

CYLINDRICALLY BOUNDED CONSTANT MEAN CURVATURE SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

LAURENT MAZET

ABSTRACT. In this paper it is proved that a properly embedded constant mean curvature surface in $\mathbb{H}^2 \times \mathbb{R}$ which has finite topology and stays at a finite distance from a vertical geodesic line is invariant by rotation around a vertical geodesic line.

1. INTRODUCTION

In 1988, Meeks [13] proved that, in \mathbb{R}^3 , a properly embedded annulus with non-vanishing constant mean curvature (cmc, in the following) must stay at a bounded distance from a straight line. Then Korevaar, Kusner and Solomon [9] proved that any properly embedded constant mean curvature surface staying at a bounded distance of a straight line is rotationally invariant and then a Delaunay surface.

These results imply that any end of a finite topology properly embedded cmc surface is asymptotic to a Delaunay surface. So this allows a description of the space of all properly embedded cmc surfaces with finite topology (see the paper of Kusner, Mazzeo and Pollack [10]).

Korevaar, Kusner and Solomon used the Alexandrov reflection procedure to prove their results. In fact, their proof works in higher dimension and also in \mathbb{H}^n [8].

Recently the theory of cmc surfaces has been developed in 3-dimensional homogeneous spaces. One interesting case is the ambient space $\mathbb{H}^2 \times \mathbb{R}$. The group of isometries of $\mathbb{H}^2 \times \mathbb{R}$ possesses rotations around vertical geodesic lines $p \times \mathbb{R}$ where p is a point of \mathbb{H}^2 . So one can look for embedded cmc surfaces invariant by rotation around such a geodesic line; this was done by Hsiang and Hsiang in [7]. For any mean curvature H_0 , they found rotationally invariant surfaces but these surfaces stay at a bounded distance from the vertical geodesic axis only for $H_0 > 1/2$. In the sequel, we will focus on the case $H_0 > 1/2$. Among these rotationally invariant surfaces, certain examples are spheres and, actually, they are the only compact embedded cmc surface in $\mathbb{H}^2 \times \mathbb{R}$ [7]. The other ones are periodic with respect to a vertical translation. For example, we have the vertical cylinder $C \times \mathbb{R}$ where C is a circle in \mathbb{H}^2 . So these surfaces correspond to the Delaunay surfaces in \mathbb{R}^3 and they are also called Delaunay surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

A subset of $\mathbb{H}^2 \times \mathbb{R}$ which is at a bounded distance from a vertical geodesic line will be called *cylindrically bounded*. So, as in the paper of Korevaar, Kusner and Solomon: can we classify properly embedded cmc surfaces that are cylindrically bounded? The main result of the paper (Theorem 15) gives an answer to this question.

Received by the editors January 22, 2013.

2010 *Mathematics Subject Classification*. Primary 53A10.

©2015 American Mathematical Society
Reverts to public domain 28 years from publication

Theorem. *Let Σ be a properly embedded cmc surface in $\mathbb{H}^2 \times \mathbb{R}$. If Σ has finite topology and is cylindrically bounded, Σ is a Delaunay surface.*

So any cylindrically bounded cmc annulus in $\mathbb{H}^2 \times \mathbb{R}$ is a Delaunay surface. Let us make a remark about the limit case $H_0 = 1/2$. In \mathbb{R}^3 , the results of Meeks and Korevaar, Kusner and Solomon imply that any properly embedded cmc $H_0 > 0$ annulus is rotational. For $H_0 = 0$, we also know that a properly embedded minimal annulus is a catenoid and thus rotational (see Collin [3]). In $\mathbb{H}^2 \times \mathbb{R}$, we do not have such a rigidity. Actually, Cartier and Hauswirth [2] have recently constructed a properly embedded cmc $H_0 = 1/2$ annulus in $\mathbb{H}^2 \times \mathbb{R}$ whose ends are asymptotic to rotational ones but which is not rotationally invariant.

We notice that a vertical cylinder $C \times \mathbb{R}$ has mean curvature larger than $1/2$. Thus, any properly embedded cmc surface which is cylindrically bounded has mean curvature $H_0 > 1/2$. This is a consequence of the half space theorem in $\mathbb{H}^2 \times \mathbb{R}$ (see [14], [12] or [5], for example). So we can focus only on the $H_0 > 1/2$ case.

The proof of the theorem is also based on the Alexandrov reflection technique but the space of planar symmetries in $\mathbb{H}^2 \times \mathbb{R}$ is smaller than in \mathbb{R}^3 (in \mathbb{R}^3 it is a 3-dimensional space and in $\mathbb{H}^2 \times \mathbb{R}$ it is only 2-dimensional). So the ideas of Korevaar, Kusner and Solomon cannot be applied.

First we remark that we already know that a compact embedded cmc surface is a rotational sphere so we only consider noncompact surfaces. For a noncompact cylindrically bounded cmc surface Σ and a foliation of $\mathbb{H}^2 \times \mathbb{R}$ by vertical planes, we define a function α on \mathbb{R} called the *Alexandrov function*. One property of this function is that, if α admits a maximum, Σ is symmetric with respect to a vertical plane of the foliation. To prove that Σ is rotationally invariant, it suffices to prove that it is symmetric with respect to a lot of vertical planes. So we want to prove that any Alexandrov function has a maximum.

If the surface Σ has finite topology, we know by previous results [6] that it has bounded curvature. So we can control the asymptotic behavior of the surface. This gives information about the behavior of the Alexandrov function α near $\pm\infty$. Then we prove that α is decreasing near $+\infty$ and increasing near $-\infty$: this implies that α has a maximum. To prove this monotonicity result we use a flux argument similar to the one of the positive flux lemma proved by Korevaar, Kusner, Meeks and Solomon in [8].

We notice that the proofs given in this paper work also in $\mathbb{S}^2 \times \mathbb{R}$: they prove that a properly embedded constant mean curvature surface with finite topology in $D \times \mathbb{R}$ is rotationally invariant where D is a geodesic disk in \mathbb{S}^2 of radius less than $\pi/2$.

The paper is divided into sections as follows. In Section 2, we recall several results concerning cylindrically bounded cmc surfaces in $\mathbb{H}^2 \times \mathbb{R}$. In Section 3, we recall the construction of the Delaunay surfaces. In Section 4, we define the Alexandrov function and give its properties, and we study the asymptotic behavior of an annular end of a cylindrically bounded cmc surface. Finally we state our main result and give the main steps of the proof. Section 5 is devoted to the study of the horizontal Killing graph, *i.e.* cmc surfaces that are transverse to the horizontal Killing vector field generating horizontal translations in $\mathbb{H}^2 \times \mathbb{R}$. The last section is devoted to the study of the monotonicity of the Alexandrov function near $\pm\infty$. The

main idea is to compare the flux along the surface with the flux along a comparison surface which is constructed as a horizontal Killing graph.

Throughout the paper, we use z to denote the real coordinate in $\mathbb{H}^2 \times \mathbb{R}$. On \mathbb{H}^2 , we consider the polar coordinates (ρ, θ) in Section 3 and an (s, r) coordinate system in Sections 5 and 6. This last coordinate system is adapted to the Killing vector field that generates translations along a geodesic line.

2. PREVIOUS RESULTS

In this section, we recall several results concerning properly embedded cmc surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Most of them can be found in the papers of Hoffman, de Lira and Rosenberg [6, 18] and the one by Korevaar, Kusner and Solomon [9]. We also explain the convergence we will consider for sequences of cmc surfaces with bounded curvature.

2.1. Flux. This first subsection is devoted to the notion of flux for cmc surfaces that was introduced in several preceding papers (see for example [6, 9]).

Let U be a bounded domain in $\mathbb{H}^2 \times \mathbb{R}$ whose boundary is the union of a smooth surface Σ and a smooth surface Q with common boundary $\partial\Sigma = \partial Q$. So the boundary ∂U is piecewise smooth. Let us denote by \vec{n} the unit outgoing normal along ∂U and by \vec{n}_Σ and \vec{n}_Q the respective restriction of \vec{n} along Σ and S (see Figure 1).

Let Y be a Killing vector field of $\mathbb{H}^2 \times \mathbb{R}$. We have

$$0 = \int_U \operatorname{div}_{\mathbb{H}^2 \times \mathbb{R}} Y = \int_\Sigma Y \cdot \vec{n}_\Sigma + \int_Q Y \cdot \vec{n}_Q$$

where $\operatorname{div}_{\mathbb{H}^2 \times \mathbb{R}}$ is the divergence operator on $\mathbb{H}^2 \times \mathbb{R}$.

Let $\vec{\nu}$ be the outgoing unit conormal to Σ along $\partial\Sigma$. Along Σ the vector field Y can be decomposed into the sum of a tangent part Y^\top and a normal part Y^\perp . We have

$$0 = \int_\Sigma \operatorname{div}_\Sigma Y = \int_\Sigma \operatorname{div}_\Sigma Y^\top + \int_\Sigma \operatorname{div}_\Sigma Y^\perp = \int_{\partial\Sigma} Y \cdot \vec{\nu} + \int_\Sigma 2HY \cdot \vec{n}_\Sigma$$

where $\operatorname{div}_\Sigma$ is the divergence operator on Σ acting on any vector field of $\mathbb{H}^2 \times \mathbb{R}$ defined along Σ and H is the mean curvature of Σ computed with respect to $-\vec{n}_\Sigma$.

Thus, if Σ has constant mean curvature H_0 , we have

$$(1) \quad 0 = \int_{\partial\Sigma} Y \cdot \vec{\nu} - 2H_0 \int_Q Y \cdot \vec{n}_Q.$$

Now let us consider Σ a cmc H_0 surface and γ a smooth closed curve in Σ . Let Q be a smooth surface in $\mathbb{H}^2 \times \mathbb{R}$ with boundary γ . For a Killing vector field Y , we define the quantity

$$F_\gamma(Y) = \int_\gamma Y \cdot \vec{\nu} - 2H_0 \int_Q Y \cdot \vec{n}_Q$$

where, as above, $\vec{\nu}$ is the conormal unit vector field of Σ along γ and \vec{n}_Q is a unit normal along Q (the choice of these normal vectors is consistent with the above computations). Because of formula (1), the quantity $F_\gamma(Y)$ does not depend on the choice of Q and depends only on the homology class of γ in Σ . $F_\gamma(Y)$ is called the flux of Σ along γ in the direction Y .

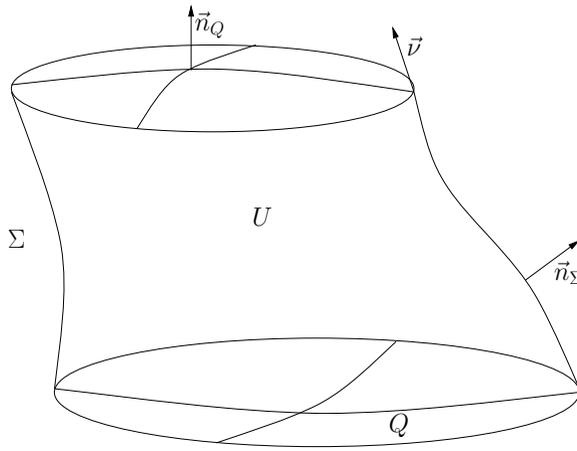


FIGURE 1. Computation of the flux

In fact, the map $Y \mapsto F_\gamma(Y)$ is linear so it can be seen as an element of the dual of the vector space of Killing vector fields. This element is called the flux of Σ along γ .

2.2. Linear area growth. In this subsection, we recall a result concerning the area growth of a cylindrically bounded properly embedded cmc surface in $\mathbb{H}^2 \times \mathbb{R}$.

Let Σ be a properly embedded cmc surface with possibly nonempty compact boundary. We say that Σ has linear area growth if there exist two constants α and β such that $Area(\Sigma \cap \{a \leq z \leq b\}) \leq \alpha(b - a) + \beta$ for any $a \leq b \in \mathbb{R}$.

We then have the following result.

Proposition 1 (Corollary 1 in [6] or Theorem 14 in [18]). *Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a properly embedded cmc surface with possibly nonempty compact boundary. If Σ is cylindrically bounded, Σ has linear area growth.*

We notice that in this result we do not make any hypothesis on the topology of the surface Σ . Although the original proof in [6] contains a mistake, it has been corrected by Rosenberg in [18].

2.3. The height function. On a properly embedded surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ the restriction to the surface of the real coordinate z is called the height function on Σ . For a cmc H_0 surface, this height function has several properties.

Lemma 2. *Let Σ be a noncompact properly embedded cylindrically bounded cmc H_0 surface in $\mathbb{H}^2 \times \mathbb{R}$. The height function cannot be either bounded from above or bounded from below.*

The idea of the proof can be found in the proof of Theorem 1.1 in [15] or Proposition 2 in [6].

Proof. For example, let us assume that the height function is bounded from below. We can apply the Alexandrov reflection technique with respect to the horizontal slice $\mathbb{H}^2 \times \{t\}$. Since Σ is bounded from below, for t close to $-\infty$, Σ does not intersect $\mathbb{H}^2 \times \{t\}$. Thus we can start the reflection procedure up to a first contact

point. But since Σ is noncompact, a first contact point cannot exist. Indeed, if there is a first contact point, the maximum principle would imply that Σ is symmetric with respect to some $\mathbb{H}^2 \times \{t_0\}$ and thus compact.

So the reflection procedure can be done for any t and this implies that the part of Σ below $\{z = t\}$ is a vertical graph with boundary in $\{z = t\}$. The height of such a vertical graph is bounded from above by a constant that depends only on H_0 (see [1]); thus the Alexandrov procedure has to stop. We get a contradiction. \square

When the surface Σ is an annulus we have the following property (see Lemma 4.1 in [9]).

Lemma 3. *Let $H_0 > 1/2$ be a real number; there exists an $M > 0$ that depends only on H_0 such that the following is true. Let A be an embedded annulus $A \subset \mathbb{H}^2 \times \mathbb{R}$ of constant mean curvature H_0 with boundary outside $\mathbb{H}^2 \times [0, M]$. Then $A \cap \mathbb{H}^2 \times [0, M]$ has at most one connected component \hat{A} such that $z(\hat{A}) = [0, M]$.*

2.4. Uniform curvature estimate. In this subsection, we recall an estimate of the norm of the second fundamental form of a cylindrically bounded properly embedded cmc surface in $\mathbb{H}^2 \times \mathbb{R}$. More precisely, Hoffman, de Lira and Rosenberg proved the following result.

Proposition 4 (Theorem 3 in [6]). *Let Σ be a properly embedded cmc surface with finite topology and possibly nonempty compact boundary. If Σ is cylindrically bounded, the norm of the second fundamental form $|A|$ is bounded on Σ .*

2.5. Convergence of sequences of cmc surfaces. In the following, we will consider sequences of cmc surfaces coming from vertical translations of a given cmc surface with bounded curvature. In this subsection, we explain how these sequences converge to a limit cmc surface. The surfaces we consider have a uniform curvature bound.

Considering the normal coordinates around a point in $\mathbb{H}^2 \times \mathbb{R}$, a constant mean curvature surface in $\mathbb{H}^2 \times \mathbb{R}$ can be viewed as an immersed surface in \mathbb{R}^3 . A bound of its second fundamental form in $\mathbb{H}^2 \times \mathbb{R}$ is equivalent to a bound in \mathbb{R}^3 . We have the following classical result.

Proposition 5 ([9, 17]). *Let Σ be an immersed surface in \mathbb{R}^3 whose second fundamental form satisfies $|A| \leq 1/(4\delta)$ for some $\delta > 0$. Then for any $x \in \Sigma$ with $d(x, \partial\Sigma) > 4\delta$ there is a neighborhood of x in Σ which is a graph of a function u over the Euclidean disk of radius $\sqrt{2}\delta$ centered at x in the tangent plane to Σ at x . Moreover*

$$|u| < 2\delta, \quad |\nabla u| < 1, \quad \text{and} \quad |\nabla^2 u| < \frac{1}{\delta}.$$

Let Σ be a cmc H_0 surface ($H_0 \neq 0$) that bounds a domain D in $\mathbb{H}^2 \times \mathbb{R}$ (D is in the mean convex side of Σ) and with a uniform curvature bound. Let $p \in \Sigma$ be a point and let γ be the geodesic line starting from p in the direction of the mean curvature vector. Let q be the first point where γ meets Σ (if it exists). If q is close to p , Proposition 5 implies that γ is close to being normal to Σ at q . Moreover, since γ is in D between p and q , the mean curvature vector to Σ at q points in the opposite direction to the velocity vector of γ at q . But since $H_0 > 0$, this implies that p and q cannot be too close to each other. More precisely, we have the following result that gives a local description of the surface.

Proposition 6. *Let Σ be an embedded cmc H_0 surface in $\mathbb{H}^2 \times \mathbb{R}$ ($H_0 > 0$) which bounds a domain D in its mean convex side. Moreover we assume that its second fundamental form satisfies $|A| \leq k$ for some $k > 0$. Then there exists $R = R(k) > 0$ such that the following is true. For any $p \in \mathbb{H}^2 \times \mathbb{R}$, there exist at most two topological disks Δ^1 and Δ^2 in $\Sigma \cap B(p, 2R)$ such that $\Sigma \cap B(p, R) = (\Delta^1 \cup \Delta^2) \cap B(p, R)$. Moreover, when there are two disks, the domain between the two disks in $B(p, R)$ is outside D .*

The above proposition says that a local description of the surface Σ is given by one of the pictures in Figure 2.

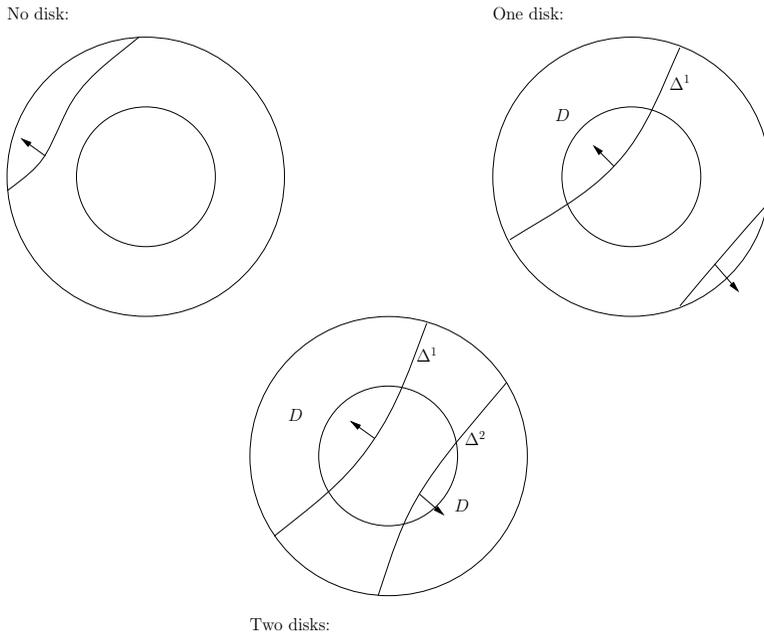


FIGURE 2. The local description of a cmc surface

Now let us assume that we have a sequence (Σ_n) of properly embedded cmc H_0 surfaces with uniformly bounded curvature and bounding a domain D_n in their mean convex side. For each compact subset K of $\mathbb{H}^2 \times \mathbb{R}$, $(K \cap \Sigma_n)$ is a sequence of compact sets so by a diagonal process we can assume that the sequence (Σ_n) converges in the Hausdorff topology restricted to each K to a limit that we call Σ . Let p be in Σ ; there exist $p_n \in \Sigma_n$ such that $p_n \rightarrow p$. From Proposition 5, p_n is the center of a geometrically controlled disk Δ_n^1 . By considering a subsequence we can assume that $\Delta_{n'}^1$ converges to a cmc H_0 disk Δ^1 . We then have $\Delta^1 \subset \Sigma$.

Let R be given by Proposition 6. If $\Sigma \cap B(p, R) = \Delta^1 \cap B(p, R)$, this implies that the whole sequence Δ_n^1 converges to Δ^1 . If there is $q \in (\Sigma \cap B(p, R)) \setminus \Delta^1$, there exist $q_n \in \Sigma_n$ and q_n is in the second disk Δ_n^2 given by Proposition 6. By considering a subsequence we can assume $\Delta_{n'}^2$ converges to a cmc H_0 disk Δ^2 . We notice that Δ^2 is different from Δ^1 since it contains q . We then have $\Delta^2 \subset \Sigma$. From Proposition 6, we can be sure that $\Sigma \cap B(p, R) = (\Delta^1 \cup \Delta^2) \cap B(p, R)$. In fact this implies that the whole sequence (Δ_n^2) converges to Δ^2 .

To complete our local description of Σ , we notice that Δ^1 and Δ^2 can touch each other but, at these contact points, the mean curvature vectors have opposite value. Besides, Σ bounds the domain D which is constructed as the limit of the domains D_n . This proves that Σ is what we call a weakly embedded cmc H_0 surface.

Definition 7. A properly immersed cmc surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ is said to be weakly embedded if there exists an open subset Ω of $\mathbb{H}^2 \times \mathbb{R}$ such that Σ is the boundary of Ω and the mean curvature vector of Σ points into Ω .

As an example, two rotational cmc spheres that are tangent form a weakly embedded surface. The union of two vertical cylinders tangent along a common vertical geodesic line is also weakly embedded.

We say that a weakly embedded surface is connected if the underlying abstract surface Σ is connected. As an example, two rotational cmc spheres tangent at a point are not a connected weakly embedded surface.

Finally, we remark that we have proved that the sequence Σ_n converges smoothly in any compact set to the surface Σ .

3. DELAUNAY SURFACES

In this section, we briefly recall the construction of the embedded cmc surfaces that are rotationally invariant around a vertical axis. We only focus on surfaces with $H > 1/2$ (see [7] and [16] for more details).

Let (ρ, θ) be the polar coordinates on \mathbb{H}^2 so the metric is $d\rho^2 + (\sinh \rho)^2 d\theta^2$. We look for surfaces of revolution; so that we can look for the graph of a function $u = f(\rho)$, we will orient this graph using the upward pointing unit normal. Since we want a constant mean curvature H graph, the function f satisfies the equation

$$\frac{f'}{\sqrt{1 + f'^2}} \sinh \rho - 2H(\cosh \rho - 1) = \tau$$

where τ is a constant. In order to have a solution, τ has to satisfy

$$-\sinh \rho - 2H(\cosh \rho - 1) \leq \tau \leq \sinh \rho - 2H(\cosh \rho - 1).$$

The graphs of the functions in the left-hand and right-hand sides are given in Figure 3. So in order to have a solution with nonempty definition set, τ has to be chosen less than $2H - \sqrt{4H^2 - 1}$. Actually, for $\tau = 2H - \sqrt{4H^2 - 1}$, we find the surface $\rho = \operatorname{argth} 2H$ which is the vertical cylinder of constant mean curvature H .

For all values of $\tau < 2H - \sqrt{4H^2 - 1}$, a solution f can be defined for $\rho \in [\rho_{\min}(\tau), \rho_{\max}(\tau)]$. This solution f has the following properties (see Figure 4):

- if $0 < \tau$, $0 < \rho_{\min}(\tau)$, then f is increasing and $f'(\rho_{\min}(\tau)) = +\infty = f'(\rho_{\max}(\tau))$.
- if $\tau = 0$, $0 = \rho_{\min}(0)$, then f is increasing, $f'(0) = 0$ and $f'(\rho_{\max}(0)) = +\infty$.
- if $\tau < 0$, $0 < \rho_{\min}(\tau)$, then $f'(\rho_{\min}(\tau)) = -\infty$ and $f'(\rho_{\max}(\tau)) = +\infty$.

In each case, the graph of the function u is a piece of a cmc H surface of revolution that can be extended along its boundary by symmetry to produce a complete rotationally invariant cmc H surface \mathcal{D}_τ . When $\tau > 0$ we produce an embedded surface called unduloid. When $\tau = 0$ we produce a cmc sphere. For $\tau < 0$, we get a nonembedded surface called a nodoid.

The parameter τ can be interpreted as a flux on the surface. More precisely, if γ is the circle $\mathcal{D}_\tau \cap \{z = t\}$, then $2\pi\tau$ is the flux of \mathcal{D}_τ along γ in the direction ∂_z .

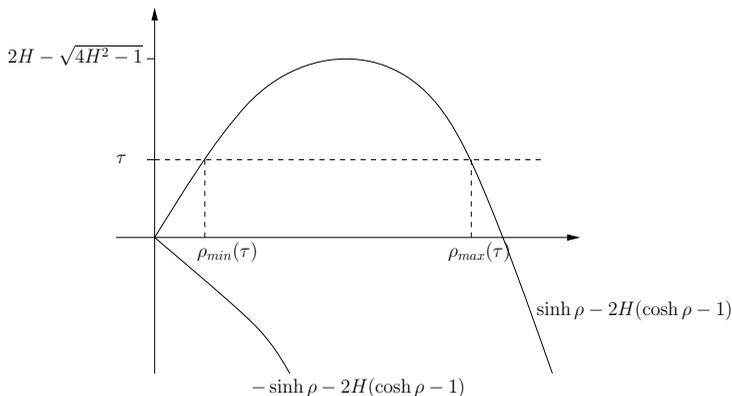


FIGURE 3. The domain of definition of solutions

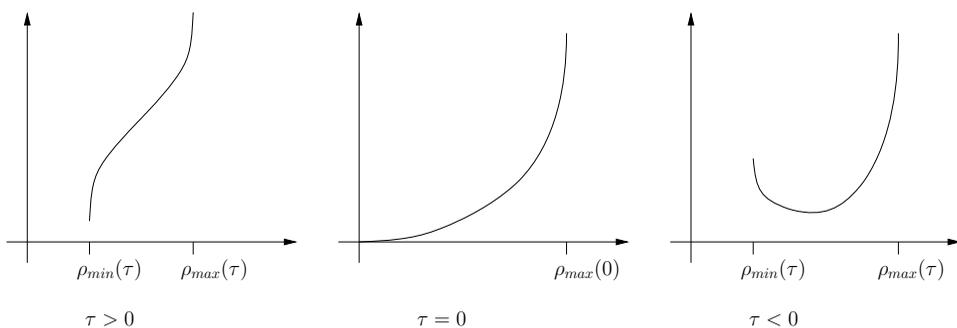


FIGURE 4. The solution of the equation

4. THE ALEXANDROV FUNCTION AND THE MAIN RESULT

In this section we introduce the notion of Alexandrov functions. Then we study the annular ends of a cylindrically bounded properly embedded cmc surface. We then explain the proof of our main theorem.

4.1. The Alexandrov function. The notion of Alexandrov function was introduced by Korevaar, Kusner and Solomon in [9] for \mathbb{R}^3 . Let us explain the situation in $\mathbb{H}^2 \times \mathbb{R}$.

Let $\Gamma = (\gamma_t)_{t \in \mathbb{R}}$ be a smooth family of geodesic lines that foliates \mathbb{H}^2 . We define $\Gamma_t^+ = \bigcup_{s > t} \gamma_s$ where Γ_t^+ is a half hyperbolic space bounded by γ_t ; we also define $\Gamma_t^- = \bigcup_{s < t} \gamma_s$. Let Π_t be the vertical plane $\gamma_t \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$. Let S_t denote the symmetry of \mathbb{H}^2 with respect to γ_t . S_t extends to the symmetry of $\mathbb{H}^2 \times \mathbb{R}$ with respect to Π_t ; we still denote this symmetry by S_t . Among all these foliations, we say that Γ is a translation foliation if all the geodesic lines γ_s are orthogonal to one common geodesic line g .

For an open interval I of \mathbb{R} , let G be a cylindrically bounded domain in $\mathbb{H}^2 \times \bar{I}$ such that $\partial G \cap (\mathbb{H}^2 \times I)$ is a smooth connected surface Σ with possibly nonempty boundary in the horizontal slices at height given by the end points of I .

Let us fix some z_0 in I and we focus our attention on what happens at height z_0 . So we denote $\Sigma_{z_0} = \Sigma \cap \{z = z_0\}$ which bounds $G_{z_0} = G \cap \{z = z_0\}$ and we consider that Γ foliates $\mathbb{H}^2 \times \{z_0\}$. Since G is cylindrically bounded, for t large $\Sigma_{z_0}(t) = \Sigma_{z_0} \cap \Gamma_t^+$ is empty. If Σ_{z_0} is nonempty, there is a largest t_1 such that $\Sigma_{z_0}(t_1) \neq \emptyset$. For $t \leq t_1$, we consider $\tilde{\Sigma}_{z_0}(t) = S_t(\Sigma_{z_0}(t))$ the symmetric of $\Sigma_{z_0}(t)$ with respect to γ_t . We notice that $\tilde{\Sigma}_{z_0}(t)$ is included in Γ_t^- and that, for t close to $-\infty$, $\tilde{\Sigma}_{z_0}(t)$ is outside G_{z_0} . So we can define

$$t_2(z_0) = \sup\{t \leq t_1 \mid \tilde{\Sigma}_{z_0}(t) \cap \Sigma_{z_0} \neq \emptyset\}.$$

We also define

$$t_3(z_0) = \sup\{t \leq t_1 \mid \exists p \in \Sigma_{z_0} \cap \gamma_t, T_p\Sigma \text{ is orthogonal to } \gamma_t \times \mathbb{R}\}.$$

Finally we define $\alpha_\Gamma(z_0) = \max(t_2(z_0), t_3(z_0))$, and if Σ_{z_0} is empty we define $\alpha_\Gamma(z_0) = -\infty$. This number can be understood as the first time where there is contact between $\tilde{\Sigma}_{z_0}(t)$ and $\Sigma_{z_0} \cap \Gamma_t^-$. In fact, for $t > \alpha_\Gamma(z_0)$, $\tilde{\Sigma}_{z_0}(t) \subset G_{z_0}$. The function $z_0 \mapsto \alpha_\Gamma(z_0)$ is called the *Alexandrov function* of Σ associated to the foliation Γ . This Alexandrov function has the following first important property.

Lemma 8. *The Alexandrov function α_Γ is upper semi-continuous.*

Proof. Let (z_n) be a sequence in I converging to z_0 in I . If Σ_{z_0} is empty, Σ_{z_n} is empty for large n , so α_Γ is upper semi-continuous at z_0 if $\alpha_\Gamma(z_0) = -\infty$. So we can assume $\alpha_\Gamma(z_n) > -\infty$ for every n . We can also assume that either $\alpha_\Gamma(z_n) = t_2(z_n)$ for all n or $\alpha_\Gamma(z_n) = t_3(z_n)$ for all n . In the second case, we have a sequence of points $p_n \in \Sigma_{z_n} \cap \gamma_{t_3(z_n)}$ such that $T_{p_n}\Sigma$ is orthogonal to $\gamma_{t_3(z_n)} \times \mathbb{R}$. We can assume that $t_3(z_n) \rightarrow \limsup \alpha_\Gamma(z_n) = t_0$ and $p_n \rightarrow p_0$. Then $p_0 \in \Sigma_{z_0} \cap \gamma_{t_0}$ and $T_{p_0}\Sigma$ is orthogonal to $\gamma_{t_0} \times \mathbb{R}$. This implies that $t_0 \leq t_3(z_0) \leq \alpha_\Gamma(z_0)$.

If we are in the first case, there is a sequence of points p_n in $\Sigma_{z_n}(t_2(z_n))$ such that $S_{t_2(z_n)}(p_n) \in \Sigma_{z_n}$. We can assume that $t_2(z_n) \rightarrow \limsup \alpha_\Gamma(z_n) = t_0$ and $p_n \rightarrow p_0$. Thus $p_0 \in \Sigma_{z_0}(t_0)$ or $p_0 \in \Sigma_{z_0} \cap \gamma_{t_0}$. In the first case $S_{t_0}(p_0) \in \Sigma_{z_0}$ so $t_0 \leq t_2(z_0) \leq \alpha_\Gamma(z_0)$. In the second, since $S_{t_2(z_n)}(p_n) \in \Sigma_{z_n}$ and these points converge to p_0 , the tangent space $T_{p_0}\Sigma$ is orthogonal to $\gamma_{t_0} \times \mathbb{R}$. So $t_0 \leq t_3(z_0) \leq \alpha_\Gamma(z_0)$. This finishes the proof. \square

The second important property is a consequence of the maximum principle when Σ has constant mean curvature.

Lemma 9. *Assume that Σ is connected, has constant mean curvature and the mean curvature vector points into G . If the Alexandrov function α_Γ has a local maximum at z , Σ is symmetric with respect to $\Pi_{\alpha_\Gamma(z)}$.*

Proof. Assume that z is a local maximum of α_Γ . Let $p \in \Sigma_z$ such that $p \in \Sigma_z(\alpha_\Gamma(z))$ and $S_{\alpha_\Gamma(z)}(p) \in \Sigma_z$ if $\alpha_\Gamma(z) = t_2(z)$ or $p \in \gamma_{\alpha_\Gamma(z)} \times \mathbb{R}$ and $T_p\Sigma$ is orthogonal to $\gamma_{\alpha_\Gamma(z)} \times \mathbb{R}$ if $\alpha_\Gamma(z) = t_3(z)$. In both cases, since z is a local maximum of α_Γ , near $S_{\alpha_\Gamma(z)}(p)$, $S_{\alpha_\Gamma(z)}(\Sigma)$ is on one side of Σ . Moreover these two surfaces have the same orientation at $S_{\alpha_\Gamma(z)}(p)$. So applying the maximum principle or the maximum principle at the boundary for cmc surface, we get that Σ is symmetric with respect to $\Pi_{\alpha_\Gamma(z)}$. \square

These two results have the following consequence.

Lemma 10. *Let Σ be as in Lemma 9. If $\alpha_\Gamma > -\infty$ on $[a, b]$, then:*

- α_Γ is monotonous, or
- there exists $c \in [a, b]$ such that α_Γ is decreasing on $[a, c]$ and increasing on $(c, b]$.

Proof. First, from Lemma 9, we remark that, if α_Γ has a local maximum at z , Σ is symmetric with respect to $\Pi_{\alpha_\Gamma(z)}$ so $\alpha_\Gamma(z)$ is a global minimum for α_Γ on $[a, b]$ so α_Γ is constant close to z .

Let x, y be in $[a, b]$. Since α_Γ is upper semi-continuous, $\sup_{[x, y]} \alpha_\Gamma$ is reached somewhere in $[x, y]$. If it is in the inside, the above remark implies that α_Γ is constant on $[x, y]$. Thus the maximum is always reached at x or y . So we have $\sup_{[x, y]} \alpha_\Gamma = \sup\{\alpha_\Gamma(x), \alpha_\Gamma(y)\}$.

Let (x_n) be a monotonous sequence converging to $c \in [a, b]$ such that $\lim \alpha_\Gamma(x_n) = \inf_{[a, b]} \alpha_\Gamma$. We assume that (x_n) decreases (the same argument can be done if it increases).

Let us consider $x < y < c$. If $\alpha_\Gamma(y) = \inf_{[a, b]} \alpha_\Gamma$, for any $z \in (y, c)$, $\alpha_\Gamma(z) \leq \alpha_\Gamma(x_n)$ so $\alpha_\Gamma(z) = \inf_{[a, b]} \alpha_\Gamma$. α_Γ is constant on (y, c) so it is constant on $[a, b]$. So we can assume $\alpha_\Gamma(y) > \lim \alpha_\Gamma(x_n)$. Since $\alpha_\Gamma(y) \leq \max\{\alpha_\Gamma(x), \alpha_\Gamma(x_n)\}$, we get $\alpha_\Gamma(x) \geq \alpha_\Gamma(y)$: α_Γ decreases on $[a, c]$.

Let us now consider $c < x < y$. As above we can assume $\alpha_\Gamma(x) > \lim \alpha_\Gamma(x_n)$. Since $\alpha_\Gamma(x) \leq \max\{\alpha_\Gamma(y), \alpha_\Gamma(x_n)\}$, we get $\alpha_\Gamma(x) \leq \alpha_\Gamma(y)$: α_Γ increases on $(c, b]$.

When $c = a$ or b , α_Γ can be monotonous. □

If Σ is a weakly embedded cmc surface which is cylindrically bounded, we notice that the Alexandrov functions can also be defined on it. These functions also satisfy Lemmas 9 and 10.

4.2. Asymptotical Delaunay ends. In this subsection we prove that a cylindrically bounded annular end of a cmc surface is asymptotic to a Delaunay surface \mathcal{D}_τ .

Let A be a properly embedded annular end with cmc H and which is cylindrically bounded. The annulus A can be viewed as the punctured disk embedded in $\mathbb{H}^2 \times \mathbb{R}$ with boundary in $\mathbb{H}^2 \times \{0\}$. The height function converges to $\pm\infty$ at the puncture. If the limit of the height function is $+\infty$, then A is said to be a top end, and if the limit is $-\infty$, then A is a bottom end. In the sequel we always study top ends, since the properties of bottom ends can be deduced by symmetry with respect to $\{z = 0\}$. The annular end A bounds a cylindrically bounded domain G so we can define Alexandrov functions on A .

From Proposition 1, A has linear area growth. Proposition 4 gives a uniform bound on the second fundamental form on A . These two properties are sufficient to study the limit of slide-back sequences $t_{-z_n}(A)$ where (z_n) is an increasing sequence going to $+\infty$ and $t_{\bar{z}}$ denotes the vertical translation by \bar{z} .

Proposition 11. *Let A be a properly embedded annular end with cmc H which is cylindrically bounded ($H > 1/2$). Let (z_n) be an increasing sequence converging to $+\infty$. There is a parameter $\tau \in (0, 2H - \sqrt{4H^2 - 1}]$ that depends only on A (not on (z_n)) and a subsequence $(z_{n'})$ of (z_n) such that $t_{-z_{n'}}(A)$ converges to a rotationally invariant Delaunay surface \mathcal{D}_τ . Moreover the axis of \mathcal{D}_τ only depends on A .*

Proof. Let $s \mapsto \gamma(s)$ be a geodesic in \mathbb{H}^2 . Let γ_s^1 denote the geodesic line of \mathbb{H}^2 orthogonal to γ at $\gamma(s)$. Then $\Gamma^1 = (\gamma_s^1)_{s \in \mathbb{R}}$ is a foliation of \mathbb{H}^2 , so we can consider the Alexandrov function α_{Γ^1} of A . From Lemma 10, α_{Γ^1} is monotonous close to $+\infty$. Moreover, it is bounded so it has a limit at $+\infty$. By changing the parametrization of γ , we assume that this limit is 0.

Let θ be an irrational angle and consider the geodesic line γ_t^2 of \mathbb{H}^2 which meets γ_0^1 with an angle θ at $\gamma_0^1(t)$. $\Gamma^2 = (\gamma_t^2)_{t \in \mathbb{R}}$ is a foliation of \mathbb{H}^2 so we consider the Alexandrov function α_{Γ^2} of A . As above, this function has a limit at $+\infty$ and we can assume it is 0. We denote by p the point where γ_0^1 and γ_0^2 meet. We notice that the position of this point p will fix the axis of the Delaunay limit surface.

Now let us consider an increasing sequence (z_n) with limit $+\infty$. From Subsection 2.5, a subsequence of $t_{-z_n}(A)$ (still denoted $t_{-z_n}(A)$) converges to a properly weakly embedded surface Σ with constant mean curvature H . Moreover Σ is cylindrically bounded.

Claim 12. *The surface Σ is connected and noncompact.*

Proof of the claim. Assume Σ has a compact connected component Σ' . This implies that, for large n , there is a part of $t_{-z_n}(A)$ that is a graph over Σ' . So A would possess a compact component, and this gives a contradiction.

Assume now that Σ has two noncompact connected components Σ' and Σ'' . We recall that the height function on Σ' and Σ'' cannot be lower or upper bounded (Lemma 2). So there is a connected component of $\Sigma' \cap \{0 \leq z \leq M\}$ and one of $\Sigma'' \cap \{0 \leq z \leq M\}$ with boundary in both $\{z = 0\}$ and $\{z = M\}$. So this implies that for large n , $A \cap \{z_n \leq z \leq z_n + M\}$ possesses at least two connected components with boundary in both $\{z = z_n\}$ and $\{z = z_n + M\}$. This is in contradiction with Lemma 3 when M is large. □

The existence of a limit for the Alexandrov functions α_{Γ^1} and α_{Γ^2} has the following consequence.

Claim 13. *The surface Σ is symmetric with respect to $\gamma_0^1 \times \mathbb{R}$ and $\gamma_0^2 \times \mathbb{R}$.*

Proof of the claim. We only write the proof for $\gamma_0^1 \times \mathbb{R}$. Let S_t^1 denote the symmetries associated to the foliation Γ^1 . We want to prove that $S_0^1(\Sigma \cap \Gamma_0^+)$ is on one side of $\Sigma \cap \Gamma_0^-$ and these surfaces touch each other (eventually in the boundary). If this last assertion holds, the maximum principle applies and we get the symmetry with respect to $\gamma_0^1 \times \mathbb{R}$.

So let us fix z_0 in \mathbb{R} to be a regular value of the height function on Σ . Because of the properness, any value t close to z_0 is also a regular value of the height function. So $\Sigma \cap \{z_0 - \varepsilon \leq z \leq z_0 + \varepsilon\}$ consists in a finite union of annuli transverse to horizontal slices of $\mathbb{H}^2 \times \mathbb{R}$.

Let A_n denote the sequence $t_{-z_n}(A)$ and let $G_n = t_{-z_n}(G)$ (where G is the domain bounded by A). We also denote by \overline{G} the domain bounded by Σ and $t_n = \alpha_{\Gamma^1}(t + z_n)$ for some $t \in (z_0 - \varepsilon, z_0 + \varepsilon)$: we have $t_n \rightarrow 0$. Because of the transversality and the convergence, we have $A_n \cap \{z = t\} \rightarrow \Sigma \cap \{z = t\}$.

Besides, from the definition of the Alexandrov function, $S_{t_n}^1(A_n \cap \Gamma_{t_n}^+ \cap \{z = t\})$ is in $G_n \cap \{z = t\}$. Thus, by taking the limit, $S_0^1(\Sigma \cap \Gamma_0^+ \cap \{z = t\})$ is in $\overline{G} \cap \{z = t\}$. Since we have this for any $t \in (z_0 - \varepsilon, z_0 + \varepsilon)$, we have

$$S_0^-(\Sigma \cap \Gamma_0^+ \cap \{z_0 - \varepsilon < z < z_0 + \varepsilon\}) \subset G$$

so it is on one side of $\Sigma \cap \Gamma_0^-$. Besides, as in the proof of Lemma 8, there is a contact point between these two surfaces at every height $t \in (-z_0 - \varepsilon, z_0 + \varepsilon)$. Thus the maximum principle applies to prove that Σ is invariant by S_0^1 (here we use the connectedness of Σ). \square

Since Σ is symmetric with respect to $\gamma_0^1 \times \mathbb{R}$ and $\gamma_0^2 \times \mathbb{R}$, it is invariant by the rotation R of angle 2θ around the vertical axis $\{p\} \times \mathbb{R}$. Since 2θ is irrational, Σ is invariant by rotation around $\{p\} \times \mathbb{R}$.

This implies that Σ is equal to a Delaunay surface \mathcal{D}_τ of axis $\{p\} \times \mathbb{R}$. The height 0 is a regular value of \mathcal{D}_τ and $2\pi\tau$ is the flux of \mathcal{D}_τ along $\mathcal{D}_\tau \cap \{z = 0\}$ in the direction ∂_z . So $2\pi\tau$ is the limit of the flux of A along $A \cap \{z = z_n\}$ in the direction ∂_z . But this flux does not depend on n and is equal to the flux of A along $A \cap \{z = 0\}$ in the direction ∂_z . So the parameter τ only depends on A . \square

4.3. The main theorem. In this section, we settle the main theorem of this paper and explain its proof.

Actually, the main theorem is based on the following proposition that will be proved in Section 6.

Proposition 14. *Let A be a properly embedded annular top end with cmc H which is cylindrically bounded. Let $\Gamma = (\gamma_s)_{s \in \mathbb{R}}$ be a translation foliation of \mathbb{H}^2 by geodesic lines. The Alexandrov function $\alpha_\Gamma : \mathbb{R}^+ \rightarrow \mathbb{R}$ is then decreasing.*

With this proposition we can prove our main result.

Theorem 15. *Let Σ be a properly embedded cmc surface in $\mathbb{H}^2 \times \mathbb{R}$. If Σ has finite topology and is cylindrically bounded, Σ is a Delaunay surface (i.e. Σ is rotationally invariant).*

If Σ is compact, the result is already known [7], so we focus on the noncompact case.

Proof. Let $\Gamma = (\gamma_s)_{s \in \mathbb{R}}$ be a translation foliation of \mathbb{H}^2 . Let us denote by E_1^+, \dots, E_p^+ the annular top ends of Σ and by E_1^-, \dots, E_q^- the annular bottom ends of Σ . We consider the Alexandrov functions $\alpha_{\Gamma, \Sigma}$ and α_{Γ, E_i^\pm} . We can assume that the Alexandrov functions α_{Γ, E_i^+} are defined on $[M, +\infty)$ and the α_{Γ, E_i^-} are defined on $(-\infty, -M]$. By Proposition 14, the functions α_{Γ, E_i^+} decrease and the functions α_{Γ, E_i^-} increase.

Besides, on $[M, +\infty)$, we have:

$$\alpha_{\Gamma, \Sigma}(z) = \max_{1 \leq i \leq p} \alpha_{\Gamma, E_i^+}(z)$$

and, on $(-\infty, -M]$, we have:

$$\alpha_{\Gamma, \Sigma}(z) = \max_{1 \leq i \leq q} \alpha_{\Gamma, E_i^-}(z).$$

So the function $\alpha_{\Gamma, \Sigma}$ increases on $(-\infty, -M]$ and decreases on $[M, +\infty)$. By Lemma 10, this implies that $\alpha_{\Gamma, \Sigma}$ is constant and Σ is symmetric with respect to some $\gamma_s \times \mathbb{R}$.

Let Γ^1 be a translation foliation of \mathbb{H}^2 ; Σ is then symmetric with respect to some γ_s^1 . We can assume that it is symmetric with respect to $\gamma_0^1 \times \mathbb{R}$. Let Γ_2 be the translation foliation of \mathbb{H}^2 composed by the geodesic line orthogonal to γ_0^1 . Σ is then symmetric with respect to some $\gamma_s^2 \times \mathbb{R}$. We can also assume it is γ_0^2 . Let p

be the intersection point of γ_0^1 and γ_0^2 . Let g be a geodesic passing by p and let Γ^3 be the translation foliation composed by the geodesic lines orthogonal to g . Σ is then symmetric with respect to some $\gamma_s^3 \times \mathbb{R}$. Since Σ is cylindrically bounded, γ_s^3 passes by p . This implies that Σ is symmetric with respect to any vertical plane passing by p , so Σ is invariant by rotation around the vertical axis $p \times \mathbb{R}$. \square

5. HORIZONTAL KILLING GRAPHS

Let us consider a new model for \mathbb{H}^2 : $\mathbb{H}^2 = \{(s, r) \in \mathbb{R}^2\}$ with the metric $dr^2 + (\cosh r)^2 ds^2$. In this model $\{r = 0\}$ is a geodesic, $\{r = c\}$ are its equidistant lines and $\{s = c\}$ are the geodesic lines orthogonal to $\{r = 0\}$. Moreover ∂_s is the Killing vector field corresponding to the translation along $\{r = 0\}$. In this section, we will use this model of \mathbb{H}^2 to describe $\mathbb{H}^2 \times \mathbb{R}$. The surfaces $\{s = c\}$ are then totally geodesic flat planes.

Let Ω be a domain in \mathbb{R}^2 and let u be a smooth function on Ω . Using the above model for \mathbb{H}^2 , we can consider the surface in $\mathbb{H}^2 \times \mathbb{R}$ parametrized by $(r, z) \mapsto (u(r, z), r, z)$. Such a surface is called the *horizontal Killing graph* of u , it is transverse to the Killing vector field ∂_s and any integral curve of ∂_s intersects the surface at most once.

5.1. The mean curvature equation. In the following, we are interested in horizontal Killing graphs with constant mean curvature H_0 . This condition implies that the function u satisfies a partial differential equation.

Let Ω and u be as above. A unit normal vector to the horizontal Killing graph of u is given by the expression

$$(2) \quad N = \frac{-\partial_s + (\cosh r)^2 \nabla u}{\cosh r \sqrt{1 + (\cosh r)^2 |\nabla u|^2}}$$

where ∇ is the Euclidean gradient operator and $|\cdot|$ is the Euclidean norm. In the sequel, we will use this unit normal vector to compute the mean curvature of a horizontal Killing graph.

Lemma 16. *Let Ω and u be as above. Then the mean curvature H of the horizontal Killing graph of u satisfies:*

$$(3) \quad -2H \cosh r = \operatorname{div} \frac{(\cosh r)^2 \nabla u}{\sqrt{1 + (\cosh r)^2 |\nabla u|^2}}$$

with div the Euclidean divergence operator.

Proof. We extend the vector field N to the whole $\mathbb{R} \times \Omega$ by using the expression given in (2). The mean curvature of the horizontal Killing graph of u is then given by

$$\begin{aligned} -2H &= \operatorname{div}_{\mathbb{H}^2 \times \mathbb{R}} N \\ &= \left(\overline{\nabla} \frac{\partial_s}{\cosh r} N, \frac{\partial_s}{\cosh r} \right) + \left(\overline{\nabla}_{\partial_r} N, \partial_r \right) + \left(\overline{\nabla}_{\partial_z} N, \partial_z \right). \end{aligned}$$

Let W denote $\sqrt{1 + (\cosh r)^2 |\nabla u|^2}$; we then have:

$$\begin{aligned} \left(\bar{\nabla}_{\frac{\partial_s}{\cosh r}} N, \frac{\partial_s}{\cosh r}\right) &= \frac{1}{(\cosh r)^2} \left(\frac{-1}{W \cosh r} (\bar{\nabla}_{\partial_s} \partial_s, \partial_s) + \frac{\cosh r}{W} (\bar{\nabla}_{\partial_s} \nabla u, \partial_s) \right) \\ &= \frac{1}{W \cosh r} (\bar{\nabla}_{\nabla u} \partial_s, \partial_s) \\ &= \frac{\nabla u \cdot \cosh r}{W} \end{aligned}$$

and for $a = r$ or $a = z$:

$$\begin{aligned} (\bar{\nabla}_{\partial_a} N, \partial_a) &= (\bar{\nabla}_{\partial_a} \frac{-\partial_s}{W \cosh r}, \partial_a) + (\nabla_{\partial_a} \frac{(\cosh r) \nabla u}{W}, \partial_a) \\ &= (\nabla_{\partial_a} \frac{(\cosh r) \nabla u}{W}, \partial_a). \end{aligned}$$

Summing all these terms, we get:

$$\begin{aligned} -2H &= \frac{\nabla u \cdot \cosh r}{W} + \operatorname{div} \left(\frac{(\cosh r) \nabla u}{W} \right) \\ &= \frac{1}{\cosh r} \left(\frac{(\nabla(\cosh r), (\cosh r) \nabla u)}{W} + (\cosh r) \operatorname{div} \left(\frac{(\cosh r) \nabla u}{W} \right) \right) \\ &= \frac{1}{\cosh r} \operatorname{div} \left(\frac{(\cosh r)^2 \nabla u}{W} \right). \end{aligned}$$

□

Thus the constant mean curvature H_0 equation for a function u can be written

$$(4) \quad \operatorname{div} \frac{(\cosh r)^2 \nabla u}{\sqrt{1 + (\cosh r)^2 |\nabla u|^2}} = -2H_0 \cosh r$$

or after expanding all the terms

$$\begin{aligned} (5) \quad &\left(\frac{1 + (\cosh r)^2 u_z^2}{2 + (\cosh r)^2 |\nabla u|^2} \right) u_{rr} - 2 \left(\frac{(\cosh r)^2 u_r u_z}{2 + (\cosh r)^2 |\nabla u|^2} \right) u_{rz} \\ &+ \left(\frac{1 + (\cosh r)^2 u_r^2}{2 + (\cosh r)^2 |\nabla u|^2} \right) u_{zz} + (\tanh r) u_r = -2H_0 \frac{(1 + (\cosh r)^2 |\nabla u|^2)^{3/2}}{(\cosh r)(2 + (\cosh r)^2 |\nabla u|^2)}. \end{aligned}$$

For $H_0 = 0$, we get the minimal surface equation:

$$(6) \quad \operatorname{div} \frac{(\cosh r)^2 \nabla u}{\sqrt{1 + (\cosh r)^2 |\nabla u|^2}} = 0.$$

We notice that the maximum principle is true for these equations. Thus we have uniqueness of a solution to the Dirichlet problem associated to these equations on bounded domains.

5.2. A gradient estimate. An important result concerning solutions of (4) is the following gradient estimate.

Proposition 17. *Let u be a nonnegative solution of (4) on a disk centered at $p = (r_p, z_p)$ and radius R . Then there is a constant M that depends only on r_p, R and H_0 such that*

$$|\nabla u|(p) \leq \max(2, 32M(u(p)/R)) e^{6Mu(p) + 4M^2(u(p)/R)^2}.$$

The proof of this result is similar to that of the gradient estimate proved by Spruck in [20]; but our result does not seem to be a corollary of his result.

Before beginning the proof, let us make some preliminary computations. So let u be as in the proposition and let Σ denote the horizontal Killing graph of u . We denote $\tilde{N} = -N$ (see (2)) and define $\nu = \langle \tilde{N}, \partial_s \rangle$ and $\mu = \nu / (\cosh r)$. We have $\nu > 0$ and, ∂_s being a Killing vector field,

$$\Delta_\Sigma \nu = -(\text{Ric}(\tilde{N}, \tilde{N}) + |A|^2)\nu$$

where $|A|^2$ is the square of the norm of the second fundamental form and Ric is the Ricci tensor.

Let us denote by h the restriction of s along Σ . We have

$$\nabla_\Sigma h = \frac{1}{(\cosh r)^2} \partial_s^\top \quad \text{and} \quad |\nabla_\Sigma h|^2 = \frac{1}{(\cosh r)^2} (1 - \mu^2).$$

If (e_1, e_2) is an orthonormal basis of $T\Sigma$, we have

$$\begin{aligned} \Delta_\Sigma h &= \text{div}_\Sigma \left(\frac{1}{(\cosh r)^2} \partial_s^\top \right) = -\frac{2 \tanh r}{(\cosh r)^2} (\partial_r^\top, \partial_s^\top) + \frac{1}{(\cosh r)^2} \text{div}_\Sigma (\partial_s - \nu \tilde{N}) \\ &= \frac{2 \tanh r}{\cosh r} \mu (\partial_r, \tilde{N}) + \frac{1}{(\cosh r)^2} (-\nu) \sum_{i=1}^2 (\bar{\nabla}_{e_i} \tilde{N}, e_i) \\ &= \frac{2 \tanh r}{\cosh r} \mu (\partial_r, \tilde{N}) - \frac{2H_0 \mu}{\cosh r}. \end{aligned}$$

Let us define the distance function $d = ((r - r_p)^2 + (z - z_p)^2)^{1/2}$. The vector field $\partial_d = ((r - r_p)\partial_r + (z - z_p)\partial_z)/d$ is well defined in $\mathbb{H}^2 \times \mathbb{R}$ outside $\mathbb{R} \times \{p\}$ and has unit length; $d\partial_d$ is well defined everywhere. We have:

$$\nabla_\Sigma d^2 = 2d\partial_d^\top \quad \text{and} \quad |\nabla_\Sigma d^2|^2 = 4d^2 |\partial_d^\top|^2.$$

We denote $\tilde{r} = r - r_p$ and $\tilde{z} = z - z_p$. We then have:

$$\begin{aligned} \Delta_\Sigma d^2 &= 2 \text{div}_\Sigma (\tilde{r} \partial_r^\top + \tilde{z} \partial_z^\top) \\ &= 2(|\partial_r^\top|^2 + |\partial_z^\top|^2) \\ &\quad + 2 \sum_{i=1}^2 \left(\tilde{r} (\bar{\nabla}_{e_i} (\partial_r - (\partial_r, \tilde{N}) \tilde{N}), e_i) + \tilde{z} (\bar{\nabla}_{e_i} (\partial_z - (\partial_z, \tilde{N}) \tilde{N}), e_i) \right) \\ &= 2(1 + \mu^2) + 2 \sum_{i=1}^2 \left(\tilde{r} (\bar{\nabla}_{e_i} \partial_r, e_i) - (\tilde{r} (\partial_r, \tilde{N}) + \tilde{z} (\partial_z, \tilde{N})) (\bar{\nabla}_{e_i} \tilde{N}, e_i) \right) \\ &= 2(1 + \mu^2) + 2\tilde{r} \sum_{i=1}^2 (\bar{\nabla}_{e_i} \partial_r, e_i) - 2H_0 (d\partial_d, \tilde{N}). \end{aligned}$$

We define $f_1 = \partial_s/(\cosh r)$, $f_2 = \partial_r$ and $f_3 = \partial_z$ to be an orthonormal basis of $T\mathbb{H}^2 \times \mathbb{R}$ and we write $e_i = \sum_j \lambda_i^j f_j$. We then have

$$\begin{aligned} \sum_{i=1}^2 (\nabla_{e_i} \partial_r, e_i) &= \sum_{i=1}^2 \sum_{k,l=1}^3 \lambda_i^k \lambda_i^l (\bar{\nabla}_{f_k} \partial_r, f_l) \\ &= \sum_{i=1}^2 (\lambda_i^1)^2 \frac{1}{(\cosh r)^2} (\bar{\nabla}_{\partial_s} \partial_r, \partial_s) \\ &= \tanh r(1 - \mu^2). \end{aligned}$$

So $\Delta_\Sigma d^2 = 2(1 + \mu^2) + 2\tilde{r} \tanh r(1 - \mu^2) - 2H_0 d(\partial_d, \tilde{N})$. Using the above computations, we are ready to write the proof.

Proof of Proposition 17. Let us introduce the second order operator $Lf = \Delta_\Sigma f - 2\nu(\nabla_\Sigma \frac{1}{\nu}, \nabla_\Sigma f)$ on Σ . We notice that the maximum principle is true for L . We have:

$$\Delta_\Sigma \frac{1}{\nu} = \operatorname{div}_\Sigma(-\frac{1}{\nu^2} \nabla_\Sigma \nu) = 2\nu |\nabla_\Sigma \frac{1}{\nu}|^2 + (\operatorname{Ric}(\tilde{N}, \tilde{N}) + |A|^2) \frac{1}{\nu}.$$

Since $\operatorname{Ric}(\tilde{N}, \tilde{N}) \geq -1$, we have $L\frac{1}{\nu} \geq -\frac{1}{\nu}$. Let us define $v = \eta\frac{1}{\nu}$ with η a positive function. We have:

$$Lv = \eta L\frac{1}{\nu} + \frac{1}{\nu} \Delta_\Sigma \eta \geq (\Delta_\Sigma \eta - \eta) \frac{1}{\nu}.$$

We define on Σ the function $\varphi = (-\frac{h}{2h_0} + 1 - \varepsilon - (\frac{d}{R})^2)^+$ which is less than 1 ($\varepsilon > 0$) where $h_0 = u(p) = h(P)$ with $P = (u(p), p)$. Moreover, $\varphi(P) = 1/2 - \varepsilon$ and $\varphi = 0$ close to $\partial\Sigma$. We define $\eta = e^{K\varphi} - 1$ with K a positive constant that will be chosen later. We then have $\max v > 0$ and it is reached inside the support of φ .

We have $\Delta_\Sigma \eta = \operatorname{div}_\Sigma(e^{K\varphi} K \nabla_\Sigma \varphi) = e^{K\varphi} (K^2 |\nabla_\Sigma \varphi|^2 + K \Delta_\Sigma \varphi)$ so:

$$\begin{aligned} \Delta_\Sigma \eta - \eta &= e^{K\varphi} (K^2 |\nabla_\Sigma \varphi|^2 + K \Delta_\Sigma \varphi - 1) + 1 \\ &\geq e^{K\varphi} (K^2 |\nabla_\Sigma \varphi|^2 + K \Delta_\Sigma \varphi - 1). \end{aligned}$$

We have

$$\begin{aligned} |\nabla_\Sigma \varphi|^2 &= |-\frac{1}{2h_0} \nabla_\Sigma h - \frac{1}{R^2} \nabla_\Sigma d^2|^2 \\ &= \frac{1}{4h_0^2 \cosh^2 r} (1 - \mu^2) + \frac{4d^2}{R^2} |\partial_d^\top|^2 + \frac{2d}{h_0 R^2 \cosh^2 r} (\partial_s^\top, \partial_d^\top) \\ &\geq \frac{1}{4h_0^2 \cosh^2 r} (1 - \mu^2) - \frac{2d}{h_0 R^2 \cosh r} \mu(\partial_d, N) \\ &\geq \frac{1}{4h_0^2 \cosh^2 r} (1 - \mu^2 - 8\frac{h_0}{R} M\mu) \end{aligned}$$

where we use $d \leq R$ and M is a constant chosen to be larger than $\cosh r \sqrt{4 + 2R + 2H_0 R}$ on the disk of center p and radius R . So:

$$\text{if } \mu \leq \min(\frac{1}{2}, \frac{R}{32Mh_0}), \text{ then } |\nabla_\Sigma \varphi|^2 \geq \frac{1}{8h_0^2 (\cosh r)^2}.$$

Besides, we have

$$\begin{aligned} \Delta_\Sigma \varphi &= -\frac{1}{2h_0} \Delta_\Sigma h - \frac{1}{R^2} \Delta_\Sigma d^2 \\ &= -\frac{1}{2h_0} \left(\frac{2 \tanh r}{\cosh r} \mu(\partial_r, \tilde{N}) - \frac{2H_0 \mu}{\cosh r} \right) \\ &\quad - \frac{1}{R^2} (2(1 + \mu^2) + 2\tilde{r} \tanh r(1 - \mu^2) - 2H_0 d(\partial_d, \tilde{N})) \\ &\geq -\frac{\mu}{h_0 \cosh r} - \frac{1}{R^2} (2(1 + \mu^2) + 2R(1 - \mu^2) + 2H_0 R) \\ &\geq -\frac{1}{h_0^2 (\cosh r)^2} \left(Mh_0 + \frac{h_0^2 (\cosh r)^2}{R^2} (4 + 2R + 2H_0 R) \right) \\ &\geq -\frac{1}{h_0^2 (\cosh r)^2} \left(Mh_0 + \frac{h_0^2 M^2}{R^2} \right). \end{aligned}$$

We deduce from the above computation that, if $\mu \leq \min(\frac{1}{2}, \frac{R}{32Mh_0})$,

$$\begin{aligned} &K^2 |\nabla_\Sigma \varphi|^2 + K \Delta_\Sigma \varphi - 1 \\ &\geq \frac{1}{8h_0^2 (\cosh r)^2} K^2 + K \left(-\frac{1}{h_0^2 (\cosh r)^2} \right) \left(Mh_0 + \frac{h_0^2 M^2}{R^2} \right) - 1 \\ &\geq \frac{1}{8h_0^2 (\cosh r)^2} \left(K^2 - 8K \left(Mh_0 + \frac{h_0^2 M^2}{R^2} \right) - 8h_0^2 M^2 \right). \end{aligned}$$

So if $K = (12Mh_0 + 8h_0^2 M^2 / R^2)$ we obtain that $K^2 |\nabla_\Sigma \varphi|^2 + K \Delta_\Sigma \varphi - 1 > 0$ and then $Lv > 0$. By the maximum principle applied to L , it implies that the maximum of v can only be attained at a point q where $\mu \geq \min(\frac{1}{2}, \frac{R}{32Mh_0})$. This implies that

$$v(p) = (e^{K(1/2-\varepsilon)} - 1) \frac{1}{\nu(p)} \leq \frac{e^K - 1}{\nu(q)} \leq \frac{e^K - 1}{\min(\frac{1}{2}, \frac{R}{32Mh_0})}.$$

So letting ε tend to 0 we get:

$$\nu(p) \geq \min\left(\frac{1}{2}, \frac{R}{32Mh_0}\right) e^{-K/2}.$$

So

$$|\nabla u|(p) \leq \max(2, 32M(h_0/R)) e^{6Mh_0 + 4M^2(h_0/R)^2}.$$

□

5.3. An existence result for the Dirichlet problem. In this subsection, we give a result about the existence of a solution of the Dirichlet problem for equation (4) on small domains. Actually, it is a consequence of the work of Serrin in [19].

Proposition 18. *Let $p = (r_p, z_p)$ be a point in \mathbb{R}^2 and let H_0 be a nonnegative constant. Then, there exists a constant $R > 0$ that depends only on H_0 and $|r_p|$ such that the Dirichlet problem for equation (4) can be solved on the disk $D(p, \tilde{R})$ centered at p and radius \tilde{R} less than R . More precisely, for any continuous function φ on the boundary of $D(p, \tilde{R})$ ($\tilde{R} \leq R$) there exists $u \in C^2(D(p, \tilde{R})) \cap C^0(\overline{D(p, \tilde{R})})$ such that u solves (4) and $u = \varphi$ on the boundary of the disk.*

Proof. If φ is C^2 , the result is a consequence of Theorem 14.3 in [19]. We notice that the hypotheses of this theorem are satisfied by equation (4). In fact in order to have the same notation as Serrin, the equation has to be written in the form (5). Moreover, since the coefficients of (5) only depend on r and H_0 , the radius R only depends on r_p and H_0 .

When φ is only continuous, we proceed by approximation. Let (φ_n) be a sequence of C^2 functions converging to φ in the C^0 norm. We denote by u_n the solution of (4) with φ_n as boundary value. The sequence u_n is uniformly bounded. So, by the gradient estimate (Proposition 17) and elliptic estimates, the sequence (u_n) converges to a solution u of (4) on the disk. Let us consider $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $\varphi_n - \varepsilon \leq \varphi - \varepsilon/2 < \varphi + \varepsilon/2 \leq \varphi_n + \varepsilon$, and for m large we have $\varphi_n - \varepsilon \leq \varphi_m \leq \varphi_n + \varepsilon$. So by the maximum principle, $u_n - \varepsilon \leq u_m \leq u_n + \varepsilon$. This implies $u_n - \varepsilon \leq u \leq u_n + \varepsilon$, so on the boundary of the disk

$$\varphi - 3\varepsilon/2 \leq \varphi_n - \varepsilon \leq \liminf_{\partial D(p, \bar{R})} u \leq \limsup_{\partial D(p, \bar{R})} u \leq \varphi_n + \varepsilon \leq \varphi + 3\varepsilon/2.$$

Letting ε go to 0, we see that u is continuous up to the boundary and $u = \varphi$ there. □

5.4. A uniqueness result. In this section we give a uniqueness result for the Dirichlet problem associated to (4) when the domain is unbounded.

Proposition 19. *Let Ω be an unbounded domain in \mathbb{R}^2 such that the r coordinate is bounded on Ω . Let u and v be two solutions of (4) on Ω which are continuous up to the boundary of Ω and such that $u = v$ along this boundary. If the function $|v - u|$ is bounded on Ω , then $u = v$.*

The proof is based on the same ideas as Theorem 2 in [4].

Proof. Let us define $\Omega_a = \{p \in \Omega \mid |p| < a\}$, $C_a = \{p \in \Omega \mid |p| = a\}$, $w = v - u$ and $X = \frac{\cosh r \nabla v}{\sqrt{1 + \cosh^2 r |\nabla v|^2}} - \frac{\cosh r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}$. We denote by $\vec{\eta}$ the outgoing normal to Ω_a .

We then have

$$\int_{C_a} w(\cosh r X) \cdot \vec{\eta} = \int_{\partial \Omega_a} w(\cosh r X) \cdot \vec{\eta} = \int_{\Omega_a} \nabla w \cdot (\cosh r X).$$

By Lemma 1 in [4],

$$\begin{aligned} \nabla w \cdot (\cosh r X) &= (\cosh r \nabla v - \cosh r \nabla u) \\ &\quad \cdot \left(\frac{\cosh r \nabla v}{\sqrt{1 + \cosh^2 r |\nabla v|^2}} - \frac{\cosh r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}} \right) \\ &\geq \left| \frac{\cosh r \nabla v}{\sqrt{1 + \cosh^2 r |\nabla v|^2}} - \frac{\cosh r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}} \right|^2 \\ &\geq |X|^2. \end{aligned}$$

Let $a_0 > 0$ be such that $\Omega_{a_0} \neq \emptyset$ and denote $\mu = \int_{\Omega_{a_0}} \nabla w \cdot (\cosh r X)$; $\mu > 0$ if $u \neq v$. Since w is bounded by a constant M and r is bounded on Ω , $\cosh r$ is bounded also by M . We then have:

$$\mu + \int_{\Omega_a \setminus \Omega_{a_0}} |X|^2 \leq M^2 \int_{C_a} |X|.$$

Let us define $I(a) = \int_{C_a} |X|$ and let l_a be the length of C_a ; we remark that $l_a \leq 2\pi a$. We have

$$I^2(a) = \left(\int_{C_a} |X| \right)^2 \leq l_a \int_{C_a} |X|^2.$$

Then

$$(7) \quad \mu + \int_{a_0}^a \frac{I^2(t)}{2\pi t} dt \leq M^2 I(a).$$

Let ζ be the function defined on $[a_0, a_0 \exp(4\pi M^4/\mu)]$ by the equation

$$-\frac{1}{\zeta} + \frac{2M^2}{\mu} = \frac{1}{2\pi M^2} \ln \frac{a}{a_0}$$

where ζ satisfies $\zeta(a_0) = \mu/(2M^2)$ and $M^2\zeta' = \zeta^2/(2\pi a)$. Equation (7) then implies that $I(a) \geq \zeta(a)$. But this is impossible since $\zeta(a)$ converges to $+\infty$ when $a \rightarrow a_0 \exp(4\pi M^4/\mu)$. Then u and v are equal. \square

6. THE MONOTONICITY OF THE ALEXANDROV FUNCTION

This section is entirely devoted to the proof of Proposition 14. This will finish the proof of our main result (Theorem 15).

6.1. The geometric configuration. Let us consider a properly embedded annular top end A with cmc H_0 which is cylindrically bounded. Let $\Gamma = (\gamma_t)_{t \in \mathbb{R}}$ be a translation foliation of \mathbb{H}^2 by geodesic lines. We assume that α_Γ is not decreasing.

By considering only $A \cap \{z \geq z_0\}$ for some $z_0 > 0$ large, we can assume that α_Γ is increasing. Because of Proposition 11, we can also assume that any horizontal section $A \cap \{z = z'\}$ is composed of one curve with curvature strictly larger than 1.

Using the model introduced in Section 5 for \mathbb{H}^2 , we can assume that the foliation Γ is given by $\gamma_t = \{s = t\}$. Moreover, by changing the origin of the s variable, we can assume that $\alpha_\Gamma(0) < 0$ and $\lim_{+\infty} \alpha_\Gamma > 0$. We also can assume that the intersection $A \cap \{s = 0\}$ is transverse. We define $A^+ = A \cap \{s > 0\}$ and $A^- = A \cap \{s < 0\}$ (see Figure 5).

The idea of the proof of Proposition 14 is to obtain a control of the flux of A along ∂A^+ in the direction ∂_s . This idea comes from the Positive Flux Lemma in [8].

6.2. A Dirichlet problem. The annulus A bounds a cylindrically bounded domain D in $\mathbb{H}^2 \times \mathbb{R}_+^*$. Let Ω denote the domain $\{(r, z) \in \mathbb{R} \times \mathbb{R}_+^* \mid (0, r, z) \in D\}$.

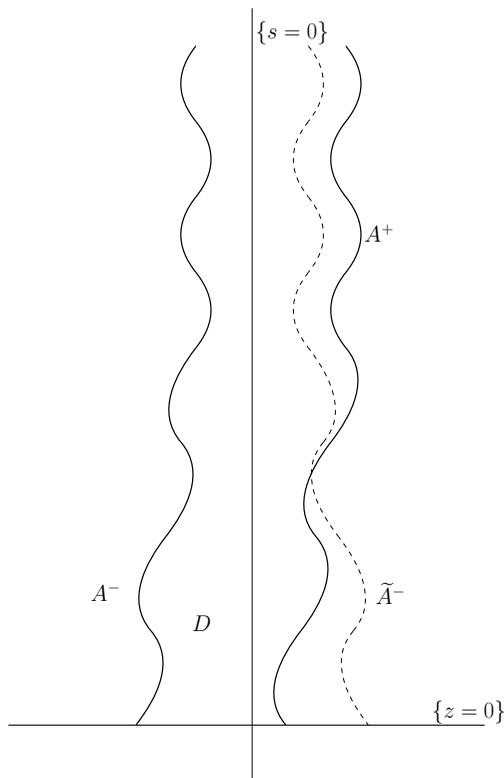
Let us denote by \tilde{A}^- the symmetric of A^- by $\{s = 0\}$. We then define on Ω the function f by $f(r, z) = \inf\{s \in \mathbb{R}_+ \mid (s, r, z) \in A^+ \cup \tilde{A}^-\}$. Since A is cylindrically bounded, f is uniformly bounded on Ω . Since the curvature of the curve $A \cap \{z = z'\}$ is larger than 1, f extends continuously to $\partial\Omega$ and this boundary value is 0 along $\partial\Omega \cap \{z > 0\}$.

Since $\alpha_\Gamma(0) < 0$, the reflection procedure described in Section 4.1 implies that $A^+ \cap \{z = 0\}$ is the horizontal Killing graph of f over $\partial\Omega \cap \{z = 0\}$.

Besides, if $p \in \partial\Omega \cap \{z > 0\}$ and the tangent space to A at $(0, p)$ is not normal to $\{s = 0\}$, for $q \in \Omega$

$$\frac{f(q)}{d(q, \partial\Omega)} \text{ is bounded in a neighborhood of } p$$

where $d(q, \partial\Omega)$ is the Euclidean distance from q to $\partial\Omega$.

FIGURE 5. The annular end A

The aim of this section is to solve a Dirichlet problem for (4): we prove the following result.

Lemma 20. *There exists a unique nonnegative solution u on Ω of equation (4) which is continuous up to the boundary with boundary value f and such that $u \leq f$ on Ω .*

Proof. The uniqueness comes from Proposition 19 since the r coordinate is bounded in Ω (D is cylindrically bounded).

For the existence part, we use the Perron method to solve the Dirichlet problem. Let us recall the framework of the Perron method.

Let v be a continuous function on $\overline{\Omega}$; v is called a subsolution for our problem if $v \leq f$ and if, for any compact subdomain $U \subset \Omega$ and any solution h of (4) with $v \leq h$ on the boundary of U , we have $v \leq h$ on U . If S denotes the set of all subsolutions, we define our solution by the following formula:

$$u(q) = \sup_{v \in S} v(q).$$

We notice that S is nonempty since the function 0 is a subsolution; thus $u \geq 0$. Moreover if v and w are subsolutions, the continuous function $\max(v, w)$ is also a subsolution. It is also clear that $u \leq f$ but it is not clear that u is a solution to our problem.

Since the r coordinate is bounded in Ω , Proposition 18 implies there is an $R > 0$ such that, for any disk $\Delta \subset \Omega$ of radius less than R , the Dirichlet problem can be solved on Δ for equation (4). Thus for any such disk Δ and subsolution v , we can define the continuous function $M_\Delta(v)$ on Ω as $M_\Delta(v) = v$ outside Δ and $M_\Delta(v)$ is equal to the solution of (4) in Δ with v as boundary value. Since v is a subsolution, $v \leq M_\Delta(v)$. The graph of v is below A^+ and \tilde{A}^- so, by the maximum principle for cmc H_0 surfaces, the graph of $M_\Delta(v)$ is below A^+ and \tilde{A}^- . This implies that $M_\Delta(v)$ is a subsolution.

Claim 21. *The function u is a solution of (4) in Ω .*

Proof of Claim 21. Let us consider $p \in \Omega$ and Δ a disk in Ω centered at p with radius less than R . Let (v_n) be a sequence of subsolutions such that $v_n(p) \rightarrow u(p)$. By considering $\max(0, v_n)$ we can assume $v_n \geq 0$. We have $M_\Delta(v_n)$ is also a sequence of subsolutions with $M_\Delta(v_n)(p) \rightarrow u(p)$. On Δ , $M_\Delta(v_n)$ is a bounded sequence of solutions of (4). So by considering a subsequence if necessary, we can assume that it converges to a solution \bar{v} on Δ with $u \geq \bar{v}$ and $\bar{v}(p) = u(p)$. Let us prove that $u = \bar{v}$ on Δ , so u will be a solution of (4).

If $u \neq \bar{v}$ on Δ , there is a point $q \in \Delta$ where $u(q) > \bar{v}(q)$. So there is a subsolution w such that $w(q) > \bar{v}(q)$. So let us consider the sequence of subsolutions $M_\Delta(\max(w, v_n))$. We have $M_\Delta(\max(w, v_n)) \geq M_\Delta(v_n)$ and $M_\Delta(\max(w, v_n)) \geq w$. Moreover on Δ , it is a sequence of solutions of (4); so considering a subsequence, it converges to a solution \bar{w} of (4) with $\bar{w} \geq \bar{v}$ and $\bar{w}(q) \geq w(q) > \bar{v}(q)$. But since $\bar{w}(p) = \bar{v}(p)$, the maximum principle gives $\bar{w} = \bar{v}$ on Δ which contradicts $\bar{w}(q) \geq w(q) > \bar{v}(q)$. The claim is proved. \square

Since $0 \leq u \leq f$, the function u is continuous up to the part of $\partial\Omega$ not in $\{z = 0\}$ and $u = 0 = f$ there. For $\partial\Omega \cap \{z = 0\}$, we need to construct some barriers for the problem.

Claim 22. *Let $r_0, M > 0$ and $\varepsilon > 0$ be real numbers. There exist a neighborhood V of $(r_0, 0)$ in $\mathbb{R} \times \mathbb{R}_+$ as small as we want and a solution h of (6) in V which is continuous up to ∂V such that $0 \leq h \leq M$ on V , $h = 0$ on $\partial V \cap (\mathbb{R} \times \mathbb{R}_+^*)$ and $h(r_0, 0) = M - \varepsilon$.*

Proof of Claim 22. Let us consider $r_0 \in \mathbb{R}, M > 0, \varepsilon > 0$ as in the claim and $R > 0$ such that the Dirichlet problem for the minimal surface equation (6) can be solved on the disk Δ centered at $(r_0, 0)$ and radius R (Proposition 18). On $\partial\Delta$, let φ_n be a continuous function such that:

- $0 \leq \varphi_n \leq 2M$.
- $0 = \varphi_n$ on $\partial\Delta \cap \{z > 1/n\}$ and $\varphi_n \leq M$ on $\{z \geq 0\}$.
- $\varphi_n(r, z) = 2M - \varphi_n(r, -z)$.

Moreover, we assume $\varphi_n \geq \varphi_{n+1}$ on $\partial\Delta \cap \{z \geq 0\}$. Let h_n be the solution of (6) on Δ such that $h_n = \varphi_n$ on $\partial\Delta$. By uniqueness of the solution and the maximum principle, we have:

- $0 \leq h_n \leq 2M$.
- $h_n(r, z) = 2M - h_n(r, -z)$ so $h_n(r, 0) = M$.
- h_n is decreasing and $h_n \leq M$ on $\Delta \cap \{z \geq 0\}$.

Since the sequence is bounded, it converges to a solution \bar{h} of (6) on Δ . Because of the monotonicity, \bar{h} is continuous up to the boundary except at the points $(r_0 + R, 0)$

and $(r_0 - R, 0)$. \bar{h} is equal to 0 on $\partial\Delta \cap \{z > 0\}$ and to M on $\Delta \cap \{z = 0\}$. So by continuity there is an $\eta > 0$ such that $\bar{h}(r_0, \eta) = M - \varepsilon$. So if we consider the restriction of \bar{h} to $\Delta \cap \{z \geq \eta\}$ we have constructed a neighborhood V of (r_0, η) in $\{z \geq \eta\}$ and a solution \bar{h} of (6) on V which is continuous up to the boundary such that $0 \leq \bar{h} \leq M$ on V , $\bar{h} = 0$ on $\partial V \cap \{z > \eta\}$ and $\bar{h}(r_0, \eta) = M - \varepsilon$. We notice that by choosing R small, we can assume V as small as we want. \square

With these barriers we can finish the proof of Lemma 20.

Claim 23. *The function u is continuous up to the boundary of Ω and takes the value f on it.*

Proof of Claim 23. The problem is only on $\partial\Omega \cap \{z = 0\}$ minus its end points; so take a point $p \in \partial\Omega \cap \{z = 0\}$. Let us consider $\varepsilon > 0$ and I a segment in $\{z = 0\}$ containing p such that $f \geq f(p) - \varepsilon$ on I . Now from our construction of barriers, we know that there exist a neighborhood V of p in Ω such that $\bar{V} \cap \{z = 0\} \subset I$ and a solution h of (6) on V continuous up to the boundary and such that $h = 0$ on $\partial V \cap \Omega$, $h \leq f(p) - \varepsilon$ on $\partial V \cap \{z = 0\}$ and $h(p) = f(p) - 2\varepsilon$. Let us extend the definition of h by 0 to the whole Ω . By the maximum principle, h is a subsolution for our problem, so $u \geq h$. This implies that $\liminf_p u \geq f(p) - 2\varepsilon$. Since $u \leq f$ on Ω we have $\limsup_p u \leq f(p)$. Then u is continuous at p and takes the value $f(p)$. \square

6.3. The asymptotic behavior of u . We know from Proposition 11 that the annular end A is asymptotic for large z to a Delaunay surface. In this subsection, we will see that this asymptotic behavior passes to the function u .

Let (z_n) be a sequence such that $z_n \nearrow +\infty$ and $t_{-z_n}(A)$ converges to a Delaunay surface \mathcal{D}_τ . Let us denote by G the cylindrically bounded domain whose boundary is \mathcal{D}_τ . We notice that by our normalization of A , the axis of \mathcal{D}_τ is $\{r = 0, s = \lim_{+\infty} \alpha_\Gamma\}$.

Let us also denote by t_{-z_n} the translation by $-z_n$ in the (r, z) plane. Because of the asymptotic behavior of A , the sequence of domains $t_{-z_n}(\Omega)$ converges to the domain Ω_0 defined by $\Omega_0 = \{(r, z) \in \mathbb{R}^2 \mid (0, r, z) \in G\}$ (the convergence is smooth on any compact). Let us define $\mathcal{D}_\tau^- = \mathcal{D}_\tau \cap \{s < 0\}$ and $\tilde{\mathcal{D}}_\tau^-$ the symmetric of \mathcal{D}_τ^- by $\{s = 0\}$. $\tilde{\mathcal{D}}_\tau^-$ is a horizontal Killing graph of a function f_0 on Ω_0 . Actually, $f_0 = \lim f \circ t_{z_n}$.

Let us consider the sequence $u \circ t_{z_n} \leq f \circ t_{z_n}$ on $t_{-z_n}(\Omega)$; it is a uniformly bounded sequence. So if we consider a subsequence, we can assume that $u \circ t_{z_n}$ converges to a solution u_0 of (4) on Ω_0 . Moreover we have $0 \leq u_0 \leq f_0$. This implies that u_0 is continuous up to $\partial\Omega_0$ and takes the value 0 there. We then have u_0 and f_0 two solutions of (4) on Ω_0 with the same vanishing boundary value; so, by Proposition 19, $u_0 = f_0$.

The uniqueness of the possible limit implies that the whole sequence $u \circ t_{z_n}$ converges to f_0 .

6.4. Computation of fluxes. The idea of this section is to compute the flux of A along the boundary of A^+ in the direction of ∂_s and find a contradiction which will prove Proposition 14.

Let (z_n) be an increasing sequence such that $z_n \rightarrow +\infty$ and let Ω_0 be the associated limit domain. This domain is either a strip if $\tau = 2H_0 - \sqrt{4H_0^2 - 1}$ or a periodic domain composed of successive ‘‘bubbles’’ if $0 < \tau < 2H_0 - \sqrt{4H_0^2 - 1}$. By

adding a constant to (z_n) , we assume that $\{z = 0\}$ is a line of symmetry of Ω_0 . If $u_0 = \lim u \circ t_{z_n}$, we get that $u_0(r, z) = u_0(r, -z)$ and \mathcal{D}_τ is symmetric with respect to $\{z = 0\}$.

The boundary of $A_n^+ = A^+ \cap \{0 < z < z_n\}$ is composed of four smooth arcs: $\gamma^1 = A^+ \cap \{z = 0\}$, $\gamma_n^2 = A^+ \cap \{z = z_n\}$ and $\gamma_n^3 = A \cap \{s = 0\} \cap \{0 \leq z \leq z_n\}$ (γ_n^3 is actually composed of two arcs). The flux of A along ∂A_n^+ in the direction of ∂_s is equal to 0 since ∂A_n^+ is homologically trivial. The idea is to use the graph of u as a barrier for the computation of $F_{\partial A_n^+}(\partial_s)$ to prove that it cannot vanish for large n .

Let us denote by Ω_{z_n} the subdomain $\Omega \cap \{0 < z < z_n\}$. In order to compute the flux, we need a surface Q bounded by ∂A_n^+ : we define Q as the union of $G \cap \{s \geq 0, z = 0\}$, $G \cap \{s \geq 0, z = z_n\}$ and $\{0\} \times \Omega_{z_n}$. The term (∂_s, \vec{n}_Q) is zero along the first two parts and is equal to $-\cosh r$ along the third part so the flux of A along ∂A_n^+ is equal to

$$(8) \quad 0 = F_{\partial A_n^+}(\partial_s) = \int_{\partial A_n^+} (\vec{\nu}, \partial_s) + 2H_0 \int_{\Omega_{z_n}} \cosh r \, dr \, dz.$$

On another hand, we have from (4)

$$(9) \quad 0 = \int_{\partial \Omega_{z_n}} \left(\frac{\cosh^2 r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}, \vec{\eta} \right) ds + \int_{\Omega_{z_n}} 2H_0 \cosh r \, dr \, dz$$

with $\vec{\eta}$ the outgoing unit normal to Ω_{z_n} . We notice that, even if we do not know that u is smooth up to the boundary of Ω , the first integral is well defined since the vector field $\frac{\cosh^2 r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}$ is bounded and equation (4) is satisfied.

Thus equations (8) and (9) give

$$(10) \quad 0 = \int_{\partial \Omega_{z_n}} \left(\frac{\cosh^2 r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}, \vec{\eta} \right) ds - \int_{\partial A_n^+} (\vec{\nu}, \partial_s).$$

In order to compare the terms in the above equality, we need to study the regularity of u at the boundary of Ω .

Let us define $c^1 = \partial \Omega_{z_n} \cap \{z = 0\}$, $c_n^2 = \partial \Omega_{z_n} \cap \{z = z_n\}$ and $c_n^3 = \partial \Omega_{z_n} \setminus (c^1 \cup c_n^2)$. We notice that $\gamma_n^3 = \{0\} \times c_n^3$, so the same notation can be used to denote the two curves.

Claim 24. *The function u is C^2 up to the boundary at each point of c^1 (except its end points) and each point in c_n^3 where A is not normal to $\{s = 0\}$.*

Proof. Let p be a point c_n^3 where A is not normal to $\{s = 0\}$. As written at the beginning of subsection 6.2, there is a neighborhood of p such that $f/d(\cdot, \partial \Omega)$ is bounded. Since $0 \leq u \leq f$, Proposition 17 gives a uniform upper bound for $|\nabla u|$ in a neighborhood of p . Since u satisfies equation (4) and the boundary data are smooth, elliptic regularity theory implies that u is C^2 up to the boundary near p (see for example [11], where Theorem 4.6.1 gives $C^{1,\alpha}$ regularity and Theorem 4.6.3 gives C^2 regularity).

For the regularity at a point in c^1 , the argument is the same but we have to prove the uniform upper bound for the gradient. We notice that, near γ^1 , A^+ is the graph of f . So, near c^1 , f is a smooth function. Since $u \leq f$, this gives a barrier

from above with bounded gradient for u . The problem is to obtain a barrier from below.

The curves γ^1 and $\{0\} \times c^1$ bound a convex domain in $\mathbb{H}^2 \times \{0\}$ whose boundary is thus composed of an arc of curvature larger than 1 and a geodesic arc. Let us denote by U this domain viewed in $\mathbb{H}^2 \times \{0\}$. Let q be the middle of $\{0\} \times c^1$. Let ε be positive and let Γ_ε be the Jordan arc in $\mathbb{H}^2 \times \mathbb{R}$ composed by γ^1 and two geodesic arcs joining the end points of γ^1 to $t_\varepsilon(q)$. When ε is small, the two geodesic arcs are included in $\{0\} \times \Omega$. Let Σ_ε be the solution of the Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$ with Γ_ε as boundary. Since Γ_ε is a vertical graph above the boundary of the convex domain U , Σ_ε is unique and is a vertical graph above U . Moreover, this graph is smooth up to the boundary (barriers from above and below can be easily found). Σ_ε is included in $\{z \geq 0\}$ and cannot be tangent to $\{z = 0\}$ by the maximum principle. The translate $t_{-a}(\Gamma_\varepsilon)$ for $a > 0$ never meets the graph of u . So by the maximum principle, $t_{-a}(\Sigma_\varepsilon)$ never meets the graph of u and then Σ_ε is inside $\{(s, r, z) \in \mathbb{R} \times \mathbb{R} \times \Omega \mid s \leq u(r, z)\}$. Since Σ_ε is not tangent to $\{z = 0\}$ along γ^1 , Σ_ε is a good barrier from below for u .

We then get a uniform bound for the gradient of u near any points of c^1 (Proposition 17); this gives us the C^2 regularity up to the boundary. \square

Using the regularity of the function u , we prove the following statement.

Claim 25. *We have:*

$$\int_{c^1} \left(\frac{\cosh^2 r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}, \vec{\eta} \right) - \int_{\gamma^1} (\vec{\nu}, \partial_s) > 0$$

and

$$\int_{c_n^3} \left(\frac{\cosh^2 r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}, \vec{\eta} \right) - \int_{\gamma_n^3} (\vec{\nu}, \partial_s) > 0.$$

Proof. The curve γ^1 is the graph of the function f over c^1 and, in fact, close to γ^1 , and A^+ is the graph of f so we have

$$\int_{\gamma^1} (\vec{\nu}, \partial_s) = \int_{c^1} \left(\frac{\cosh^2 r \nabla f}{\sqrt{1 + \cosh^2 r |\nabla f|^2}}, \vec{\eta} \right).$$

Now, since $u \leq f$ and by the maximum principle u and f cannot have the same gradient on c^1 , this implies that along c^1 we have

$$\left(\frac{\cosh^2 r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}, \vec{\eta} \right) > \left(\frac{\cosh^2 r \nabla f}{\sqrt{1 + \cosh^2 r |\nabla f|^2}}, \vec{\eta} \right).$$

This give the first inequality.

For a point p in γ_n^3 , if A is normal to $\{s = 0\}$, then $(\vec{\nu}, \partial_s) = -\cosh r$. Since $\frac{\cosh^2 r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}$ has a norm less than $\cosh r$ everywhere, we get a large inequality at p between the two integrands.

If A is not normal to $\{s = 0\}$ at p , we remark that the term $(\vec{\nu}, \partial_s)$ gives the same result if $\vec{\nu}$ is the conormal to A^+ or \tilde{A}^- . Besides the graph of u is regular up to the boundary and is below A^+ and \tilde{A}^- . So, with $\vec{\nu}_u$ the conormal to the graph of u and $\vec{\nu}_A$ the one for A^+ , the maximum principle implies that $(\vec{\nu}_u, \partial_s) > (\vec{\nu}_A, \partial_s)$

at p . After integration we get the second inequality since there is always a point where A is not normal to $\{s = 0\}$ on γ_n^3 . \square

We have the following limits for the integrals along γ_n^2 and c_n^2 .

Claim 26. *We have the following limits:*

$$\lim_{n \rightarrow \infty} \int_{\gamma_n^2} (\vec{\nu}, \partial_s) = 0$$

and

$$\lim_{n \rightarrow \infty} \int_{c_n^2} \left(\frac{\cosh^2 r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}, \vec{\eta} \right) = 0.$$

Proof. Since $t_{-z_n}(A)$ converges to \mathcal{D}_τ , the first limit is equal to the integral of $(\vec{\nu}, \partial_s)$ along the curve $\mathcal{D}_\tau \cap \{s \geq 0, z = 0\}$. By our choice of (z_n) , the surface \mathcal{D}_τ is symmetric with respect to $\{z = 0\}$. The conormal is then equal to ∂_z and the scalar product vanishes. The limit is then 0.

For the second limit, we know that $u \circ t_{z_n}$ converges to u_0 . This implies that the limit of the integral is equal to

$$\int_{\Omega_0 \cap \{z=0\}} \left(\frac{\cosh^2 r \nabla u_0}{\sqrt{1 + \cosh^2 r |\nabla u_0|^2}}, \partial_z \right).$$

We notice that *a priori* the convergence of $u \circ t_{z_n}$ is smooth only on compact subdomains of Ω_0 but since the integrand is uniformly bounded it is sufficient to take the limit of the integral. Now, we have $u_0(r, z) = u_0(r_0, -z)$ so ∇u_0 is normal to ∂_z on $\Omega_0 \cap \{z = 0\}$ and the limit integral vanishes. \square

Now using equation (10) and Claims 25 and 26, we get our contradiction which finishes the proof of Proposition 14

$$\begin{aligned} 0 &= \lim \left(\int_{\partial\Omega_{z_n}} \left(\frac{\cosh^2 r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}, \vec{\eta} \right) ds - \int_{\partial A_n^+} (\vec{\nu}, \partial_s) \right) \\ &\geq \int_{c^1} \left(\frac{\cosh^2 r \nabla u}{\sqrt{1 + \cosh^2 r |\nabla u|^2}}, \vec{\eta} \right) ds - \int_{\gamma^1} (\vec{\nu}, \partial_s) \\ &> 0. \end{aligned}$$

REFERENCES

[1] Juan A. Aledo, José M. Espinar, and José A. Gálvez, *Height estimates for surfaces with positive constant mean curvature in $\mathbb{M}^2 \times \mathbb{R}$* , Illinois J. Math. **52** (2008), no. 1, 203–211. MR2507241 (2010e:53006)

[2] Sébastien Cartier and Laurent Hauswirth, *Deformations of constant mean curvature-1/2 surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with vertical ends at infinity*, Comm. Anal. Geom. **22** (2014), no. 1, 109–148, DOI 10.4310/CAG.2014.v22.n1.a2. MR3194376

[3] Pascal Collin, *Topologie et courbure des surfaces minimales proprement plongées de \mathbf{R}^3* (French), Ann. of Math. (2) **145** (1997), no. 1, 1–31, DOI 10.2307/2951822. MR1432035 (98d:53010)

[4] P. Collin and R. Krust, *Le problème de Dirichlet pour l'équation des surfaces minimales sur des domaines non bornés* (French, with English summary), Bull. Soc. Math. France **119** (1991), no. 4, 443–462. MR1136846 (92m:53007)

- [5] José M. Espinar, José A. Gálvez, and Harold Rosenberg, *Complete surfaces with positive extrinsic curvature in product spaces*, Comment. Math. Helv. **84** (2009), no. 2, 351–386, DOI 10.4171/CMH/165. MR2495798 (2010c:53086)
- [6] David Hoffman, Jorge H. S. de Lira, and Harold Rosenberg, *Constant mean curvature surfaces in $M^2 \times \mathbf{R}$* , Trans. Amer. Math. Soc. **358** (2006), no. 2, 491–507, DOI 10.1090/S0002-9947-05-04084-5. MR2177028 (2006e:53016)
- [7] Wu-Teh Hsiang and Wu-Yi Hsiang, *On the uniqueness of isoperimetric solutions and imbedded soap bubbles in noncompact symmetric spaces. I*, Invent. Math. **98** (1989), no. 1, 39–58, DOI 10.1007/BF01388843. MR1010154 (90h:53078)
- [8] Nicholas J. Korevaar, Rob Kusner, William H. Meeks III, and Bruce Solomon, *Constant mean curvature surfaces in hyperbolic space*, Amer. J. Math. **114** (1992), no. 1, 1–43, DOI 10.2307/2374738. MR1147718 (92k:53116)
- [9] Nicholas J. Korevaar, Rob Kusner, and Bruce Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. **30** (1989), no. 2, 465–503. MR1010168 (90g:53011)
- [10] R. Kusner, R. Mazzeo, and D. Pollack, *The moduli space of complete embedded constant mean curvature surfaces*, Geom. Funct. Anal. **6** (1996), no. 1, 120–137, DOI 10.1007/BF02246769. MR1371233 (97b:58022)
- [11] Olga A. Ladyzhenskaya and Nina N. Ural'tseva, *Linear and quasilinear elliptic equations*, Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis, Academic Press, New York, 1968. MR0244627 (39 #5941)
- [12] Laurent Mazet, *A general halfspace theorem for constant mean curvature surfaces*, Amer. J. Math. **135** (2013), no. 3, 801–834, DOI 10.1353/ajm.2013.0027. MR3068403
- [13] William H. Meeks III, *The topology and geometry of embedded surfaces of constant mean curvature*, J. Differential Geom. **27** (1988), no. 3, 539–552. MR940118 (89h:53025)
- [14] Barbara Nelli and Harold Rosenberg, *Global properties of constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$* , Pacific J. Math. **226** (2006), no. 1, 137–152, DOI 10.2140/pjm.2006.226.137. MR2247859 (2007i:53010)
- [15] Barbara Nelli and Harold Rosenberg, *Simply connected constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$* , Michigan Math. J. **54** (2006), no. 3, 537–543, DOI 10.1307/mmj/1163789914. MR2280494 (2008f:53007)
- [16] Renato H. L. Pedrosa and Manuel Ritoré, *Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems*, Indiana Univ. Math. J. **48** (1999), no. 4, 1357–1394, DOI 10.1512/iumj.1999.48.1614. MR1757077 (2001k:53120)
- [17] Joaquín Pérez and Antonio Ros, *Properly embedded minimal surfaces with finite total curvature*, The global theory of minimal surfaces in flat spaces (Martina Franca, 1999), Lecture Notes in Math., vol. 1775, Springer, Berlin, 2002, pp. 15–66, DOI 10.1007/978-3-540-45609-4_2. MR1901613
- [18] Harold Rosenberg, *Some recent developments in the theory of minimal surfaces in 3-manifolds*, Publicações Matemáticas do IMPA. [IMPA Mathematical Publications], Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2003. 24° Colóquio Brasileiro de Matemática. [24th Brazilian Mathematics Colloquium]. MR2028922 (2005b:53015)
- [19] J. Serrin, *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Philos. Trans. Roy. Soc. London Ser. A **264** (1969), 413–496. MR0282058 (43 #7772)
- [20] Joel Spruck, *Interior gradient estimates and existence theorems for constant mean curvature graphs in $M^n \times \mathbf{R}$* , Pure Appl. Math. Q. **3** (2007), no. 3, Special Issue: In honor of Leon Simon., 785–800. MR2351645 (2009b:58025)

LABORATOIRE D'ANALYSE ET MATHÉMATIQUES APPLIQUÉES, UNIVERSITÉ PARIS-EST, CNRS UMR8050, UFR DES SCIENCES ET TECHNOLOGIE, BÂTIMENT P3 4EME ÉTAGE, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 CRÉTEIL CEDEX, FRANCE

E-mail address: laurent.mazet@math.cnrs.fr