

## ON CUBIC ELLIPTIC VARIETIES

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ABSTRACT. Let  $\pi: X \rightarrow \mathbb{P}^{n-1}$  be an elliptic fibration obtained by resolving the indeterminacy of the projection of a cubic hypersurface  $Y$  of  $\mathbb{P}^{n+1}$  from a line  $L$  not contained in  $Y$ . We prove that the Mordell-Weil group of  $\pi$  is finite if and only if the Cox ring of  $X$  is finitely generated. We also provide a presentation of the Cox ring of  $X$  when it is finitely generated.

### INTRODUCTION

Let  $\pi: X \rightarrow Z$  be an elliptic fibration between smooth projective complex varieties which admits a section. The generic fiber  $X_\eta$  of  $\pi$  is an elliptic curve over the function field of  $Z$  and its group of rational points (the Mordell-Weil group of  $\pi$ ) reflects into the geometry of  $X$ . It is thus interesting to explore the relation between the Mordell-Weil group of  $\pi$  and the Cox ring of the variety  $X$ . In this paper we focus on a class of elliptic fibrations defined by the linear system  $|- \frac{1}{n-1}K_X|$ , where  $X$  is the blowing-up of a smooth cubic  $n$ -dimensional hypersurface  $Y$  along its intersection points with a line  $L$ . Inspired by the recent work [Cos12], in Theorem 2.4 we determine the structure of the Mordell-Weil group of such fibrations in terms of local information about the intersection of  $L$  and  $Y$  (see Table 2.1). We rely on this result to prove the following.

**Theorem.** *Let  $\pi: X \rightarrow \mathbb{P}^{n-1}$  be an elliptic fibration, with  $n \geq 3$ , obtained by resolving the indeterminacy of the projection of a smooth cubic hypersurface from a line. Then the following are equivalent:*

- (1) *the Cox ring of  $X$  is finitely generated;*
- (2) *the Mordell-Weil group of the elliptic fibration  $\pi$  is finite.*

Moreover, when (1) and (2) hold we provide an explicit presentation for the Cox ring of  $X$  in Theorem 4.1.

Observe that if  $X$  is as in the statement of the preceding theorem,  $D$  is a general divisor in the linear system  $|-mK_X|$ , for  $m > 1$  and  $\Delta = \frac{1}{m}D$ , then  $(X, \Delta)$  is a *klt Calabi-Yau pair* (see [Cos12]). As a byproduct of the theorem we obtain that if the fibration  $X$  has finite Mordell-Weil group, then the Morrison-Kawamata cone conjecture for klt Calabi-Yau pairs holds.

The paper is structured as follows. In Section 1 we prove some facts about elliptic fibrations  $\pi: X \rightarrow \mathbb{P}^{n-1}$  with a section and such that  $-K_X$  is a multiple of the preimage of a general hyperplane of  $\mathbb{P}^{n-1}$ . In Section 2 we introduce a

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particular case of elliptic fibration, i.e., the blowing-up of a cubic hypersurface  $Y$  of  $\mathbb{P}^{n+1}$  along the intersection  $Y \cap L$  with a line not contained in  $Y$  and we state some general results about it. In Section 3 we study the nef and moving cones of these varieties and we finally prove that, for these cubic elliptic varieties, the finite generation of the Cox ring is equivalent to the finiteness of the Mordell-Weil group of  $\pi$ . Finally in Section 4 we give a presentation for the Cox ring of  $X$  when it is finitely generated.

### 1. ELLIPTIC FIBRATIONS

Let  $X$  be a smooth projective variety and let  $\pi: X \rightarrow \mathbb{P}^{n-1}$  be an *elliptic fibration*, that is, a general fiber of  $\pi$  is a smooth irreducible curve of genus one.

**Definition 1.1.** The fibration  $\pi$  is *jacobian* if it admits a rational section. If this is the case, the *Mordell-Weil group* of  $\pi$  is the group of rational sections  $\sigma: \mathbb{P}^{n-1} \dashrightarrow X$ , that is,

$$\text{MW}(\pi) := \{ \sigma: \mathbb{P}^{n-1} \dashrightarrow X : \pi \circ \sigma = \text{id} \}.$$

We say that the fibration  $\pi: X \rightarrow \mathbb{P}^{n-1}$  is *extremal* if its Mordell-Weil group is finite. Moreover, we say that  $\pi$  is *relatively minimal* if, for a general line  $R$  of  $\mathbb{P}^{n-1}$ , the restriction of  $\pi$  to the elliptic surface  $S = \pi^{-1}(R)$  does not contract  $(-1)$ -curves.

Observe that by the Riemann-Roch theorem the set of rational sections of  $\pi$  is in bijection with the group  $\text{Pic}^0(X_\eta)$ , where  $X_\eta$  is the generic fiber of  $\pi$ .

**Proposition 1.2.** *Let  $\pi: X \rightarrow \mathbb{P}^{n-1}$  be a jacobian elliptic fibration and assume that  $K_X$  is linearly equivalent to  $\alpha \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ , where  $\alpha$  is a rational number. Then  $\pi$  is relatively minimal and  $\alpha$  is integer. Moreover, if  $S$  is the preimage of a general line of  $\mathbb{P}^{n-1}$ , then the following are equivalent:*

- (1)  $S$  is a rational surface;
- (2)  $\alpha = 1 - n$ .

*Proof.* Consider a general flag of linear subspaces of  $\mathbb{P}^{n-1}$ . The corresponding preimages via  $\pi$  give a flag of subvarieties  $F_i$  of  $X$

$$X \supset F_{n-1} \supset \cdots \supset F_2 = S \supset F_1 = f,$$

where  $\dim F_i = i$  for any  $i$  and  $f$  is a fiber of  $\pi$ . Observe that each  $F_i$  is smooth by applying inductively Bertini's second theorem [Ber] since each  $F_i$  is the general element of a spanned linear system on  $F_{i+1}$ . By hypothesis and the adjunction formula we get  $K_S \sim (n - 2 + \alpha)f$ . If  $C$  is a  $(-1)$ -curve of  $S$ , then  $-1 = C \cdot K_S = C \cdot (n - 2 + \alpha)f$  implies that  $C$  cannot be contained in a fiber of  $\pi$ , so that  $\pi$  is relatively minimal. Moreover, observe that given a section  $\sigma$ , the curve  $\Gamma = \sigma(\mathbb{P}^{n-1}) \cap S$  is a section of  $\pi|_S$ , so that  $\Gamma \cdot f = 1$ . Hence  $n - 2 + \alpha = \Gamma \cdot K_S$  is integer so that  $\alpha$  is integer too.

(1)  $\Rightarrow$  (2). Since  $S$  is a rational surface and  $K_S \sim (n - 2 + \alpha)f$ , we have  $\alpha \leq 1 - n$  and in particular  $\Gamma \cdot K_S < 0$ . Observe that the divisor  $K_S - \Gamma$  cannot be linearly equivalent to an effective divisor since  $(K_S - \Gamma) \cdot f = -1$ . Hence  $h^2(S, \Gamma) = 0$ , by Serre's duality. Moreover, since  $\Gamma$  is a section of  $\pi|_S$  we have  $h^0(S, \Gamma) = 1$ . Hence by Riemann-Roch

$$1 = h^0(S, \Gamma) \geq \chi(S, \Gamma) = \frac{\Gamma^2 - \Gamma \cdot K_S}{2} + 1$$

which implies  $\Gamma^2 \leq \Gamma \cdot K_S < 0$ . Thus  $\Gamma$  is a  $(-1)$ -curve and in particular  $n - 2 + \alpha = \Gamma \cdot K_S = -1$  giving  $\alpha = 1 - n$ .

(2)  $\Rightarrow$  (1). Since  $\alpha = 1 - n$ , we have  $K_S \sim -f$ , so that  $S$  has negative Kodaira dimension. By the classification theory of surfaces  $S$  is either rational or the blowing-up of a ruled surface. Since  $K_S^2 = 0$ , by [Har77, Corollary V.2.11] we conclude that  $S$  is rational.  $\square$

**Proposition 1.3.** *Let  $\pi: X \rightarrow \mathbb{P}^{n-1}$  be a jacobian elliptic fibration and assume that  $K_X$  is linearly equivalent to a negative multiple of the pull-back of  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ . Then any nef effective divisor of  $X$  is semiample.*

*Proof.* Let  $D$  be a nef effective divisor of  $X$ . Since both  $D$  and  $-K_X$  are nef,  $D - K_X$  is nef. If  $D - K_X$  is also big, then  $D$  is semiample by the Kawamata-Shokurov base point free theorem (see [KMM87] and [Sho86]). If  $D - K_X$  is not big, then  $(D - K_X)^n = 0$  and in particular  $D \cdot (-K_X)^{n-1} = 0$ . By hypothesis  $-(K_X)^{n-1}$  is rationally equivalent to a positive multiple of a fiber of  $\pi$ . Hence, since  $D$  is effective, its support is the preimage of a hypersurface of  $\mathbb{P}^{n-1}$ .

We conclude by showing that  $D$  is the pull-back of a divisor of  $\mathbb{P}^{n-1}$ , so that it is semiample. Indeed if this is not the case, let  $S$  be the preimage of a general line of  $\mathbb{P}^{n-1}$ . Then  $(D|_S)^2 < 0$ , by [BHPVdV04, Lemma III.8.2], a contradiction since  $D$  is nef.  $\square$

## 2. GENERALITIES ON CUBIC ELLIPTIC VARIETIES

From now on we will concentrate on the case in which  $X$  is obtained from a cubic hypersurface  $Y$  of  $\mathbb{P}^{n+1}$  by resolving the indeterminacy of the projection map from a line  $L$  not contained in  $Y$ . Therefore, the variety  $X$  comes with two morphisms:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \mathbb{P}^{n-1} \\ \sigma \downarrow & & \\ Y \subset \mathbb{P}^{n+1} & & \end{array}$$

where  $\pi$  is the elliptic fibration while  $\sigma$  is the resolution of the indeterminacy. Observe that the fibers of  $\pi$  are the strict transforms of the plane cubics cut out on  $Y$  by planes containing  $L$ .

*Remark 2.1.* The birational morphism  $\sigma$  is a composition of three blowing-ups

$$X \xrightarrow{\sigma_3} Y_2 \xrightarrow{\sigma_2} Y_1 \xrightarrow{\sigma_1} Y$$

$\sigma$  (curved arrow from  $X$  to  $Y$ )

at the points  $p_1, p_2, p_3$ . There are three possibilities (modulo a relabelling of the three points):

- (1) the points  $p_2$  and  $p_3$  do not lie on the exceptional divisors;
- (2)  $p_2$  lies on the exceptional divisor of  $\sigma_1$ , while  $p_3$  does not lie on any exceptional divisor;
- (3)  $p_2$  lies on the exceptional divisor of  $\sigma_1$  and  $p_3$  on that of  $\sigma_2$ .

In what follows we denote by  $H$  the pull-back of a hyperplane of  $Y$  and by  $E_i$  the pull-back of the exceptional divisor of  $\sigma_i$ , for  $i \in \{1, 2, 3\}$ . In case (1) each  $E_i$  is isomorphic to  $\mathbb{P}^{n-1}$ . In case (2) the prime divisor  $E_1 - E_2$  is isomorphic to the projectivization  $\mathbb{F}$  of the vector bundle  $\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)$  while  $E_2$  and  $E_3$  are both

isomorphic to  $\mathbb{P}^{n-1}$ . Finally in case (3) the prime divisors  $E_1 - E_2$  and  $E_2 - E_3$  are isomorphic to  $\mathbb{F}$  while  $E_3$  is isomorphic to  $\mathbb{P}^{n-1}$ . In each case

$$\text{Pic}(X) = \langle H, E_1, E_2, E_3 \rangle,$$

where, with abuse of notation, we are adopting the same symbols for the divisors and for their classes.

**2.1. Cubic elliptic varieties.** Let us recall the following definition (see [CC10]):

**Definition 2.2.** Given a hypersurface  $Y$  of  $\mathbb{P}^{n+1}$  of degree  $d$ , a smooth point  $p$  of  $Y$  is said to be a *star point* if  $\mathbb{T}_p Y \cap Y$  has multiplicity  $d$  at  $p$ .

Let us consider now the local study of a cubic  $Y \subset \mathbb{P}^{n+1}$  at a smooth point  $p$ . In what follows we denote by  $T_1, \dots, T_{n+2}$  the coordinates of  $\mathbb{P}^{n+1}$ . After applying a linear change of coordinates we can assume  $p = (0 : \dots : 0 : 1)$  and the equation of the tangent space to  $Y$  at  $p$  to be  $T_{n+1} = 0$ . Hence a defining equation for  $Y$  is

$$(2.1) \quad T_{n+1} a + T_{n+2} b + c = 0,$$

where  $a$  is a degree two homogeneous polynomial while  $b$  and  $c$  are homogeneous polynomials of  $\mathbb{C}[T_1, \dots, T_n]$  of degrees two and three respectively. Observe that  $c$  cannot be the zero polynomial, since otherwise  $Y$  would contain the linear space  $V(T_{n+1}, T_{n+2})$  and hence it would be singular.

Observe that any line  $R$  of  $Y$  through  $p$  is contained in the tangent space  $\mathbb{T}_p Y$  and so it is contained in the intersection of the two cones  $V(b) \cap V(c)$ .

**Proposition 2.3.** *Let  $Y$  be a smooth cubic hypersurface of  $\mathbb{P}^{n+1}$  and let  $p$  be a point of  $Y$ . Assume that a local equation of  $Y$  at  $p$  is (2.1). Then the following properties hold:*

- (1)  $p$  is a star point of  $Y$  if and only if  $b$  is the zero polynomial;
- (2) if  $p$  is not a star point, then there is an  $(n - 3)$ -dimensional family of lines of  $Y$  passing through it;
- (3) a line through two star points of  $Y$ , intersects  $Y$  at a third star point.

*Proof.* Point (2) is an immediate consequence of our previous discussion, while (1) follows by observing that a general line tangent to  $Y$  at  $p$  has parametric equation

$$u(0, \dots, 0, 1) + v(t_1, \dots, t_n, 0, t_{n+2}),$$

where the  $t_i$  are general complex numbers. By substituting in (2.1) it follows that the left hand side is a cubic polynomial in  $u$  and  $v$  and it has a root of multiplicity three for any choice of the  $t_i$  if and only if  $b$  vanishes identically.

To prove (3) first consider the case when  $L$  is tangent to  $Y$  at the star point  $p_1$ . Then  $L$  intersects  $Y$  at  $p_1$  with multiplicity three by definition of star point. Assume now that  $L$  intersects  $Y$  at three distinct points  $p_1, p_2, p_3$  such that  $p_1$  and  $p_2$  are star points. After a linear change of coordinates we can assume  $p_1 = (0 : \dots : 0 : 1)$  with  $\mathbb{T}_{p_1} Y$  of equation  $T_{n+1} = 0$  and  $p_2 = (0 : \dots : 0 : 1 : 0)$  with  $\mathbb{T}_{p_2} Y$  of equation  $T_{n+2} = 0$ . Using equation (2.1) and (1) we get that a defining equation for  $Y$  is

$$T_{n+1} T_{n+2} \ell + c = 0,$$

where  $\ell$  is a linear form. Hence  $p_3 = (0 : \dots : 0 : \alpha : \beta)$ , where  $\ell(p_3) = 0$ . The fact that  $p_3$  is a star point follows immediately from the previous equation for  $Y$ , being  $\ell = 0$  the equation of  $\mathbb{T}_{p_3} Y$ . □

As a consequence of this result and of Remark 2.1 we have that there are seven different possibilities concerning the points  $L \cap Y$ . We are now going to construct a table in which we list the seven types of cubic elliptic varieties we can obtain (see Table 2.1). In the first column we write the type of the variety using a symbol that records which points we are blowing up and in which order. For example if  $X$  is a blowing-up at three distinct non-star points, then we will denote it by  $X_{111}$ , while if  $X$  is a blowing-up of one star point and two non-star infinitely near points we will denote it by  $X_{S2}$ . The second column contains the defining equations of  $Y$  and the line  $L$  while the third column is for the Mordell-Weil groups of the elliptic fibrations.

TABLE 2.1. The seven types of cubic elliptic varieties

Type	Defining equations for $Y$ and $L$	Mordell-Weil group
$X_3$	$T_{n+1}(a' + T_{n+2} a_1) + T_{n+2} b' + b_1 = 0$ $T_1 = \dots = T_{n-1} = T_{n+1} = 0$	$\langle 0 \rangle$
$X_S$	$T_{n+1} a_2 + b_2 = 0$ $T_1 = \dots = T_{n-1} = T_{n+1} = 0$	$\langle 0 \rangle$
$X_{S2}$	$T_{n+1} a_3 + b_3 = 0$ $T_1 = \dots = T_n = 0$	$\mathbb{Z}/2\mathbb{Z}$
$X_{SSS}$	$T_{n+1} T_{n+2} a_4 + b_4 = 0$ $T_1 = \dots = T_n = 0$	$\mathbb{Z}/3\mathbb{Z}$
$X_{12}$	$T_{n+1} a_5 + T_{n+2} b_5 + c_5 = 0$ $T_1 = \dots = T_{n-1} = T_{n+1} = 0$	$\mathbb{Z}$
$X_{S11}$	$T_{n+1} a_6 + b_6 = 0$ $T_1 = \dots = T_n = 0$	$\mathbb{Z}$
$X_{111}$	$T_{n+1} a_7 + T_{n+2} b_7 + c_7 = 0$ $T_1 = \dots = T_n = 0$	$\mathbb{Z} \oplus \mathbb{Z}$

The polynomials appearing in the table satisfy the following conditions:  $b', b_i, c_i \in \mathbb{C}[T_1, \dots, T_n]$ ,  $a' \in \mathbb{C}[T_1, \dots, T_{n+1}]$ ,  $a_i \in \mathbb{C}[T_1, \dots, T_{n+2}]$ ; moreover,  $b'$  does not contain  $T_n^2$  and  $a_3$  does not contain  $T_{n+1}^2$  and  $T_{n+1} T_{n+2}$ . The equations appearing in the table can be obtained from (2.1) with a case by case study of the tangency conditions at the points of  $L \cap Y$  (as we did in the proof of Proposition 2.3 for  $X_{SSS}$ ).

**2.2. Mordell-Weil groups.** Recall that the Mordell-Weil group of the elliptic fibration  $\pi$  is the group of rational sections of  $\pi$  or equivalently the group of  $K = \mathbb{C}(\mathbb{P}^{n-1})$ -rational points  $X_\eta(K)$  of the generic fiber  $X_\eta$  of  $\pi$  once we choose one of such points  $O$  as an origin for the group law. Let  $\mathcal{S}$  be the subgroup of  $\text{Pic}(X)$  generated by the classes of *vertical divisors*, that is, divisors mapped to hypersurfaces by  $\pi$ , and by the class of the section  $O$ . There is an exact sequence [Waz04, Section 3.3]:

$$(2.2) \quad 0 \longrightarrow \mathcal{S} \longrightarrow \text{Pic}(X) \longrightarrow X_\eta(K) \longrightarrow 0.$$

**Theorem 2.4.** *The Mordell-Weil group of the elliptic fibration for each type in Table 2.1 is the one given in the third column.*

*Proof.* Let  $X$  be one of the cubic elliptic varieties appearing in the first column of Table 2.1. As already observed in Section 2, the Picard group of  $X$  is free of rank four and is generated by the classes of  $H, E_1, E_2, E_3$ . Observe that since  $p_3$  is the last point that we blow up,  $E_3$  gives a section of the elliptic fibration  $\pi$  so that from now on we take  $O = E_3$ . The subgroup  $\mathcal{T}$  has rank at least two, since it contains the subgroup

$$\mathcal{L} = \langle H - E_1 - E_2 - E_3, E_3 \rangle,$$

where the first class is that of the pull-back of  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ . Hence by (2.2) the Mordell-Weil group of  $X$  has rank at most two.

Consider now a prime vertical divisor  $D$  of  $\pi$ . By identifying  $D$  with its support we have that  $\pi(D)$  is a hypersurface  $B$  of  $\mathbb{P}^{n-1}$ . If  $D$  equals the pull-back  $\pi^*B$ , then it is linearly equivalent to a multiple of  $H - E_1 - E_2 - E_3$ . If not, then any fiber  $\Gamma = \pi^{-1}(q)$  over a point  $q$  of  $B$  is reducible and has a component contained in  $D$ . There are two possibilities for the curve  $C = \sigma(\Gamma)$ , where  $\sigma: X \rightarrow Y$  is the blowing-up map:

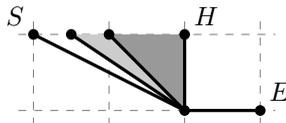
- (1)  $C$  is a reducible cubic curve;
- (2)  $C$  is an irreducible singular cubic curve, with singular point at one of the points of  $L \cap Y$ .

In the first case  $C$  must contain a line, so that one of the points  $p$  of  $L \cap Y$  is a star point and, denoting by  $E$  the corresponding exceptional divisor, one of the irreducible components of  $\pi^*B$  is linearly equivalent to  $H - 3E$ . This shows that  $\mathcal{T} = \mathcal{L}$  for  $X_{111}$ , that  $\mathcal{T} = \mathcal{L} + \langle H - 3E_1 \rangle$  for  $X_{S11}$  and that  $\mathcal{T} = \mathcal{L} + \langle H - 3E_1, H - 3E_2 \rangle$  for  $X_{SSS}$ .

In the second case  $L$  is tangent to  $Y$  at a point  $p$  of  $L \cap Y$ . The elliptic fibrations corresponding to the varieties  $X_{12}, X_3$  and  $X_S$  belong to this case. We have  $\mathcal{T} = \mathcal{L} + \langle E_1 - E_2 \rangle$  for  $X_{12}$  and  $\mathcal{T} = \mathcal{L} + \langle E_1 - E_2, E_2 - E_3 \rangle$  for  $X_3$  and  $X_S$  respectively.

Finally  $X_{S2}$  belongs to both cases and we have  $\mathcal{T} = \mathcal{L} + \langle H - 3E_1, E_2 - E_3 \rangle$ . We conclude by observing that the Mordell-Weil group of each such elliptic fibration is isomorphic to  $\text{Pic}(X)/\mathcal{T}$ . □

**2.3. A flop.** In this subsection we study a flop image of the blowing-up  $Y_1$  of a smooth cubic hypersurface  $Y$  of  $\mathbb{P}^{n-1}$  at a non-star point  $p_1$ . The Picard group of  $Y_1$  is free of rank two generated by the classes of the exceptional divisor  $E$  and the pull-back  $H$  of a hyperplane section of  $Y$ . Inside  $\text{Pic}(Y_1) \otimes_{\mathbb{Z}} \mathbb{Q}$  we have the following cones:



The cone generated by the classes of  $H$  and  $H - E$  is the nef cone of  $Y_1$ , while the moving cone is generated by the classes of  $H$  and  $H - \frac{3}{2}E$ . To prove this consider the birational map

$$(2.3) \quad \psi: Y \dashrightarrow Y \quad q \mapsto (\text{line}(p, q) \cap Y) - \{p, q\}.$$

Denote by  $\psi_1: Y_1 \dashrightarrow Y_1$  the lift of  $\psi$  to  $Y_1$ . Let  $V$  be the strict transform of the union of lines of  $Y$  through  $p$ . Since  $\psi_1$  is an involution whose indeterminacy locus is  $V$  and  $V$  has codimension two in  $Y_1$ ,  $\psi_1$  is an isomorphism in codimension one. In particular it induces by pull-back an isomorphism  $\psi_1^*$  on the Picard group of  $Y_1$ . To calculate the representative matrix of  $\psi_1^*$  with respect to the basis  $(H, E)$ , observe that  $\psi$  maps points of the strict transform of  $\mathbb{T}_p Y \cap Y$  to points of the exceptional divisor  $E$  and vice versa. The first divisor is linearly equivalent to  $H - 2E$ . Hence the representative matrix for  $\psi_1^*$  is:

$$\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}.$$

The previous matrix explains the  $\mathbb{Z}/2\mathbb{Z}$ -symmetry of the moving and effective cones of  $Y_1$ . If we blow up a set of points  $Q$  on  $Y_1$ , then  $\psi_1$  lifts to a birational map which is an isomorphism in codimension one if and only if  $\psi_1(Q) = Q$ . This is exactly what happens for the cubic elliptic varieties  $X_3$  and  $X_{S2}$ . In the first case each point is fixed by  $\psi_1$ , while in the second case the points  $p_2$  and  $p_3$  are exchanged. This implies the following.

**Proposition 2.5.** *Let  $X$  be a cubic elliptic variety of type  $X_3$ , or  $X_{S2}$ , and let  $\varphi: X \rightarrow X$  be the flop induced by (2.3). Then the action of  $\varphi^*$  on  $\text{Pic}(X)$  with respect to the basis  $(H, E_1, E_2, E_3)$  is described by the following two matrices respectively:*

$$M_3 := \begin{pmatrix} 2 & 1 & 0 & 0 \\ -3 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{S2} := \begin{pmatrix} 2 & 1 & 0 & 0 \\ -3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

### 3. NEF AND MOVING CONES

As a general reference about the cones discussed in this section see [Laz04].

**Construction 3.1.** In what follows we will write the classes of  $\text{Pic}(X)$  with respect to the basis  $(H, E_1, E_2, E_3)$ . Recall that  $N_1(X)$  is the group of numerical equivalence classes of irreducible and reduced one-cycles. We fix the basis  $(h, e_1, e_2, e_3)$  of  $N_1(X)$  such that the intersection pairing  $\text{Pic}(X) \times N_1(X) \rightarrow \mathbb{Z}$  in these coordinates is given by

$$((a, a_1, a_2, a_3), (b, b_1, b_2, b_3)) \mapsto ab - a_1b_1 - a_2b_2 - a_3b_3.$$

Observe that  $h$  is the class of the pull-back of a line of  $Y$  and  $e_3$  is the class of a line in the exceptional divisor  $E_3$ . The geometric interpretation of the remaining elements is the following. If we are blowing-up one point of  $Y$  (cases  $X_3, X_S$ )  $e_2 - e_3$  and  $e_1 - e_2$  are fibers of the  $\mathbb{P}^1$ -bundles  $E_2 - E_3$  and  $E_1 - E_2$  respectively. If we are blowing-up two points of  $Y$  (cases  $X_{12}, X_{S2}$ )  $e_2$  is the class of a line in the exceptional divisor  $E_2$  while  $e_1 - e_2$  is a fiber of the  $\mathbb{P}^1$ -bundle  $E_1 - E_2$ . If we are blowing-up three points of  $Y$  (cases  $X_{111}, X_{S11}, X_{SSS}$ ) each  $e_i$  is the class of a line in the exceptional divisor  $E_i$ .

**3.1. Nef cones.** Let us compute now the nef cones of the cubic elliptic varieties of Table 2.1. In each case we will proceed as follows. We take some classes of nef divisors and we consider the cone  $N$  they span. Since the nef cone of  $X$  is the dual of the Mori cone  $\text{NE}(X) \subset N_1(X)_\mathbb{Q} := N_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$  of  $X$  and  $N$  is contained in

the former, we deduce that the dual  $N^*$  contains  $\text{NE}(X)$ . We conclude by proving that the classes which generate  $N^*$  are indeed classes of effective curves and hence  $\text{NE}(X) = N^*$  so that  $\text{Nef}(X) = N$ .

**Proposition 3.2.** *Let  $X$  be one of the cubic elliptic varieties of Table 2.1. Then the nef cone of  $X$  is generated by the semiample classes whose coordinates with respect to the basis  $(H, E_1, E_2, E_3)$  of  $\text{Pic}(X)$  are the columns of the corresponding matrix in Table 3.1.*

TABLE 3.1

Type	Generators of the nef cone
$X_3, X_S$	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$
$X_{12}, X_{S2}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$
$X_{111}, X_{S11}, X_{SSS}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix}$

*Proof.* First of all observe that all the columns of the previous matrices are degrees of nef divisors (indeed semiamples) since the class of  $F = H - E_1 - E_2 - E_3$  is semiample being the pull-back of  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$  and all the remaining columns are of the form  $\gamma^* \gamma_* F$  for some birational morphism  $\gamma$  which is a composition of some of the contractions  $\sigma_i$ .

We conclude by showing that the dual of each cone generated by the columns of the given matrices is contained in the Mori cone of  $X$ , that is, consists of classes of effective curves. In the first case the dual cone is generated by the following classes:  $e_1 - e_2, e_2 - e_3, h - e_1, e_3$ , where  $h - e_1$  is the class of the strict transform of a line through the first point. In the second case the dual cone is generated by the classes  $e_1 - e_2, e_2, e_3, h - e_1, h - e_3$ , while in the third case it is generated by the classes  $e_1, e_2, e_3$  and  $h - e_1, h - e_2, h - e_3$ . □

**3.2. Moving cones.** We are now going to study the moving cones of the first four cubic elliptic varieties appearing in Table 2.1.

**Proposition 3.3.** *If  $X$  is of type  $X_S$  or  $X_{SSS}$ , then  $\text{Mov}(X) = \text{Nef}(X)$ .*

*Proof.* Let  $D$  be an effective divisor whose class does not lie in  $\text{Nef}(X)$ . Hence  $D \cdot C < 0$  for some curve  $C$  which spans an extremal ray of the Mori cone of  $X$ . We claim that the curves numerically equivalent to any such  $C$  span a divisor. Since this divisor must be contained into the stable base locus of  $D$  we get that the class of  $D$  does not belong to  $\text{Mov}(X)$  and this, together with the inclusion  $\text{Nef}(X) \subset \text{Mov}(X)$ , gives the thesis.

By the proof of Proposition 3.2 the Mori cone of a variety of type  $X_S$  is generated by the following effective classes:  $e_1 - e_2, e_2 - e_3, h - e_1, e_3$ . The curves numerically equivalent to these classes span respectively  $E_1 - E_2, E_2 - E_3$ , the strict transform of the cubic cone  $\mathbb{T}_{p_1}Y \cap Y$  and the exceptional divisor  $E_3$ .

The Mori cone of a variety of type  $X_{SSS}$  is generated by the following effective classes:  $e_i$  and  $h - e_i$ , for  $i \in \{1, 2, 3\}$ . In these cases we obtain the divisors  $E_i$  and the strict transforms of the cubic cones  $\mathbb{T}_{p_i}Y \cap Y$  for any  $i \in \{1, 2, 3\}$ , respectively.  $\square$

**Proposition 3.4.** *For any cubic elliptic variety  $X$  of type  $X_3$  or  $X_{S_2}$ , the moving cone is  $\text{Mov}(X) = \text{Nef}(X) \cup \varphi^* \text{Nef}(X)$ , where  $\varphi$  is the flop of  $X$  described in Proposition 2.5.*

*Proof.* Observe that the curves numerically equivalent to one of the generators of the Mori cone of  $X$  span either a divisor or a variety of codimension two. For both types  $X_3$  and  $X_{S_2}$  the only class which spans a variety of codimension two is  $h - e_1$ . Let  $X$  be of type  $X_3$  and consider the following cone of  $N_1(X)_{\mathbb{Q}}$ :

$$(3.1) \quad \text{Cone}(e_2 - e_3, e_3, 3h - 2e_1 - e_2, e_1 - h).$$

We claim that if  $D$  is a movable non-nef class of  $X$ , then it belongs to the dual of this cone. First of all, since  $D$  is not nef, it has negative intersection with  $h - e_1$ . The curves numerically equivalent to one of the first two classes span divisors of  $X$ . The same holds for the curves equivalent to  $3h - 2e_1 - e_2$ . Indeed consider the divisor linearly equivalent to  $\pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ ,

$$\pi^*(\pi_*(E_1 - E_2)) = (E_1 - E_2) + (E_2 - E_3) + V,$$

where  $V$  is the strict transform of the hyperplane section  $\mathbb{T}_{p_1}Y \cap Y$ . The fiber over a point of  $\pi(E_1 - E_2)$  has a component in  $V$  whose class is  $3h - 2e_1 - e_2$  since  $3h - e_1 - e_2 - e_3 = (e_1 - e_2) + (e_2 - e_3) + (3h - 2e_1 - e_2)$ . This proves the claim. To conclude we observe that the dual of the cone of (3.1) is  $\varphi^* \text{Nef}(X)$  and thus it is generated by movable classes.

Let  $X$  be of type  $X_{S_2}$  and consider the following cone of  $N_1(X)_{\mathbb{Q}}$ :

$$(3.2) \quad \text{Cone}(e_2, e_3, 2h - e_1 - e_2, 3h - 2e_1 - e_3, e_1 - h).$$

As before we claim that if  $D$  is a movable non-nef class of  $X$ , then it belongs to the dual of this cone. First of all, since  $D$  is not nef, it has negative intersection with  $h - e_1$ . The curves numerically equivalent to one of the first two classes span divisors of  $X$ . Concerning the third class, observe that the class of a fiber of  $\pi$  is  $3h - e_1 - e_2 - e_3$  and its push-forward in  $Y$  is the class of the plane cubic obtained intersecting  $Y$  with a plane  $\Pi$  containing the line  $L$ . If we take the plane  $\Pi$  to be tangent to  $Y$  at the star point  $p_3$ , then the cubic splits as the union of the line and the conic corresponding to  $h - e_3$  and  $2h - e_1 - e_2$  respectively. When  $\Pi$  moves, the curves equivalent to  $2h - e_1 - e_2$  span a prime vertical divisor. If we now take a plane  $\Pi$  tangent to  $Y$  at  $p_1$ , the fiber decomposes as the sum of a curve in  $e_1 - e_2$  and one in  $3h - 2e_1 - e_3$ . As before, if we let  $\Pi$  move, the curves equivalent to  $3h - 2e_1 - e_3$  span a prime vertical divisor. Since  $D$  is movable, it must have non-negative intersection with the first four classes. This proves the claim. To conclude we observe that the dual of the cone of (3.2) is  $\varphi^* \text{Nef}(X)$  and thus it is generated by movable classes.  $\square$

**3.3. Finitely generated Cox rings.** Recall that a  $\mathbb{Q}$ -factorial projective variety is *Mori dream* if its Cox ring is finitely generated [HK00]. We conclude the section by showing which cubic elliptic varieties appearing in Table 2.1 are Mori dream.

**Lemma 3.5.** *Let  $X$  be a complex normal variety with finitely generated class group and  $\Gamma(X, \mathcal{O}^*) = \mathbb{C}^*$ . Let  $\pi: X \rightarrow Z$  be a jacobian elliptic fibration. If the Cox ring of  $X$  is finitely generated, then the Mordell-Weil group of  $\pi$  is finite.*

*Proof.* First of all observe that the hypotheses on  $X$  are needed just to define its Cox ring [ADHL15]. Let  $X_\eta$  be the generic fiber of  $\pi$ , let  $\sigma: Z \dashrightarrow X$  be a rational section of  $\pi$  and let  $E$  be the closure of  $\sigma(Z)$  in  $X$ . The Riemann-Roch space  $H^0(X, E)$  is one dimensional since  $E$  is effective and it cannot move in a linear series because it corresponds to a point on the elliptic curve  $X_\eta$ . Since  $E$  is irreducible and reduced, any set of homogeneous generators of the Cox ring of  $X$  must contain a basis of  $H^0(X, E)$ . Thus the Mordell-Weil group of  $\pi$  must be finite. □

**Theorem 3.6.** *Let  $X$  be one of the cubic elliptic varieties of Table 2.1. Then the following are equivalent:*

- (1) *the Cox ring of  $X$  is finitely generated;*
- (2) *the Mordell-Weil group of  $\pi: X \rightarrow \mathbb{P}^{n-1}$  is finite.*

*Proof.* (1)  $\Rightarrow$  (2). Follows from Lemma 3.5.

(2)  $\Rightarrow$  (1). Since the Mordell-Weil group of  $\pi$  is finite, looking at Table 2.1 we have that  $X$  must be either  $X_3, X_S, X_{S_2}$  or  $X_{SSS}$ . In each of these cases we are going to use [HK00], showing that the moving cone  $\text{Mov}(X)$  is the union of finitely many polyhedral chambers, each of which is pull-back via a small  $\mathbb{Q}$ -factorial modification  $\phi: X \rightarrow X_i$  of  $\text{Nef}(X_i)$ , the last being generated by a finite number of semiample classes.

In the cases  $X_{SSS}$  and  $X_S$  the moving cone  $\text{Mov}(X) = \text{Nef}(X)$  by Proposition 3.3. In cases  $X_3$  and  $X_{S_2}$ , by Proposition 3.4 the moving cone  $\text{Mov}(X)$  is the union of the two polyhedral chambers  $\text{Nef}(X)$  and  $\varphi^* \text{Nef}(X)$ , where  $\varphi: X \rightarrow X$  is the small  $\mathbb{Q}$ -factorial modification defined in Proposition 2.5. In all the cases we conclude by Proposition 3.2. □

*Remark 3.7.* Theorem 3.6 is the converse of Lemma 3.5 for the cubic elliptic varieties of Table 2.1. The converse of the lemma is not true in general: given a jacobian elliptic fibration  $X \rightarrow Z$ , with finite Mordell-Weil group and  $Z$  Mori dream, the variety  $X$  is not necessarily Mori dream.

For example, consider the lattice  $\Lambda = U \oplus 3A_1 \oplus A_2$ . Since  $\Lambda$  is an even hyperbolic lattice of rank  $7 \leq 10$ , it embeds into the K3 lattice by [Nik79]. Thus by the global Torelli theorem there exists a K3 surface  $X$  whose Picard lattice is isometric to  $\Lambda$ . We observe that the surface  $X$  admits a jacobian elliptic fibration with finite (indeed trivial) Mordell-Weil group by [Shi00, Table 1, n.19]. Moreover, the automorphism group of  $X$  is infinite since the lattice  $\Lambda$  is not 2-elementary and does not appear in the list of [Dol83, Theorem 2.2.2]. Hence  $X$  is not Mori dream by [AHL10, Theorem 2.7, Theorem 2.11].

We could not find an example of a variety  $X$  which admits a unique jacobian elliptic fibration  $X \rightarrow Z$  with finite Mordell-Weil group,  $Z$  Mori dream and such that  $X$  is not Mori dream as well.

4. COX RINGS

In this section we provide a presentation for the Cox rings of the cubic elliptic varieties of type  $X_3$ ,  $X_S$ ,  $X_{S_2}$  and  $X_{SSS}$ . Without loss of generality we can assume that the defining polynomial of a smooth cubic hypersurface is one of the polynomials listed in Table 2.1. We recall here the geometric meaning of the given homogeneous generators of the Cox ring appearing in Theorem 4.1 for varieties of type  $X_3$ , i.e.,  $\{p\} = L \cap Y$  (the other cases admitting a similar interpretation). The generators  $T_1, \dots, T_{n-1}$  and  $T_{n+1}S_1S_2$  are pull-backs of homogeneous coordinates of  $\mathbb{P}^{n-1}$ , where  $T_{n+1}$  corresponds to the strict transform of  $\mathbb{T}_pY \cap Y$ . The generator  $T_n$  is a section corresponding to a hyperplane containing the point  $p$  but not the line  $L$ , while  $T_{n+2}$  corresponds to a hyperplane not containing the point  $p$ . Finally the generator  $T_{n+3}$  is a flop image of  $T_{n+1}$  and  $S_1, S_2, S_3$  correspond to the three exceptional divisors.

We now define four homomorphisms of rings which will be used in Theorem 4.1 (see Table 4.1). These are defined in such a way that the value  $\beta_i(T_j)$  is the section corresponding to the strict transform of  $T_j$  via the blowing-up map  $\sigma : X \rightarrow Y$ .

TABLE 4.1

Homomorphism	Defined by
$\beta_1 : \mathbb{C}[T_1, \dots, T_{n+3}] \rightarrow \mathbb{C}[T_1, \dots, T_{n+3}, S_1, S_2, S_3]$	$T_k \mapsto T_k S_1 S_2^2 S_3^3$
	$T_n \mapsto T_n S_1 S_2 S_3$
	$T_{n+1} \mapsto T_{n+1} S_1^3 S_2^3 S_3^3$
	$T_{n+2} \mapsto T_{n+2}$
	$T_{n+3} \mapsto T_{n+3} S_1^3 S_2^3 S_3^3$
$\beta_2 : \mathbb{C}[T_1, \dots, T_{n+2}] \rightarrow \mathbb{C}[T_1, \dots, T_{n+2}, S_1, S_2, S_3]$	$T_k \mapsto T_k S_1 S_2^2 S_3^3$
	$T_n \mapsto T_n S_1 S_2 S_3$
	$T_{n+1} \mapsto T_{n+1} S_1^3 S_2^3 S_3^3$
	$T_{n+2} \mapsto T_{n+2}$
$\beta_3 : \mathbb{C}[T_1, \dots, T_{n+3}] \rightarrow \mathbb{C}[T_1, \dots, T_{n+3}, S_1, S_2, S_3]$	$T_k \mapsto T_k S_1 S_2^2 S_3$
	$T_n \mapsto T_n S_1^2 S_2^2 S_3$
	$T_{n+1} \mapsto T_{n+1} S_3^3$
	$T_{n+2} \mapsto T_{n+2} S_1 S_2$
	$T_{n+3} \mapsto T_{n+3} S_1^3 S_2^6$
$\beta_4 : \mathbb{C}[T_1, \dots, T_{n+3}] \rightarrow \mathbb{C}[T_1, \dots, T_{n+3}, S_1, S_2, S_3]$	$T_k \mapsto T_k S_1 S_2 S_3$
	$T_n \mapsto T_n S_1 S_2 S_3$
	$T_{n+1} \mapsto T_{n+1} S_1^3$
	$T_{n+2} \mapsto T_{n+2} S_2^3$
	$T_{n+3} \mapsto T_{n+3} S_3^3$

The following theorem is the main result of this section. We postpone its proof until the end of the section.

**Theorem 4.1.** *Let  $Y$  and  $L$  be a smooth cubic hypersurface and a line of  $\mathbb{P}^{n+1}$  whose defining equations are given in Table 2.1. Let  $X$  be the corresponding cubic elliptic variety of type  $X_3, X_S, X_{S_2}, X_{SSS}$ . Then the Cox ring of  $X$  is one of the following:*

- (1) *Type  $X_3$ : the Cox ring is  $\mathbb{C}[T_1, \dots, T_{n+3}, S_1, S_2, S_3]/\mathfrak{I}_1$ , where  $\mathfrak{I}_1$  is generated by*

$$\frac{\beta_1(T_{n+3} - T_{n+1} a_1 - b')}{S_1^2 S_2^3 S_3^3} \quad \frac{\beta_1(T_{n+2} T_{n+3} + T_{n+1} a' + b_1)}{S_1^3 S_2^3 S_3^3}$$

*with the  $\mathbb{Z}^4$ -grading given by the grading matrix*

$$\begin{bmatrix} 1 & \dots & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ -1 & \dots & -1 & -1 & -2 & 0 & -3 & 1 & 0 & 0 \\ -1 & \dots & -1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ -1 & \dots & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

- (2) *Type  $X_S$ : the Cox ring is  $\mathbb{C}[T_1, \dots, T_{n+2}, S_1, S_2, S_3]/\mathfrak{I}_2$ , where  $\mathfrak{I}_2$  is generated by*

$$\frac{\beta_2(T_{n+1} a_2 + b_2)}{S_1^3 S_2^3 S_3^3}$$

*with the  $\mathbb{Z}^4$ -grading given by the grading matrix*

$$\begin{bmatrix} 1 & \dots & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & \dots & -1 & -1 & -3 & 0 & 1 & 0 & 0 \\ -1 & \dots & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & \dots & -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

- (3) *Type  $X_{S_2}$ : the Cox ring is  $\mathbb{C}[T_1, \dots, T_{n+3}, S_1, S_2, S_3]/\mathfrak{I}_3$ , where  $\mathfrak{I}_3$  is generated by*

$$\frac{\beta_3(T_{n+3} - a_3)}{S_1 S_2^2} \quad \frac{\beta_3(T_{n+1} T_{n+3} + b_3)}{S_1^3 S_2^3 S_3^3}$$

*with the  $\mathbb{Z}^4$ -grading given by the grading matrix*

$$\begin{bmatrix} 1 & \dots & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ -1 & \dots & -1 & -2 & 0 & -1 & -3 & 1 & 0 & 0 \\ -1 & \dots & -1 & 0 & 0 & 0 & -3 & -1 & 1 & 0 \\ -1 & \dots & -1 & -1 & -3 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

- (4) *Type  $X_{SSS}$ : the Cox ring is  $\mathbb{C}[T_1, \dots, T_{n+3}, S_1, S_2, S_3]/\mathfrak{I}_4$ , where  $\mathfrak{I}_4$  is generated by*

$$\beta_4(T_{n+3} - a_4) \quad \frac{\beta_4(T_{n+1} T_{n+2} T_{n+3} + b_4)}{S_1^3 S_2^3 S_3^3}$$

*with the  $\mathbb{Z}^4$ -grading given by the grading matrix*

$$\begin{bmatrix} 1 & \dots & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & \dots & -1 & -3 & 0 & 0 & 1 & 0 & 0 \\ -1 & \dots & -1 & 0 & -3 & 0 & 0 & 1 & 0 \\ -1 & \dots & -1 & 0 & 0 & -3 & 0 & 0 & 1 \end{bmatrix}.$$

*Remark 4.2.* Observe that for varieties of type  $X_S$  the Cox ring admits only one relation which is the equation of the strict transform of the corresponding cubic hypersurface. For all the remaining types the Cox ring admits two relations: one of them comes from the defining equation of  $T_{n+3}$  in  $\mathbb{P}^{n+1}$  and the other one comes from the the strict transform of the corresponding cubic hypersurface.

**4.1. Algebraic preliminaries.** We follow the construction given in Section 3 of [BHK12]. Let  $\mathbb{C}[T, S]$  be a polynomial ring in the variables  $T_1, \dots, T_r, S_1, \dots, S_s$ , graded by an abelian group  $K_T \oplus K_S$ . Let  $\mathbb{C}[T, S^{\pm 1}]$  be its localization with respect to all the  $S$  variables and let  $\mathbb{C}[T]$  be the polynomial ring in the first  $r$  variables graded by  $K_T$ . Denote by  $\mathbb{C}[T, S^{\pm 1}]_0$  the degree zero part of  $\mathbb{C}[T, S^{\pm 1}]$  with respect to the  $K_S$  grading. Assume the following diagram of homomorphisms is given:

$$(4.1) \quad \begin{array}{ccccc} I & \longrightarrow & \mathbb{C}[T, S] & \xrightarrow{\psi} & R \\ & & \downarrow j & \swarrow \beta & \uparrow \rho \\ & & \mathbb{C}[T, S^{\pm 1}] & & \mathbb{C}[T] \\ & & \uparrow j_0 & \nwarrow \alpha \cong & \uparrow \\ & & \mathbb{C}[T, S^{\pm 1}]_0 & & J \end{array}$$

such that  $R$  is a  $K_T \oplus K_S$ -graded domain,  $\psi$  is a graded surjective homomorphism with kernel  $I$ ,  $\rho$  a graded homomorphism with kernel  $J$ , both  $j$  and  $j_0$  are inclusions,  $\rho = \psi \circ \beta$  and  $\alpha(T_i) = T_i \cdot m_i(S) = \beta(T_i)$ , for any  $i$ , where  $m_i(S) \in \mathbb{C}[T, S]$  is a monomial in the variables  $S$ .

**Proposition 4.3.** *Under the above assumptions, let  $J' \subset \mathbb{C}[T, S]$  be the extension and contraction of the ideal  $\alpha(J)$ . Then  $J' \subset I$ .*

*Proof.* Observe that  $\beta(J) \subset I$  since  $\rho = \psi \circ \beta$ . Moreover, from

$$\beta(J) \cdot \mathbb{C}[T, S^{\pm 1}] = \alpha(J) \cdot \mathbb{C}[T, S^{\pm 1}],$$

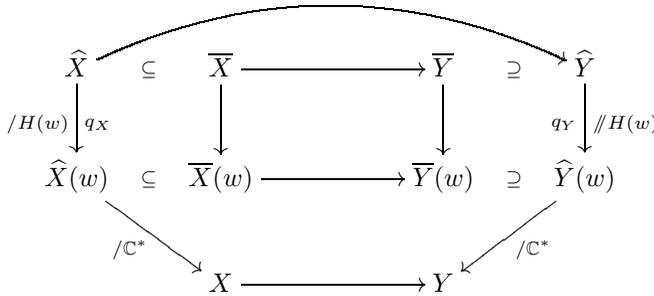
we get that  $J'$  is contained in the saturation of  $I$  with respect to the variables  $S$ . Since  $R$  is a domain,  $I$  is saturated, hence we get the statement.  $\square$

The following statement identifies a Cox ring with certain subalgebras. Consider a factorially  $K$ -graded normal affine algebra  $R = \bigoplus_K R_w$  with pairwise non-associated  $K$ -prime generators  $f_1, \dots, f_r$  and set  $w_i := \deg(f_i) \in K$ . The  $K$ -grading is *almost free* if any  $r - 1$  of the  $w_i$  generate  $K$  as a group. The *moving cone*  $\text{Mov}(R) \subseteq K_{\mathbb{Q}}$  is the intersection over all cones in  $K_{\mathbb{Q}}$  generated by any  $r - 1$  of the degrees  $w_i$ . Recall that  $\text{Mov}(R)$  comes with a subdivision into finitely many polyhedral *GIT-cones*  $\lambda(w)$  associated to the classes  $w \in \text{Mov}(R)$ ; see [Hau08, Proposition 3.9].

**Proposition 4.4.** *Let  $X$  be a  $\mathbb{Q}$ -factorial projective variety with finitely generated Cox ring  $\mathcal{R}(X)$  and  $R \subseteq \mathcal{R}(X)$  a finitely generated normal almost freely factorially  $\text{Cl}(X)$ -graded subalgebra such that  $R$  and  $\mathcal{R}(X)$  have the same quotient field. If there is a very ample divisor  $D$  on  $X$  such that  $R_{[D]} = \mathcal{R}(X)_{[D]}$  holds and  $\lambda([D]) \subseteq \text{Mov}(R)$  is of full dimension, then we have  $R = \mathcal{R}(X)$ .*

*Proof.* Consider the total coordinate space  $\overline{X} := \text{Spec } \mathcal{R}(X)$  and  $\overline{Y} := \text{Spec } R$ . Both come with an action of the characteristic quasitorus  $H := \text{Spec } \mathbb{C}[\text{Cl}(X)]$  and we have a canonical  $H$ -equivariant morphism  $\overline{X} \rightarrow \overline{Y}$ . Moreover, for  $w := [D] \in \text{Cl}(X)$ ,

the inclusion  $R(w) \subseteq \mathcal{R}(X)(w)$  defines a morphism  $\overline{X}(w) \rightarrow \overline{Y}(w)$ . Altogether we arrive at a commutative diagram:



Here  $\widehat{X} \subseteq \overline{X}$  and  $\widehat{Y} \subseteq \overline{Y}$  are the respective unions of all localizations  $\overline{X}_f$  and  $\overline{Y}_f$ , where  $f$  is of degree  $w$ , and the subsets  $\widehat{X}(w) \subseteq \overline{X}(w)$  and  $\widehat{Y}(w) \subseteq \overline{Y}(w)$  are defined analogously. The downwards maps are quotients with respect to the action of the subgroup  $H(w) \subseteq H$  corresponding to the map of character groups  $\text{Cl}(X) \rightarrow \text{Cl}(X)/\mathbb{Z}w$ . Note that by ampleness of  $D$ , the composition  $\widehat{X} \rightarrow X$  is the characteristic space.

Since  $R$  is almost freely factorially  $\text{Cl}(X)$ -graded and  $w$  lies in the relative interior of  $\text{Mov}(R)$ , we infer from [Hau08, Theorem 3.6] that also  $\widehat{Y} \rightarrow Y$  is a characteristic space. The resulting variety  $Y$  is projective [Hau08, Proposition 3.9]. As a dominant morphism of projective varieties, the induced map  $X \rightarrow Y$  is surjective. Since the GIT-cone  $\lambda(w)$  is of full dimension, the fibers of  $\widehat{Y} \rightarrow Y$  are precisely the  $H$ -orbits [Hau08, Lemma 3.10]. The commutative diagram then yields that the  $H$ -equivariant morphism  $\widehat{X} \rightarrow \widehat{Y}$  is surjective. Moreover, the complement  $\overline{Y} \setminus \widehat{Y}$  is of codimension at least two in  $\overline{Y}$  [Hau08, Construction 3.11]. Thus, by Richardson’s Lemma, the birational morphism  $\overline{X} \rightarrow \overline{Y}$  of normal affine varieties is an isomorphism. The assertion follows.  $\square$

**4.2. Proof of Theorem 4.1.** Let us give here all the necessary preliminary lemmas to prove the main result of the section. For each cubic elliptic variety  $X$  in Theorem 4.1 we construct the  $\mathbb{Z}^4$ -graded ring

$$R_n := \mathbb{C}[T, S]/\mathfrak{J},$$

where  $\mathfrak{J}$  is one of the four ideals  $\mathfrak{J}_1, \dots, \mathfrak{J}_4$ . Consider a factorially  $K$ -graded normal affine algebra  $R = \bigoplus_K R_w$  with pairwise non-associated  $K$ -prime generators  $f_1, \dots, f_r$  and set  $w_i := \text{deg}(f_i) \in K$ .

*Remark 4.5.* Given an effective divisor  $D$  we consider the subspace  $V$  of  $H^0(X, D)$ , generated by all the sections corresponding to reducible divisors. Observe that any system of homogeneous generators of the Cox ring of  $X$  must contain all the elements of a basis of  $H^0(X, D)$  which are not in  $V$ .

**Lemma 4.6.** *Let  $R_n$  be as before. Then the following hold:*

- (1)  $R_n$  is a subalgebra of  $\mathcal{R}(X)$ .
- (2)  $R_n$  and  $\mathcal{R}(X)$  have the same quotient field.
- (3)  $R_n$  is almost free factorially graded.

*Proof.* Each grading matrix is written with respect to the basis  $(H, E_1, E_2, E_3)$ . We are going to show that the columns of any such matrix are degrees of generators of the Cox ring according to Remark 4.5. We will proceed in two steps. First of all we will construct in each case a homomorphism of rings:

$$\psi: \mathbb{C}[T, S] \rightarrow \mathcal{R}(X)$$

which maps the generators  $T_j$  and  $S_k$  to certain sections of the Cox ring. Then we will show that the kernel of  $\psi$  is the defining ideal of  $R_n$ .

Any prime divisor of Riemann-Roch dimension one gives a homogeneous generator of the Cox ring. Among these there are the exceptional divisors corresponding to the last three columns of each grading matrix. We also have the strict transforms of the intersections of  $Y$  with a tangent hyperplane, corresponding to the columns whose first entry is 1, and at least one of the other entries is smaller than  $-1$ . Moreover, by Proposition 2.5 the classes  $[2, -3, 0, 0]$  of type  $X_3$  and  $[2, -3, -3, 0]$  of type  $X_{S_2}$  are the flop images of  $[1, -2, -1, 0]$  and  $[1, -2, 0, -1]$  respectively.

We now claim that if  $D$  is an effective irreducible divisor with class  $H - m_1 E_1 - m_2 E_2 - m_3 E_3$ , then either  $m_i \leq 1$  for each  $i = 1, 2, 3$  or  $D$  is the intersection of  $Y$  with a tangent hyperplane. Indeed if the three points are distinct, then the claim is obvious. Otherwise let us assume for example that  $p_2$  is a point of the exceptional divisor on  $p_1$ . Then  $e_1 - e_2$  is the class in  $N_1(X)$  of a fiber of the  $\mathbb{P}^1$ -bundle  $E_1 - E_2$  and  $D \cdot (e_1 - e_2) = m_1 - m_2 \geq 0$  since  $D$  is irreducible and distinct from  $E_1 - E_2$ . Hence the biggest multiplicities are those of the points in  $L \cap Y$  and the claim follows.

Since we already considered the sections of  $Y$  with a tangent hyperplane, from the previous claim we now concentrate on the case in which all the  $m_i$  are less than or equal to 1. Observe that  $H^0(X, \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1))$  contains no reducible sections when  $X$  is of type  $X_{SSS}$  and just one reducible section for the remaining three types. Hence by Remark 4.5 we get the columns of degree  $[1, -1, -1, -1]$  (they are  $n$  in type  $X_{SSS}$  and  $n - 1$  otherwise). Moreover, when  $X$  is of type  $X_3$ ,  $X_S$  or  $X_{S_2}$ , the Riemann-Roch dimension of a divisor of degree  $[1, -1, 0, 0]$  is  $n + 1$ , while with the previous generators one can only form an  $n$ -dimensional subspace. Hence, again by Remark 4.5 we add a generator in this degree and a similar argument applies to  $[1, 0, 0, 0]$  for  $X_3$  and  $X_S$ .

We have thus defined the homomorphism  $\psi$ . Since we are considering four cases, let us denote by  $\psi_i$ , for  $i \in \{1, \dots, 4\}$ , these homomorphisms. By the definition of  $\beta_i$ , the homomorphism  $\rho_i := \psi_i \circ \beta_i$ , is just the composition of the natural map  $\mathbb{C}[T] \rightarrow \mathcal{R}(Y)$  with the pull-back map  $\mathcal{R}(Y) \rightarrow \mathcal{R}(X)$ . If we denote by  $J_i$  the kernel of  $\rho_i$ , we have that

$$\begin{aligned} J_1 &= \langle T_{n+3} - T_{n+1} a_1 - b', T_{n+2} T_{n+3} + T_{n+1} a' + b_1 \rangle, \\ J_2 &= \langle T_{n+1} a_2 + b_2 \rangle, \\ J_3 &= \langle T_{n+3} - a_3, T_{n+1} T_{n+3} - b_3 \rangle, \\ J_4 &= \langle T_{n+3} - a_4, T_{n+1} T_{n+2} a_4 + b_4 \rangle. \end{aligned}$$

By the generality assumptions on the polynomials  $a_i, b_i, c_i$  and  $d_i$  we have that each  $J_i$  is prime. For each of the four cases we now refer to diagram (4.1) where the ring  $R$  in the diagram is the image of  $\psi_i$ . By Proposition 4.3 the contraction and extension  $J'_i$  of the ideal  $\alpha_i(J_i)$  is contained in  $I_i := \ker(\psi_i)$ . By [BHK12, Proposition 3.3] and the fact that  $J_i$  is prime, we deduce that also  $J'_i$  is prime. We are now going

to prove that

$$\mathfrak{J}_i = J'_i \quad \text{for } i \in \{1, \dots, 4\},$$

where  $\mathfrak{J}_i$  is the  $i$ -th ideal appearing in Theorem 4.1. This is equivalent to showing that each ideal  $\mathfrak{J}_i$  is saturated with respect to the variables  $S$ . For  $\mathfrak{J}_2$  this is straightforward since it is principal and the generator is irreducible. In the remaining cases, since each ideal  $\mathfrak{J}_i$  is generated by two elements it is enough to prove that there are no components of codimension one in  $V(S_1 S_2 S_3)$ . The second generator of  $\mathfrak{J}_4$  is a polynomial in the  $T_j$  and hence there is nothing to prove. The second generator of  $\mathfrak{J}_1$  can be written as  $f := T_{n+2} T_{n+3} + \beta_1(T_{n+1}a' + b_1)S_1^{-3}S_2^{-3}S_3^{-3}$ . The first monomial is  $T_{n+2} T_{n+3}$ , while the sum of the remaining monomials does not contain these two variables and does not vanish identically on  $V(S_i)$ . Hence  $V(f, S_i)$  is irreducible and since  $f$  does not divide the first generator, there are no components of codimension one in  $V(S_1 S_2 S_3)$ . A similar analysis applies to  $\mathfrak{J}_3$ .

We proved that each  $R_n = \mathbb{C}[T, S]/\mathfrak{J}_i = \mathbb{C}[T, S]/J'_i$  is a domain. Moreover,  $J'_i \subset I_i$  implies that  $R \subset R_n$  and we claim that  $R_n = R$ . By construction we know that  $\dim R_n = n + 4$ . Observe now that the field of rational functions of  $Y$  has dimension  $n$  and is equal to the field of homogeneous fractions of  $R$ . Since  $R$  is graded by  $\mathbb{Z}^4$  we conclude that also  $\dim R = n + 4$ . Moreover,  $R$  is a domain too since it is contained in  $\mathcal{R}(X)$ . We conclude by observing that  $R$  and  $R_n$  are domains of the same dimension and hence the inclusion  $R \subset R_n$  implies that  $R = R_n$ . This proves (1).

Part (2) of the statement follows from the fact that both  $R_n$  and  $\mathcal{R}(X)$  contain the homogeneous coordinate ring of the cubic hypersurface  $Y$  as a subring. According to (4.1) the ideal  $\mathfrak{J}$  of  $\mathbb{C}[T, S]$  is obtained by extending and contracting the homogeneization  $\alpha(J)$  of the ideal  $J$ . Hence  $R_n$  is factorially graded by [BHK12, Theorem 3.2] and it is almost free graded since by [BHK12, Corollary 3.4] it is the Cox ring of a toric ambient modification of  $Y$ . This proves (3). □

According to Lemma 4.6 the algebra  $R_n$  is a subalgebra of the Cox ring  $\mathcal{R}(X)$ . Let  $f_1$  be the generator of  $R_n$  corresponding to the variable  $T_1$  and let  $D$  be the divisor of  $X$  defined by  $f_1$ . In what follows with abuse of notation we will denote by the same symbol the divisor  $D$  and its support.

**Lemma 4.7.** *The following properties hold:*

- (1)  $R_{n-1}$  is isomorphic to  $R_n/\langle T_1 \rangle$  for any  $n > 3$ ;
- (2)  $D$  is a cubic elliptic  $(n - 1)$ -dimensional variety of the same type of  $X$ ;
- (3) there is a surjective morphism  $\mathcal{R}(D) \rightarrow \mathcal{R}(D')$ , where  $D'$  is the image of  $D$  via some composition of the  $\sigma_i$ 's.

*Proof.* (1) follows directly from the definition of  $R_n$ , while (2) is implied by the fact that  $D$  is the pull-back of a hyperplane of  $\mathbb{P}^{n-1}$  via the elliptic fibration  $\pi: X \rightarrow \mathbb{P}^{n-1}$ . (3) follows from the fact that each composition of the  $\sigma_i$  is a blow-down and then it is a toric ambient modification in the sense of [BHK12, Remark 3.6]. □

**Lemma 4.8.** *Let  $X$  be a cubic elliptic  $n$ -dimensional variety of type  $X_3, X_S, X_{S_2}$  or  $X_{SSS}$  and let  $W = 4H - 3E_1 - 2E_2 - E_3$ . Then the following hold:*

- (1) The divisor  $W$  is very ample.
- (2) The GIT chamber  $\lambda([W]) \subseteq \text{Mov}(R_n)$  is full-dimensional.

*Proof.* We begin by proving (1). Writing  $W$  as

$$W = (H - E_1 - E_2 - E_3) + (H - E_1 - E_2) + (H - E_1) + H,$$

we observe that it is ample since it lies in the interior of the nef cone of  $X$  by Proposition 3.2. The linear system  $|W|$  is base point free since all the summands in the above sum are base point free. Finally the morphism defined by the linear system  $|H|$  is birational, since it is just the contraction  $X \rightarrow Y$ . Hence  $|W|$  is an ample and spanned linear system which defines a degree one morphism and thus  $W$  is very ample.

In order to prove (2) let us denote by  $w_i$  the degree of the  $i$ -th generator of  $R_n$ , that is, the  $i$ -th column of the corresponding grading matrix given in Theorem 4.1. A direct calculation shows that the class  $w$  of  $W$  is not contained in any two-dimensional cone spanned by the  $w_i$ . The three-dimensional cones  $\text{cone}(w_\alpha, w_\beta, w_\gamma)$  which contain  $w$  into their relative interiors correspond to the sets of indices  $I = \{\alpha, \beta, \gamma\}$  given in the table below, where  $i \in \{1, \dots, n - 1\}$ . Let  $T_{n+2+j} = S_j$  if  $X$  is of type  $X_3$  and  $T_{n+3+j} = S_j$  otherwise. For any subset of indices  $I$  let us put

$$f^I := f(U_1, \dots, U_s) \quad \text{where} \quad U_k = \begin{cases} T_k & \text{if } k \in I, \\ 0 & \text{otherwise,} \end{cases}$$

and  $s = n + 5$  if  $X$  is of type  $X_3$  while  $s = n + 6$  in the remaining cases. Let us define the ideal  $\mathfrak{J}^I := \langle f^I : f \in \mathfrak{J} \rangle$ , and let us denote by  $f_j$  the  $j$ -th generator of the ideal  $\mathfrak{J}$  given in Theorem 4.1. Then for any set of indices  $I$  in the table below the ideal  $\mathfrak{J}^I$  contains one monomial  $f_j^I$ :

$X_3$	$\{i, n + 1, n + 2\}$ $f_1^I = T_{n+1}T_{n+2}^2$	$\{i, n, n + 4\}$ $f_2^I = T_n^3$	$\{n + 3, n + 4, n + 5\}$ $f_1^I = T_{n+3}S_1$
$X_S$	$\{i, n, n + 3\}$ $f_1^I = T_n^3$		
$X_{S2}$	$\{i, n + 2, n + 4\}$ $f_1^I = T_{n+2}^2S_1$	$\{n, n + 2, n + 4\}$ $f_1^I = T_{n+2}^2S_1$	

This allows us to conclude that the corresponding cone  $\text{cone}(w_\alpha, w_\beta, w_\gamma)$  with  $I = \{\alpha, \beta, \gamma\}$  is not an orbit cone. Since  $\lambda(w)$  is the intersection of all the orbit cones which contain  $w$  into their relative interior and since all such cones are full-dimensional, we conclude that  $\lambda(w)$  is full-dimensional as well. □

**Lemma 4.9.** *Let  $X$  be a cubic elliptic threefold of type  $X_3, X_S, X_{S2}$  or  $X_{SSS}$ . Then the Cox ring of  $X$  is isomorphic to  $R_3$ .*

*Proof.* Denote by  $A$  the polynomial ring  $\mathbb{C}[T, S]$ . If  $X = X_S$ , then a presentation of  $R_3$  is the Koszul complex:

$$0 \longrightarrow A(-w_1) \longrightarrow A \longrightarrow 0,$$

where  $w_1 = [3, -3, 0, 0]$  is the degree of the generator of  $\mathfrak{I}_2$ . If  $X$  is one of the remaining types, then a presentation of  $R_3$  is the Koszul complex:

$$0 \longrightarrow A(-w_1 - w_2) \longrightarrow A(-w_1) \oplus A(-w_2) \longrightarrow A \longrightarrow 0,$$

where  $w_1$  and  $w_2$  are the degrees of the generators of the ideal  $\mathfrak{I}_i$  for  $i \in \{1, 3, 4\}$ . A computer calculation done by using the previous exact sequences shows that the dimension of the degree  $w$  part of  $R_3$  is 66, 53, 64 and 75 for the types  $X_3$ ,  $X_S$ ,  $X_{S_2}$  and  $X_{SS}$  respectively. In each case this dimension equals the Riemann-Roch dimension of the class  $w$ . Hence  $(R_3)_w = \mathcal{R}(X)_w$  and we conclude by Lemma 4.6, Lemma 4.8 and Proposition 4.4.  $\square$

*Proof of Theorem 4.1.* We proceed by induction on  $n$ . The case  $n = 3$  is proved in Lemma 4.9. Assume  $n > 3$ . Observe that  $H - E_1 - E_2 - E_3$  is linearly equivalent to the divisor  $D$  of  $X$  defined by  $f_1$  and that its push-forwards via  $\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_3$ ,  $\sigma_2 \circ \sigma_3$  and  $\sigma_3$  equal those of  $H$ ,  $H - E_1$  and  $H - E_1 - E_2$  respectively. According to Proposition 4.4 it is enough to show that the degree  $w$  part of  $R_n$  equals that of  $\mathcal{R}(X)$ . To this aim we consider the exact sequence

$$0 \longrightarrow H^0(X, W - D) \xrightarrow{f_1} H^0(X, W) \longrightarrow H^0(D, W|_D) \longrightarrow 0,$$

$\longleftarrow \quad \quad \quad \longleftarrow$   
 $\gamma$

where the last 0 is due to Kawamata-Viehweg and the fact that  $W - D - K_X$  is nef and big. By the induction hypothesis and our choice of  $D$  we have a surjective map  $R_n \rightarrow R_{n-1} = \mathcal{R}(D)$ . This allows us to construct a section  $\gamma$  whose image is contained in  $R_n$ .

We claim that any section of  $H^0(X, W - D)$  is in  $R_n$  and this is enough to conclude. The divisor  $W - D = (H - E_1 - E_2) + (H - E_1) + H$  is the pull-back of a divisor  $W_2$  of  $Y_2$ . Denote by  $D_2$  the divisor of  $Y_2$  which is the image of  $D$  via  $\sigma_3$ . As before there is an exact sequence

$$0 \longrightarrow H^0(Y_2, W_2 - D_2) \longrightarrow H^0(Y_2, W_2) \longrightarrow H^0(D_2, W_2|_{D_2}) \longrightarrow 0,$$

$\longleftarrow \quad \quad \quad \longleftarrow$   
 $\gamma_2$

where the last 0 is due to Kawamata-Viehweg and the fact that  $W_2 - D_2 - K_{Y_2}$  is linearly equivalent to  $\sigma_{3*}(H - E_1 + H) - K_{Y_2}$  which is nef and big. By Lemma 4.7 and the fact that  $\sigma_3: X \rightarrow Y_2$  is a toric ambient modification we get the following diagram, where all the maps but the inclusion  $R_n \rightarrow \mathcal{R}(X)$  are surjective:

$$\begin{array}{ccccc} R_n & \longrightarrow & \mathcal{R}(X) & \longrightarrow & \mathcal{R}(Y_2) \\ \downarrow & & & & \\ R_{n-1} & \xlongequal{\quad} & \mathcal{R}(D) & \longrightarrow & \mathcal{R}(D_2) \end{array}$$

This allows us to construct a section  $\gamma_2: \mathcal{R}(D_2)_{w_2} \rightarrow \mathcal{R}(Y_2)_{w_2}$  whose image is contained in the image of  $R_n$ . Now we proceed in a similar way with the divisor  $W_2 - D_2 = \sigma_{3*}(2H - E_1)$  obtaining the divisors  $W_1 = (\sigma_2 \circ \sigma_3)_*(2H - E_1)$  and  $D_1 = (\sigma_2 \circ \sigma_3)_*(H - E_1)$ , so that  $W_1 - D_1$  is pull-back of the divisor  $\sigma_*(H)$  on  $Y$ .

This last divisor is a hyperplane section of  $Y$  and thus a Riemann-Roch basis consists of elements of the coordinate ring of  $Y$  which is a homomorphic image of  $R_n$ . This proves the claim. Hence  $(R_n)_w = \mathcal{R}(X)_w$  and we conclude by Lemma 4.6, Lemma 4.8 and Proposition 4.4.  $\square$

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