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A GEOMETRIC SETTING FOR QUANTUM $\mathfrak{osp}(1|2)$

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ABSTRACT. A geometric categorification is given for arbitrary-large-finite-dimensional quotients of quantum $\mathfrak{osp}(1|2)$ and tensor products of its simple modules. The modified quantum $\mathfrak{osp}(1|2)$ of Clark-Wang, a new version in this paper and the modified quantum $\mathfrak{sl}(2)$ are shown to be isomorphic to each other over a field containing $\mathbb{Q}(v)$ and $\sqrt{-1}$.

CONTENTS

1. INTRODUCTION

1.1. In the classical work [L90, L91, L93] of Lusztig, he gives a geometric construction of the negative half of the quantum algebra associated to a Kac-Moody Lie algebra. It was later shown by Vasserot and Varagnolo in [VV11] that the extension algebra of Lusztig's complexes is isomorphic to the KLR algebra, a.k.a. quiver Hecke algebra, of symmetric type introduced independently by Khovanov-Lauda and Rouquier in [KL09] and [R08]. The KLR algebras admit an odd/super analogue, the so-called quiver Hecke superalgebras by Kang-Kashiwara-Tsuchioka [KKT11] (see also [EKL14] and [W09]). By using representation theory of quiver Hecke superalgebras, Hill and Wang [HW15] give a categorification of the negative half of a covering algebra involving two parameters (q, π), which specializes to the negative half of a quantum algebra at $\pi = 1$ and that of a quantum superalgebra at $\pi = -1$. See also [EKL14, EL13, KKO13, KKO14] for further progress in this active research direction.

To this end, it is natural to ask if one can categorify the negative part of Hill-Wang's covering algebra and, moreover, the covering algebra itself (or its modified form) by using representation theory of KLR alegbras, or equivalently from

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Lusztig's geometric setting. This question was first raised by Weiqiang Wang and answered affirmatively for the negative half of the covering algebra by the authors in [FL14] and [CFLW14] together with Clark and Wang. A new idea in answering this question is that the Tate twist (mod 4) categorifies the square root of the second parameter π in the covering algebra.

After the negative half of the covering algebra is categorified in Lusztig's geometric setting for negatives halves of quantum algebras, we are forced intuitively to search for a categorification of the covering algebra in a geometric setting analogous to the one for the negative halves. Indeed, there is such a setting for quantum $\mathfrak{osp}(1|2)$, one of the smallest quantum superalgebras. It is in Beilinson-Lusztig-MacPherson's geometric construction [BLM90] of the q-Schur algebra, a quotient of a quantum algebra of type A.

As one of the main results in this paper, we give a geometric construction of an arbitrary large finite-dimensional quotient of quantum $\mathfrak{osp}(1|2)$, as well as tensor products of its highest weight modules. We follow the approach taken in our previous work by adding the Tate twist, or the mixed structure, to the geometric setting laid out in [BLM90] involving the geometry of two copies of Grassmannians. With a slight, though non-trivial, modification of the generators for the geometric construction of quantum $\mathfrak{sl}(2)$ in [BLM90], we obtain a quotient of quantum $\mathfrak{osp}(1|2)$. In this realization, the Tate twist corresponds to the imaginary unit $t = \sqrt{-1}$. We tend to call this quotient the q-Schur algebra of quantum $\mathfrak{osp}(1|2)$ because it gets identified with that of quantum $\mathfrak{sl}(2)$ immediately from the construction. Along the way, we also obtain a geometric construction of tensor products of finite-dimensional simple modules of quantum $\mathfrak{osp}(1|2)$ following [Z07] (see also [GL92]) using perverse sheaves on Grassmannians.

Just like the authors' previous work [FL14], the simple perverse sheaves of weight zero arising from this construction form a basis for the categorified quotients and tensor product modules. The structure constants with respect to this basis possess again a positivity property in an appropriate sense (see Theorem 4.12). We provide with an algebraic characterization, up to a sign, of the basis by using a bilinear form, integrality and bar invariant properties.

In the last section, we formulate a new version of modified quantum $\mathfrak{osp}(1|2)$ following [BLM90] and [L93]. As far as we can tell, this is the most natural definition from our presentation of quantum $\mathfrak{osp}(1|2)$ and its geometric construction. We further observe that our modified quantum $\mathfrak{osp}(1|2)$ is isomorphic to that of Clark-Wang in [CW13], and, surprisingly, to Lusztig's modified quantum $\mathfrak{sl}(2)$ over a field containing $\mathbb{Q}(v)$ and $t = \sqrt{-1}$. We arrive at the latter isomorphism by observing the facts that quantum $\mathfrak{osp}(1|2)$ and $\mathfrak{sl}(2)$ have the same q-Schur algebras from the geometric construction and that modified versions of quantum algebras sit inside the limit of a projective system of q-Schur algebras. The proof turns out to be extremely easy. A first consequence of the isomorphism of modified quantum $\mathfrak{osp}(1|2)$ and $\mathfrak{sl}(2)$ is that there exists a basis in the modified quantum $\mathfrak{osp}(1|2)$, coming from the canonical basis of quantum $\mathfrak{sl}(2)$, whose structure constants are in $\mathbb{N}[v, v^{-1}]$. Such a positivity property in quantum $\mathfrak{osp}(1|2)$ is rather mysterious, given the fact that the super sign "-1" is essentially used in the definition of the quantum $\mathfrak{osp}(1|2)$. In other words, in modified quantum $\mathfrak{osp}(1|2)$, the super sign "-1" (or t^2 for the modified covering algebra) can be moved outside the structure. A second consequence of the isomorphism is that the categories of weight modules of quantum

$\mathfrak{osp}(1|2)$ and $\mathfrak{sl}(2)$ are isomorphic to each other. Moreover, we are able to construct very explicit and simple functors of isomorphism between the two categories of weight modules. A third consequence is that Lauda's categorification ([Lau10]) of quantum $\mathfrak{sl}(2)$ can serve as a version of categorifications of modified quantum $\mathfrak{osp}(1|2)$.

We remark that the results obtained in this paper can be rephrased in the setting of the covering algebras (or their modified versions). We stick to quantum $\mathfrak{osp}(1|2)$ for simplicity.

The coincidence of modified quantum $\mathfrak{sl}(2)$ and $\mathfrak{osp}(1|2)$ is somehow predicted by various results in literature and, in turn, explains why the representation theories of the two algebras are identical. In [CFLW14], we will show that modified quantum algebras and superalgebras (or covering algebras) are isomorphic in general cases.

Meanwhile, Weiqiang Wang informed us that the equivalence of categories of weight modules has been known to him for more than a year by using the work [Lan02]. This equivalence is also proved independently by Kang-Kashiwara-Oh ([KKO14]) in a completely different way and a more general setting.

2. Preliminaries

2.1. Let v be an indeterminate and t the imaginary unit such that $t^2 = -1$. For any $k \leq n \in \mathbb{N}$, we set

$$[n]_v = \tfrac{v^n - v^{-n}}{v - v^{-1}}, \qquad [n]^!_v = \prod_{k=1}^n [k]_v, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_v = \tfrac{[n]^!_v}{[k]^!_v [n-k]^!_v},$$

$$\begin{bmatrix} n \end{bmatrix}_{v,t} = \tfrac{(vt)^n - (vt^{-1})^{-n}}{vt - (vt^{-1})^{-1}}, \quad [n]^!_{v,t} = \prod_{k=1}^n [k]_{v,t}, \quad \begin{bmatrix} n \\ k \end{bmatrix}_{v,t} = \tfrac{[n]^!_{v,t}}{[k]^!_{v,t} [n-k]^!_{v,t}}.$$

One can easily check that

$$[n]_{v,t} = t^{n-1}[n]_v, \qquad [n]^!_{v,t} = t^{\frac{n(n-1)}{2}} [n]^!_v, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{v,t} = t^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_v.$$

The quantum algebra $\mathbf{U}$ associated to the ortho-symplectic Lie algebra $\mathfrak{osp}(1|2)$ is, by definition, an associative $\mathbb{Q}[t](v)$-algebra with 1 generated by the symbols E, F, K and K^{-1}, subject to the following defining relations:

$$KK^{-1} = 1 = K^{-1}K, \tag{S1}$$

$$KE = v^2 t^{-2} EK, \quad KF = v^{-2} t^2 FK, \tag{S2}$$

$$EF - t^2 FE = \frac{K - K^{-1}}{v - v^{-1}}. \tag{S3}$$

The above presentation of the algebra $\mathbf{U}$ is new. Note that the algebra $\mathbf{U}$ is isomorphic to the algebra U_1 in [CW13, 2.3] if the ground field is extended to $\mathbb{Q}[t](v)$. For the reader's convenience, we provide an isomorphism defined by the following correspondence.

$\mathbf{U}$	E	F	$t^{-1}K$	vt^{-1}	t^2	$[n]_{v,t}$
U_1	E	F	K	q	π	$[n]^{\pi}_{(vt^{-1})}$

The algebra $\mathbf{U}$ admits a superalgebra structure by setting the parity function p to be $p(E) = p(F) = 1$ and $p(K) = p(K^{-1}) = 0$. By convention, the multiplication on $\mathbf{U} \otimes \mathbf{U}$ is defined by

$$(x \otimes y)(x' \otimes y') = t^{2p(y)p(x')} xx' \otimes yy',$$

where x, y, x' and y' are homogeneous elements in $\mathbf{U}$. This gives a superalgebra structure on $\mathbf{U} \otimes \mathbf{U}$. Moreover, a straightforward calculation yields the following proposition.

Proposition 2.2. *There is a unique superalgebra homomorphism $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ defined by*

$$\Delta(K) = K \otimes K, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = 1 \otimes F + F \otimes K^{-1}.$$

A standard induction gives rise to the following lemma.

Lemma 2.3. *Let $F^{(n)} = \frac{F^n}{[n]^!_{v,t}}$, for any $n \in \mathbb{N}$. We have*

$$EF^{(n)} = t^{2n} F^{(n)} E + t^{n-1} F^{(n-1)} \frac{v^{1-n}K - v^{n-1}K^{-1}}{v - v^{-1}}.$$

By applying Lemma 2.3, we have the following proposition.

Proposition 2.4. *For any $d \in \mathbb{N}$, there exist only two non-isomorphic $(d+1)$-dimensional simple highest weight $\mathbf{U}$-modules $\Lambda_d^{\pm}$. More precisely, the modules $\Lambda_d^{\pm}$ are spanned by vectors $\xi_0, \xi_1, \cdots, \xi_d$, as vector spaces, and the action of $\mathbf{U}$ on Λ_d^+ is given by*

(1) $F \cdot \xi_r = t^r [r+1]_v \xi_{r+1}, \quad E \cdot \xi_r = t^{r-1}[d+1-r]_v \xi_{r-1}, \quad K \cdot \xi_r = t^{2r} v^{d-2r} \xi_r,$

while the action on Λ_d^- is given by

(2) $F \cdot \xi_r = t^r [r+1]_v \xi_{r+1}, \quad E \cdot \xi_r = -t^{r-1}[d+1-r]_v \xi_{r-1}, \quad K \cdot \xi_r = -t^{2r} v^{d-2r} \xi_r.$

Remark 2.5. By rewriting the last identity in (1) into $K \cdot \xi_r = t^d q^{d-2r} \xi_r$, Proposition 2.4 is compatible with the classification of simple modules of $U_1 \otimes_{\mathbb{Q}[t](v)} \mathbb{C}(v)$ in [CW13, Proposition 3.2] and [Zou98, Theorem 3.1].

In Sections 3 and 4, we will give a geometric categorification of Λ_d^+. The module Λ_d^- can be categorified similarly. *From now on, we write Λ_d for Λ_d^+ for simplicity.* The module Λ_d becomes a $\mathbb{Z}_2$-graded $\mathbf{U}$-module by setting the parity function p to be $p(\xi_r) = 1$ if r is odd and $p(\xi_r) = 0$ otherwise. For any $\mathbf{U}$-modules M and N, the $\mathbf{U} \otimes \mathbf{U}$-module structure on $M \otimes N$ is defined by

$$(a \otimes b) \cdot (m \otimes n) = (-1)^{p(b)p(m)} am \otimes bn,$$

for any homogenous element $b \in \mathbf{U}$ and $m \in M$. Moreover, the $\mathbf{U}$-module structure on $M \otimes N$ is defined by

(3) $$a \cdot (m \otimes n) = \Delta(a)(m \otimes n), \quad \forall a \in \mathbf{U}, m \otimes n \in M \otimes N.$$

For any $\mathbf{d} = (d_1, d_2, \cdots, d_m) \in \mathbb{N}^m$, let

$$\Lambda_{\mathbf{d}} = \Lambda_{d_1} \otimes \Lambda_{d_2} \otimes \cdots \otimes \Lambda_{d_m}. \tag{4}$$

Let

$$\mathcal{A} = \mathbb{Z}[v^{\pm 1}, t]$$

denote the subring in $\mathbb{Q}(v)[t]$ of Laurent polynomials. We denote by ${}_{\mathcal{A}}\mathbf{U}$ the $\mathcal{A}$-subalgebra of $\mathbf{U}$ generated by $E^{(n)} := \frac{E^n}{[n]^!_{v,t}}$, $F^{(n)}$ and $K^{\pm 1}$ for $n \in \mathbb{N}$. It is clear that the comultiplication Δ induces a comultiplication on ${}_{\mathcal{A}}\mathbf{U}$. By combining this with (1), we can define the integral form ${}_{\mathcal{A}}\Lambda_{\mathbf{d}}$ of the module $\Lambda_{\mathbf{d}}$.

2.6. In this section, we recall notation and facts in the theory of mixed perverse sheaves. We refer to [L93, Chapter 8] and [BBD82] for more details.

Let k be an algebraically closed field of positive characteristic. Let l be a fixed prime number invertible in k, and $\bar{\mathbb{Q}}_l$ be the algebraic closure of the field of l-adic numbers. Denote by $\mathcal{D}(X) = \mathcal{D}^b_c(X)$ the bounded derived category of $\bar{\mathbb{Q}}_l$-constructible sheaves on the algebraic variety X over k. Let $\mathcal{D}_m(X)$ be the full subcategory of $\mathcal{D}(X)$ consisting of all mixed complexes. If two complexes K and L in $\mathcal{D}(X)$ are isomorphic, we write $K = L$.

Let $\mathrm{wt}(L)$ denote the weight of a pure complex L. Let $[-]$, $(-)$, $\mathbb{D}$ and $\otimes$ denote the shift functor, Tate twist functor, Verdier duality functor and tensor product functor, respectively. To a morphism $f : X \to Y$ of algebraic varieties over k, we can associate four of Grothendieck's six operations $f^*, f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$ and $f_*, f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$. We recall some facts which we will use freely later:

(1) Simple perverse sheaves are pure.
(2) Functors $(-)$ and $[-]$ commute with each other and with all functors $f^*, f_!, f^!, f_*$.
(3) $\mathbb{D}f_! = f_*\mathbb{D}$, $\quad \mathbb{D}f^! = f^*\mathbb{D}$, $\quad \mathbb{D}(L[n](m)) = (\mathbb{D}L)[-n](-m)$.
(4) $\mathrm{wt}(L[n]) = \mathrm{wt}(L) + n$, $\mathrm{wt}(L(n)) = \mathrm{wt}(L) - 2n$, $\mathrm{wt}(\mathbb{D}L) = -\mathrm{wt}(L)$ for any pure complex L.
(5) If $f : X \to Y$ is smooth with connected fibers of equal dimension, then $\mathrm{wt}(f^*L) = \mathrm{wt}(L)$ for any pure complex L.
(6) If $f : X \to Y$ is a proper morphism, then $\mathrm{wt}(f_!L) = \mathrm{wt}(L)$ for any pure complex L.
(7) $f_!(L \otimes f^*M) = f_!L \otimes M$, $\quad f^*(L \otimes M) = f^*L \otimes f^*M$.
(8) If the following square is cartesian and f is a proper map

$$\begin{tikzcd} Z \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ X' \arrow[r, "g"] & Y, \end{tikzcd}$$

then we have $g^* f_! L = f'_! g'^* L$ for any complex $L \in \mathcal{D}(X)$.

2.7. Suppose that X_1, X_2 and X_3 are three algebraic varieties over k. Let $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ be the projection to the (i,j)-factor, for $(i,j) = (1,2), (2,3), (1,3)$. For any $L \in \mathcal{D}(X_1 \times X_2)$ and $M \in \mathcal{D}(X_2 \times X_3)$, we set

$$L \circ M = (p_{13})_!(p_{12}^* L \otimes p_{23}^* M) \in \mathcal{D}(X_1 \times X_3). \tag{5}$$

Lemma 2.8. *Assume the above setup; we have $(L \circ M) \circ N = L \circ (M \circ N)$, where N is any complex in $\mathcal{D}(X_3 \times X_4)$ and X_4 is a fourth variety over k.*

The proof is standard and left to the reader.

3. A geometric categorification of $\mathbf{U}$

3.1. We fix a positive integer d and we write $\mathcal{F}_r$ for the Grassmannian of all dimension r subspaces in k^d. It is clear that $\mathcal{F}_r$ is empty unless r subjects to $0 \leq r \leq d$. The group $G = GL(k^d)$ naturally acts on $\mathcal{F}_r$ from the left. Let G act on $\mathcal{F}_r \times \mathcal{F}_{r'}$ diagonally. By [BLM90, Section 1], the G-orbits in $\mathcal{F}_r \times \mathcal{F}_{r'}$ are parametrized by the set $\Theta_d(r, r')$ of all 2×2 matrices (a_{ij}) such that $a_{11}+a_{12} = r$, $a_{11}+a_{21} = r'$ and $\sum_{i,j=1,2} a_{ij} = d$. In fact, a bijection is defined by sending a pair $(V, V') \in \mathcal{F}_r \times \mathcal{F}_{r'}$ to the following matrix:

$$\begin{bmatrix} |V \cap V'| & |V/V \cap V'| \\ |V + V'/V| & d - |V + V'| \end{bmatrix}, \quad \text{where } |V| = \dim V.$$

We set

$$\Theta_d = \bigsqcup_{r,r'} \Theta_d(r, r').$$

We define the following closed subvarieties in $\mathcal{F}_r \times \mathcal{F}_{r'}$ for appropriate r and r':

$$\begin{aligned} \mathcal{F}_{r,r+a} &= \{(V, V') \in \mathcal{F}_r \times \mathcal{F}_{r+a} | V \subset V'\}, \quad \forall 0 \leq r, a, r+a \leq d, \\ \mathcal{F}_{r,r-a} &= \{(V, V') \in \mathcal{F}_r \times \mathcal{F}_{r-a} | V \supset V'\}, \quad \forall 0 \leq r, a, r-a \leq d. \end{aligned} \tag{6}$$

We denote

$$\begin{aligned} E_{r,r+a} &= (\bar{\mathbb{Q}}_l)_{\mathcal{F}_{r,r+a}}[a(d-(r+a))]\left(\frac{a(d-a)}{2}\right) \in \mathcal{D}(\mathcal{F}_r \times \mathcal{F}_{r+a}), \\ F_{r,r-a} &= (\bar{\mathbb{Q}}_l)_{\mathcal{F}_{r,r-a}}[a(r-a)](a(r-a)) \in \mathcal{D}(\mathcal{F}_r \times \mathcal{F}_{r-a}), \\ 1_r &= (\bar{\mathbb{Q}}_l)_{\mathcal{F}_{r,r}} \in \mathcal{D}(\mathcal{F}_r \times \mathcal{F}_r), \end{aligned}$$

where $a > 0$ and $(\bar{\mathbb{Q}}_l)_{X_1} \in \mathcal{D}(X)$ denotes the extension by zero of the constant sheaf $(\bar{\mathbb{Q}}_l)_{X_1}$ in $\mathcal{D}(X_1)$ for a given subvariety X_1 in X. If the variety $\mathcal{F}_{r,r'}$ is empty, the associated complex is defined to be zero.

Lemma 3.2. (*a*) $E_{r,r+a}E_{r+a,r+a+1} = \bigoplus_{j=0}^{a} E_{r,r+a+1}[a-2j](a-j)$.
(*b*) $F_{r,r-a}F_{r-a,r-a-1} = \bigoplus_{j=0}^{a} F_{r,r-a-1}[a-2j](a-j)$.

Proof. The support of the complex $p_{12}^* E_{r,r+a} \otimes p_{23}^* E_{r+a,r+a+1}$ is

$$S = \{(V, V', V'') \in \mathcal{F}_r \times \mathcal{F}_{r+a} \times \mathcal{F}_{r+a+1} | V \subset V' \subset V''\}.$$

The restriction of p_{13} to S is a $\mathbb{P}^a$-bundle. So we have

$$\begin{aligned} &E_{r,r+a}E_{r+a,r+a+1} = (p_{13})_!(p_{12}^* E_{r,r+a} \otimes p_{23}^* E_{r+a,r+a+1}) \\ &= (p_{13})_!(\bar{\mathbb{Q}}_l)_S[a(d-r-a)+d-(r+a)-1]\left(\frac{a(d-a)}{2}+\frac{d-1}{2}\right) \\ &= \bigoplus_{j=0}^{a}(\bar{\mathbb{Q}}_l)_{\mathcal{F}_{r,r+a+1}}[-2j](-j)[(a+1)(d-(r+a))-1]\left(\frac{(a+1)(d-1-a)}{2}+a\right) \\ &= \bigoplus_{j=0}^{a} E_{r,r+a+1}[a-2j](a-j). \end{aligned}$$

Similarly, the support of $p_{12}^*F_{r,r-a}\otimes p_{23}^*F_{r-a,r-a-1}$ is

$$S=\{(V,V',V'')\in\mathcal{F}_r\times\mathcal{F}_{r-a}\times\mathcal{F}_{r-a-1}|V\supset V'\supset V''\}.$$

The restriction of p_{13} to S is again a $\mathbb{P}^a$-bundle. By a similar argument as above, we have the second identity. $\square$

Lemma 3.3. *(a)* $1_r1_{r'}=\delta_{r,r'}1_r$.

$$\begin{aligned}
&(b)\quad E_{r,r+1}1_{r'}=\delta_{r+1,r'}E_{r,r+1}, \qquad 1_{r'}E_{r,r+1}=\delta_{r',r}E_{r,r+1}.\\
&(c)\quad F_{r,r-1}1_{r'}=\delta_{r-1,r'}F_{r,r-1}, \qquad 1_{r'}F_{r,r-1}=\delta_{r',r}F_{r,r-1}.\\
&(d)\quad E_{r,r+1}F_{r+1,r}\oplus\textstyle\bigoplus_{0\le j<2r-d}1_r[2r-2j-1-d]\left(2r-j-\frac{d+1}{2}\right)\\
&\qquad =F_{r,r-1}E_{r-1,r}(1)\oplus\textstyle\bigoplus_{0\le j<d-2r}1_r[d-2j-1-2r]\left(\frac{d-1}{2}-j\right).
\end{aligned}$$

Proof. Let us show that $1_r1_{r'}=\delta_{r,r'}1_r$. Assume that $r=r'$. As before, let $p_{ij}:\mathcal{F}_r\times\mathcal{F}_r\times\mathcal{F}_r\to\mathcal{F}_r\times\mathcal{F}_r$ be the projection to the (i,j)-factor. By the definition of 1_r, the support of $p_{12}^*1_r\otimes p_{23}^*1_r$ is the variety $S=\{(V,V',V'')\in\mathcal{F}_r\times\mathcal{F}_r\times\mathcal{F}_r|V=V'=V''\}$, i.e., the diagonal of the variety $\mathcal{F}_r\times\mathcal{F}_r\times\mathcal{F}_r$. So the image of S under p_{ij} is exactly $\mathcal{F}_{r,r}$ for $(i,j)=(1,2),(2,3),(1,3)$. Moreover, the restriction of p_{ij} to S is an isomorphism. Thus the restriction of 1_r1_r to $\mathcal{F}_{r,r}$ is the constant sheaf. Therefore, we have $1_r1_r=1_r$. If $r\neq r'$, then the support of $p_{12}^*1_r\otimes p_{23}^*1_r$ is empty. So we have $1_r1_{r'}=0$.

Next, let us show that $E_{r,r+1}1_{r'}=\delta_{r+1,r'}E_{r,r+1}$. Assume that $r'=r+1$. The support of $p_{12}^*E_{r,r+1}\otimes p_{23}^*1_r$ is $S=\{(V,V',V'')\in\mathcal{F}_r\times\mathcal{F}_{r+1}\times\mathcal{F}_{r+1}|V\subset V',V'=V''\}$. By definition, the restriction of $p_{12}^*E_{r,r+1}\otimes p_{23}^*1_r$ to S is $(\bar{\mathbb{Q}}_l)_S[d-(r+1)](\frac{d-1}{2})$. Note that the restriction of p_{13} to S is again an isomorphism, and the image of p_{13} is $\mathcal{F}_{r,r+1}$. Therefore we have $E_{r,r+1}1_{r+1}=E_{r,r+1}$. For the case of $r'\neq r$, the identity holds by definitions. One may show similar identities in the lemma in a comparable way.

Finally, let us show the last identity in the lemma. Let us compute the complex $E_{r,r+1}F_{r+1,r}$. The support of $p_{12}^*E_{r,r+1}\otimes p_{23}^*F_{r+1,r}$ is

$$S=\{(V,V',V'')\in\mathcal{F}_r\times\mathcal{F}_{r+1}\times\mathcal{F}_r|V\subset V'\supset V''\}.$$

Let $S_1=\{(V,V',V'')\in S|V=V''\}$ and $S_2=S\backslash S_1$. Then

$$S_2\simeq S_2'\overset{\text{def.}}{=}\{(V,V'')\in\mathcal{F}_r\times\mathcal{F}_r||(V+V'')/V|=1=|(V+V'')/V''|\}.$$

Observe that the restriction p_{13}' of p_{13} to S_1 is a fiber bundle of fiber isomorphic to the projective space $\mathbb{P}^{d-r-1}$, while the restriction p_{13}'' of p_{13} to S_2 is an isomorphism. Further, the image of p_{13}' is $\mathcal{F}_{r,r}$, and the image of p_{13}'' is S_2'. The restriction of $p_{12}^*E_{r,r+1}\otimes p_{23}^*F_{r+1,r}$ to S is

$$(\bar{\mathbb{Q}}_l)_S[d-(r+1)](\frac{d-1}{2})\otimes(\bar{\mathbb{Q}}_l)_S[r](r)=(\bar{\mathbb{Q}}_l)_S[d-1]\left(\frac{d-1}{2}+r\right).$$

So

$$\begin{aligned} E_{r,r+1}F_{r+1,r} &= (p_{13})_!(p_{12}^*E_{r,r+1}\otimes p_{23}^*F_{r+1,r}) = (p_{13})_!(\bar{\mathbb{Q}}_l)_S[d-1]\left(\frac{d-1}{2}+r\right) \\ &= (p'_{13})_!(\bar{\mathbb{Q}}_l)_{S_1}[d-1](\frac{d-1}{2}+r)\oplus(p''_{13})_!(\bar{\mathbb{Q}}_l)_{S_2}[d-1]\left(\frac{d-1}{2}+r\right) \\ &= (\bigoplus_{j=0}^{d-r-1}(\bar{\mathbb{Q}}_l)_{\mathcal{F}_{r,r}}[-2j](-j)\oplus(\bar{\mathbb{Q}}_l)_{S_2'})[d-1]\left(\frac{d-1}{2}+r\right), \end{aligned}$$

where the third equation is due to [L93, 8.1.6].

Similarly, we compute $F_{r,r-1}E_{r-1,r}$ and get

$$F_{r,r-1}E_{r-1,r}(1) = (\bigoplus_{j=0}^{r-1}(\bar{\mathbb{Q}}_l)_{\mathcal{F}_{r,r}}[-2j](-j)\oplus(\bar{\mathbb{Q}}_l)_{S_2''})[d-1]\left(\frac{d-1}{2}+r\right),$$

where $S_2''=\{(V,V'')\in\mathcal{F}_r\times\mathcal{F}_r \| V/(V\cap V'')|=1=|V''/(V\cap V'')|\}$. Observe that $S_2'=S_2''$ and

$$(\bigoplus_{j=0}^{d-r-1}1_r\oplus\bigoplus_{d-r\le j<r}1_r)[-2j](-j) = (\bigoplus_{j=0}^{r-1}1_r\oplus\bigoplus_{r\le j<d-r}1_r)[-2j](-j).$$

We have the last identity in the lemma. The lemma follows. $\square$

For any $A=(a_{ij})_{1\le i,j\le 2}$, we set

$$\{A\} = IC(O_A)[-r(A)](-r(A)/2),\quad \forall A\in\Theta_d,$$

where O_A is the corresponding G-orbit of A, $IC(O_A)$ is the intersection cohomology complex attached to the closure of O_A ([BBD82]), and $r(A)=(a_{11}+a_{12})(a_{21}+a_{22})$ is the dimension of the image of O_A under the first projection.

Lemma 3.4. *(a)* $E_{r-a,r}1_rF_{r,r-b}(n_1)=\left\{\begin{matrix} r-a-b & b \\ a & d-r\end{matrix}\right\}$, $\forall d\le(r-a)+(r-b)$.

(b) $F_{r+b,r}1_rE_{r,r+a}(n_2)=\left\{\begin{matrix} r & b \\ a & d-r-a-b\end{matrix}\right\}$, $\forall d\ge(r+a)+(r+b)$.

(c) $E_{r-a,r}1_rF_{r,r-b}=F_{d-r+b,d-r}1_{d-r}E_{d-r,d-r+a}(ab)$, *if* $d=(r-a)+(r-b)$,

where $n_1=-\frac{1}{2}(a(r-a)+b(r-b))$ *and* $n_2=-\frac{1}{2}(ar+br)$.

Proof. We prove the first equation. The support of the complex $p_{12}^*E_{r-a,r}\otimes p_{23}^*F_{r,r-b}$ is

$$S=\{(V,V'',V')\in\mathcal{F}_{r-a}\times\mathcal{F}_r\times\mathcal{F}_{r-b}|V\subset V''\supset V'\}.$$

By definition, we have

$$E_{r-a,r}F_{r,r-b}=(p_{13})_!(\bar{\mathbb{Q}}_l)_S[a(d-r)+b(r-b)]\left(\frac{a(d-a)}{2}+b(r-b)\right). \tag{7}$$

Consider the restriction of p_{13} to S. The image of S under p_{13} consists of the pairs $(V,V')\in\mathcal{F}_{r-a}\times\mathcal{F}_{r-b}$ such that $|V+V'|\le r$. Thus we have

$$r-a-b\le|V\cap V'|\le\min\{r-a,r-b\}.$$

Recall from [BLM90, 2.3] that

$$\begin{aligned} d(A) - r(A) &= a_{11}a_{12} + a_{21}a_{12} + a_{21}a_{22} \\ &= (r-a-b-|V\cap V'|)(d-r+b+|V\cap V'|) + b|V\cap V'| + a(d-r+b). \end{aligned} \tag{8}$$

In particular,

$$d(A) - r(A) = a(d-r) + b(r-b), \quad \text{if} \quad |V\cap V'| = r-a-b. \tag{9}$$

From (8), we see that $p_{13}(S)$ is the orbit closure of the G-orbit whose associated matrix is

$$A_0 = \begin{bmatrix} r-a-b & b \\ a & d-r \end{bmatrix}.$$

We claim that

$$\text{the restriction of } p_{13} \text{ to } S \text{ is a small resolution.} \tag{10}$$

Recall that smallness means that the following two conditions are satisfied:

(a) $2|p_{13}^{-1}(x)| \leq d(A_0) - d(A)$, for any $x \in O_A \subseteq O_{A_0}$.

(b) The equality holds if and only if $A = A_0$.

We show that p_{13} satisfies (a). Given any pair (V, V') in $p_{13}(S)$, the dimension of the fiber $p_{13}^{-1}(V, V')$ is

$$|p_{13}^{-1}(V,V')| = |Gr(d-|V+V'|, r-|V+V'|)| = (r-|V+V'|)(d-r). \tag{11}$$

Since $r(A_0) = r(A)$, we have

$$2|p_{13}^{-1}(x)| - (d(A_0) - d(A)) = 2|p_{13}^{-1}(x)| - (d(A_0) - r(A_0)) - (d(A) - r(A)).$$

By (8) and (11), we have

$$2|p_{13}^{-1}(x)| - (d(A_0) - d(A)) = (r-a-b-|V\cap V'|)(-d+r+|V\cap V'|) \leq 0. \tag{12}$$

This shows (a). The inequality (12) becomes an equality if and only if $|V\cap V'| = r-a-b$. So (b) holds for p_{13}. It is clear that the restriction of p_{13} to $p_{13}^{-1}(O_{A_0})$ is an isomorphism. The claim follows.

By (7) , (9) and (10), we have $E_{r-a,r}F_{r,r-b} = \{A_0\}$ up to a Tate twist. Since $\mathrm{wt}(\{A_0\}) = 0$, we have $n_1 = -\frac{1}{2}(a(r-a) + b(r-b))$ by checking the weight of $E_{r-a,r}F_{r,r-b}$. The first equation in the lemma follows. The second equation can be shown similarly. The third one follows from the first two equations. □

3.5. Consider the complex

$$L_1 \circ L_2 \circ \cdots \circ L_m, \quad m \in \mathbb{N}, \tag{13}$$

where the L_i's are either $E_{r,r+1}$, $F_{r,r-1}$, or 1_r. Assume that $L_i \in \mathcal{D}(\mathcal{F}_{r_i} \times \mathcal{F}_{r_{i+1}})$ for $i = 1, \cdots, m$. Let $s_{ij} : \prod_{k=1}^{m+1} \mathcal{F}_{r_k} \to \mathcal{F}_{r_i} \times \mathcal{F}_{r_j}$ be the projection to the (i,j)-factor. By applying (7) and (8), we get

$$L_1 \circ L_2 \circ \cdots \circ L_m = (s_{1,m+1})_!(\bigotimes_{i=1}^{m} s_{i,i+1}^*(L_i)).$$

Observe that $s_{1,m+1}$ is proper, and the restriction of the complex $\bigotimes_{i=1}^m s_{i,i+1}^*(L_i)$ to its support, which is smooth and irreducible, is a constant sheaf with a shift and a Tate twist. By the decomposition theorem ([BBD82]), we see that the complex (13) is semisimple.

3.6. Let $\mathcal{Q}_d^{r,r'}$ be the full subcategory of $\mathcal{D}(\mathcal{F}_r \times \mathcal{F}_{r'})$ consisting of semisimple complexes whose simple constitutes are direct summands of the complex (13) up to shifts and twists.

Let $\mathbf{Q}_d^{r,r'}$ be the split Grothendieck group of $\mathcal{Q}_d^{r,r'}$. More precisely, $\mathbf{Q}_d^{r,r'}$ is the abelian group generated by the isomorphism classes of objects in $\mathcal{Q}_d^{r,r'}$ and subject to the following relation:

$$\langle C \oplus C' \rangle = \langle C \rangle + \langle C' \rangle, \quad \forall C, C' \in \mathcal{Q}_d^{r,r'}. \tag{14}$$

Let $\mathbf{Q}_d = \bigoplus_{r,r' \in \mathbb{Z}_{\geq 0}} \mathbf{Q}_d^{r,r'}$ and $\tilde{\mathcal{A}} = \mathbb{Z}[v^{\pm 1}, \tau^{\pm 1}]$, where τ is an indeterminate. We define an $\tilde{\mathcal{A}}$-module structure on $\mathbf{Q}_d$ as follows:

$$v \cdot \langle C \rangle = \langle C[1](\frac{1}{2}) \rangle, \quad \tau \cdot \langle C \rangle = \langle C(\frac{1}{2}) \rangle, \quad \forall \langle C \rangle \in \mathbf{Q}_d. \tag{15}$$

By the property of the shift and Tate twist functors, this action is well-defined. Recall that

$$\mathcal{A} = \mathbb{Z}[v^{\pm 1}, t].$$

There is an obvious ring homomorphism $\tilde{\mathcal{A}} \to \mathcal{A}$ by sending τ to t. Let

$${}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d) = \mathcal{A} \otimes_{\tilde{\mathcal{A}}} \mathbf{Q}_d.$$

By the τ-action in (15), we have

$$1 \otimes \langle C(2) \rangle = 1 \otimes \tau^4 \langle C \rangle = t^4 \otimes \langle C \rangle = 1 \otimes \langle C \rangle, \quad \forall \langle C \rangle \in \mathbf{Q}_d. \tag{16}$$

Let

$$\mathbf{S}_{v,t}(2,d) = \mathbb{Q}[t](v) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d). \tag{17}$$

By (14), (16) and Lemma 3.4, the convolution product "$\circ$" in (5) descends to a bilinear map on ${}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d)$:

$$\circ : {}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d) \times {}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d) \to {}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d).$$

It is associative due to Lemma 2.8. Together with "$\circ$", the space ${}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d)$ becomes an associative algebra over $\mathcal{A}$. By an abuse of notation, we write C instead of $1 \otimes \langle C \rangle$ for elements in ${}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d)$.

Lemma 3.7. *The following identities hold in* ${}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d)$*:*

(*a*) $E_{r,r+a} E_{r+a,r+a+1} = [a+1]_{v,t} E_{r,r+a+1}$.
(*b*) $F_{r,r-a} F_{r-a,r-a-1} = [a+1]_{v,t} F_{r,r-a-1}$.

Proof. By Lemma 3.2 and $\mathcal{A}$-action on ${}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d)$ defined above, we have

$$E_{r,r+a} \circ E_{r+a,r+a+1} = \sum_{j=0}^{a} v^{a-2j} t^a E_{r,r+a+1} = [a+1]_{v,t} E_{r,r+a+1}.$$

The second identity can be proved similarly. □

Let $K_r = 1_r[d-2r](\frac{d}{2})$, $K_r^{-1} = 1_r[2r-d](-\frac{d}{2})$ and

$$E = \sum_{r=0}^{d-1} E_{r,r+1}, \quad F = \sum_{r=1}^{d} F_{r,r-1}, \quad K = \sum_{r=0}^{d} K_r, \quad K^{-1} = \sum_{r=0}^{d} K_r^{-1}.$$

By Lemmas 3.3, 3.4, 3.7 and using $t^2 = -1$, we have

Theorem 3.8. *There exists a unique surjective algebra homomorphism $\chi : \mathbf{U} \to \mathbf{S}_{v,t}(2,d)$ by sending the generators in $\mathbf{U}$ to the respective elements in $\mathbf{S}_{v,t}(2,d)$. Moreover, it induces a surjective $\mathcal{A}$-algebra homomorphism from the integral form ${}_{\mathcal{A}}\mathbf{U}$ of $\mathbf{U}$ to the algebra ${}_{\mathcal{A}}\mathbf{S}_{v,t}(2,d)$.*

3.9. Let $\mathbf{S}_v(2,d)$ be the q-Schur algebra associated to $\mathfrak{sl}(2)$. By [BLM90] and [D95, 1.3], $\mathbf{S}_{v,t}(2,d)$ is isomorphic to $\mathbb{Q}[t](v) \otimes_{\mathbb{Q}(v)} \mathbf{S}_v(2,d)$. We define a $\mathbb{Q}[t](v)$-linear map

$$\psi_{d,d+2} : \mathbf{S}_{v,t}(2,d+2) \to \mathbf{S}_{v,t}(2,d) \tag{18}$$

by

$$\{A\} \mapsto \begin{cases} \{A - I_{2\times 2}\}, & \text{if } A - I_{2\times 2} \in \Theta_d, \\ 0, & \text{otherwise}, \end{cases} \tag{19}$$

where $I_{2\times 2}$ is the identity matrix of rank 2.

Proposition 3.10. *For any $d \in \mathbb{Z}_{>0}$, $\psi_{d,d+2}$ in* (18) *is a surjective algebra homomorphism.*

Proof. By Lemma 3.4, $\mathbf{S}_{v,t}(2,d)$ is generated by $1_r, E_{r,r+1}$ and $F_{r,r-1}$, $\forall 0 \le r \le d$ and subject to the relations given by Lemma 3.3. By (19), we have

$$\psi_{d,d+2}(E_{r,r+1}) = tE_{r-1,r}, \quad \psi_{d,d+2}(F_{r,r-1}) = tF_{r-1,r-2}, \quad \text{and} \quad \psi_{d,d+2}(1_r) = 1_{r-1}.$$

$\psi_{d,d+2}$ is surjective, since $1_r, E_{r,r+1}$ and $F_{r,r-1}$ are algebraic generators of $\mathbf{S}_{v,t}(2,d)$. The rest is to show that $\psi_{d,d+2}$ is compatible with the defining relations of $\mathbf{S}_{v,t}(2,d+2)$. We only check the relation $E_{r,r+1}F_{r+1,r} - t^2F_{r,r-1}E_{r-1,r} = t^{2r}[d+2-2r]_v 1_r$ and the remaining relations can be checked similarly. By the definition of K_r, we have $\psi_{d,d+2}(K_r) = v^{d+2-2r}t^{2r}\psi_{d,d+2}(1_r) = v^{d+2-2r}t^{2r}1_{r-1} = t^2K_{r-1}$. So

$$\psi_{d,d+2}(E_{r,r+1}F_{r+1,r} - t^2F_{r,r-1}E_{r-1,r}) = \psi_{d,d+2}(\frac{K_r - K_r^{-1}}{v - v^{-1}}) = t^2\frac{K_{r-1} - K_{r-1}^{-1}}{v - v^{-1}}.$$

On the other hand, we have

$$\begin{aligned} & \psi_{d,d+2}(E_{r,r+1})\psi_{d,d+2}(F_{r+1,r}) - t^2\psi_{d,d+2}(F_{r,r-1})\psi_{d,d+2}(E_{r-1,r}) \\ = \; & t^2(E_{r-1,r}F_{r,r-1} - t^2F_{r-1,r-2}E_{r-2,r-1}) = t^2\frac{K_{r-1} - K_{r-1}^{-1}}{v - v^{-1}}. \end{aligned}$$

The proposition follows. $\square$

4. A geometric categorification of $\mathbf{U}$-modules

4.1. Let $\mathcal{C}_{r,r'}$ be the category of triangulated functors from $\mathcal{D}(\mathcal{F}_{r'})$ to $\mathcal{D}(\mathcal{F}_r)$. Consider the diagram

$$\mathcal{F}_r \xleftarrow{p_1} \mathcal{F}_r \times \mathcal{F}_{r'} \xrightarrow{p_2} \mathcal{F}_{r'},$$

where p_1 and p_2 are projections to the first and second components, respectively. Define a functor

$$\Psi_{r,r'} : \mathcal{D}(\mathcal{F}_r \times \mathcal{F}_{r'}) \to \mathcal{C}_{r,r'}$$

by $\Psi_{r,r'}(L) = p_{1!}(L \otimes p_2^*(-))$ for any object L in $\mathcal{D}(\mathcal{F}_r \times \mathcal{F}_{r'})$.

Proposition 4.2. *$\Psi_{r'',r}(L \circ M) = \Psi_{r'',r'}(L)\Psi_{r',r}(M)$ for any objects L in $\mathcal{D}(\mathcal{F}_{r''} \times \mathcal{F}_{r'})$ and M in $\mathcal{D}(\mathcal{F}_{r'} \times \mathcal{F}_r)$.*

This is a special case of Proposition 7.2 in [Li10].

Consider the diagram

$$\mathcal{F}_r \xleftarrow{p} \mathcal{F}_{r,r+a} \xrightarrow{p'} \mathcal{F}_{r+a},$$

where $\mathcal{F}_{r,r+a}$ is defined in (6) and p, p' are projections. For any $0 \leq r, r+a \leq d$ and $a > 0$, we define

$$\begin{aligned}
&\mathfrak{K}_r = \mathrm{Id}[d-2r]\left(\frac{d}{2}\right) : \mathcal{D}(\mathcal{F}_r) \to \mathcal{D}(\mathcal{F}_r),\\
&\mathfrak{E}_{r,r+a} = p_! p'^*[a(d-a-r)]\left(\frac{a(d-a)}{2}\right) : \mathcal{D}(\mathcal{F}_{r+a}) \to \mathcal{D}(\mathcal{F}_r),\\
&\mathfrak{F}_{r+a,r} = p'_! p^*[ar](ar) : \mathcal{D}(\mathcal{F}_r) \to \mathcal{D}(\mathcal{F}_{r+a}).
\end{aligned} \tag{20}$$

Lemma 4.3. *For any $0 \leq r, r+a \leq d$ and $a > 0$, we have*

$$\Psi_{r,r+a}(E_{r,r+a}) = \mathfrak{E}_{r,r+a},\ \Psi_{r,r-a}(F_{r,r-a}) = \mathfrak{F}_{r,r-a},\ \text{and } \Psi_{r,r}(K_r) = \mathfrak{K}_r.$$

Proof. We show that $\Psi_{r,r+a}(E_{r,r+a}) = \mathfrak{E}_{r,r+a}$. The other identities can be proved similarly. We notice that the shift and Tate twist are the same in the complex $E_{r,r+a}$ and in the functor $\mathfrak{E}_{r,r+a}$, respectively. So it is enough to show that $p_! p'^*(C) = p_{1!}(\iota_!(\bar{\mathbb{Q}}_l)_{\mathcal{F}_{r,r+a}} \otimes p_2^*(C))$ for any object C in $\mathcal{D}(\mathcal{F}_{r+a})$, where p, p', p_1 and p_2 are the obvious maps in the following commutative diagram and ι is the closed embedding:

$$\begin{tikzcd}
\mathcal{F}_r & \mathcal{F}_{r,r+a} \arrow[l, "p"'] \arrow[r, "p'"] \arrow[d, "\iota"] & \mathcal{F}_{r+a} \\
& \mathcal{F}_r \times \mathcal{F}_{r+a}. \arrow[ul, "p_1"] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

By the projection formula (7) and the commutativity of the diagram, we have

$$p_{1!}(\iota_!(\bar{\mathbb{Q}}_l)_{\mathcal{F}_{r,r+a}} \otimes p_2^*(C)) = p_{1!}\iota_!((\bar{\mathbb{Q}}_l)_{\mathcal{F}_{r,r+a}} \otimes \iota^* p_2^*(C)) = p_{1!}\iota_!\iota^* p_2^*(C)) = p_! p'^*(C).$$

The lemma follows. ☐

By Proposition 4.2 and Lemma 4.3, we can transport results on the complexes E, F and K to the corresponding functors $\mathfrak{E}$, $\mathfrak{F}$, and $\mathfrak{K}$. In particular, we have

Lemma 4.4. (*a*) $\mathfrak{E}_{r,r+a}\mathfrak{E}_{r+a,r+a+1} = \bigoplus_{j=0}^{a} \mathfrak{E}_{r,r+a+1}[a-2j](a-j)$.

$$(b) \quad \mathfrak{F}_{r,r-a}\mathfrak{F}_{r-a,r-a-1} = \bigoplus_{j=0}^{a} \mathfrak{F}_{r,r-a-1}[a-2j](a-j).$$

$$\begin{aligned}
(c) \quad &\mathfrak{E}_{r,r+1}\mathfrak{F}_{r+1,r} \oplus \bigoplus_{0\leq j<2r-d} \mathrm{Id}\ [2r-2j-1-d]\left(2r-j-\frac{d+1}{2}\right)\\
&= \mathfrak{F}_{r,r-1}\mathfrak{E}_{r-1,r}(1) \oplus \bigoplus_{0\leq j<d-2r} \mathrm{Id}\ [d-2j-1-2r]\left(\frac{d-1}{2}-j\right).
\end{aligned}$$

Let

$$\mathfrak{K} = \bigoplus_{r=0}^{d} \mathfrak{K}_r, \quad \mathfrak{E}^{(a)} = \bigoplus_{r=0}^{d} \mathfrak{E}_{r,r+a}, \quad \mathfrak{F}^{(a)} = \bigoplus_{r=0}^{d} \mathfrak{F}_{r,r-a}.$$

These are endofunctors on $\bigoplus_{r=0}^{d} \mathcal{D}(\mathcal{F}_r)$.

4.5. We fix a sequence $\mathbf{d} = (d_1, d_2, \cdots, d_m)$ of integers such that $\sum_{l=1}^{m} d_l = d$. To such a sequence $\mathbf{d}$, we associate a fixed partial flag in k^d of the form

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = k^d, \quad |V_l/V_{l-1}| = d_l, \ \forall\, l. \tag{21}$$

Denote by $P_{\mathbf{d}}$ the parabolic subgroup of $G = GL(k^d)$ which fixes all subspaces $V_l, \forall\, l = 1, \cdots, m$, in the fixed flag (21). Let $\mathcal{Q}^r_{\mathbf{d}}$ be the full subcategory of $\mathcal{D}(\mathcal{F}_r)$ consisting of $P_{\mathbf{d}}$-equivariant semisimple complexes and $\mathcal{Q}_{\mathbf{d}} = \bigoplus_r \mathcal{Q}^r_{\mathbf{d}}$. It is clear that the functors $\mathfrak{K}^{\pm 1}$, $\mathfrak{E}^{(a)}$ and $\mathfrak{F}^{(a)}$ induce functors

$$\mathfrak{K}^{\pm 1}, \mathfrak{E}^{(a)}, \mathfrak{F}^{(a)} : \mathcal{Q}_{\mathbf{d}} \to \mathcal{Q}_{\mathbf{d}}.$$

Let $\mathbf{Q}_{\mathbf{d}}$ be the split Grothendieck group of $\mathcal{Q}_{\mathbf{d}}$. It admits a left $\tilde{\mathcal{A}}$-module structure similar to (15). We set

$$\mathbf{V}_{\mathbf{d}} = \mathcal{A} \otimes_{\tilde{\mathcal{A}}} \mathbf{Q}_{\mathbf{d}} \quad \text{and} \quad \mathsf{V}_{\mathbf{d}} = \mathbb{Q}[t^{\pm 1}](v) \otimes_{\mathcal{A}} \mathbf{V}_{\mathbf{d}}.$$

The functors $\mathfrak{K}^{\pm 1}$, $\mathfrak{E}^{(a)}$, and $\mathfrak{F}^{(a)}$ descend to linear maps

$$\mathbf{K}^{\pm 1}, \mathbf{E}^{(a)}, \mathbf{F}^{(a)} : \mathsf{V}_{\mathbf{d}} \to \mathsf{V}_{\mathbf{d}}.$$

We also use the same for the respective linear maps on $\mathbf{V}_{\mathbf{d}}$. By Lemma 4.4, we have

Lemma 4.6. *The quadruple* $(\mathsf{V}_{\mathbf{d}}, \mathbf{K}^{\pm 1}, \mathbf{E}, \mathbf{F})$ *defines a* $\mathbf{U}$*-module. It is a* ${}_{\mathcal{A}}\mathbf{U}$*-module if* $\mathsf{V}_{\mathbf{d}}$ *is replaced by* $\mathbf{V}_{\mathbf{d}}$*.*

Moreover, the quadruple $(\mathbf{V}_{\mathbf{d}}, \mathbf{K}^{\pm 1}, \mathbf{E}^{(a)}, \mathbf{F}^{(a)})$ defines a module of the integral form of $\mathbf{U}$. We shall show that $\mathsf{V}_{\mathbf{d}} \equiv (\mathsf{V}_{\mathbf{d}}, \mathbf{K}^{\pm 1}, \mathbf{E}, \mathbf{F})$ is isomorphic to the tensor product module $\Lambda_{\mathbf{d}}$ in Section 2 (4).

4.7. In this subsection, we treat the special case $\mathbf{d} = (d)$. In particular, the category $\mathcal{Q}_{\mathbf{d}}$ is nothing but the category $\mathcal{Q}_d^{r,0}$ in Section 3.6. Again, by Proposition 4.2 and Lemma 4.3, we have

Proposition 4.8. $\mathsf{V}_d \simeq \Lambda_d$ *and* $\mathbf{V}_d \simeq {}_{\mathcal{A}}\Lambda_d$*, as modules of* $\mathbf{U}$ *and* ${}_{\mathcal{A}}\mathbf{U}$*, respectively. In particular,*

$$\mathfrak{E}\xi_r = t^{r-1}[d+1-r]_v \xi_{r-1}, \quad \mathfrak{F}\xi_r = t^r [r+1]_v \xi_{r+1}, \quad \mathfrak{K}\xi_r = v^{d-2r} t^{2r-1} \xi_r, \tag{22}$$

where $\xi_r = (\bar{\mathbb{Q}}_l)_{\mathcal{F}_r}$*.*

Proof. By Proposition 2.4, it is enough to show (22). Since $\xi_r = \Psi_{r,0}(F_{r,0})(\bar{\mathbb{Q}}_l)_{\mathcal{F}_0}$, by Proposition 4.2, it remains to show that

$$KF_{r,0} = v^{d-2r} t^{2r} F_{r,0}, \ FF_{r,0} = t^r [r+1]_v F_{r+1,0}, \quad EF_{r,0} = t^{r-1}[d+1-r]_v F_{r-1,0}. \tag{23}$$

The first two identities in (23) follow from Lemma 3.3 and Lemma 3.7. We now show the last one in (23). By Lemma 3.3, we have

$$EF_{r,0} \quad = \textstyle\sum_{r'} E_{r',r'+1} F_{r,0} = E_{r-1,r} F_{r,0}.$$

Let us compute the complex $E_{r-1,r}F_{r,0}$. The support of $p_{12}^*E_{r-1,r}\otimes p_{23}^*F_{r,0}$ is

$$S=\{(V,V',V'')\in\mathcal{F}_{r-1}\times\mathcal{F}_r\times\mathcal{F}_0|V''\subset V'\supset V\}.$$

Observe that p_{13} is a fiber bundle of fiber isomorphic to the projective space $\mathbb{P}^{d-r}$. Further, the image of p_{13} is $\mathcal{F}_{r-1,0}$. The restriction of $p_{12}^*E_{r-1,r}\otimes p_{23}^*F_{r,0}$ to S is

$$(\bar{\mathbb{Q}}_l)_S[d-r](\frac{d-1}{2})\otimes(\bar{\mathbb{Q}}_l)_S=(\bar{\mathbb{Q}}_l)_S[d-r]\left(\frac{d-1}{2}\right).$$

So

$$\begin{aligned}E_{r-1,r}F_{r,0}&=(p_{13})_!(p_{12}^*E_{r-1,r}\otimes p_{23}^*F_{r,0})=(p_{13})_!(\bar{\mathbb{Q}}_l)_S[d-r]\left(\frac{d-1}{2}\right)\\&=\bigoplus_{k=0}^{d-r}(\bar{\mathbb{Q}}_l)_{\mathcal{F}_{r-1,0}}[d-r-2k]\left(\frac{d-1}{2}-k\right)=t^{r-1}[d-r+1]_vF_{r-1,0}.\end{aligned}$$

The proposition follows. □

4.9. In this subsection, we treat the general case. Let

$$\Xi_r^{\mathbf{d}}=\{\mathbf{r}=(r_1,r_2,\cdots,r_m)\in\mathbb{Z}_{\geq 0}^m|r=\sum_{1\leq l\leq 1}r_l,r_l\leq d_l,\forall l\}\quad\text{and}\quad\Xi^{\mathbf{d}}=\bigcup_{0\leq r\leq d}\Xi_r^{\mathbf{d}}.$$

To each $\mathbf{r}\in\Xi_r^{\mathbf{d}}$, we associate a $P_{\mathbf{d}}$-orbit in $\mathcal{F}_r$ as follows:

$$O_{\mathbf{r}}=\{W\in\mathcal{F}_r\mid |W\cap V_l/W\cap V_{l-1}|=r_l,\forall\ l\}.$$

The intersection complexes $IC(O_{\mathbf{r}})$ are all possible $P_{\mathbf{d}}$-equivariant simple perverse sheaves on $\mathcal{F}_r$. So we have

Lemma 4.10. *The intersection complexes $IC(O_{\mathbf{r}})$, $\forall\ \mathbf{r}\in\Xi^{\mathbf{d}}$, form an $\mathcal{A}$-basis of $\mathbf{V}_{\mathbf{d}}$ and a $\mathbb{Q}(v)[t]$-basis of $\mathsf{V}_{\mathbf{d}}$.*

Next, we want to define the restriction functor "Res". In the following, we use the notation $\mathcal{F}_r^d$ instead of $\mathcal{F}_r$ to avoid ambiguities. We fix pairs (r',r'') and (d',d'') of non-negative integers such that $r'+r''=r$ and $d'+d''=d$. We fix a vector subspace W in k^d such that $|W|=d'$. Consider the diagram

$$\mathcal{F}_{r'}^{d'}\times\mathcal{F}_{r''}^{d''}\xleftarrow{\kappa}Y^{r',r''}\xrightarrow{\iota}\mathcal{F}_r^d,\tag{24}$$

where $Y^{r',r''}=\{W'\in\mathcal{F}_r^d\mid |W\cap W'|=r'\}$, $\kappa(W')=(W\cap W',W'/(W\cap W'))$ and ι is the closed embedding. We define

$$\mathrm{Res}_{d',d''}^{r',r''}=\kappa_!\iota^*[(d''-r'')r']\left(\frac{d''r'}{2}\right):\mathcal{D}(\mathcal{F}_r^d)\to\mathcal{D}(\mathcal{F}_{r'}^{d'}\times\mathcal{F}_{r''}^{d''}),\quad\text{and}$$

$$\mathrm{Res}_{d',d''}=\bigoplus_{r',r''}\mathrm{Res}_{d',d''}^{r',r''}:\bigoplus_{0\leq r\leq d}\mathcal{D}(\mathcal{F}_r^d)\to\bigoplus_{0\leq r'\leq d',0\leq r''\leq d''}\mathcal{D}(\mathcal{F}_{r'}^{d'}\times\mathcal{F}_{r''}^{d''}).$$

We define

$$\mathfrak{E}'_{r,r+1}=\mathfrak{E}_{r',r'+1}\times\mathrm{Id}:\quad\mathcal{D}(\mathcal{F}_{r+1})\otimes\bigoplus_{r'}\mathcal{D}(\mathcal{F}_{r'})\to\mathcal{D}(\mathcal{F}_r)\otimes\bigoplus_{r'}\mathcal{D}(\mathcal{F}_{r'}),$$

$$\mathfrak{E}''_{r,r+1}=\mathrm{Id}\times\mathfrak{E}_{r,r+1}:\quad\bigoplus_{r'}\mathcal{D}(\mathcal{F}_{r'})\otimes\mathcal{D}(\mathcal{F}_{r+1})\to\bigoplus_{r'}\mathcal{D}(\mathcal{F}_{r'})\otimes\mathcal{D}(\mathcal{F}_r).$$

Similarly, we define the notation $\mathfrak{K}'_r,\mathfrak{F}'_{r,r-1}$, $\mathfrak{K}''_r$ and $\mathfrak{F}''_{r,r-1}$. The following proposition is a mixed version of Proposition 3.8.3 in [Z07].

Proposition 4.11. *For any $C_r \in \mathcal{Q}^r_{\mathbf{d}}$, we have*

$$
\begin{aligned}
&\mathrm{Res}^{r_1,r_2}_{d',d''}\mathfrak{K}_r C_r = \mathfrak{K}'_{r_1}\mathfrak{K}''_{r_2}\mathrm{Res}^{r_1,r_2}_{d',d''}C_r,\\
&\mathrm{Res}^{r_1,r_2}_{d',d''}\mathfrak{E}_{r-1,r}C_r = \mathfrak{E}'_{r_1,r_1+1}\mathrm{Res}^{r_1+1,r_2}_{d',d''}C_r \oplus \mathfrak{K}'_{r_1}\mathfrak{E}''_{r_2,r_2+1}(r_1)\mathrm{Res}^{r_1,r_2+1}_{d',d''}C_r,\\
&\mathrm{Res}^{r_1,r_2}_{d',d''}\mathfrak{F}_{r+1,r}C_r = \mathfrak{F}'_{r_1,r_1-1}\mathfrak{K}''^{-1}_{r_2}\mathrm{Res}^{r_1-1,r_2}_{d',d''}C_r \oplus \mathfrak{F}''_{r_2,r_2-1}(r_1)\mathrm{Res}^{r_1,r_2-1}_{d',d''}C_r.
\end{aligned}
\tag{25}
$$

Proof. The first identity is obvious. We now prove the second one. We only need to prove it for C_r a simple perverse sheaf.

We claim that for any simple perverse sheaf $C_r \in \mathcal{Q}^r_{\mathbf{d}}$, there exists a proper map $\pi : \widetilde{\mathcal{F}} \to \mathcal{F}_r$, where $\widetilde{\mathcal{F}}$ is a smooth irreducible variety, such that C_r is a direct summand of $\pi_!(\bar{\mathbb{Q}}_l)_{\widetilde{\mathcal{F}}}$ up to a shift. Let $\widetilde{\iota} : \mathcal{F}_r \to \mathcal{F}_r \times \mathcal{F}_{d_1}$ be the embedding map sending $V' \mapsto (V', V_1)$, where V_1 is the fixed vector space in (21). By the argument in [BL94, Section 2.6.2], the functor $\widetilde{\iota}^* : \mathcal{D}_{P_{\widetilde{\mathbf{d}}}}(\mathcal{F}_r \times \mathcal{F}_{d_1}) \to \mathcal{D}_{P_{\mathbf{d}}}(\mathcal{F}_r)$ is an equivalence, where $\widetilde{\mathbf{d}} = (d_2, \cdots, d_m)$. By the argument in Section 3.5, for any object C' in $\mathcal{D}_{P_{\widetilde{\mathbf{d}}}}(\mathcal{F}_r \times \mathcal{F}_{d_1})$, there exist a smooth irreducible variety $\widetilde{\mathcal{F}}'$ and a proper map $\pi' : \widetilde{\mathcal{F}}' \to \mathcal{F}_r \times \mathcal{F}_{d_1}$ such that C' is a direct summand of $\pi'_!(\bar{\mathbb{Q}}_l)_{\widetilde{\mathcal{F}}'}$. Let $\widetilde{\mathcal{F}} = \mathcal{F}_r \times_{(\mathcal{F}_r \times \mathcal{F}_{d_1})} \widetilde{\mathcal{F}}'$ and $C_r = \widetilde{\iota}^* C'$. By the base change formula (8), C_r is a direct summand of $\pi_!(\bar{\mathbb{Q}}_l)_{\widetilde{\mathcal{F}}}$, where $\pi : \widetilde{\mathcal{F}} \to \mathcal{F}_r$ is the pull back map of π'. This proves the claim.

By the above claim, we may and will assume that $C_r = \pi_!(\bar{\mathbb{Q}}_l)_{\widetilde{\mathcal{F}}}$ for some smooth irreducible variety $\widetilde{\mathcal{F}}$ and a proper map $\pi : \widetilde{\mathcal{F}} \to \mathcal{F}_r$. Now consider the diagram

$$
\begin{tikzcd}
 & & X' \arrow[dl, "\iota'"'] \arrow[drr, "s"] & & & \\
\mathcal{F}_r & \mathcal{F}_{r-1,r} \arrow[l, "p'"'] \arrow[r, "p"] & \mathcal{F}_{r-1} & & X'' \arrow[ll, "\iota''"'] \arrow[r, "\kappa''"] & \mathcal{F}^{d'}_{r_1} \times \mathcal{F}^{d''}_{r_2}
\end{tikzcd}
$$

where $X'' = \{V \in \mathcal{F}_{r-1} \mid |V \cap W| = r_1\}$ and $X' = \mathcal{F}_{r-1,r} \times_{\mathcal{F}_{r-1}} X''$. By the base change formula (8), we have $\kappa''_! \iota''^* p_! p'^* C_r = \kappa''_! s_! \iota'^* p'^* C_r$.

Let $X = \{V' \in \mathcal{F}_r \mid |V' \cap W| = r_1 \text{ or } r_1+1\}$. Then X has a partition $X = X_1 \sqcup X_2$ with $X_j = \{V' \in \mathcal{F}_r \mid |V' \cap W| = r_1 + j - 1\}$ for $j = 1, 2$. Let $Y = Y_1 \sqcup Y_2$ with $Y_1 = \mathcal{F}_{r_1} \times \mathcal{F}_{r_2+1}$ and $Y_2 = \mathcal{F}_{r_1+1} \times \mathcal{F}_{r_2}$. Let $Y' = Y'_1 \sqcup Y'_2$ with $Y'_1 = \mathcal{F}_{r_1} \times \mathcal{F}_{r_2,r_2+1}$ and $Y'_2 = \mathcal{F}_{r_1,r_1+1} \times \mathcal{F}_{r_2}$. Let $Z_j = X_j \times_{Y_j} Y'_j$ and $Z = Z_1 \sqcup Z_2$. Consider the diagram

$$
\begin{tikzcd}
\widetilde{\mathcal{F}} \arrow[d, "\pi"'] & & \widetilde{Z} \arrow[ll] \arrow[d, "\widetilde{\pi}'"] & \widetilde{X'} \arrow[l, "\widetilde{b}"'] \arrow[d, "\widetilde{\pi}"] \\
\mathcal{F}_r & & Z \arrow[ll, "\iota s'"'] \arrow[dll, "s'"'] \arrow[d, "\kappa'"'] & X' \arrow[l, "b"'] \arrow[d, "\kappa'' s"] \\
X \arrow[u, "\iota"] \arrow[r, "\kappa"'] & Y & Y' \arrow[l, "q'"'] \arrow[r, "q"] & \mathcal{F}^{d'}_{r_1} \times \mathcal{F}^{d''}_{r_2}
\end{tikzcd}
$$

where b is the map such that $p'\iota' = \iota s' b$, $\widetilde{Z} = Z \times_{\mathcal{F}_r} \widetilde{\mathcal{F}}$ and $\widetilde{X'} = X' \times_{\mathcal{F}_r} \widetilde{\mathcal{F}}$. By the base change formula, the commutativity of the diagram and the assumption $C_r = \pi_!(\bar{\mathbb{Q}}_l)_{\widetilde{\mathcal{F}}}$, we have

$$
\kappa''_! s_! \iota'^* p'^* C_r = q_! \kappa'_! b_! b^* s'^* \iota^* C_r = q_! \kappa'_! b_! \widetilde{\pi}_! (\bar{\mathbb{Q}}_l)_{\widetilde{X'}}.
$$

We further have a partition of $\widetilde{X'} = \widetilde{X'_1} \sqcup \widetilde{X'_2}$ and $\widetilde{Z} = \widetilde{Z_1} \sqcup \widetilde{Z_2}$. For a map in the above diagram, if its domain has a partition, we shall use a subscript j ($j = 1, 2$) to indicate the restriction map to the corresponding part. We can check that b_1 is a vector bundle of rank r_1 (resp. b_2 is an identity map), so is $\widetilde{b}_1$ (resp. $\widetilde{b}_2$). By [L93, 8.1.6], we have

$$\begin{aligned} & b_! b^* s'^* \iota^* C_r = b_! \widetilde{\pi}_! (\bar{\mathbb{Q}}_l)_{\widetilde{X'}} = b_{1!} \widetilde{\pi}_{1!} (\bar{\mathbb{Q}}_l)_{\widetilde{X'_1}} \oplus b_{2!} \widetilde{\pi}_{2!} (\bar{\mathbb{Q}}_l)_{\widetilde{X'_2}} \\ = \quad & \widetilde{\pi}'_1 (\bar{\mathbb{Q}}_l)_{\widetilde{Z}_1} [-2r_1] \oplus \widetilde{\pi}'_2 (\bar{\mathbb{Q}}_l)_{\widetilde{Z}_2} = s'^*_1 \iota^*_1 C_r [-2r_1] \oplus s'^*_2 \iota^*_2 C_r. \end{aligned}$$

If we ignore Tate twists, then we have

$$\begin{aligned} & \mathrm{Res}^{r_1, r_2}_{d', d''} \mathfrak{E}_{r-1, r} C_r = q_{1!} \kappa'_{1!} s'^*_1 \iota^*_1 C_r [N - 2r_1] \oplus q_{2!} \kappa'_{2!} s'^*_2 \iota^*_2 C_r [N] \\ = \quad & \mathfrak{K}'_{r_1} \mathfrak{E}''_{r_2, r_2+1} \mathrm{Res}^{r_1, r_2+1}_{d', d''} C_r [N - 2r_1 - N_1] \oplus \mathfrak{E}'_{r_1, r_1+1} \mathrm{Res}^{r_1+1, r_2}_{d', d''} C_r [N - N_2], \end{aligned}$$

where $N = d - r + (d'' - r_2) r_1$, $N_1 = d' - 2r_1 + d'' - r_2 - 1 + (d'' - r_2 - 1) r_1$ and $N_2 = d' - r_1 - 1 + (d'' - r_2)(r_1 + 1)$. By using $d = d' + d''$ and $r_1 + r_2 = r - 1$, we have $N - 2r_1 - N_1 = 0 = N - N_2$. This shows that the complexes on both sides in the second identity are isomorphic to each other if the Tate twists are ignored.

Next, we check that the weights of complexes on both sides in the second identity are the same. By [B03], we see that $\kappa_! \iota^*$ in diagram (24) is equivalent to a hyperbolic localization functor. By [B03, Theorem 8], the functor $\kappa_! \iota^*$ preserves purities and weights of equivariant complexes. Thus

$$\mathrm{wt}(\mathrm{Res}^{r_1, r_2}_{d', d''} \mathfrak{E}_{r-1, r} (\bar{\mathbb{Q}}_l)_{\mathcal{F}_r}) = -r + 1 - r_1 r_2. \tag{26}$$

On the other hand, we have

$$\begin{aligned} & \mathrm{wt}(\mathfrak{E}'_{r_1, r_1+1} \mathrm{Res}^{r_1+1, r_2}_{d', d''} (\bar{\mathbb{Q}}_l)_{\mathcal{F}_r}) = -(r_1 + 1) r_2 - r_1, \\ & \mathrm{wt}(\mathfrak{K}'_{r_1} \mathfrak{E}''_{r_2, r_2+1} (r_1) \mathrm{Res}^{r_1, r_2+1}_{d', d''} (\bar{\mathbb{Q}}_l)_{\mathcal{F}_r}) = -r_1 (r_2 + 1) - r_2 - 4r_1. \end{aligned} \tag{27}$$

(26) and (27) are the same by using $r_1 + r_2 = r - 1$ and $4r_1 \equiv 0 \bmod 4$. This shows that the second equality holds. The third one can be proved similarly. $\square$

Let $\mathbf{d}' = (d_1, d_2, \cdots, d_{m'})$ and $\mathbf{d}'' = (d_{m'+1}, d_{m'+2}, \cdots, d_m)$. Let $d' = \sum_{l=1}^{m'} d_l$ and $d'' = \sum_{l=m'+1}^{m} d_l$ so that $d' + d'' = d$. Let $\mathcal{Q}^{r', r''}_{\mathbf{d}', \mathbf{d}''}$ be the full subcategory of $\mathcal{D}(\mathcal{F}^{d'}_{r'} \times \mathcal{F}^{d''}_{r''})$ consisting of all $P_{\mathbf{d}'} \times P_{\mathbf{d}''}$-equivariant semisimple complexes and $\mathcal{Q}_{\mathbf{d}', \mathbf{d}''} = \bigoplus \mathcal{Q}^{r', r''}_{\mathbf{d}', \mathbf{d}''}$. It is clear from Proposition 4.11 that $\mathrm{Res}_{d', d''}$ restricts to a functor

$$\mathrm{Res}_{d', d''} : \mathcal{Q}_{\mathbf{d}} \to \mathcal{Q}_{\mathbf{d}', \mathbf{d}''}.$$

Let $\mathbf{Q}_{\mathbf{d}', \mathbf{d}''}$ be the split Grothendieck group of $\mathcal{Q}_{\mathbf{d}', \mathbf{d}''}$ and $\mathbf{V}_{\mathbf{d}', \mathbf{d}''} = \mathcal{A} \otimes_{\tilde{\mathcal{A}}} \mathbf{Q}_{\mathbf{d}', \mathbf{d}''}$. From the definitions, we have that $\mathbf{V}_{\mathbf{d}', \mathbf{d}''} \simeq \mathbf{V}_{\mathbf{d}'} \otimes \mathbf{V}_{\mathbf{d}''}$. The functor $\mathrm{Res}_{d', d''}$ induces an $\mathcal{A}$-linear map

$$r_{d', d''} : \mathbf{V}_{\mathbf{d}} \to \mathbf{V}_{\mathbf{d}'} \otimes \mathbf{V}_{\mathbf{d}''}.$$

An argument similar to the proof of Proposition 3.8.1 in [Z07] shows that $r_{d', d''}$ is an $\mathcal{A}$-linear isomorphism. Moreover, we have

Theorem 4.12. *(a) The maps $r_{d', d''}$ induce isomorphisms $\mathsf{V}_{\mathbf{d}} \simeq \Lambda_{\mathbf{d}}$ and $\mathbf{V}_{\mathbf{d}} \simeq {}_{\mathcal{A}}\Lambda_{\mathbf{d}}$ of modules of $\mathbf{U}$ and ${}_{\mathcal{A}}\mathbf{U}$, respectively.*

(b) The image of $\{IC(O_{\mathbf{r}}) \mid \mathbf{r} \in \Xi^{\mathbf{d}}\}$ under the above isomorphism form a $\mathbb{Q}[t](v)$-basis of $\Lambda_{\mathbf{d}}$. Moreover, the structure constants of the actions of E, F and K on $IC(O_{\mathbf{r}})$ are in $t^a \mathbb{N}[v^{\pm 1}]$ for various $a \in \mathbb{Z}$ with respect to this basis.

Proof. (a) follows from Propositions 4.8, 4.11 and (3). The first statement of part (b) follows from the fact that $\mathbf{r} \in \Xi_r^{\mathbf{d}}$ parameterizes the $P_{\mathbf{d}}$-orbits of $\mathcal{F}_r$. The second statement of (b) follows from the first one and Lemma 4.10. □

4.13. Recall that the Ext groups of any two objects L, M in $\mathcal{D}(X)$ are defined by

$$\mathrm{Ext}^n(L, M) = \mathrm{Hom}_{\mathcal{D}(X)}(L, M[n]).$$

We will use the following properties of Ext groups:

(a) $\mathrm{Ext}^j(L[n], M[m]) = \mathrm{Ext}^{j-n+m}(L, M)$.
(b) If both L and M are perverse sheaves, then $\mathrm{Ext}^j_{\mathcal{D}(X)}(L, M) = 0$ for any $j < 0$.
(c) Suppose L and M are both simple perverse sheaves; then $\dim \mathrm{Ext}^0(L, M) = 1$ if $L = M$ and 0 otherwise.

Given any two pure complexes L, M in $\mathcal{Q}_{\mathbf{d}}$, we define

$$(L, M) = \sum_{j \in \mathbb{Z}} \dim \mathrm{Ext}^j(L, \mathbb{D}M) v^{-j} t^{-\mathrm{wt}(L) - \mathrm{wt}(M)}. \tag{28}$$

Since any complex in $\mathcal{Q}_{\mathbf{d}}$ is semisimple, the above definition can be extended to any two complexes in $\mathcal{Q}_{\mathbf{d}}$. This defines a bilinear form on $\mathbf{V}_{\mathbf{d}}$.

Proposition 4.14. *For any two complexes L, M in $\mathcal{Q}_{\mathbf{d}}$, we have*

$$\begin{aligned}
(\mathfrak{K}_r L, M) &= (L, \mathfrak{K}_r M),\\
(\mathfrak{E}_{r,r+1} L, M) &= (L, \mathfrak{K}_{r+1}\mathfrak{F}_{r+1,r} M[1](-\frac{2r+1}{2})),\\
(\mathfrak{F}_{r+1,r} L, M) &= (L, \mathfrak{K}_r^{-1}\mathfrak{E}_{r,r+1} M[1](\frac{2r+1}{2})).
\end{aligned}$$

Proof. The first equality is obvious. We now show the second equality and the third one can be proved similarly. By the definition of $\mathfrak{E}_{r,r+1}$ in (20), we have

$$\begin{aligned}
&\mathrm{Ext}^j(p_! p'^* L[d-1-r](\tfrac{d-1}{2}), \mathbb{D}M) = \mathrm{Ext}^j(L, p'_* p^! [-d+1+r](-\tfrac{d-1}{2})\mathbb{D}M)\\
&= \mathrm{Ext}^j(L, \mathbb{D}(p'_! p^*[d-1-r](\tfrac{d-1}{2})M)) = \mathrm{Ext}^j(L, \mathbb{D}(\mathfrak{K}_{r+1}\mathfrak{F}_{r+1,r}[1]M)).
\end{aligned}$$

Without loss of generality, we assume that both L and M are pure complexes. By (28), we have

$$\begin{aligned}
(\mathfrak{E}_{r,r+1} L, M) &= \sum_{j \in \mathbb{Z}} \dim \mathrm{Ext}^j(\mathfrak{E}_{r,r+1} L, \mathbb{D}M) v^{-j} t^{-\mathrm{wt}(L) - \mathrm{wt}(M) + r}\\
&= \sum_{j \in \mathbb{Z}} \dim \mathrm{Ext}^j(L, \mathbb{D}(\mathfrak{K}_{r+1}\mathfrak{F}_{r+1,r}[1]M)) v^{-j} t^{-\mathrm{wt}(L) - \mathrm{wt}(M) + r}\\
&= (L, \mathfrak{K}_{r+1}\mathfrak{F}_{r+1,r} M[1](-\frac{2r+1}{2})).
\end{aligned}$$

The proposition follows. □

We define an algebra isomorphism $\rho : \mathbf{S}_{v,t}(2,d) \to (\mathbf{S}_{v,t}(2,d))^{op}$ by

$$\rho(1_r) = 1_r, \quad \rho(E_{r,r+1}) = vt^{-2r-2} K_r F_{r+1,r}, \quad \rho(F_{r+1,r}) = vt^{2r} K_r E_{r,r+1}.$$

The following corollary follows directly from Proposition 4.14.

Corollary 4.15. *For any two isomorphism classes L and M in $\mathbf{V}_{\mathbf{d}}$ and any $x \in \mathbf{S}_{v,t}(2,d)$, we have $(xL, M) = (L, \rho(x)M)$.*

Given any pure complex $L \in \mathcal{Q}_{\mathbf{d}}$, let

$$\mathfrak{D}(L) = (\mathbb{D}L)(-\mathrm{wt}(L)).$$

Since objects in $\mathcal{Q}_{\mathbf{d}}$ are semisimple, this defines a functor $\mathfrak{D} : \mathcal{Q}_{\mathbf{d}} \to \mathcal{Q}_{\mathbf{d}}$. We notice that $\mathfrak{D}^2$ is the identity functor. Let $^- : \mathbf{V_d} \to \mathbf{V_d}$ be the $\mathbb{Z}[t]$-linear map defined by $L \mapsto \mathfrak{D}L$ and $v \mapsto v^{-1}$. Let $\mathbf{B_d}$ be the subset of $\mathbf{V_d}$ consisting of all x satisfying

$$\overline{x} = x, \quad (x, x) \in 1 + v^{-1}\mathbb{Z}[v^{-1}].$$

Recall that a signed basis of a module M is a subset, say B, of M such that $B = B' \cup (-B')$ for some basis B' of M.

Proposition 4.16. $\mathbf{B_d}$ *is the canonical signed basis of* $\mathbf{V_d}$.

Proof. For any $x \in \mathbf{B_d}$, let L be a representative complex of x. Then we have $\mathfrak{D}(L) \simeq L$. Let $L \simeq \bigoplus_{i=1}^m L'_i$, where L'_i are all simple complexes. For each L'_i, there exists $a_i \in \mathbb{Z}$ such that $L_i := L'_i[-a_i](-\frac{1}{2}a_i)$ is a simple perverse sheaf. Let $a' = \max_i a_i$. Denote by x_i the isomorphic classes of L_i for each i. We have $(x, x) = \sum_{i,j} v^{a_i+a_j}(x_i, x_j)$. Since $(x, x) \in 1 + v^{-1}\mathbb{Z}[v^{-1}]$, by Section 4.13 (b), we have $a' \leq 0$. Hence $a_i \leq 0$ for all i.

On the other hand, $\mathfrak{D}(L) = \bigoplus_i \mathfrak{D}(L_i)[-a_i](-\frac{1}{2}a_i)$ and $\mathfrak{D}(L_i)$ is still a simple perverse sheaf. Let $a'' = \min_i a_i$. By a similar argument, we have $-a'' \leq 0$. Hence $a_i \geq 0$ for all i. So $a_i = 0$ for all i and, therefore, $L = \bigoplus_{i=1}^m L_i$ is a perverse sheaf. By Section 4.13 (c), we have $(L, L) \in m + v^{-1}\mathbb{Z}[v^{-1}]$. Hence L is a simple perverse sheaf.

If the weight of L is odd, then $(L, L) \in -1 + v^{-1}\mathbb{Z}[v^{-1}]$. It is a contradiction. Therefore, L is a simple perverse sheaf of weight 0 or 2. $\square$

5. Modified forms of $\mathbf{U}$ and weight modules

5.1. A $\mathbf{U}$-module M is called a *weight module* if there is a decomposition of vector spaces $M = \bigoplus_{\lambda \in \mathbb{Z}} M_\lambda^\pm$ such that

$$M_\lambda^+ = \{m \in M \mid K \cdot m = v^\lambda t^{-\lambda - p(\lambda)} m\},$$

and

$$M_\lambda^- = \{m \in M \mid K \cdot m = -v^\lambda t^{-\lambda - p(\lambda)} m\},$$

where $p(\lambda) = 0$ if λ is even and 1 otherwise. The subspaces $M_\lambda^\pm$ are called the weight spaces of M. Let $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) be the category whose objects are weight modules of the form $M = \bigoplus_\lambda M_\lambda^+$ (resp. $M = \bigoplus_\lambda M_\lambda^-$) of $\mathbf{U}$ and morphisms are $\mathbf{U}$-linear maps.

The *modified quantum superalgebra* $\dot{\mathbf{U}}$ associated to $\mathfrak{osp}(1|2)$ is defined to be the associative $\mathbb{Q}[t](v)$-algebra without unit, generated by $1_\lambda, E_{\lambda,\lambda-2}$ and $F_{\lambda,\lambda+2}$, $\forall \lambda \in \mathbb{Z}$, and subject to the following defining relations:

$$\begin{aligned} &1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda, \\ &E_{\lambda,\lambda-2} 1_{\lambda'} = \delta_{\lambda-2,\lambda'} E_{\lambda,\lambda-2}, \qquad 1_{\lambda'} E_{\lambda,\lambda-2} = \delta_{\lambda',\lambda} E_{\lambda,\lambda-2}, \\ &F_{\lambda,\lambda+2} 1_{\lambda'} = \delta_{\lambda+2,\lambda'} F_{\lambda,\lambda+2}, \qquad 1_{\lambda'} F_{\lambda,\lambda+2} = \delta_{\lambda',\lambda} F_{\lambda,\lambda+2}, \end{aligned} \tag{29}$$

$$E_{\lambda,\lambda-2} F_{\lambda-2,\lambda} - t^2 F_{\lambda,\lambda+2} E_{\lambda+2,\lambda} = t^{-\lambda-p(\lambda)} [\lambda]_v 1_\lambda. \tag{30}$$

Let $\dot{\mathcal{C}}$ be the category of unital $\dot{\mathbf{U}}$-modules in the sense of Lusztig [L93, 23.1.4]. Given a weight $\mathbf{U}$-module $M = \bigoplus_\lambda M_\lambda^+$, we define a $\dot{\mathbf{U}}$-module structure on M as follows:

$$E_{\lambda'+2,\lambda'} \cdot m = \delta_{\lambda,\lambda'} E \cdot m,\ F_{\lambda'-2,\lambda'} \cdot m = \delta_{\lambda,\lambda'} F \cdot m$$

and

$$1_{\lambda'} \cdot m = \delta_{\lambda,\lambda'} m, \text{ for any } m \in M_\lambda^+.$$

This $\dot{\mathbf{U}}$-module structure is well-defined. To prove this, we only need to check the relation (30). The rest are obvious. For any $m \in M_\lambda^+$, we have

$$\begin{aligned}&(E_{\lambda,\lambda-2}F_{\lambda-2,\lambda} - t^2 F_{\lambda,\lambda+2}E_{\lambda+2,\lambda}) \cdot m = (EF - t^2 FE) \cdot m \\ &= \frac{K - K^{-1}}{v - v^{-1}} m = \frac{v^\lambda t^{-\lambda-p(\lambda)} - v^{-\lambda} t^{\lambda+p(\lambda)}}{v - v^{-1}} m = t^{-\lambda-p(\lambda)}[\lambda]_v 1_\lambda m,\end{aligned}$$

where the last equality follows from $t^{\lambda+p(\lambda)} = t^{-\lambda-p(\lambda)}$. It is clear that a homomorphism $f : M \to N$ in $\mathcal{C}^+$ becomes a homomorphism in $\dot{\mathcal{C}}$ if M and N are regarded as $\dot{\mathbf{U}}$-modules. The above analysis provides us with a functor

$$\eta : \mathcal{C}^+ \to \dot{\mathcal{C}}. \tag{31}$$

Conversely, given a $\dot{\mathbf{U}}$-module M, let $M_\lambda^+ = 1_\lambda \cdot M$. By using Lemma 3.3 (a), one can easily show that $M_\lambda^+ \cap M_{\lambda'}^+ = \{0\}$ if $\lambda \neq \lambda'$. So we have $M = \bigoplus_\lambda M_\lambda^+$ as a vector space. We now define a $\mathbf{U}$-module structure on M by $E \cdot m = E_{\lambda+2,\lambda} \cdot m,\ K \cdot m = v^\lambda t^{-\lambda-p(\lambda)} m$ and $F \cdot m = F_{\lambda-2,\lambda} \cdot m$ for any $m \in M_\lambda^+$. Similarly, we can check that this $\mathbf{U}$-module structure is well-defined. This defines a functor

$$\eta' : \dot{\mathcal{C}} \to \mathcal{C}^+. \tag{32}$$

It is clear that $\eta\eta'$ and $\eta'\eta$ are identity functors on $\dot{\mathcal{C}}$ and $\mathcal{C}^+$, respectively. We have the following proposition.

Proposition 5.2. *The functors η and η' in* (31) *and* (32) *establish an isomorphism of categories between $\mathcal{C}^+$ and $\dot{\mathcal{C}}$.*

Note that the notion of an isomorphism of categories is stronger than the notion of an equivalence of categories. We thank Jon Kujawa for pointing this out to us.

5.3. Recall that the *modified quantum algebra* $\dot{\mathbf{U}}(\mathfrak{sl}(2))$ associated to $\mathfrak{sl}(2)$ is a $\mathbb{Q}[t](v)$-algebra without unit, generated by $\widetilde{1}_\lambda, \widetilde{E}_{\lambda,\lambda-2}$ and $\widetilde{F}_{\lambda,\lambda+2}$, $\forall \lambda \in \mathbb{Z}$, subject to the analogous relations of (29) and the following:

$$\widetilde{E}_{\lambda,\lambda-2}\widetilde{F}_{\lambda-2,\lambda} - \widetilde{F}_{\lambda,\lambda+2}\widetilde{E}_{\lambda+2,\lambda} = [\lambda]_v \widetilde{1}_\lambda.$$

Theorem 5.4. *(a) The assignments $\widetilde{E}_{\lambda,\lambda-2} \mapsto t^{\lambda+p(\lambda)} E_{\lambda,\lambda-2}$, $\widetilde{F}_{\lambda,\lambda+2} \mapsto F_{\lambda,\lambda+2}$ and $\widetilde{1}_\lambda \mapsto 1_\lambda$, for any $\lambda \in \mathbb{Z}$, define a unique algebra isomorphism $\varphi : \dot{\mathbf{U}}(\mathfrak{sl}(2)) \to \dot{\mathbf{U}}$.*

(b) The algebra $\dot{\mathbf{U}}$ is isomorphic to the algebra in the same notation in [CW13, Section 6] *over $\mathbb{Q}[t](v)$.*

(c) There is a basis in $\dot{\mathbf{U}}$ whose structure constants are in $\mathbb{N}[v, v^{-1}]$.

Proof. We have

$$\begin{aligned}&\varphi(\widetilde{E}_{\lambda,\lambda-2}\widetilde{F}_{\lambda-2,\lambda} - \widetilde{F}_{\lambda,\lambda+2}\widetilde{E}_{\lambda+2,\lambda} - [\lambda]_v \widetilde{1}_\lambda) \\ = \ &t^{\lambda+p(\lambda)}(E_{\lambda,\lambda-2}F_{\lambda-2,\lambda} - t^2 F_{\lambda,\lambda+2}E_{\lambda+2,\lambda}) - [\lambda]_v 1_\lambda = 0.\end{aligned}$$

Similarly, one can show that the other defining relations of $\dot{\mathbf{U}}(\mathfrak{sl}(2))$ get sent to zero by φ. This shows that φ is an algebra homomorphism. Similarly, there is a unique algebra homomorphism $\varphi' : \dot{\mathbf{U}} \to \dot{\mathbf{U}}(\mathfrak{sl}(2))$ defined by $E_{\lambda,\lambda-2} \mapsto t^{-\lambda-p(\lambda)}\widetilde{E}_{\lambda,\lambda-2}$, $F_{\lambda,\lambda+2} \mapsto \widetilde{F}_{\lambda,\lambda+2}$ and $1_\lambda \mapsto \widetilde{1}_\lambda$. Clearly, $\varphi\varphi' = \mathrm{Id}$ and $\varphi'\varphi = \mathrm{Id}$. This finishes the proof of (a). Statement (c) follows by taking the basis to be the image of the canonical basis of $\dot{\mathbf{U}}(\mathfrak{sl}(2))$ under the isomorphism in (a). The commutator relation (30) can be rewritten as

$$(t^{\lambda+p(\lambda)}E_{\lambda,\lambda-2})(t^{\lambda-1}F_{\lambda-2,\lambda}) - t^2(t^{\lambda+1}F_{\lambda,\lambda+2})(t^{\lambda+2+p(\lambda+2)}E_{\lambda+2,\lambda}) = [\lambda]_{v,t}1_\lambda.$$

By comparing with the commutator relation for the modified quantum $\mathfrak{osp}(1|2)$ in [CW13, 6.3], we have (b). $\square$

Let $\mathbf{U}(\mathfrak{sl}(2))$ be the quantum algebra associated to $\mathfrak{sl}(2)$ defined over the field $\mathbb{Q}[t](v)$. To avoid any confusion, we shall denote by $\widetilde{E}, \widetilde{F}, \widetilde{K}^{\pm 1}$ the standard generators of $\mathbf{U}(\mathfrak{sl}(2))$. Recall that a $\mathbf{U}(\mathfrak{sl}(2))$-module M is called a *weight module of type 1* if there is a decomposition of vector spaces $M = \bigoplus_{\lambda\in\mathbb{Z}} M_\lambda$ such that $M_\lambda = \{m \in M \mid \widetilde{K}\cdot m = v^\lambda m\}$. Let $\mathcal{C}^+(\mathfrak{sl}(2))$ be the category whose objects are weight $\mathbf{U}(\mathfrak{sl}(2))$-modules of type 1 and morphisms are $\mathbf{U}(\mathfrak{sl}(2))$-linear maps. Similarly, we can define the category $\mathcal{C}^-(\mathfrak{sl}(2))$ of weight modules of type -1. By a similar argument, $\mathcal{C}^+(\mathfrak{sl}(2))$ is equivalent to the category of unital $\dot{\mathbf{U}}(\mathfrak{sl}(2))$-modules. By Theorem 5.4, we have

Proposition 5.5. *The category $\mathcal{C}^+$ is isomorphic to the category $\mathcal{C}^+(\mathfrak{sl}(2))$.*

5.6. By a similar argument as in the proof of Proposition 5.2, the category $\mathcal{C}^-$ is equivalent to $\mathcal{C}^-(\mathfrak{sl}(2))$. Let $\mathcal{C} = \mathcal{C}^+ \oplus \mathcal{C}^-$. Note that the highest weight simple modules $\Lambda_d^{\pm}$, for all $d \in \mathbb{N}$, are objects in $\mathcal{C}$. Let $\mathcal{C}(\mathfrak{sl}(2)) = \mathcal{C}^+(\mathfrak{sl}(2)) \oplus \mathcal{C}^-(\mathfrak{sl}(2))$. Then we have the following theorem.

Theorem 5.7. *The category $\mathcal{C}$ is isomorphic to $\mathcal{C}(\mathfrak{sl}(2))$.*

5.8. By Lemma 3.3, there is a unique surjective algebra homomorphism

$$\phi_d : \dot{\mathbf{U}} \to \mathbf{S}_{v,t}(2,d)$$

defined by

$$E_{\lambda,\lambda-2} \mapsto \begin{cases} t^{-\frac{d-p(d)}{2}}E_{r,r+1}, & \text{if } \lambda = d-2r, \\ 0, & \text{otherwise}, \end{cases} \quad 1_\lambda \mapsto \begin{cases} 1_r, & \text{if } \lambda = d-2r, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$F_{\lambda,\lambda+2} \mapsto \begin{cases} t^{-\frac{d-p(d)}{2}}F_{r,r-1}, & \text{if } \lambda = d-2r, \\ 0, & \text{otherwise}. \end{cases}$$

By checking the image of the generators, we have

$$\psi_{d,d+2}\phi_{d+2} = \phi_d,$$

where $\psi_{d,d+2}$ is (18) in Section 3.9.

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