculescu (see [V1], [V2], [V3]), a special role is played by a distinguished natural Dirichlet form.

Let (\mathfrak{M}, τ) be a noncommutative probability space, i.e. a von Neumann algebra \mathfrak{M} with a faithful, normal trace state τ on it. Let $1 \in B \subseteq \mathfrak{M}$ be a $*$ -subalgebra and let $X = X^* \in \mathfrak{M}$ be a noncommutative random variable.

Denote by $B[X] \subseteq \mathfrak{M}$ the $*$ -subalgebra generated by B and X and consider on $B[X] \otimes_{\text{alg}} B[X]$ the $B[X]$ -bimodule structure given by

$$c \cdot (a \otimes b) := ca \otimes b, \quad (a \otimes b) \cdot c := a \otimes bc.$$

If X and B are algebraically free, in the sense that no algebraic relation exists between them, there exists a unique derivation $\partial_X : B[X] \rightarrow B[X] \otimes_{\text{alg}} B[X]$ such that

$$\partial_X X = 1 \otimes 1, \quad \partial_X b = 0, \quad b \in B,$$

which, more explicitly, acts as follows:

$$\begin{aligned} \partial_X (b_0 X b_1 X \dots X b_n) &= \sum_{k=1}^n b_0 X b_1 \dots b_{k-1} (1 \otimes 1) b_k X \dots X b_n \\ &= \sum_{k=1}^n b_0 X b_1 \dots b_{k-1} \otimes b_k X \dots X b_n, \quad b_0, \dots, b_n \in B. \end{aligned}$$

In other words, if $B[X]$ is regarded as the algebra of noncommutative polynomials in the variable X with coefficients belonging to the algebra B , the operator ∂_X acts as the partial derivation with respect to the noncommutative variable X and the elements of B play the role of *constants*.

Denoting by $W^*(B[X])$ the von Neumann subalgebra of \mathfrak{M} generated by $B[X]$ we may consider on it the restriction of the trace τ and the standard representation in $L^2(W^*(B[X]), \tau)$. The derivation may be considered as a densely defined map from the Hilbert space $L^2(W^*(B[X]), \tau)$ to the Hilbert bimodule $L^2(W^*(B[X]) \otimes W^*(B[X]), \tau \otimes \tau)$.

Under the assumption that $1 \otimes 1 \in D(\partial_X^*)$ it may be proved that

- the derivation is closable,
- $B[X] \subset D(\partial_X^* \overline{\partial_X})$ and in particular that $X \in D(\partial_X^* \overline{\partial_X})$.

The element

$$\mathcal{J}(X : B) := \partial_X^*(1 \otimes 1) \in L^2(W^*(B[X]), \tau)$$

is then called the *noncommutative Hilbert transform of X with respect to B* and the square of its norm

$$\Phi^*(X : B) := \|\mathcal{J}(X : B)\|_2^2 = \|\partial_X(1 \otimes 1)\|_2^2 = \|\partial_X^* \partial_X(X)\|_2^2$$

is by definition the *relative free information* of X with respect to B .

In case $B = \mathbb{C}$ and μ_X is the distribution of X given by

$$\int_{\mathbb{R}} f(t) \mu_X(dt) := \tau(f(X)), \quad f \in C_0(\mathbb{R}),$$

the algebra $W^*(\mathbb{C}[X])$ is identified with $L^\infty(\mathbb{R}, \mu_X)$ and the Hilbert space $L^2(W^*(\mathbb{C}[X]), \tau)$ becomes $L^2(\mathbb{R}, \mu_X)$. The elements $f \in \mathbb{C}[X]$ are just polynomials on \mathbb{R} and $\partial_X f$ coincides with the difference quotient (4.3),

$$\partial_X f(s, t) = \begin{cases} \frac{f(s)-f(t)}{s-t} & \text{if } s \neq t, \\ f'(s) & \text{if } s = t. \end{cases}$$

In case the Radon-Nikodym derivative $p := \frac{d\mu_X}{d\lambda}$ with respect to the Lebesgue measure λ exists in $L^3(\mathbb{R}, \lambda)$, $\mathcal{J}(X : \mathbb{C})$ is, up to a factor 2π , the Hilbert transform H_p of p :

$$H_p(t) := \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{p(s)}{t-s} ds.$$

In general, the condition $1 \otimes 1 \in D(\partial_X^*)$ assures that the closure of the quadratic form

$$\mathcal{E}_X[\xi] := \|\partial_X \xi\|^2, \quad \xi \in B[X],$$

is a Dirichlet form on $L^2(W^*(B[X]), \tau)$. The Dirichlet form \mathcal{E}_X is deeply connected with the relative free Fisher information and with the free entropy. In fact it has been shown by Ph. Biane [Bi] that the Hessian of the free entropy coincides with \mathcal{E}_X on the domain where the relative free Fisher information is finite. Moreover, the Dirichlet form \mathcal{E}_X satisfies the Poincaré inequality if and only if the random variable X is centered, has unital covariance and a semi-circular distribution ([Bi]).

The following lemma contains consequences of the crucial observation that a Dirichlet form which is regular with respect to the C^* -algebra A is also automatically regular with respect to the fine C^* algebra \mathcal{C} .

Lemma 5.7. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(A, \tau)$ which is regular with respect to the C^* -algebra A . Then the trace τ on A naturally extends to a trace on the fine C^* -algebra \mathcal{C} so that the G.N.S. representation of (\mathcal{C}, τ) is an extension of the G.N.S. representation of (A, τ) and, in particular, the G.N.S. Hilbert spaces coincide: $L^2(\mathcal{C}, \tau) = L^2(A, \tau) = L^2(\mathcal{M}, \tau)$.*

Moreover, since $\mathcal{C} \cap \mathcal{F} \supseteq \tilde{\mathcal{B}} \cap \mathcal{F} = \tilde{\mathcal{B}}$, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is also regular with respect to the C^ -algebra \mathcal{C} .*

As a consequence, the differential calculus $(\tilde{\partial}, \tilde{\mathcal{B}}, \tilde{\mathcal{H}}, \tilde{\mathcal{J}})$ associated to $(\mathcal{E}, \mathcal{F})$ on (\mathcal{C}, τ) is an extension of the corresponding one $(\partial, \mathcal{B}, \mathcal{H}, \mathcal{J})$ on (A, τ) . In particular, once these calculi have been identified, the Leibniz rule holds true on the extended Dirichlet algebra $\tilde{\mathcal{B}}$,

$$\partial(ab) = (\partial a) \cdot b + a \cdot (\partial b), \quad a, b \in \tilde{\mathcal{B}}.$$

Proof. Notice that, even if the fine C^* -algebra \mathcal{C} need not be separable, it acts, by definition, on a separable Hilbert space so that it admits a faithful state and the framework of [CS1] applies.

The first statement concerning the trace comes from the fact that, by definition, $A \subseteq \mathcal{C} \subseteq \mathcal{M}$ so that the normal extension of the trace τ to the von Neumann algebra \mathcal{M} reduces to a trace on the subalgebra \mathcal{C} . The second one follows because, by definition, the Dirichlet algebra $\mathcal{C} \cap \mathcal{F}$ of $(\mathcal{E}, \mathcal{F})$ with respect to (\mathcal{C}, τ) contains the extended Dirichlet algebra $\tilde{\mathcal{B}}$ and this one is, again by definition, dense in \mathcal{C} . \square

As announced before, using the derivation associated to a Dirichlet space, one can readily give a definition of the energy functional $\Gamma[a]$ for all elements $a \in \mathcal{F}$ by

$$(5.2) \quad \langle \Gamma[a], b \rangle_{\mathcal{C}^*, \mathcal{C}} = \langle \partial a, (\partial a) \cdot b \rangle_{\mathcal{H}}, \quad b \in \mathcal{C}.$$

Using the Leibniz rule one can check that the above formula reduces to (5.1) whenever $a, b \in \mathcal{B}$.

The following result shows that the family of finite-energy functionals include some natural functional deeply connected to the structure of the Dirichlet space.

Proposition 5.8. *If G is a bounded potential, $G \in \mathcal{P}_+ \cap \mathcal{M} = \mathcal{P}_+ \cap \tilde{\mathcal{B}}$, its carré du champ $\Gamma[G] \in \mathcal{C}_+^*$ is a finite-energy functional.*

Proof. Let us consider on the extended Dirichlet algebra the functional $\omega_G : \tilde{\mathcal{B}} \rightarrow \mathbb{C}$ defined by the potential $G \in \mathcal{P}_+ \cap \tilde{\mathcal{B}}$:

$$\omega_G : \tilde{\mathcal{B}} \rightarrow \mathbb{C}, \quad \omega_G(b) := \langle G, b \rangle_{\mathcal{F}}.$$

Since the Dirichlet form is completely positive, the functional ω_G is completely positive with respect to the cone $\mathcal{P}_+ \subset \tilde{\mathcal{B}}$. Therefore a Cauchy-Schwartz inequality holds true:

$$(5.3) \quad |\omega_G(b^*c)|^2 \leq \omega_G(b^*b) \cdot \omega_G(c^*c), \quad b, c \in \tilde{\mathcal{B}}.$$

Hence we have

$$(5.4) \quad |\langle G, Gb \rangle_{\mathcal{F}}|^2 \leq \langle G, G^2 \rangle_{\mathcal{F}} \cdot \langle G, b^*b \rangle_{\mathcal{F}}, \quad b \in \tilde{\mathcal{B}},$$

and by Corollary 4.5 we have also

$$(5.5) \quad |\langle G, Gb \rangle_{\mathcal{F}}| \leq \|G\|_{\mathcal{M}} \cdot \|G\|_{\mathcal{F}} \cdot \|b\|_{\mathcal{F}}, \quad b \in \tilde{\mathcal{B}}.$$

Then we compute for $b \in \tilde{\mathcal{B}}_+$

$$\begin{aligned} \Gamma[G](b) &= \langle \partial(G), \partial(G)b \rangle_{\mathcal{H}} \\ &= \langle \partial(G), \partial(Gb) \rangle_{\mathcal{H}} - \langle \partial(G), G\partial(b) \rangle_{\mathcal{H}} \\ &= \mathcal{E}(G, Gb) - \langle G\partial(G), \partial(b) \rangle_{\mathcal{H}} \\ &\leq \langle G, Gb \rangle_{\mathcal{F}} + \|G\|_{\mathcal{M}} \sqrt{\mathcal{E}[G]} \cdot \sqrt{\mathcal{E}[b]} \quad (b \in \tilde{\mathcal{B}}) \\ &\leq \langle G, Gb \rangle_{\mathcal{F}} + \|G\|_{\mathcal{M}} \|G\|_{\mathcal{F}} \cdot \|b\|_{\mathcal{F}} \\ &\leq \|G\|_{\mathcal{M}} \cdot \|G\|_{\mathcal{F}} \cdot \|b\|_{\mathcal{F}} + \|G\|_{\mathcal{M}} \|G\|_{\mathcal{F}} \cdot \|b\|_{\mathcal{F}} \\ &= 2\|G\|_{\mathcal{M}} \cdot \|G\|_{\mathcal{F}} \cdot \|b\|_{\mathcal{F}}, \end{aligned}$$

which provides the result. □

6. MULTIPLIERS OF DIRICHLET SPACES

We define in this section *multipliers of Dirichlet spaces* and, as a final application of the results obtained so far, we prove their existence and the fact that they give rise to a form core. This notion extends the one investigated for the Sobolev space $H^{1,2}$ of Euclidean domains [MS], [R], [Str].

Definition 6.1 (Multipliers of a Dirichlet space). An element $b \in \mathcal{M}$ is called a *multiplier* of the Dirichlet space $(\mathcal{E}, \mathcal{F})$ if

$$b\xi \in \mathcal{F} \quad \text{and} \quad \xi b \in \mathcal{F} \quad \forall \xi \in \mathcal{F}.$$

A direct application of the closed-graph theorem implies that multipliers are bounded maps on the Dirichlet space \mathcal{F} and form an involutive algebra, denoted by $\mathcal{M}(\mathcal{E}, \mathcal{F})$, which is both a subalgebra of the algebra $\mathcal{B}(L^2(A, \tau))$ of all bounded operators on $L^2(A, \tau)$ and a subalgebra of the algebra $\mathcal{B}(\mathcal{F})$ of all bounded operators on \mathcal{F} .

Notice that if the Dirichlet space contains the unit $1_{\mathcal{M}} \in \mathcal{F}$, then the multiplier algebra is a subalgebra of the extended Dirichlet algebra: $\mathcal{M}(\mathcal{E}, \mathcal{F}) \subseteq \tilde{\mathcal{B}}$.

Example 6.2 (Multipliers on group C*-algebras). In the framework of Example 2.7, let us consider the Dirichlet form associated to a positive, conditionally negative type function $\ell : \Gamma \rightarrow [0, +\infty)$ on a discrete group Γ ,

$$\mathcal{E}_{\ell}[a] := \sum_{s \in \Gamma} |a(s)|^2 \ell(s),$$

defined on the Dirichlet space \mathcal{F}_{ℓ} of those $a \in l^2(\Gamma)$ where the quadratic form is finite. Let us check that a unitary $\delta_s \in c_c(\Gamma) \subseteq C_{red}^*(\Gamma)$, defined for a fixed $s \in \Gamma$ by $\delta_s(t) := \delta_{st}$ for $t \in \Gamma$, is a multiplier. Since for all $a \in \mathcal{F}_{\ell}$

$$\begin{aligned} \|\delta_s * a\|_{\mathcal{F}_{\ell}}^2 &= \sum_{t \in \Gamma} |a(ts^{-1})|^2 (1 + \ell(t)) \\ &= \sum_{t \in \Gamma} |a(t)|^2 (1 + \ell(ts)) \\ &= \sum_{t \in \Gamma} |a(t)|^2 (1 + \ell(t)) \frac{(1 + \ell(ts))}{(1 + \ell(t))} \\ &\leq \|a\|_{\mathcal{F}_{\ell}}^2 \cdot \sup_{t \in \Gamma} \frac{(1 + \ell(ts))}{(1 + \ell(t))}, \end{aligned}$$

we have that δ_s is a multiplier and that its norm is bounded by

$$\|\delta_s\|_{\mathcal{B}(\mathcal{F}_\ell)} = \sup_{t \in \Gamma} \sqrt{\frac{(1 + \ell(ts))}{(1 + \ell(t))}} \leq \sup_{t \in \Gamma} \sqrt{\frac{(1 + 2(\ell(t) + \ell(s)))}{(1 + \ell(t))}} \leq \sqrt{2} \sqrt{1 + \ell(s)}.$$

In the special cases where ℓ is the length function associated to a system of generators, by the triangular inequality we have instead

$$\|\delta_s\|_{\mathcal{B}(\mathcal{F}_\ell)} \leq \sup_{t \in \Gamma} \sqrt{\frac{(1 + \ell(ts))}{(1 + \ell(t))}} \leq \sup_{t \in \Gamma} \sqrt{\frac{(1 + \ell(t) + \ell(s))}{(1 + \ell(t))}} = \sqrt{1 + \ell(s)}.$$

We prove below that multipliers exist on any Dirichlet space.

Proposition 6.3. *Let $g \in \mathcal{P}_+ \cap \mathcal{M}$ be a bounded potential and suppose that its carré du champ $\Gamma[g] \in \mathcal{C}_+^*$ has a bounded potential $G(\Gamma[g]) \in \mathcal{P}_+ \cap \mathcal{M}$. Then g is a multiplier of the Dirichlet space with a norm bounded by*

$$(6.1) \quad \|g\|_{\mathcal{B}(\mathcal{F})} \leq \sqrt{[(\|G(\Gamma[g])\|_{\mathcal{M}}^{1/2} + \|g\|_{\mathcal{M}})^2 + \|g\|_{\mathcal{M}}^2]}.$$

Proof. By Proposition 5.8, $\Gamma[g]$ is a finite-energy functional. Applying the generalized Deny embedding Theorem 4.1 we get, for $b \in \mathcal{B}$:

$$\begin{aligned} \|(\partial g)b\|_{\mathcal{H}}^2 &= \langle \Gamma[g], bb^* \rangle_{\mathcal{C}^*, \mathcal{C}} \\ &\leq \|G(\Gamma[g])\|_{\mathcal{M}} \|b^*\|_{\mathcal{F}}^2 = \|G(\Gamma[g])\|_{\mathcal{M}} \|b\|_{\mathcal{F}}^2. \end{aligned}$$

Hence we have

$$\|\partial(gb)\|_{\mathcal{H}} = \|\partial(g)b + g\partial(b)\|_{\mathcal{H}} \leq (\|G(\Gamma[g])\|_{\mathcal{M}}^{1/2} + \|g\|_{\mathcal{M}}) \|b\|_{\mathcal{F}}$$

from which we obtain

$$(6.2) \quad \|gb\|_{\mathcal{F}}^2 = \|\partial(gb)\|_{\mathcal{H}}^2 + \|gb\|_{L^2(A, \tau)}^2 \leq [(\|G(\Gamma[g])\|_{\mathcal{M}}^{1/2} + \|g\|_{\mathcal{M}})^2 + \|g\|_{\mathcal{M}}^2] \|b\|_{\mathcal{F}}^2.$$

Since the Dirichlet algebra \mathcal{B} is a form core, for a fixed $b \in \mathcal{F}$ there exists a Cauchy net $\{b_i \in \mathcal{B} : i \in I\}$ converging to it in the norm of \mathcal{F} . The above bound implies that also $\{gb_i \in \mathcal{B} : i \in I\} \subset \mathcal{F}$ is a Cauchy net in \mathcal{F} , hence converging to an element $c \in \mathcal{F}$. Since \mathcal{F} is continuously embedded in $L^2(A, \tau)$, we have that $c = gb$ so that (6.2) holds true for all $b \in \mathcal{F}$. An analogous computation shows that $bg \in \mathcal{F}$ for all $b \in \mathcal{F}$ so that g is a multiplier and from (6.2) the bound (6.1) follows. \square

Let us consider the closed ideal $\mathcal{M}^1 \subset \mathcal{M}$ of the von Neumann algebra \mathcal{M} , defined as the operator norm closure of the ideal $L^1(A, \tau) \cap \mathcal{M}$. For instance, if the trace τ is finite, then $\mathcal{M}^1 = \mathcal{M}$, and if \mathcal{M} is the hyperfinite type I_∞ factor $\mathcal{B}(\mathcal{K})$, acting on a separable Hilbert space \mathcal{K} , then \mathcal{M}^1 coincides with the elementary C^* -algebra $\mathbb{K}(\mathcal{K})$ of compact operators. Notice that, in general, $L^p(A, \tau) \cap \mathcal{M} \subset \mathcal{M}^1$ for all $p \geq 1$.

Proposition 6.4. *If $h \in \mathcal{M}^1$, then $g = (I + L)^{-1}h \in \mathcal{C}$ is a multiplier of the Dirichlet space \mathcal{F} and the following bound holds true:*

$$(6.3) \quad \|g\|_{\mathcal{B}(\mathcal{F})} \leq 2\sqrt{5}\|h\|_{\mathcal{M}}.$$

If moreover $h \in L^1(A, \tau) \cap \mathcal{M}$, the potential of the carré du champ $G(\Gamma[g])$ is bounded. When h is also self-adjoint this potential is given by

$$(6.4) \quad G(\Gamma[g]) = \frac{1}{2} \left((I + L)^{-1}(hg + gh) - g^2 \right).$$

Proof. Let us assume first that $h \in L^1(A, \tau) \cap \mathcal{M}$ so that $h \in L^2(A, \tau)$ and $g = (I + L)^{-1}h \in \mathcal{P}_+ \cap \mathcal{M}$ is a bounded potential (see Example 3.2). Since $g = g^*$, we have for all $b \in \tilde{\mathcal{B}}$

$$\langle \partial g, (\partial g)b \rangle_{\mathcal{H}} = \langle \mathcal{J}((\partial g)b), \mathcal{J}(\partial g) \rangle_{\mathcal{H}} = \langle b^*(\partial g^*), \partial g^* \rangle_{\mathcal{H}} = \langle \partial g, b(\partial g) \rangle_{\mathcal{H}}$$

and then

$$\begin{aligned} 2\langle \Gamma[g], b \rangle_{\mathcal{C}^*, \mathcal{C}} &= 2\langle \partial g, (\partial g)b \rangle_{\mathcal{H}} \\ &= 2\langle \partial g, b(\partial g) \rangle_{\mathcal{H}} \\ &= \langle \partial g, (\partial g)b + b(\partial g) \rangle_{\mathcal{H}} \\ &= \langle \partial g, \partial(gb) + \partial(bg) - g(\partial b) - (\partial b)g \rangle_{\mathcal{H}} \\ &= \tau(h(gb + bg)) - \langle g(\partial g) + (\partial g)g, \partial b \rangle_{\mathcal{H}} \\ &= \tau((hg + gh)b) - \langle \partial g^2, \partial b \rangle_{\mathcal{H}} \\ &= \langle (I + L)^{-1}(hg + gh) - g^2, b \rangle_{\mathcal{F}}. \end{aligned}$$

This proves that $\Gamma[g]$ is a finite-energy functional with a bounded potential given by (6.4). By polarization the formula remains true when $h \in L^1(A, \tau) \cap \mathcal{M}$ is self-adjoint.

Applying Proposition 6.3 we conclude that in this case g is a multiplier. To bound its multiplier norm notice first $G(\Gamma(g))$ and g^2 are positive operators so that $\|G(\Gamma[g])\| \leq \frac{1}{2}\|(I + L)^{-1}(hg + gh)\|$. Applying twice the contractivity of the resolvent $(I + L)^{-1}$, we get $\|g\|_{\mathcal{M}} \leq \|h\|_{\mathcal{M}}$ and

$$\|G(\Gamma[g])\|_{\mathcal{M}} \leq \frac{1}{2}\|(I + L)^{-1}(hg + gh) - g^2\|_{\mathcal{M}} \leq \frac{1}{2}\|(I + L)^{-1}(hg + gh)\|_{\mathcal{M}} \leq \|h\|_{\mathcal{M}}^2.$$

Finally, by (6.1) we have

$$\|g\|_{\mathcal{B}(\mathcal{F})} \leq \sqrt{(\sqrt{\|G(\Gamma[g])\|_{\mathcal{M}} + \|g\|_{\mathcal{M}}^2} + \|g\|_{\mathcal{M}})^2 + \|g\|_{\mathcal{M}}^2} \leq \sqrt{5}\|h\|_{\mathcal{M}}.$$

For the general case decompose $h \in L^1(A, \tau) \cap \mathcal{M}$ as $h = h_1 + ih_2$ by its real and imaginary parts $h_{1,2} \in L^1_+(A, \tau) \cap \mathcal{M}$, to which there correspond a similar decomposition of $g = (I + L)^{-1}h = (I + L)^{-1}h_1 + i(I + L)^{-1}h_2g_1 + ig_2$. We have the inequality

$$\Gamma[g] \leq 2(\Gamma[g_1] + \Gamma[g_2])$$

to which there corresponds the inequality in the cone of potentials (by Proposition 3.11)

$$G(\Gamma[g]) \leq 2(G(\Gamma[g_1]) + G(\Gamma[g_2])).$$

So we can conclude first that g is a multiplier and that (6.3) holds true; and then that its carré du champ has bounded potential with uniform norm less than $2\|h\|_{\mathcal{M}}^2$. Since, by definition, $L^1(A, \tau) \cap \mathcal{M}^1$ is norm dense in \mathcal{M}^1 , we have that for any $h \in \mathcal{M}^1$, $g = (I + L)^{-1}h \in \mathcal{C}$ is a multiplier such that (6.3) holds true. \square

As a straightforward application of the above result, a natural class of multipliers emerges. Recall that by the Beurling-Deny theory [BeDe2] and its successive generalizations to von Neumann algebras with trace ([AHK], [C1], [C2], [DL], [G1], [S2]), the Markovian semigroup $\{e^{-tL} : t \geq 0\}$ on $L^2(A, \tau)$ extends as a Markovian semigroup to each space $L^p(A, \tau)$ for each $p \in [1, +\infty]$, strongly continuous for $p \in [0, +\infty)$ and weakly*-continuous for $p = +\infty$. The closed, self-adjoint, non-negative operator L on $L^2(A, \tau)$ whose quadratic form is the Dirichlet form \mathcal{E} (see (2.4)) and which generates the Markovian semigroup on $L^2(A, \tau)$ is closable in any $L^p(A, \tau)$.

Corollary 6.5. *A bounded eigenvector $h \in \text{dom}_{L^p}(L) \in L^p(A, \tau) \cap \mathcal{M}$ of the generator L , considered as a closed operator on $L^p(A, \tau)$ for all $p \in [1, +\infty]$, corresponding to the eigenvalue λ (whenever it exists)*

$$Lh = \lambda h,$$

is a multiplier of the Dirichlet space \mathcal{F} and the following bound holds true:

$$(6.5) \quad \|h\|_{\mathcal{B}(\mathcal{F})} \leq 2\sqrt{5}(1 + \lambda)\|h\|_{\infty}.$$

Proof. To apply Proposition 6.4 just notice that $L^p(A, \tau) \cap \mathcal{M} \subseteq \mathcal{M}^1$ for any $p \in [1, +\infty]$. □

Example 6.6 (Eigenfunctions on p.c.f. self-similar fractals). On post critically finite, self-similar fractal sets K , a natural class of the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ has been constructed by J. Kigami, using the notion of harmonic structure (see [Ki], Definition 3.1.2). With respect to a wide range of positive Borel finite measures μ on them, the corresponding nonnegative, self-adjoint operators on $L^2(K, \mu)$ have discrete spectrum (see [Ki], Theorem 3.4.6 and Corollary 3.4.7) so that, by the previous corollary, the corresponding eigenfunctions are multipliers of the Dirichlet space. Notice that in case the harmonic structure is regular (see [Ki], Definition 3.1.2) the Dirichlet space \mathcal{F} is automatically a subalgebra of the algebra of continuous functions $C(K)$ on the compact topological fractal space K . Then the Dirichlet algebra \mathcal{B} and the extended Dirichlet algebra $\tilde{\mathcal{B}}$ coincide with the whole Dirichlet space \mathcal{F} . The fine C*-algebra \mathcal{C} coincides with the whole algebra of continuous functions $\mathcal{C} = C(K)$ while the ideal $L^1(K, \mu) \cap L^\infty(K, \mu)$ and its uniform closure coincide with the (commutative) von Neumann algebra $L^\infty(K, \mu)$, since the measures μ are finite. Moreover, since $1 \in \mathcal{F}$, the algebra of multipliers coincides with \mathcal{F} and is therefore uniformly dense in $C(K)$.

Example 6.7 (Eigenfunctions of ultracontractive semigroups). In commutative frameworks, the situations where the positive operator L associated to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ generate an ultracontractive semigroup are largely studied (see for example [Dav]). This means that the semigroup e^{-tL} continuously maps $L^2(X, m)$ into $L^\infty(X, m)$ for all $t > 0$. This is the case, for example, if the Dirichlet form satisfies a suitable family of logarithmic Sobolev inequalities (see [G3]) or suitable Sobolev or Nash inequalities (see [Dav], Corollary 2.2.8 and Sections 2.3, 2.4). If the measure m is finite, then the spectrum of L on $L^p(X, m)$ is discrete (see [Dav], Theorem 2.1.4) and, by the above corollary, all eigenfunctions are multipliers.

- Corollary 6.8.**
- i) $(I + \varepsilon L)^{-1}h$ is a multiplier of the Dirichlet space \mathcal{F} for any $\varepsilon > 0$ and any $h \in L^1(A, \tau) \cap \mathcal{M}^1$.
 - ii) $(I + \varepsilon L)^{-1}h$ is a multiplier of the Dirichlet space \mathcal{F} for any $h \in \tilde{\mathcal{B}}$. As a consequence, multipliers are dense in \mathcal{F} . Hence the algebra $\mathcal{M}(\mathcal{E}, \mathcal{F}) \cap \mathcal{F}$ of multipliers lying in \mathcal{F} is a form core for the Dirichlet space.
 - iii) On the norm closure $\mathcal{A} := \overline{\mathcal{M}(\mathcal{E}, \mathcal{F}) \cap \mathcal{F}}$ in \mathcal{M} of the algebra of multipliers lying in \mathcal{F} , the Dirichlet form is regular.
 - iv) The algebra of multipliers $\mathcal{M}(\mathcal{E}, \mathcal{F})$ is dense in A provided the semigroup is strongly continuous on A which means

$$\lim_{\varepsilon \downarrow 0^+} \|e^{-\varepsilon L}a - a\|_{\mathcal{M}} = 0, \quad a \in A.$$

Proof. Item i) is stated and proved in Proposition 6.4. As far as ii) is concerned, again by Proposition 6.4, since $\tilde{\mathcal{B}} \subset L^2 \cap \mathcal{M} \subset \mathcal{M}^1$, we have that $(I + L)^{-1}h$ is a multiplier for $h \in \tilde{\mathcal{B}}$. The core property then follows by Lemma 2.3. To prove iii) notice first that any element $(I + \varepsilon L)^{-1}b$, for $b \in \tilde{\mathcal{B}}$, belongs to \mathcal{A} , and the vector space they generate is form core for $(\mathcal{E}, \mathcal{F})$ (by Lemma 2.3). Item iv): the density of $\mathcal{F} \cap \mathcal{A}$ in \mathcal{A} follows by construction; its density in \mathcal{F} is ii). \square

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