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ARITHMETIC RESULTS ON ORBITS OF LINEAR GROUPS

MICHAEL GIUDICI, MARTIN W. LIEBECK, CHERYL E. PRAEGER, JAN SAXL, AND PHAM HUU TIEP

Abstract. Let p be a prime and G a subgroup of $GL_d(p)$. We define G to be *p-exceptional* if it has order divisible by p, but all its orbits on vectors have size coprime to p. We obtain a classification of p-exceptional linear groups. This has consequences for a well-known conjecture in representation theory, and also for a longstanding question concerning $\frac{1}{2}$-transitive linear groups (i.e. those having all orbits on nonzero vectors of equal length), classifying those of order divisible by p.

1. Introduction

The study of orbits of linear groups acting on finite vector spaces has a long history. Zassenhaus [47] investigated linear groups for which all orbits on nonzero vectors are regular, classifying the insoluble examples, i.e., the insoluble Frobenius complements. If one merely assumes that there is at least one regular orbit, there are many examples and the investigation and classification of these is a lively area of current research. For example, if p is the characteristic and G is a quasisimple irreducible p'-group, there is almost always a regular orbit, the exceptions being classified in [19, 28]; this played a major role in the solution of the $k(GV)$-problem [16]. In a different direction, linear groups acting transitively on the set of nonzero vectors were determined by Hering [23], leading to the classification of 2-transitive permutation groups of affine type. Results on groups with few orbits, or a long orbit, or orbits with coprime lengths, can be found in [11, 30, 33]. A much weaker assumption than transitivity is that of $\frac{1}{2}$-*transitivity* – namely, that all orbits on nonzero vectors have the same size. The soluble linear groups with this property were classified by Passman [38, 39].

In this paper we study linear groups with the following property.

Definition. Let $V = V_d(p)$ be a vector space of dimension d over $\mathbb{F}_p$ with p prime, and let $G \leq GL_d(p) = GL(V)$. We say that G is *p-exceptional* if p divides $|G|$ and G has no orbits on V of size divisible by p.

Note that if $d = ab$ for positive integers a, b, and $q = p^a$, then $\Gamma L_b(q) \leq GL_d(p)$, so the above definition also applies to subgroups of $\Gamma L_b(q)$.

If $G \leq GL_d(p)$ has a regular orbit on vectors, then G is not p-exceptional. On the other hand, if G is transitive (or $\frac{1}{2}$-transitive) on nonzero vectors and has order divisible by p, then G is p-exceptional.

We shall obtain a classification of all p-exceptional linear groups, up to some undecided questions in the imprimitive case. We also give applications to $\frac{1}{2}$-transitive groups, and to a conjecture in representation theory.

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We begin with our result for primitive linear groups. In the statement, by the *deleted permutation module* over $\mathbb{F}_p$ (p prime) for a symmetric group S_c, we mean the irreducible $\mathbb{F}_p\mathsf{S}_c$-module $S/S\cap T$, where $S=\{(a_1,\dots,a_c): a_i\in\mathbb{F}_p, \sum a_i=0\}$ and $T=\{(a,\dots,a): a\in\mathbb{F}_p\}$, and S_c acts by permuting coordinates in the obvious way. Denote by $V^\sharp$ the set of nonzero vectors in a vector space V.

Theorem 1. *Let G be an irreducible p-exceptional subgroup of $GL_d(p)=GL(V)$, and suppose G acts primitively on V. Then one of the following holds:*

(i) *G is transitive on $V^\sharp$ (a list can be found in* [30, Appendix 1]*);*
(ii) *$G\le \Gamma L_1(q)$ ($q=p^d$), determined in Lemma* 2.7*;*
(iii) *G is one of the following:*
  (a) *$G=\mathsf{A}_c$, S_c where $c=2^r-2$ or 2^r-1, with V the deleted permutation module over $\mathbb{F}_2$, of dimension $c-2$ or $c-1$ respectively (see Lemma* 9.4*);*
  (b) *$SL_2(5)\trianglelefteq G<\Gamma L_2(9)<GL_4(3)$, orbit sizes* $1,40,40$*;*
  (c) *$L_2(11)\trianglelefteq G<GL_5(3)$, orbit sizes* $1,22,110,110$*;*
  (d) *$M_{11}\trianglelefteq G<GL_5(3)$, orbit sizes* $1,22,220$*;*
  (e) *$M_{23}=G<GL_{11}(2)$, orbit sizes* $1,23,253,1771$.

For the imprimitive case we first require a result on permutation groups. For a prime p, we say a subgroup $K\le\mathsf{S}_n$ is *p-concealed* if it has order divisible by p, and all its orbits on the power set of $\{1,\dots,n\}$ have size coprime to p. The following result is an extension of [5, 41], which classify primitive groups having no regular orbit on the power set.

Theorem 2. *Let H be a primitive subgroup of S_n of order divisible by a prime p. Then H is p-concealed if and only if one of the following holds:*

(i) *$\mathsf{A}_n\trianglelefteq H\le\mathsf{S}_n$, and $n=ap^s-1$ with $s\ge1$, $a\le p-1$ and $(a,s)\ne(1,1)$; also $H\ne\mathsf{A}_3$ if $(n,p)=(3,2)$;*
(ii) *$(n,p)=(8,3)$, and $H=AGL_3(2)=2^3:SL_3(2)$ or $H=A\Gamma L_1(8)=2^3:7:3$;*
(iii) *$(n,p)=(5,2)$ and $H=D_{10}$, a dihedral group of order* 10.

Theorem 2 will be proved in Section 3.

Here is our main result on imprimitive p-exceptional linear groups.

Theorem 3. *Suppose $G\le GL_d(p)=GL(V)$ is irreducible, p-exceptional and imprimitive, and also $G=O^{p'}(G)$. Let $V=V_1\oplus\dots\oplus V_n$ ($n>1$) be any imprimitivity decomposition for G. Then G_{V_1} is transitive on $V_1^\sharp$, and G induces a primitive p-concealed subgroup of S_n on $\{V_1,\dots,V_n\}$.*

There is a partial converse: if $X\le GL(V_1)$ is transitive on $V_1^\sharp$ and $H\le\mathsf{S}_n$ is primitive and p-concealed, then the full wreath product $X\wr H$ acting on $V=V_1^n$ is p-exceptional (see Lemma 2.5).

The following is a general structure theorem for irreducible p-exceptional groups.

Theorem 4. *Let $G\le GL_d(p)=GL(V)$ be an irreducible p-exceptional group, and let $G_0=O^{p'}(G)$. Write $V\downarrow G_0=V_1\oplus\cdots\oplus V_t$ with V_i irreducible G_0-modules. Then $G_0^{V_1}$ is either a primitive p-exceptional group (given by Theorem* 1*), or an imprimitive p-exceptional group (given by Theorem* 3*). Moreover, the V_i are pairwise nonisomorphic G_0-modules, and G acts on $\{V_1,\dots,V_t\}$ as a transitive p'-group.*

Again, there is a partial converse (Lemma 2.5): the full wreath product of a p-exceptional group and a transitive p'-group is p-exceptional.

The next result has applications in the modular representation theory of finite groups. Recall that, if p is any prime and B is a Brauer p-block of any finite group G with defect group P, then the *Brauer height zero conjecture* asserts that all irreducible complex characters in B have height zero if and only if P is abelian. One of the significant results of the representation theory of finite groups in the 1980s was to prove that if G is p-soluble and $\lambda \in \mathrm{Irr}(Z)$ is an irreducible complex character of a normal subgroup $Z \lhd G$ such that $\chi(1)/\lambda(1)$ is not divisible by p for all $\chi \in \mathrm{Irr}(G)$ lying over λ, then G/Z has abelian Sylow p-subgroups. This theorem, established by D. Gluck and T. Wolf in [17, 18] led to a proof of the Brauer height zero conjecture for p-soluble groups. As shown by very recent results on the Brauer height zero conjecture, in particular, the proof [36] of the conjecture in the case $p = 2$ and $P \in \mathrm{Syl}_p(G)$, as well as reduction theorems for the conjecture [34] and [35], one of the main obstacles to proving the conjecture in full generality is to obtain a proof of the Gluck-Wolf theorem for arbitrary finite groups. This has now been achieved in [37], which uses Theorem 5 in a crucial way.

Theorem 5. *Let G be a nonidentity finite group and let p be an odd prime. Assume that $G = O^{p'}(G) = O^p(G)$ and G has abelian Sylow p-subgroups. Suppose that V is a finite-dimensional, faithful, irreducible $\mathbb{F}_pG$-module such that every orbit of G on V has length coprime to p. Then one of the following holds:*

(i) *$G = SL_2(q)$ and $|V| = q^2$ for some $q = p^a$;*

(ii) *G acts transitively on the n summands of a decomposition $V = \bigoplus_{i=1}^n V_i$, where $p < n < p^2$, $n \equiv -1 \bmod p$. Furthermore, G_{V_1} acts transitively on $V_1^\sharp$, and the action of G on $\{V_1, \ldots, V_n\}$ induces either A_n, or the affine group $2^3 : SL_3(2)$ for $(n, p) = (8, 3)$;*

(iii) *$(G, |V|) = (SL_2(5), 3^4)$, $(2_-^{1+4} \cdot \mathsf{A}_5, 3^4)$, $(L_2(11), 3^5)$, $(M_{11}, 3^5)$, or $(SL_2(13), 3^6)$.*

Here is a further consequence concerning $\frac{1}{2}$-transitive linear groups.

Theorem 6. *Let $G \leq GL_d(p)$ (p prime) be a $\frac{1}{2}$-transitive linear group, and suppose p divides $|G|$. Then one of the following holds:*

(i) *G is transitive on the set of nonzero vectors (given by* [30, Appendix]*);*

(ii) *$G \leq \Gamma L_1(p^d)$;*

(iii) *$SL_2(5) \trianglelefteq G \leq \Gamma L_2(9) < GL_4(3)$ and G has two orbits on nonzero vectors of size 40.*

Concerning part (ii) of the theorem, some examples of $\frac{1}{2}$-transitive subgroups of $\Gamma L_1(p^d)$ are given in Lemma 2.7.

Recall that a finite transitive permutation group is said to be $\frac{3}{2}$-*transitive* if all nontrivial orbits of a point stabiliser have the same size, this size being greater than 1. By [45, Theorem 10.4], such groups are either Frobenius groups or primitive. Steps towards the classification of the primitive examples were taken in [2], where it was shown that they must be either almost simple or affine, and the former were classified. The next result deals with the modular affine case. The nonmodular case will be the subject of a future paper.

Corollary 7. *If $G \leq AGL_d(p)$ is a $\frac{3}{2}$-transitive affine permutation group with point-stabiliser G_0 of order divisible by p, then one of the following holds:*

(i) *G is 2-transitive;*

(ii) *$G \leq A\Gamma L_1(p^d)$;*

(iii) *$SL_2(5) \trianglelefteq G_0 \leq \Gamma L_2(9) < GL_4(3)$ and G has rank 3 with subdegrees 1, 40, 40.*

The layout of the paper is as follows. Theorems 2 and 3 are proved in Section 3; then the proof of Theorem 1 is given in the next nine sections, culminating in Section 12. The deductions of Theorems 4, 5 and 6 can be found in the final Section 13.

Notation. The following notation will be used throughout the paper. For a vector space V with subspace U, an element $g \in GL(V)$ and a subgroup $H \leq GL(V)$,

$$\begin{aligned}
&V^\sharp = V \setminus \{0\},\\
&C_V(g) = \{v \in V \mid vg = v\},\\
&C_V(H) = \{v \in V \mid vh = v \text{ for all } h \in H\},\\
&V \downarrow H = \text{ restriction of } V \text{ to } H,\\
&H_U = \{h \in H \mid Uh = U\}, \text{ the setwise stabiliser of } U \text{ in } H.
\end{aligned}$$

Moreover, if H stabilises the subspace U, then H^U is the subgroup of $GL(U)$ induced by H. Also, x^G denotes the conjugacy class of an element x in a group G, and J_k denotes a unipotent Jordan block of size k.

2. Preliminaries

We begin with a simple observation.

Lemma 2.1. *Let $H \leq GL_d(p)$ be p-exceptional on $V = \mathbb{F}_p^d$.*

(i) *If K is a normal subgroup of H and p divides $|K|$, then K is p-exceptional on V.*

(ii) *If $N \leq GL_d(p)$ has order coprime to p and N is normalised by H, then NH is p-exceptional on V.*

Proof. (i) If K had an orbit, say Δ, in V of length a multiple of p, then the H-orbit containing Δ would have length divisible by $|\Delta|$ since K is normal in H, contradicting the fact that H is p-exceptional.

(ii) Let $L := NH$, let $v \in V^\sharp$, and consider v^L and v^N, the L-orbit and N-orbit containing v, respectively. Since N is normal in L, v^L is the set theoretic union of a subset $\mathcal{B}_0$ of the set $\mathcal{B}$ of N-orbits in V, and $\mathcal{B}_0$ is an H-orbit in its induced action on $\mathcal{B}$. Moreover, $v^N \in \mathcal{B}_0$ and $|v^L| = |v^N|.|\mathcal{B}_0|$. As $|N|$ is coprime to p, also $|v^N|$ is coprime to p. Since H acts on $\mathcal{B}$, the H-orbit v^H consists of a constant number of vectors from each N-orbit in $\mathcal{B}_0$. Thus $|\mathcal{B}_0|$ divides $|v^H|$ and hence $|\mathcal{B}_0|$ is coprime to p. □

Lemma 2.2. *Let $q = p^a$ with p prime, let $Z = Z(GL_n(q))$ and let H be a subgroup of $\Gamma L_n(q)$. Then H is p-exceptional if and only if ZH is p-exceptional.*

Proof. If ZH is p-exceptional, then so is H, by Lemma 2.1(i). The converse follows from Lemma 2.1(ii). □

Lemma 2.3. *Let $G \leq \Gamma L(V) = \Gamma L_n(q)$ $(q = p^a)$ be p-exceptional, and let $G_0 = G \cap GL(V)$. Then one of the following holds:*

(i) *p divides $|G_0|$ and G_0 is p-exceptional;*

(ii) *G_0 is a p'-group, and G contains a p-exceptional normal subgroup of the form $G_0\langle\sigma\rangle$, where $\sigma \in \Gamma L(V)\backslash GL(V)$ is a field automorphism of order p.*

Proof. If p divides $|G_0|$, then G_0 is p-exceptional by Lemma 2.1, so (i) holds. Now assume p does not divide $|G_0|$. As G/G_0 is cyclic, we have $G = G_0\langle x\rangle$ for some x of order divisible by p. Taking σ to be a power of x of order p, we obtain (ii) by applying Lemma 2.1 to $G_0\langle\sigma\rangle$. □

The next lemma will be used many times in the proof of Theorem 1.

Lemma 2.4. *Let $G \leq GL(V) = GL_d(q)$ with $q = p^a$ (p prime), and suppose that G is p-exceptional and $C_V(O^{p'}(G)) = 0$. Let t be an element of G of order p, and let $P \in \mathrm{Syl}_p(G)$.*

(i) *Then $d = \dim V \leq r_p \log_q |G : N_G(P)|$, where r_p is the minimal number of conjugates of P generating $O^{p'}(G)$.*

(ii) *We have $|V| \leq |C_V(t)| \cdot |t^G|$.*

(iii) *Suppose $O^{p'}(G)$ is generated by α conjugates of t. Then $q^{d/\alpha} \leq |t^G|$.*

Proof. As G is p-exceptional, every nonzero vector is fixed by some conjugate of P, so

$$V = \bigcup_{g \in G} C_V(P^g).$$

Moreover, $\dim C_V(P) \leq d(1 - \frac{1}{r_p})$, since otherwise the group generated by r_p conjugates of P would have a nonzero centraliser in V, contrary to the hypothesis. Hence

$$q^d = |V| \leq |G : N_G(P)|\, q^{d(1-\frac{1}{r_p})}.$$

This gives (i).

For (ii), observe that every nonzero vector in V is fixed by a conjugate of t (as G is p-exceptional), so $V = \bigcup_{g \in G} C_V(t^g)$, which implies (ii). Finally, (iii) follows from (ii) together with the fact that $\dim C_V(t) \leq d(1 - \frac{1}{\alpha})$ (which follows from the argument of the first paragraph). □

The next lemma proves the existence of many examples of imprimitive p-exceptional linear groups, giving partial converse statements to Theorems 3 and 4.

Lemma 2.5. *Let $V_1 = \mathbb{F}_p^k$, let n be a positive integer, and let $V = V_1^n = \mathbb{F}_p^{kn}$. Suppose $G_1 \leq GL(V_1)$ and $H \leq \mathsf{S}_n$ are such that one of the following conditions holds:*

(i) *G_1 is transitive on $V_1^\sharp$ and H is p-concealed,*

(ii) *G_1 is p-exceptional and H is a p'-group.*

Then the wreath product $G = G_1$ wr H, acting naturally on V, is p-exceptional.

Proof. Suppose (i) holds, and let $0 \neq v = (v_1, \ldots, v_n) \in V_1^n = V$. Let $i_1, \ldots, i_k$ be the positions i for which $v_i \neq 0$. Then the orbit v^G has size $|V_1^\sharp|^k \cdot \delta$, where δ is the size of the orbit of H on k-sets containing $\{i_1, \ldots, i_k\}$. As H is p-concealed, p does not divide δ, and so $|v^G|$ is coprime to p. The argument for (ii) is similar. □

We shall need the following upper bounds on the order of p'-subgroups of $GL_m(q)$ for $m = 2, 3$.

Lemma 2.6. *Let $q = p^f \geq 4$ and let A be a p'-subgroup of $GL_m(q)$.*
(i) *If $m = 2$ and $q \neq 5, 7, 11$ or 19, then $|A| \leq (q^2 - 1) \cdot (2, q - 1)$.*
(ii) *If $m = 3$, then $|A| \leq (q - 1)(q^3 - 1)$.*

Proof. (i) It suffices to show that any p'-subgroup of $PGL_2(q)$ ($q \neq 5, 7, 11, 19$) has order at most $(q + 1) \cdot (2, q - 1)$. From the list of subgroups of $PGL_2(q)$ in [10, Chapter XII], any subgroup of order at least $(q+1)\cdot(2, q-1)$ has order dividing one of $2(q + 1)$, $q(q - 1)$, 24 (if $q = 4$ or 9), or 60 (if $q = 4, 9$ or 29). The assertion follows.

(ii) The bound can be checked directly using [8] for $q \leq 11$, so we will assume $q \geq 13$. If A is reducible on $\mathbb{F}_q^3$, then A is contained in a maximal parabolic subgroup P of $GL_3(q)$, and so $|A| \leq |P|_{p'} = (q^2 - 1)(q - 1)^2$. If A is irreducible but imprimitive, then $|A| \leq 6(q - 1)^3$. Finally, if A is irreducible and primitive, then $|A| \leq q^3 \cdot \log_2 q^3$ by the main result of [14]. In all cases $|A| < (q - 1)(q^3 - 1)$ since $q \geq 13$. □

Now we consider the case of p-exceptional 1-dimensional semilinear groups. Here we identify $V = \mathbb{F}_p^d$ with the field $\mathbb{F}_{p^d}$. Let ω be a primitive element of $\mathbb{F}_{p^d}$ and let $\varphi : x \to x^p$ denote the Frobenius automorphism. The 1-dimensional semilinear groups are subgroups of $\Gamma L_1(p^d) = \langle \hat{\omega}, \varphi \rangle$, where $\hat{\omega}$ denotes the multiplication map $x \mapsto x\omega$. We determine all such p-exceptional groups and show that p divides d and there exists a unique minimal example.

Lemma 2.7. *Suppose that $H \leq \Gamma L_1(p^d)$ and H is p-exceptional on $V = \mathbb{F}_{p^d}$. Then p divides d and there is a factorisation $d = p^k s$ for some $k \geq 1$ such that H has a normal subgroup $K = \langle \hat{\omega}^{(p^s-1)/j}, \varphi^s \rangle$ of index coprime to p, for some j dividing $p^s - 1$. Moreover, all such subgroups H and K are p-exceptional and each contains the p-exceptional group $\langle \hat{\omega}^{p^{d/p}-1}, \varphi^{d/p} \rangle$. The group K is $\frac{1}{2}$-transitive on $V^\sharp$, having $\frac{p^s-1}{j}$ orbits of length $\frac{p^d-1}{p^s-1} \cdot j$.*

Proof. Write $H_0 = H \cap \langle \hat{\omega} \rangle = \langle \hat{\omega}^c \rangle$, say, where c divides $p^d - 1$. Then $|H_0| = (p^d - 1)/c$ is coprime to p, and $H/H_0 \cong H\langle \hat{\omega} \rangle / \langle \hat{\omega} \rangle$ is isomorphic to a subgroup of $\langle \varphi \rangle$ and hence is cyclic of order dividing d. Since p divides $|H|$ it follows that p divides d. Let $d = p^k s$ where p^k is the p-part of $|H|$. Then H has a unique normal subgroup K containing H_0 such that $|K/H_0| = p^k$. The group K is generated by $\hat{\omega}^c$ and some element τ of the form $\varphi^s \hat{\omega}^b$. We may assume that $|\tau| = p^k$. A routine computation shows that $\tau^{p^k} = \hat{\omega}^{b(p^d-1)/(p^s-1)}$, and hence $p^s - 1$ divides b, say $b = (p^s - 1)b'$.

By Lemma 2.1, K is p-exceptional. This implies in particular that τ fixes setwise each of the H_0-orbits in $V^\sharp$, and these orbits are the multiplicative cosets $\omega^i \langle \omega^c \rangle$ for $0 \leq i \leq c - 1$. Now τ maps ω^i to ω^{ip^s+b} and this element must therefore lie in $\omega^i \langle \omega^c \rangle$. It follows that $i(p^s - 1) + b = (i + b')(p^s - 1)$ is divisible by c. Choosing i such that $i + b' \equiv 1 \pmod{c}$, we conclude that c divides $p^s - 1$, say $c = (p^s - 1)/j$. This means that $\hat{\omega}^b \in H_0$, and hence that $K = \langle \hat{\omega}^{(p^s-1)/j}, \varphi^s \rangle$.

The computation in the previous paragraph shows that φ^s fixes each H_0-orbit setwise and hence that K has c orbits of length $(p^d - 1)/c$ on nonzero elements of V. In particular, K is p-exceptional and hence any subgroup H containing K with index coprime to p, and intersecting $\langle \hat{\omega} \rangle$ in H_0 is also p-exceptional. Finally, each of these subgroups K contains the group $\langle \hat{\omega}^{p^{d/p}-1}, \varphi^{d/p} \rangle$, and our arguments (with $k = 1$) show that this group is p-exceptional. □

Next we analyse the possibilities for 2-dimensional semilinear p-exceptional groups. We use the following notation: Z denotes the group of scalar matrices in $GL_2(p^f)$; the group of diagonal matrices is denoted by T; and φ denotes the Frobenius map $(a_{ij}) \mapsto (a_{ij}^p)$.

Lemma 2.8. *Suppose that $H \leq \Gamma L_2(p^f)$ and H is p-exceptional on $V = \mathbb{F}_{p^f}^2$. Then one of the following holds:*

(i) *H contains $SL_2(p^f)$.*
(ii) *p divides $2f$, $H \cap GL_2(p^f)$ is contained in $\langle T, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\rangle$ if p is odd, or T if $p = 2$.*
(iii) *p divides $2f$, and $H \leq \Gamma L_1(p^{2f})$ is as in Lemma 2.7.*
(iv) *$p^f = 9$ and $SL_2(5) \trianglelefteq H \cap GL_2(9)$.*

Proof. If H contains $SL_2(p^f)$, then H is transitive on $V^\sharp$ so H is p-exceptional. Suppose now that this is not the case, and let $H_0 = H \cap GL_2(p^f)$.

Observe that, for a proper subfield $\mathbb{F}_{p^c}$ of $\mathbb{F}_{p^f}$ the group $SL_2(p^c)$ acts regularly on the orbit containing $(1, \omega)$ where ω is a primitive element of $\mathbb{F}_{p^f}$. If $SL_2(p^c)$ were normal in H, then the H-orbit containing $(1, \omega)$ would have length a multiple of p. Thus H has no normal subgroup conjugate to $SL_2(p^c)$ for any proper divisor c of f. Moreover, if $f \geq 3c$, then the stabiliser in $Z \circ SL_2(p^c)$ of the 1-space $\langle (1, \omega)\rangle$ is Z. Hence, if p divides $|H_0|$, then H_0 is not contained in a conjugate of $Z \circ SL_2(p^c)$ for any divisor c of f such that $f \geq 3c$.

Observe that the T-orbits in V have lengths $1, q-1, q-1, (q-1)^2$. Thus if p divides f, then any subgroup of $T.\langle\varphi\rangle$ containing T and of order divisible by p is p-exceptional. If p is odd the same is true for such subgroups of $T.2.\langle\varphi\rangle$. These examples and some of their subgroups are listed in (ii). So suppose that H_0 is not conjugate to a subgroup of $T.2$.

Also, if H_0 preserves on V the structure of a 1-dimensional space over $\mathbb{F}_{p^{2f}}$, then H is a 1-dimensional semilinear group and we obtain the examples in (iii) by Lemma 2.7.

If H_0 has a nontrivial normal p-subgroup K, then for a vector v not fixed by K, the H-orbit containing v is a union of some K-orbits, each of length a nontrivial power of p. Thus no such subgroup exists.

Consider $\bar{H}_0 \cong H_0Z/Z \leq PGL_2(p^f)$. From our arguments so far, and the classification of subgroups of $PGL_2(p^f)$ [10, Chapter XII], we may assume that $\bar{H}_0 \cong \mathsf{A}_4, \mathsf{S}_4$ (with p odd) or A_5 (with $p^f \equiv \pm 1 \bmod 10$), and that H_0 is not realisable modulo scalars over a proper subfield $\mathbb{F}_{p^c}$ with $f \geq 3c$. In particular then, p is odd and f divides 4. Thus p divides $|H_0|$ and hence $p = 3$ (as $p \neq 5$ if $\bar{H}_0 = \mathsf{A}_5$). If $\bar{H}_0 = \mathsf{A}_4$ or S_4, then $H_0 \triangleright SL_2(3)$ which is not the case, so $\bar{H}_0 = \mathsf{A}_5$, and $p^f = 9$ or 81. In the latter case one checks that the orbit of H_0 containing the vector $(1, \omega)$ has size divisible by 3. Hence $p^f = 9$, which leads to the examples in (iv) since $Z \circ SL_2(5)$ is transitive on the nonzero vectors. □

The next two lemmas concern the usual action of a group G on a quotient group G/V defined by $(Vx)^g = Vx^g$.

Lemma 2.9. *Let G be a finite group, p a prime, and suppose G has a normal subgroup V which is an elementary abelian p-group. If $t \in G$ is a p'-element, then $C_G(t)/C_V(t) \cong C_{G/V}(t)$.*

Proof. Assume first that $[V,t] = V$. Let $g \in G$ be a preimage of an element of $C_{G/V}(t)$, so that $t^g = tv$ for some $v \in V$. By assumption there exists $u \in V$ such that $[t,u] = v$. Then $t^u = tv = t^g$, so $g \in VC_G(t)$. This shows that $C_{G/V}(t) = VC_G(t)/V$, as required.

Now consider the general case. Writing V additively, we have $V = [V,t] \oplus C_V(t)$, by coprime action. Again let $g \in G$ with $t^g = tv$, $v \in V$, and write $v = v_1 + v_2$ with $v_1 \in [V,t], v_2 \in C_V(t)$. If $v_2 \neq 0$, then $tv = t(v_1 + v_2)$ has order divisible by p, a contradiction (as t is a p'-element). Hence $v_2 = 0$, and now we argue as in the first paragraph of the proof. □

Corollary 2.10. *Let G be a finite group, p a prime, and suppose G has a normal p-subgroup V such that $V/Z(V)$ is elementary abelian and $Z(V) \leq Z(G)$. If $t \in G$ is a p'-element, then $C_G(t)/C_V(t) \cong C_{G/V}(t)$.*

Proof. Write $\bar{G} = G/Z(V)$. By Lemma 2.9, $\bar{V}C_{\bar{G}}(t)/\bar{V} = C_{\bar{G}/\bar{V}}(t)$. If $g \in G$ is a preimage of an element of $C_{\bar{G}}(t)$, then $t^g = tz$ for some $z \in Z(V)$. Since this has p' order, we must have $z = 1$, and the conclusion follows. □

3. Imprimitive groups

In this section we prove Theorems 2 and 3. First, for Theorem 2, we determine the primitive *p-concealed* groups, i.e., primitive subgroups H of S_n, such that the prime p divides $|H|$ and every orbit of H on the set of all subsets of $\{1,\dots,n\}$ has length coprime to p.

Proof of Theorem 2. Let $\Omega = \{1,\dots,n\}$, and define $\Omega_k := \{X \subseteq \Omega \mid |X| = k\}$ for $0 \leq k \leq n$. Let H be a primitive subgroup of S_n of order divisible by a prime p.

Assume first that $H \geq \mathsf{A}_n$. Since p divides $|H|$, we have $n \geq p$ and $H \neq \mathsf{A}_3$ if $(n,p) = (3,2)$, and H has exactly one orbit on Ω_{p-1}. Furthermore, p is coprime to $|\Omega_{p-1}|$ precisely when p does not divide any of the $p-1$ consecutive integers $n-p+2, n-p+3, \dots, n$, that is, $p|(n+1)$. Now we can write $n = \sum_{i=1}^{s} a_i p^i - 1$, where $s \geq 1$, $a_s > 0$, and $p-1 \geq a_i \geq 0$. Suppose that $n \neq a_s p^s - 1$. Choosing $k := p^s - 1$, we see that

$$\left\lfloor \frac{n}{p^s} \right\rfloor - \left\lfloor \frac{k}{p^s} \right\rfloor - \left\lfloor \frac{n-k}{p^s} \right\rfloor = a_s - 0 - (a_s - 1) = 1,$$

and so p divides $|\Omega_k|$. Next suppose that $n = a_s p^s - 1$. Write any ℓ between 0 and n as $\ell = \sum_{i=0}^{s} b_i p^i$ with $0 \leq b_i \leq p-1$. Then $b_s \leq a_s - 1$ and $n - \ell = (a_s - b_s - 1)p^s + \sum_{i=0}^{s-1}(p - b_i - 1)p^i$. Hence, for $0 \leq r \leq s$,

$$\left\lfloor \frac{n}{p^r} \right\rfloor - \left\lfloor \frac{\ell}{p^r} \right\rfloor - \left\lfloor \frac{n-\ell}{p^r} \right\rfloor = (a_s p^{s-r} - 1) - \sum_{i=r}^{s} b_i p^{i-r} - (a_s - b_s - 1)p^{s-r}$$
$$- \sum_{i=r}^{s-1} (p - b_i - 1)p^{i-r} = 0,$$

and so p does not divide $|\Omega_\ell|$. Since $n \geq 3$, H is transitive on Ω_ℓ. We have shown that H is p-concealed if $n = a_s p^s - 1$.

From now on we will assume that $H \ngeq \mathsf{A}_n$. Clearly, if H contains a normal subgroup K which is also primitive of order divisible by p and has a regular orbit Δ on 2^Ω, then the H-orbit containing Δ has length divisible by p. Hence, we may assume that H has no regular orbit on 2^Ω and apply [41, Theorem 2] to H. In all

but three of the cases listed in [41, Theorem 2] for H, either we can find a subgroup K with the prescribed properties, or we can use GAP [15] or MAGMA [4] to show directly that H is not a p-concealed group. The three exceptional cases give the examples in parts (ii) and (iii) of Theorem 2. □

The main result of this section is the following:

Theorem 3.1. *Assume $G < GL(V)$ is a (not necessarily irreducible) p-exceptional group which acts primitively as a permutation group on the n summands of the direct sum decomposition*

$$\mathbb{F}_q^{mn} = V = V_1 \oplus V_2 \oplus \ldots \oplus V_n,$$

where $\dim_{\mathbb{F}_q} V_i = m \geq 1$ and $n \geq 2$. Let $H \leq \mathsf{S}_n$ be the subgroup induced by this primitive action. Then one of the following holds:

(i) *H is a p'-group.*

(ii) *One of* (i), (ii) *and* (iii) *of Theorem* 2 *holds for H. Moreover, G_{V_1} is transitive on $V_1^\sharp$.*

Proof. Assume that p divides $|H|$. First we show that H is p-concealed, and so (i), (ii) or (iii) of Theorem 2 holds for H. Indeed, suppose that an H-orbit Δ of H on the subsets of $\Omega := \{V_1, \ldots, V_n\}$ has length divisible by p. Pick $X = \{V_{j_1}, \ldots, V_{j_t}\} \in \Delta$, $0 \neq v_i \in V_{j_i}$ for $1 \leq i \leq t$, and let $v := v_{j_1} + \ldots + v_{j_t}$. Then $I := G_v$ preserves X and so $IK/K \leq H_X$ for $K := \bigcap_{r=1}^n G_{V_r}$. Since p divides $|\Delta|$, it must divide $[G : I] = |v^G|$, contrary to the p-exceptionality of G.

Now we assume that H satisfies one of the conclusions of Theorem 2, but $G_1 := G_{V_1}$ has at least two orbits u^{G_1} and v^{G_1} on $V_1^\sharp$. For each $1 \leq i \leq n$, fix some $g_i \in G$ such that $V_1 g_i = V_i$ (and $g_1 = 1$), and set $u_i = ug_i$, $v_i = vg_i$. Choose any nonempty subsets $X, Y \subset \Omega$ with $X \cap Y = \emptyset$. Also consider $w = \sum_{i=1}^n w_i$, with $w_i = u_i$ if $V_i \in X$, $w_i = v_i$ if $V_i \in Y$, and $w_i = 0$ otherwise. Observe that any $h \in G_w$ stabilises both X and Y. (Indeed, h fixes $X \cup Y$. Assume that $V_i h = V_j \in Y$ for some $V_i \in X$. Comparing the V_j-component of $wh = w$, we see that $vg_j = v_j = u_i h = ug_i h$, and so $ug_i h g_j^{-1} = v$. But $g_i h g_j^{-1}$ stabilises V_1, so we conclude that $v \in u^{G_1}$, a contradiction.) Now the p-exceptionality of G implies that p does not divide $|H : J|$ for $J := H_{X,Y}$, the subgroup of H consisting of all elements that stabilise X and stabilise Y.

It remains to exhibit a pair (X, Y) such that p divides $|H : J|$ to get the desired contradiction. In the case (i) of Theorem 2, we choose $X := \{V_1\}$, $Y := \{V_2, \ldots, V_p\}$. Then

$$|H : J| = |\mathsf{S}_n : (\mathsf{S}_{p-1} \times \mathsf{S}_{n-p})| = p \cdot \frac{n!}{p! \cdot (n-p)!}$$

is divisible by p. In the case (ii) of Theorem 2, we can choose $X := \{a\}$ and $Y := \{b, c\}$ for some distinct $a, b, c \in \Omega$, and check that $|H : J| = 168$ is divisible by $p = 3$. In the case (iii) of Theorem 2, we can choose $X := \{a\}$ and $Y := \{b, c\}$ for some distinct $a, b, c \in \Omega$, and check that $|H : J| = 10$ is divisible by $p = 2$. □

Deduction of Theorem 3. Theorem 3 follows very quickly from the above theorem. Indeed, suppose $G \leq GL(V)$ is irreducible, imprimitive and p-exceptional, with $G = O^{p'}(G)$. Let $V = W_1 \oplus \cdots \oplus W_m$ be an imprimitivity decomposition for G. Coarsen this to a decomposition $V = V_1 \oplus \cdots \oplus V_n$ such that G acts as a primitive permutation group on $\{V_1, \ldots, V_n\}$, where V_1 is, say, $W_1 \oplus \cdots \oplus W_k$. As $G = O^{p'}(G)$,

conclusion (ii) of Theorem 3.1 holds for the action of G on $\{V_1, \ldots, V_n\}$. In particular, G_{V_1} acts transitively on the nonzero vectors of V_1. But it also permutes $W_1, \ldots, W_k$, so $k = 1$. Theorem 3 follows.

4. Tensor products I: $\mathcal{C}_4$ case

In this section we handle p-exceptional groups preserving tensor product decompositions. These correspond to subgroups of groups in class $\mathcal{C}_4$ in Aschbacher's classification of maximal subgroups of classical groups [1], hence the title of this (and forthcoming) sections. If U and W are vector spaces over a field $\mathbb{F}_q$, then a central product $GL(U) \circ GL(W)$ acts naturally on $V = U \otimes W$. We also denote by $\Gamma L(V)_{U\otimes W}$ the stabiliser of the tensor decomposition, which is a group $(GL(U) \circ GL(W))\langle\sigma\rangle$ where σ is a field automorphism fixing both factors. As usual, write Z for the group $\mathbb{F}_q^*$ of scalars in $GL(V)$.

Theorem 4.1. *Let V be a vector space over $\mathbb{F}_q$ of characteristic p, and write $V = U \otimes W$, a tensor product over $\mathbb{F}_q$ with $\dim U, \dim W \geq 2$. Let $H \leq GL(U) \circ GL(W) < GL(V)$ and suppose H is p-exceptional.*

Then $p = 2$, $\dim U = \dim W = 2$, and $ZH = (GL_1(q^2) \circ GL_1(q^2)).2$, where involutions in H act nontrivially on both factors $GL_1(q^2)$. This group is p-exceptional, and acts reducibly on V.

We also need a result for the semilinear case.

Theorem 4.2. *Let V be a vector space over $\mathbb{F}_{q^p}$ (of characteristic p), and write $V = U \otimes W$ with $2 \leq \dim U \leq \dim W$. Let $H \leq \Gamma L(V)_{U\otimes W}$, and assume H is p-exceptional and $H \cap GL(V)$ is a p'-group.*

Then $p = \dim U = 2$. In particular, $H \cap GL(V)$ is not absolutely irreducible on V.

The proofs are given in the following three subsections.

4.1. Some theory of tensor decompositions.

First we give some general theory for tensor decompositions $V = U \otimes W$ of a vector space $V = \mathbb{F}_q^n$, where $q = p^f$ for a prime p, $a = \dim U \geq 2, b = \dim W \geq 2$, and $n = ab$.

Let $\{u_1, \ldots, u_a\}$ be a basis for U, $\{w_1, \ldots, w_b\}$ be a basis for W, and write elements of $GL(U), GL(W)$ as matrices with respect to these bases respectively. Then V has an associated basis $\mathcal{B} := \{u_i \otimes w_j \,|\, 1 \leq i \leq a, 1 \leq j \leq b\}$, which we refer to as the *standard basis*. For elements $u = \sum_i a_i u_i \in U$ and $w = \sum_j b_j w_j \in W$ we denote the element $\sum_{i,j}(a_i b_j)(u_i \otimes w_j)$ of V by $u \otimes w$. A vector $v \in V$ is called *simple* if it can be expressed as $v = u \otimes w$ for some $u \in U, w \in W$.

The stabiliser $X := GL(V)_{U\otimes W}$ in $GL(V)$ of this decomposition is a central product of $X := GL(U) \circ GL(W)$ and we view elements of X as ordered pairs $(A, B) \in GL(U) \times GL(W)$ modulo the normal subgroup $Z_0 = \{(\lambda I, \lambda^{-1} I) \,|\, \lambda \in \mathbb{F}_q^*\}$, where $(A, B) : u_i \otimes w_j \mapsto u_i A \otimes w_j B$ (extending linearly). The stabiliser $\hat{X} := \Gamma L(V)_{U\otimes W}$ in $\Gamma L(V)$ is a semidirect product of X and the group $\langle\sigma\rangle$ of field automorphisms, where $\sigma : \sum_{ij} a_{ij} u_i \otimes w_j \mapsto \sum_{ij} a_{ij}^p u_i \otimes w_j$.

For an arbitrary $v \in V$, the *weight of* v is defined as the minimum number k such that v can be written as a sum of k simple vectors. It is not difficult to prove that elements of $\hat{X}$ map weight k vectors to weight k vectors; in particular, the notion of a simple vector does not depend on the choice of standard basis. Also,

the weight of a vector is well defined since any vector can be written as a sum of n simple vectors (each a scalar multiple of an element of $\mathcal{B}$).

For subspaces U_0 of U and W_0 of W (not necessarily proper subspaces), $X_{U_0\otimes W_0}$ consists of all elements (A,B) of X such that U_0 is A-invariant and W_0 is B-invariant, and $\hat{X}_{U_0\otimes W_0}$ is generated by $X_{U_0\otimes W_0}$ and a conjugate of σ.

Lemma 4.3. *Let $v \in V$ of weight k, and suppose that $v = \sum_{i=1}^k x_i \otimes y_i$, where the $x_i \in U, y_i \in W$. Write $U_0 = \langle x_1, \dots, x_k\rangle$ and $W_0 = \langle y_1, \dots, y_k\rangle$. Then*

(i) $\dim(U_0) = \dim(W_0) = k$, *so* $k \le \min\{a, b\}$.

(ii) *If also $v = \sum_{i=1}^k x_i' \otimes y_i'$, where the $x_i' \in U, y_i' \in W$, then $\langle x_1', \dots, x_k'\rangle = U_0$ and $\langle y_1', \dots, y_k'\rangle = W_0$.*

(iii) *Let A, B be the matrices representing the linear transformations $A : x_i \mapsto x_i'$, $B : y_i \mapsto y_i'$ (for all i) of U_0, W_0 with respect to the bases $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ respectively, where x_i', y_i' are as in* (ii). *Then $B = A^{-T} \in GL_k(q)$.*

(iv) *$\hat{X}_v \le \hat{X}_{U_0\otimes W_0}$, the group induced by X_v on $U_0 \otimes W_0$ is a diagonal subgroup of the group $GL_k(q) \circ GL_k(q)$ induced by $X_{U_0\otimes W_0}$, consisting of the pairs (A, A^{-T}) (modulo Z_0) for $A \in GL_k(q)$ (with respect to the bases in (iii)), and $\hat{X}_v = \langle X_v, \sigma'\rangle$, where σ' is conjugate to σ and induces a generator of the group of field automorphisms of $GL(U_0 \otimes W_0)$.*

Proof. Part (i) follows almost from the definition of k, since, for example, if $x_k = \sum_{i=1}^{k-1} a_i x_i$, then $v = \sum_{i=1}^{k-1} x_i \otimes (y_i + a_i y_k)$. For the rest of the proof we will assume without loss of generality that $x_i = u_i$ and $y_i = w_i$ for $1 \le i \le k$.

Suppose also that v has an expression as in part (ii). For $1 \le i \le k$, in terms of the basis $\mathcal{B}$ we have $x_i' = \sum_{j=1}^a a_{ij} u_j$ and $y_i' = \sum_{j=1}^b b_{ij} w_j$. Write $A_0 = (a_{ij}) \in \mathrm{M}_{k\times a}(q)$ and $B_0 = (b_{ij}) \in \mathrm{M}_{k\times b}(q)$. Then

$$v = \sum_{i=1}^k \sum_{j=1}^a \sum_{\ell=1}^b a_{ij} b_{i\ell} u_j \otimes w_\ell = \sum_{j,\ell} \left(\sum_{i=1}^k a_{ij} b_{i\ell} \right) u_j \otimes w_\ell = \sum_{j,\ell} (A_0^T B_0)_{j,\ell} u_j \otimes w_\ell.$$

Then since $v = \sum_{i=1}^k u_i \otimes w_i$, and since $\mathcal{B}$ is a basis for V, we deduce that $(A_0^T B_0)_{j\ell} = 0$ if at least one of $j > k, \ell > k, j \ne \ell$, and $(A_0^T B_0)_{jj} = 1$ for $1 \le j \le k$. In particular, this implies that each $x_i' \in U_0$ and each $y_i' \in W_0$. Moreover, by part (i) the x_i' are linearly independent, and also the y_i' are linearly independent. Thus part (ii) follows. We have also proved part (iii).

Finally, if $g = (A_1, B_1) \in X_v$ (modulo Z_0), with $A_1 = (a_{ij}) \in GL(U)$ and $B_1 = (b_{ij}) \in GL(W)$, then we have $v = \sum_{i=1}^k (u_i A_1) \otimes (w_i B_1)$. By part (ii), for each $i \le k$, $u_i A_1 \in U_0$ and $w_i B_1 \in W_0$, so $g \in X_{U_0\otimes W_0}$. Moreover, if $A = A_1|_{U_0}, B = B_1|_{W_0}$, written as matrices with respect to the bases $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ respectively, then by part (iii), $B = A^{-T}$ so g is one of the elements described in part (iv). For each $A \in GL_k(q)$, there exists an element $(A_1, B_1) \in X$ with $A = A_1|_{U_0}, A^{-T} = B_1|_{W_0}$, and the fact that this element fixes v follows from the displayed computation above. Finally $\hat{X}_v$ contains a conjugate of σ which induces on $U_0 \otimes W_0$ the natural field automorphism with respect to the basis formed by the $x_i \otimes y_j$. □

By Lemma 4.3(ii), the subspaces U_0 and W_0 are determined uniquely by v, and we denote them by $U_0(v)$ and $W_0(v)$ respectively.

4.2. **Proof of Theorem 4.1.** Suppose $H \leq \hat{X}$ preserves a nontrivial tensor decomposition $V = U \otimes W$ of $V = \mathbb{F}_q^n$, and that $\hat{X}$, $X, Z_0, q = p^f, a, b$ are as in Subsection 4.1. Suppose also that H is p-exceptional on V. By Lemma 2.2 we may assume that H contains $Z := Z(GL(V)) = (Z_U \times Z_W)/Z_0$, where $Z_U = Z(GL(U)), Z_W = Z(GL(W))$.

For Theorem 4.1 we will have $H \leq X$, but Lemma 4.4 is more general and will be used also for the proof of Theorem 4.2. The natural projection maps $\phi_U : \hat{X} \to P\Gamma L(U)$ and $\phi_W : \hat{X} \to P\Gamma L(W)$ have kernels $K_U = Z_U \circ GL(W) \cong GL(W)$ and $K_W = GL(U) \circ Z_W \cong GL(U)$ respectively. Also, for subspaces $U_0 \leq U, W_0 \leq W$, we have maps $\phi_{U_0} : \hat{X}_{U_0 \otimes W} \to P\Gamma L(U_0)$ and $\phi_{W_0} : \hat{X}_{U \otimes W_0} \to P\Gamma L(W_0)$ with kernels K_{U_0}, K_{W_0} respectively. If an element $x \in \hat{X}$ or subgroup $L \leq \hat{X}$ lies in $\hat{X}_{U_0 \otimes W}$, then we write $x^{U_0} = \phi_{U_0}(x)$, $L^{U_0} = \phi_{U_0}(L)$ for the corresponding element or subgroup of $P\Gamma L(U_0)$; also, by the *fixed point subspace* of L^U we mean the largest L^U-invariant subspace U_0 of U such that $L^{U_0} = 1$. Similarly for subspaces of W. A subgroup $L \leq \hat{X}_{U_0 \otimes W_0}$ is said to *act diagonally on* $U_0 \otimes W_0$ if $L \cap K_{U_0}$ and $L \cap K_{W_0}$ both induce only scalar transformations on $U_0 \otimes W_0$.

Lemma 4.4. *Suppose that $H \leq \hat{X}$ and H is p-exceptional. Let U_0, W_0 be 2-dimensional subspaces of U, W respectively, and let P be a Sylow p-subgroup of $H_{U_0 \otimes W_0}$. Then P is a Sylow p-subgroup of H (and so $P \neq 1$), P acts diagonally on $U_0 \otimes W_0$, and moreover $P \cap K_{U_0} = P \cap K_{W_0} = 1$, and $P \cap X$ is elementary abelian of order at most q.*

Proof. Choose a weight 2 vector $v \in V$ such that $U_0(v) = U_0$ and $W_0(v) = W_0$. Then by Lemma 4.3, $H_v \leq H_{U_0 \otimes W_0}$, and by our assumption $|H : H_v|$ is coprime to p. Thus P is a Sylow p-subgroup of H, and in particular $P \neq 1$. Also, P is conjugate in $H_{U_0 \otimes W_0}$ to a Sylow p-subgroup P' of H_v. Since P' induces a diagonal action on $U_0 \otimes W_0$ by Lemma 4.3(iv), it follows that also P induces a diagonal action on $U_0 \otimes W_0$.

Let $Q = P \cap K_{U_0}$ and assume that $Q \neq 1$. Since P acts diagonally on $U_0 \otimes W_0$, it follows that $Q^{W_0} = 1$ (since Q is a p-group), and we deduce that $Q = P \cap K_{W_0}$. Further, since $Q \neq 1$, we may assume without loss of generality that $Q^U \neq 1$. We produce an element $g \in Q$ and a 2-dimensional $\langle g^U \rangle$-invariant subspace U_0' such that $g^{U_0'} \neq 1$ as follows: the fixed point subspace U_1 of the nontrivial p-group Q^U contains U_0 and $U_1 \neq U$, and there is a 1-dimensional subspace U_2/U_1 of U/U_1 left invariant by Q^U in its induced quotient action on U/U_1. Since $Q^{U_1} = 1$ and Q^{U_2} is a nontrivial p-group, there exists $u \in U_2 \setminus U_1$ and $g \in Q$ such that $\langle u^g \rangle = \langle u + x \rangle$ for some nonzero $x \in U_1$. Let $U_0' = \langle u, x \rangle$, and note that $x^g = x$ since $x \in U_1$. Thus U_0' is invariant under $\langle g^U \rangle$ and has dimension 2, and $g^{U_0'} \neq 1$. Moreover, since $g \in Q \subset P$, it follows that $g \in H_{U_0' \otimes W_0}$ and that g does not act diagonally on $U_0' \otimes W_0$ since $g^{W_0} = 1$. This contradicts the diagonal action of a Sylow p-subgroup of $H_{U_0' \otimes W_0}$ on $U_0' \otimes W_0$. Thus $P \cap K_{U_0} = P \cap K_{W_0} = 1$. In particular P is isomorphic to a subgroup of $P\Gamma L(U_0)$ and hence $P \cap X$ is elementary abelian of order at most q. □

Corollary 4.5. *Assume that $H \leq X$, and let $1 \neq g \in H$ be a p-element. Then g^U and g^W are both regular unipotent of order p.*

Proof. We may assume that $1 \neq g \in P \leq H_{U_0 \otimes W_0}$ with U_0, W_0, P as in Lemma 4.4. Then P is elementary abelian so $|g| = p$, and g^{U_0}, g^{W_0} are both nontrivial. Also,

g^U, g^W have nonzero fixed point subspaces. Suppose that U_1 is a 2-dimensional subspace of the fixed point subspace of g^U. Then g is a nontrivial p-element in $H_{U_1 \otimes W_0}$, contradicting the diagonal action of p-elements proved in Lemma 4.4. Thus the fixed point subspace of g^U has dimension 1, so g^U is regular unipotent. Similarly, g^W is regular unipotent. □

In the next lemma we use the following notation. For a vector space V of dimension at least k, let $P_k(V)$ denote the set of k-dimensional subspaces of V. Also, let $\Gamma L_1^*(q^p) = GL_1(q^p).p$, where the cyclic group of order p on top is generated by a field automorphism (so that $\Gamma L_1^*(q^p) \leq GL_p(q)$).

Lemma 4.6. *Assume that $H \leq X$ and let $k \leq \min\{\dim(U), \dim(W)\}$. Then H acts transitively on $P_k(U) \times P_k(W)$ (in its natural product action). Moreover, $p = \dim(U) = \dim(W) \in \{2, 3\}$, and $H \leq \Gamma L_1^*(q^p) \circ \Gamma L_1^*(q^p)$.*

Proof. By Corollary 4.5, each element of order p in H fixes unique k-dimensional subspaces of U and of W, and this property is true also for each Sylow p-subgroup of H. For $i = 1, 2$, let U_i, W_i be a k-dimensional subspace of U, W respectively, and let $v_i = \sum_{j=1}^k x_{ij} \otimes y_{ij} \in U_i \otimes W_i$ be a weight k-vector so that the x_{ij} span U_i and the y_{ij} span W_i. Let P_i be a Sylow p-subgroup of H_{v_i}. Since H is p-exceptional, P_i is a Sylow p-subgroup of H, and by Lemma 4.3, $H_{v_i} \leq H_{U_i \otimes W_i}$. Thus there is an element $x \in H$ such that $P_1^x = P_2$, and hence P_2 fixes the k-subspaces U_1^x, U_2 of U and W_1^x, W_2 of W. By uniqueness, we have $U_1^x = U_2$ and $W_1^x = W_2$. This proves the first assertion.

By [31, Lemma 4.1], for the group $\phi_U(H)$ to be transitive on k-subspaces, one of the following holds: (i) $\phi_U(H) \geq PSL(U)$, or (ii) $k \in \{1, \dim(U) - 1\}$ and $\phi_U(H) \ngeq PSL(U)$, or (iii) $\dim(U) = 5, p = 2$ and $|\phi_U(H)| = 31.5$. Since p divides $|\phi_U(H)|$, case (iii) does not arise, and by Corollary 4.5, $\dim(U) = 2$ in case (i). The same comments apply to $\phi_W(H)$. Since we may always take $k = 2$ in the previous paragraph, it follows that each of $a := \dim(U)$ and $b := \dim(W)$ is at most 3. Then by the classification of transitive linear groups (see [30, Appendix]), and noting that p divides $|\phi_U(H)|$, either $\phi_U(H) \leq \Gamma L_1(q^a)/Z_U$, or $a = 2$ and $\phi_U(H) \geq PSL(U)$, or $\mathsf{A}_5 \leq \phi_U(H) \leq P\Gamma L_2(9)$. We have the same three possibilities for $\phi_W(H)$.

Suppose first that $\phi_U(H) \geq PSL(U)$ with $a = 2$. Then since P acts diagonally by Lemma 4.4, it follows that $b = 2$ and H has a single composition factor $PSL_2(q)$. However, in this case H is not transitive on $P_1(U) \times P_1(W)$. A similar argument rules out the third possibility. Thus $\phi_U(H) \leq \Gamma L_1(q^a)/Z_U$, and similarly $\phi_W(H) \leq \Gamma L_1(q^b)/Z_W$. Since p divides the order of each of these groups we must have $p = a = b \in \{2, 3\}$. □

Next we show that the case $p = 3$ does not yield a p-exceptional group. The proof and the proof of Lemma 4.10 use the following simple fact.

Remark 4.7. Suppose that $a = \dim(U) = \dim(W)$. Let $\{u_1, u_2, \ldots, u_a\}$ be a basis for U. By Lemma 4.3, each weight a vector v in $U \otimes W$ has a unique representation of the form $v = \sum_{i=1}^a u_i \otimes w_i$ where the w_i form a basis for W. Thus the number of weight a vectors is $|GL_a(q)|$.

Lemma 4.8. *If $p = 3$, and $H \leq \Gamma L_1^*(q^p) \circ \Gamma L_1^*(q^p)$, then H is not p-exceptional.*

Proof. Suppose that H is as stated and that H is p-exceptional. By Lemma 2.2, we may assume that $Z < H$, and by Lemma 4.4 it follows that a Sylow 3-subgroup P

of H has order 3 and acts diagonally on $U \otimes W$. Let $r := q^2+q+1$. By Lemma 4.6, r^2 divides $|H|$. Also, since $\gcd(r, q-1) = 1$, it follows that $H = Z \times (Z_r^2.P)$. Now P centralises Z and $N_H(P) = ZP$ so that H has exactly r^2 Sylow 3-subgroups.

The group P in its action on U leaves invariant a unique 2-subspace U_2. The same is true for the P-action on W. Let $\{u_1, u_2, u_3\}$ be a P-orbit forming a basis for U. By Lemma 4.3, each weight 3 vector v in $U \otimes W$ has a unique representation of the form $v = \sum_{i=1}^{3} u_i \otimes w_i$ where the w_i form a basis for W, and it is straightforward to show that v is fixed by P if and only if the w_i form a P-orbit in W; now each such P-orbit yields three weight 3 vectors fixed by P. Hence the number of weight 3 vectors fixed by P is exactly q^3-q^2. By Remark 4.7, the number of weight 3 vectors is $|GL_3(q)|$, and since each weight 3 vector is fixed by some Sylow 3-subgroup of H, it follows that H has at least $|GL_3(q)|/(q^3-q^2)$ Sylow 3-subgroups. Since $r^2 < |GL_3(q)|/(q^3-q^2)$, this is a contradiction. □

Finally, we show that the case $p = 2$ does lead to (reducible) p-exceptional examples. This result completes the proof of Theorem 4.1.

Lemma 4.9. *If $p = 2$, and $H \leq \Gamma L_1^*(q^2) \circ \Gamma L_1^*(q^2)$, then ZH is 2-exceptional if and only if $ZH = (GL_1(q^2) \circ GL_1(q^2)).2$, an index 2 subgroup of $\Gamma L_1^*(q^2) \circ \Gamma L_1^*(q^2)$, with the Sylow 2-subgroups acting diagonally. This group ZH is reducible and 2-exceptional, with two orbits of length q^2-1 and $q-1$ orbits of length $(q^2-1)(q+1)$ on nonzero vectors.*

Proof. Suppose that H is as stated and that H is 2-exceptional. By Lemma 2.2, we may assume also that $Z < H$. Arguing as in the proof of Lemma 4.8, a Sylow 2-subgroup P has order 2 and acts diagonally on $U \otimes W$, $N_H(P) = ZP$, and $H = Z \times (Z_{q+1}^2.P) = (GL_1(q^2) \circ GL_1(q^2)).2$.

Conversely suppose that $H = (GL_1(q^2) \circ GL_1(q^2)).2$ with each Sylow 2-subgroup P acting diagonally. Identify U and W with $\mathbb{F}_{q^2}$ and let H_1 be the index 2 subgroup of H so that H_1 acts by field multiplication on both factors. Then $H = H_1\langle\tau\rangle$ where τ acts on both factors as the field automorphism of order 2. Let $\omega \in \mathbb{F}_{q^2}$ have order $q+1$ and note that $\omega^2 = 1 + \lambda\omega$ where $\lambda = \omega + \omega^q \in \mathbb{F}_q$.

Let $v = 1 \otimes 1 + \omega \otimes \omega$ and $w = 1 \otimes \omega + \omega \otimes 1$. Let $X = GL_1(q^2) \circ I$ and note that $v^X = \{\xi \otimes 1 + \xi\omega \otimes \omega \mid \xi \in \mathbb{F}_{q^2}^*\}$ is a set of size q^2-1 and forms the set of nonzero vectors of an $\mathbb{F}_q$-subspace U_1 of V. Similarly, $w^X = \{\xi \otimes \omega + \xi\omega \otimes 1 \mid \xi \in \mathbb{F}_{q^2}^*\}$ is also the set of q^2-1 nonzero vectors of an $\mathbb{F}_q$-subspace U_2. Moreover, $U_1 \cap U_2 = \{0\}$. Note that an element of X induces multiplication by the same element of $\mathbb{F}_{q^2}^*$ on each of U_1 and U_2.

Let $g \in H$ be the element that multiplies by ω in both the first and second factors. Then

$$\begin{aligned} w^g &= \omega \otimes \omega^2 + \omega^2 \otimes \omega \\ &= \omega \otimes (1+\lambda\omega) + (1+\lambda\omega) \otimes \omega \\ &= \omega \otimes 1 + (\lambda\omega + 1 + \lambda\omega) \otimes \omega = w. \end{aligned}$$

Thus $|w^{H_1}| \leq q^2-1$, but since $w^X \subseteq w^{H_1}$, it follows that w^{H_1} has size q^2-1. Also,

$$\begin{aligned} (\xi \otimes \omega + \xi\omega \otimes 1)^\tau &= \xi^q \otimes \omega^q + \xi^q\omega^q \otimes 1 \\ &= \xi^q \otimes (\lambda+\omega) + \xi^q(\lambda+\omega) \otimes 1 \\ &= \xi^q \otimes \omega + (\xi^q\lambda + \xi^q\lambda + \xi^q\omega) \otimes 1 \\ &= \xi^q \otimes \omega + \xi^q\omega \otimes 1 \in U_2. \end{aligned}$$

Hence U_2 is H-invariant.

Similar calculations, taking g to be the element that multiplies the first factor by ω and the second by ω^q, show that U_1 is also H-invariant.

Let C be the subgroup of $GL_1(q^2) \times GL_1(q^2)$ acting on $U_1 \oplus U_2$ given by

$$C := \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{F}_{q^2}^*,\ (\alpha\beta^{-1})^{q+1} = 1\}.$$

Note that $|C| = (q^2-1)(q+1) = |H_1|$. We have already seen that elements of X are elements of C. Now consider elements $y = (1, \xi)$ of $Y = I \circ GL_1(q^2) \leq H_1$ and let $a, b \in \mathbb{F}_q$ such that $\xi = a + b\omega$. Then

$$\begin{aligned}(1 \otimes 1 + \omega \otimes \omega)^y &= 1 \otimes (a + b\omega) + \omega \otimes (a + b\omega)\omega \\ &= a \otimes 1 + b \otimes \omega + a\omega \otimes \omega + b\omega \otimes (1 + \lambda\omega) \\ &= (a + b\omega) \otimes 1 + (b + a\omega + b\lambda\omega) \otimes \omega \\ &= \xi \otimes 1 + \xi\omega \otimes \omega\end{aligned}$$

and so y induces multiplication by ξ on U_1. Similarly, $(1 \otimes \omega + \omega \otimes 1)^y = \xi^q \otimes \omega + \xi^q\omega \otimes 1$ and so y induces multiplication by ξ^q on U_2. Thus the elements of Y are also elements of C. Since each element of H_1 is the product of an element of X and an element of Y, comparing orders yields $C = H_1$. We have already seen that H_1 has two orbits of length $q^2 - 1$ ($U_1^\sharp$ and $U_2^\sharp$), and using the fact that $H_1 = C$ we see that it also has $q - 1$ orbits of length $(q^2-1)(q+1)$; namely,

$$\Delta_\lambda := \{(u, v) \mid u, v \in \mathbb{F}_{q^2}^*,\ (uv^{-1})^{q+1} = \lambda\}$$

for $\lambda \in \mathbb{F}_q^*$, on $V^\sharp$. Also consider $s_1, s_2 \in GL(V)$, where s_1 sends $(u, v) \in V$ to (u^q, v) and s_2 sends (u, v) to (u, v^q). Then $\tilde{C} := \langle C, s_1, s_2\rangle \cong C : 2^2$ has the same orbits as C does on V. Moreover, $H = C : \langle s_1 s_2\rangle$. □

4.3. **Proof of Theorem 4.2.** Suppose now that $H \leq \Gamma L_n(q^p)$ acting on $V = \mathbb{F}_{q^p}^n$, is p-exceptional, that $p = |H : H \cap GL(V)|$, and that $|H \cap GL(V)|$ is coprime to p. Suppose moreover that $H \leq X := \Gamma L(V)_{U\otimes W}$, where $a = \dim(U) \geq 2, b = \dim(W) \geq 2$ with $a \leq b$, and set $r = q^p$. By Lemma 2.2, we may assume that H contains $Z = Z(GL(V))$. Theorem 4.2 follows from the following lemma.

Lemma 4.10. $p = a = 2$.

Proof. Choose 2-dimensional subspaces U_0, W_0 of U, W respectively, set $V_0 := U_0 \otimes W_0$, and consider the subgroup L of $\Gamma L(V_0)_{U_0\otimes W_0}$ induced by $H_{U_0\otimes W_0}$. Let Δ denote the set of weight 2 vectors of $U_0 \otimes W_0$ (considered as vectors of V), and let $v \in \Delta$. By Lemma 4.3, $H_v \leq H_{U_0\otimes W_0}$, and since H is p-exceptional $|H : H_v|$ is coprime to p. It follows from Lemma 4.4 that p divides $|L|$ and a Sylow p-subgroup P of L acts diagonally on $U_0 \otimes W_0$. We may assume that P acts as a group of field automorphisms of order p. Then the set of fixed points of P in $U_0 \otimes W_0$ forms an $\mathbb{F}_q$-space $U_0' \otimes W_0' = \mathbb{F}_q^2 \otimes \mathbb{F}_q^2$. In particular, we may choose a basis u_1, u_2 for U_0 from U_0'.

By Remark 4.7, $|\Delta| = |GL_2(r)|$. Also, each $v' \in \Delta$ has a unique expression as $v' = u_1 \otimes w_1 + u_2 \otimes w_2$ where w_1, w_2 span W_0, and it is straightforward to prove that P fixes v' if and only if P fixes w_1 and w_2, that is to say, if and only if $v' \in U_0' \otimes W_0'$. Thus P fixes exactly $|GL_2(q)|$ vectors in Δ. Since each $v' \in \Delta$ is fixed by at least one Sylow p-subgroup of L (by p-exceptionality), it follows that the number $|L : N_L(P)|$ of Sylow p-subgroups of L is at least

$$y := \frac{|GL_2(r)|}{|GL_2(q)|} = \frac{(r^2-1)(r^2-r)}{(q^2-1)(q^2-q)}.$$

Let $Z_0 = Z(GL(V_0))$, and note that Z_0 is contained in $L_0 := L \cap GL(V_0)$ since H contains Z. Now $N_L(P) \cap Z_0 \cong Z_{q-1}$, and $N_L(P)$ contains $L \cap (GL(U_0') \circ GL(W_0'))$.

Recall the definitions of the maps ϕ_{U_0}, ϕ_{W_0} at the beginning of Section 4.2. From the classification of subgroups of $PGL_2(r)$ (see [10, Chapter XII]), it follows that the p'-group L_0 is such that each of $\phi_{U_0}(L_0), \phi_{W_0}(L_0)$ is either a subgroup of $D_{2(r\pm1)}$ or equals one of $\mathsf{A}_4, \mathsf{S}_4, \mathsf{A}_5$. In the latter three cases, p would be odd, and such subgroups would lie in a subfield subgroup $PGL_2(q)$, and be centralised by P.

Suppose for instance that $\phi_{U_0}(L_0)$ is A_4, S_4, or A_5. As L_0 is p'-group, we get $p \geq 5$, $r = q^p \geq 32$, so $2(r+1)$ is the largest possible order for $\phi_{W_0}(L_0)$. Also, $L_0 \geq Z_0$. Hence for $K_0 = \mathrm{Ker}(\phi_{U_0}) \cap L_0$, we have that $|K_0/Z_0| \leq 2(r+1)$. On the other hand, we noted that L_0/K_0 is centralised by P, and $L = PL_0$. It follows that $L = NK_0 = NZ_0K_0$ for $N = N_L(P)$. Now

$$\begin{aligned} |L:N| &= |L:NZ_0||NZ_0:N| = |NZ_0K_0:NZ_0||NZ_0:N| \\ &= |K_0/(NZ_0 \cap K_0)||Z_0/(N \cap Z_0)| \\ &\leq |K_0/Z_0||Z_0/(N \cap Z_0)| \\ &\leq 2(r+1)(r-1)/(q-1) \end{aligned}$$

which is strictly less than y, giving a contradiction. Thus $L_0/Z_0 \leq D_{2(r+\varepsilon)} \times D_{2(r+\varepsilon')}$, for some $\varepsilon, \varepsilon' \in \{1, -1\}$.

Suppose first that p is odd. Then $N_{L_0}(P)/N_{Z_0}(P) \leq D_{2(q+\varepsilon)} \times D_{2(q+\varepsilon')}$, and we find

$$|L:N_L(P)| \leq \frac{r-1}{q-1}.\frac{r+\varepsilon}{q+\varepsilon}.\frac{r+\varepsilon'}{q+\varepsilon'} \leq \left(\frac{r-1}{q-1}\right)^3$$

which is less than y, a contradiction. Hence $p = 2$.

Suppose now that $a = \min\{a, b\} \geq 3$. We may repeat the above analysis with 3-dimensional subspaces U_0, W_0 and Δ the set of weight 3 vectors in $V_0 = U_0 \otimes W_0$: the cardinality $|\Delta|$ is $|GL_3(r)|$, P fixes $|GL_3(q)|$ vectors in Δ, and the number of Sylow 2-subgroups of L is at least

$$y' := \frac{|GL_3(r)|}{|GL_3(q)|} = \frac{(r^3-1)(r^3-r)(r^3-r^2)}{(q^3-1)(q^3-q)(q^3-q^2)}.$$

Again $L_0 := L \cap GL(V_0)$ is such that each of $\phi_{U_0}(L_0)$ and $\phi_{W_0}(L_0)$ is an odd order subgroup of $GL_3(r)$, and hence is completely reducible. Thus each of these subgroups is a subgroup of one of $Z_{r^2+r+1}.3$, $(Z_{r^2-1} \times Z_{r-1})/Z_0$ or Z_{r-1}^3/Z_0, and it follows that

$$|L:N_L(P)| \leq \frac{r-1}{q-1}.\left(\max\left\{\frac{r^2+r+1}{q^2+q+1}, (r+1)\frac{r-1}{q-1}, \left(\frac{r-1}{q-1}\right)^2\right\}\right)^2$$

which equals $(q^2+1)^2(q+1)^3$ (recall that $r = q^2$ here). However, this quantity is less than y' and we have a contradiction. Thus $a = 2$. □

5. Tensor products II: $\mathcal{C}_7$ case

In this section we classify p-exceptional groups which preserve tensor-induced decompositions. By this we mean the following. Let V_1 be a vector space over $\mathbb{F}_q$, and let $V = V_1^{\otimes t} = V_1 \otimes V_2 \otimes \cdots \otimes V_t$, a tensor product of t spaces isomorphic to V_1. The group $(GL(V_1) \circ \cdots \circ GL(V_t)).\mathsf{S}_t$ acts on V, where all centres are identified in the central product and the group S_t permutes the tensor factors. If G is a

subgroup of this group we say that G preserves the tensor-induced decomposition $V = V_1^{\otimes t}$.

Theorem 5.1. *Assume $G < GL(V)$ is a (not necessarily irreducible) p-exceptional group which preserves a tensor-induced decomposition*

$$\mathbb{F}_q^n = V = (V_1)^{\otimes t} = V_1 \otimes V_2 \otimes \cdots \otimes V_t,$$

where $\dim_{\mathbb{F}_q} V_i = m \geq 2$ and $t \geq 2$. Then $p = 2$, and one of the following holds:

(i) *$t = 4$ and $m = q = 2$.*

(ii) *$t = 3$, and $m = 2, 3$. Moreover, if $m = 3$, then $q = 2$, and if $m = 2$, then $q \leq 4$.*

(iii) *$t = 2$, and $m = 2, 3$. Moreover, if $m = 3$, then $q \leq 8$.*

We shall also need the following result for the semilinear case.

Theorem 5.2. *Let $G \leq \Gamma L(V)$, and assume that $G_0 \trianglelefteq G = \langle G_0, \sigma \rangle$, where*

(i) *G_0 is an absolutely irreducible p'-subgroup of $GL(V)$ which preserves a tensor-induced decomposition*

$$V = (V_1)^{\otimes t} = V_1 \otimes V_2 \otimes \cdots \otimes V_t,$$

with $V = \mathbb{F}_q^n$, $\dim_{\mathbb{F}_q} V_i = m \geq 2$, $t \geq 2$, and

(ii) *$q = p^{pf}$, and σ induces the field automorphism $x \mapsto x^{p^f}$ of V modulo $GL(V)$.*

Then G is not p-exceptional.

The following result classifies the p-exceptional examples occurring in the cases left over by Theorem 5.1.

Proposition 5.3. *Let $G < GL(V)$ be p-exceptional as in the hypothesis of Theorem 5.1, and suppose t, m, q are as in one of conclusions* (i)–(iii) *of the theorem. Suppose also that G is irreducible on V. Then one of the following holds:*

(a) *$m = 3, t = 2, q = 2$: there are two irreducible 2-exceptional groups G, of the form $7^2.\mathsf{S}_3$ (orbit lengths $1, 21, 49^7, 147$) and $(7.3)^2.2$ (orbit lengths $1, 21, 49, 147^3$); both are imprimitive.*

(b) *$m = 2, t = 2$: any irreducible 2-exceptional group G in this case is conjugate to a subgroup of $GL_2(q^2)$, hence is given by Lemma 2.8.*

The proofs of these results are presented in the following three subsections.

5.1. **Proof of Theorem 5.1.** Throughout this section we assume that $G \leq GL(V)$ is a (not necessarily irreducible) p-exceptional group which preserves a tensor-induced decomposition

$$V = (V_1)^{\otimes t} = V_1 \otimes V_2 \otimes \cdots \otimes V_t,$$

where $V = \mathbb{F}_q^n$, $\dim_{\mathbb{F}_q} V_i = m \geq 2$, $t \geq 2$, and $(m, t, p) \neq (2, 2, 2)$. Let $B := G \cap (GL(V_1) \circ \ldots \circ GL(V_t))$ be the base group and let $H = G/B \leq \mathsf{S}_t$ be the permutation group induced by the action of G on the t tensor factors V_i.

5.1.1. *First reduction.* We begin with some elementary observations. Recall that a *rational* element of a finite group is an element which is conjugate to all of its powers which have the same order.

Lemma 5.4. *Under the above hypothesis, the following statements hold:*

(i) *B is a p'-group.*

(ii) *H is a transitive subgroup of S_t of order divisible by p. In particular, $t \geq p$.*

(iii) *Let $1 \neq h \in G$ be any p-element and let $Q \leq G$ be any p-subgroup containing h. Then*

$$|G : N_G(Q)| > |V/C_V(h)|.$$

(iv) *$G \setminus B$ contains an element g of order p, and for such an element,*

$$\frac{|G|}{p \cdot |C_B(g)|} > |V/C_V(g)|.$$

If in addition the element gB is rational in $H = G/B$, then

$$\frac{|G|}{p(p-1) \cdot |C_B(g)|} > |V/C_V(g)|.$$

Proof. If p divides $|B|$, then B is p-exceptional by Lemma 2.1, which contradicts Theorem 4.1 (since we are assuming that $(m, t, p) \neq (2, 2, 2)$). Part (i) follows. Likewise, if H is intransitive, then G preserves a nontrivial tensor decomposition of V and we get a contradiction by the same result; hence (ii) holds. Next, the p-exceptionality of G implies that any nonzero element $v \in V$ is fixed by a Sylow p-subgroup of G and so by a conjugate of Q as well. Hence,

$$|V| - 1 = |V^\sharp| \leq |G : N_G(Q)| \cdot |C_V(Q)| \leq |G : N_G(Q)| \cdot |C_V(h)|,$$

and (iii) follows. Since p divides $|G|$ and B is a p'-group, we can find $g \in G \setminus B$ of order p. Now we choose $h := g$ and $Q := \langle g \rangle$ in (iii). Observe that $C_G(g)$ contains g and the p'-subgroup $C_B(g)$, whence the first inequality in (iv) follows. Finally, since B is a p'-group, the rationality of gB in G/B implies g is rational in G (see e.g. [42, Lemma 4.11]), in which case we have $|N_G(Q)| = (p-1)|C_G(g)|$. Hence the second inequality in (iv) follows. □

We fix the element g in Lemma 5.4(iv) and bound $\kappa := (\dim C_V(g))/(\dim V)$. Observe that $|V/C_V(g)| = |V|^{1-\kappa}$. Replacing G by some conjugate subgroup, we may assume that g permutes $V_1, \ldots, V_p$ cyclically:

$$g \ : \ V_1 \mapsto V_2 \mapsto V_3 \mapsto \ldots \mapsto V_p \mapsto V_1.$$

Choose a basis $(e_j^1 \mid 1 \leq j \leq m)$ of V_1 and let $e_j^i := (e_j^1)g^{i-1}$ for $1 \leq i \leq p$. Since $|g| = p$, we see that $(e_j^p)g = e_j^1$ and $(e_j^i \mid 1 \leq j \leq m)$ is a basis of V_i for $1 \leq i \leq p$. Clearly,

$$(e_{j_1}^1 \otimes e_{j_2}^2 \otimes \cdots \otimes e_{j_p}^p \mid 1 \leq j_1, \ldots, j_p \leq m)$$

is a basis of $U := V_1 \otimes \cdots \otimes V_p$. Let J_k denote the Jordan block of size k with eigenvalue 1. Then $g \downarrow U$ permutes the above basis vectors of U in $(m^p - m)/p$ cycles of length p, hence has Jordan canonical form (J_1^a, J_p^b), where $a := m$, $b := (m^p - m)/p$. Also, let $W := V_{p+1} \otimes \cdots \otimes V_t$ so that $V = U \otimes W$.

Lemma 5.5. *We have*

$$\kappa = \frac{\dim C_V(g)}{\dim V} \leq \frac{1}{p} + \frac{1 - \frac{1}{p}}{m^{p-1}}.$$

Proof. Consider any indecomposable direct summand W' of the $\langle g\rangle$-module W. Suppose g acts on W' via Jordan block J_k. Then g acts on $U \otimes W'$ with Jordan canonical form $(J_1^a, J_p^b) \otimes J_k = (J_k^a, J_p^{bk})$. It follows, using the values of a, b above, that

$$\frac{\dim C_{U\otimes W'}(g)}{\dim(U \otimes W')} = \frac{a+bk}{k(a+bp)} \leq \frac{a+b}{a+bp} = \frac{\dim C_U(g)}{\dim U} = \frac{1}{p} + \frac{1-\frac{1}{p}}{m^{p-1}}.$$

Applying this observation to every indecomposable direct summand W' of the $\langle g\rangle$-module W, we get $\kappa \leq (a+b)/(a+bp)$, yielding the desired inequality. □

Next we estimate $|B : C_B(g)|$.

Lemma 5.6. *Let X be a p'-subgroup of $PGL_m(q)$ of largest possible order.*

(i) *Let $h \in G$ be an arbitrary element. Then*

$$|B : C_B(h)| \leq |X|^t \leq (|PGL_m(q)|_{p'})^t.$$

(ii) *If, in addition, g acts trivially on W, then*

$$|B : C_B(g)| \leq |X|^{p-1} \leq (|PGL_m(q)|_{p'})^{p-1}.$$

Proof. Recall that for a tensor product space $\mathbb{F}_q^k \otimes \mathbb{F}_q^l$, if $A, C \in GL_k(q)$ and $D, E \in GL_l(q)$ are such that $A \otimes D = C \otimes E$, then $C = \alpha A$ and $E = \alpha^{-1} D$ for some $\alpha \in \mathbb{F}_q^*$. It follows that the map

$$f \ : \ B \to PGL(V_1) \times \cdots \times PGL(V_t),$$

defined by $f(x) = (\bar{A}_1, \ldots, \bar{A}_t)$ if $x = A_1 \otimes A_2 \otimes \cdots \otimes A_t \in B$, $A_i \in GL(V_i)$, and $\bar{A}_i$ denotes the coset containing A_i in $PGL(V_i)$, is a well-defined homomorphism. Observe that each fibre of f is contained in exactly one $C_B(h)$-coset in B. Indeed, if $f(x) = f(x')$, then $x' = \beta x$ for some $\beta \in \mathbb{F}_q^*$, and so $x'x^{-1} \in C_B(h)$. Furthermore, since each $x \in B$ is a p'-element by Lemma 5.4(i), the elements $\bar{A}_i$ are p'-elements in $PGL(V_i)$. Composing f with the projection $PGL(V_1) \times \cdots \times PGL(V_t) \to PGL(V_i)$, we therefore get a homomorphism $f_i \ : \ B \to PGL(V_i)$ with $f_i(B)$ being a p'-group. It follows that $|f_i(B)| \leq |X| \leq |PGL_m(q)|_{p'}$. Now $|f(B)| \leq \prod_{i=1}^t |f_i(B)|$, whence (i) follows.

For (ii), notice that

$$Y_i := \{C \in GL(V_i) \mid C = A_i \text{ for some } h = A_1 \otimes \cdots \otimes A_t \in B\} \tag{1}$$

is a p'-subgroup of $GL(V_i)$. Given the action of g on $V_1, \ldots, V_p$, we can identify Y_i, $1 \leq i \leq p$, with Y_1. Consider the homomorphism

$$f^* \ : \ Y := Y_1 \times \cdots \times Y_t \to Y_1 \circ \cdots \circ Y_t$$

given by $f^*(A_1, \ldots, A_t) = A_1 \otimes \cdots \otimes A_t$, and note that $B \leq f^*(Y)$. Let

$$\begin{aligned} K &:= \{(\underbrace{A, \ldots, A}_{p}, D_{p+1}, \ldots, D_t) \mid A \in Y_1, D_i \in Y_i\}, \\ Z_0 &:= \left\{(a_1 I_m, \ldots, a_t I_m) \mid a_i \in \mathbb{F}_q^*\right\}. \end{aligned}$$

Since the element g of order p acts trivially on W and permutes $V_1, \ldots, V_p$ cyclically, $f^*(KZ_0)$ centralises g. Thus $f^*(KZ_0) \leq C_{f^*(Y)}(g)$ and

$$|B : C_B(g)| = |g^B| \leq |g^{f^*(Y)}| \leq \frac{|f^*(Y)|}{|f^*(KZ_0)|} = \frac{|Y|}{|KZ_0|} = \left(\frac{|Y_1|}{q-1}\right)^{p-1}.$$

It remains to observe that $|Y_1| \leq (q-1)|X|$. □

Lemma 5.7. *Under the above assumptions, one of the following holds:*

(i) $p = 3$ *and* $(m, t) = (2, 4)$, $(2, 3)$.

(ii) $p = 2$. *Furthermore, either* $t = 2$ *or* $(m, t) = (4, 3)$, $(3, 3)$, $(2, 6)$, $(2, 5)$, $(2, 4)$, $(2, 3)$.

Proof. By Lemma 5.6, for g the element defined before Lemma 5.5 we have

$$\begin{aligned} |G : C_B(g)| &\leq |H| \cdot |B : C_B(g)] \leq (t!) \cdot (|PGL_m(q)|_{p'})^t \\ &< \left(\tfrac{t+1}{2} \cdot q^{m(m+1)/2-1}\right)^t < |V|^{f(m,t,q)}, \end{aligned}$$

where

$$f(m, t, q) = t \cdot \frac{m(m+1)/2 + \log_q \frac{t+1}{2} - 1}{m^t}.$$

In particular, if $t \geq 5$, then $f(m, t, q) < 0.6$. By Lemma 5.4(iv),

$$|G : C_B(g)| > p|V/C_V(g)| > q^{\dim V - \dim C_V(g)} = |V|^{1-\kappa},$$

and so $f(m, t, q) + \kappa > 1$.

If $p \geq 5$, then $t \geq 5$ by Lemma 5.4(ii) and so $f(m, t, q) < 0.6$. Then by Lemma 5.5, $\kappa \leq 1/4$, a contradiction.

Now assume that $p = 3$, so $t \geq 3$ by Lemma 5.4(ii). If $t \geq 5$, then $f(m, t, q) < 0.47$. If $t = 4$ and $m \geq 3$, then $f(m, t, q) < 0.3$. If $t = 3$ and $m \geq 4$, then $f(m, t, q) < 0.46$. In all these cases $\kappa \leq 1/2$, and we arrive at a contradiction as above.

Consider the case $m = t = 3$ (still with $p = 3$). If $q \geq 9$, then $f(m, t, q) + \kappa < 0.5907 + 11/27 < 1$, again a contradiction. Assume $q = 3$. Then $|C_V(g)| \leq 3^{11}$ by Lemma 5.5. On the other hand, by Lemma 5.6 we have

$$|G : C_B(g)| \leq 2 \cdot (|PGL_3(3)|_{3'})^3 < 3^{16},$$

contradicting Lemma 5.4(iv). The remaining cases are listed in (i).

Now we consider the case $p = 2$. If $t \geq 7$, then $f(m, t, q) + \kappa < 0.22 + 0.75 < 1$. If $m \geq 3$ and $t \geq 4$, then $f(m, t, q) + \kappa < 0.32 + 2/3 < 1$. And if $m \geq 5$ and $t \geq 3$, then $f(m, t, q) + \kappa < 0.36 + 0.6 < 1$. The remaining cases are listed in (ii). □

5.1.2. *The case* $p = 3$.

Proposition 5.8. *The case* $p = 3$ *is impossible.*

Proof. Observe that the subgroup X in Lemma 5.6 has order $2(q + 1)$ by Lemma 2.6(i), and $\kappa \leq \frac{1}{2}$ by Lemma 5.5. So $|V/C_V(g)| \geq |V|^{1-\kappa} \geq q^{m^t/2}$. By Lemma 5.7 we need to consider two cases.

(1) Suppose $(m, t) = (2, 4)$. In this case, Lemma 5.4(iv) and Lemma 5.6 imply

$$q^8 \leq |V/C_V(g)| < 8 \cdot (2(q + 1))^4,$$

whence $q = 3$. Assume in addition that g acts nontrivially on $V_4 = \mathbb{F}_3^2$, i.e., $V_4 \downarrow g = J_2$. Then

$$V \downarrow g = (J_1^2, J_3^2) \otimes J_2 = (J_2^2, J_3^4)$$

and so Lemma 5.4(iv) implies

$$q^{10} \leq |V/C_V(g)| < 8 \cdot (2(q + 1))^4,$$

again a contradiction.

Thus g acts trivially on $V_4 = W$. In this case, by Lemma 5.6 we have $q^8 < |G : C_G(g)| \leq 8 \cdot (2(q + 1))^2$, again a contradiction.

(2) Assume now that $(m,t) = (2,3)$. In particular, $H = \mathsf{A}_3$ or S_3, and in the latter case g is rational. Also, g acts trivially on W. Hence by Lemma 5.4(iv) we have

$$q^4 \le |V/C_V(g)| < |B : C_B(g)| \le (2(q+1))^2,$$

a contradiction as $q \ge 3$. □

5.1.3. *The case* $p = 2$.

Lemma 5.9. *Assume* $p = 2$ *and* $t \ge 4$. *Then* $t = 4$ *and* $m = q = 2$, *i.e.,* $V = \mathbb{F}_2^{16}$.

Proof. Recall that H is an even-order transitive subgroup of S_t. We claim that H contains an involution $h = dk$ with $d \in \mathsf{S}_4$ a double transposition and $k \in \mathsf{S}_{t-4}$ disjoint from d. (If not, then we may assume $H \ni x = (1,t)$. Since H is transitive, for any $2 \le i \le t-1$ we can find $u \in H$ with $1^u = i$, and so $H \ni x^u = (i, t^u)$. If $t^u \ne 1, t$, then $H \ni x \cdot x^u = (1,t)(i,t^u)$, contrary to our assumption. If $t^u = t$, then $x^u = (i,t) \in H$. If $t^u = 1$, then $H \ni x^{ux} = (i,t)$. We have shown that $(i,t) \in H$ for all i with $1 \le i \le t-1$, and so $H = \mathsf{S}_t \ni (12)(34)$, again a contradiction.)

Without loss of generality we may now assume that G contains an involution h which permutes V_1 with V_2, V_3 with V_4, and acts on $\{V_5, \dots, V_t\}$. Arguing as in the discussion about g preceding Lemma 5.5, we see that

$$(V_1 \otimes V_2) \downarrow h = (V_3 \otimes V_4) \downarrow h = (J_1^a, J_2^b).$$

Then setting $M := V_1 \otimes \cdots \otimes V_4$ and arguing as in the proof of Lemma 5.5 we obtain

$$\gamma := \frac{\dim C_V(h)}{\dim V} \le \frac{\dim C_M(h)}{\dim M} = \frac{a^2 + 2ab + 2b^2}{m^4} = \frac{1}{2} + \frac{1}{2m^2}.$$

In particular, $\gamma \le 5/8$. Now we will apply Lemmas 5.4 and 5.6(i) to h instead of g, and treat the cases described in Lemma 5.7 separately.

Assume first that $(m,t) = (2,6)$. Then $|C_V(h)| \le |V|^\gamma \le q^{40}$, so $|V/C_V(h)| \ge q^{24}$. On the other hand, by Lemmas 5.6 and 2.6(i), we have

$$\frac{1}{2}|G : C_B(h)| \le |G : C_G(h)| \le 360 \cdot (q^2-1)^6 < q^{24},$$

contradicting Lemma 5.4(iv).

Next assume that $(m,t) = (2,5)$. Then $|C_V(h)| \le |V|^\gamma \le q^{20}$, so $|V/C_V(h)| \ge q^{12}$. Hence by Lemmas 5.4(iv) and 5.6 we must have

$$q^{12} < \frac{1}{2}|G : C_B(h)| \le |G : C_G(h)| \le 60 \cdot (q^2-1)^5,$$

and so $q = 2$ or 4. If $q = 4$, then by Lemmas 2.6 and 5.6 we have $|B : C_B(h)| \le (q+1)^5$, whence

$$|G : C_G(h)| \le 60 \cdot (q+1)^5 < q^{12},$$

a contradiction. If $q = 2$, then for $Q \in \mathrm{Syl}_2(G)$ we have

$$|G : N_G(Q)| \le 15 \cdot 3^5 < 2^{12},$$

again a contradiction by Lemma 5.4(iii).

Finally, we assume that $(m,t) = (2,4)$ and $q \ge 4$. Then $|C_V(h)| \le |V|^\gamma \le q^{10}$. Also, any $2'$-subgroup of $PGL_2(q)$ has order at most $q+1$ by Lemma 2.6(i), and $|H| \le |\mathsf{S}_4|$. In particular, the involution h is central in some $Q \in \mathrm{Syl}_2(G)$.

Furthermore, $C_B(h)$ has odd order, so $|C_G(h)| \geq |Q| \cdot |C_B(h)|$. It follows that

$$|G : C_G(h)| \leq |H : Q| \cdot |B : C_B(h)| \leq 3 \cdot (q+1)^4 < q^6,$$

a contradiction for $q \geq 4$. □

Lemma 5.10. *Suppose that $p = 2$ and $t = 3$. Then either $m = 3$ and $q = 2$, or $m = 2$ and $q \leq 4$.*

Proof. By Lemma 5.7 we need to distinguish two cases.

(1) Assume that $(m, t) = (4, 3)$. Then $\kappa \leq 5/8$ by Lemma 5.5 and so $|C_V(g)| \leq q^{40}$. Observe that any $2'$-subgroup X of $PGL_4(q) \cong SL_4(q)$ has order $\leq (q^3-1)(q^2-1)(q-1)$. (Indeed, this follows from Lemma 2.6 if X acts reducibly on $V_1 = \mathbb{F}_q^4$. Suppose that this action is irreducible. Then $\mathrm{Hom}_X(V_1) \cong \mathbb{F}_{q^a}$ for some $a|4$, and V_1 is a $(4/a)$-dimensional absolutely irreducible $\mathbb{F}_{q^a}X$-module. Since X is soluble, any irreducible Brauer character of X lifts to a complex character by the Fong-Swan Theorem [12, 72.1]; in particular, $4/a$ divides $|X|$ and so $a = 4$. This in turn implies that $X \leq GL_1(q^4)$, and so we are done.) Hence, applying Lemma 5.6 to the element g defined before Lemma 5.5, we have

$$q^{24} < |G : C_G(g)| \leq 3 \cdot ((q^3-1)(q^2-1)(q-1))^3,$$

a contradiction.

(2) Consider the case $(m, t) = (3, 3)$ and $q \geq 4$. Note that any $2'$-subgroup of $PGL_3(q)$ has order $\leq q^3 - 1$ by Lemma 2.6(ii). Assume in addition that the involution g acts nontrivially on $V_3 = \mathbb{F}_q^3$, i.e., $V_3 \downarrow g = (J_1, J_2)$. Then

$$V \downarrow g = (J_1^3, J_2^3) \otimes (J_1, J_2) = (J_1^3, J_2^{12})$$

and so $|C_V(g)| = q^{15}$. Lemmas 5.4(iv) and 5.6 now imply that

$$q^{12} < |G : C_G(g)| \leq 3 \cdot (q^3-1)^3,$$

a contradiction.

Thus g acts trivially on $V_3 = W$. In this case, by Lemma 5.6(ii) we have

$$q^9 \leq |V/C_V(g)| < |G : C_G(g)| \leq 3 \cdot (q^3-1),$$

again a contradiction.

(3) Now assume $(m, t) = (2, 3)$ and $q \geq 8$. Note that any $2'$-subgroup of $PGL_2(q)$ has order $\leq q+1$ by Lemma 2.6(i). Assume in addition that the involution g acts nontrivially on $V_3 = \mathbb{F}_q^2$, i.e., $V_3 \downarrow g = J_2$. Then

$$V \downarrow g = (J_1^2, J_2) \otimes J_2 = J_2^4$$

and so $|C_V(g)| = q^4$. Lemmas 5.4(iv) and 5.6 now imply that

$$q^4 < |G : C_G(g)| \leq 3 \cdot (q+1)^3,$$

a contradiction. Thus g acts trivially on $V_3 = W$. In this case, by Lemma 5.6(ii) we have

$$q^2 \leq |V/C_V(g)| < |G : C_G(g)| \leq 3 \cdot (q+1),$$

again a contradiction. □

The rest of this subsection is devoted to the case $t = 2$, so $V = V_1 \otimes V_2$ and $p = 2$ by Lemma 5.4(ii), and V_1, V_2 are interchanged by g. As before, we fix the basis $(e_j := e_j^1 \mid 1 \leq j \leq m)$ of V_1 and $(f_j := e_j^2 = e_j g \mid 1 \leq j \leq m)$ of V_2. Then we

can identify both V_1 and V_2 with $\mathbb{F}_q^m$. Consider the subgroups $Y_1 \cong Y_2$ of $GL_m(q)$ defined in (1).

The key observation in the case $t = 2$ is the following:

Lemma 5.11. *Suppose $t = 2$. Then the subgroup $Y_1 < GL_m(q)$ is transitive on k-dimensional subspaces of $\mathbb{F}_q^m$ for any $k \le m - 1$.*

Proof. Recall that if $0 \neq v \in V$ has weight k: $v = \sum_{i=1}^k x_i \otimes y_i$, then

$$[v]_1 := \langle x_1, \dots, x_k \rangle_{\mathbb{F}_q},\ [v]_2 := \langle y_1, \dots, y_k \rangle_{\mathbb{F}_q}$$

are k-dimensional subspaces of $\mathbb{F}_q^m$ uniquely determined by v. In particular, if $vg = v$, then

$$\sum_{i=1}^k x_i \otimes y_i = v = vg = \sum_{i=1}^k y_i \otimes x_i,$$

and so $([v]_1)g = [v]_2$.

Next we show that if $0 \neq v \in V$ has weight k and is fixed by some involution g' of G, then there is some element $a \otimes b \in B$ such that $[v]_2 b = ([v]_1 a)g$. Indeed, since $|G| = 2|B|$ and $|B|$ is odd, we must have $g' = xgx^{-1}$ for some $x = a \otimes b \in B$. Writing $v = \sum_{i=1}^k x_i \otimes y_i$, we see that $w := vx = \sum_{i=1}^k x_i a \otimes y_i b$ is fixed by g. Hence, according to the first paragraph we then have

$$([v]_1 a)g = ([w]_1)g = [w]_2 = ([v]_2)b,$$

as stated.

Now we set $L := \langle e_1, \dots, e_k \rangle_{\mathbb{F}_q}$, $M := \langle f_1, \dots, f_k \rangle_{\mathbb{F}_q}$, and consider any k-dimensional subspace $N := \langle u_1, \dots, u_k \rangle_{\mathbb{F}_q}$ of V_1. Then $v = \sum_{i=1}^k u_i \otimes f_i \in V$ has weight k, with $[v]_1 = N$, $[v]_2 = M$. The p-exceptionality of G implies that v is fixed by some involution $g' \in G$. By the previous paragraph, there is some $x = a \otimes b \in B$ such that

$$(Na)g = ([v]_1 a)g = ([v]_2)b = Mb = (Lg)b,$$

i.e., $Na = Lgbg^{-1}$. Writing $f_j b = \sum_{i=1}^m b_{ij} f_i$ for some $b_{ij} \in \mathbb{F}_q$, we have $(e_j)gbg^{-1} = \sum_{i=1}^m b_{ij} e_i$. Thus, under our identification of V_1 and V_2 with $\mathbb{F}_q^m$, we have $Na = Lgbg^{-1} = Lb$, and so $Lba^{-1} = N$. It remains to observe that B contains $x^g x^{-1} = (ba^{-1}) \otimes (ab^{-1})$, whence $ba^{-1} \in Y_1$. □

Proposition 5.12. *Assume $t = 2$ (so $p = 2$). Then either $m = 2$, or $m = 3$ and $q \le 8$.*

Proof. By Lemma 5.11, Y_1 is transitive on k-spaces for all k, and has odd order. Hence [31, Lemma 4.1] shows that if $m > 2$, then either $m = 3$ or $(m, q) = (5, 2)$. In the latter case $Y_1 = \Gamma L_1(32)$, $V = \mathbb{F}_2^{25}$, and $|C_V(g)| \le 2^{15}$ by Lemma 5.5. Applying Lemma 5.4(iv), $|B : C_B(g)| > 2^{10}$. On the other hand, the proof of Lemma 5.6(ii) shows that $|B : C_B(g)| \le |Y_1| = 155$, a contradiction.

Next suppose that $m = 3$. By Lemma 5.5 and Lemma 5.4(iv) we have $|B : C_B(g)| > |V/C_V(g)| \ge q^3$. Now the proof of Lemma 5.6(ii) shows that $q^3 < |B : C_B(g)| \le |Y_1|/(q-1) \le 3f(q^2+q+1)$, which can happen only when $q \le 8$. □

This completes the proof of Theorem 5.1.

5.2. **Proof of Theorem 5.2.** Throughout this section we assume that $G \leq \Gamma L(V)$ is a p-exceptional group such that $G_0 \lhd G = \langle G_0, \sigma \rangle$, where

(i) G_0 is an absolutely irreducible p'-subgroup of $GL(V)$ which preserves a tensor-induced decomposition

$$V = (V_1)^{\otimes t} = V_1 \otimes V_2 \otimes \cdots \otimes V_t,$$

with $V = \mathbb{F}_q^n$, $\dim_{\mathbb{F}_q} V_i = m \geq 2$, $t \geq 2$, and

(ii) $q = q_0^p$, $q_0 = p^f$, and σ induces the field automorphism $x \mapsto x^{q_0}$ of V modulo $GL(V)$.

Note that $|C_V(\sigma)| \leq |V|^{1/p}$. Indeed, since $\mathcal{G} := GL(V \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)$ is connected (where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$), σ is $\mathcal{G}$-conjugate to the standard Frobenius morphism $\sigma_0 \ : \ x \mapsto x^{q_0}$. Hence

$$|C_V(\sigma)| \leq |C_{(V \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)}(\sigma)| = |C_{(V \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)}(\sigma_0)| = |V|^{1/p}.$$

5.2.1. *First reductions.* The following lemma simplifies further computations.

Lemma 5.13. *Under the above assumptions,*

(i) *m^t divides $|G_0|$; in particular, p does not divide m, and*

(ii) *$|G_0| > |V|^{1-1/p}$.*

Proof. Part (i) follows from the assumptions that G_0 is a p'-group and absolutely irreducible. For (ii), one can argue as in the proof of Lemma 5.4(iii), taking $Q = \langle \sigma \rangle$. □

Next we rule out most of the cases using Lemma 5.13:

Proposition 5.14. *Under the above assumptions, $t = 2$.*

Proof. Assume to the contrary that $t \geq 3$. The proof of Lemma 5.6 implies that the base subgroup $G_0 \cap (GL(V_1) \circ \ldots \circ GL(V_t))$ of G_0 has order at most $(q-1) \cdot |X|^t$, where X is a p'-subgroup of largest possible order of $PGL_m(q)$. Hence

$$|G_0| \leq t! \cdot (q-1) \cdot (|PGL_m(q)|_{p'})^t < \left(\frac{t+1}{2}\right)^t \cdot q^{t(\frac{m(m+1)}{2}-1)+1},$$

and Lemma 5.13(ii) yields that

$$f(m,t,p) := m^t \left(1 - \frac{1}{p}\right) - t \cdot \frac{m^2+m-2}{2} - 1 - t \cdot \log_p \frac{t+1}{2} < 0.$$

Note that the function $f(m,t,p)$ is nondecreasing for each of its variables. Now direct computations show that the latter inequality is impossible unless one of the following holds:

(a) $t = 5$, $m = p = 2$. This is ruled out by Lemma 5.13(i).

(b) $t = 4$, $m = 2$, $p \leq 3$. By Lemma 5.13(i) we must have $p = 3$ and so $q = q_o^p \geq 27$. Since for $m = 2$ we have $|X| \leq 2(q+1)$ by Lemma 2.6(i),

$$|G_0| \leq 24 \cdot (2(q+1))^4 \cdot (q-1) < q^{32/3} = |V|^{2/3},$$

a contradiction by Lemma 5.13(ii).

(c) $(t,m,p) = (3,3,2)$. In this case $q \geq 4$, and $|X| \leq q^3 - 1$ as $m = 3$ by Lemma 2.6(ii). It follows that

$$|G_0| \leq 6 \cdot (q^3-1)^3 \cdot (q-1) < q^{27/2} = |V|^{1/2},$$

again contradicting Lemma 5.13(ii).

(d) $(t, m) = (3, 2)$. By Lemma 5.13(i) we must have $p \geq 3$ and so $q \geq 27$. Also, by Lemma 2.6(i) we have $|X| \leq 2(q+1)$ as $m = 2$. Hence

$$|G_0| \leq 6 \cdot (2(q+1))^3 \cdot (q-1) < q^{16/3} \leq |V|^{1-1/p},$$

again a contradiction. □

5.2.2. *The case* $t = 2$. Throughout this subsection we assume $t = 2$.

Lemma 5.15. *We have* $G_0 \leq GL(V_1) \circ GL(V_2)$ *and* $p = 2$.

Proof. Let $B = G_0 \cap (GL(V_1) \circ GL(V_2))$, and suppose $G_0 \neq B$. Then $G_0 = B\langle s\rangle = B.2$, where s interchanges V_1 and V_2. Since G_0 is a p'-group this implies that $p > 2$. As σ has order p it therefore fixes V_1 and V_2, and so $B\langle\sigma\rangle$ is a normal subgroup of index 2 in G, hence is p-exceptional. This contradicts Theorem 4.2.

Hence $G_0 = B$. If $p > 2$, then again σ fixes V_1, V_2 and we contradict Theorem 4.2. Hence $p = 2$, completing the proof. □

Proposition 5.16. *The case* $t = p = 2$ *cannot occur.*

Proof. Assume to the contrary that $t = p = 2$. By Lemma 5.15 we have $G_0 = B$ and $G = B\langle\sigma\rangle$ with σ of order 2, and G_0 is an absolutely irreducible $2'$-group on V. By Lemma 5.13, m is odd; in particular, $m \geq 3$. If σ fixes both V_1 and V_2, then Theorem 4.2 gives a contradiction. So σ interchanges V_1 and V_2 and also it is semilinear: $(\lambda v)^\sigma = \lambda^r v$ with $q = 2^{2f} = r^2$.

We will now follow the arguments in Subsection 5.1.3 for the corresponding case in Theorem 4.1, and indicate necessary modifications because of the semilinearity of σ. We fix the basis $(e_j := e_j^1 \mid 1 \leq j \leq m)$ of V_1 and $(f_j := e_j^2 = \sigma(e_j) \mid 1 \leq j \leq m)$ of V_2. Then we can identify both V_1 and V_2 with $\mathbb{F}_q^m$. Consider the subgroups $Y_1 \cong Y_2$ of $GL_m(q)$ defined in (1). Note that if $x = X \otimes Y \in B$, then $x^\sigma = Y^{(r)} \otimes X^{(r)}$, where $X^{(r)} = (x_{ij}^r)$ if $X = (x_{ij})$. In particular, if $X \in Y_1$, then $X^{(r)} \in Y_2$ and vice versa, whence $Y_1 \cong Y_2$. Now the proof of Lemma 5.11 can be carried over verbatim, except that we have to replace ba^{-1} by $b^{(r)}a^{-1}$. Thus Y_1 (and Y_2) is transitive on k-spaces of $\mathbb{F}_q^m$ for all k, and $|Y_1|$ is odd (and contains all scalar matrices). Now [31, Lemma 4.1] implies that $m = 3$ or $(m, q) = (5, 2)$. The latter is impossible as $q = r^2$.

Hence $m = 3$. Now we consider the homomorphism $f^* : Y_1 \times Y_2 \to Y_1 \otimes Y_2$ defined by $f^*(X, Y) = X \otimes Y$, and the subgroup $K := \{(X, X^{(r)}) \mid X \in Y_1\}$ of $Y_1 \times Y_2$. Note that σ centralises $f^*(K)$. As in the proof of Lemma 5.6(ii), this implies that $|B : C_B(\sigma)|$ has order at most $|(Y_1 \times Y_2) : K| = |Y_1|$. On the other hand, $|C_V(\sigma)| = |V|^{1/2} = q^{9/2}$. Hence the 2-exceptionality of G implies $|G : C_G(\sigma)| \geq |V/C_V(\sigma)| = q^{9/2}$ by Lemma 2.4(ii). Since $|C_G(\sigma)| = 2|C_B(\sigma)|$, we get

$$q^{9/2} \leq |Y_1| \leq 6f(q^3 - 1),$$

again a contradiction. □

This completes the proof of Theorem 5.2.

5.3. **Proof of Proposition 5.3.** Now we prove Proposition 5.3 by treating the cases left in conclusions (i)-(iii) of Theorem 5.1. Let G be as in the hypothesis of the proposition. If $(m, t) \neq (2, 2)$, then there are just a small number of cases for (m, t, q) to consider, all with $q \leq 8$ and $m^t \leq 16$, and a MAGMA computation shows that the only irreducible p-exceptional examples are those in part (a) of

Proposition 5.3. (For $(m,t) = (3,2)$ and $q = 4, 8$ we first use Lemmas 5.4(i) and 5.11 to reduce to G being a subgroup of $(\Gamma L_1(q^m) \circ \Gamma L_1(q^m)).2$ before doing the MAGMA computations.)

The remaining case $(m,t) = (2,2)$ is handled by the following result.

Proposition 5.17. *Suppose $G < GL(V)$ is an irreducible p-exceptional subgroup satisfying the assumptions of Theorem 5.1 with $m = t = p = 2$. Then G is conjugate to a subgroup of $GL_2(q^2)$ (and so is known by Lemma 2.8).*

Proof. Without loss of generality, we may assume that G contains $Z = Z(GL(V))$. Consider the base subgroup $B = G \cap (GL(V_1) \circ GL(V_2))$ of index at most 2 in G. Then all B-orbits on V have odd lengths. Hence, by Theorem 4.1, either $|B|$ is odd, or B is (conjugate to) the group H appearing in the conclusion of that theorem.

Consider the latter case. We may identify B with a subgroup of index 2 in the group $\tilde{C}$ defined in the proof of Lemma 4.9. Adopt the notation of that proof. Since G normalises $C = O_{2'}(B)$, G permutes the two C-orbits $U_1^\sharp$ and $U_2^\sharp$ of length $q^2 - 1$. Since G is irreducible, G cannot fix either of them. Thus G interchanges them, and so G has an orbit of length $2(q^2 - 1)$, contradicting the 2-exceptionality of G.

Hence $|B|$ is odd. Thus $|G|$ is not divisible by 4 and so G is soluble. Also, G is irreducible on V. Hence by the Fong-Swan Theorem, the dimension d of V over $\mathrm{End}_G(V) \supseteq \mathbb{F}_q$ cannot be 4 (but divides 4), and so it is either 1 or 2. If $d = 1$, then $|G|$ divides $|GL_1(q^4)|$ and so it is odd, a contradiction. Thus $d = 2$ and $G \leq GL_2(q^2)$ as in the conclusion. □

This completes the proof of Proposition 5.3.

6. Subfields

It turns out that a p-exceptional group $H \leq \Gamma L_n(q)$ cannot be realisable modulo scalars over a proper subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$. Such groups are conjugate to subgroups of $Z \circ GL_n(q_0).\langle\phi\rangle$, for some proper subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$, where ϕ generates the group of field automorphisms of $GL_n(q)$ and $Z = Z(GL_n(q))$. We prove

Theorem 6.1. *Let $V = \mathbb{F}_q^n$, and $q = q_0^s$ with $s > 1$. Suppose $H \leq (Z \circ GL_n(q_0))\langle\phi\rangle < \Gamma L(V)$, where ϕ generates the group of field automorphisms of $GL_n(q)$. Then H is not p-exceptional.*

Proof. Suppose that H is p-exceptional on $V = \mathbb{F}_q^n$. By Lemma 2.1, ZH is also p-exceptional, so we may assume that $Z \subseteq H$. Then $H \cap GL_n(q) = Z \circ H_0$, where $H_0 = H \cap GL_n(q_0)$, and note that H_0 is normal in H. Let $\{b_1 \ldots, b_s\}$ be a basis for $\mathbb{F}_q$ as an $\mathbb{F}_{q_0}$-vector space. Then each $v \in V$ may be written as a sum $\sum_{i=1}^s b_i v_i$ with $v_i \in V_0 = \mathbb{F}_{q_0}^n$. If the nonzero v_i are linearly independent in V_0, then the stabiliser $(H_0)_v$ must fix each of the v_i.

Suppose first that p divides $|H_0|$, or equivalently that p divides $|H \cap GL_n(q)|$. Then H_0 is p-exceptional on V, and hence also on V_0, by Lemma 2.1. Let $x \in H_0$ have order p. Then there exists a 2-dimensional x-invariant $\mathbb{F}_{q_0}$-subspace $\langle v_1, v_2\rangle_{\mathbb{F}_{q_0}}$ of V_0 which is not fixed pointwise by x. Let $v = b_1 v_1 + b_2 v_2$. Then as we observed in the previous paragraph, $(H_0)_v$ fixes both v_1 and v_2 and hence leaves the $\mathbb{F}_q$-space $U := \langle v_1, v_2\rangle_{\mathbb{F}_q}$ invariant, fixing it pointwise. Moreover, $(H_0)_v = (H_0)_{v_1,v_2}$ is the kernel of the action of $(H_0)_U$ on U. It follows that $x \in (H_0)_U \setminus (H_0)_v$ and

hence that p divides $|(H_0)_U : (H_0)_v|$, which divides $|H_0 : (H_0)_v|$, a contradiction to p-exceptionality.

Thus $|H_0|$ is coprime to p, and H has a normal subgroup which contains $H \cap GL_n(q)$ as a subgroup of index p. By Lemma 2.1, this normal subgroup is p-exceptional and so, without loss of generality, we may assume that $H = (H \cap GL_n(q)).\langle x\rangle = (Z \circ H_0).\langle x\rangle$, where x is a field automorphism of order p, and so x induces a (possibly trivial) field automorphism on $GL_n(q_0)$. Thus each element of H has the form $x^i ch$, for some i, with $c \in Z$ and $h = (a_{ij}) \in GL_n(q_0)$. Recall that $q = q_0^s$, and note that x fixes pointwise (at least) an $\mathbb{F}_p$-subspace $\mathbb{F}_p^n$ of V_0. Let $q_1 := q^{1/p}$, so $\mathbb{F}_{q_1}$ is the fixed field of x.

Choose v_1, v_2 linearly independent vectors in V_0 fixed by x, let $U := \langle v_1, v_2\rangle_{\mathbb{F}_q}$, and define

$$X(U) := \{c_1v_1 + c_2v_2 \,|\, c_1, c_2 \in \mathbb{F}_q^*, c_1c_2^{-1} \notin \mathbb{F}_{q_0}\}.$$

First we show that (i) for $v \in X(U)$, the stabiliser $H_v \subseteq H_U$, and (ii) $X(U)$ is H_U-invariant. Let $v = c_1v_1+c_2v_2 \in X(U)$ and extend $\{v_1, v_2\}$ to a basis $\{v_1, v_2, \ldots, v_n\}$ for V over $\mathbb{F}_q$.

(i) Let $x^i ch \in H_v$ with c, h as in the previous paragraph, with h represented with respect to the basis $\{v_1, \ldots, v_n\}$. Since x and Z leave U invariant it is sufficient to show that h does also. Now

$$v = v^{x^i ch} = c\sum_{j=1}^{n}(c_1^{x^i}a_{1j} + c_2^{x^i}a_{2j})v_j$$

and so, for each $j \geq 3$, $c_1^{x^i}a_{1j} + c_2^{x^i}a_{2j} = 0$ which implies that $(c_1c_2^{-1})^{x^i}a_{1j} = -a_{2j}$. Since $a_{1j}, a_{2j} \in \mathbb{F}_{q_0}$ while $(c_1c_2^{-1})^{x^i} \notin \mathbb{F}_{q_0}$, it follows that $a_{1j} = a_{2j} = 0$. Thus h leaves U invariant, proving part (i). Also we have, for $j = 1, 2$, that $c_j = c(c_1^{x^i}a_{1j} + c_2^{x^i}a_{2j})$, whence

$$c^{-1} = c_1^{-1}c_1^{x^i}a_{11} + c_1^{-1}c_2^{x^i}a_{21} = c_2^{-1}c_1^{x^i}a_{12} + c_2^{-1}c_2^{x^i}a_{22} \tag{2}$$

and since h is nonsingular, $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

(ii) Since $c_1^{x^i}(c_2^{x^i})^{-1} = (c_1c_2^{-1})^{x^i} \notin \mathbb{F}_{q_0}$, and since $(cc_1)(cc_2)^{-1} = c_1c_2^{-1} \notin \mathbb{F}_{q_0}$, it follows that x^i and each element of Z leaves $X(U)$ invariant. It remains to consider v^h where $h = (a_{ij}) \in H_U$. Now

$$v^h = (c_1a_{11} + c_2a_{21})v_1 + (c_1a_{12} + c_2a_{22})v_2.$$

If the coefficient of v_2 were 0 we would have $a_{22} = -a_{12}(c_1c_2^{-1}) \in \mathbb{F}_{q_0}$ and hence $a_{22} = a_{12} = 0$. However, $a_{1j} = a_{2j} = 0$ for each $j \geq 3$, and this would imply that h is singular, a contradiction. A similar argument shows that the coefficient of v_1 is also nonzero. Suppose for a contradiction that $v^h \notin X(U)$. Then

$$d := (c_1a_{11} + c_2a_{21})(c_1a_{12} + c_2a_{22})^{-1} \in \mathbb{F}_{q_0}^*$$

and we have $c_1(a_{11} - da_{12}) = c_2(da_{22} - a_{21})$. If $da_{22} - a_{21} \neq 0$, then $c_2c_1^{-1} = (da_{22} - a_{21})^{-1}(a_{11} - da_{12}) \in \mathbb{F}_{q_0}$ which is a contradiction. Thus $a_{21} = da_{22}$, and hence also $a_{11} = da_{12}$. This again implies that h is singular, and finally we conclude that $v^h \in X(U)$, proving (ii).

We have $\langle x\rangle \leq H_U$, and since the p-part of $|H|$ is p, it follows that $\langle x\rangle$ is a Sylow p-subgroup of H_U. By p-exceptionality, $|H_v|$ is divisible by p and hence, by Sylow's theorem, it follows that x must fix at least one vector of $X(U)$. Without loss of

generality we may assume that x fixes v. Now observe that $v = v^x$ if and only if $c_j = c_j^x$ for $j = 1, 2$, and hence $(c_1c_2^{-1})^x = c_1c_2^{-1} \in \mathbb{F}_{q_1} \setminus \mathbb{F}_{q_0}$. Thus $\mathbb{F}_{q_1} \not\subseteq \mathbb{F}_{q_0}$.

Now we consider a special element of $X(U)$, namely $v' = v_1 + bv_2$, where b is a primitive element of $\mathbb{F}_{q_1}$ (which we have just shown does not lie in $\mathbb{F}_{q_0}$). Suppose that x^ich fixes v', and note that $b^{x^i} = b$ since b lies in the fixed field $\mathbb{F}_{q_1}$ of x. Then equation (2) in the computation in (i) shows that

$$c^{-1} = a_{11} + ba_{21} = b^{-1}a_{12} + a_{22}$$

which implies that $b^2a_{21} + b(a_{11} - a_{22}) - a_{12} = 0$. Since $b \notin \mathbb{F}_{q_0}$, it follows that b has minimal polynomial over $\mathbb{F}_{q_0}$ of degree 2. The extension field $\mathbb{F}_{q_0}(b)$ contains the maximal subfield $\mathbb{F}_{q_1}$ of $\mathbb{F}_q$ as a proper subfield, and hence $\mathbb{F}_{q_0}(b) = \mathbb{F}_q$ and $s = 2$. Also, since $\mathbb{F}_{q_1} \not\subseteq \mathbb{F}_{q_0}$, p is odd.

We may therefore write $q = r^{2p}, q_0 = r^p, q_1 = r^2$, and then $\mathbb{F}_{q_1} \cap \mathbb{F}_{q_0} = \mathbb{F}_r$. Now $|X(U)| = (q-1)(q-q_0) = r^p(r^{2p}-1)(r^p-1)$. We showed above that the vectors of $X(U)$ fixed by x are $c_1v_1 + c_2v_2$ with each $c_i \in \mathbb{F}_{q_1}$ but $c_1c_2^{-1} \notin \mathbb{F}_{q_0}$. Thus x fixes precisely $(r^2-1)(r^2-r)$ vectors in $X(U)$. Since each Sylow p-subgroup of H_U fixes the same number of vectors in $X(U)$, the number of Sylow p-subgroups of H_U is at least $\frac{r^p(r^{2p}-1)(r^p-1)}{r(r^2-1)(r-1)} > r^{p-1}r^{2p-2}r^{p-1} = r^{4(p-1)}$.

We now consider the induced group $H_U^U \leq \Gamma L(U)$. Since $|H \cap GL(V)|$ is coprime to p, and since x acts nontrivially on U, it follows that H_U^U is of the form $(Z^U \circ L).\langle x^U\rangle$ with L a p'-subgroup of $GL_2(q_0)$, where Z^U is the group induced by Z. Note that the normaliser N of $\langle x^U\rangle$ in H_U^U intersects Z^U in a subgroup of order r^2-1. Thus $|H_U^U : NZ^U| = |H_U^U : N|/(\frac{r^{2p}-1}{r^2-1})$, which is at least $\frac{r^p(r^p-1)}{r(r-1)} > r^{2p-2}$. Since NZ^U contains $Z^U.\langle x^U\rangle$, we have $|L/(L \cap Z^U)| \geq |H_U^U : NZ^U| > r^{2p-2} \geq 3^4$. It then follows from the classification of subgroups of $PGL_2(q_0)$, that the p'-group $L/(L \cap Z^U)$ is contained in a dihedral group $D_{2(q_0-1)}$ or $D_{2(q_0+1)}$. Thus the number of Sylow p-subgroups of H_U^U is at most the number of Sylow p-subgroups of H_U^U in the case where $L/(L \cap Z^U) = D_{2(q_0\pm1)}$. In this case, the normaliser of $\langle x^U\rangle$ in H_U^U is $Z_{r^2-1} \cdot D_{2(r\pm1)} \cdot \langle x^U\rangle$, and so the number of Sylow p-subgroups is $\frac{(r^{2p}-1)(r^p\pm1)}{(r^2-1)(r\pm1)} < 4r^{3(p-1)}$, which is less than the lower bound $r^{4(p-1)}$. This contradiction completes the proof. □

7. Extraspecial type normalisers: $\mathcal{C}_6$ subgroups

Let r be a prime, m a positive integer, and let R be an r-group of symplectic type such that $|R/Z(R)| = r^{2m}$, R is of exponent $r \cdot (2, r)$, and R is as in Table 1. Let $V = V_d(q)$ be a faithful, absolutely irreducible $\mathbb{F}_qR$-module, where r does not divide q. Then $d = \dim V = r^m$, and $N_{GL(V)}(R)$ is as in the table. Assume further that R is not realised over a proper subfield of $\mathbb{F}_q$. Then q is a minimal power of the characteristic p, subject to the conditions in the last column of the table.

Table 1

R	d	$N_{GL(V)}(R)/RZ$	q
r^{1+2m}, r odd	r^m	$Sp_{2m}(r)$	$q \equiv 1 \bmod r$
$4 \circ 2^{1+2m}$	2^m	$Sp_{2m}(2)$	$q \equiv 1 \bmod 4$
$2_{\pm}^{1+2m}$	2^m	$O_{2m}^{\pm}(2)$	$q \equiv 1 \bmod 2$

Here we prove

Theorem 7.1. *Let r be a prime, and assume that R and $V = V_d(q)$ are as above. Suppose $R \trianglelefteq G \leq N_{\Gamma L(V)}(R)$ and G is p-exceptional, and is not transitive on $V^\sharp$. Then G is imprimitive on V, and one of the following holds:*

(i) *$r = 2, q = 3, d = 4$ and $G = 2^{1+4}\mathsf{A}_4$ or $2^{1+4}\mathsf{S}_4$, with orbits on vectors of sizes $1, 16, 64$.*

(ii) *$r = 3, q = 4, d = 3$: here there are five 2-exceptional groups of the form $3^{1+2}.X$; they are $3^{1+2}.2$, $3^{1+2}.6$, $3^{1+2}.\mathsf{S}_3$ (two such groups), and $3^{1+2}.D_{12}$; the first has orbit sizes $1, 9^4, 27$, the rest have $1, 9, 27^2$.*

(iii) *$r = 2, q = 3, d = 8$: here there are five 3-exceptional groups; they are $2_+^{1+6}.X$ with $X = L_3(2)$, $2^3.L_3(2)$ or $2^3.7.3$ (all with orbit sizes 1, 16, 112, 128^2, 224, 448, 896, 1024, 1792^2), and $2_-^{1+6}.Y$ with $Y = 2^4.\mathsf{A}_5$ or $2^4.\mathsf{S}_5$ (all with orbit sizes 1, 160, 1280, 5120).*

7.1. **Reductions.** Let G be a p-exceptional group as in the hypothesis of Theorem 7.1, and assume that $G \leq GL(V)$. We shall handle the case where $G \leq \Gamma L(V)$ and $G \not\leq GL(V)$ at the end of the proof in Section 7.4.

We begin with a technical lemma concerning the Jordan block structure of certain elements of $N_{GL(V)}(R)$. Write J_k for a unipotent Jordan block of size k.

Lemma 7.2. *Let R be as in Table 1. Assume that the characteristic p is a primitive prime divisor of $r^{2m} - 1$ or a primitive prime divisor of $r^m - 1$ with m odd, and also that $p = 7$ when $(r, m) = (2, 3)$. Let t be an element of order p in $N_{GL(V)}(R)$. Then t acts on V as (J_p^k, J_ℓ), where $r^m = kp + \ell$ and $0 \leq \ell < p$.*

Proof. First we consider the case where m is odd and p is a primitive prime divisor of $r^m - 1$ (so $\ell = 1$). If $R = 2_\pm^{1+2m}$, then this in particular implies that $R = 2_+^{1+2m}$. In all cases, embedding tRZ in a subgroup $GL_m(r)$ of $N_{GL(V)}(R)/RZ$, one can check that there exists a t-stable elementary abelian r-subgroup $A < R$ of order r^m such that $V \downarrow A$ affords the regular representation of A and moreover t acts fixed-point-freely on the nontrivial irreducible characters of A. Thus t also permutes fixed-point-freely the nontrivial A-eigenspaces in V. Since $|t| = p$ and $\dim C_V(A) = 1$, it follows that t acts on V as (J_p^k, J_1) as stated.

Assume now that p is a primitive prime divisor of $r^{2m}-1$ (so $\ell = p-1$). Note that $Z(R)Z$ acts trivially on $V \otimes V^*$, $B := R/Z(R)$ is elementary abelian of order r^{2m}, $(V \otimes V^*) \downarrow B$ affords the regular representation of B, and t acts fixed-point-freely on the nontrivial irreducible characters of B. Thus t also permutes fixed-point-freely the nontrivial B-eigenspaces in $V \otimes V^*$. As before, it follows that t acts on $V \otimes V^*$ as (J_p^n, J_1) with $n := (r^{2m} - 1)/p$.

Recall [13, Theorem VIII.2.7] that the Jordan canonical form of $J_a \otimes J_b$ equals

$$(J_{a+b-1}, J_{a+b-3}, \ldots, J_{b-a+1})$$

if $1 \leq a \leq b < a + b \leq p$, and

$$(J_p^{(a+b-p)}, J_{2p-a-b-1}, J_{2p-a-b-3}, \ldots, J_{b-a+1})$$

if $1 \leq a \leq b < p < a+b$. It follows, that $J_a \otimes J_b$ contains no block of size between 2 and $p - 1$ only when $a = b \in \{1, p - 1\}$. Applying this observation to $(V \otimes V^*) \downarrow t$, we see that $V \downarrow t = (J_p^c, J_e^d)$ for some c, d and some $e \in \{1, p - 1\}$. In fact, since $(V \otimes V^*) \downarrow t$ contains J_1 only once, $d \leq 1$. But $pc + de = r^m = kp + (p - 1)$, whence $(d, e) = (1, p - 1)$, as stated. $\square$

The next lemma reduces the number of possibilities for R to a finite number.

Lemma 7.3. *The possibilities for R are as follows:*

$$\begin{array}{l} 4 \circ 2^{1+2m},\ m \leq 10, \\ 2_{\pm}^{1+2m},\ m \leq 11, \\ 3^{1+2m},\ m \leq 5, \\ 5^{1+2m},\ m \leq 2, \\ 7^{1+2m},\ m \leq 2, \\ 11^{1+2}. \end{array}$$

Proof. First consider $R = 4 \circ 2^{1+2m}$. Assume $m \geq 11$. Here $N_{GL(V)}(R)/RZ = Sp_{2m}(2)$ and $G \leq N_{GL(V)}(R)$. Let $t \in G$ be an element of order p. By [22, 4.3], there are $m+3$ conjugates of t which generate a subgroup of $N_{GL(V)}(R)$ covering $Sp_{2m}(2)$. In fact these conjugates generate the whole of $N_{GL(V)}(R)$ since otherwise they would generate a covering group of $Sp_{2m}(2)$; but there is no such group in dimension 2^m by [29]. It follows that $\dim C_V(t) \leq 2^m(1 - \frac{1}{m+3})$. Hence, as G is p-exceptional, Lemma 2.4(ii) implies that

$$q^{2^m} \leq q^{2^m(1-\frac{1}{m+3})}|t^G|. \tag{3}$$

It follows that $q^{2^m/(m+3)} \leq |t^G| \leq 2^{2m}|Sp_{2m}(2)|$. Since $q \geq 5$ in this case (as it is 1 mod 4), this is a contradiction for $m \geq 11$.

An entirely similar argument shows that $m \leq 11$ in the case where $R = 2_{\pm}^{1+2m}$.

Now consider $R = r^{1+2m}$ with r odd. As above, take an element t of order p in G. By [22], there are $m+3$ conjugates of t which generate a subgroup of $N_{GL(V)}(R)$ which covers $Sp_{2m}(r)$. Such a subgroup fixes no nonzero vectors in V (note that the restriction of V to a subgroup $Sp_{2m}(r)$ is the sum of two irreducible Weil modules – see [44]), and so again $\dim C_V(t) \leq r^m(1 - \frac{1}{m+3})$, giving

$$q^{r^m/(m+3)} \leq |t^G| \leq r^{2m}|Sp_{2m}(r)|. \tag{4}$$

Also r divides $q-1$. The bound (4) implies that R is as in the conclusion, except that $R = 13^{1+2}$ is also possible; but this can be ruled out by noting that only 3 conjugates of t are required (rather than 4), by [22, 3.1]. □

Lemma 7.4. *We have $r < 5$.*

Proof. Suppose $r \geq 5$. Then $r = 5, 7$ or 11 by Lemma 7.3.

First consider $r = 5$. Here $m \leq 2$. Suppose $m = 1$, so $G/Z \leq 5^2.Sp_2(5)$, $d = \dim V = 5$ and $q \equiv 1 \bmod 5$. As p divides $|G|$ we have $p = 2$ or 3 and $q = 16$ or 81. For $p = 2$, Lemma 7.2 shows that an involution $t \in G$ acts on V as (J_2^2, J_1), so that $\dim C_V(t) = 3$. Hence Lemma 2.4(ii) gives $16^2 \leq |t^G|$. However, $|t^G| \leq 5^2$, so this is a contradiction. And for $p = 3$, an element $t \in G$ of order 3 acts on V as (J_3, J_2) and we argue similarly.

Now suppose $m = 2$ (still with $r = 5$). Here $G/Z \leq 5^4.Sp_4(5)$, $d = \dim V = 25$ and $q \equiv 1 \bmod 5$. As p divides $|G|$ we have $p = 2, 3$ or 13 and $q = 16$, 81 or 13^4. Let $t \in G$ be an element of order p. For $p = 2$, involutions in G lie in a subgroup $5^{1+2}Sp_2(5) \circ 5^{1+2}Sp_2(5)$ acting on V as a tensor product of two 5-spaces, and hence act on V as either $(J_2^2, J_1) \otimes (J_2^2, J_1)$ or $(J_2^2, J_1) \otimes I_5$; hence $\dim C_V(t) \leq 15$. For $p = 3$, elements of order 3 in G also lie in this subgroup, so act as $(J_3, J_2) \otimes (J_3, J_2)$ or $(J_3, J_2) \otimes I_5$; hence $\dim C_V(t) \leq 10$. And for $p = 13$, elements of order 13 in

G act as (J_{13}, J_2), hence $\dim C_V(t) = 2$. We conclude that $\dim C_V(t) \leq 15$ in any characteristic. Now Lemma 2.4(ii) gives a contradiction.

The cases $r = 7$ and $r = 11$ are ruled out in an entirely similar fashion. □

7.2. **The case** $r = 2$**.** In this section we handle the case where $r = 2$. By Lemma 7.3, R is either $4 \circ 2^{1+2m}$ with $m \leq 10$, or $2_{\pm}^{1+2m}$ with $m \leq 11$.

Lemma 7.5. *We have* $m \leq 5$.

Proof. Consider the case $m = 6$. Here $d = \dim V = 64$, $R = 2_{\pm}^{1+12}$ or $4 \circ 2^{1+12}$ and $G/RZ \leq O_{12}^{\pm}(2)$ or $Sp_{12}(2)$, respectively. Moreover, $p = 3, 5, 7, 11, 13, 17$ or 31.

Suppose first that $p = 3$ and $R = 2_{\pm}^{1+12}$. Then $q = 3$. We shall use Lemma 2.4(ii). Let $t \in G$ have order 3. Modulo R, t is conjugate to an element t_k of order 3 in a subgroup $O_2^-(2)^k$ of $O_{12}^{\pm}(2)$, where $1 \leq k \leq 6$, projecting nontrivially to each factor. We can work out the Jordan form of t_k on V as an element of $2^{1+2}O_2^-(2) \otimes \cdots \otimes 2^{1+2}O_2^-(2) \otimes I_{2^{6-k}}$: so on V, t_k acts as $J_2 \otimes \cdots \otimes J_2 \otimes I_{2^{6-k}}$ where there are k factors J_2. The Jordan forms of these tensor products are easily worked out, and we find that the number of Jordan blocks of t_k is as follows:

k	1	2	3	4	5	6
$\dim C_V(t_k)$	32	32	24	24	22	22

By Corollary 2.10, modulo R the centraliser of t_k in $R.O_{12}^{\pm}(2)$ is $C_{O_{12}^-(2)}(t_k)$, which contains $O_{12-2k}(2) \times 3^k.\mathsf{S}_k$. Hence

$$|t_k^G| \leq \frac{2^{12}|O_{12}^{\pm}(2)|}{2^{12-2k}|O_{12-2k}(2)| \cdot 3^k \cdot k!}.$$

For each k between 1 and 6, we check that $|V : C_V(t_k)| = 3^{64-\dim C_V(t_k)}$ is greater than $|t_k^G|$. So this contradicts Lemma 2.4(ii).

When $p = 3$ and $R = 4 \circ 2^{1+12}$ we have $q = 9$ (as $q \equiv 1 \bmod 4$ in this case). Here an element $t \in G$ of order 3 lies in a subgroup $Sp_2(2)^k$ of $Sp_{12}(2)$ (modulo R) for some k with $1 \leq k \leq 6$, and we argue as above.

Now suppose $p = 5$. Here $q = 5$ for both cases $R = 4 \circ 2^{1+12}$ and $R = 2_{\pm}^{1+10}$. In the former case, an element $t \in G$ of order 5 is conjugate to an element t_k in a subgroup $Sp_4(2)^k$ of $Sp_{12}(2)$, where $1 \leq k \leq 3$, projecting nontrivially to each factor. Then t_k acts on V as an element of a tensor product of k factors $4 \circ 2^{1+4}.Sp_4(2)$ acting in dimension 4. By Lemma 7.2, an element of order 5 in such a factor acts as J_4, so t_k acts on V as $J_4 \otimes \cdots \otimes J_4 \otimes I_{64/4^k}$. Working out these tensor products of Jordan blocks, it follows that $\dim C_V(t_k) = 16, 16, 13$ according as $k = 1, 2, 3$ respectively. Moreover, as above

$$|t_k^G| \leq \frac{2^{12}|Sp_{12}(2)|}{2^{12-4k}|Sp_{12-4k}(2)| \cdot 5^k \cdot k!}.$$

It now follows that $|V : C_V(t_k)| = 5^{64-\dim C_V(t_k)}$ is greater than $|t_k^G|$ for each k, contradicting Lemma 2.4(ii) again. The $R = 2_{\pm}^{1+10}$ case is entirely similar.

For $p = 7$, we see as above that an element $t \in G$ of order 7 is conjugate to an element t_k ($k = 1$ or 2) acting on V as $(J_7, J_1) \otimes I_8$ or $(J_7, J_1) \otimes (J_7, J_1)$ (using Lemma 7.2 to see that the action of an element of order 7 in $4 \circ 2^{1+6}.Sp_6(2) < SL_8(7^a)$ is (J_7, J_1)). Hence $\dim C_V(t_k) = 16$ or 10 and we contradict Lemma 2.4 as before.

For larger values of p there is only one class of elements of order p in $N_{GL(V)}(R)$, and its action on V can be computed as above using Lemma 7.2. We find that for $t \in G$ of order p, $\dim C_V(t)$ is $6, 5, 4$ or 4 according as $p = 11, 13, 17$ or 31. In each case $|V : C_V(t)|$ is much larger than $|N_{GL(V)}(R)|$, contradicting Lemma 2.4 once more.

This completes the argument for $m = 6$. For $m \geq 7$ the same method applies. □

Lemma 7.6. *m is not* 5.

Proof. Suppose $m = 5$. The method is very similar to the previous proof, but the bounds are tighter and more work is needed for the case $q = 3$.

Consider first the case where $p = 3$. Let $R = 2^{1+10}_{\pm}$. Then $q = 3$. If $t \in G$ is an element of order 3, then t is conjugate to an element t_k lying in a subgroup $O^-_2(2)^k$ of $O^{\pm}_{10}(2)$ and acting on V as $J_2 \otimes \cdots \otimes J_2 \otimes I_{32/2^k}$. We compute that $\dim C_V(t_k) = 16, 16, 12, 12, 11$ for $k = 1, 2, 3, 4, 5$ respectively. If $k = 1$, then Lemma 2.4(ii) gives $3^{16} < |t_1^G|$; however, $|t_1^G| \leq |O_{10}(2) : 2^8.(3 \times O_8(2)|$ which is less than 3^{16}. Hence (again by Lemma 2.4(ii)), G/R is a subgroup of $O^{\pm}_{10}(2)$ containing no conjugates of t_1 and at least $3^{32-\dim C_V(t_k)}/2^{2k}$ conjugates of t_k for some $k \geq 2$. However, a MAGMA computation shows that there is no such subgroup.

In the case where $p = 3$ and $R = 4 \circ 2^{1+12}$ we have $q = 9$. Here an element $t \in G$ of order 3 is conjugate to some t_k as in the previous paragraph but no MAGMA computation is required as $9^{32-\dim C_V(t_k)} > |t_k^G|$ for all k, which contradicts Lemma 2.4.

Now consider $p = 5$. Here $q = 5$ and an element $t \in G$ of order 5 is conjugate to an element t_k ($k = 1$ or 2) lying in a subgroup $O^-_4(2)^k$ of $O_{10}(2)$ or $Sp_{10}(2)$ and acting on V as $J_4 \otimes I_8$ (for $k = 1$) or $J_4 \otimes J_4 \otimes I_2$ (for $k = 2$). So $\dim C_V(t_k) = 8$ for $k = 1, 2$. However, we check as above that $5^{24} > |t_k^G|$ for $k = 1, 2$, which contradicts Lemma 2.4(ii).

The other possible values of p are $7, 11, 17$ and 31. For these values there is only one class of elements of order p in $N_{GL(V)}(R)$, and its action on V can be computed as above using Lemma 7.2. In each case $|V : C_V(t)|$ is much larger than $|N_{GL(V)}(R)|$, contradicting Lemma 2.4 once more. □

Lemma 7.7. *m is not* 4.

Proof. Suppose $m = 4$. Then p divides $|Sp_8(2)|$, so is $3, 5, 7$ or 17. The cases $p = 7$ or 17 are easily handled as in the last paragraph of the previous proof.

Consider now the case $p = 5$. Here $q = 5$ and $G \leq (4 \circ 2^{1+8}).Sp_8(2) < GL_{16}(5)$. An element $t \in G$ of order 5 is conjugate to an element t_k ($k = 1$ or 2) lying in $Sp_4(2)^k$ and acting on V as $J_4 \otimes I_4$ or $J_4 \otimes J_4$; so $\dim C_V(t_k) = 4$ for both $k = 1, 2$. Hence $5^{12} < |t^G|$ by Lemma 2.4(ii). It follows that G/R contains no conjugates of t_1 and contains at least $5^{12}/2^8$ conjugates of t_2. Using [8], one checks that the only possible maximal subgroup of $Sp_8(2)$ containing G/R is $O^+_8(2)$, and then that the only subgroup of this containing enough conjugates of t_2 are $\Omega^+_8(2)$ and $O^+_8(2)$. But these contain conjugates of t_1, a contradiction.

Now suppose $p = 3$. Consider first the case where $R = 4 \circ 2^{1+8}$. Here $q = 9$. An element $t \in G$ of order 3 is conjugate to some t_k ($1 \leq k \leq 4$) lying in $Sp_2(2)^k$ and acting on V as $J_2 \otimes \cdots \otimes J_2 \otimes I_{16/2^k}$, and we find that $\dim C_V(t_k) = 8, 8, 6, 6$ according as $k = 1, 2, 3, 4$ respectively. We compute that $9^{\dim V - \dim C_V(t_k)} > |t_k^G|$ for $k = 1, 3, 4$. Hence G/R is a subgroup of $Sp_8(2)$ containing no conjugates of

t_1, t_3, t_4, and at least $9^8/2^4$ conjugates of t_2. One checks using [8] that there are no such subgroups.

Finally suppose $p = 3$ and $R = 2_\epsilon^{1+8}$, so $G \leq 2_\epsilon^{1+8}.O_8^\epsilon(2) < GL_{16}(3)$. Here the usual bounding methods do not work and we use MAGMA computation along the following lines. For $\epsilon = -$, we compute that the group R has a regular orbit on vectors that has 48960 images under $R.O_8^-(2)$. Moreover, $O_8^-(2)$ has 4683 subgroups of order divisible by 3, and all of them have an orbit of length divisible by 3 in the action on 48960 points. Similarly, for $\epsilon = +$ the group R has an orbit of length 128 that has 1575 images under $R.O_8^+(2)$, and $O_8^+(2)$ has 5988 subgroups with order divisible by 3; all of them have an orbit of length divisible by 3 on the set of size 1575. □

Lemma 7.8. *If $m = 3$, then $q = 3$ and G is as in conclusion* (iii) *of Theorem* 7.1.

Proof. Suppose $m = 3$. Then p divides $|Sp_6(2)|$, so is $3, 5$ or 7. We make heavy use of computation in this proof.

Let $p = 3$. First consider $R = 2_+^{1+6}$, so $G \leq 2_+^{1+6}.O_6^+(2) < GL_8(3)$. We compute that the group R has an orbit of length 16 on vectors, that has 30 images under $X := R.O_6^+(2)$. Let Δ be this set of 30 images, and let $G = R.Y \leq X$ with $Y \leq O_6^+(2)$. Then $|Y|$ is divisible by 3. If Y has an orbit Δ_0 on Δ of length divisible by 3, then G has an orbit on vectors of length $16|\Delta_0|$, contrary to 3-exceptionality. We compute that the group $O_6^+(2)$ has 176 subgroups of order divisible by 3, and all but 12 have an orbit of length divisible by 3 in the action on Δ. Hence Y is one of the remaining 12 groups. When pulled back to subgroups of X containing R, all but three of these have an orbit on nonzero vectors of length divisible by 3. The three remaining groups are 3-exceptional – they are $R.L_3(2)$, $R.2^3.L_3(2)$ and $R.2^3.7.3$. All three are imprimitive on V, preserving a decomposition into eight 1-dimensional spaces, and are as in Theorem 7.1(iii).

Now consider $R = 2_-^{1+6}$, so $G \leq 2_-^{1+6}.O_6^-(2) < GL_8(3)$. Here R has an orbit on vectors of length 32 that has 45 images under $X := R.O^-(6,2)$. Now $O^-(6,2)$ has 238 subgroups of order divisible by 3 and all but 7 have an orbit of length divisible by 3 in the action on 45 points. Pulling these remaining 7 subgroups back to subgroups of X containing R we see that all but two have an orbit on nonzero vectors of length divisible by 3. The two groups are $R.(2^4.\mathsf{A}_5)$ and $R.(2^4.\mathsf{S}_5)$, as in Theorem 7.1(iii). Both are imprimitive on V with a decomposition into two 4-dimensional spaces.

Finally, if $R = 4 \circ 2^{1+6}$, then $q = 9$ and R has an orbit of length 32 that has 270 images under $X := R.Sp(6,2)$. Then $Sp_6(2)$ has 2777 subgroups of order divisible by 3, and all but 13 of these have an orbit of length divisible by 3 in the action on 270 points. Pulling back these 13 subgroups to subgroups of X containing R, we find that they all have an orbit of length divisible by 3 on vectors.

Now let $p = 5$, so $G \leq X := 4 \circ 2^{1+6}.Sp_6(2) < GL_8(5)$. Here R has an orbit on vectors of length 32 that has 135 images under $X := R.Sp_6(2)$. Now $Sp_6(2)$ has 82 subgroups with order divisible by 5 and all have an orbit of length divisible by 5 in the action on 135 points.

Lastly let $p = 7$. If $q = 49$ the usual method using Lemma 2.4 yields a contradiction, so suppose $q = 7$ and $R = 2_+^{1+6}$ (note that 7 does not divide $|O_6^-(2)|$ so only the $+$ type is possible). An element $t \in G$ of order 7 acts on V as (J_7, J_1), so Lemma 2.4(ii) implies that G/R is a subgroup of $O_6^+(2)$ containing at least $7^6/2^6$

conjugates of t. This implies that G/R contains $\Omega_6^+(2)$. Now one computes that $R.\Omega_6^+(2)$ has an orbit on vectors of length divisible by 7. □

Lemma 7.9. *If $m = 2$, then $q = 3$ and G is a subgroup $2_-^{1+4}.\mathsf{A}_4$ or $2_-^{1+4}.\mathsf{S}_4$ of $GL_4(3)$. These subgroups are imprimitive and 3-exceptional, with orbits on vectors of sizes $1, 16, 64$.*

Proof. Here G is a subgroup of one of $2_\epsilon^{1+4}.O_4^\epsilon(2) < GL_4(3)$, or $4 \circ 2^{1+4}.Sp_4(2) < GL_4(9)$ or $GL_4(5)$. A routine computation of all subgroups of $R.O_4^\epsilon(2)$ and $R.Sp_4(2)$ containing R gives the conclusion. □

This completes the proof of Theorem 7.1 in the case where $G \leq GL(V)$ and $r = 2$.

7.3. **The case $r = 3$.** Now suppose $r = 3$. By Lemma 7.3, we have $R = 3^{1+2m}$ with $m \leq 5$. Also $G \leq R.Sp_{2m}(3) < GL(V) = GL_{3^m}(q)$ with $q \equiv 1 \bmod 3$.

Lemma 7.10. *We have $m \leq 2$.*

Proof. First consider $m = 5$. The bound (4) forces $q = 4$ or 7. If $q = 7$ and $t \in G$ is an element of order 7, then modulo R, t lies in a subgroup $Sp_6(3) \times Sp_4(3)$ of $Sp_{10}(3)$. Hence by Lemma 7.2, t acts on V as (J_7^{27}, J_6^9) and $\dim C_V(t) = 36$. Now Lemma 2.4 gives a contradiction. Similarly, when $q = 4$ an involution $t \in G$ lies in a subgroup $Sp_2(3)^k$ and acts as a tensor product of k factors (J_2, J_1) with an identity matrix, whence we see that $\dim C_V(t) \leq 162$, and again Lemma 2.4 is violated.

Now let $m = 4$. Then p divides $|Sp_8(3)|$. If $t \in G$ has order p, we see using Lemma 7.2 in the usual way that $\dim C_V(t) \leq 54$. Now Lemma 2.4(ii) forces $p = 2$ and $q = 4$. The involution t is conjugate to an element t_k lying in $Sp_2(3)^k$ $(1 \leq k \leq 4)$ and acting on V as a tensor product of k factors (J_2, J_1) with an identity matrix. Hence $\dim C_V(t_k) = 54, 45, 42, 41$ according as $k = 1, 2, 3, 4$ respectively. We find that for each k we have $|V : C_V(t)| > |t_k^G|$, contrary to Lemma 2.4.

The case $m = 3$ is dealt with in exactly the same fashion, and we leave this to the reader. □

Lemma 7.11. *We have $m = 1$, $q = 4$ and $G = 3^{1+2}.2$ or $3^{1+2}.6$ in $GL_3(4)$, as in Theorem 7.1(iii). Both are imprimitive on V, with orbit sizes $1, 9^4, 27$ or $1, 9, 27^2$, respectively.*

Proof. Suppose $m = 2$. Then p divides $|Sp_4(3)|$, so $p = 2$ or 5. The usual arguments rule out $p = 5$ (for which $q = 25$), so $p = 2$, $q = 4$ and $G \leq R.Sp_4(3) < GL_9(4)$. A computation of all the subgroups of $Sp_4(3)$ of even order shows that there are no 3-exceptional groups in this case.

Hence $m = 1$. Then p divides $|Sp_2(3)|$, so $p = 2$, $q = 4$ and a computation gives the examples in the lemma. □

This completes the proof of Theorem 7.1 in the case where $G \leq GL(V)$.

7.4. **The semilinear case.** We now complete the proof of Theorem 7.1 by handling the case where the p-exceptional group G in the hypothesis lies in $\Gamma L(V)$ and not in $GL(V)$, where $V = V_d(q)$. Let $G_0 = G \cap GL(V)$. If p divides $|G_0|$, then G_0 is p-exceptional, hence is given by the linear case of the theorem which we have already proved. The only possibility is that $d = 3, q = 4$ and G_0 is one of the two groups in the conclusion of Lemma 7.11. Computation in $\Gamma L_3(4)$ reveals one further 2-exceptional group G in this case, the group $3^{1+2}.D_{12}$ in Theorem 7.1(iii).

Assume now that p does not divide $|G_0|$. By Lemma 2.3, G has a p-exceptional normal subgroup $G_0\langle\sigma\rangle$, where $\sigma \in \Gamma L(V)\backslash GL(V)$ is a field automorphism of order p. Hence $q = p^{kp}$ for some integer k. If $r = 2$, then $q = p$ or p^2 (as G is not realisable over a proper subfield of $\mathbb{F}_q$), and also q is odd; so it is impossible to have $q = p^{kp}$. Therefore, $r \geq 3$. Also, the field automorphism σ acts on V, fixing pointwise a subset $V_d(q^{1/p})$, and so $|V : C_V(\sigma)| = q^{(1-\frac{1}{p})r^m}$. Hence Lemma 2.4(ii) gives

$$q^{(1-\frac{1}{p})r^m} \leq |\sigma^G| \leq r^{2m}|Sp_{2m}(r)|. \tag{5}$$

If $p \neq 2$, then $q \geq 27$ and one checks that (5) cannot hold. Hence $p = 2$ and $q = 4^k$.

If $k \geq 2$, then (5) implies that $m = 1$, $q = 16$ and $r = 5$. For $k = 1$, we have $r = 3$ and (5) gives $m \leq 3$. Moreover, if $m = 3$, then G_0/R is an odd order subgroup of $Sp_6(3)$. Computation shows that the largest such subgroup has order $3^7 \cdot 13$. Hence $|G_0/R| \leq 3^7 \cdot 13$, and so (5) gives $2^{27} \leq 3^{14} \cdot 13$, which is false. Hence $m \leq 2$. The possibilities remaining are as follows, where we write K for the odd order group $G_0/R \leq Sp_{2m}(r)$:

(1) $G = 5^{1+2}.K.2 < \Gamma L_5(16)$,
(2) $G = 3^{1+2}.K.2 < \Gamma L_3(4)$,
(3) $G = 3^{1+4}.K.2 < \Gamma L_9(4)$.

In cases (1) and (3) we check computationally that there are no 2-exceptional examples, and in case (2) we get the two examples $3^{1+2}.\mathsf{S}_3$ in the conclusion of Theorem 7.1(iii).

This completes the proof of Theorem 7.1.

8. $\mathcal{C}_9$ case I: Preliminaries and Lie(p)

Define $G \leq \Gamma L(V) = \Gamma L_d(q)$ $(q = p^a)$ to be in class $\mathcal{C}_9$ if G/Z is almost simple, with socle absolutely irreducible and not realisable over a proper subfield of $\mathbb{F}_q$. Let $L = \mathrm{soc}(G/Z)$, a simple group, and let $\hat{L}$ be a perfect preimage of L in G.

The case where the simple group L is of Lie type in characteristic p turns out to be very easy. Define Lie(p) to be the set of simple groups of Lie type in characteristic p, excluding $Sp_4(2)'$, $G_2(2)'$, ${}^2F_4(2)'$ in characteristic 2, and ${}^2G_2(3)'$ in characteristic 3. The first two of these and the last are dealt with in Sections 9 and 11 (in their guises as A_6, $U_3(3)$ and $L_2(8)$), and ${}^2F_4(2)'$ in the remark below after Corollary 8.2.

Lemma 8.1. *Suppose that $G \leq \Gamma L(V) = \Gamma L_d(q)$ $(q = p^a)$ is in class $\mathcal{C}_9$ and $L = \mathrm{soc}(G/Z) \in$ Lie(p). If G is p-exceptional, then G (and $\hat{L}$) is transitive on $P_1(V)$.*

Proof. From the structure of the exceptional Schur multipliers of simple groups in Lie(p) (see [27, Table 5.1.D] for example), we see that the perfect group $\hat{L}$ must be an image of the simply connected group of type L (since the extra parts of such multipliers are always p-groups which hence act trivially on irreducible modules in characteristic p). Therefore, it follows from [9, Theorem 4.3(c)] that a Sylow p-subgroup P of $\hat{L}$ fixes a unique 1-space in V. If G is p-exceptional, then so is $\hat{L}$, and so every nonzero vector in V is fixed by an $\hat{L}$-conjugate of P. For 1-spaces $\langle v\rangle$, $\langle w\rangle$ fixed by P^g, P^h, we then have $\langle v\rangle g^{-1}h = \langle w\rangle$, and the conclusion follows. $\square$

Corollary 8.2. *If G is as in Lemma* 8.1 *and is p-exceptional, then G is transitive on $V^\sharp$.*

Proof. By Lemma 8.1, $\hat{L}$ is transitive on $P_1(V)$. Hence $\hat{L}$ is given by Hering's Theorem (see [30, Appendix 1]): so $\hat{L}$ is $SL_d(q)$, $Sp_d(q)$ or $G_2(q)$ (q even, $d = 6$), and in each case $\hat{L}$ is transitive on $V^\sharp$. □

Remark. The case where $L = \mathrm{soc}(G/Z) = {}^2F_4(2)'$ and $p = 2$ is quickly ruled out, as follows. Suppose G is 2-exceptional. The 2-modular characters of L are given in [26], and the bound in Lemma 2.4(i) implies that $d = 26, q = 2$. As 13 does not divide $|V^\sharp| = 2^{26} - 1$, an element $t \in L$ of order 13 fixes a nonzero vector v. By 2-exceptionality, v is also fixed by a Sylow 2-subgroup P of L. But t and P generate L (see [8, p. 74]), so this is impossible.

9. $\mathcal{C}_9$ case II: Alternating groups

In this section we deal with the case where the simple group $L = \mathrm{soc}(G/Z)$ is an alternating group. We prove

Theorem 9.1. *Let $G \leq \Gamma L(V) = \Gamma L_d(q)$ ($q = p^a$) be such that G/Z is almost simple with socle $L \cong \mathsf{A}_c$, an alternating group with $c \geq 5$, and suppose the perfect preimage $\hat{L}$ of L in G is absolutely irreducible on V and realisable over no proper subfield of $\mathbb{F}_q$. Assume G is p-exceptional and not transitive on $V^\sharp$. Then one of the following holds:*

(i) *$G = \mathsf{A}_c$ or S_c where $c = 2^r - 2$ or $2^r - 1$, with V the deleted permutation module for G over $\mathbb{F}_2$, of dimension $d = c - 2$ or $c - 1$ respectively;*

(ii) *$\hat{L} = SL_2(5) \trianglelefteq G < \Gamma L_2(9)$, with orbit sizes $1, 40, 40$ on vectors.*

Conversely, the groups G in (i) *are p-exceptional.*

We shall need the following.

Proposition 9.2. *If r is a prime with $r \leq c$ (and $c \geq 5$), then A_c is generated by two of its Sylow r-subgroups.*

Proof. If $r = 2$ this follows from [21], so assume that $r > 2$. Write $c = kr + s$ for integers k, s with $0 \leq s \leq r - 1$.

First consider the case where $k = 1$. If $s = 0$, observe that $(1\,2 \cdots r)$ and $(1\,2 \cdots r)\,(1\,2\,3)$ are both r-cycles, and the group they generate contains a 3-cycle, hence is A_r. If $s = 1$, then $c = r + 1$ and the r-cycles $(1\,2 \cdots r)$, $(2\,3 \cdots r + 1)$ generate A_c as their commutator contains a 3-cycle. A similar argument applies for $s = 2$, taking $(1\,2 \cdots r)$ and $(3\,4 \cdots r + 2)$: the group these generate is a primitive group containing both an r-cycle and a 5-cycle, hence is A_c. Finally, if $s > 2$, then $(1\,2 \cdots r)$ and $(s + 1\, s + 2 \cdots c)$ generate a primitive group containing an r-cycle fixing more than 2 points, hence generate A_c by Jordan's Theorem [45, 13.9].

Now suppose $k \geq 2$. Take R to be a Sylow r-subgroup of A_c containing the r-cycles $(1 \cdots r)$, $(2r-1 \cdots 3r-2)$, …, $(2ir-2i+1 \cdots (2i+1)r-2i)$ up to the maximal i such that $(2i+1)r - 2i \leq c$, and take S to be a Sylow r-subgroup containing the r-cycles $(r \cdots 2r-1)$, $(3r-2 \cdots 4r-3)$, …, $((2i-1)r - 2i + 2 \cdots 2ir - 2i + 1)$. Then the group generated by R and S contains a subgroup A_{c-r+1}, and now we can include a further r-cycle in S to generate A_c. □

Now we embark upon the proof of Theorem 9.1. Let $G \leq \Gamma L(V) = \Gamma L_d(q)$ ($q = p^a$) be as in the hypothesis, with $L = \mathrm{soc}(G/Z) \cong \mathsf{A}_c$. Note that $|\mathrm{Out}(L)| = 2$ or 4, so p divides $G \cap GL(V)$ which is therefore p-exceptional by Lemma 2.1. Hence we may replace G by $G \cap GL(V)$ and assume that $G \leq GL(V)$. By Lemma 2.4, together with Proposition 9.2, we have

$$d = \dim V \leq 2\log_q(c!) - 2\log_q((\frac{c!}{2})_p), \tag{6}$$

where n_p denotes the p-part of an integer n.

Since $\mathsf{A}_5 \cong L_2(5) \cong L_2(4)$, $\mathsf{A}_6 \cong L_2(9)$ and $\mathsf{A}_8 \cong L_4(2)$, Lemma 8.1 shows that we may exclude these groups from consideration in these characteristics.

Lemma 9.3. *One of the following holds:*
(i) $\hat{L} = L$ *and* V *is the deleted permutation module over* $\mathbb{F}_p$*;*
(ii) $c \leq 13$ *and* $d \leq 32$.

Proof. Suppose V is not the deleted permutation module. Assume that $d > 250$. Then the bound (6) forces $c > 40$. By [25, Theorem 7] (for $\hat{L} = \mathsf{A}_c$) and [43] (for $\hat{L} = 2.\mathsf{A}_c$), we have $d \geq \frac{1}{4}c(c-5)$. But this is greater than the upper bound for d given by (6) when $c > 40$. Hence $d \leq 250$. Now all the possibilities for the $\mathbb{F}_q\hat{L}$-module V are given by [24]. We check that the only possibilities for which the bound (6) is satisfied have $c \leq 13$ and $d \leq 32$. □

Lemma 9.4. *Suppose* V *is the deleted permutation module for* $\hat{L} = \mathsf{A}_c$ *over* $\mathbb{F}_p$*. Then* $p = 2$*,* $c = 2^r - 1$ *or* $2^r - 2$*, and* V *has dimension* $d = c-1$ *or* $c-2$ *respectively. Moreover, in these representations* A_c *and* S_c *are* 2*-exceptional.*

Proof. Here $V = S/(S \cap T)$ where $S = \{(a_1, \ldots, a_c) \in \mathbb{F}_p^c \mid \sum a_i = 0\}$, $T = \{(a, \ldots, a) \mid a \in \mathbb{F}_p\}$. If $p \geq 5$ define

$$v = (1, 2, -3, 1, -1, 2, -2, \ldots, \frac{p-3}{2}, -\frac{p-3}{2}, 0, \ldots, 0) \in S$$

and let $x = v + (S \cap T) \in V$. Then $(\hat{L})_x = \mathsf{A}_{c-p} \times H$ where $H \leq \mathsf{A}_p$ and $|H|$ is coprime to p; hence p divides $|x^{\hat{L}}|$, contradicting the fact that G is p-exceptional. If $p = 3$ and $c \neq 6$ the same argument applies, taking $v = (1, 1, -1, -1, 0, \ldots, 0)$; and if $p = 3$ and $c = 6$, then taking the same v, we have $|\hat{L}_x|_3 = 3$, so $|x^{\hat{L}}|$ is divisible by $|\hat{L}|_3/3 = 3$, again a contradiction.

Hence $p = 2$. Here $d = \dim V = c - (c, 2)$. For c odd, the orbit sizes of $\hat{L} = \mathsf{A}_c$ on nonzero vectors are $\binom{c}{i}$ for $1 \leq i \leq c-1$; and for c even the orbit sizes are $\binom{c}{2i}$ ($i \neq c/2$) and also $\frac{1}{2}\binom{c}{c/2}$ when $4|c$. It follows that we get 2-exceptional examples when $c = 2^r - 1$ or $2^r - 2$. And when c is not of one of these forms, let 2^e be the smallest power of 2 that is missing in the binary expansion of c; then $\binom{c}{2^e}$ is even, so $\hat{L}$ (and hence G) is not 2-exceptional, a contradiction. □

Assume from now on that V is not the deleted permutation module.

Lemma 9.5. *If* V *is not the deleted permutation module, then* $c \leq 7$.

Proof. From [24] and (6), we see that the possibilities for c, d, p and q are as in the following table:

Case	c	d	p	q
1	13	32	3	3
2	12	16	$2, 3$	$4, 3$
3	11	16	$2, 3, 5, 7$	p or p^2
4	10	8	5	5
5		16	$2, 3$	$2, 3$
6	9	8	$2, 3, 5, 7$	p
7		21	3	3
8	8	8	$3, 5, 7$	p
9		13	3	3

In cases 1, 2, 3, 4 and 7 we adopt a unified approach: for the respective primes $r = 13, 11, 11, 7$ and 7 we observe that r does not divide $q^d - 1$, and hence there is a nonzero vector $v \in V$ fixed by an element $t \in \hat{L}$ of order r. As G is p-exceptional, v is also fixed by a Sylow p-subgroup P of $\hat{L}$. However, it is easy to see that $\langle t, P\rangle = \hat{L}$ in all these cases, so this is a contradiction.

Now consider case 5. Here $c = 10$, $p = 2$ or 3 and $d = \dim V = 16$. The Brauer character χ of V is given in [26]. For $p = 3$ we have $\chi(t) = 1$ where t is a $5B$-element of $\hat{L} = 2.\mathsf{A}_{10}$; this implies that t fixes a nonzero vector v – however, $\langle t, P\rangle = \hat{L}$ for any Sylow 3-subgroup P, so this is not possible. For $p = 2$ we apply the same argument with t a $9A$-element of $\hat{L} = \mathsf{A}_{10}$, noting that t and any Sylow 2-subgroup generate $\hat{L}$.

Next consider case 6. For $p = 2$ we have

$$\hat{L} = \mathsf{A}_9 < O_8^+(2) = O(V),$$

and $\hat{L}$ has two orbits on nonzero vectors (see [8]), one of which has even size, contrary to 2-exceptionality. For $p = 3$ or 5 we argue as above that an element t in class $7A$ or $9A$ fixes a nonzero vector, and generates $\hat{L}$ with any Sylow p-subgroup, a contradiction. The case $p = 7$ requires a little more argument. Let S be a Sylow 3-subgroup of $\hat{L} = 2.\mathsf{A}_9$. Since $|V| = 7^8 \equiv 4 \bmod 9$, there is an orbit of S on nonzero vectors of size 1 or 3, and hence there is a nonzero vector v which is fixed by a subgroup S_0 of S of index 3. However, S_0 and any Sylow 7-subgroup generate $\hat{L}$, so this is impossible.

Now consider case 8. For $p = 5$ we argue with the Brauer character that an element in class $4A$ in $\hat{L} = 2.\mathsf{A}_8$ fixes a nonzero vector; but this element generates $\hat{L}$ with any Sylow 5-subgroup. For $p = 3$, the Brauer character χ of V shows that an element $t \in \hat{L}$ of order 7 fixes a nonzero vector v. Then v is stabilised by $\langle t, P\rangle$ for some Sylow 3-subgroup P of $\hat{L}$, and this contains $2.\mathsf{A}_7$. However, $\chi \downarrow 2.\mathsf{A}_7$ is a sum of two irreducibles of degree 4, so $2.\mathsf{A}_7$ does not fix a nonzero vector, a contradiction. For $p = 7$ we argue similarly using a $4B$-element t.

Finally, consider case 9. Here a $7A$ element $t \in \hat{L} = \mathsf{A}_8$ fixes a nonzero vector v, and so v is fixed by $\langle t, P\rangle$ for some Sylow 3-subgroup P of $\hat{L}$. This contains A_7. However, we see from [26] that A_7 acts irreducibly on V, so this is impossible. □

Lemma 9.6. *If V is not the deleted permutation module, then c is not 7.*

Proof. Suppose $c = 7$. Here [24] and (6) show that d, q are as in the following table of possibilities:

Case	d	q
1	15	3
2	13	$3, 5$
3	9	7
4	8	5
5	6	$4, 9, 7$
6	4	$2, 9, 25, 7$
7	3	25

In cases 1–4, an element t of order 5 or 7 fixes a nonzero vector, and generates $\hat{L}$ with any Sylow p-subgroup, giving a contradiction as usual.

Consider case 5. For $p = 7$, we argue as before with an element of order 5. For $p = 3$, the Brauer character shows that an element t of order 5 in $\hat{L} = 2.\mathsf{A}_7$ fixes a nonzero vector v, and so v is fixed by $\langle t, P\rangle$ for some Sylow 3-subgroup P of $\hat{L}$. This contains $2.\mathsf{A}_6$, which is irreducible on V, a contradiction. Finally for $p = 2$, we have $\hat{L} = 3.\mathsf{A}_7 < 3.M_{22} < SU_6(2) < SL(V)$ (see [8, p. 39]). From [8] we see that M_{22} is transitive on the set of 672 nonisotropic 1-spaces in V, and A_7 has 3 orbits on these, so that one orbit must have even size, contrary to p-exceptionality.

Now consider case 6. For $p = 3$ or 5, we argue as usual with an element of order 7 which fixes a vector. For $p = 2$ we have $\hat{L} = \mathsf{A}_7 < SL_4(2) = SL(V)$ and $\hat{L}$ is transitive on $V^\sharp$, contrary to assumption. And for $p = 7$ we have $\hat{L} = 2.\mathsf{A}_7 < SL_4(7)$; by [30, Appendix 2], $\hat{L}$ has two orbits on nonzero vectors, one of which has size divisible by 7.

Finally, in case 7 the Brauer character shows that there is an element t of order 3 fixing a nonzero vector v, so v is fixed by $\langle t, P\rangle$ for some Sylow 5-subgroup P, and this contains A_5; but a subgroup A_5 must be irreducible on V, a contradiction. □

Lemma 9.7. *If V is not the deleted permutation module, then we have $c = 5$, $d = 2$, $q = 9$ and $\hat{L} = SL_2(5) < SL_2(9) = SL(V)$, with orbit sizes $40, 40$ on nonzero vectors.*

Proof. We know that $c = 5$ or 6. Recall that $p = 3$ when $c = 5$, and $p \neq 3$ when $c = 6$. Hence [24] and (6) show that d, q are as in the following table of possibilities:

Case	c	d	q
1	6	4	$2, 5$
2	6	3	$4, 25$
3	5	6	3
4	5	3	9
5	5	2	9

Consider case 1. If $p = 2$, then $\hat{L} < SL_4(2)$ is transitive on nonzero vectors. And if $p = 5$, the Brauer character shows that there is an element t of order 3 fixing a nonzero vector v, so the stabiliser of v contains $\langle t, P\rangle$ for some Sylow 5-subgroup P of $\hat{L} = 2.\mathsf{A}_6$. So this stabiliser contains $2.\mathsf{A}_5$; but a subgroup $2.\mathsf{A}_5$ is irreducible on V, a contradiction. The same argument deals with case 2 for $p = 5$; and in case 2 for $p = 2$ we have $\hat{L} = 3.\mathsf{A}_6 < SL_3(4) = SL(V)$, and [30] shows that $\hat{L}$ has 2 orbits on nonzero vectors, one of which has even size.

In cases 3 and 4 we observe that an element of order 5 in $\hat{L}$ fixes a vector and generates $\hat{L}$ with any Sylow 3-subgroup. Finally, case 5 gives the example in the conclusion. □

This completes the proof of Theorem 9.1.

10. $\mathcal{C}_9$ case III: Sporadic groups

In this section we deal with the case where the simple group $L = \mathrm{soc}(G/Z)$ is a sporadic group. We prove

Theorem 10.1. *Let $G \leq \Gamma L(V) = \Gamma L_d(q)$ $(q = p^a)$ be such that G/Z is almost simple with socle L a sporadic group, and suppose the perfect preimage $\hat{L}$ of L in G is absolutely irreducible on V and realisable over no proper subfield of $\mathbb{F}_q$. Assume G is p-exceptional on V. Then one of the following holds:*
(i) $M_{11} \trianglelefteq G < GL_5(3)$, *orbit sizes* $1, 22, 220$;
(ii) $M_{23} = G < GL_{11}(2)$, *orbit sizes* $1, 23, 253, 1771$.

We shall need the following result, analogous to Proposition 9.2.

Proposition 10.2. *Let T be a sporadic simple group and p a prime dividing T. Then there exists Sylow p-subgroups S_1, S_2 such that $T = \langle S_1, S_2 \rangle$.*

Proof. For all sporadic simple groups except for Th, J_4, Ly, B and M, permutation representations exist in Magma [4] (for the larger ones using the generators given in the online Atlas [46]). This allows a Sylow p-subgroup to be constructed and a conjugate that generates T can then be found.

For the remaining five sporadics we use three strategies. Let H be a maximal subgroup of T such that p divides $|T : H|$ and suppose that H has Sylow p-subgroups S_1 and S_2 such that $\langle S_1, S_2 \rangle = H$ or a normal subgroup of index coprime to p. Then for each $i \in \{1, 2\}$ we can find a Sylow p-subgroup S_i' of T properly containing S_i and since H is maximal in T it follows that $\langle S_1', S_2' \rangle = T$. Here we list the sporadic groups, primes p and maximal subgroups H for which we used this method:

T	p	H
Th	$3, 5, 7$	$2^5.L_5(2)$
	2	S_5
J_4	$2, 3, 11$	$L_2(32) : 5$
Ly	$2, 3$	$5^3.L_3(5)$
	5	$2\mathsf{A}_{11}$
B	$2, 3, 5$	Th
	7	Fi_{23}
M	$2, 3, 5, 7, 11, 13$	$2B$

Next, if the only maximal subgroup of T containing a Sylow p-subgroup S normalises S, then T will be generated by any pair of Sylow p-subgroups. This is true for $(T, p) = (J_4, 43)$, $(Ly, 67)$, and $(B, 47)$. Finally, if the order of a Sylow p-subgroup of T is p and there are two elements of order p whose product has order a prime r such that there are no maximal subgroups of T with order divisible by pr, then T is generated by these two elements of order p. The existence of such primes can be checked by either doing random searches using the matrix representations

available in the online Atlas [46] or by doing character table calculations in GAP [15]. This method was used for the groups and primes listed below.

T	p	r
Th	$13, 19$	31
	31	19
J_4	$5, 23, 29, 31, 37$	43
	7	37
Ly	$7, 31, 37$	67
	11	37
B	$11, 13, 17, 19, 31$	47
	23	31
M	$17, 19, 23, 29, 31, 41, 47, 59$	71
	71	59

□

Now we embark upon the proof of Theorem 10.1. Let $G \leq \Gamma L(V) = \Gamma L_d(q)$ ($q = p^a$) be as in the hypothesis, with $L = \mathrm{soc}(G/Z)$ a sporadic group. Note that $|\mathrm{Out}(L)| \leq 2$, so as in the previous section we may assume that $G \leq GL(V)$. By Lemma 2.4, together with Proposition 10.2, we have

$$d = \dim V \leq 2 \log_q |G : N_G(P)|, \tag{7}$$

where $P \in \mathrm{Syl}_p(G)$.

Lemma 10.3. *L is not M, BM, Fi_{24}, Fi_{23}, Th, Ly, HN or $O'N$.*

Proof. Suppose L is one of these groups other than M. Then (7) implies that $d < 250$, whence L and V are in the list in [24]. We check that for all possibilities, (7) fails. And if $L = M$, then (7) gives $d < 360$, whereas any nontrivial representation of M has dimension greater than this (see [27, 5.3.8]). □

Lemma 10.4. *L is not J_4, Fi_{22}, Co_1, Co_2, Co_3, Suz, Ru or He.*

Proof. Suppose L is one of these groups. Then $d < 250$ by (7), so [24] together with (7) imply that L, d, q are as in the following table:

L	d	q
J_4	112	2
Fi_{22}	27, 78	$4, 2$
Co_1	24	p (any)
Co_2	22	2
	23	$p\,(p > 2)$
Co_3	22	$2, 3$
	23	$p\,(p > 3)$
Suz	12	p or p^2
Ru	28	2
He	51	2

We deal with each of these in turn.

Let $L = J_4$. From [8, p. 190] we see that there is a maximal subgroup $H = 2^{10}.SL_5(2)$ fixing a nonzero vector $v \in V$. Hence v is stabilised by the subgroup generated by H and a Sylow 2-subgroup of L, which is L, a contradiction.

Now let $L = Fi_{22}$. Here $(d, q) = (27, 4)$ or $(78, 2)$. Since $q^d - 1$ is not divisible by 13, there is an element t of order 13 fixing a vector, and L is generated by t together with any Sylow 2-subgroup.

Next consider $L = Co_1$. A subgroup $H = Co_2$ of L fixes a nonzero vector v, and together with any Sylow p-subgroup generates L, provided $p \neq 11$ or 23. For $p = 11, 23$ consider $V \downarrow M_{24}$. Using [26] we see that this is $V_1 \oplus V_{23}$, where V_{23} is the deleted permutation module for M_{24}. Hence for an element $t \in L$ of order p we have $\dim C_V(t) = 4$ or 2 according as $p = 11$ or 23. Now Lemma 2.4(ii) gives a contradiction.

Now let $L = Co_2$. If $p = 2$ we see from [8, p. 154] that there is a vector stabilised by $U_6(2).2$, and this generates L with any Sylow 2-subgroup. If $p = 3$ or 5, then there is a vector stabilised by an element of order 23, and this generates L with any Sylow 2-subgroup. Finally, for $p \geq 7$, consideration of $V \downarrow M_{23}$ shows that $\dim C_V(t) \leq 5$ for an element t of order p, and now Lemma 2.4 gives a contradiction.

Now consider $L = Co_3$. For $p = 2$, there is a cyclic subgroup $H = C_{11}$ fixing a vector; and for $p = 3, 5$ subgroups $H = McL, M_{23}$, respectively, fix a vector. Now observe that in each case H generates L with any Sylow p-subgroup. And for $p \geq 7$ we argue just as for Co_2.

Next let $L = Suz$. For $p \neq 3, 11$, there is an element of order 11 fixing a vector, and this generates L with any Sylow p-subgroup. For $p = 3$ there is a subgroup $U_5(2)$ fixing a vector and generating L with any Sylow 3-subgroup. Finally, if $p = 11$, then $q = 121$ and (7) fails.

Now let $L = Ru$. Here a 13-element fixes a vector and generates L with any Sylow 2-subgroup.

Finally, for $L = He$, a 17-element fixes a vector and generates L with any Sylow 2-subgroup. □

Lemma 10.5. *L is not McL, HS, J_1, J_2 or J_3.*

Proof. Suppose L is one of these groups. Then [24] together with (7) imply that L, d, q are as in the following table:

L	d	q
McL	21	$3, 5$
	22	$2, 7$
HS	20	2
	21	5
	22	3
J_3	9	4
	18	4
J_2	6	$4, 9, 5, 49$
	14	$4, 3, 5, 7, 49$
	36	2
J_1	7	11
	20	2

Let $L = McL$. For each q we produce a subgroup H of L stabilising a nonzero vector: for $q = 2, 3$, take $H = 5^{1+2}$ (as 5 is coprime to $|V^\sharp|$); for $q = 5$ take $H = C_{11}$; and for $q = 7$ take $H = M_{11}$, noting that from [26] the Brauer character of L on V restricts to M_{11} as $1 + \chi_{10} + \chi_{11}$. From [8, p. 100], we check that each

of these subgroups H generates L together with any Sylow p-subgroup, which is a contradiction.

Now consider $L = HS$. For $q = 2$, we see from [26] that the value of the Brauer character of L on elements in classes $5B, 5C$ is 0, and hence such elements fix a nonzero vector in V. By [8, p. 80], L has only two classes of maximal subgroups of odd index, and between them they meet only one class of elements of order 5. This is a contradiction. For $q = 3$ observe that the restriction of V to a subgroup M_{22} is $V_1 + V_{21}$ where V_{21} is the deleted permutation module. Now M_{22} contains a Sylow 3-subgroup of L, and if $t \in M_{22}$ is an element of order 3 acting on 22 points with cycle-type $(3^6, 1^4)$, then $\dim C_V(t) = 10$. This leads to a contradiction via Lemma 2.4(ii). Finally for $q = 5$, V restricts to M_{22} as the deleted permutation module, and hence $M_{21} = L_3(4)$ fixes a vector; but $L_3(4)$ generates L with any Sylow 5-subgroup.

Next let $L = J_3$. Here an element of order 17 fixes a vector, and generates L with any Sylow 2-subgroup.

Now let $L = J_2$. For $d = 14$ or 36 we produce a subgroup H of L fixing a vector and generating L with any Sylow p-subgroup, as follows: if $p = 2$ or 5, $H = C_7$ (if $p = 2$ use the Brauer character value); if $p = 3$, $H = 5^2$; and if $p = 7$, $H = C_5$ generated by a $5A$-element (which fixes a vector by consideration of its Brauer character value). For $d = 6$ we have $L < PSp_6(q)$ for $q = 4, 5, 9$ or 49. In the first three cases the orbit sizes of $\hat{L}$ on vectors can be found in [30] ($q = 4, 5$) and [3, Table 5] ($q = 9$), and in each case there is a size divisible by p, a contradiction. And for $q = 49$ a $5A$-element fixes a vector and generates L with any Sylow 7-subgroup.

Finally let $L = J_1$. Here an element of order 19 fixes a vector in both modules, and generates L with any Sylow p-subgroup. □

Lemma 10.6. *If L is a Mathieu group, then one of the following holds:*
(i) $M_{11} \trianglelefteq G < GL_5(3)$, *orbit sizes* $1, 22, 220$;
(ii) $M_{23} = G < GL_{11}(2)$, *orbit sizes* $1, 23, 253, 1771$.

Proof. Here (7) and [24] imply that the possibilities for L, d, q are as follows:

L	d	q
M_{24}	$11, 44$	2
	22	3
	23	5
M_{23}	$11, 44$	2
	22	3
M_{22}	10	$2, 9, 25, 7, 11$
	$6, 15, 34$	$4, 4, 2$
	21	3
M_{12}	10	2
	$6, 10, 15$	3
	11	5
M_{11}	10	2
	$5, 10$	3

Let $L = M_{24}$. For $q = 2$, an element of order 11 fixes a vector and generates L with any Sylow 2-subgroup. And for $q = 3$ or 5, V is the deleted permutation module and the orbits of the vectors $(1, 1, -1, -1, 0^{20})$ ($q = 3$) and $(1, 1, 2, 2, -1, 0^{19})$ ($q = 5$) have size divisible by p.

Now consider $L = M_{23}$. For $q = 2$ and $d = 11$ there are two possible modules. In both cases the orbit sizes are given in [3, p. 170], and one of them gives the 2-exceptional example in conclusion (ii) of the lemma. For $d = 44$ we argue as follows. The group L has one class of involutions, and can be generated by three of them; hence $\dim C_V(t) \leq \frac{2}{3} \dim V < 30$. Now Lemma 2.4(ii) gives a contradiction. Finally, for $q = 3$, V is the deleted permutation module and the orbit of the vector $(1, 1, -1, -1, 0^{19})$ has size divisible by 3.

Next let $L = M_{22}$. Consider $p = 2$. For $d = 10, 15$ or 34, an element of order 7 fixes a vector and generates L with any Sylow 2-subgroup. And for $d = 6$ we have $\hat{L} = 3.M_{22} < SU_6(2)$ and from [8, p. 39] we see that $\hat{L}$ is transitive on the 672 nonisotropic vectors in V, contrary to 2-exceptionality. For $p = 3$ and $d = 21$, V is the deleted permutation module, dealt with in the usual way. It remains to consider $d = 10$ in characteristics $3, 5, 7$ and 11. For $p = 11$, an element of order 7 fixes a vector and generates L with any Sylow p-subgroup. And for $p = 3, 5, 7$ we have $\hat{L} = 2.M_{22} < SL_{10}(q)$ with $q = 9, 25, 7$ respectively, and a MAGMA computation reveals the existence of orbits of size divisible by p.

Now let $L = M_{12}$. For $q = 2$ or 5, V is the deleted permutation module, dealt with in the usual way. Next let $q = 3$. For $d = 10$ or 15, an element of order 5 fixes a vector and generates L with any Sylow 3-subgroup. And for $d = 6$ we have $\hat{L} = 2.M_{12} < SL_6(3)$ and [3, p. 170] shows that there is an orbit size divisible by 3.

Finally, let $L = M_{11}$. For $q = 2$, V is the deleted permutation module. For $q = 3$ and $d = 5$ the orbit sizes are given by [30], giving the example in part (i) of the lemma. And for $q = 3$, $d = 10$ there are three possible modules V, and a MAGMA computation shows that for each of these there is an orbit size divisible by 3. □

This completes the proof of Theorem 10.1.

11. $\mathcal{C}_9$ CASE IV: Lie(p')

In this section we deal with the case where the simple group $L = \mathrm{soc}(G/Z)$ is a simple group of Lie type in p'-characteristic. We prove

Theorem 11.1. *Let $G \leq \Gamma L(V) = \Gamma L_d(q)$ $(q = p^a)$ be such that G/Z is almost simple with socle L a simple group of Lie type in p'-characteristic, and L is not isomorphic to an alternating group. Suppose the perfect preimage $\hat{L}$ of L in G is absolutely irreducible on V and realisable over no proper subfield of $\mathbb{F}_q$. Assume G is p-exceptional on V and is not transitive on $V^\sharp$. Then $L_2(11) \trianglelefteq G < GL_5(3)$, with orbit sizes $1, 22, 110, 110$.*

As in previous sections, for the proof it is useful to know that the group L is generated by any two of its Sylow p-subgroups. For $p = 2$ this is true by [21]. It is presumably true for all primes, but we do not prove this here, and just cover the following groups for which we need the result.

Lemma 11.2. *Let L be one of the following simple groups:*

$L_n(r)$: $n = 2, r \leq 37$; *or* $n = 3, r \leq 5$; *or* $n \leq 5, r = 3$; *or* $n \leq 8, r = 2$,
$PSp_{2n}(r)$: $n = 2, r \leq 11$; *or* $n = 3, r \leq 7$; *or* $n = 4, r = 5$; *or* $n \leq 6, r \leq 3$,
$U_n(r)$: $n = 3, r \leq 5$; *or* $n = 4, r \leq 4$; *or* $n = 5, r \leq 3$; *or* $n \leq 11, r = 2$,
and $\Omega_7(3), \Omega_8^{\pm}(2), \Omega_{10}^{\pm}(2), F_4(2), {}^2F_4(2)', {}^3D_4(2), G_2(3), G_2(4), {}^2B_2(8)$.

Let X be a group such that $L \leq X \leq \mathrm{Aut}(L)$. If p is a prime dividing $|X|$, then $O^{p'}(X)$ is generated by two of its Sylow p-subgroups.

Proof. This was verified computationally using MAGMA. □

Now we embark upon the proof of Theorem 9.1. Let $G \leq \Gamma L(V) = \Gamma L_d(q)$ ($q = p^a$) be as in the hypothesis, with $L = \mathrm{soc}(G/Z)$ a simple group of Lie type in p'-characteristic not isomorphic to an alternating group. We assume now that $G \leq GL(V)$; we shall handle the case where $G \leq \Gamma L(V)$ at the end of the proof.

Lemma 11.3. *The simple group L is one of the following:*

$L_2(r)$ $(r \leq 37)$, $L_3(4)$, $L_3(5)$, $L_4(3)$,
$PSp_4(r)$ $(r \leq 9, r \neq 8)$, $PSp_6(r)$ $(r = 2, 3, 5)$, $PSp_8(3)$, $PSp_{10}(3)$,
$U_3(r)$ $(r = 3, 4, 5)$, $U_4(2)$, $U_5(2)$, $U_6(2)$,
$\Omega_7(3)$, $\Omega_8^{\pm}(2)$,
$F_4(2)$, ${}^2F_4(2)'$, ${}^3D_4(2)$, $G_2(3)$, $G_2(4)$, ${}^2B_2(8)$.

Proof. By Lemma 2.4(i) we have $d = \dim V \leq r_p \log_q |G : N_G(P)|$, where r_p is the minimal number of Sylow p-subgroups required to generate $O^{p'}(G)$. We have $r_2 = 2$ by [21], while upper bounds for r_p with p odd are provided by [22, Sections 3,4,5]. On the other hand, lower bounds for d are given in [29, 40]. These, together with the above bound, imply that L is one of the groups listed in Lemma 11.2. By that lemma, we have $r_p = 2$ for these groups, and so we now have the bound

$$d = \dim V \leq 2 \log_q |G : N_G(P)|. \tag{8}$$

Applying this with the above-mentioned lower bounds for d eliminates many of the groups in the list, and leaves just the groups in the conclusion. □

Lemma 11.4. *If $L = L_2(r)$, then $r = 11$ and $L_2(11) \trianglelefteq G < GL_5(3)$, with orbit sizes* $1, 22, 110, 110$.

Proof. Assume $L = L_2(r)$. By Lemma 11.3, $r \leq 37$, so certainly $\dim V < 250$ and we can use [24] together with (8) to identify the possibilities for L, d, q:

L	d	q
$L_2(31)$	15	2
$L_2(25)$	12	2
$L_2(23)$	11	$2, 3$
$L_2(17)$	8	2
$L_2(13)$	6	$4, 3$
	7	3
$L_2(11)$	10	2
	5	$4, 3, 5$
	6	3
$L_2(7)$	3	9

The case where $r = 11$ and $(d, q) = (5, 3)$ gives the example in the conclusion; the orbit sizes are given in [7]. (Note that there are two 5-dimensional representations of L over $\mathbb{F}_3$, but they are quasiequivalent, hence have the same orbit sizes.) Also, when $r = 13$ and $(d, q) = (6, 3)$, $\hat{L} = SL_2(13)$ is transitive on nonzero vectors (see [30, Appendix 1]), contrary to our assumption in Theorem 11.1.

When $r = 11, 13, 17, 23$ or 25 and $(d, q) = (5, 5)$, $(6, 4)$, $(8, 2)$, $(11, 2)$ or $(12, 2)$, the orbit sizes are given by [7, Section 5] and [30, Appendix 2]; there is an orbit size divisible by p in all cases.

Now consider the cases where $r = 11, 13, 23$ or 31 and $(d, q) = (6, 3)$, $(7, 3)$, $(11, 3)$ or $(15, 2)$. For these, we observe that there is a nonzero vector fixed by a subgroup H of order 11,7,11 or 5 respectively, and H generates L together with any Sylow p-subgroup, a contradiction.

This leaves the cases where $r = 7$ or 11 and $(d, q) = (3, 9)$, $(5, 4)$ or $(10, 2)$. For these cases a MAGMA computation shows that there is an orbit of size divisible by p. □

Lemma 11.5. *L is not $L_3(4)$, $L_3(5)$ or $L_4(3)$.*

Proof. Suppose L is one of these groups. By [24] and (8), the possibilities for L, d, q are:

L	d	q
$L_3(4)$	4	9
	6	$3, 7$
	8	5
$L_4(3)$	26	2

Consider the case where $L = L_3(4)$ and $(d, q) = (6, 3)$. Here $L < P\Omega_6^-(3)$ and we see from [8, p. 52] that L is transitive on the 126 nonisotropic points in V, contrary to p-exceptionality.

For the remaining cases $(d, q) = (4, 9)$, $(8, 5)$, $(6, 7)$, $(26, 2)$, there is a nonzero vector fixed by a subgroup H of order 7, 7, 5 or 13 respectively, and H generates L together with any Sylow p-subgroup, a contradiction. □

Lemma 11.6. *L is not $PSp_{2m}(r)$ for $m \geq 2$.*

Proof. Suppose $L = PSp_{2m}(r)$, so that L is as in Lemma 11.3. Using [24] and (8), we see that the possibilities for L, d, q are:

L	d	q
$PSp_4(3)$	4	25
	6	5
$PSp_4(5)$	12	4
$PSp_4(7)$	24	2
$PSp_4(9)$	40	2
$PSp_6(3)$	13	$4, 7$
	14	7
$PSp_8(3)$	40	4
$Sp_4(4)$	18	3
$Sp_6(2)$	7	$3, 5, 7$
	8	$3, 5, 7$
	14	3

The three cases $PSp_4(3) < L_6(5)$, $Sp_6(2) < L_7(5)$ and $Sp_6(2) < L_8(7)$ were handled using a MAGMA computation, and in each case there is an orbit size divisible by p.

Now consider all the other cases in the table with r odd. In each case r^2 does not divide $q^d - 1$, so there is a subgroup H fixing a nonzero vector, where H is of index r in a Sylow r-subgroup of L (replacing r by 3 when $L = PSp_4(9)$). However, one checks that H generates L with any Sylow p-subgroup, contradicting p-exceptionality (use [8], or [32] for $p = 2$).

If $L = Sp_4(4)$, an element of order 17 fixes a vector and generates L with any Sylow 3-subgroup.

Finally, let $L = Sp_6(2)$. For the $p = 3$ cases, an element of order 7 in L fixes a vector and generates L with any Sylow 3-subgroup. And when $(d, q) = (7, 7)$ or $(8, 5)$ there are subgroups $\Omega_6^-(2)$ or $U_3(3)$, respectively, fixing a vector, and these generate L with any Sylow p-subgroup. □

Lemma 11.7. *L is not $U_n(r)$ for $n \geq 3$.*

Proof. Suppose $L = U_n(r)$ is as in Lemma 11.3. By [24] and (8), the possibilities are:

L	d	q
$U_3(3)$	6	2
	14	2
$U_3(4)$	12	3
$U_3(5)$	20	2
$U_5(2)$	10	$3, 5, 11$
$U_6(2)$	21	3
	22	5

The group $U_3(3) < L_6(2)$ is transitive on nonzero vectors (see [30, Appendix 1]), contrary to assumption. For the cases $(d, q) = (14, 2)$, $(20, 2)$, $(10, 3)$, $(10, 11)$ and $(21, 3)$, there is a subgroup H or order $7, 7, 2^7, 2^7, 11$, respectively, fixing a vector and generating L with any Sylow p-subgroup. In the remaining three cases with $(d, q) = (12, 3)$, $(10, 5)$ and $(22, 5)$, a MAGMA computation shows that there is an orbit of size divisible by p. □

Lemma 11.8. *L is not an orthogonal group in Lemma* 11.3.

Proof. Suppose $L = \Omega_7(3)$ or $\Omega_8^{\pm}(2)$. Using [24] and (8), we see that $L = \Omega_8^+(2)$, $d = 8$ and $q = 3, 5$ or 7. Here $\hat{L} = 2.L$ and these 8-dimensional representations arise from the action of the Weyl group of E_8 on the root lattice. This is transitive on the 240 root vectors, and hence there is an orbit of size 120 on 1-spaces. Hence $\hat{L}$ is not p-exceptional for $p = 3, 5$. For $p = 7$, the total number of 1-spaces in V is not divisible by 3, so there is a 1-space fixed by a Sylow 3-subgroup of L, which generates L with any Sylow 7-subgroup of L. □

Lemma 11.9. *L is not an exceptional group in Lemma* 11.3.

Proof. Suppose L is an exceptional group. By [24] and (8), the possibilities are:

L	d	q
$G_2(3)$	14	2
$G_2(4)$	12	$3, 5, 7$
${}^2B_2(8)$	8	5

For $(d, q) = (14, 2), (12, 3), (12, 5), (12, 7)$, there is a subgroup H of L of order $3^5, 2^9, 2^{10}, 2^8$, respectively, fixing a 1-space, and H generates L with any Sylow p-subgroup, a contradiction. For the remaining case $(d, q) = (8, 5)$, a MAGMA computation gives an orbit of size divisible by 5. □

We now complete the proof of Theorem 11.1 by handling the case where the p-exceptional group G in the hypothesis lies in $\Gamma L(V)$ and not in $GL(V)$, where $V = V_d(q)$. Let $G_0 = G \cap GL(V)$. If p divides $|G_0|$, then G_0 is p-exceptional, hence is given by the linear case of the theorem which we have already proved; but this implies that $GL(V) = GL_5(3)$, in which case $GL(V) = \Gamma L(V)$.

Hence p does not divide $|G_0|$, and by Lemma 2.3, G has a p-exceptional normal subgroup $G_0\langle\sigma\rangle$ (hence also $\hat{L}\langle\sigma\rangle$), where $\sigma \in \Gamma L(V)\backslash GL(V)$ is a field automorphism of order p. Hence $q = p^{kp}$ for some integer k. Since G_0 is a p'-group, we have $p > 2$. Moreover, σ induces an automorphism of order p of the simple p'-group L, which must be a field automorphism, so $L = L(r^p)$ is a group of Lie type over $\mathbb{F}_{r^p}$ for some r.

Let ℓ be the untwisted Lie rank of L. Then $d \geq \frac{1}{2}(r^{p\ell} - 1)$ by [29], and also $|\hat{L}| < r^{4p\ell^2}$ and $|C_V(\sigma)| = q^{d(1-\frac{1}{p})}$. By Lemma 2.4(ii), $|V : C_V(\sigma)| < |\sigma^L|$, which implies

$$q^{\frac{1}{2}(r^{p\ell}-1)\cdot(1-\frac{1}{p})} < r^{4p\ell^2}.$$

This cannot hold.

Hence the case where G lies in $\Gamma L(V)$ and not in $GL(V)$ does not occur. This completes the proof of Theorem 11.1.

12. Proof of Theorem 1

Let G be an irreducible subgroup of $GL_n(p)$ with p prime, and suppose that G acts primitively on $V_n(p)$. Suppose also that G is p-exceptional, so that p divides $|G|$ and every orbit of G on V has size coprime to p.

Choose $q = p^k$ maximal such that $G \leq \Gamma L_d(q) \leq GL_n(p)$, where $n = dk$. Write $V = V_d(q)$, $G_0 = G \cap GL_d(q)$ and $Z = G_0 \cap \mathbb{F}_q^*$, the group of scalar multiples of the identity in G_0. Note that $G_0 \trianglelefteq G$ and G/G_0 is cyclic. Write $K = \mathbb{F}_q$.

If $d = 1$, then $G \leq \Gamma L_1(q)$, as in Theorem 1(ii). So assume that $d \geq 2$.

Lemma 12.1. *G_0 is absolutely irreducible on $V = V_d(q)$.*

Proof. We have $G \leq N_{GL_n(p)}(K) = \Gamma L_d(q)$. So $G_0 = C_H(K) \leq GL_d(q)$.

Let $E = \mathrm{End}_G(V) = \mathbb{F}_r \subseteq K$, and write $q = p^k = r^b$. Viewing V as $V_{bd}(r)$, it is an absolutely irreducible $\mathbb{F}_rG$-module. Now view V as an $\mathbb{F}_qG_0$-module. Then $U := V \otimes_{\mathbb{F}_r} \mathbb{F}_q$, as an $\mathbb{F}_qG_0$-module, is the sum of b Frobenius twists of V. However, G/G_0 is cyclic of order at most b, so if $V \downarrow G_0$ were reducible, then $U \downarrow G$ would be reducible. But G is absolutely irreducible, so this is a contradiction.

Hence $V \downarrow G_0$ is irreducible. As $C_{\mathrm{End}(V)}(G_0)$ is a field extension of K, the choice of K implies that $C_{\mathrm{End}(V)}(G_0) = K$, and so V is an absolutely irreducible KG_0-module. □

If G preserves a tensor product decomposition $V = U \otimes W$ over $\mathbb{F}_q$, where $\dim U \geq \dim W \geq 2$ (i.e. $G \leq \Gamma L(V)_{U\otimes W}$), then Theorems 4.1 and 4.2 (together with Lemma 2.3) give a contradiction. So assume that G does not preserve a nontrivial tensor decomposition of V.

Lemma 12.2. *Let N be a normal subgroup of G such that $N \leq G_0$ and $N \not\leq Z$. Then $V \downarrow N$ is absolutely irreducible.*

Proof. By Clifford's Theorem $V \downarrow N$ is a direct sum of homogeneous components; these are permuted by G, and hence by the primitivity of G, $V \downarrow N$ is homogeneous. Say $V \downarrow N \cong W \oplus \cdots \oplus W$, a direct sum of k copies of an irreducible KN-module W. Let $K_0 = C_{\mathrm{End}(W)}(N)$, a field extension of K. By the first few lines of the proof of [1, 5.7], K_0^* can be identified with $Z(C_{GL(V)}(N))$ and $G \leq N_{\Gamma L(V)}(K_0)$,

so $K_0 = K$ by choice of K. Hence W is an absolutely irreducible KN-module. At this point [1, 3.13] shows that there is a K-space A such that V can be identified with $W \otimes A$ in such a way that $N \leq GL(W) \otimes 1_A$, $G_0 \leq GL(W) \otimes GL(A)$ and $G \leq N_{\Gamma L(V)}(GL(W) \otimes GL(A))$. By our assumption that G preserves no nontrivial tensor decomposition, this implies that $W = V$, completing the proof. □

Now let S be the socle of G/Z, and write $S = M_1 \times \cdots \times M_k$ where each M_i is a minimal normal subgroup of G/Z. Let R be the full preimage of S in G, and P_i the preimage of M_i, so that $R = P_1 \dots P_k$, a commuting product. If some P_i, say P_k, is not contained in G_0, then M_k is generated by a field automorphism ϕ of prime order, and so $G/Z \leq C_{P\Gamma L(V)}(\phi) = PGL_d(q_0)\langle\psi\rangle$, where ψ generates the Galois group of $\mathbb{F}_q/\mathbb{F}_p$ and $\mathbb{F}_{q_0}$ is a proper subfield of $\mathbb{F}_q$. This contradicts Theorem 6.1.

Hence $R = P_1 \dots P_k \leq G_0$. As $P_1 \trianglelefteq G$, Lemma 12.2 implies that $V \downarrow P_1$ is absolutely irreducible, hence $C_{G_0}(P_1) = Z$. It follows that $k = 1$ and $R = P_1$. Also, G is not realised (modulo scalars) over a proper subfield of $\mathbb{F}_q$, by Theorem 6.1.

Suppose first that $M_1 = R/Z$ is an elementary abelian r-group for some prime r, and replace R by a minimal preimage of M_1 in G. By Lemma 12.2 and since $d \geq 2$, any maximal abelian characteristic subgroup of R must be contained in Z. Hence R is of symplectic type, and now we argue as in [1, 11.8] that R can be taken to be as in Section 7. Since G is primitive, Theorem 7.1 now gives a contradiction.

Now suppose $M_1 = R/Z$ is nonabelian, so $M_1 \cong T^l$ for some nonabelian simple group T, and $R = R_1 \dots R_l$ where $R_i/Z \cong T$ and the factors are permuted transitively by G. If $l > 1$, then [1, 3.16, 3.17] implies that R preserves a tensor decomposition $V = V_1 \otimes \cdots \otimes V_l$ with $\dim V_i$ independent of i, and $G \leq N_{\Gamma L(V)}(\bigotimes GL(V_i))$; then Theorems 5.1 and 5.2 give a contradiction.

It remains to consider the case where $l = 1$, so that $M_1 = \mathrm{Soc}(G/Z)$ is simple. Then G/Z is almost simple, and has socle absolutely irreducible on V and not realisable over a proper subfield of $\mathbb{F}_q$. In other words G is in the class $\mathcal{C}_9$, and so G is given by the results in Sections 8-11.

This completes the proof of Theorem 1.

13. Deduction of Theorems 4, 5 and 6

Proof of Theorem 4. Let $G \leq GL_d(p) = GL(V)$ be irreducible and p-exceptional, and let $G_0 = O^{p'}(G)$. Then G_0 is also p-exceptional by Lemma 2.1. By Clifford's Theorem, $V \downarrow G_0 = V_1 \oplus \cdots \oplus V_t$, where the V_i are irreducible G_0-modules, conjugate under G. Note that $O^{p'}(G_0) = G_0$ and p divides $|G_0^{V_1}|$, so $G_0^{V_1}$ is p-exceptional. If $G_0^{V_1}$ is primitive, it is given by Theorem 1, and if it is imprimitive it is given by Theorem 3.

We now claim that the V_i are pairwise nonisomorphic G_0-modules. Suppose false, and let $W = V_1 \oplus \cdots \oplus V_k$ be a homogeneous component for G_0 with $k > 1$. Then $G_0^W \leq GL(V_1) \otimes 1 \leq GL(V_1) \otimes GL_k(p)$. Since G_0 is p-exceptional on V, it is also p-exceptional on W. We now apply Theorem 4.1, which classifies p-exceptional groups which preserve tensor product decompositions. From this theorem, it is clear that such a group cannot act just as scalars on one of the tensor factors, which is what G_0^W does. This is a contradiction, proving the claim.

By the claim, G permutes the summands V_i. Finally, the kernel K of the action of G on the set of summands $\{V_1, \dots, V_t\}$ contains G_0, so G/K is a p'-group.

Proof of Theorem 5. Let G be a finite group and let $p > 2$ be a prime. Assume that $G = O^{p'}(G) = O^p(G)$, that G has abelian Sylow p-subgroups, and that V is a faithful irreducible $\mathbb{F}_pG$-module such that every orbit of G on V has length coprime to p. These assumptions imply that $G = G'$ and also that p divides $|G|$.

Suppose first that G acts primitively on V. Then G is given by Theorem 1. As G is insoluble, it is either transitive on $V^\sharp$ or one of the examples in (iii) of the theorem. In the first case we refer to the list of transitive linear groups in [30, Appendix]: the only examples where $G = G'$ and G has abelian Sylow p-subgroups occur in conclusions (i) and (iii) of Theorem 5; and the examples in Theorem 1(iii) with $p > 2$ are also in conclusion (iii).

Now suppose G is imprimitive on V. As $G = O^{p'}(G)$, G is given by Theorem 3. Theorem 2 and the assumptions that $p > 2$ and G has abelian Sylow p-subgroups, now force G to be as in conclusion (ii) of Theorem 5. This completes the proof of the corollary.

Proof of Theorem 6. Let $G \leq GL_d(p)$ be a $\frac{1}{2}$-transitive linear group of order divisible by p, and write $V = V_d(p)$. Let $H = VG \leq AGL_d(p)$, the corresponding affine permutation group acting on V. Since G has order divisible by p, it does not act semiregularly on $V^\sharp$, and so H is $\frac{3}{2}$-transitive on V, and hence is a primitive permutation group on V by [45, 10.4]. This implies that G acts irreducibly on V.

Since G is $\frac{1}{2}$-transitive and has order divisible by p, it is p-exceptional. So if G acts primitively as a linear group on V, then it is given by Theorem 1. For $G = A_c$ or S_c as in (iii)(a) of the theorem, the orbit sizes are given in the proof of Lemma 9.4, and we see that the only $\frac{1}{2}$-transitive example is for $c = 6$ with $(d, p) = (4, 2)$, in which case G is transitive on $V^\sharp$. Hence G is as in the conclusion of Theorem 6. Finally, if G acts imprimitively on V, then it is given by [39, Theorem 1.1] (which determines all imprimitive $\frac{1}{2}$-transitive linear groups). The only example of order divisible by p is $D_{18} < \Gamma L_1(2^6) < GL_6(2)$, as in (ii) of Theorem 6.

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Centre for the Mathematics of Symmetry and Computation, School of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009

E-mail address: michael.giudici@uwa.edu.au

Department of Mathematics, Imperial College, London SW7 2BZ, England

E-mail address: m.liebeck@imperial.ac.uk

Centre for the Mathematics of Symmetry and Computation, School of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009

E-mail address: Cheryl.Praeger@uwa.edu.au

Department of Pure Mathematics and Mathematical Statistics, CMS, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, England

E-mail address: saxl@dpmms.cam.ac.uk

Department of Mathematics, University of Arizona, 617 N. Santa Rita Avenue, Tucson, Arizona 85721-0089

E-mail address: tiep@math.arizona.edu