

## INVARIANCE OF $R$ -GROUPS BETWEEN $p$ -ADIC INNER FORMS OF QUASI-SPLIT CLASSICAL GROUPS

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ABSTRACT. We study the reducibility of parabolically induced representations of non-split inner forms of quasi-split classical groups. The isomorphism of Arthur  $R$ -groups, endoscopic  $R$ -groups and Knapp-Stein  $R$ -groups is established, as well as showing these  $R$ -groups are isomorphic to the corresponding ones for the quasi-split form. This shows  $R$ -groups are an invariant of the  $L$ -packets. The results are applied to classify the elliptic spectrum.

### 1. INTRODUCTION

We continue our study of  $L$ -packets and reducibility for non-quasi-split inner forms of reductive  $p$ -adic groups. In [CG15] we examined the case of inner forms of  $SL_n$ . There we found a relation between the structure of the tempered  $L$ -packets of  $SL_n(F)$  and those of  $SL_r(D)$ , where  $D$  is a central division algebra of dimension  $d^2$  over  $F$ , and  $r = n/d$ . Our method was to use the transfer of Plancherel measures via a Jacquet-Langlands type correspondence developed by the first named author in [Cho14]. This allows us to determine the Knapp-Stein  $R$ -groups for  $SL_r(D)$  from those of  $SL_n(F)$ . The same result was proved by Chao and Li [CL14] using a different method, and they showed there are  $L$ -packets of the non-quasi-split inner form whose  $R$ -groups are strictly smaller than those of the corresponding  $L$ -packet of  $SL_n(F)$ . Here we turn to the case of the split classical groups  $SO_n$  and  $Sp_{2n}$ , as well as the non-split, quasi-split special orthogonal group  $SO_{2n}^*$ . In these cases we can show the  $R$ -groups are invariant under the transfer to the non-quasi-split inner form. Further, we can show the intertwining algebras are preserved. This shows the theory of reducibility of induced from discrete series representations, and elliptic representations for these cases transfers from [Gol94, Her93].

We fix a  $p$ -adic field,  $F$ , of characteristic zero, and consider a reductive connected quasi-split algebraic group  $\mathbf{G}$  defined over  $F$ . We assume  $\mathbf{G}'$  is an inner form of  $\mathbf{G}$ . The tempered spectrum of  $G = \mathbf{G}(F)$  consists of the discrete series, as well as the irreducible components of representations parabolically induced from discrete series representations of proper  $F$ -Levi subgroups. If  $\mathbf{M}'$  is an  $F$ -Levi subgroup of  $\mathbf{G}'$ , then there is an  $F$ -Levi subgroup  $\mathbf{M}$  of  $\mathbf{G}$  with  $\mathbf{M}'$  an inner form of  $\mathbf{M}$ . Since the root datum of  $\mathbf{G}$  determines the Langlands  $L$ -group  ${}^L G$ , we have an  $\bar{F}$ -isomorphism  $\psi : \mathbf{G}' \xrightarrow{\sim} \mathbf{G}$  and we see  $\hat{G} = \hat{G}'$ , where these are the connected components of the  $L$ -groups. The local Langlands correspondence gives a parameterization of the tempered spectrum, in the form of  $L$ -packets, via admissible

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homomorphisms  $\varphi : W'_F \rightarrow {}^L G$ . This, then, gives a correspondence between the  $L$ -packets  $\Pi_\varphi(G)$  of  $G$  and  $\Pi_\varphi(G')$  of  $G'$ . Since  $\mathbf{M}$  and  $\mathbf{M}'$  are inner forms, we have a similar correspondence between  $L$ -packets on the Levi subgroups.

For the groups under consideration,  $SO_n$ ,  $Sp_{2n}$  and  $SO_{2n}^*$ , the theory of induced representations is well understood (see [Gol94] and Appendix A). Furthermore, from the work of Harris-Taylor, [HT01], Henniart, [Hen00] and Arthur, [Art13], we know the local Langlands correspondence has been established for all these groups, as well as all of their Levi subgroups (see also [JS04, MW03] for  $SO_{2n+1}$ ). Here we investigate the induced representations for the non-quasi-split inner forms  $G'$  of these groups  $G$ . Suppose  $\sigma'$  is a discrete series representation of some Levi subgroup  $M' = \mathbf{M}'(F)$  of  $G'$ . We suppose  $\sigma' \in \Pi_\varphi(M')$ , and take an element  $\sigma \in \Pi_\varphi(M)$ . We show there is a one-to-one correspondence between the components of the induced representations  $i_{G,M}(\sigma)$  and  $i_{G',M'}(\sigma')$  (cf. Theorem 3.9, and its corollaries). The structure of  $i_{G,M}(\sigma)$  and  $i_{G',M'}(\sigma')$  are determined by the intertwining algebras  $\mathcal{C}(\sigma)$  and  $\mathcal{C}(\sigma')$ . These in turn are determined by the Knapp-Stein  $R$ -groups  $R_\sigma$  and  $R_{\sigma'}$ , along with 2-cocycles of each of these groups. We show  $R_\sigma \simeq R_{\sigma'}$ . Herb showed the cocycle of  $R_\sigma$  is trivial, [Her93], and we show the argument there can be adapted to the inner form, showing the cocycle of  $R_{\sigma'}$  is also trivial. This, then gives the correspondence between components we referred to above. Furthermore, we show the components of  $i_{G',M'}(\sigma')$  are elliptic if and only if the components of  $i_{G,M}(\sigma)$  are elliptic.

Our approach is through the theory of endoscopic and Arthur  $R$ -groups. These are  $R$ -groups attached to the packet through the  $\hat{G}$ -centralizer of the image of the parameter,  $\varphi$  (see sections 2 and 3). In order to establish our results we need to work under a pair of hypotheses (cf. Hypotheses 3.4). Namely, we have to assume the stabilization of the twisted trace formula, and also assume the classification for the inner forms, which are expected to follow from the work of Arthur [Art13], though this has not been completed as of now. In the case of a split group the endoscopic  $R$ -group  $R_\varphi$  is the quotient of the Weyl group  $W_\varphi$  of the centralizer modulo that of the connected component  $W_\varphi^\circ$ . The Arthur  $R$ -group is then defined by identifying  $W_\varphi$  with a subgroup of the Weyl group  $W_M$ . Then  $R_{\varphi,\sigma} = W_{\varphi,\sigma}/W_{\varphi,\sigma}^\circ$ , where the two factors are determined by intersection with the  $W_M$ -stabilizer of  $\sigma$ . Then our main result is  $R_\sigma \simeq R_{\varphi,\sigma} \simeq R_{\varphi,\sigma'} \simeq R_{\sigma'}$  (cf. Theorem 3.9). From this all the results on reducibility and components now follow directly.

In section 2 we give our definitions and notation. We give an explicit description of all the inner forms we will study, as well as preliminary results we need. Section 3 then contains the main results on elliptic tempered  $A$ -packets, endoscopic, and Arthur  $R$ -groups. In section 4, we apply the results of section 3 to the study of elliptic representations. Finally, Appendix A contains an extension of the results of [Gol94] to the case of the non-split, quasi-split even special orthogonal group.

## 2. PRELIMINARIES

**2.1. Basic notation.** Let  $F$  denote a  $p$ -adic field of characteristic 0, that is, a finite extension of  $\mathbb{Q}_p$ . Fix an algebraic closure  $\bar{F}$  of  $F$ . We let  $\mathbf{G}$  denote a connected reductive algebraic group defined over  $F$ . We use the notation  $G$  to denote the group  $\mathbf{G}(F)$  of  $F$ -points and likewise for other algebraic groups defined over  $F$ .

Fix a minimal  $F$ -parabolic subgroup  $\mathbf{P}_0$  of  $\mathbf{G}$  with Levi component  $\mathbf{M}_0$  and unipotent radical  $\mathbf{N}_0$ . Let  $\mathbf{A}_0$  be the split component of  $\mathbf{M}_0$ , that is, the maximal

$F$ -split torus in the center of  $\mathbf{M}_0$ . Let  $\Delta$  be the set of simple roots of  $\mathbf{A}_0$  in  $\mathbf{N}_0$ . Let  $\mathbf{P} \subseteq \mathbf{G}$  be a standard (that is, containing  $\mathbf{P}_0$ )  $F$ -parabolic subgroup of  $\mathbf{G}$ . Write  $\mathbf{P} = \mathbf{M}\mathbf{N}$  with its Levi component  $\mathbf{M} = \mathbf{M}_\Theta \supseteq \mathbf{M}_0$  generated by a subset  $\Theta \subseteq \Delta$  and its unipotent radical  $\mathbf{N} \subseteq \mathbf{N}_0$ . We note that the split component  $\mathbf{A}_\mathbf{M} = \mathbf{A}_{\mathbf{M}_\Theta}$  of  $\mathbf{M} = \mathbf{M}_\Theta$  is equal to the identity component  $(\bigcap_{\alpha \in \Theta} \ker \alpha)^\circ$  in  $\mathbf{A}_0$ , so that  $\mathbf{M} = Z_\mathbf{G}(\mathbf{A}_\mathbf{M})$ , where  $Z_\mathbf{G}(\mathbf{A}_\mathbf{M})$  is the centralizer of  $\mathbf{A}_\mathbf{M}$  in  $\mathbf{G}$  (see [Bor91, Proposition 20.4] and [Spr98, Section 15.1]). We denote by  $\delta_P$  the modulus character of  $P$ .

Let  $\text{Irr}(M)$  denote the set of isomorphism classes of irreducible admissible representations of  $M$ . Denote by  $\Pi_{\text{disc}}(M)$  the set of discrete series representations of  $M$ . By abuse of terminology, we do not make a distinction between each isomorphism class and its representative. For any  $\sigma \in \text{Irr}(M)$ , we write  $i_{G,M}(\sigma)$  for the normalized (twisted by  $\delta_P^{1/2}$ ) induced representation. We denote by  $\tilde{\sigma}$  the contragredient of  $\sigma$ .

We let  $X^*(\mathbf{M})_F$  denote the group of  $F$ -rational characters of  $\mathbf{M}$ . We set  $\mathfrak{a}_M := \text{Hom}(X^*(\mathbf{M})_F, \mathbb{R})$  the real Lie algebra of  $\mathbf{A}_\mathbf{M}$ . We denote by  $\Phi(P, A_M)$  the set of reduced roots of  $\mathbf{P}$  with respect to  $\mathbf{A}_\mathbf{M}$ . Let  $W_M = W(\mathbf{G}, \mathbf{A}_\mathbf{M}) := N_\mathbf{G}(\mathbf{A}_\mathbf{M})/Z_\mathbf{G}(\mathbf{A}_\mathbf{M})$  denote the Weyl group of  $\mathbf{A}_\mathbf{M}$  in  $\mathbf{G}$ , where  $N_\mathbf{G}(\mathbf{A}_\mathbf{M})$  is the normalizer of  $\mathbf{A}_\mathbf{M}$  in  $\mathbf{G}$ . For simplicity, we write  $\mathbf{A}_0 = \mathbf{A}_{\mathbf{M}_0}$ .

We denote by  $W_F$  the Weil group of  $F$  and  $\Gamma := \text{Gal}(\bar{F}/F)$ . Fixing  $\Gamma$ -invariant splitting data, we define the  $L$ -group,  ${}^L G$ , of  $\mathbf{G}$  as a semi-direct product  ${}^L G := \widehat{G} \rtimes \Gamma$  (see [Bor79, Section 2]). By an  $L$ -parameter for  $G$ , we mean an admissible homomorphism  $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L G$ . Two  $L$ -parameters are said to be equivalent if they are conjugate by  $\widehat{G}$ . We denote by  $\Phi(G)$  the set of equivalence classes of  $L$ -parameters for  $G$  (see [Bor79, Section 8.2]). We write a subset  $\Pi_\phi(G)$  of  $\text{Irr}(G)$  for an  $L$ -packet, attached to a given  $L$ -parameter  $\phi$ . For any topological group  $H$ , we denote by  $Z(H)$  the center of  $H$ . We let  $\pi_0(H)$  denote the group  $H/H^\circ$  of connected components of  $H$ , where  $H^\circ$  is the identity component of  $H$ .

**2.2. Structure of  $F$ -inner forms of quasi-split classical groups.** Let  $\mathbf{G} = \mathbf{G}(n)$  denote a quasi-split classical group of rank  $n$  over  $F$ . More precisely, there are the following cases:

- (1) Type  $\mathbf{B}_n : \mathbf{G}(n) = SO_{2n+1}$ , the split special orthogonal group in  $2n+1$  variables defined over  $F$ .
- (2) Type  $\mathbf{C}_n : \mathbf{G}(n) = Sp_{2n}$ , the symplectic group in  $2n$  variables defined over  $F$ .
- (3) Type  ${}^1 \mathbf{D}_n : \mathbf{G}(n) = SO_{2n}$ , the split special orthogonal group in  $2n$  variables defined over  $F$ .
- (4) Type  ${}^2 \mathbf{D}_n : \mathbf{G}(n) = SO_{2n}^*$ , the quasi-split special orthogonal group associated to a quadratic extension  $E$  over  $F$ , where  $\mathbf{G}(n)$  is split over  $E$ .

We denote by  $\mathbf{M}$  an  $F$ -Levi subgroup of  $\mathbf{G}$ . Then,  $M$  is of the form

$$(2.1) \quad GL_{n_1}(F) \times \cdots \times GL_{n_k}(F) \times G_-(m),$$

where  $\sum_{i=1}^k n_i + m = n$  and  $\mathbf{G}_-(m)$  denotes  $SO_{2m+1}$ ,  $Sp_{2m}$ ,  $SO_{2m}$  or  $SO_{2m}^*$  with the same type as  $\mathbf{G}$ . We refer the reader to [Gol94] for split cases and Appendix A for the quasi-split case  $SO_{2n}^*$ .

We let  $\mathbf{G}' = \mathbf{G}'(n)$  denote an  $F$ -inner form of  $\mathbf{G}$ . Recall that  $\mathbf{G}$  and  $\mathbf{G}'$  are  $F$ -inner forms with respect to an  $\bar{F}$ -isomorphism  $\varphi : \mathbf{G}' \xrightarrow{\sim} \mathbf{G}$  if  $\varphi \circ \tau(\varphi)^{-1}$  is an

inner automorphism ( $g \mapsto xgx^{-1}$ ) defined over  $\bar{F}$  for all  $\tau \in \Gamma$ . We often omit  $\varphi$  with no danger of confusion. Note that  $\mathbf{G}'$  can be  $\mathbf{G}$  itself by taking  $\varphi$  to be the identity map. We denote by  $\mathbf{M}'$  an  $F$ -Levi subgroup of  $\mathbf{G}'$  such that  $\mathbf{M}'$  is an  $F$ -inner form of  $\mathbf{M}$ .

For the rest of the section, we discuss the structure of  $M'$ , which turns out to be of the form

$$(2.2) \quad GL_{m_1}(D) \times \cdots \times GL_{m_k}(D) \times G'_-(m).$$

Here  $D$  denotes a central division algebra of dimension 1 (hence,  $D = F$ ) or 4 over  $F$ , and  $\mathbf{G}'_-(m)$  denotes an  $F$ -inner form of  $SO_{2m+1}, Sp_{2m}, SO_{2m}$  or  $SO_{2m}^*$  with the same type as  $\mathbf{G}'$ . Further,  $\sum_{i=1}^k dm_i + m = n$ , where  $d = 1$  or 2 depending on whether the dimension of  $D$  is 1 or 4. Note that  $d m_i = n_i$ .

In what follows, based on the Satake classification [Sat71, pp. 119-120], we describe every  $F$ -inner form  $\mathbf{G}'$  and every possible maximal  $F$ -Levi subgroup  $\mathbf{M}'$  of  $\mathbf{G}'$ . In the diagram below (Satake diagram), a black vertex indicates a root in the set of simple roots of a fixed minimal  $F$ -Levi subgroup  $\mathbf{M}'_0$  of  $\mathbf{G}'$ . So, we remove only a subset  $\vartheta$  of white vertices to obtain an  $F$ -Levi subgroup  $\mathbf{M}'$  (see [Sat71, Section 2.2] and [Bor79, Section I.3]). As discussed in section 2.1, the  $F$ -Levi subgroup  $\mathbf{M}'$ , corresponding to  $\Theta = \Delta \setminus \vartheta$ , is the centralizer in  $\mathbf{G}'$  of the split component  $\mathbf{A}_{\mathbf{M}'} = (\bigcap_{\alpha \in \Theta} \ker \alpha)^\circ$ . One thus notices that there is an  $F$ -isomorphism between two split components

$$(2.3) \quad \mathbf{A}_\mathbf{M} \simeq \mathbf{A}_{\mathbf{M}'} \simeq (GL_1)^k.$$

We write  $D_2$  and  $D_4$  for a division algebra of dimension 4 and 16 over  $F$ , respectively.

**B<sub>n</sub>-type:** There is only one (up to isomorphism) non-split  $F$ -inner form  $\mathbf{G}' = \mathbf{G}'(n)$  of  $SO_{2n+1}$  with the following diagram.



Set  $\Theta = \Delta \setminus \alpha_i$ , where  $i = 1, 2, \dots, n-1$  and  $\alpha_i = e_i - e_{i+1}$ . Then  $M' = M'_\Theta$  is of the form  $GL_i(F) \times G'(n-i)$ . For the case  $i = n-1$ ,  $G'(n-i) = G'(1) = PSL_1(D_2)$ .

**C<sub>n</sub>-type:** According to the parity of  $n$ , there is only one (up to isomorphism) non-split  $F$ -inner form  $\mathbf{G}' = \mathbf{G}'(n)$  of  $Sp_{2n}$  with the following diagram.

- **n: odd**



(every other dot black)

Set  $\Theta = \Delta \setminus \alpha_i$ , where  $i = 2, 4, \dots, n-1$  and  $\alpha_i = e_i - e_{i+1}$ . Then  $M' = M'_\Theta$  is of the form  $GL_{i/2}(D_2) \times G'(n-i)$ . For the case  $i = n-1$ ,  $G'(n-i) = G'(1) = SL_1(D_2)$ .

- **n: even**



(every other dot black)

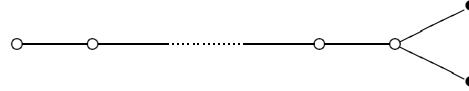
Set  $\Theta = \Delta \setminus \alpha_i$ , where  $i = 2, 4, \dots, n-2, n$  and  $\alpha_i = e_i - e_{i+1}$  for  $i \neq n$ ;  $\alpha_n = 2e_n$  for  $i = n$ . Then  $M' = M'_\Theta$  is of the form  $GL_{i/2}(D_2) \times SU_{n-i}^+(D_2)$ . Here,  $SU_{2k}^+(D_2)$  is defined as

$$SU_{2k}^+(D_2) := \{g \in GL_{2k}(D_2) : g^* J^+ g = J^+\},$$

where  $g^* = (g_{ij})^* = (\bar{g}_{ji})$  with the usual involution  $\bar{g}$  on  $D$ ,  $J^+ = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}$ , and  $SU_0^+(D_2) = G'(0) = 1$  (see [PR94, Section 2.3.3]). For the case  $i = n$ ,  $M' = M'_\Theta$  is of the form  $GL_{n/2}(D_2)$ . We remark that, for even  $n$ , the  $F$ -inner form  $\mathbf{G}' = \mathbf{G}'(n)$  of  $Sp_{2n}$  is of the form  $SU_n^+(D_2)$ .

**${}^1\mathbf{D}_n$ -type:** For each  $n$ , there are two (up to isomorphism) non-split  $F$ -inner forms  $\mathbf{G}' = \mathbf{G}'(n)$  of a split group  $SO_{2n}$  with the following diagrams.

- **n: any**



Set  $\Theta = \Delta \setminus \alpha_i$ , where  $i = 1, 2, \dots, n-2$  and  $\alpha_i = e_i - e_{i+1}$ . Then  $M' = M'_\Theta$  is of the form  $GL_i(F) \times G'(n-i)$ . For the case  $i = n-2$ ,  $G'(n-i) = G'(2)$ , and we have an exact sequence of groups

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow SL_1(D_2) \times SL_1(D_2) \rightarrow G'(2) \rightarrow 1.$$

Further, for the case  $i = n-3$ , we have  $G'(n-i) = G'(3)$ , with the exact sequence

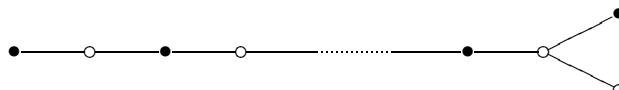
$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow SL_2(D_2) \rightarrow G'(3) \rightarrow 1.$$

- **n: odd**



Set  $\Theta = \Delta \setminus \alpha_i$ , where  $i = 2, 4, \dots, n-3$  and  $\alpha_i = e_i - e_{i+1}$ . Then  $M' = M'_\Theta$  is of the form  $GL_{i/2}(D_2) \times G'(n-i)$ . For the case  $i = n-3$ ,  $G'(n-i) = G'(3) = PSL_1(D_4)$ .

- **n: even**



Set  $\Theta = \Delta \setminus \alpha_i$ , where  $i = 2, 4, \dots, n-2, n$  and  $\alpha_i = e_i - e_{i+1}$  for  $i \neq n$ ;  $\alpha_n = e_{n-1} + e_n$  for  $i = n$ . Then  $M' = M'_\Theta$  is of the form  $GL_{i/2}(D_2) \times SU_{n-i}^-(D_2)$ . Here,  $SU_{2k}^-(D_2)$  is defined as

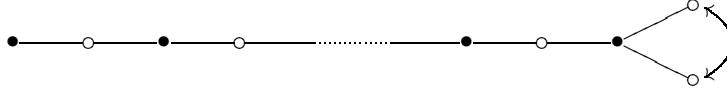
$$SU_{2k}^-(D_2) := \{g \in GL_{2k}(D_2) : g^* J^- g = J^-\},$$

where  $g^* = (g_{ij})^* = (\bar{g}_{ji})$  with the usual involution  $\bar{g}$  on  $D$ ,  $J^- = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$ , and  $SU_0^-(D_2) = G(0) = 1$  (see [PR94, Section 2.3.3]). We remark that, for even  $n$ , the  $F$ -inner form  $\mathbf{G}' = \mathbf{G}'(n)$  of  $SO_{2n}$  is of the form  $SU_n^-(D_2)$ . For the case  $i = n$ ,  $M' = M'_\Theta$  is of the form  $GL_{n/2}(D_2)$ . Further, for the case  $i = n-2$ , we have  $M'_\Theta \cong GL_{(n-2)/2}(D_2) \times SU_2^-(D_2)$  and we have an exact sequence of groups

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow SL_1(D_2) \times SL_2 \rightarrow SU_2^-(D_2) \rightarrow 1.$$

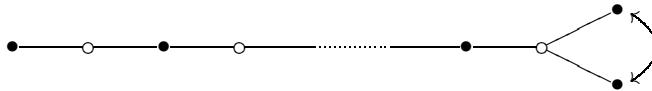
**$^2\mathbf{D}_n$ -type:** According to the parity of  $n$ , there is only one (up to isomorphism) non-split  $F$ -inner form  $\mathbf{G}' = \mathbf{G}'(n)$  of a quasi-split group  $SO_{2n}^*$  over a quadratic extension  $E/F$  with the following diagrams.

- **n: odd**



Set  $\Theta = \Delta \setminus \alpha_i$ , where  $i = 2, 4, \dots, n-1, n$  and  $\alpha_i = e_i - e_{i+1}$  for  $i \neq n$ ;  $\alpha_n = e_{n-1} + e_n$  for  $i = n$ . When  $i = 2, 4, \dots, n-3$ ,  $M' = M'_\Theta$  is of the form  $GL_{i/2}(D_2) \times G'(n-i)$ . When  $i = n-1$ ,  $M' = M'_\Theta$  is of the form  $GL_{i/2}(D_2) \times E^\times$ . When  $i = n$ ,  $M' = M'_\Theta$  is of the form  $GL_{(n-1)/2}(D_2)$ .

- **n: even**



Set  $\Theta = \Delta \setminus \alpha_i$ , where  $i = 2, 4, \dots, n-2$  and  $\alpha_i = e_i - e_{i+1}$ . Then  $M' = M'_\Theta$  is of the form  $GL_{i/2}(D_2) \times G'(n-i)$ .

**2.3.  $R$ -groups.** In this section, we recall the definitions of Knapp-Stein, Arthur and endoscopic  $R$ -groups. For  $\sigma \in \text{Irr}(M)$  and  $w \in W_M$ , we let  ${}^w\sigma$  be the representation given by  ${}^w\sigma(x) = \sigma(w^{-1}xw)$ . (Note, for the purpose of computing  $R$ -groups, we need not worry about the representative we choose for  $w$ .) Given  $\sigma \in \Pi_{\text{disc}}(M)$ , we define

$$W(\sigma) := \{w \in W_M : {}^w\sigma \simeq \sigma\}.$$

Set  $\Delta'_\sigma = \{\alpha \in \Phi(P, A_M) : \mu_\alpha(\sigma) = 0\}$ , where  $\mu_\alpha(\sigma)$  is the rank one Plancherel measure for  $\sigma$  attached to  $\alpha$  [Gol94, p. 1108]. The *Knapp-Stein  $R$ -group* is defined by

$$R_\sigma := \{w \in W(\sigma) : w\alpha > 0, \forall \alpha \in \Delta'_\sigma\}.$$

Denote by  $W'_\sigma$  the subgroup of  $W(\sigma)$ , generated by the reflections in the roots of  $\Delta'_\sigma$ . We write  $C(\sigma) := \text{End}_G(i_{G,M}(\sigma))$  for the algebra of  $G$ -endomorphisms of  $i_{G,M}(\sigma)$ , known as the commuting algebra of  $i_{G,M}(\sigma)$ .

**Theorem 2.1** (Knapp-Stein [KS72]; Silberger [Sil78, Sil79]). *For any  $\sigma \in \Pi_{\text{disc}}(M)$ , we have*

$$W(\sigma) = R(\sigma) \ltimes W'_\sigma.$$

Moreover,  $C(\sigma) \simeq \mathbb{C}[R(\sigma)]_\eta$ , the group algebra of  $R(\sigma)$  twisted by a 2-cocycle  $\eta$ , which is explicitly defined in terms of the group  $W(\sigma)$ .

Let  $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L M$  be an  $L$ -parameter for  $M$ . Through composition with the inclusion,  $\widehat{M} \hookrightarrow \widehat{G}$ , we have  $\phi$  is an  $L$ -parameter for  $G$  as well. We denote by  $C_\phi(\widehat{G})$  the centralizer of the image of  $\phi$  in  $\widehat{G}$  and by  $C_\phi(\widehat{G})^\circ$  its identity component. Fix a maximal torus  $T_\phi$  in  $C_\phi(\widehat{G})^\circ$ .

*Remark 2.2.* From [Art13, Chapter 2.4] we set a maximal torus  $T_\phi$  in  $C_\phi(\widehat{M})^\circ$  to be the identity component

$$A_{\widehat{M}} = (Z(\widehat{M})^\Gamma)^\circ$$

of the  $\Gamma$ -invariants of the center  $Z(\widehat{M})$ .

We set

$$W_\phi^\circ := N_{C_\phi(\widehat{G})^\circ}(T_\phi)/Z_{C_\phi(\widehat{G})^\circ}(T_\phi), \quad W_\phi := N_{C_\phi(\widehat{G})}(T_\phi)/Z_{C_\phi(\widehat{G})}(T_\phi).$$

The *endoscopic  $R$ -group*  $R_\phi$  is defined as follows:

$$R_\phi := W_\phi/W_\phi^\circ.$$

Note that  $W_\phi$  can be identified with a subgroup of  $W_M$  (see [Art89, p. 45]). Let  $\Pi_\phi(M) \subset \text{Irr}(M)$  be the  $L$ -packet associated to the  $L$ -parameter  $\phi$ . For  $\sigma \in \Pi_\phi(M)$ , we set

$$(2.4) \quad W_{\phi,\sigma}^\circ := W_\phi^\circ \cap W(\sigma), \quad W_{\phi,\sigma} := W_\phi \cap W(\sigma) \text{ and } R_{\phi,\sigma} := W_{\phi,\sigma}/W_{\phi,\sigma}^\circ.$$

We call  $R_{\phi,\sigma}$  the *Arthur  $R$ -group*.

### 3. TRANSFER OF $R$ -GROUPS

We continue with the notation in section 2. Let  $\mathbf{G} = \mathbf{G}(n)$  denote a quasi-split classical group,  $SO_{2n+1}$ ,  $Sp_{2n}$ ,  $SO_{2n}$ , or  $SO_{2n}^*$ , of rank  $n$  over  $F$ , and let  $\mathbf{G}' = \mathbf{G}'(n)$  denote an  $F$ -inner form of  $\mathbf{G}$  ( $\mathbf{G}'$  can be  $\mathbf{G}$  itself). In this section, we describe Weyl group actions for  $\mathbf{G}'$ . Further, we prove the three  $R$ -groups for  $\mathbf{G}'$ , Knapp-Stein, Arthur, and endoscopic, are identical if they are associated to the same elliptic tempered  $A$ -parameter for an  $F$ -Levi subgroup of  $\mathbf{G}'$ . For these cases, all three  $R$ -groups are thus invariant on  $A$ -packets and preserved by inner forms. As a corollary, we identically transfer all the description of Knapp-Stein  $R$ -groups for the quasi-split  $\mathbf{G}$  (see [Her93, Gol94] and Appendix A) to its inner form  $\mathbf{G}'$ .

Following Arthur's endoscopic classification for  $G$  and  $G'$  in [Art13], we take an  $L$ -group

$${}^L G = {}^L G' = \widehat{G} \rtimes \text{Gal}(E/F),$$

where  $E = F$  when  $\mathbf{G}$  is split, or  $E$  a quadratic extension  $E$  over  $F$  when  $\mathbf{G} = SO_{2n}^*$ . To be precise, we have the following  $L$ -groups:

- (1) Type  $\mathbf{B}_n$ :  $\widehat{G} = \widehat{G'} = Sp_{2n}(\mathbb{C}) = {}^L G = {}^L G'$ .
- (2) Type  $\mathbf{C}_n$ :  $\widehat{G} = \widehat{G'} = SO_{2n+1}(\mathbb{C}) = {}^L G = {}^L G'$ .
- (3) Type  ${}^1 \mathbf{D}_n$ :  $\widehat{G} = \widehat{G'} = SO_{2n}(\mathbb{C}) = {}^L G = {}^L G'$ .
- (4) Type  ${}^2 \mathbf{D}_n$ :  $\widehat{G} = \widehat{G'} = SO_{2n}(\mathbb{C})$ , and  ${}^L G = {}^L G' = SO_{2n}(\mathbb{C}) \rtimes \text{Gal}(E/F)$ .

Note that  $\text{Gal}(E/F) \simeq O_{2n}(\mathbb{C})/SO_{2n}(\mathbb{C})$ , where  $O_{2n}(\mathbb{C})$  denotes the even orthogonal group of size  $2n$ . Further, the non-trivial outer automorphism (denoted by  $c_0$ ) exchanging the roots  $\alpha_n$  and  $\alpha_{n-1}$  is induced by conjugation by an element in  $O_{2n}(\mathbb{C}) \setminus SO_{2n}(\mathbb{C})$ , cf. [CPSS11, Section 7].

Let  $\mathbf{M}$  and  $\mathbf{M}'$  be  $F$ -Levi subgroups of  $\mathbf{G}$  and  $\mathbf{G}'$ , respectively, such that  $\mathbf{M}'$  is an  $F$ -inner form of  $\mathbf{M}$ . We identify

$$\widehat{M} = \widehat{M}', \quad {}^L M = {}^L M'.$$

**3.1. Tempered  $A$ -packets.** In this section, we describe elliptic tempered  $A$ -packets for  $M$  and  $M'$ , and tempered  $A$ -packets for  $G$  and  $G'$ . We follow Arthur's local results in [Art13, Chapters 1 and 9], cf. [JS04, MW03], and the local Langlands correspondence for  $GL_n$  in [Hen00, HT01] and for  $GL_m(D)$  in [HS12].

Let  $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L M$  be an  $L$ -parameter for  $M$ . We recall from [Art89, Art13] that  $\phi$  is *elliptic* if the quotient group  $C_\phi(\widehat{M})/Z(\widehat{M})$  is finite, and  $\phi$  is *tempered* if the image  $\phi(W_F)$  is relatively compact (or bounded).

For our purpose of studying  $R$ -groups, we assume that the  $L$ -parameter  $\phi$  for  $M$  is both elliptic and tempered. Thus,  $\phi$  is  $M$ -relevant (see [Bor79, Section 8.2]) and becomes an  $L$ -parameter for  $M'$  as well. Considering (2.1) and (2.2), we note that  $\phi$  is of the form  $\phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_k \oplus \phi_-$ , where  $\phi_i$  is an elliptic tempered  $L$ -parameter for  $GL_{n_i}(F)$  as well as  $GL_{m_i}(D)$ , and  $\phi_-$  is one for  $G_-(m)$  as well as  $G'_-(m)$ . When  $G$  is of type  $\mathbf{D}_n$  (including both  ${}^1\mathbf{D}_n$  and  ${}^2\mathbf{D}_n$ ), as in [Art13], we must consider conjugation by  $O_{2n}(\mathbb{C})$ . For this case, we actually work with the  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -orbit of  $\phi_-$ , rather than the  $L$ -parameter  $\phi_-$  itself. In this sense, we should make the following remarks.

- Remark 3.1.* 1. Following the notion in [Art13], an  $A$ -parameter is an  $L$ -homomorphism from  $W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$  to the  $L$ -group whose restriction to the first two factors  $W_F \times SL_2(\mathbb{C})$  is bounded, and  $O_{2n}(\mathbb{C})$ -conjugation has to be considered for even orthogonal cases. We write  $\tilde{\Phi}(G)$  for the set of  $O_{2n}(\mathbb{C})/SO_{2n}(\mathbb{C})$ -orbits in  $\Phi(G)$  under the action of  $O_{2n}(\mathbb{C})$ . For  $G$  of type  $\mathbf{B}_n$  or type  $\mathbf{C}_n$ , we have  $\tilde{\Phi}(G) = \Phi(G)$ . The same notation and argument apply to the  $F$ -inner form  $\mathbf{G}'$ .
2. Note that an  $A$ -parameter is called *generic* if it is trivial on the last factor  $SL_2(\mathbb{C})$  [Art13, Chapter 1.4]. Thus, generic  $A$ -parameters coincide with  $L$ -parameters except for considering  $O_{2n}(\mathbb{C})$ -conjugation for even orthogonal cases.
3. Since the  $L$ -parameter  $\phi_-$  for  $G_-(m)$ , above, is elliptic and tempered, we canonically have an  $A$ -parameter, which is the  $O_{2n}(\mathbb{C})/SO_{2n}(\mathbb{C})$ -orbit of  $\phi_-$ , by setting its restriction to the last factor  $SL_2(\mathbb{C})$  to be trivial. This argument is also true for tempered  $L$ -parameters (see [Art13, Chapter 1.3]). Thus, tempered  $L$ -parameters automatically equal generic  $A$ -parameters, except even orthogonal cases, in which their  $O_{2n}(\mathbb{C})/SO_{2n}(\mathbb{C})$ -orbits are  $A$ -parameters.

Now, we are ready to construct  $A$ -packets. For each  $\phi_i$ , due to [Hen00], [HT01], and [HS12], we construct  $L$ -packets  $\Pi_{\phi_i}(GL_{n_i}(F))$  and  $\Pi_{\phi_i}(GL_{m_i}(D))$  consisting of discrete series representations of  $GL_{n_i}(F)$  and  $GL_{m_i}(D)$ , respectively. Note that  $\Pi_{\phi_i}(GL_{n_i}(F))$  and  $\Pi_{\phi_i}(GL_{m_i}(D))$  are singletons. For  $\phi_-$  of type  $\mathbf{B}_m$  or type  $\mathbf{C}_m$ , due to [Art13, Theorems 1.5.1 and 9.4.1], we construct  $L$ -packets  $\Pi_{\phi_-}(G_-)$  and  $\Pi_{\phi_-}(G'_-)$  consisting of discrete series representations of  $G_-$  and  $G'_-$ , respectively. On the other hand, for  $\phi_-$  of type  $\mathbf{D}_m$ , as in [Art13] again, we have to consider conjugation by  $O_{2m}(F)$ . Thus, we work with the  $O_{2m}(F)/SO_{2m}(F)$ -orbits in  $\text{Irr}(G)$ , rather than individual irreducible representations of  $G$ . We make the following remarks.

- Remark 3.2.* 1. Only for even orthogonal cases,  $A$ -packets consist of  $O_{2n}(F)/SO_{2n}(F)$ -orbits of order 2 or 1 (hence, not necessarily individual representations) of irreducible representations under the  $O_{2n}(F)$ -action by conjugation on  $G$ . The same is true for  $F$ -inner forms  $\mathbf{G}'$  of  $SO_{2n}$  and  $SO_{2n}^*$  (see [Art13, Chapters 1.5 and 9.4]).
2. We write  $\tilde{\text{Irr}}(G)$  for the set of  $O_{2n}(F)/SO_{2n}(F)$ -orbits in  $\text{Irr}(G)$  under the action of  $O_{2n}(F)$  by conjugation on  $G$ . Further, we write  $\tilde{\Pi}_{\phi}(G)$ , which is a subset of  $\tilde{\text{Irr}}(G)$ , for an  $A$ -packet attached to the  $A$ -parameter (the  $O_{2n}(\mathbb{C})/SO_{2n}(\mathbb{C})$ -orbit of  $\phi$ ) for  $G$ . As stated in 1 above, for  $G$  of type  $\mathbf{B}_n$  or type  $\mathbf{C}_n$ , we have  $\tilde{\text{Irr}}(G) = \text{Irr}(G)$  and  $\tilde{\Pi}_{\phi}(G)$  coincides with the  $L$ -packet  $\Pi_{\phi}(G)$ . The same notation and argument apply to the  $F$ -inner form  $\mathbf{G}'$ .

3. Even in the case of the even orthogonal groups, we have the following special cases in which conjugation by  $O_{2m}(F)$  is no longer necessary.

- (1) For  $\mathbf{G} = SO_{2n}, SO_{2n}^*$ , we let  $\phi$  be a tempered  $L$ -parameter for  $G$ . Suppose that  $\phi$  is  $O_{2n}(\mathbb{C})/SO_{2n}(\mathbb{C})$ -stable, i.e.,  $c_0\phi \simeq \phi$ . Then, by [Art13, Theorem 2.2.4(b)], every member in the  $A$ -packet  $\tilde{\Pi}_\phi(G)$  is also  $O_{2n}(F)/SO_{2n}(F)$ -stable. Thus,  $\tilde{\Pi}_\phi(G)$  consists of individual irreducible tempered representations of  $G$ , i.e.,  $\tilde{\Pi}_\phi(G) = \Pi_\phi(G)$ . This has also been discussed in [Art13, Chapter 8.4].
- (2) For the non-quasi-split  $F$ -inner form  $\mathbf{G}'$  of  $SO_{2n}$  or  $SO_{2n}^*$ , we let  $\phi$  be a tempered  $L$ -parameter for  $G'$ . It automatically follows from [Art13, Theorem 9.4.1] that the  $A$ -packet  $\tilde{\Pi}_\phi(G')$  consists of individual irreducible tempered representations of  $G'$ , i.e.,  $\tilde{\Pi}_\phi(G') = \Pi_\phi(G')$ . We should mention that two subscripts “dis” and “bdd” are identified in Chapter 9.4, so [Art13, Theorem 9.4.1] applies to our case. Further, this theorem, for the non-quasi-split  $F$ -inner form  $\mathbf{G}'$  of  $SO_{2n}$  or  $SO_{2n}^*$ , is based on the local refinements of [Art13, Chapter 8.4], where an  $\tilde{O}ut_{2n}(G)$ -torsor  $T(\phi)$  plays a role in parameterizing the  $L$ -packet  $\Pi_\phi(G)$  for a generic  $A$ -parameter  $\phi$ . Thus, the argument in (1) above applies to  $\mathbf{G}'$  as well, i.e., if  $\phi$  is  $O_{2n}(\mathbb{C})/SO_{2n}(\mathbb{C})$ -stable, then the torsor  $T(\phi)$  turns out to be a singleton and  $c_0\pi' \simeq \pi'$  for all  $\pi' \in \Pi_\phi(G')$ .

Let us return to the  $A$ -packet attached to the  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -orbit of  $\phi_-$  of type  $\mathbf{D}_m$ . Due to [Art13, Theorems 1.5.1 and 9.4.1] and Remark 3.2, we construct the  $A$ -packet  $\tilde{\Pi}_{\phi_-}(G_-)$  consisting of  $O_{2m}(F)/SO_{2m}(F)$ -orbits of order 2 or 1 of discrete series representations of  $G_-$ , and the  $L$ -packet  $\Pi_{\phi_-}(G'_-)$  consisting of discrete series representations of  $G'_-$ . Note that the action of  $O_{2m}(\mathbb{C})$  or  $O_{2m}(F)$  to be trivial except  $G_-$  of type  $\mathbf{D}_m$ . By taking the tensor product of members in packets for each  $\phi_i$  and  $\phi_-$ , we thus construct  $A$ -packets  $\tilde{\Pi}_\phi(M)$  of  $M$  and  $\tilde{\Pi}_\phi(M')$  of  $M'$ , associated to the elliptic tempered  $A$ -parameter which is the  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -orbit of the elliptic tempered  $L$ -parameter  $\phi$ . We should mention that the  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -orbit of  $\phi$  stands for the product of  $\phi_1, \dots, \phi_k$ , and the  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -orbit of  $\phi_-$ . Note that the group  $O_{2m}(\mathbb{C})$  acts only on the factor  $\phi_-$ .

*Remark 3.3.* We make the following remarks on the elliptic tempered  $A$ -packets.

1. Let  $\phi$  be an elliptic tempered  $L$ -parameter for  $M$  and  $M'$ . If  $c_0\phi \simeq \phi$ , it is always true that the elliptic tempered  $A$ -packet  $\tilde{\Pi}_\phi(M)$  consists of individual discrete series representations of  $M$ , so that  $\tilde{\Pi}_\phi(M) = \Pi_\phi(M)$  (see Remark 3.2.2). Further, regardless of the condition  $c_0\phi \simeq \phi$ , except for the case  $G_- = SO_{2m}, SO_{2m}^*$ , we have  $\tilde{\Pi}_\phi(M) = \Pi_\phi(M)$  (see Remark 3.2.3(1)). On the other hand, for the non-quasi-split inner form  $\mathbf{M}'$ , it is always true that  $\tilde{\Pi}_\phi(M') = \Pi_\phi(M')$  (see Remark 3.2.3(2)).
2. Arthur's classifications for  $SO_{2n}$  and  $SO_{2n}^*$  rely on some expected arguments on the stabilization of the twisted trace formula. So, as Arthur did in [Art13], we shall assume this in Hypothesis 3.4.a below.
3. The classification for non-quasi-split  $F$ -inner forms  $\mathbf{G}'$  of  $\mathbf{G}$  is simply stated in [Art13, Theorem 9.4.1] and the proof is expected to be delivered in future work [Art13, Chapter 9.1]. So, we will assume this in Hypothesis 3.4.b below.

**Hypotheses 3.4.** *a. We assume that the twisted trace formula for  $GL_n$  and a twisted even orthogonal group (whose identity component is  $SO_{2n}$  or  $SO_{2n}^*$ ) can be stabilized [Art13, Chapter 3.2].*

*b. We assume that the classification for non-quasi-split  $F$ -inner forms  $\mathbf{G}'$  of  $\mathbf{G}$  is known [Art13, Chapter 9.4].*

Following [Art13, Chapters 9.2 and 9.4], we review the component groups parameterizing  $A$ -packets and their connection with endoscopic  $R$ -groups. We borrow notation from [Art13] for the subscripts ‘ad’ and ‘sc’ to mean the adjoint and simply connected groups, respectively. Let  $S_\phi(\widehat{M})$  denote the group  $\pi_0(C_\phi(\widehat{M})) := C_\phi(\widehat{M})/C_\phi(\widehat{M})^\circ$  of connected components. Since  $\phi$  is elliptic, the identity component  $S_\phi(\widehat{M})^\circ$  is contained in  $Z(\widehat{M})^\Gamma$  (see [Kot84, Lemma 10.3.1] and [Art89, Section 7]). Thus, we have

$$S_\phi(\widehat{M}) = C_\phi(\widehat{M})/Z(\widehat{M})^\Gamma,$$

which can be considered as a finite subgroup of  $(\widehat{M})_{\text{ad}} := \widehat{M}/Z(\widehat{M})$ . It turns out that  $S_\phi(\widehat{M})$  is a finite abelian 2-group for our cases [Art13, Chapter 1.4]. Let  $S_{\phi,\text{sc}}(\widehat{M})$  be the full pre-image of  $C_\phi(\widehat{M})/Z(\widehat{M})^\Gamma$  in the simply connected cover  $\widehat{M}_{\text{sc}}$  of the derived group  $\widehat{M}_{\text{der}}$  of  $\widehat{M}$  under the isogeny  $\widehat{M}_{\text{sc}} \twoheadrightarrow (\widehat{M})_{\text{ad}}$ . One thus has the following exact sequence:

$$(3.1) \quad 1 \rightarrow Z(\widehat{M}_{\text{sc}}) \rightarrow S_{\phi,\text{sc}}(\widehat{M}) \rightarrow S_\phi(\widehat{M}) \rightarrow 1.$$

It is well known [Kot97, p. 280] that the  $F$ -inner form  $\mathbf{M}'$  of  $\mathbf{M}$  determines a unique character  $\zeta_{\mathbf{M}'}$  on  $Z(\widehat{M}_{\text{sc}})$  whose restriction to  $Z(\widehat{M}_{\text{sc}})^\Gamma$  corresponds to the isomorphism class of  $\mathbf{M}'$  via the Kottwitz isomorphism [Kot86, Theorem 1.2]. We denote by  $\text{Irr}(S_{\phi,\text{sc}}(\widehat{M}), \zeta_{\mathbf{M}'})$  the set of irreducible representations of  $S_{\phi,\text{sc}}(\widehat{M})$  whose restriction to  $Z(\widehat{M}_{\text{sc}})$  is equal to the character  $\zeta_{\mathbf{M}'}$ . We note that, for the case  $\mathbf{M}' = \mathbf{M}$ , the character  $\zeta_{\mathbf{M}}$  turns out to be the trivial character  $\mathbb{1}$ . Given an elliptic tempered  $L$ -parameter  $\phi$  for  $M'$ , there is a one-one bijection between the  $A$ -packet  $\widetilde{\Pi}_\phi(M')$ , attached to the  $A$ -parameter (the  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -orbit of  $\phi$ ), and  $\text{Irr}(S_{\phi,\text{sc}}(\widehat{M}), \zeta_{\mathbf{M}'})$  (see [Hen00], [HT01], [HS12], and [Art13, Theorems 1.5.1 and 9.4.1]). The following lemma is due to this bijection.

**Lemma 3.5.** *Let  $\mathbf{M}$  and  $\mathbf{M}'$  be as above. Given an elliptic tempered  $L$ -parameter  $\phi$  for  $M'$ , if the  $A$ -packet  $\widetilde{\Pi}_\phi(M)$ , attached to the  $A$ -parameter (the  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -orbit of  $\phi$ ), is a singleton, then so is the  $A$ -packet  $\widetilde{\Pi}_\phi(M')$ .*

*Proof.* It suffices to show that  $\text{Irr}(S_{\phi,\text{sc}}(\widehat{M}), \zeta_{\mathbf{M}'})$  is a singleton. Note that, if  $\mathbf{M} = \mathbf{M}'$ , then  $\text{Irr}(S_{\phi,\text{sc}}(\widehat{M}), \zeta_{\mathbf{M}}) = \text{Irr}(S_\phi(\widehat{M}))$ . Since  $\widetilde{\Pi}_\phi(M)$  is a singleton, we have  $\#\text{Irr}(S_\phi(\widehat{M})) = 1$ , which implies  $S_\phi(\widehat{M}) = \{1\}$ . From the exact sequence (3.1), we have  $S_{\phi,\text{sc}}(\widehat{M}) \simeq Z(\widehat{M}_{\text{sc}})$ , which is finite and abelian. Thus, we have  $\text{Irr}(S_{\phi,\text{sc}}(\widehat{M}), \zeta_{\mathbf{M}'}) = \{\zeta_{\mathbf{M}'}\}$ . This completes the proof.  $\square$

Through the natural embedding  $\widehat{M} \hookrightarrow \widehat{G}$ , we have

$$(3.2) \quad \phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L M \hookrightarrow {}^L G,$$

and  $\phi$  is considered a tempered  $L$ -parameter for  $G$ . From [Art13, Theorem 1.5.1] and the argument following [Art13, (1.5.1)], the set of inequivalent irreducible (hence, tempered) constituents of induced representations from all members in  $\widetilde{\Pi}_\phi(M)$

forms an  $A$ -packet  $\tilde{\Pi}_\phi(G)$  ( $= \Pi_\phi(G)$ ) of  $G$  associated to  $\phi$ , except for even orthogonal groups, in which case each member in  $\tilde{\Pi}_\phi(G)$  represents an  $O_{2n}(F)/SO_{2n}(F)$ -orbit of order 2 or 1 of irreducible tempered representations, rather than an individual representation (see Remarks 3.2 and 3.3.1). On the other hand, for the non-quasi-split inner form  $\mathbf{G}'$ , the set of inequivalent irreducible (hence, tempered) constituents of induced representations from all members in  $\tilde{\Pi}_\phi(M')$  forms an  $A$ -packet  $\tilde{\Pi}_\phi(G')$  ( $= \Pi_\phi(G')$ ) of  $G$  associated to  $\phi$ , for all cases (see [Art13, Theorem 9.4.1] and Remarks 3.2 and 3.3.1).

We now consider  $C_\phi(\widehat{G})/Z(\widehat{G})^\Gamma$  as a subgroup of  $(\widehat{G})_{\text{ad}}$ . Write  $C_{\phi,\text{sc}}(\widehat{G})$  for the full pre-image of  $C_\phi(\widehat{G})/Z(\widehat{G})^\Gamma$  in  $\widehat{G}_{\text{sc}}$  under the isogeny  $\widehat{G}_{\text{sc}} \twoheadrightarrow (\widehat{G})_{\text{ad}}$ . We then have the following exact sequence:

$$1 \rightarrow Z(\widehat{G}_{\text{sc}}) \rightarrow C_{\phi,\text{sc}}(\widehat{G}) \rightarrow C_\phi(\widehat{G})/Z(\widehat{G})^\Gamma \rightarrow 1.$$

Since  $\phi$  may not be elliptic for  $G$ , there is no guarantee that  $C_\phi(\widehat{G})/Z(\widehat{G})^\Gamma$  is finite. We let

$$\begin{aligned} S_\phi(\widehat{G}) &:= \pi_0(C_\phi(\widehat{G})/Z(\widehat{G})^\Gamma), \\ S_{\phi,\text{sc}}(\widehat{G}) &:= \pi_0(C_{\phi,\text{sc}}(\widehat{G})), \\ \widehat{Z}_{\phi,\text{sc}} &:= Z(\widehat{G}_{\text{sc}})/(Z(\widehat{G}_{\text{sc}}) \cap C_{\phi,\text{sc}}(\widehat{G})^\circ). \end{aligned}$$

We then have a central extension

$$1 \rightarrow \widehat{Z}_{\phi,\text{sc}} \rightarrow S_{\phi,\text{sc}}(\widehat{G}) \rightarrow S_\phi(\widehat{G}) \rightarrow 1;$$

cf. [Art13, (9.2.2)]. Let  $\zeta_{\mathbf{G}'}$  be a unique character on  $Z(\widehat{G}_{\text{sc}})$  whose restriction to  $Z(\widehat{G}_{\text{sc}})^\Gamma$  corresponds to the class of the  $F$ -inner form  $\mathbf{G}'$  of  $\mathbf{G}$  via the Kottwitz isomorphism [Kot86, Theorem 1.2]. We denote by  $\text{Irr}(S_{\phi,\text{sc}}(\widehat{G}), \zeta_{\mathbf{G}'})$  the set of irreducible representations of  $S_{\phi,\text{sc}}(\widehat{G})$  with central character  $\zeta_{\mathbf{G}'}$  on  $Z(\widehat{G}_{\text{sc}})$ . Given an elliptic tempered  $L$ -parameter  $\phi$  for  $M'$  which is tempered for  $G'$  via the composite (3.2), there is a one-one bijection between the  $A$ -packet  $\tilde{\Pi}_\phi(G')$ , attached to the  $A$ -parameter (the  $O_{2n}(\mathbb{C})/SO_{2n}(\mathbb{C})$ -orbit of  $\phi$ ), and  $\text{Irr}(S_{\phi,\text{sc}}(\widehat{G}), \zeta_{\mathbf{G}'})$  [Art13, Theorem 9.4.1]. Note that Lemma 3.5 is also true for tempered  $L$ -parameters for  $G$  and  $G'$ . Further, the diagrams in [Art13, (2.4.3) and (9.2.16)] yield

$$(3.3) \quad 1 \longrightarrow S_\phi(\widehat{M}) \longrightarrow S_\phi(\widehat{G}) \longrightarrow R_\phi \longrightarrow 1$$

and

$$(3.4) \quad 1 \longrightarrow S_{\phi,\text{sc}}(\widehat{M}) \longrightarrow S_{\phi,\text{sc}}(\widehat{G}) \longrightarrow R_\phi \longrightarrow 1.$$

**3.2. Weyl group actions.** In this subsection, we describe the action of the Weyl group  $W_M$  (respectively,  $W_{M'}$ ,  $W_{\widehat{M}}$ ) on  $M$  (respectively,  $M'$ ,  $\widehat{M}$ ) and  $\sigma$  (respectively,  $\sigma'$ ,  $\phi$ ). We identify  $\widehat{G} = \widehat{G}'$  and  $\widehat{M} = \widehat{M}'$ . Recall that the Weyl groups are

$$W_M = W(\mathbf{G}, \mathbf{A}_M) := N_{\mathbf{G}}(\mathbf{A}_M)/Z_{\mathbf{G}}(\mathbf{A}_M),$$

$$W_{M'} = W(\mathbf{G}', \mathbf{A}_{M'}) := N_{\mathbf{G}'}(\mathbf{A}_{M'})/Z_{\mathbf{G}'}(\mathbf{A}_{M'}),$$

and

$$W_{\widehat{M}} = W(\widehat{G}, A_{\widehat{M}}) := N_{\widehat{G}}(A_{\widehat{M}})/Z_{\widehat{G}}(A_{\widehat{M}}).$$

Through the duality

$$(3.5) \quad s_\alpha \mapsto s_{\alpha^\vee}$$

between simple reflections for  $\alpha \in \Delta$ , we have

$$(3.6) \quad W_M \simeq W_{\widehat{M}} \quad W_{\widehat{M}'} \simeq W_{M'};$$

cf. [Art13, Chapter 2.4]. We thus identify

$$(3.7) \quad W_M = W_{\widehat{M}} = W_{\widehat{M}'} = W_{M'}.$$

We start with the following lemma which will be used to describe the actions of Weyl groups.

**Lemma 3.6.** *Denote by  $D$  a central division algebra (possibly,  $F$  itself) of dimension  $d^2$  over  $F$ . Let  $\pi \in \text{Irr}(GL_m(D))$  be given. Then we have*

$$(3.8) \quad \tilde{\pi} \simeq {}^t\bar{\pi}^{-1}$$

as representations of  $GL_m(D)$ , where  ${}^t\bar{\pi}^{-1}(g) := \pi({}^t\bar{g}^{-1})$ , and  $\bar{g}$  denotes the usual involution of  $g$  in  $GL_m(D)$ , unless  $D = F$ , in which  $\bar{g} = g$ .

*Proof.* Following the idea in the proof of [MS00, Lemma 1.1], it suffices to prove that all representations in (3.8) have the same (Harish-Chandra) character over regular semi-simple elements. Given a regular semi-simple element  $g \in GL_m(D)$ , we note that  $g$  and  ${}^t\bar{g}$  are conjugate over  $\bar{F}$  because they are regular semi-simple and have the same characteristic polynomial over  $F$ . Since the character  $\Theta_{\tilde{\pi}}(g)$  equals  $\Theta_{\pi}(g^{-1})$  [DKV84, Introduction d.] and Harish-Chandra characters are invariant under conjugation by  $F$ -points, it thus remains to show that  $g$  and  ${}^t\bar{g}$  are conjugate over  $F$ .

Let  $x$  and  $y$  be two  $\bar{F}$ -conjugate regular semi-simple elements in  $GL_m(D)$ . We denote by  $\mathbf{G}'$  the  $F$ -inner form of  $GL_n$  with  $\mathbf{G}'(F) = GL_m(D)$ . Since  $\mathbf{G}'(\bar{F}) = GL_n(\bar{F})$  with  $n = md$ , we then have  $h \in GL_n(\bar{F})$  such that  $x = hyh^{-1}$ . It then follows that

$$hyh^{-1} = x = {}^v x = ({}^v h)y({}^v h^{-1})$$

for any  $v \in \Gamma$ , so that  $h^{-1}({}^v h)$  lies in the centralizer  $Z_{GL_n(\bar{F})}(y)$  of  $y$  in  $GL_n(\bar{F})$ . Thus, the mapping  $v \mapsto h^{-1}({}^v h)$  from  $\Gamma$  to  $Z_{GL_n(\bar{F})}(y)$  gives a 1-cocycle class in the Galois cohomology  $H^1(F, Z_{GL_n(\bar{F})}(y))$ . We further note from [Kot82, Section 3] that

$$\ker \left( H^1(F, Z_{GL_n(\bar{F})}(y)) \rightarrow H^1(F, \mathbf{G}') \right)$$

is in the bijection with the set of  $F$ -conjugacy classes in  $\bar{F}$ -conjugacy classes of  $y$ . Since  $y$  is a regular semi-simple element in  $GL_m(D)$ , it follows that  $Z_{GL_n(\bar{F})}(y)$  forms a torus which is isomorphic to  $\prod_j E_j^\times$  over  $F$ , where  $E_j$  is an extension of  $F$  of degree  $[E_j : F]$  with  $\sum [E_j : F] = m$ . By Hilbert's Theorem 90, we have  $H^1(F, Z_{GL_n(\bar{F})}(y)) = 1$ , which implies that  $x$  and  $y$  are conjugate over  $F$ . We thus conclude that  $g$  and  ${}^t\bar{g}$  are conjugate over  $F$ .  $\square$

**3.2.1. Weyl group actions on Levi subgroups and their representations.** For simplicity, in section 3.2.1, we will write  $\mathbf{G}$  for both quasi-split classical groups  $SO_{2n+1}$ ,  $Sp_{2n}$ ,  $SO_{2n}$ ,  $SO_{2n}^*$  and their non-quasi-split  $F$ -inner forms. We describe the action of  $W_M$  on an irreducible representation  $\sigma \in \text{Irr}(M)$ , based on the results in [Gol94] for split cases. Recall from section 2.2 that any of  $M$  is of the form

$$GL_{n_1}(D) \times GL_{n_2}(D) \times \cdots \times GL_{n_k}(D) \times G_-(m),$$

where  $D$  is a central division algebra of dimension 1 or 4 over  $F$  and  $\mathbf{G}_-$  is the same type as  $\mathbf{G}$  with lower rank  $m$ . We denote by  $S_k$  the symmetric group in  $k$  letters. Since we have from [Gol94, Section 2] and Appendix A

$$(3.9) \quad W_M \subset S_k \ltimes \mathbb{Z}_2^k,$$

the identification (3.7) implies that (3.9) is true for non-quasi-split cases. More precisely,  $W_M \simeq S \ltimes \mathcal{C}$ , where  $S = \langle (ij) | n_i = n_j \rangle$ , and  $\mathcal{C} \subset \mathbb{Z}_2^k$ . For  $g \in M$ , write

$$g = (g_1, \dots, g_i, \dots, g_j, \dots, g_k, h) \in GL_{n_1}(D) \times GL_{n_2}(D) \times \cdots \times GL_{n_k}(D) \times G_-(m).$$

The permutation  $(ij)$  acts on  $g \in M$

$$(3.10) \quad (ij) : (g_1, \dots, g_i, \dots, g_j, \dots, g_k, h) \mapsto (g_1, \dots, g_j, \dots, g_i, \dots, g_k, h).$$

The finite 2-group  $\mathbb{Z}_2^k$  is generated by elements  $C_i$  which acts on  $g \in M$

$$(3.11) \quad C_i : (g_1, \dots, g_i, \dots, g_k, h) \mapsto (g_1, \dots, {}^t\bar{g}_i^{-1}, \dots, g_k, h),$$

where  ${}^-$  denotes the involution in Lemma 3.6, which is trivial for quasi-split cases. Moreover, if  $\mathbf{G}$  is of type  $\mathbf{B_n}$  or  $\mathbf{C_n}$ , then  $\mathcal{C} = \mathbb{Z}_2^k$ , cf. [Her93, Section 2]. If  $\mathbf{G}$  is of type  $\mathbf{D_n}$  (i.e., either  ${}^1\mathbf{D_n}$  or  ${}^2\mathbf{D_n}$ ), then  $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ , where  $\mathcal{C}_1 = \langle C_i | n_i \text{ is even} \rangle$ , and

$$(3.12) \quad \mathcal{C}_2 = \begin{cases} \langle C_i C_j | n_i, n_j \text{ are odd} \rangle, & \text{if } m = 0; \\ \langle C_i c_0 | n_i \text{ is odd} \rangle, & \text{if } m > 0, \end{cases}$$

where  $c_0$  is the outer automorphism given by the reflection  $\alpha_{n-1} \leftrightarrow \alpha_n$  of the Dynkin diagram, cf. [Her93, Section 3]. Set  $\sigma$  to be  $\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_k \otimes \tau$ . From Lemma 3.6, (3.10), and (3.11), we then have

$$\begin{aligned} (ij)\sigma &= \sigma_1 \otimes \sigma_j \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes \sigma_k \otimes \tau; \\ C_i\sigma &= \sigma_1 \otimes \cdots \otimes \tilde{\sigma}_i \otimes \cdots \otimes \sigma_k \otimes \tau; \\ C_i c_0 \sigma &= \sigma_1 \otimes \cdots \otimes \tilde{\sigma}_i \otimes \cdots \otimes \sigma_k \otimes c_0 \tau, \end{aligned}$$

and these describe the action of  $W_M$  on  $\sigma$ .

**3.2.2. Weyl group actions on  $L$ -group  $\widehat{M}$  and  $L$ -parameter  $\phi$ .** In this section we describe the action of the Weyl group  $W_{\widehat{M}}$  on  $L$ -group  $\widehat{M}$  and on the elliptic tempered parameter  $\phi$ . From (3.7) and (3.9), we have

$$W_{\widehat{M}} \subset S_k \ltimes \mathbb{Z}_2^k.$$

More precisely,  $W_{\widehat{M}} \simeq S \ltimes \mathcal{C}$ , where  $S = \langle (ij) | n_i = n_j \rangle$ , and  $\mathcal{C} \subset \mathbb{Z}_2^k$ . For  $\hat{g} \in \widehat{M}$ , write

$$\hat{g} = (\hat{g}_1, \dots, \hat{g}_i, \dots, \hat{g}_j, \dots, \hat{g}_k, \hat{h}) \in GL_{n_1}(\mathbb{C}) \times GL_{n_2}(\mathbb{C}) \times \cdots \times GL_{n_k}(\mathbb{C}) \times \widehat{G}_-(m).$$

By the duality (3.5), the permutation  $(ij)$  acts on  $\hat{g} \in \widehat{M}$

$$(3.13) \quad (ij) : (\hat{g}_1, \dots, \hat{g}_i, \dots, \hat{g}_j, \dots, \hat{g}_k, \hat{h}) \mapsto (\hat{g}_1, \dots, \hat{g}_j, \dots, \hat{g}_i, \dots, \hat{g}_k, \hat{h}).$$

The finite 2-group  $\mathbb{Z}_2^k$  is generated by elements  $C_i$  which acts on  $\hat{g} \in \widehat{M}$

$$C_i : (\hat{g}_1, \dots, \hat{g}_i, \dots, \hat{g}_k, \hat{h}) \mapsto (\hat{g}_1, \dots, {}^t\hat{g}_i^{-1}, \dots, \hat{g}_k, \hat{h}).$$

Moreover, if  $\mathbf{G}$  is of type  $\mathbf{B}_n$  or  $\mathbf{C}_n$ , then  $\mathcal{C} = \mathbb{Z}_2^k$ . If  $\mathbf{G}$  is of type  $\mathbf{D}_n$  (i.e., either  ${}^1\mathbf{D}_n$  or  ${}^2\mathbf{D}_n$ ), then  $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ , where  $\mathcal{C}_1 = \langle C_i | n_i \text{ is even} \rangle$ , and

$$(3.14) \quad \mathcal{C}_2 = \begin{cases} \langle C_i C_j | n_i, n_j \text{ are odd} \rangle, & \text{if } m = 0; \\ \langle C_i c_0 | n_i \text{ is odd} \rangle, & \text{if } m > 0, \end{cases}$$

where  $c_0$  is the outer automorphism given by the reflection  $\alpha_{n-1}^\vee \leftrightarrow \alpha_n^\vee$  of the Dynkin diagram. Set  $\phi$  to be  $\phi_1 \oplus \phi_2 \oplus \dots \oplus \phi_k \oplus \phi_-$ . Since  $\phi_i({}^t\hat{g}_i^{-1}) = \tilde{\phi}(\hat{g}_i)$  for  $\hat{g}_i \in GL_{n_i}(\mathbb{C})$  [HT01, Hen00, AV12], where  $\tilde{\phi}$  is the contragredient of  $\phi$ , we have

$$\begin{aligned} (ij)\phi &= \phi_1 \oplus \phi_j \oplus \dots \oplus \phi_i \oplus \dots \oplus \phi_k \oplus \phi_-; \\ C_i \phi &= \phi_1 \oplus \dots \oplus \tilde{\phi}_i \oplus \dots \oplus \phi_k \oplus \phi_-; \\ C_i c_0 \phi &= \phi_1 \oplus \dots \oplus \tilde{\phi}_i \oplus \dots \oplus \phi_k \oplus c_0 \phi_-, \end{aligned}$$

and these describe the action of  $W_{\widehat{M}}$  on  $\phi$ .

*Remark 3.7.* For the set  $\mathcal{C}_2$  defined in (3.12) and (3.14), we have either  $m > 0$  or each  $m_i$  is even due to section 2.2. It hence follows that  $\{C_i C_j | n_i, n_j \text{ are odd}\}$  is empty, and that  $\{C_i c_0 | n_i \text{ is odd}\}$  is non-empty if and only if  $\mathbf{G}$  is of type labelled “n: any” of  ${}^1\mathbf{D}_n$  in section 2.2.

**3.3. Main theorem.** In this section, we prove the following theorem, which asserts that the three  $R$ -groups, Knapp-Stein, Arthur and endoscopic, are identical if they are attached to the same elliptic tempered  $L$ -parameter. Before we state our main theorem, it is worth making the following remark.

*Remark 3.8.* For the purpose of studying  $R$ -groups,  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -orbits and  $O_{2m}(F)/SO_{2m}(F)$ -orbits for  $SO_{2n}$  or  $SO_{2n}^*$  can be ignored. To be precise, we let  $\mathbf{M}$  be an  $F$ -Levi subgroup of  $SO_{2n}$  or  $SO_{2n}^*$ , and let  $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L M$  be an elliptic tempered  $L$ -parameter for  $M$ . Given  $\sigma \in \Pi_{\text{disc}}(M)$ , we let the  $O_{2m}(F)/SO_{2m}(F)$ -orbit of  $\sigma$  be in the  $A$ -packet  $\widetilde{\Pi}_\phi(M)$ , attached to the  $A$ -parameter (the  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -orbit of  $\phi$ ). If  $w\sigma \simeq \sigma$  with  $w \in W_M$ , then  $w\phi \simeq \phi$ , which implies that  $c_0\phi_- \simeq \phi_-$  (see section 3.2). So,  $\phi_-$  turns out to be  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -stable (hence, so is  $\phi$ ). Further, from Remark 3.3.1,  $\widetilde{\Pi}_\phi(M) = \Pi_\phi(M)$ , i.e., the  $O_{2m}(F)/SO_{2m}(F)$ -orbit of  $\sigma$  is just a singleton. For the non-quasi-split  $F$ -inner form  $\mathbf{G}'$  of  $SO_{2n}$  or  $SO_{2n}^*$ , as in Remark 3.3.1 again, we do not need to consider these orbits.

Due to Remark 3.8, throughout section 3.3, we can no longer distinguish elliptic tempered  $L$ -parameters and their  $O_{2m}(\mathbb{C})/SO_{2m}(\mathbb{C})$ -orbits. Likewise, given an elliptic tempered  $L$ -parameter, we can assume that its  $L$ -packet coincides with its  $A$ -packet.

**Theorem 3.9.** *Let  $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L M$  be an elliptic tempered  $L$ -parameter. Given any  $\sigma \in \Pi_\phi(M)$  and  $\sigma' \in \Pi_\phi(M')$ , we have*

$$R_\sigma \simeq R_{\phi, \sigma} \simeq R_\phi \simeq R_{\phi, \sigma'} \simeq R_{\sigma'}.$$

*Remark 3.10.* 1. For the quasi-split case, the isomorphism

$$R_\sigma \simeq R_{\phi,\sigma} \simeq R_\phi$$

is proved in [BG12] and in [Art13, Chapters 6.5 & 6.6]. However, we will develop our own proof which works for both quasi-split and non-quasi-split cases, except the argument in [Art13, p. 346] by Arthur that

$$(3.15) \quad W'_\sigma \supset W_\phi^\circ.$$

This will be used to verify the containment (3.17) below.

2. For  $\mathbf{G} = SO_{2n}$  or  $SO_{2n}^*$ , and its non-quasi-split  $F$ -inner form  $\mathbf{G}'$ , as mentioned in Remark 3.2.3, we have

$$c_0\phi \simeq \phi \implies c_0\sigma \simeq \sigma \quad \text{and} \quad c_0\sigma' \simeq \sigma'.$$

Combining Theorem 3.9 with the second named author's result for  $R_\sigma$  in [Gol94], we obtain the following corollaries which give a description of  $R_{\sigma'}$ .

**Corollary 3.11.** *Let  $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L M$  be an elliptic tempered  $L$ -parameter. Let  $\sigma \in \Pi_\phi(M)$  and  $\sigma' \in \Pi_\phi(M')$  be given. Then,  $i_{G,M}(\sigma)$  is irreducible if and only if  $i_{G',M'}(\sigma')$  is irreducible.*

**Corollary 3.12.** *Let  $\mathbf{M}'$  be an  $F$ -Levi subgroup of  $\mathbf{G}'$  of type  $\mathbf{B}_n$  or  $\mathbf{C}_n$ . Let  $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L M$  be an elliptic tempered  $L$ -parameter. Let  $\sigma' \in \Pi_\phi(M')$  be given. Set*

$I(\sigma') = \{1 \leq i \leq k : \sigma'_i \simeq \tilde{\sigma}'_i \text{ and } i_{G'(n_i+m), GL(n_i) \times G'(m)}(\sigma' \otimes \tau') \text{ is reducible}\}$ ,  
and  $d$  is the number of inequivalent  $\sigma'_i$  such that  $i \in I(\sigma')$ . Then, we have

$$R_{\sigma'} \simeq \mathbb{Z}_2^d,$$

and  $R_{\sigma'}$  is generated by the  $d$  sign changes

$$\{C_i | i \in I_{\sigma'}, \text{ and } \sigma'_j \not\simeq \sigma'_i \text{ for all } j > i\}.$$

**Corollary 3.13.** *Let  $\mathbf{M}'$  be an  $F$ -Levi subgroup of  $\mathbf{G}'$  of type  $\mathbf{D}_n$  (i.e., either  ${}^1\mathbf{D}_n$  or  ${}^2\mathbf{D}_n$ ). Let  $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L M$  be an elliptic tempered  $L$ -parameter. Let  $\sigma' \in \Pi_\phi(M')$  be given. Denote  $I_1 = \{1, 2, \dots, k\}$  if  $m \geq 2$  and  $c_0\tau' \simeq \tau'$ , otherwise,  $I_1 = \{1 \leq i \leq k : n_i \text{ is even}\}$ . Denote  $I_2 = \{1, 2, \dots, k\} - I_1$ . Set*

$$I_1(\sigma') = \{i \in I_1 : \sigma'_i \simeq \tilde{\sigma}'_i \text{ and } i_{G'(n_i+m), GL(n_i) \times G'(m)}(\sigma' \otimes \tau') \text{ is reducible}\},$$

$$I_2(\sigma') = \{i \in I_2 : \sigma'_i \simeq \tilde{\sigma}'_i\}.$$

Let  $d_j$  denote the number of inequivalent  $\sigma'_i$  such that  $i \in I_j(\sigma')$ , and let  $d = d_1 + d_2$ . If  $d_2 = 0$ , then

$$R_{\sigma'} \simeq \mathbb{Z}_2^d.$$

If  $d_2 > 0$ , then

$$R_{\sigma'} \simeq \mathbb{Z}_2^{d-1}.$$

In either case,  $R_{\sigma'}$  is a subgroup of  $W_{M'}$  generated by sign changes.

The rest of section 3.3 is devoted to the proof of Theorem 3.9.

**Lemma 3.14.** *Let  $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L M$  be an elliptic tempered  $L$ -parameter. Under the identity (3.7), we have*

$$W(\sigma) = W_\phi = W(\sigma').$$

*Proof.* Let  $w \in W_\phi$  be given. Then by abuse of notation its representative  $w \in C_\phi(\widehat{G})$  satisfies  ${}^w\phi \simeq \phi$ . Set  $\phi$  to be  $\phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_k \oplus \phi_-$ . By the action of  $W_{\widehat{M}}$  on  $L$ -parameters in section 3.2.2, there exist  $i, j$  with  $1 \leq i, j \leq k$  (possibly,  $i = j$ ) such that

- (i)  $\phi_i \simeq \phi_j$ , or  $\phi_i \simeq \tilde{\phi}_j$  if  $\mathbf{G} = SO_{2n+1}, Sp_{2n}$ ,
- (ii)  $\phi_i \simeq \phi_j$ , or  $\phi_i \simeq \phi_j$ , or  $\phi_i \simeq \tilde{\phi}_i$  and  $c_0\phi_- \simeq \phi_-$  if  $\mathbf{G} = SO_{2n}, SO_{2n}^*$ .

Due to the local Langlands correspondence for  $GL_n$  for the case  $\phi_i, \phi_j$  ([Hen00, HT01]) and for  $G$  for the case  $\phi_-$  ([Art13, Theorem 1.5.1] and Remark 3.10.2), we have  $i, j$  with  $1 \leq i, j \leq k$  (possibly,  $i = j$ ) such that

- (a)  $\sigma_i \simeq \sigma_j$  or  $\sigma_i \simeq \tilde{\sigma}_j$  if  $\mathbf{G} = SO_{2n+1}, Sp_{2n}$  (see [Gol94, Section 4], [Her93, Section 2]),
- (b)  $\sigma_i \simeq \sigma_j$ , or  $\sigma_i \simeq \tilde{\sigma}_j$ , or  $\sigma_i \simeq \tilde{\sigma}_i$  and  $c_0\sigma_- \simeq \sigma_-$  if  $\mathbf{G} = SO_{2n}, SO_{2n}^*$  (see [Her93, Section 3], Appendix A).

By the action of  $W_M$  on  $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_k \otimes \tau$  in section 3.2.2, we have

$${}^w\sigma \simeq \sigma,$$

which implies  $w \in W(\sigma)$ . Likewise, by the action of  $W_{M'}$  on  $\sigma'$  in section 3.2.1 we have

$${}^w\sigma' \simeq \sigma',$$

which implies  $w \in W(\sigma')$ . Note that, for the local Langlands correspondence of a non-split  $F$ -inner form  $M'$ , we apply [HS12] and [Art13, Theorems 1.5.1 and 9.4.1].

Let  $w \in W(\sigma)$  be given. It then follows from the action of  $W_M$  on  $\sigma$  in section 3.2.1 that there exist  $i, j$  with  $1 \leq i, j \leq k$  (possibly,  $i = j$ ) satisfying (a) and (b) above. Thus, by the local Langlands correspondence, we have (i) and (ii) above. The action of  $W_{\widehat{M}}$  on  $L$ -parameters in section 3.2.2 then yields

$${}^w\phi \simeq \phi,$$

which implies  $w \in C_\phi(\widehat{G})$ . Therefore, we have proved Lemma 3.14.  $\square$

**Lemma 3.15.** *Let  $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L M$  be an elliptic tempered  $L$ -parameter. Under the identity (3.7), we then have*

$$W'_\sigma = W_\phi^\circ = W'_{\sigma'}.$$

*Proof.* Let us start with the case that  $M$  is maximal. Since  $M$  is maximal, we have  $W_M$  is either trivial or isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . It thus suffices to consider the case  $W_M \simeq \mathbb{Z}/2\mathbb{Z}$ . Let  $w \in W_\phi^\circ$  be given. If  $w$  is a trivial element, it is clear that  $w$  sits in  $W'(\sigma)$  as well as  $W'(\sigma')$ . Suppose that  $w$  is a non-trivial element. We then have  $R_\phi = 1$ . From (3.3) and (3.4), we have

$$S_\phi(\widehat{G}) = S_\phi(\widehat{M}) \quad \text{and} \quad S_{\phi, \text{sc}}(\widehat{G}) = S_{\phi, \text{sc}}(\widehat{M}).$$

Thus, we have equalities of the sizes of  $L$ -packets ( $A$ -packets coincide with  $L$ -packets by Remarks 3.2 and 3.3.1, since  $\phi$  is generic and  $c_0\phi \simeq \phi$  (because  $w\phi \simeq \phi$  and  $w = C_1c_0$ , see section 3.2)), i.e.,

$$\#\Pi_\phi(G) = \#\Pi_\phi(M) \quad \text{and} \quad \#\Pi_\phi(G') = \#\Pi_\phi(M'),$$

which imply that both induced representations  $i_{G,M}(\sigma)$  and  $i_{G',M'}(\sigma')$  are irreducible for all  $\sigma \in \Pi_\phi(M)$  and  $\sigma' \in \Pi_\phi(M')$ , respectively. Hence, we have  $R_\sigma = R_{\sigma'} = 1$ . So,  $w$  must be in  $W'_\sigma$  and  $W'_{\sigma'}$ .

Let  $w \in W'_\sigma$  be given. If  $w$  is a trivial element, it is clear that  $w$  sits in  $W_\phi^\circ$ . Suppose  $w$  is a non-trivial element. We then have  $R_\sigma = 1$ . Thus, all induced representations  $i_{G,M}(\sigma)$ ,  $\sigma \in \Pi_\phi(M)$ , are irreducible, which implies  $\#\Pi_\phi(G) = \#\Pi_\phi(M)$  and thus  $R_\phi = 1$  by (3.3). So,  $w$  must be in  $W_\phi^\circ$ . The same argument applies to  $W'_{\sigma'}$ . Therefore, we have proved Lemma 3.15 for the maximal case  $M$  and  $M'$ .

We return to an arbitrary  $F$ -Levi subgroup. We shall complete the proof by verifying two containments

$$(3.16) \quad W'_\sigma \subset W_\phi^\circ \quad (W'_{\sigma'} \subset W_\phi^\circ)$$

and

$$(3.17) \quad W'_\sigma \supset W_\phi^\circ \quad (W'_{\sigma'} \supset W_\phi^\circ).$$

To see (3.16), we let  $w \in W'_\sigma$  such that  $w$  equals the element  $w_\alpha$  representing the reflection with respect to some  $\alpha \in \Phi(P, A_M)$ . By definition, we have  $\mu_\alpha(\sigma) = 0$ . We let  $A_\alpha := (\ker \alpha \cap A_M)^\circ$  be the identity component of  $(\ker \alpha \cap A_M)$ . Set  $\mathbf{M}_\alpha := Z_{\mathbf{G}}(A_\alpha)$ , then  $\mathbf{M}_\alpha$  contains  $\mathbf{M}$  as a maximal  $F$ -Levi subgroup. From Remark 2.2, we have

$$T_\phi \subset \widehat{M} \subset \widehat{M}_\alpha.$$

Set

$$(3.18) \quad W_{\phi,\alpha}^\circ := N_{C_\phi(\widehat{M}_\alpha)^\circ}(T_\phi)/Z_{C_\phi(\widehat{M}_\alpha)^\circ}(T_\phi).$$

By the above argument for the maximal case, we have

$$w \in W_{\phi,\alpha}^\circ.$$

It follows that

$$w \in N_{C_\phi(\widehat{M}_\alpha)^\circ}(T_\phi) \subset N_{C_\phi(\widehat{G})^\circ}(T_\phi)$$

since

$$C_\phi(\widehat{M}_\alpha)^\circ \subset C_\phi(\widehat{G})^\circ.$$

Thus, we must have  $w \in W_\phi^\circ$ . The same argument applies to  $W'_{\sigma'}$ . By the definitions of  $W'_\sigma$  and  $W'_{\sigma'}$ , we have proved the containment (3.16).

To see (3.17), we recall the argument (3.15). So, it remains to show that

$$(3.19) \quad W'_\sigma = W'_{\sigma'}.$$

We note that  $\Phi(P, A_M) = \Phi(P', A_{M'})$  since  $\mathbf{M}$  and  $\mathbf{M}'$  are inner forms of each other and their  $F$ -points are respectively of the form (2.1) and (2.2). For given  $\alpha \in \Phi(P, A_M) = \Phi(P', A_{M'})$ , by the argument for the maximal case, we have

$$\mu_\alpha(\sigma) = 0 \quad \text{if and only if} \quad \mu_\alpha(\sigma') = 0$$

since these Plancherel measures are those attached to the induced representations  $i_{M_\alpha, M}(\sigma)$  and  $i_{M'_\alpha, M'}(\sigma')$  from maximal  $F$ -Levi subgroups  $M$  and  $M'$ , respectively. By the definitions of  $\Delta'_\sigma$  and  $\Delta'_{\sigma'}$  in section 2.3, we then have

$$\Delta'_\sigma = \Delta'_{\sigma'},$$

which implies the equation (3.19). Thus, we have proved the containment (3.17). This completes the proof of Lemma 3.15.  $\square$

*Proof of Theorem 3.9.* From Lemmas 3.14 and 3.15, we have

$$R_\sigma \simeq R_\phi \simeq R_{\sigma'}.$$

Therefore, the definition (2.4) yields that

$$(3.20) \quad R_{\phi,\sigma} \simeq R_\phi \simeq R_{\phi,\sigma'}.$$

□

#### 4. TRANSFER OF ELLIPTIC SPECTRA

In this section, we show that the elliptic spectrum of  $\mathbf{G} = SO_{2n+1}, Sp_{2n}, SO_{2n}, SO_{2n}^*$  is identically transferred to its  $F$ -inner form  $\mathbf{G}'$ . An elliptic representation of  $G$  is the one whose Harish-Chandra character (see [HC81]) does not vanish on the elliptic regular set of  $G$ , cf. [Art93]. The elliptic spectrum is studied for  $SO_{2n+1}$ ,  $Sp_{2n}$ , and  $SO_{2n}$  by Herb [Her93], and for certain  $F$ -inner forms of  $Sp_{4n}$  and  $SO_{4n}$  by Hanzer [Han04], in which both rely on the second named author's work in [Gol94]. We start with the following result of Arthur for a general connected reductive group.

**Theorem 4.1** (Arthur, [Art93, Section 2.1]). *Let  $\mathbf{G}$  be a connected reductive group over  $F$  and let  $\mathbf{M}$  be an  $F$ -Levi subgroup of  $\mathbf{G}$ . Let  $\sigma \in \Pi_{\text{disc}}(M)$  be given. Suppose that  $R_\sigma$  is abelian and that  $C(\sigma) \simeq \mathbb{C}[R_\sigma]$ . Then  $i_{G,M}(\sigma)$  has an elliptic constituent if and only if all constituents of  $i_{G,M}(\sigma)$  are elliptic, and if and only if there is  $w \in R$  such that  $\mathfrak{a}_w = \mathfrak{z}$ . Here  $\mathfrak{a}_w := \{H \in \mathfrak{a}_M : wH = H\}$ , and  $\mathfrak{z}$  denotes the real Lie algebra of the split component  $A_{\mathbf{G}}$  of  $\mathbf{G}$ .*

We recall Herb's results on the elliptic spectrum for  $\mathbf{G} = SO_{2n+1}, Sp_{2n}, SO_{2n}$  as follows.

**Proposition 4.2** (Herb, [Her93, Proposition 2.3 and 3.3]). *Let  $\mathbf{G}$  be  $SO_{2n+1}, Sp_{2n}$  or  $SO_{2n}$  over  $F$  and let  $\mathbf{M}$  be an  $F$ -Levi subgroup of  $\mathbf{G}$ . Let  $\sigma \in \Pi_{\text{disc}}(M)$  be given. Then*

$$C(\sigma) \simeq \mathbb{C}[R_\sigma].$$

The result that  $R_\sigma$  is abelian for  $\mathbf{G}$  [Gol94] is combined with Theorem 4.1 and Proposition 4.2 to give the elliptic spectrum for  $\mathbf{G}$  as follows.

**Theorem 4.3** (Herb, [Her93, Theorem 2.5]). *Let  $\mathbf{G}$  be  $SO_{2n+1}$  or  $Sp_{2n}$  over  $F$  and let  $\mathbf{M}$  be an  $F$ -Levi subgroup of  $\mathbf{G}$  such that  $M \simeq GL_{n_1}(F) \times GL_{n_2}(F) \times \cdots \times GL_{n_k}(F) \times G_-(m)$ . Let  $\sigma \in \Pi_{\text{disc}}(M)$  be given. Then  $i_{G,M}(\sigma)$  has an elliptic constituent if and only if all constituents of  $i_{G,M}(\sigma)$  are elliptic if and only if*

$$R_\sigma \simeq \mathbb{Z}_2^k.$$

**Theorem 4.4** (Herb, [Her93, Theorem 3.5]). *Let  $\mathbf{G}$  be  $SO_{2n}$  over  $F$  and let  $\mathbf{M}$  be an  $F$ -Levi subgroup of  $\mathbf{G}$  of the form (2.1). Let  $\sigma \in \Pi_{\text{disc}}(M)$  be given. Then  $i_{G,M}(\sigma)$  has an elliptic constituent if and only if all constituents of  $i_{G,M}(\sigma)$  are elliptic if and only if either*

$$R_\sigma \simeq \mathbb{Z}_2^k \text{ or } R_\sigma \simeq (\mathbb{Z}_2)^{k-1} \text{ with } d_2 \text{ even.}$$

Here  $d_2$  is defined unless  $m \leq 1$  and  $c_0\tau \simeq \tau$ , in which case  $d_2$  denotes the number of inequivalent  $\sigma_i$  such that  $\sigma_i \simeq \tilde{\sigma}_i$  and  $n_i$  is odd.

Applying our results in section 3.3 and the above arguments, we will present the elliptic spectrum for  $\mathbf{G}'$ . We will first deal with the case of an  $F$ -inner form  $\mathbf{G}'$  of  $SO_{2n+1}$  or  $Sp_{2n}$ .

**Proposition 4.5.** *Let  $\mathbf{G}'$  be an  $F$ -inner form of  $SO_{2n+1}$  or  $Sp_{2n}$ , and let  $\mathbf{M}'$  be an  $F$ -Levi subgroup of  $\mathbf{G}'$ . Let  $\sigma' \in \Pi_{\text{disc}}(M')$  be given. Then*

$$C(\sigma') \simeq \mathbb{C}[R_{\sigma'}].$$

*Proof.* We follow the proof of [Her93, Proposition 2.3]. Recall from Corollary 3.12 that  $C_1, \dots, C_d$  are the generators of  $R_{\sigma'}$ . For  $1 \leq i \leq d$ , let  $V_i$  be the representation space of  $\sigma'_i$ . We note from Lemma 3.6 that

$$\sigma'_i \simeq \tilde{\sigma}'_i \simeq {}^t \bar{\sigma}'_i^{-1}.$$

Let  $T_i : V_i \rightarrow V_i$  be an intertwining operator between  $\sigma'_i$  and  ${}^t \bar{\sigma}'_i$ . Again, from Lemma 3.6, the composite  $T_i^2 = T_i \circ T_i$  is a non-zero complex scalar. Thus we normalize  $T_i$  so that  $T_i^2 = 1$ . Now we extend  $T_i$  to an endomorphism  $T_i^V$  of the representation space  $V$  of  $\sigma' = \sigma'_1 \otimes \dots \otimes \sigma'_k \otimes \tau'$  such that  $T_i^V$  acts trivially on the space  $V_j$  for all  $1 \leq j \neq i \leq d$ . Then  $T_i^V$  intertwines  $C_i \sigma'$  and  $\sigma'$ . Further, we have  $(T_i^V)^2 = 1$  from our normalization of  $T_i$ . For  $1 \leq i \neq j \leq d$ , we also notice, from the definition of  $T_i^V$ , that  $T_i^V T_j^V = T_j^V T_i^V$ . Then, the map  $C_i \mapsto T_i^V$  gives rise to a homomorphism from  $\mathbb{C}[R_{\sigma'}]$  to  $\text{End}_{G'}(\sigma')$ . Thus, Proposition 4.5 follows from Theorem 2.1.  $\square$

**Corollary 4.6.** *Let  $G', M', \sigma'$  be as in Proposition 4.5. Then each constituent of  $i_{G', M'}(\sigma')$  appears with multiplicity one.*

Combining Corollary 3.12, Theorem 4.1, and Proposition 4.5, we have the following description of the elliptic spectrum for an  $F$ -inner form  $\mathbf{G}'$  of  $SO_{2n+1}$  or  $Sp_{2n}$ .

**Theorem 4.7.** *Let  $\mathbf{G}'$  be an  $F$ -inner form of  $SO_{2n+1}$  or  $Sp_{2n}$ , and let  $\mathbf{M}'$  be an  $F$ -Levi subgroup of  $\mathbf{G}'$  of the form (2.2). Let  $\sigma' \in \Pi_{\text{disc}}(M')$  be given. Then,  $i_{G', M'}(\sigma')$  has an elliptic constituent if and only if all constituents of  $i_{G', M'}(\sigma')$  are elliptic if and only if*

$$R_{\sigma'} \simeq \mathbb{Z}_2^k.$$

$\square$

Now, we will deal with the case of an  $F$ -inner form  $\mathbf{G}'$  of  $SO_{2n}$  or  $SO_{2n}^*$ .

**Proposition 4.8.** *Let  $\mathbf{G}'$  be an  $F$ -inner form of  $SO_{2n}$  or  $SO_{2n}^*$ , and let  $\mathbf{M}'$  be an  $F$ -Levi subgroup of  $\mathbf{G}'$ . Let  $\sigma' \in \Pi_{\text{disc}}(M')$  be given. Then*

$$C(\sigma') \simeq \mathbb{C}[R_{\sigma'}].$$

*Proof.* We follows the proof of [Her93, Proposition 3.3]. Suppose that  $m = 0$ , or that  $m \geq 2$  but  $c_0 \tau' \not\simeq \tau'$ . In this case,  $I_1$  (respectively,  $I_2$ ) consist of all  $i$ 's such that  $n_i$  is even (respectively, odd), cf. Remark 3.7. If  $d_2 \leq 1$ , then  $R_{\sigma'}$  is generated by  $d_1$  sign changes in indices  $i \in I_1(\sigma')$ , and the proof is the same as that of Proposition 4.5. Assume that  $d_2 \geq 2$ . Then, Corollary 3.13 tells us that  $R_{\sigma'} \simeq \mathbb{Z}_2^{d_1+d_2-1}$ . Note that the set

$$\{C_1 C_{d_2}, C_2 C_{d_2}, \dots, C_{d_2-1} C_{d_2}, C_{d_2+1}, \dots, C_d\}$$

forms a complete set of generators for  $R_{\sigma'}$ . As in the proof of Proposition 4.5, for  $1 \leq i \leq d$ , we obtain  $T_i : V_i \rightarrow V_i$  intertwining  $\sigma'_i$  and  $\tilde{\sigma}'_i$ , such that  $T_i^2 = 1$  and

such that each  $T_i$  is extended to an endomorphism  $T_i^V$  of the representation space  $V$  of  $\sigma'$ . Again, we note that  $(T_i^V)^2 = 1$  and  $T_i^V T_j^V = T_j^V T_i^V$ . Then, the map

$$\begin{aligned} C_i C_{d_2} &\mapsto T_i^V T_{d_2}^V, \quad \text{if } 1 \leq i \leq d_2 - 1, \\ C_i &\mapsto T_i^V, \quad \text{if } d_2 + 1 \leq i \leq d, \end{aligned}$$

defines a homomorphism from  $\mathbb{C}[R_{\sigma'}]$  to  $\text{End}_{G'}(\sigma')$ .

Suppose  $m \geq 2$  and  $c_0 \tau' \simeq \tau'$ . Then, Corollary 3.13 tells us that  $d_1 = d$  and  $R_{\sigma'} \simeq \mathbb{Z}_2^d$ . Note that  $C_1, \dots, C_d$  form a complete set of generators for  $R_{\sigma'}$ . We renumber indices so that

$$\begin{aligned} I_1(\sigma') \cap \{i : n_i \text{ odd}\} &= \{1, \dots, p\}, \\ I_1(\sigma') \cap \{i : n_i \text{ even}\} &= \{p + 1, \dots, d\}. \end{aligned}$$

From the condition  $c_0 \tau' \simeq \tau'$ , we choose an intertwining operator  $T_- : V_- \rightarrow V_-$ , where  $V_-$  is the representation space of  $\tau'$ . As in the proof of Proposition 4.5, we normalize  $T_-$  so that  $T_-^2 = 1$ , and we extend it to an operator  $T_-^V$  on  $V$  such that  $T_-^V$  acts trivially on the space  $V_i$  for all  $1 \leq i \leq d$ . Again, the map

$$\begin{aligned} C_i &\mapsto T_i^V T_-^V, \quad \text{if } 1 \leq i \leq p, \\ C_i &\mapsto T_i^V, \quad \text{if } p + 1 \leq i \leq d, \end{aligned}$$

defines a homomorphism from  $\mathbb{C}[R_{\sigma'}]$  to  $\text{End}_{G'}(\sigma')$ . Thus, Proposition 4.8 follows from Theorem 2.1.  $\square$

**Corollary 4.9.** *Let  $G', M', \sigma'$  be as in Proposition 4.8. Then each constituent of  $i_{G', M'}(\sigma')$  appears with multiplicity one.*

Corollary 3.13, Theorem 4.1, and Proposition 4.8 gives the following description of the elliptic spectrum for an  $F$ -inner form  $\mathbf{G}'$  of  $SO_{2n}$  or  $SO_{2n}^*$ .

**Theorem 4.10.** *Let  $\mathbf{G}'$  be an  $F$ -inner form of  $SO_{2n}$  or  $SO_{2n}^*$ , and let  $\mathbf{M}'$  be an  $F$ -Levi subgroup of  $\mathbf{G}'$  of the form (2.2). Let  $\sigma' \in \Pi_{\text{disc}}(\mathbf{M}')$  be given. Then,  $i_{G', M'}(\sigma')$  has an elliptic constituent if and only if all constituents of  $i_{G', M'}(\sigma')$  are elliptic if and only if either*

$$R_{\sigma'} \simeq \mathbb{Z}_2^k \text{ or } R_{\sigma'} \simeq (\mathbb{Z}_2)^{k-1} \text{ with even } d_2.$$

Here  $d_2$  is defined as in Corollary 3.13, unless  $m \leq 1$  and  $c_0 \tau' \simeq \tau'$ , in which case  $d_2$  denotes the number of inequivalent  $\sigma'_i$  such that  $\sigma'_i \simeq \tilde{\sigma}'_i$  and  $n_i$  is odd.

## APPENDIX A. $R$ -GROUPS FOR A QUASI-SPLIT EVEN ORTHOGONAL GROUP

We give a brief description of the Knapp-Stein  $R$ -groups for the non-split quasi-split classical group  $SO_{2n}^*$ . In short, the arguments used for the split case are valid here as well. Let  $\mathbf{G} = \mathbf{G}(n) = SO_{2n}^*$  be defined over  $F$ , and  $G = \mathbf{G}(F)$ . We also let  $\tilde{\mathbf{G}} = \tilde{\mathbf{G}}(n) = O_{2n}^*$ . Let  $\mathbf{T}$  be the maximal torus of diagonal elements and  $\mathbf{T}_d$  the maximal split subtorus, as described in [GS98]. Let  $\mathbf{P} = \mathbf{M}\mathbf{N}$  be a standard parabolic subgroup with Levi component  $\mathbf{M}$  and unipotent radical  $\mathbf{N}$ . Then

$$\mathbf{M} \simeq GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_k} \times \mathbf{G}(m),$$

where  $n_1 + n_2 + \dots + n_k + m = n$ , and  $m \geq 1$ . Let  $\mathbf{A} = \mathbf{A}_M$  be the split component of  $M$ . Fix an element  $c_0 \in O_{2m}^* \setminus SO_{2m}^*$ . Then  $W(\mathbf{G}, \mathbf{A}) \simeq S \rtimes \mathbb{Z}_2^k$ , where  $S \subset S_k$ . More precisely, since  $m > 0$ , we have  $S = \langle (ij) | n_i = n_j \rangle$ . The (block) sign changes are given by  $\mathcal{C} = \langle C_i | n_i \text{ is even} \rangle \times \langle C_i c_0 | n_i, n_j \text{ are odd} \rangle$ . With this in mind we define

$$\bar{C}_i = \begin{cases} C_i & \text{if } n_i \text{ is even;} \\ C_i c_0 & \text{if } n_i \text{ is odd.} \end{cases}$$

We assume, without loss of generality, that  $n_1, \dots, n_t$  are all even, and  $n_{t+1}, \dots, n_k$  are odd. For  $1 \leq i \leq k$ , we set  $m_i = \sum_{j=1}^i n_j$ . We have the reduced root system,

$$\Phi(\mathbf{P}, \mathbf{A}_M) = \{e_{m_i} \pm e_{m_j} \mid 1 \leq i < j \leq k\} \cup \{e_{m_i} + e_n \mid 1 \leq i \leq k\}.$$

We let  $\alpha_{ij} = e_{m_i} - e_{m_j}$ ,  $\beta_{ij} = e_{m_i} + e_{m_j}$ , and  $\gamma_i = e_{m_i} + e_n$ .

Let  $\sigma \in \Pi_{\text{disc}}(M)$ . Then we have

$$\sigma \simeq \sigma_1 \otimes \sigma_2 \otimes \dots \otimes \sigma_k \otimes \tau,$$

with each  $\sigma_i \in \Pi_{\text{disc}}(GL_{n_i}(F))$ , and  $\tau \in \Pi_{\text{disc}}(G(m))$ .

**Lemma A.1.** *For  $1 \leq i < j \leq k$ , we have  $\mu_{\alpha_{ij}}(\sigma) = 0$  if and only if  $\sigma_i \simeq \sigma_j$ , and  $\mu_{\beta_{ij}}(\sigma) = 0$  if and only if  $\sigma_i \simeq \tilde{\sigma}_j$ .*

*Proof.* As in the cases of all split classical groups this follows from the result of Bernstein and Zelevinsky [BZ77] and Ol'sanskiĭ [O74].  $\square$

**Lemma A.2.** *Suppose  $w \in R_\sigma$  and  $w = sc$ , with  $s \in S$  and  $c \in \mathcal{C}$ . Then  $w = 1$ .*

*Proof.* We assume the cycle  $(12\dots j)$  appears in the disjoint cycle decomposition of  $s$ . By conjugation, we may assume  $c$  changes at most two of the block signs among  $1, 2, \dots, j$ . That is, we may assume  $1$ ,  $C_j$ , or  $C_{j-1}C_j$  are all the sign changes  $C_\ell$  appearing in  $c$  with  $1 \leq \ell \leq j$ . First, suppose  $c$  changes no signs among  $1, 2, \dots, j$ . Then  $w\sigma \simeq \sigma$  implies  $\sigma_1 \simeq \sigma_2 \simeq \dots \simeq \sigma_j$ . By the lemma,  $\alpha_{1j} \in \Delta'_\sigma$  and  $w\alpha_{1j} = -\alpha_{12} < 0$ . This contradicts  $w \in R_\sigma$ . If  $c$  changes just the sign of  $j$ , then we have  $\sigma_1 \simeq \sigma_2 \simeq \dots \simeq \sigma_j \simeq \tilde{\sigma}_1$ . Since  $\sigma_1 \simeq \tilde{\sigma}_j$ , we have  $\beta_{1j} \in \Delta'$ , and  $w\beta_{1j} = -\alpha_{12} < 0$ , again contradicting  $w \in R_\sigma$ . Finally, if  $c$  changes the sign of  $j-1$  and  $j$ , then, again  $\sigma_1 \simeq \tilde{\sigma}_j$ , and this  $\beta_{ij} \in \Delta'$ . Now  $w\beta_{1j} = -\beta_{12} < 0$ , so again, contradicting  $w \in R_\sigma$ . Thus,  $s = 1$ .  $\square$

**Corollary A.3.** *The  $R$ -group  $R_\sigma$  is an elementary 2-group.*

If  $c_0\tau \not\simeq \tau$ , then we set  $d_1$  to be the number of inequivalent classes among  $\sigma_1, \dots, \sigma_t$  such that

$$i_{G(m+n_i), GL_{n_i}(F) \times G(m)}(\sigma_i \otimes \tau)$$

is reducible, and set  $d_2$  to be the number of inequivalent classes among  $\sigma_{t+1}, \dots, \sigma_k$  satisfying  $\sigma_i \simeq \tilde{\sigma}_i$ . We then let

$$d = \begin{cases} d_1 + d_2 - 1 & \text{if } d_2 > 0; \\ d_1 & \text{otherwise.} \end{cases}$$

On the other hand, if  $c_0\tau \simeq \tau$ , we set  $d$  to be the number of inequivalent classes among  $\sigma_1, \dots, \sigma_k$  such that  $i_{GL_{n_i}(F) \times G(m), G(m+n_i)}(\sigma_i \otimes \tau)$  is reducible.

We then have the following result.

**Theorem A.4.** *Let  $G$ ,  $M$ ,  $\sigma$  and  $d$  be as above. Then  $R_\sigma \simeq \mathbb{Z}_2^d$ . Moreover, we can give a precise description of the  $R$ -group which mirrors that for the split groups  $SO_{2n}(F)$ .*

The proof can be given verbatim as in [Gol94], but is simpler if we follow the proof of [BG14].

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