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BELLMAN FUNCTION FOR EXTREMAL PROBLEMS IN BMO

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Abstract. We develop a general method for obtaining sharp integral estimates on BMO. Each such estimate gives rise to a Bellman function, and we show that for a large class of integral functionals, this function is a solution of a homogeneous Monge–Ampère boundary-value problem on a parabolic plane domain. Furthermore, we elaborate an essentially geometric algorithm for solving this boundary-value problem. This algorithm produces the exact Bellman function of the problem along with the optimizers in the inequalities being proved. The method presented subsumes several previous Bellman-function results for BMO, including the sharp John–Nirenberg inequality and sharp estimates of L^p-norms of BMO functions.

Contents

1. History of the problem and description of our results

1.1. **History and formulation of the problem.** The Bellman function method in analysis and probability is a technique for proving sharp inequalities. Roughly speaking, one reformulates the inequality as an extremal problem and then finds or estimates its solution, usually using a differential equation. The method was first used by Burkholder in his groundbreaking paper [1] on sharp estimates for martingale transforms. Later, Nazarov, Treil, and Volberg expanded his ideas to many settings in harmonic analysis [9, 10]. We refer the reader to the survey [11]

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for the history of the method in harmonic analysis and to the book [12] for its implementation in probabilistic settings.

Until 2003, no exact Bellman function had been computed for a harmonic analysis problem (a caveat: Burkholder's original work did not use explicit extremal formulations, but his "special functions" can now be understood as Bellman functions). Instead, in employing the method one would typically rely on a *supersolution*, meaning an appropriate majorant of the true Bellman function. In 2003, the sharp constants in the John–Nirenberg inequality were found; see [20] and [15]. Since then there have been many works applying similar procedures to various inequalities, e.g. [2, 14, 16–18, 21–25]. These applications have had many unifying features, but no matter how closely related they might seem, the solution of each specific problem required a separate solid paper. Arguably, extremal problems on BMO have yielded the largest number of exact Bellman functions. They thus provide an appropriate foundation to the present study, which is concerned with giving a theoretical basis for a method that allows one to calculate Bellman functions (and, as a consequence, prove sharp inequalities) for a general integral functional on BMO. In this case the Bellman function solves a homogeneous planar Monge–Ampère equation, and its graph is a so-called developable surface. This means that the domain of the function is foliated by straight lines along which the function is linear. The geometry of these lines plays a central role in our study; in particular, in many cases simple geometrical arguments replace unwieldy analytical calculations. To further simplify and streamline the presentation, we impose some regularity conditions on the class of functionals we estimate.

The first steps in constructing a general Bellman function on BMO were made in [18]. Here we develop that approach systematically without restricting ourselves by any specific extremal problem (the article [18] was primarily concerned with L^p-estimates for BMO functions). Our considerations, in particular, include the cases treated in [17, 18]. The results presented here were announced in [3, 4].

Lastly, we note that the tools developed in these pages have recently been applied to other questions about BMO, such as that of weakening its defining condition; see [8].

We now introduce some notation. By I and J we always denote intervals on $\mathbb{R}$. By $\langle\varphi\rangle_J$ we denote the average of a function φ over an interval J:

$$\langle\varphi\rangle_J \stackrel{\text{def}}{=} \frac{1}{|J|}\int_J \varphi,$$

where $|J|$ is the length of the interval. We consider the BMO space on I endowed with the quadratic norm:

$$\mathrm{BMO}(I) \stackrel{\text{def}}{=} \Big\{\varphi \in L^1(I) \;\Big|\; \|\varphi\|^2_{\mathrm{BMO}(I)} \stackrel{\text{def}}{=} \sup_{J\subset I}\langle|\varphi - \langle\varphi\rangle_J|^2\rangle_J < \infty\Big\}.$$

We call the expression $\|\varphi\|_{\mathrm{BMO}(I)}$ a norm, although we must factorize by the constant functions in order to obtain a normed space. Details on BMO can be found in [6] or [19]. By $\mathrm{BMO}_\varepsilon(I)$ we denote the ball of radius ε in this space.

We consider several well-known inequalities for functions in $\mathrm{BMO}(I)$. First, there is a double estimate claiming the equivalence of any p-norm $(0<p<\infty)$ and the initial quadratic norm:

$$c_p\|\varphi\|_{\mathrm{BMO}(I)} \le \sup_{J\subset I}\langle|\varphi - \langle\varphi\rangle_J|^p\rangle_J^{1/p} \le C_p\|\varphi\|_{\mathrm{BMO}(I)}. \tag{1.1}$$

Second, the weak-form John–Nirenberg inequality claims that the measure of the set where some function $\varphi \in \mathrm{BMO}(I)$ deviates from its average by more than a certain value $\lambda > 0$ decreases exponentially in λ:

$$\frac{1}{|I|}\big|\big\{t \in I \mid |\varphi(t) - \langle\varphi\rangle_I| \geq \lambda\big\}\big| \leq c_1 e^{-c_2\lambda/\|\varphi\|_{\mathrm{BMO}(I)}}. \tag{1.2}$$

And the third inequality can be obtained from the previous one by integration. It is called the integral John–Nirenberg inequality and may be treated as the reverse Jensen inequality for functions in $\mathrm{BMO}_\varepsilon(I)$ and the exponent. Namely, there exist a number $\varepsilon_0 > 0$ and a positive function $C(\varepsilon)$, $0 < \varepsilon < \varepsilon_0$, such that

$$\langle e^\varphi\rangle_I \leq C(\varepsilon)e^{\langle\varphi\rangle_I} \tag{1.3}$$

for all $\varphi \in \mathrm{BMO}_\varepsilon(I)$.

There exist various proofs of these inequalities. For example, in Koosis' book [6], Garnett's martingale proof is presented. In Stein's book [19], the proof, based on the duality of BMO and H^1, can be found. We are interested in sharp constants in inequalities of this kind. One of the methods that is employed to obtain sharp constants is called the Bellman function method.

We consider the following Bellman function:

$$\boldsymbol{B}_\varepsilon(x_1, x_2;\, f) \stackrel{\text{def}}{=} \sup_{\varphi\in\mathrm{BMO}_\varepsilon(I)} \big\{\langle f \circ \varphi\rangle_I \mid \langle\varphi\rangle_I = x_1, \langle\varphi^2\rangle_I = x_2\big\}, \tag{1.4}$$

where f is some function on $\mathbb{R}$ (we postpone the discussion of the class that f may belong to). The functions over which the supremum is taken are called *test functions*. We often omit f in the notation and merge two variables into one; i.e. we write $\boldsymbol{B}_\varepsilon(x_1, x_2)$ or $\boldsymbol{B}_\varepsilon(x;\, f)$ or simply $\boldsymbol{B}_\varepsilon(x)$, where $x = (x_1, x_2)$.

There are two points worth noting. First, $\boldsymbol{B}_\varepsilon$ does not depend on the interval I participating in the definition above. Second, if we replace supremum by infimum in (1.4), we will obtain the function $-\boldsymbol{B}_\varepsilon(x_1, x_2;\, -f)$. In the beginning of Section 2.1, all this will be discussed in detail.

If we set $f(u) = |u|^p$, then after obtaining analytical expressions for $\boldsymbol{B}_\varepsilon(x;\, f)$ and $-\boldsymbol{B}_\varepsilon(x;\, -f)$, we will get estimate (1.1) with the sharp constants c_p and C_p as a corollary. All this was done in [18]. Setting $f(u) = \chi_{(-\infty,-\lambda]\cup[\lambda,\infty)}(u)$, we obtain the Bellman function that gives us the sharp constants for the weak John–Nirenberg inequality (see (1.2)). This function was found in [23]. Finally, setting $f(u) = e^u$, we obtain the Bellman function for the integral John–Nirenberg inequality (see (1.3)). The analytical expression for this function was found independently in [20] and [15] (see [17]); the sharp constants $\varepsilon_0 = 1$ and $C(\varepsilon) = e^{-\varepsilon}(1-\varepsilon)^{-1}$ were obtained as a corollary.

In this paper, we construct the function $\boldsymbol{B}_\varepsilon(x_1, x_2;\, f)$ not for a function f fixed, but for some wide class of functions, which is described in the next section.

1.2. **Description of our results.** We will see later that in the formulas for $\boldsymbol{B}_\varepsilon$ the integrals of the following expressions participate:

$$f^{(r)}(t)e^{\pm t/\varepsilon}, \quad r = 0, 1, 2, 3.$$

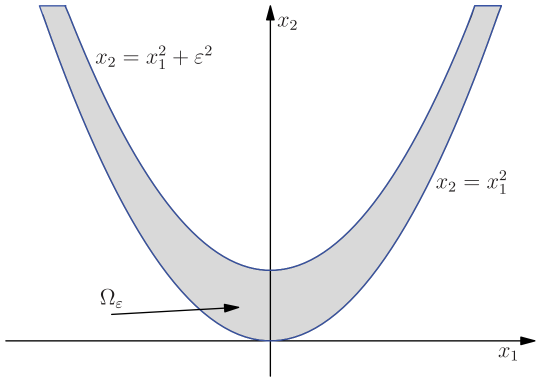

FIGURE 1. The parabolic strip Ω_ε.

Therefore, the following space is required:

$$\mathfrak{W}_{\varepsilon_0} \stackrel{\text{def}}{=} C^2(\mathbb{R}) \cap W_3^1(\mathbb{R}, w_{\varepsilon_0}),$$

where $\varepsilon_0 > 0$ and $w_{\varepsilon_0}(t) \stackrel{\text{def}}{=} e^{-|t|/\varepsilon_0}$. The space on the right of the intersection sign is a weighted Sobolev space. Functions in this space, together with their first three derivatives, are integrable with the weight w_{ε_0}.

Also, we will see that the behavior of $\boldsymbol{B}_\varepsilon$ depends strongly on the sign of f'''. We introduce a subset $\mathfrak{W}_{\varepsilon_0}^N \subset \mathfrak{W}_{\varepsilon_0}$ of functions we deal with. Any function of this class has $2N+1$ points:

$$-\infty \le c_0 < v_1 < c_1 < v_2 < \ldots < v_N < c_N \le +\infty$$

on the extended real line such that

1) $f''' > 0$ a.e. on (v_k, c_k) and on $(-\infty, c_0)$. Also, $f''' < 0$ a.e. on (c_k, v_{k+1}) and on (c_N, ∞);
2) $|c_k - v_j| \ge 2\varepsilon_0$.

We build the function $\boldsymbol{B}_\varepsilon(x; f)$ for $f \in \mathfrak{W}_{\varepsilon_0}^N$ and $\varepsilon < \varepsilon_0$.

Next, consider the parabolic strip (see Figure 1):

$$\Omega_\varepsilon \stackrel{\text{def}}{=} \left\{(x_1, x_2) \in \mathbb{R}^2 \mid \ x_1^2 \le x_2 \le x_1^2 + \varepsilon^2\right\}. \tag{1.5}$$

In the beginning of the next section we prove that Ω_ε is the domain of $\boldsymbol{B}_\varepsilon$ (in the sense that Ω_ε consists of all the points (x_1, x_2) such that the supremum in (1.4) is taken over a non-empty set for them) and $\boldsymbol{B}_\varepsilon$ satisfies the boundary condition $\boldsymbol{B}_\varepsilon(x_1, x_1^2) = f(x_1)$ on the lower parabola.

We will also see that $\boldsymbol{B}_\varepsilon$ is *locally concave*, i.e. concave on every convex subset of Ω_ε. We give the definition of the local concavity in another form that is more suitable for our purposes.

Definition 1.1. A function G, defined on some set $\Omega \subset \mathbb{R}^n$, is called locally concave in Ω if the inequality

$$G(\alpha_- x^- + \alpha_+ x^+) \ge \alpha_- G(x^-) + \alpha_+ G(x^+)$$

is fulfilled for every straight-line segment $[x^-, x^+] \subset \Omega$ and every pair of numbers $\alpha_-, \alpha_+ \ge 0$ such that $\alpha_- + \alpha_+ = 1$.

Now we are ready to describe our results. Since our goal is not to solve some specific problem but to describe the procedure how to solve a wide class of extremal problems, we have no main theorem. Thus we can say that the main result consists of two parts:

(1) Description of possible local patterns for a general Bellman function (Propositions 3.1, 4.1, 5.1, 5.3, 6.1, and 6.2) and
(2) an algorithm describing how to compose a global solution from these local patterns (the existence of such a composed solution is stated in Theorem 6.4, but what is much more important is not the statement itself but the whole machinery developed for its proof, especially the constructive algorithm described in the last subsection of the paper, Subsection 6.5).

As a byproduct we get the very important property of the Bellman function: it is the minimal locally concave function with given boundary values.

To speak more formally we need to introduce the class $\Lambda_{\varepsilon,f}$ of all continuous functions that are locally concave in Ω_ε and satisfy the boundary condition mentioned above:

$$\Lambda_{\varepsilon,f} \stackrel{\text{def}}{=} \bigl\{G \in C(\Omega_\varepsilon) \mid \ G \text{ is locally concave; } G(u,u^2) = f(u)\ \forall\, u \in \mathbb{R}\bigr\}.$$

Suppose $f \in \mathfrak{W}^N_{\varepsilon_0}$, where $\varepsilon_0 > 0$ and $N \in \mathbb{Z}_+$. Then for each ε, $0 < \varepsilon < \varepsilon_0$,

(a) *we construct an expression for $\boldsymbol{B}_\varepsilon$ in terms of f;*
(b) *the function $\boldsymbol{B}_\varepsilon(x;\, f)$ belongs to $\Lambda_{\varepsilon,f}$; moreover,*

$$\boldsymbol{B}_\varepsilon(x;\, f) = \inf_{G\in\Lambda_{\varepsilon,f}} G(x).$$

Statement (b) means that the problem of finding the Bellman function $\boldsymbol{B}_\varepsilon$ can be reformulated in geometric terms: it is equivalent to the problem of finding the minimal locally concave function in Ω_ε that satisfies a certain boundary condition.

Concerning (a), by *an expression in terms of f* we mean a rather complicated construction, which consists of various integral and differential transformations of f. Roots of some equations that cannot be solved in elementary functions also participate. However, to justify the practical value of our results, we give applications for various boundary functions f; see [4] for these examples.

2. General principles

Throughout this section, we assume that $0 < \varepsilon < \varepsilon_0$, $f \in \mathfrak{W}_{\varepsilon_0}$, and $\boldsymbol{B}_\varepsilon$ is the Bellman function defined by (1.4).

2.1. Main properties of function $\boldsymbol{B}_\varepsilon$.

Statement 2.1. The function $\boldsymbol{B}_\varepsilon(x;\, f)$ has the following properties:

(i) its domain is the parabolic strip Ω_ε defined by formula (1.5); i.e. the set over which the supremum in (1.4) is taken is non-empty for those and only those $x = (x_1, x_2)$ that lie in Ω_ε (however, the function $\boldsymbol{B}_\varepsilon$ can take the value $+\infty$ there);
(ii) the boundary condition $\boldsymbol{B}_\varepsilon(x_1, x_1^2) = f(x_1)$ is satisfied.

Proof. Consider statement (i). It is easy to see that the estimate $x_1^2 \le x_2$ is fulfilled due to the Cauchy–Schwarz inequality and the estimate $x_2 \le x_1^2 + \varepsilon^2$ follows from the requirement $\varphi \in \mathrm{BMO}_\varepsilon(I)$. Therefore, Ω_ε contains the domain of $\boldsymbol{B}_\varepsilon$. On the other hand, if the estimate $x_1^2 \le x_2 \le x_1^2 + \varepsilon^2$ is fulfilled, we can easily construct a function $\varphi \in \mathrm{BMO}_\varepsilon(I)$ whose average equals x_1 and square deviation equals $\sqrt{x_2 - x_1^2}$. For example, we may take the function

$$\varphi(t) = \begin{cases} x_1 + \sqrt{x_2 - x_1^2}, & t \in I_-; \\ x_1 - \sqrt{x_2 - x_1^2}, & t \in I_+, \end{cases}$$

where I_- and I_+ are the left and right halves of I, respectively. This means that Ω_ε is contained in the domain of $\boldsymbol{B}_\varepsilon$.

Statement (ii) is trivial. Indeed, the identity $x_2 = x_1^2$ means that all the functions φ over which the supremum is taken do not deviate from their average x_1. Therefore, the set of such functions consists of a single element $\varphi(t) \equiv x_1$. This implies the condition required. □

Also we postulate the fundamental property of $\boldsymbol{B}_\varepsilon$.

(iii) *The function $\boldsymbol{B}_\varepsilon$ is locally concave in the parabolic strip Ω_ε.*

In [4], the reader can find some heuristic reasoning leading to this principle.

2.2. **Locally concave majorants.** In this section, we prove that every function in $C(\Omega_\varepsilon)$ with properties (ii) and (iii) (we recall that the set of such functions is denoted by $\Lambda_{\varepsilon,f}$) majorizes $\boldsymbol{B}_\varepsilon$. Namely, we verify the following statement.

Statement 2.2. Suppose $0 < \varepsilon < \varepsilon_0$, $f \in \mathfrak{W}_{\varepsilon_0}$, and $G \in \Lambda_{\varepsilon,f}$. Then $\boldsymbol{B}_\varepsilon(x;\, f) \le G(x)$ for all $x \in \Omega_\varepsilon$.

In order to prove this statement, we need some preparation.

Auxiliary lemmas. First, we need the following geometric lemma, which was proved in both [20] and [17].

Lemma 2.3. *Suppose $\varepsilon_1 > \varepsilon$. Then for any interval $I \subset \mathbb{R}$ and any function $\varphi \in \mathrm{BMO}_\varepsilon(I)$ there exists a partition $I = I_- \cup I_+$ such that the line segment with the endpoints $x^\pm = \big(\langle\varphi\rangle_{I_\pm}, \langle\varphi^2\rangle_{I_\pm}\big)$ lies in Ω_{ε_1} entirely. Moreover, the parameters $\alpha_\pm = |I_\pm|/|I|$ can be chosen to be separated from 0 and 1 uniformly in I and φ.*

Now we discuss how a function $f \in \mathfrak{W}_{\varepsilon_0}$ and its first two derivatives behave at infinity.

Lemma 2.4. *If $f \in \mathfrak{W}_{\varepsilon_0}$, then the following limit relations are fulfilled*:

$$f^{(r)}(u)e^{-|u|/\varepsilon_0} \to 0 \quad \text{as} \quad u \to \pm\infty \quad \text{for} \quad r = 0, 1, 2. \tag{2.1}$$

Proof.

$$\begin{aligned} f''(u)e^{-|u|/\varepsilon_0} - f''(0) &= \int_0^u \Big(f''(t)e^{-|t|/\varepsilon_0}\Big)'\, dt \\ &= \int_0^u f'''(t)e^{-|t|/\varepsilon_0}\, dt - \varepsilon_0^{-1} \operatorname{sign} u \int_0^u f''(t)e^{-|t|/\varepsilon_0}\, dt. \end{aligned}$$

Since $f \in \mathfrak{W}_{\varepsilon_0}$, we have the existence of the limits

$$\lim_{u\to\pm\infty} f''(u)e^{-|u|/\varepsilon_0}.$$

But if such limits exist, they must be equal to zero (because $f''(u)e^{-|u|/\varepsilon_0}$ is integrable). Similar reasoning for f' and f gives (2.1). □

We are ready to prove Statement 2.2. It is worth noting that statements of this kind are commonplace in the theory and they can be found in almost every article on the Bellman function method in analysis (a classical example is the paper [9]).

Proof of Statement 2.2. Let $0 < \tau < 1$. Consider the function

$$G_\tau(x_1, x_2) \stackrel{\mathrm{def}}{=} G(\tau x_1, \tau^2 x_2).$$

We also define $f_\tau(x_1) \stackrel{\mathrm{def}}{=} f(\tau x_1)$. It is easily seen that G_τ is continuous and locally concave in $\Omega_{\varepsilon/\tau}$. This function also satisfies the boundary condition

$$G_\tau(x_1, x_1^2) = f_\tau(x_1).$$

Next, consider a point $x \in \Omega_\varepsilon$. Fix a function $\varphi \in \mathrm{BMO}_\varepsilon(I)$ such that $x = \big(\langle\varphi\rangle_I, \langle\varphi^2\rangle_I\big)$. By x^σ we denote the Bellman point generated by the same function φ and a subinterval $\sigma \subset I$, i.e. $x^\sigma \stackrel{\mathrm{def}}{=} \big(\langle\varphi\rangle_\sigma, \langle\varphi^2\rangle_\sigma\big)$. By Lemma 2.3, there exists a partition $I = I_- \cup I_+$ such that the segment with the endpoints x^{I_-} and x^{I_+} lies in $\Omega_{\varepsilon/\tau}$ entirely. Note that $x = x^I = \alpha_- x^{I_-} + \alpha_+ x^{I_+}$, where $\alpha_\pm = |I_\pm|/|I|$. Using the local concavity of G_τ, we get the inequality

$$|I|\, G_\tau(x) \geq |I_-|\, G_\tau\big(x^{I_-}\big) + |I_+|\, G_\tau\big(x^{I_+}\big). \tag{2.2}$$

We repeat the procedure described above for each subinterval $I_\pm$ (treating φ as a function on the corresponding subinterval); after that we repeat it again for each of four subintervals obtained in the previous step, and so on. After n steps we have a collection D_n of 2^n subintervals that divide I. Using the local concavity of G_τ in each step, we get the estimate

$$|I|\, G_\tau(x) \geq \sum_{\sigma \in D_n} |\sigma|\, G_\tau(x^\sigma) = \int\limits_I G_\tau\big(x^n(t)\big)\, dt,$$

where $x^n(t)$ is the step function taking the value x^σ on each interval $\sigma \in D_n$.[1] Since $\alpha_\pm$ can be chosen to be separated from 0 and 1 uniformly, the lengths of the intervals tend to zero as n tends to infinity: $\max_{\sigma \in D_n} |\sigma| \to 0$ as $n \to \infty$. By the Lebesgue differentiation theorem, this implies that

$$x^n(t) \to \big(\varphi(t), \varphi^2(t)\big)$$

for almost all $t \in I$. Suppose for a while that $\varphi \in L^\infty(I)$. Then the values of the functions $x^n(t)$ lie in some compact subset of Ω_ε. Therefore, since $G_\tau(x)$ is continuous, the sequence of functions $G_\tau\big(x^n(t)\big)$ is uniformly bounded. Passing to the limit and using the boundary condition, we get

$$|I|\, G_\tau(x) \geq \int\limits_I G_\tau\big(\varphi(t), \varphi^2(t)\big)\, dt = \int\limits_I f_\tau\big(\varphi(t)\big)\, dt.$$

Now we lift the boundedness of φ and pass to the limit in τ. Consider the truncations

$$\varphi_m(t) \stackrel{\mathrm{def}}{=} \begin{cases} m, & \varphi(t) > m; \\ \varphi(t), & |\varphi(t)| \leq m; \\ -m, & \varphi(t) < -m. \end{cases}$$

[1] The procedure just described is often called *the Bellman induction*.

In [17, 18], it is proved that they lie in the same ball $\mathrm{BMO}_\varepsilon(I)$ as φ. Thus, since the functions φ_m are bounded, the estimate proved earlier is true for them:

$$|I|\, G_\tau\big(\langle\varphi_m\rangle_I, \langle\varphi_m^2\rangle_I\big) \geq \int\limits_I f_\tau\big(\varphi_m(t)\big)\, dt.$$

Since G is continuous, the left part tends to $G(x)$ as $m \to \infty$ and $\tau \to 1-$. Thus, it remains to pass to the corresponding limits in the right part of the inequality. The continuity of f implies that the integrands converge to $f\big(\varphi(t)\big)$ pointwise. Therefore, in order to establish the convergence of the integrals, it remains to find an integrable majorant. Due to relation (2.1) for $r = 0$ and the continuity of f, the estimate $|f(s)| \leq Ce^{|s|/\varepsilon_0}$ is fulfilled. Then we have

$$\big|f_\tau\big(\varphi_m(t)\big)\big| \leq C \exp \tfrac{\tau|\varphi_m(t)|}{\varepsilon_0} \leq C \exp \tfrac{|\varphi(t)|}{\varepsilon_0} \leq C\left(\exp \tfrac{\varphi(t)}{\varepsilon_0} + \exp \tfrac{-\varphi(t)}{\varepsilon_0}\right).$$

The last expression is integrable by the integral John–Nirenberg inequality (see [20] or [17]), because $\varepsilon < \varepsilon_0$, and both φ and $-\varphi$ are in $\mathrm{BMO}_\varepsilon(I)$. Passing to the limits, we finally get $G(x) \geq \boldsymbol{B}_\varepsilon(x)$. □

2.3. **Monge–Ampère equation.** Let B be the minimal function in $\Lambda_{\varepsilon,f}$. Properties (i) and (ii), together with property (iii) being assumed and Statement 2.2, imply that we may treat B as a candidate for the Bellman function $\boldsymbol{B}_\varepsilon$. In this subsection, we present some reasoning (not intended to be rigorous) that allows us to reduce the problem of finding such a function to solving the homogeneous Monge–Ampère equation.

As we will see later, for each point $x \in \Omega_\varepsilon$, there exists a function $\varphi \in \mathrm{BMO}_\varepsilon(I)$ that realizes the supremum for the point x in the Bellman function definition (see (1.4)); i.e. $x = \big(\langle\varphi\rangle_I, \langle\varphi^2\rangle_I\big)$ and $\boldsymbol{B}_\varepsilon(x) = \langle f(\varphi)\rangle_I$. If the functions G_τ from Statement 2.2 approximate $\boldsymbol{B}_\varepsilon$, then for the optimizer φ there exists a partition of I such that in (2.2) the equality is almost attained. In view of the local concavity of G_τ, this means that this function is almost linear on the segment $\big[x^{I_-}, x^{I_+}\big] \subset \Omega_{\varepsilon/\tau}$. This yields that our candidate B must be linear along some vector Θ_x; i.e. its second derivative along Θ_x vanishes at x:

$$\frac{\partial^2 B}{\partial \Theta_x^2} = \left(\frac{d^2B}{dx^2}\,\Theta_x, \Theta_x\right) = 0, \tag{2.3}$$

where

$$\frac{d^2B}{dx^2} = \begin{pmatrix} B_{x_1x_1} & B_{x_1x_2} \\ B_{x_2x_1} & B_{x_2x_2} \end{pmatrix}$$

(here all the functions are evaluated at x). On the other hand, since the function B is locally concave, it follows that the matrix of its second derivatives is negative semidefinite:

$$\frac{d^2B}{dx^2} \leq 0.$$

Next, by virtue of (2.3), it cannot be *strictly* negative definite, so

$$\det\left(\frac{d^2B}{dx^2}\right) = B_{x_1x_1}B_{x_2x_2} - B_{x_1x_2}^2 = 0. \tag{2.4}$$

This is the homogeneous Monge–Ampère equation for B. Besides (2.4), the boundary condition $B(x_1, x_1^2) = f(x_1)$ and the inequalities $B_{x_1x_1} \leq 0$, $B_{x_2x_2} \leq 0$ must be fulfilled.

In order to solve equation (2.4), we will use the following consideration: the integral curves of the vector field Θ_x are straight lines and, what is more, all the partial derivatives of B are constant along them. We formulate this principle in the following statement, which has been proved, for example, in [25].

Statement 2.5. Suppose Ω is a domain in $\mathbb{R}^2$ and $G \in C^2(\Omega)$ is a function satisfying the homogeneous Monge–Ampère equation on Ω:

$$G_{x_1x_1}G_{x_2x_2} - G^2_{x_1x_2} = 0.$$

Let

$$t_1 = G_{x_1}, \quad t_2 = G_{x_2}, \quad \text{and} \quad t_0 = G - t_1x_1 - t_2x_2.$$

Suppose $G_{x_1x_1} \neq 0$ or $G_{x_2x_2} \neq 0$ at every point of Ω. Then the functions t_1, t_2, and t_0 are constant along the integral curves of the vector field that annihilates the quadratic form $\frac{d^2G}{dx^2}$ on Ω. The integral curves mentioned above (the extremals) are segments of the straight lines defined by the equation

$$x_1dt_1 + x_2dt_2 + dt_0 = 0. \tag{2.5}$$

Graphs of solutions of the homogeneous Monge–Ampère equation are called *developable* surfaces. All the properties of such solutions can be formulated in geometric terms. For example, the theorem presented above states that a developable surface is *ruled*. Concerning geometric interpretation, see e.g. [13].

In view of Statement 2.5, we can assume that our domain Ω_ε can be split into subdomains of two kinds: domains where $\frac{d^2G}{dx^2} = 0$ (B is a linear function there) and domains where $\dim \operatorname{Ker} \frac{d^2G}{dx^2} = 1$. Latter domains are foliated by straight-line segments such that the partial derivatives of B are constant along them. We will look for our Bellman function among the functions B corresponding to such foliations. The following definition fixes the notion of *a Bellman candidate*.

Definition 2.6. Consider a subdomain $\widetilde{\Omega} \subset \Omega_\varepsilon$ and a finite collection of pairwise disjoint subdomains[2] $\widetilde{\Omega}^1, \dots, \widetilde{\Omega}^m \subset \widetilde{\Omega}$ whose union is $\widetilde{\Omega}$. Consider some function $B \in C(\widetilde{\Omega})$ that is locally concave in $\widetilde{\Omega}$ and satisfies the boundary condition $B(x_1, x_1^2) = f(x_1)$. Suppose $B \in C^1(\widetilde{\Omega}^i)$, $i = 1 \dots m$, and those subdomains $\widetilde{\Omega}^i$ where B is not linear are foliated by non-intersecting straight-line segments such that the partial derivatives of B are constant along them. Then we say that B is a Bellman candidate in $\widetilde{\Omega}$.

Another useful observation helping us to construct Bellman candidates is that the extremals, intersecting the upper boundary of Ω_ε, must be tangents to it (see *Principle 2* on page 8 of [18]).

All of the above allows us to believe that our Bellman function can be found among the functions described in Definition 2.6. If we find some Bellman candidate B on the whole domain Ω_ε, the inequality $\boldsymbol{B}_\varepsilon \leq B$ will follow immediately from Statement 2.2. In order to verify the converse estimate $\boldsymbol{B}_\varepsilon \geq B$, we will construct, for each point $x \in \Omega_\varepsilon$, a function $\varphi \in \mathrm{BMO}_\varepsilon(I)$ such that $x = \big(\langle\varphi\rangle_I, \langle\varphi^2\rangle_I\big)$ and $B(x) = \langle f(\varphi)\rangle_I$. Such functions are called *optimizers*. General considerations on the construction of optimizers are stated in the next section.

[2]It is worth noting that here the notion of a domain has a wider meaning than usually: a domain is the union of a connected open set and any part of its boundary.

2.4. **Optimizers.** First, we fix the notion of an optimizer.

Definition 2.7. Let B be a Bellman candidate in the whole domain Ω_ε. A function φ defined on some interval I is called an optimizer for a point $x \in \Omega_\varepsilon$ if the following conditions are satisfied:

(1) $\varphi \in \mathrm{BMO}_\varepsilon(I)$;
(2) $\big(\langle\varphi\rangle_I, \langle\varphi^2\rangle_I\big) = x$;
(3) $\langle f(\varphi)\rangle_I = B(x)$.

We notice that it suffices to consider only non-decreasing optimizers. Indeed, if we replace a function by its *increasing rearrangement*, the BMO-norm does not increase (an increasing rearrangement of a function φ is a non-decreasing function φ^* such that the measure of the set $\{t \in I \mid \varphi(t) > \lambda\}$ is equal to the measure of the set $\{t \in I \mid \varphi^*(t) > \lambda\}$ for any $\lambda \in \mathbb{R}$). This statement was proved in [5]. In [7], it was employed for the calculation of the sharp constant c_2 in John–Nirenberg inequality (1.2). It is also clear that averages of the form $\langle h(\varphi)\rangle_I$ do not change when φ is replaced by its increasing rearrangement. All this implies that the supremum in (1.4) may be taken over the set of the non-decreasing functions satisfying the same conditions.

We will construct optimizers using the notion of *delivery curves*. The following reasoning, which is not intended to be rigorous, will lead us to the corresponding definition. Consider a non-decreasing optimizer φ. For it, each inequality in the Bellman induction (see the proof of Statement 2.2) turns into an equality. Thus, we must split the interval in such a way that the corresponding points move along the extremals that foliate the subdomains where the Bellman function is not linear (inside domains of the linearity, every segment is an extremal). If at each step of the Bellman induction we manage to choose an infinitesimal partition, i.e. cut off an arbitrarily small part from one side of the interval, then we get some curve inside the domain (the coordinates of its points are, in fact, the averages of φ and φ^2 over the larger of two intervals that are obtained after each cutting). If we cut off from the right side of the interval, then a resulting curve is called a *left* delivery curve (since we consider an increasing test function, this curve lies on the *left* of the point at which we begin the induction). This heuristic reasoning leads us to the following rigorous definition.

Definition 2.8. Suppose φ is some test function on $I = [l, r]$. A curve γ is called a left delivery curve if it is defined by the formula

$$\gamma(s) = \big(\langle\varphi\rangle_{[l,s]}, \langle\varphi^2\rangle_{[l,s]}\big), \quad s \in (l, r], \tag{2.6}$$

and for all $s \in (l, r]$ the following equation is fulfilled:

$$B(\gamma(s)) = \langle f(\varphi)\rangle_{[l,s]}. \tag{2.7}$$

Cutting from the other side, we come to the notion of a *right* delivery curve.

Definition 2.9. Suppose φ is some test function on $I = [l, r]$. A curve γ is called a right delivery curve if it is defined by the formula

$$\gamma(s) = \big(\langle\varphi\rangle_{[s,r]}, \langle\varphi^2\rangle_{[s,r]}\big), \quad s \in [l, r), \tag{2.8}$$

and for all $s \in [l, r)$ the following equation is fulfilled:

$$B(\gamma(s)) = \langle f(\varphi)\rangle_{[s,r]}. \tag{2.9}$$

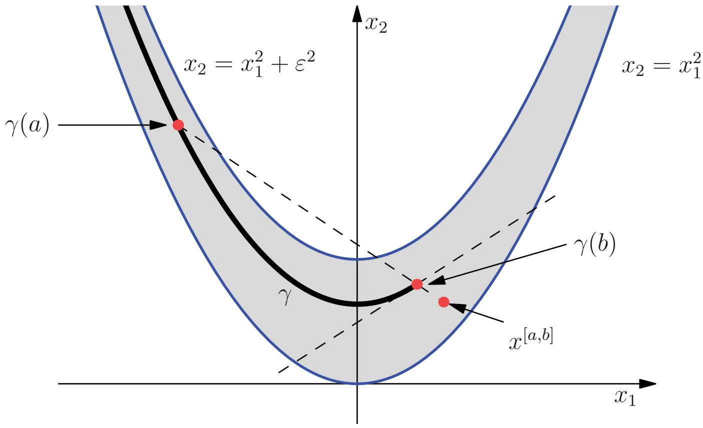

FIGURE 2. Illustration to the proof of Lemma 2.10.

Definitions 2.8 and 2.9 postulate that the restrictions $\varphi|_{[l,s]}$ are optimizers for the corresponding points $\gamma(s)$ of the left delivery curve (which lies, of course, in Ω_ε entirely), and the restrictions $\varphi|_{[s,r]}$ are optimizers for the points $\gamma(s)$ of the right delivery curve. Therefore, if we build a delivery curve, we automatically obtain the optimizers for all the points of this curve.

According to the procedure described above, delivery curves run along extremals. Thus, they can consist of some parts of extremals and arcs of the upper parabola. Also, if we take only non-decreasing test functions, then left delivery curves will run from left to right and right delivery curves will run from right to left (for right delivery curves, we assume that the "time" s runs backwards, i.e. from r to l).

We will build optimizers for some Bellman candidate B as follows. We will draw various curves along the extremals corresponding to our candidate. After that, we will construct functions $\varphi \in L^2(I)$ that generate these curves in the sense of (2.6) or (2.8). Next, we will verify that the obtained functions belong to $\mathrm{BMO}_\varepsilon(I)$ and satisfy (2.7) or (2.9). The condition $\varphi \in \mathrm{BMO}_\varepsilon(I)$ can be derived from general geometric considerations. The fact is that all our delivery curves turn out to be convex. In addition, their curvatures will not be too large: as a rule, any tangent to such a curve will lie under the upper boundary of Ω_ε. These properties can be explained by the fact that these curves must run along the upper parabola or straight extremals which intersect the upper boundary tangentially. It turns out that if some function $\varphi \in L^2(I)$ generates a curve with the properties described above, then $\varphi \in \mathrm{BMO}_\varepsilon(I)$. We formulate the corresponding statement in the local form, which is more convenient for further applications.

Lemma 2.10. *Let φ be an integrable function on $I = [l, r]$ and let γ be the curve generated by this function in the sense of* (2.6)*. Suppose γ lies in Ω_ε entirely, coincides with the graph of a convex function, and is differentiable in some point $b \in I$. If the tangent to γ at the point $\gamma(b)$ lies below the upper boundary of Ω_ε, then all the Bellman points $x^{[a,b]} = \big(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]}\big)$, $l \le a < b$, belong to Ω_ε.*

If the curve γ is generated by φ in the sense of (2.8)*, then the Bellman point $x^{[a,b]}$ is in Ω_ε provided the tangent to γ at the point $\gamma(a)$ lies below the upper parabola.*

Proof. We prove only the first half of the lemma; the proof of the second is similar. Since the curve γ is convex, the point $\gamma(a)$ must lie above the tangent to γ at the point $\gamma(b)$. The points $\gamma(a)$, $x^{[a,b]}$, and $\gamma(b)$ lie on one line and the last lies between the first two, because it is their convex combination. Thus, the point $x^{[a,b]}$ must lie below the tangent and, therefore, below the upper boundary of Ω_ε. On the other hand, by the Cauchy–Schwartz inequality, the point $x^{[a,b]}$ lies above the lower boundary. As we have already mentioned, the symmetric situation when γ and φ satisfy relation (2.8) can be treated in a similar way. $\square$

3. Homogeneous families of extremals

As already noted, an extremal intersecting the upper parabola must touch it. In this section, we assume that some subdomain of Ω_ε is foliated by extremals that are tangential to the upper boundary, and look for a Bellman candidate in such a subdomain.

3.1. **Family of tangents to the upper boundary.**

Consider the tangent to the upper parabola at a point $(w, w^2+\varepsilon^2)$. Its segment lying in Ω_ε is given by the following relation:

$$x_2 = 2wx_1 + \varepsilon^2 - w^2 \quad \text{for} \quad x_1 \in [w-\varepsilon, w+\varepsilon]. \tag{3.1}$$

Consider some hypothetical family of extremals (they are segments of straight lines) such that each of them is a tangent to the upper parabola. Parameterize this family by the first coordinate of tangency points $w \in (w_1, w_2)$. If the corresponding Bellman candidate B is not linear in both variables, then an extremal cannot contain the whole segment (3.1). Moreover, a tangency point $(w, w^2+\varepsilon^2)$ is not an inner point of an extremal; otherwise such an extremal intersects with others. Thus, each extremal line of our family lies either on the right of the point $(w, w^2+\varepsilon^2)$ or on the left of it. Consider two families of extremals. The first consists of segments of tangents to the upper parabola that lie on the right of their tangency points. The second consists of those segments that lie on the left of the tangency points. We make the substitution $w = u - \varepsilon$ in the first case and $w = u + \varepsilon$ in the second; i.e. we parameterize the extremals by the first coordinate u of those points where they intersect the lower parabola. The parameter u runs over some interval $(u_1, u_2) = (w_1 \pm \varepsilon, w_2 \pm \varepsilon)$. Therefore, our families of the right and left tangents are described, respectively, by the following equations:

$$\begin{aligned}
&\text{(R)}\ \ x_2 - 2(u-\varepsilon)x_1 + u^2 - 2u\varepsilon = 0, \qquad u \in (u_1, u_2), \quad x_1 \in [u-\varepsilon, u];\\
&\text{(L)}\ \ x_2 - 2(u+\varepsilon)x_1 + u^2 + 2u\varepsilon = 0, \qquad u \in (u_1, u_2), \quad x_1 \in [u, u+\varepsilon].
\end{aligned}$$

We look for a Bellman candidate on subdomains of Ω_ε that are foliated by families (R) or (L). We define such subdomains by $\Omega_{\mathrm{R}}(u_1, u_2)$ and $\Omega_{\mathrm{L}}(u_1, u_2)$, respectively (see Figures 3 and 4). Expressing u in terms of x_1 and x_2 for the tangents (R) and (L), we obtain, respectively, the following relations:

$$u = u_{\mathrm{R}}(x_1, x_2) = x_1 + \Big(\varepsilon - \sqrt{\varepsilon^2 - (x_2 - x_1^2)}\Big), \tag{3.2}$$

$$u = u_{\mathrm{L}}(x_1, x_2) = x_1 - \Big(\varepsilon - \sqrt{\varepsilon^2 - (x_2 - x_1^2)}\Big). \tag{3.3}$$

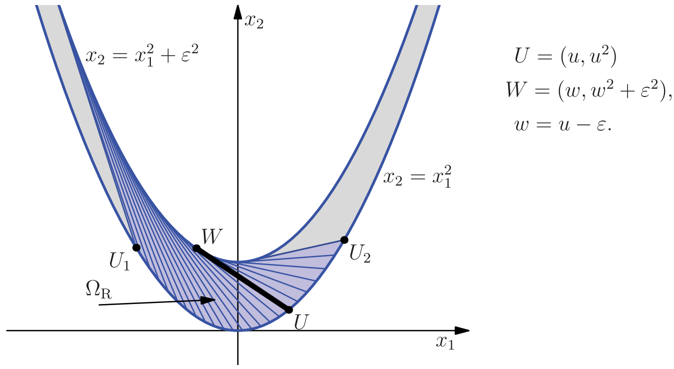

FIGURE 3. A domain Ω_{R} with the right tangents.

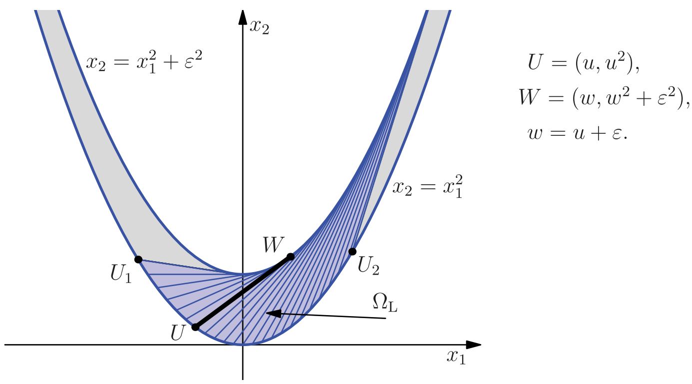

FIGURE 4. A domain Ω_{L} with the left tangents.

From now on, we establish the following rule for our notation. Any point on the lower boundary is denoted by a capital Latin letter and the first coordinate of this point is denoted by the corresponding small letter. For example, we write U for (u, u^2) (see Figures 3 and 4).

Let B be a Bellman candidate on Ω_{R} or Ω_{L}. Since the function B must be linear on the linear extremals and satisfy the boundary condition $B(U) = f(u)$, it follows that B can be written as

$$B(x_1, x_2) = m(u)(x_1 - u) + f(u). \tag{3.4}$$

Consider case (R). Using representation (3.4) and the equation

$$u_{x_2} = \frac{1}{2(x_1 - u + \varepsilon)}, \tag{3.5}$$

by direct calculation we obtain the following identity:

$$t_2 = B_{x_2} = \frac{m'(u)}{2} - \frac{\varepsilon m'(u) + m(u) - f'(u)}{2(x_1 - u + \varepsilon)}.$$

If u is fixed, the function t_2 must be constant. Therefore,

$$\varepsilon m'(u) + m(u) - f'(u) = 0; \tag{3.6}$$

$$t_2 = \frac{m'(u)}{2}. \tag{3.7}$$

All the solutions of equation (3.6) are of the form

$$m_{\mathrm{R}}(u) = e^{-u/\varepsilon}\Big(A + \varepsilon^{-1}\int_{u_1}^{u} f'(t)e^{t/\varepsilon}\,dt\Big), \tag{3.8}$$

where A is an integration constant. Substituting solution (3.8) into representation (3.4) and expressing u in terms of x by (3.2), we obtain a family of functions (we still have a free parameter A) whose derivatives are constant along extremals (R) foliating $\Omega_{\mathrm{R}}(u_1,u_2)$. We denote such functions by $B^{\mathrm{R}}(x;\,u_1,u_2)$.

Next, by virtue of (3.7), we can write

$$B^{\mathrm{R}}_{x_2x_2} = t_2'(u)u_{x_2} = \frac{m''_{\mathrm{R}}(u)u_{x_2}}{2}.$$

Using this equation and identity (3.5), we see that the condition $B^{\mathrm{R}}_{x_2x_2} \le 0$ is equivalent to $m''_{\mathrm{R}}(u) \le 0$, $u \in (u_1,u_2)$. This condition is necessary and sufficient for the local concavity of $B^{\mathrm{R}}(x;\,u_1,u_2)$.

Now we obtain a formula for m''_{R}. Differentiating equation (3.6) twice and solving it with respect to m'', we get

$$m''_{\mathrm{R}}(u) = e^{(u_1-u)/\varepsilon}m''_{\mathrm{R}}(u_1) + \varepsilon^{-1}e^{-u/\varepsilon}\int_{u_1}^{u} f'''(t)e^{t/\varepsilon}\,dt. \tag{3.9}$$

Reasoning for extremals (L) in a similar way, we get the following relations:

$$-\varepsilon m'(u) + m(u) - f'(u) = 0; \tag{3.10}$$

$$t_2 = B_{x_2} = \frac{m'(u)}{2}. \tag{3.11}$$

All the solutions of equation (3.10) have the following form:

$$m_{\mathrm{L}}(u) = e^{u/\varepsilon}\Big(A + \varepsilon^{-1}\int_{u}^{u_2} f'(t)e^{-t/\varepsilon}\,dt\Big). \tag{3.12}$$

Setting $m(u) = m_{\mathrm{L}}(u)$ in (3.4) and expressing u in terms of x by relation (3.3), we obtain the function $B(x) = B^{\mathrm{L}}(x;\,u_1,u_2)$ on $\Omega_{\mathrm{L}}(u_1,u_2)$ with the partial derivatives that are constant along extremals (L). The local concavity of the function B^{L} is equivalent to the condition $m''_{\mathrm{L}}(u) \ge 0$ for all $u \in (u_1,u_2)$, and $m''_{\mathrm{L}}(u)$ satisfies

$$m''_{\mathrm{L}}(u) = e^{(u-u_2)/\varepsilon}m''_{\mathrm{L}}(u_2) + \varepsilon^{-1}e^{u/\varepsilon}\int_{u}^{u_2} f'''(t)e^{-t/\varepsilon}\,dt. \tag{3.13}$$

We summarize this section.

Proposition 3.1. *Suppose the subdomain $\Omega_{\mathrm{R}}(u_1,u_2)\subset\Omega_\varepsilon$ is foliated by extremals* (R) *entirely. Then any Bellman candidate in this subdomain has the form*

$$B^{\mathrm{R}}(x;\,u_1,u_2)=m_{\mathrm{R}}(u)(x_1-u)+f(u), \tag{3.14}$$

where $u=u_{\mathrm{R}}(x_1,x_2)$. Besides, the function $m_{\mathrm{R}}''(u)$ must satisfy $m_{\mathrm{R}}''(u)\le 0$, $u\in(u_1,u_2)$ (in order for B^{R} to be a Bellman candidate indeed).

The same is true for the domain $\Omega_{\mathrm{L}}(u_1,u_2)$ and the function

$$B^{\mathrm{L}}(x;\,u_1,u_2)=m_{\mathrm{L}}(u)(x_1-u)+f(u), \tag{3.15}$$

where $u=u_{\mathrm{L}}(x_1,x_2)$. The estimate $m_{\mathrm{L}}''(u)\ge 0$ must be fulfilled.

3.2. **Family of tangents coming from $\pm\infty$.** Consider a domain $\Omega_{\mathrm{R}}(-\infty,u_2)$ unbounded on the left and foliated by the right tangents. It turns out that the Bellman candidate $B^{\mathrm{R}}(x;\,-\infty,u_2)$ in it can be chosen uniquely by minimality considerations. Similarly, if we consider a subdomain $\Omega_{\mathrm{L}}(u_1,+\infty)$ unbounded on the right, the minimal Bellman candidate $B^{\mathrm{L}}(x;\,u_1,+\infty)$ can also be chosen uniquely. Namely, arguing as in [18], we can prove that our Bellman candidate in $\Omega_{\mathrm{R}}(-\infty,u_2)$ becomes minimal when $A=0$ (see (3.8)). So we have $m_{\mathrm{R}}(u)=m_{\mathrm{R}}(u;\,-\infty)$, where

$$m_{\mathrm{R}}(u;\,-\infty)=\varepsilon^{-1}e^{-u/\varepsilon}\int_{-\infty}^{u}f'(t)e^{t/\varepsilon}\,dt; \tag{3.16}$$

$$m_{\mathrm{R}}''(u;\,-\infty)=\varepsilon^{-1}e^{-u/\varepsilon}\int_{-\infty}^{u}f'''(t)e^{t/\varepsilon}\,dt. \tag{3.17}$$

And for the case of left extremals (L) for $u_2=+\infty$, we have $m_{\mathrm{L}}(u)=m_{\mathrm{L}}(u;\,+\infty)$, where

$$m_{\mathrm{L}}(u;\,+\infty)=\varepsilon^{-1}e^{u/\varepsilon}\int_{u}^{+\infty}f'(t)e^{-t/\varepsilon}\,dt; \tag{3.18}$$

$$m_{\mathrm{L}}''(u;\,+\infty)=\varepsilon^{-1}e^{u/\varepsilon}\int_{u}^{+\infty}f'''(t)e^{-t/\varepsilon}\,dt. \tag{3.19}$$

We sum up all this in the following proposition.

Proposition 3.2. *Suppose the subdomain $\Omega_{\mathrm{R}}(-\infty,u_2)\subset\Omega_\varepsilon$ is foliated by extremals* (R) *entirely. In this domain, we define the function B^{R} by the formula*

$$B^{\mathrm{R}}(x;\,-\infty,u_2)=m_{\mathrm{R}}(u;\,-\infty)\,(x_1-u)+f(u), \tag{3.20}$$

where $u=u_{\mathrm{R}}(x_1,x_2)$. Assume that $m_{\mathrm{R}}''(u;\,-\infty)\le 0$, $u\in(-\infty,u_2)$. Then $B^{\mathrm{R}}(x;\,-\infty,u_2)$ is the minimal Bellman candidate in $\Omega_{\mathrm{R}}(-\infty,u_2)$.

The same is true for the domain $\Omega_{\mathrm{L}}(u_1,+\infty)$ and the function

$$B^{\mathrm{L}}(x;\,u_1,+\infty)=m_{\mathrm{L}}(u;\,+\infty)\,(x_1-u)+f(u) \tag{3.21}$$

(here $u=u_{\mathrm{L}}(x_1,x_2)$), provided $m_{\mathrm{L}}''(u;\,+\infty)\ge 0$, $u\in(u_1,+\infty)$.

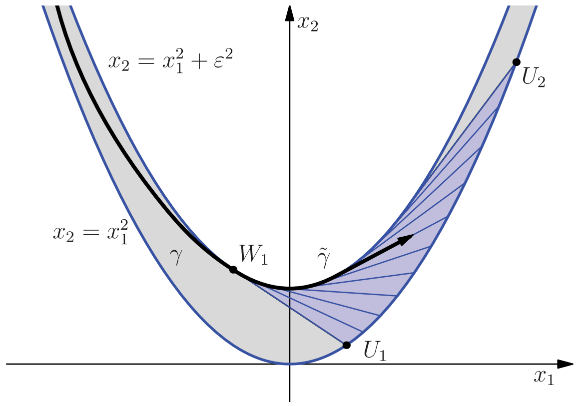

FIGURE 5. Delivery curves in Ω_{R}.

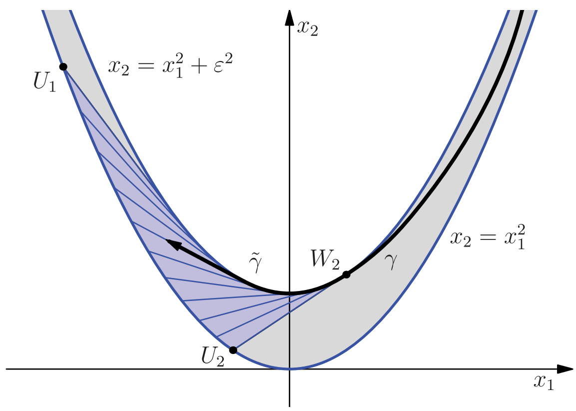

FIGURE 6. Delivery curves in Ω_{L}.

3.3. **Optimizers for the families of tangents.** Let B be a Bellman candidate in the whole domain Ω_ε. We also assume that some part $\Omega_{\mathrm{R}}(u_1, u_2)$ of Ω_ε is foliated by the right extremal tangents.

From Section 2.4, it follows that delivery curves in Ω_{R} run along the upper parabola or along the tangents. Also, it can be easily seen that these delivery curves must be left.

Consider the point $W_1 = \big(u_1 - \varepsilon, (u_1-\varepsilon)^2 + \varepsilon^2\big)$ on the upper parabola. Suppose some convex delivery curve γ runs from a neighbor subdomain and ends at W_1 (i.e. $\gamma(r) = W_1$). We will see that this curve can be continued up to each point of Ω_{R} in the way shown in Figure 5: we continue it along the upper parabola and, after that, along the tangent leading to the destination point. Therefore, we will obtain optimizers for all the points of Ω_{R}. The point W_1 is called *an entry node*: the information from the neighbor subdomain is transmitted through it only.

For a subdomain $\Omega_{\mathrm{L}}(u_1, u_2)$ foliated by the left tangents, the situation is symmetric. The point $W_2 = \big(u_2 + \varepsilon, (u_2+\varepsilon)^2 + \varepsilon^2\big)$ is its entry node. If a convex right delivery curve γ reaches this point (i.e. $\gamma(l) = W_2$), then γ can be continued up to each point in $\Omega_{\mathrm{L}}(u_1, u_2)$ (see Figure 6).

Points on the upper parabola. Let γ be a convex left delivery curve that is generated by a test function φ defined on the segment $I=[l,r]$. Also, suppose it ends at the entry node W_1 of the domain $\Omega_{\mathrm{R}}(u_1,u_2)$. First, we prove that this curve can be continued up to any point $W\in\Omega_{\mathrm{R}}(u_1,u_2)$, lying on the upper boundary, in such a way that the resulting curve $\widetilde{\gamma}$ will also be a left delivery curve.

Since delivery curves run either along extremals or along the upper parabola and extremals touch the upper parabola, we may assume that the convex curve γ also touches the upper parabola at the point W_1. Thus, the curve $\widetilde{\gamma}$ cannot avoid being convex.

Now, continue the left delivery curve γ along the upper parabola with preservation of the convexity. In order to prove that the continuation $\widetilde{\gamma}$ is also a left delivery curve, we must construct a test function $\widetilde{\varphi}$ defined on some segment $[l,\widetilde{r}]$, $\widetilde{r}>r$, such that $\widetilde{\gamma}$ is generated by this function in the sense of (2.6) and relation (2.7) is fulfilled for $\widetilde{\varphi}$ and $\widetilde{\gamma}$.

We set $\widetilde{\varphi}(s)=\varphi(s)$ for $s\in I$. The question is how to define $\widetilde{\varphi}(s)$ for $s>r$. For $s>r$, the curve $\widetilde{\gamma}(s)=\big(\widetilde{\gamma}_1(s),\widetilde{\gamma}_2(s)\big)$ runs along the upper parabola, so

$$\widetilde{\gamma}_1(s)=\frac{1}{s-l}\int_l^s\widetilde{\varphi}(t)\,dt\quad\text{and}\quad\widetilde{\gamma}_2(s)=\frac{1}{s-l}\int_l^s\widetilde{\varphi}^2(t)\,dt=\widetilde{\gamma}_1^2(s)+\varepsilon^2.$$

Therefore,

$$\widetilde{\varphi}^2(s)=\big([(s-l)\widetilde{\gamma}_1]'\big)^2=\big((s-l)(\widetilde{\gamma}_1^2+\varepsilon^2)\big)',$$

i.e.

$$\widetilde{\gamma}_1^2+2(s-l)\widetilde{\gamma}_1\widetilde{\gamma}_1'+((s-l)\widetilde{\gamma}_1')^2=\widetilde{\gamma}_1^2+2(s-l)\widetilde{\gamma}_1\widetilde{\gamma}_1'+\varepsilon^2.$$

Since we build the left delivery curve, we expect the function $\widetilde{\gamma}_1$ to be non-decreasing. Therefore, taking the square root, we obtain

$$\widetilde{\gamma}_1'(s)=\frac{\varepsilon}{s-l}. \tag{3.22}$$

Note that the other root gives us the backwards motion along the parabola. Solving the equation (3.22), we get

$$\widetilde{\gamma}_1(s)=\varepsilon\log(s-l)+c \tag{3.23}$$

and

$$\widetilde{\varphi}(s)=\big((s-l)\widetilde{\gamma}_1\big)'=\varepsilon\log(s-l)+c+\varepsilon. \tag{3.24}$$

Now, using the continuity of the delivery curve at $s=r$, we obtain the constant in (3.23) and (3.24):

$$u_1-\varepsilon=\gamma_1(r)=\widetilde{\gamma}_1(r)=\varepsilon\log(r-l)+c.$$

Therefore, $c=u_1-\varepsilon\log(r-l)-\varepsilon$ and equation (3.24) takes the form

$$\widetilde{\varphi}(s)=\varepsilon\log\frac{s-l}{r-l}+u_1,\quad s\in(r,\widetilde{r}], \tag{3.25}$$

where the choice of $\widetilde{r}$ depends on the point we want to reach.

Now we verify that $\widetilde{\varphi}$ is an admissible test function and $\widetilde{\gamma}$ is a left delivery curve generated by this function; i.e. we prove the following statement.

Proposition 3.3. *Consider a subdomain $\Omega_{\mathrm{R}}(u_1,u_2)$, $u_1>-\infty$, foliated by the right tangents. Suppose some test function φ defined on $I=[l,r]$ generates a convex left delivery curve γ that lies on the left of Ω_{R} and ends at the entry node $W_1=\big(u_1-\varepsilon,(u_1-\varepsilon)^2+\varepsilon^2\big)$ (i.e. $\gamma(r)=W_1$). We continue this curve to the right along the upper parabola without leaving Ω_{R}. If the resulting curve $\widetilde{\gamma}$ is convex, then it is a left delivery curve generated by the test function*

$$\widetilde{\varphi}(s)=\begin{cases}\varphi(s), & s\in I;\\ \varepsilon\log\dfrac{s-l}{r-l}+u_1, & s\in[r,\widetilde{r}].\end{cases}$$

Proof. The fact that $\widetilde{\varphi}$ generates $\widetilde{\gamma}$ in the sense of (2.6) follows from the construction of $\widetilde{\varphi}$ (see the above considerations). It remains to verify two points. First, it must be proved that $\widetilde{\varphi}$ belongs to $\mathrm{BMO}_\varepsilon([l,\widetilde{r}])$. Second, we must verify relation (2.7) for the function $\widetilde{\varphi}$, the curve $\widetilde{\gamma}$, and the candidate B.

The fact that $\widetilde{\varphi}\in\mathrm{BMO}_\varepsilon([l,\widetilde{r}])$ follows from the geometric Lemma 2.10. Indeed, if $[a,b]\subset I$, then the Bellman point $x^{[a,b]}$ is in Ω_ε, because $\varphi\in\mathrm{BMO}_\varepsilon(I)$. If $b>r$, then the conditions of the lemma just mentioned are fulfilled.

We turn to verification of (2.7). In Ω_{R}, the Bellman candidate B coincides with B^{R} (see Proposition 3.1). Therefore, we must check that

$$B^{\mathrm{R}}(\widetilde{\gamma}(s);u_1,u_2)=\langle f(\widetilde{\varphi})\rangle_{[l,s]},\quad s\in(r,\widetilde{r}].$$

By (3.14) and (3.8), we have

$$B^{\mathrm{R}}(\widetilde{\gamma}(s))=-e^{-u/\varepsilon}\left(\varepsilon A+\int_{u_1}^{u}f'(t)e^{t/\varepsilon}\,dt\right)+f(u), \tag{3.26}$$

where $u=\widetilde{\gamma}_1(s)+\varepsilon$. On the other hand, using the same relations and the continuity of B, we get

$$B(\gamma(r))=-\varepsilon e^{-u_1/\varepsilon}A+f(u_1).$$

Now, expressing A in terms of $B(\gamma(r))$, substituting the resulting expression into (3.26), and then integrating by parts, we have

$$\begin{aligned}B^{\mathrm{R}}(\widetilde{\gamma}(s))&=e^{(u_1-u)/\varepsilon}B(\gamma(r))-\int_{u_1}^{u}f'(t)e^{(t-u)/\varepsilon}\,dt-e^{(u_1-u)/\varepsilon}f(u_1)+f(u)\\&=e^{(u_1-u)/\varepsilon}B(\gamma(r))+\int_{u_1}^{u}f(t)\,d(e^{(t-u)/\varepsilon}).\end{aligned}$$

Further, using (3.23), we obtain

$$e^{(u_1-u)/\varepsilon}=e^{(\gamma_1(r)-\gamma_1(s))/\varepsilon}=\frac{r-l}{s-l}.$$

We make the substitution $t=\widetilde{\varphi}(\tau)$. Using formula (3.25) and the previous equation, we get

$$e^{(t-u)/\varepsilon}=\frac{\tau-l}{r-l}e^{(u_1-u)/\varepsilon}=\frac{\tau-l}{s-l}.$$

It follows from the above that τ runs over $(r,s]$ provided t runs over $(u_1,u]$. Using the substitution just described and the fact that $B(\gamma(r)) = \langle f(\varphi)\rangle_{[l,r]}$, we have

$$B^{\mathrm{R}}(\widetilde{\gamma}(s)) = \frac{r-l}{s-l}B(\gamma(r)) + \frac{1}{s-l}\int\limits_r^s f(\widetilde{\varphi}(\tau))\,d\tau$$

$$= \frac{1}{s-l}\int\limits_l^s f(\widetilde{\varphi}(\tau))\,d\tau = \langle f(\widetilde{\varphi})\rangle_{[l,s]}.$$

This concludes the proof. □

Similarly, we can prove a symmetric proposition for $\Omega_{\mathrm{L}}(u_1,u_2)$, $u_2 < +\infty$.

Proposition 3.4. *Consider a subdomain $\Omega_{\mathrm{L}}(u_1,u_2)$, $u_2 < +\infty$. Suppose some test function φ generates a convex right delivery curve γ that lies on the right of Ω_{L} and ends at the entry node W_2. We continue this curve to the left along the upper parabola without leaving Ω_{L}. If the resulting curve $\widetilde{\gamma}$ is convex, then it is a right delivery curve.*

Points inside the domain. We have explained how to continue delivery curves from entry node W_1 (or W_2) to a point in Ω_{R} (respectively, in Ω_{L}) lying on the upper parabola. It occurs that each of the other points in these subdomains (but the points on the lower boundary) can be reached if we continue the delivery curve along the corresponding tangent that contains this point.

Now, let γ be a left delivery curve that is generated by a test function φ defined on $I=[l,r]$. Suppose we have continued this curve from the point $\gamma(r)$ along some straight-line segment that ends at some point U on the lower boundary (e.g. along an extremal tangent). We want to find a function $\widetilde{\varphi}$ that generates the resulting curve $\widetilde{\gamma}$. Set $\widetilde{\varphi}(s) = \varphi(s)$ for $s \in I$ and consider the case $s > r$. Since three points $\tilde{\gamma}(r)$, $\tilde{\gamma}(s)$, and U lie on a single line, we have

$$\frac{u^2 - \widetilde{\gamma}_2(s)}{u - \widetilde{\gamma}_1(s)} = \frac{u^2 - \gamma_2(r)}{u - \gamma_1(r)},$$

i.e.

$$\big(u - \gamma_1(r)\big)\Bigg((s-l)u^2 - \int\limits_l^s \widetilde{\varphi}^2\Bigg) = \big(u^2 - \gamma_2(r)\big)\Bigg((s-l)u - \int\limits_l^s \widetilde{\varphi}\Bigg).$$

Differentiating this identity with respect to s, we obtain the quadratic equation on $\widetilde{\varphi}(s)$:

$$\big(u-\gamma_1(r)\big)\big(u^2 - \widetilde{\varphi}^2(s)\big) = \big(u^2 - \gamma_2(r)\big)\big(u - \widetilde{\varphi}(s)\big).$$

We will see that its solution $\widetilde{\varphi}(s) = u$, $s > r$, is suitable for us. The second solution corresponds to the reverse motion along the straight line containing the segment $[\gamma(r), U]$.

We prove the following general proposition.

Proposition 3.5. *Let γ be a convex left delivery curve that is generated by a test function φ defined on $I=[l,r]$. We draw a straight-line segment from the point $\gamma(r)$ to some point U on the lower boundary with preservation of the convexity. Suppose B is linear on the segment $[\gamma(r),U]$ and the line containing this segment lies below the upper boundary. Then we can continue γ up to any point inside $[\gamma(r),U]$ so* *that the resulting curve* $\widetilde{\gamma}$ *will also be a left delivery curve. In this case, the curve* $\widetilde{\gamma}$ *is generated by the test function*

$$\widetilde{\varphi}(s) = \begin{cases} \varphi(s), & s \in I; \\ u, & s \in [r, \widetilde{r}]. \end{cases}$$

Proof. We must verify (2.6) and (2.7) for $\widetilde{\varphi}$, $\widetilde{\gamma}$, and B. We must also make sure that $\widetilde{\varphi} \in \mathrm{BMO}_\varepsilon([l, \widetilde{r}])$.

Let $s \in (r, \widetilde{r}]$. For such s, we verify that the points of the curve $\widetilde{\gamma}(s) = \big(\langle \widetilde{\varphi} \rangle_{[l,s]}, \langle \widetilde{\varphi}^2 \rangle_{[l,s]}\big)$ get into $[\gamma(r), U]$. We also check that we can reach any point inside $[\gamma(r), U]$ provided $\widetilde{r}$ is sufficiently large. Indeed, we have the identity

$$\int_l^s \widetilde{\varphi}^k(t)\, dt = \int_l^r \varphi^k(t)\, dt + (s-r)u^k, \quad \text{for} \quad k = 1, 2,$$

which implies the representation

$$\widetilde{\gamma}(s) = \alpha_- \gamma(r) + \alpha_+ U, \quad \text{where} \quad \alpha_- = \frac{r-l}{s-l} \quad \text{and} \quad \alpha_+ = \frac{s-r}{s-l}. \tag{3.27}$$

Thus, we have proved that $\widetilde{\gamma}$ and $\widetilde{\varphi}$ are related by (2.6).

The fact that $\widetilde{\varphi}$ belongs to $\mathrm{BMO}_\varepsilon([l, \widetilde{r}])$ follows from the geometric Lemma 2.10.

It remains to verify equation (2.7). Using the linearity of B on $[\gamma(r), U]$ and representation (3.27), we obtain

$$\begin{aligned} B(\widetilde{\gamma}(s)) &= \alpha_- B(\gamma(r)) + \alpha_+ B(U) \\ &= \frac{r-l}{s-l} \langle f(\varphi) \rangle_{[l,r]} + \frac{s-r}{s-l} f(u) \\ &= \frac{1}{s-l} \Big(\int_l^r f(\varphi(t))\, dt + \int_r^s f(u)\, dt \Big) \\ &= \langle f(\widetilde{\varphi}) \rangle_{[l,s]}. \end{aligned}$$

The proposition is proved. □

Similarly, we can prove a symmetric statement for right delivery curves.

Proposition 3.6. *Let* γ *be a convex right delivery curve. Then it can be continued by a straight-line segment on which* B *is linear. This can be done in the manner that is symmetric to that described in Proposition* 3.5.

Applying Propositions 3.3 and 3.5 for the case $\Omega_{\mathrm{R}}(u_1, u_2)$ or Propositions 3.4 and 3.6 for the case $\Omega_{\mathrm{L}}(u_1, u_2)$, we can continue delivery curves from entry nodes up to any points of these domains, except the points on the lower boundary. But for each point U on the lower boundary, we can take the optimizer φ to be equal to u on the whole interval I because of the boundary condition (although it is clear without any optimizers that $\boldsymbol{B}_\varepsilon$ and B coincide on the lower boundary).

It is worth mentioning that we only continue delivery curves already constructed, but do not build new ones; i.e. we require some information from the left neighbor of Ω_{R} or from the right neighbor of Ω_{L}. In [18], the domain $\Omega_{\mathrm{R}}(u_1, u_2)$ with $u_1 \neq -\infty$ was called *left-incomplete*, and the domain $\Omega_{\mathrm{L}}(u_1, u_2)$ with $u_2 \neq +\infty$ was called *right-incomplete*.

Unbounded domains. Discuss domains $\Omega_{\mathrm{R}}(-\infty, u_2)$ and $\Omega_{\mathrm{L}}(u_1, +\infty)$ unbounded on one side. As usual, we treat in detail only $\Omega_{\mathrm{R}}(-\infty, u_2)$ and left delivery curves in it. It turns out that we can draw a left delivery curve from $-\infty$ to every point of this domain (except the points on the lower boundary). At this time, we do not need any extra information.

Consider some curve $\gamma = (\gamma_1, \gamma_2)$ that runs along the upper parabola from $-\infty$ up to some point $W = (w, w^2 + \varepsilon^2)$. According to the arguments preceding Proposition 3.3, such a curve is generated by the function

$$\varphi(s) = \varepsilon \log(s - l) + c + \varepsilon,$$

defined on $I = [l, r]$, and

$$\gamma_1(s) = \varepsilon \log(s - l) + c.$$

We set $I = [l, r] = [0, 1]$ and calculate c:

$$w = \gamma_1(1) = c.$$

As usual, the fact that φ lies in $\mathrm{BMO}_\varepsilon(I)$ follows from Lemma 2.10. In order to prove equation (2.7), we must repeat the corresponding reasoning from the proof of Proposition 3.3. But now we must integrate from $-\infty$, and the constant A is equal to zero. We get the following statement.

Proposition 3.7. *Consider a subdomain $\Omega_{\mathrm{R}}(-\infty, u_2)$ foliated by the right tangents. If a point $W = (w, w^2 + \varepsilon^2)$ of the upper parabola lies in this subdomain, then we can construct a left delivery curve running along the upper parabola from $-\infty$ to W. Such a curve is generated by the test function*

$$\varphi(s) = \varepsilon \log s + w + \varepsilon, \quad s \in [0, 1].$$

For $\Omega_{\mathrm{L}}(u_1, +\infty)$, we can formulate a symmetric proposition about right delivery curves running along the upper parabola.

Proposition 3.8. *In $\Omega_{\mathrm{L}}(u_1, +\infty)$, we can construct a right delivery curve running along the upper parabola from $+\infty$ to a point of the upper boundary.*

Concerning the points of $\Omega_{\mathrm{R}}(-\infty, u_2)$ and $\Omega_{\mathrm{L}}(u_1, +\infty)$ not lying on the upper boundary, we can continue our delivery curves up to them using Propositions 3.5 and 3.6. Thus, we have obtained the optimizers for all the points of this domain. In [18], if no extra information from neighbors was required for a domain, it was called *complete*.

3.4. **Function f''' does not change its sign.** It is stated in Proposition 3.2 that the function $B^{\mathrm{R}}(x; -\infty, +\infty)$, defined by (3.20), is a Bellman candidate in the whole domain Ω_ε provided f satisfies some integral condition. Thus, from Statement 2.2, it follows that $\boldsymbol{B}_\varepsilon \le B^{\mathrm{R}}$. On the other hand, we have constructed (see the previous section) optimizers for all the points of the domain $\Omega_{\mathrm{R}}(-\infty, +\infty) = \Omega_\varepsilon$. This gives us the converse inequality $\boldsymbol{B}_\varepsilon \ge B^{\mathrm{R}}$. We come to the following theorem.

Theorem 3.9. *Suppose $0 < \varepsilon < \varepsilon_0$, $f \in \mathfrak{W}_{\varepsilon_0}$, and*

$$\int_{-\infty}^{u} f'''(t) e^{t/\varepsilon}\, dt \le 0, \quad \forall u \in \mathbb{R}.$$

Then $\boldsymbol{B}_\varepsilon(x; f) = B^{\mathrm{R}}(x; -\infty, +\infty)$.

Using the second part of Proposition 3.2 and the optimizers constructed in the previous section, we get the symmetric theorem.

Theorem 3.10. *Suppose* $0 < \varepsilon < \varepsilon_0$, $f \in \mathfrak{W}_{\varepsilon_0}$, *and*

$$\int_u^{+\infty} f'''(t)e^{-t/\varepsilon}\,dt \geq 0, \quad \forall u \in \mathbb{R}.$$

Then $\boldsymbol{B}_\varepsilon(x;\, f) = B^{\mathrm{L}}(x;\, -\infty, +\infty)$.

Obviously, these theorems treat the case where f''' has one and the same sign a.e. on $\mathbb{R}$.

Corollary 3.11. *Suppose* $0 < \varepsilon < \varepsilon_0$ *and* $f \in \mathfrak{W}^0_{\varepsilon_0}$. *If* $c_0 = -\infty$, *then*

$$\boldsymbol{B}_\varepsilon(x;\, f) = B^{\mathrm{R}}(x;\, -\infty, +\infty),$$

and if $c_0 = +\infty$, *then*

$$\boldsymbol{B}_\varepsilon(x;\, f) = B^{\mathrm{L}}(x;\, -\infty, +\infty).$$

Example: *the exponential function.* The Bellman functions for $f(t) = \pm e^t$ were already constructed in [17], but we can calculate them using our general theorems. Note that the function $f(t) = e^t$ belongs to $\mathfrak{W}_{\varepsilon_0}$ only if $\varepsilon_0 < 1$. Therefore, all the further formulas are reasonable only for $\varepsilon < 1$. Applying Corollary 3.11 and calculating, we have

$$\boldsymbol{B}_\varepsilon(x_1, x_2;\, e^t) = \frac{1 - \sqrt{x_1^2 - x_2 + \varepsilon^2}}{1 - \varepsilon}\, e^u,$$

where u is defined by (3.3). Similarly,

$$\boldsymbol{B}_\varepsilon(x_1, x_2;\, -e^t) = -\frac{1 + \sqrt{x_1^2 - x_2 + \varepsilon^2}}{1 + \varepsilon}\, e^u,$$

where u is defined by (3.2).

4. Transition from right tangents to left ones

In this section, we treat the case when there are two domains of left and right tangents simultaneously. There is also a triangle domain between them, where our Bellman candidate is linear. The reader can glance at Figure 7 to understand what is meant.

4.1. **Angle.** Let $u_1 < v < u_2$. Consider two subdomains $\Omega_{\mathrm{R}}(u_1, v)$ and $\Omega_{\mathrm{L}}(v, u_2)$ foliated by extremals (R) and (L), respectively. We can see a subdomain in the form of *an angle* lying between Ω_{R} and Ω_{L}. It is bounded by the upper parabola and by the right and left tangents coming from the point $V = (v, v^2)$ (see Figure 7):

$$\Omega_{\mathrm{ang}}(v) \stackrel{\mathrm{def}}{=} \bigl\{x \in \mathbb{R}^2 \mid v - \varepsilon \leq x_1 \leq v + \varepsilon,\ 2vx_1 - v^2 + 2\varepsilon|v - x_1| \leq x_2 \leq x_1^2 + \varepsilon^2\bigr\}.$$

Now we construct a Bellman candidate in the subdomain

$$\Omega_{\mathrm{RL}}(u_1, v, u_2) \stackrel{\mathrm{def}}{=} \Omega_{\mathrm{R}}(u_1, v) \cup \Omega_{\mathrm{ang}}(v) \cup \Omega_{\mathrm{L}}(v, u_2).$$

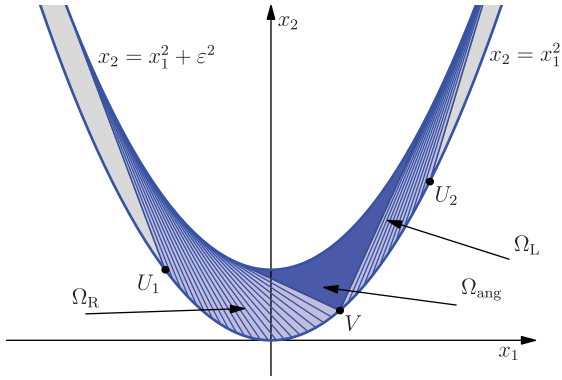

FIGURE 7. Angle Ω_{ang} between Ω_{R} and Ω_{L}.

We denote this candidate by $B^{\mathrm{RL}}(x;\, u_1, v, u_2)$. The candidates in Ω_{R} and Ω_{L} have been constructed already:

$$\begin{aligned} B^{\mathrm{RL}}(x;\, u_1, v, u_2) &= B^{\mathrm{R}}(x;\, u_1, v) \qquad \text{if} \quad x \in \Omega_{\mathrm{R}}(u_1, v);\\ B^{\mathrm{RL}}(x;\, u_1, v, u_2) &= B^{\mathrm{L}}(x;\, v, u_2) \qquad \text{if} \quad x \in \Omega_{\mathrm{L}}(v, u_2). \end{aligned}$$

We recall that B^{R} and B^{L} are, in fact, families of functions. Concerning the domain $\Omega_{\mathrm{ang}}(v)$, the function we are looking for must be linear on it. Therefore, if $x \in \Omega_{\mathrm{ang}}(v)$, then

$$B^{\mathrm{RL}}(x;\, u_1, v, u_2) = B^{\mathrm{ang}}(x;\, v) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_0.$$

Calculating the values of B^{ang} in the vertices of the angle $\Omega_{\mathrm{ang}}(v)$, we have

$$\begin{cases} \alpha_1 v + \alpha_2 v^2 + \alpha_0 = f(v);\\ \alpha_1 (v-\varepsilon) + \alpha_2\big((v-\varepsilon)^2 + \varepsilon^2\big) + \alpha_0 = -m_{\mathrm{R}}(v)\varepsilon + f(v);\\ \alpha_1 (v+\varepsilon) + \alpha_2\big((v+\varepsilon)^2 + \varepsilon^2\big) + \alpha_0 = m_{\mathrm{L}}(v)\varepsilon + f(v). \end{cases}$$

Solving this system, we obtain

$$\begin{aligned} \alpha_1 &= \frac{m_{\mathrm{R}}(v) + m_{\mathrm{L}}(v)}{2} - \frac{m_{\mathrm{L}}(v) - m_{\mathrm{R}}(v)}{2\varepsilon}\, v;\\ \alpha_2 &= \frac{m_{\mathrm{L}}(v) - m_{\mathrm{R}}(v)}{4\varepsilon};\\ \alpha_0 &= \frac{m_{\mathrm{L}}(v) - m_{\mathrm{R}}(v)}{4\varepsilon}\, v^2 - \frac{m_{\mathrm{R}}(v) + m_{\mathrm{L}}(v)}{2}\, v + f(v). \end{aligned} \tag{4.1}$$

Now we discuss the concavity of B^{RL}. As has already been verified, the local concavity of $B^{\mathrm{R}}(x;\, u_1, v)$ and $B^{\mathrm{L}}(x;\, v, u_2)$ is equivalent, respectively, to the inequalities

$$\begin{aligned} m_{\mathrm{R}}''(u) &\le 0 \quad \text{for} \quad u \in (u_1, v);\\ m_{\mathrm{L}}''(u) &\ge 0 \quad \text{for} \quad u \in (v, u_2). \end{aligned} \tag{4.2}$$

Suppose these inequalities are fulfilled. We want to obtain some conditions on v that are necessary and sufficient for the concatenation of B^{R}, B^{ang}, and B^{L} to be locally concave. In order for the function B^{RL} to be concave along the direction x_2, its derivative $B^{\mathrm{RL}}_{x_2}$ must be monotonically decreasing in x_2. Therefore, the jumps of $B^{\mathrm{RL}}_{x_2}$ on the boundary of Ω_{ang} must be non-positive. They are

$$\delta_{\mathrm{R}} = \alpha_2 - \lim_{u\to v-} t_2(u) \quad \text{and} \quad \delta_{\mathrm{L}} = \alpha_2 - \lim_{u\to v+} t_2(u).$$

Using (3.6) and (3.7), we obtain

$$\lim_{u\to v-} t_2(u) = \frac{m'_{\mathrm{R}}(v)}{2} = \frac{f'(v) - m_{\mathrm{R}}(v)}{2\varepsilon},$$

and due to (3.10) and (3.11), we have

$$\lim_{u\to v+} t_2(u) = \frac{m'_{\mathrm{L}}(v)}{2} = \frac{m_{\mathrm{L}}(v) - f'(v)}{2\varepsilon}.$$

Now, using formula (4.1) for α_2, we get the expressions for the jumps:

$$\begin{aligned} \delta_{\mathrm{R}} &= \frac{1}{4\varepsilon}\big(m_{\mathrm{R}}(v) + m_{\mathrm{L}}(v) - 2f'(v)\big); \\ \delta_{\mathrm{L}} &= -\frac{1}{4\varepsilon}\big(m_{\mathrm{R}}(v) + m_{\mathrm{L}}(v) - 2f'(v)\big). \end{aligned}$$

We see that their signs are always different. On the other hand, both jumps are non-positive and, therefore, are equal to zero. Thus, the condition

$$m_{\mathrm{R}}(v) + m_{\mathrm{L}}(v) = 2f'(v)$$

is necessary for the function B^{RL} to be locally concave. Thus, if our concatenation is locally concave, then its derivative $B^{\mathrm{RL}}_{x_2}$ must be continuous. The partial derivatives of B^{RL} along the tangents bounding $\Omega_{\mathrm{ang}}(v)$ are also continuous (constant). Therefore, the function B^{RL} has continuous derivatives along two non-collinear directions, so the derivatives along all the directions are continuous. But a C^1-smooth concatenation of locally concave functions is locally concave. Hence, the condition $m_{\mathrm{R}}(v) + m_{\mathrm{L}}(v) = 2f'(v)$ is also sufficient for the local concavity of the concatenation B^{RL} provided its components B^{R}, B^{ang} and B^{L} are locally concave. Finally, by (3.6) and (3.10), the resulting condition is equivalent to the identity

$$m''_{\mathrm{R}}(v) + m''_{\mathrm{L}}(v) = 0. \tag{4.3}$$

We summarize this section.

Proposition 4.1. *Let $u_1 < v < u_2$. Consider the subdomains $\Omega_{\mathrm{R}}(u_1, v)$ and $\Omega_{\mathrm{L}}(v, u_2)$ foliated by extremals* (R) *and* (L), *respectively. We also suppose that the domain $\Omega_{\mathrm{ang}}(v)$ lying between them is a domain of linearity. Then a Bellman candidate in the union $\Omega_{\mathrm{RL}}(u_1, v, u_2)$ of these domains has the form*

$$B^{\mathrm{RL}}(x;\, u_1, v, u_2) = \begin{cases} B^{\mathrm{R}}(x;\, u_1, v), & x \in \Omega_{\mathrm{R}}(u_1, v); \\ B^{\mathrm{ang}}(x;\, v), & x \in \Omega_{\mathrm{ang}}(v); \\ B^{\mathrm{L}}(x;\, v, u_2), & x \in \Omega_{\mathrm{L}}(v, u_2), \end{cases} \tag{4.4}$$

where $B^{\mathrm{ang}}(x;\, v) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_0$, and the coefficients α_1, α_2, and α_0 are calculated by (4.1). *In addition, relations* (4.2) *and* (4.3) *must be fulfilled.*

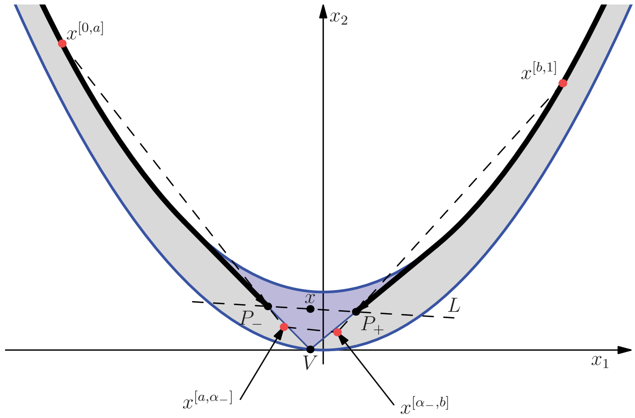

FIGURE 8. Optimizers in $\Omega_{\mathrm{ang}}(v)$.

4.2. **Optimizers in angle.** Now we construct optimizers for the points inside an angle. Suppose B is a Bellman candidate in Ω_ε and some part of Ω_ε is represented by the construction $\Omega_{\mathrm{RL}}(u_1, v, u_2)$ described in Proposition 4.1. We have already learned (see Section 3.3) how to build delivery curves and optimizers in $\Omega_{\mathrm{R}}(u_1, v)$ and $\Omega_{\mathrm{L}}(v, u_2)$. It turns out that we need information from both right and left neighbors of the angle in order to obtain optimizers for its points. To be more precise, we need two delivery curves already built: a left delivery curve γ_- that reaches some point P^- on the right boundary of $\Omega_{\mathrm{R}}(u_1, v)$, and a right delivery curve γ_+ that reaches some point P^+ on the left boundary of $\Omega_{\mathrm{L}}(v, u_2)$. If we have optimizers in two points of $\Omega_{\mathrm{ang}}(v)$, then we can construct an optimizer for any points of the segment that connects them provided this segment lies in $\Omega_{\mathrm{ang}}(v)$ entirely.

Let $x \in \Omega_{\mathrm{ang}}(v)$. We draw some straight line L that passes through x and does not intersect the upper parabola. This line intersects both sides of the angle. We denote the points of intersection by $P^\pm$. Then x will be a convex combination of $P^\pm$: $x = \alpha_- P^- + \alpha_+ P^+$, where $\alpha_\pm \geq 0$ and $\alpha_- + \alpha_+ = 1$. We build the optimizer φ_- for P^- on $I_- = [0, \alpha_-]$ and the optimizer φ_+ for P^+ on $I_+ = [\alpha_-, 1]$ (see Section 3.3). Concatenating φ_- and φ_+, we obtain the function φ on $[0,1]$. It is easy to see that φ satisfies conditions (2) and (3) of Definition 2.7. This follows immediately from the representation of x as a convex combination of $P^\pm$ and from the linearity of B in Ω_{ang}:

$$x_k = \alpha_- P_k^- + \alpha_+ P_k^+ = \int\limits_0^{\alpha_-} \varphi_-^k(s)\, ds + \int\limits_{\alpha_-}^1 \varphi_+^k(s)\, ds = \int\limits_0^1 \varphi^k(s)\, ds = \langle \varphi^k \rangle_{[0,1]} \qquad k = 1, 2;$$

$$B(x) = \alpha_- B(P^-) + \alpha_+ B(P^+) = \int\limits_0^{\alpha_-} f(\varphi_-(s))\, ds + \int\limits_{\alpha_-}^1 f(\varphi_+(s))\, ds$$

$$= \int\limits_0^1 f(\varphi(s))\, ds = \langle f(\varphi) \rangle_{[0,1]}.$$

In order to prove that φ is an optimizer for x, it remains to verify that $\varphi \in \mathrm{BMO}_\varepsilon([0,1])$. Consider some subinterval $[a,b] \subset [0,1]$ and the Bellman point $x^{[a,b]} = \big(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]}\big)$. If $\alpha_- \notin (a,b)$, then $x^{[a,b]}$ gets into Ω_ε, because $\varphi_\pm \in \mathrm{BMO}_\varepsilon(I_\pm)$. Thus, we only need to consider the intervals $[a,b]$ such that $\alpha_- \in (a,b)$. Note that $P_- = x^{[0,\alpha_-]}$ is a convex combination of $x^{[0,a]}$ and $x^{[a,\alpha_-]}$ and, therefore, lies on the segment connecting them. The point $x^{[0,a]}$ lies somewhere on the delivery curve coming from above and ending at P_- (we already know how this curve is arranged: it is a convex curve that runs along the upper parabola and then descends along the right tangent down to the point P_-). Consequently, $x^{[0,a]}$ lies above L, and so $x^{[a,\alpha_-]}$ lies below L. Similarly, we can verify that $x^{[\alpha_-,b]}$ lies below L. But the point $x^{[a,b]}$ is a convex combination of $x^{[a,\alpha_-]}$ and $x^{[\alpha_-,b]}$. Therefore, it lies below L and, consequently, in Ω_ε. As a result, we have constructed optimizers φ for all the points x in $\Omega_{\mathrm{ang}}(v)$.

4.3. **Function f''' changes its sign from minus to plus.** Propositions 3.2 and 4.1, together with the existence of optimizers in the domains $\Omega_{\mathrm{R}}(-\infty, v)$, $\Omega_{\mathrm{L}}(v,+\infty)$, and $\Omega_{\mathrm{ang}}(v)$, imply the following theorem.

Theorem 4.2. *Let $0<\varepsilon<\varepsilon_0$ and $f \in \mathfrak{W}_{\varepsilon_0}$. Suppose there exists $v \in \mathbb{R}$ such that*

$$\begin{aligned} &m''_{\mathrm{R}}(u;\,-\infty) \le 0 \quad \textit{for} \quad u \in (-\infty, v);\\ &m''_{\mathrm{L}}(u;\,+\infty) \ge 0 \quad \textit{for} \quad u \in (v, +\infty);\\ &m''_{\mathrm{R}}(v;\,-\infty) + m''_{\mathrm{L}}(v;\,+\infty) = 0, \end{aligned}$$

where $m''_{\mathrm{R}}(u;\,-\infty)$ and $m''_{\mathrm{L}}(u;\,+\infty)$ are expressed by (3.17) and (3.19). Then

$$\boldsymbol{B}_\varepsilon(x;\,f) = B^{\mathrm{RL}}(x;\,-\infty, v, +\infty),$$

where the function on the right hand side is defined by (4.4) and its parts $B^{\mathrm{R}}(x;\,-\infty,v)$ and $B^{\mathrm{L}}(x;\,v,+\infty)$ are defined by (3.20) and (3.21).

It turns out that the conditions of Theorem 4.2 can be satisfied if f''' changes its sign from minus to plus.

Theorem 4.3. *Let $0<\varepsilon<\varepsilon_0$ and $f \in \mathfrak{W}^1_{\varepsilon_0}$ with $c_0=-\infty$, $c_1=+\infty$. We denote*

$$g_\varepsilon(u) \stackrel{\mathrm{def}}{=} (f''' * w_\varepsilon)(u),$$

where $w_\varepsilon(t) = e^{-|t|/\varepsilon}$. The function g_ε is continuous, and

1) *if $g_\varepsilon < 0$ on $\mathbb{R}$, then the conditions of Theorem 3.9 are satisfied;*
2) *if $g_\varepsilon > 0$ on $\mathbb{R}$, then the conditions of Theorem 3.10 are satisfied;*
3) *if $g_\varepsilon(v) = 0$ for some $v \in \mathbb{R}$, then the conditions of Theorem 4.2 are satisfied.*

Proof. First, we note that

$$\varepsilon^{-1} g_\varepsilon(u) = m''_{\mathrm{R}}(u;\,-\infty) + m''_{\mathrm{L}}(u;\,+\infty),$$

where $m''_{\mathrm{R}}(u;\,-\infty)$ and $m''_{\mathrm{L}}(u;\,+\infty)$ are given by (3.17) and (3.19).

Recall the point where f''' changing its sign was denoted by v_1. Consider case 1). It is clear that for $u \le v_1$ the inequality $m''_{\mathrm{R}}(u;\,-\infty) < 0$ is always fulfilled, because $f'''(u) < 0$ a.e. on $(-\infty, v_1)$. On the other hand, for $u \ge v_1$ we use the condition $g_\varepsilon < 0$:

$$m''_{\mathrm{R}}(u;\,-\infty) = \varepsilon^{-1} g_\varepsilon - m''_{\mathrm{L}}(u;\,+\infty) < -m''_{\mathrm{L}}(u;\,+\infty).$$

Since $f'''(u) > 0$ a.e. for $u \ge v_1$, the inequality $-m''_{\mathrm{L}}(u;\, +\infty) < 0$ is valid. Thus, we see that $m''_{\mathrm{R}}(u;\, -\infty) < 0$ for all $u \in \mathbb{R}$, and the conditions of Theorem 3.9 are fulfilled. Case 2) can be treated similarly.

Finally, we consider case 3). We treat only the case $v \ge v_1$ (the case $v \le v_1$ can be treated similarly). The sign of f''' is known, and so $m''_{\mathrm{L}}(u;\, +\infty) > 0$ for $u \ge v$ (this is one of the conditions of Theorem 4.2). We also know that $m''_{\mathrm{R}}(u;\, -\infty) < 0$ for $u \le v_1$. Thus, it remains to verify that $m''_{\mathrm{R}}(u;\, -\infty) < 0$ for $u \in (v_1, v)$. On the one hand, we have

$$m''_{\mathrm{R}}(v;\, -\infty) = \varepsilon^{-1} g_\varepsilon(v) - m''_{\mathrm{L}}(v;\, +\infty) = -m''_{\mathrm{L}}(v;\, +\infty) < 0.$$

On the other hand, the function $e^{u/\varepsilon}\, m''_{\mathrm{R}}(u;\, -\infty)$ increases monotonically on (v_1, v), because

$$e^{u/\varepsilon}\, m''_{\mathrm{R}}(u;\, -\infty) = \varepsilon^{-1} \int\limits_{-\infty}^{u} f'''(t) e^{t/\varepsilon}\, dt$$

and f''' is positive on this interval. Consequently, $e^{u/\varepsilon}\, m''_{\mathrm{R}}(u;\, -\infty)$ is negative for all $u \in (v_1, v)$. As a result, all the conditions of Theorem 4.2 are fulfilled. □

Example: *the power function.* The function $f(t) = |t|^p$ was treated in [18]. For $p > 2$, it gets into the class being considered: $f \in \mathfrak{W}^1_{\varepsilon_0}$ with $c_0 = -\infty$, $c_1 = +\infty$. Here, we do not write an explicit expression for the Bellman function, but merely verify that the conditions of case 3) in Theorem 4.3 are satisfied. Indeed, the expression

$$\begin{aligned} \varepsilon^{-1} g_\varepsilon(u) &= m''_{\mathrm{R}}(u;\, -\infty) + m''_{\mathrm{L}}(u;\, +\infty) \\ &= \varepsilon^{-1} \int\limits_{-\infty}^{\infty} \operatorname{sign} t \cdot p(p-1)(p-2)|t|^{p-3} e^{-|u-t|/\varepsilon}\, dt \end{aligned}$$

has the unique root $u = 0$. Therefore, the vertex of the angle has coordinates $(0, 0)$ for any $\varepsilon \in (0, \infty)$; i.e. it does not depend on ε.

5. Transition from left tangents to right ones

In this section, we consider a transition from left tangents to right ones. Such a transition is performed through a subdomain foliated by extremal chords whose endpoints lie on the lower parabola. The reader can look at Figure 10 to understand what is meant.

Before continuing, we recall our agreement on the notation. If a point on the lower boundary is denoted by a capital Latin letter, then the corresponding small letter denotes the first coordinate of this point (and vice versa). Throughout this section, we use this rule very often.

5.1. **Family of chords.** Let A_0, A_1, B_1, and B_0 be four points on the lower boundary of Ω_ε with the abscissas a_0, a_1, b_1, and b_0 such that $a_0 < a_1 < b_1 < b_0$ and $b_0 - a_0 \le 2\varepsilon$. We draw two segments $[A_0, B_0]$ and $[A_1, B_1]$. We consider the subdomain bounded by these segments and two arcs of the lower parabola: the one connects A_0 and A_1 and the other connects B_1 and B_0. Suppose this subdomain is foliated entirely by a family of non-intersecting chords with the endpoints lying on the different arcs of the lower parabola. We denote such a subdomain by $\Omega_{\mathrm{ch}}(a_0, b_0, a_1, b_1)$ (see Figure 9). We see that for any point x in $\Omega_{\mathrm{ch}}(a_0, b_0, a_1, b_1)$

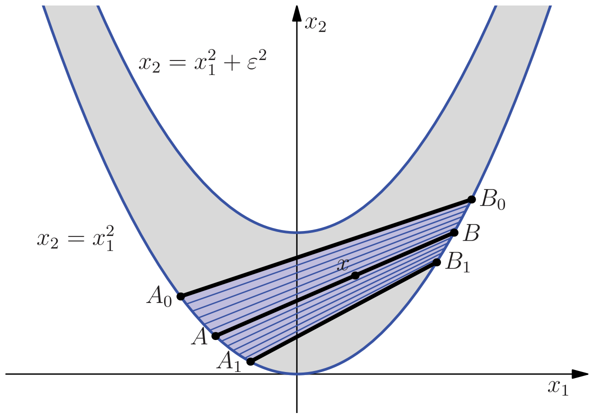

FIGURE 9. A domain Ω_{ch} with the chords.

there are two numbers $a \in [a_0, a_1]$ and $b \in [b_1, b_0]$ such that the chord $[A, B]$ belongs to our family and contains x. We want to construct a Bellman candidate in $\Omega_{\mathrm{ch}}(a_0, b_0, a_1, b_1)$ whose partial derivatives are constant along the chords in our family. What is more, we derive some conditions on the chords that allow such a candidate to exist at all. We denote the function required by $B^{\mathrm{ch}}(x; a_0, b_0, a_1, b_1)$ (sometimes we write $B^{\mathrm{ch}}(x)$ for short).

First, we note that the principal difference between the cases of extremal chords and extremal tangents lies in the fact that using the linearity along the chords, we can restore B^{ch} in Ω_{ch} uniquely. Indeed, if we know that $B^{\mathrm{ch}}(A) = f(a)$, $B^{\mathrm{ch}}(B) = f(b)$, and B^{ch} is linear along the chord $[A, B]$, then we can calculate the value of B^{ch} at any point x lying on this chord:

$$B^{\mathrm{ch}}(x) = \frac{f(b) - f(a)}{b - a} x_1 + \frac{b f(a) - a f(b)}{b - a}. \tag{5.1}$$

However, the function B^{ch} built in this way is a Bellman candidate only if its derivatives $B^{\mathrm{ch}}_{x_1}$ and $B^{\mathrm{ch}}_{x_2}$ are constant along the extremals. We will get some condition on the chords that guarantees the constancy of $B^{\mathrm{ch}}_{x_1}$ and $B^{\mathrm{ch}}_{x_2}$ on them.

We parametrize our chords $[A, B]$ by the values $\ell = b - a$. We assume that the functions a and b are differentiable and the inequalities $a' < 0$ and $b' > 0$ are fulfilled. The last requirement implies that the chords $[A, B]$ do not intersect. Domains foliated by extremal chords that share a common point on the boundary can arise if the boundary function f is not smooth enough (see [18] or [23]). We will not encounter such domains due to our assumptions on the smoothness of f. In its turn, ℓ can be treated as a function of $x \in \Omega_{\mathrm{ch}}$; i.e. we consider the function $\ell(x)$. For short, we often omit the arguments of the functions a, b, and ℓ. We write the equation of the line passing through points A and B:

$$x_2 = (a + b)x_1 - ab.$$

Now we calculate ℓ_{x_2}. By the last relation, if x_1 is fixed, then x_2 is a differentiable function of ℓ with

$$x_2' = x_1(a' + b') - (ab' + ba').$$

But x_1 takes values from a to b. Therefore, x_2' runs between $(a - b)a'$ and $(b - a)b'$. Each of this two values is greater than zero, and so $x_2'(\ell) > 0$. Consequently, the inverse function ℓ is differentiable in x_2, and

$$\ell_{x_2}=\frac{1}{x_1(a'+b')-(ab'+ba')}. \tag{5.2}$$

We are ready to calculate the partial derivatives of $B^{\rm ch}$. As we have already mentioned, we are searching for a condition on the chords under which $B^{\rm ch}_{x_1}$ and $B^{\rm ch}_{x_2}$ are constant along them. Since $B^{\rm ch}$ is linear along the chords, it is sufficient to obtain a condition that guarantees the constancy of $B^{\rm ch}_{x_2}$ along them. Differentiating identity (5.1) in x_2, we get

$$B^{\rm ch}_{x_2}(x_1,x_2)=\frac{\alpha x_1+\beta}{(b-a)^2}\,\ell_{x_2}, \tag{5.3}$$

where

$$\begin{aligned}\alpha =&\big(f'(b)b'-f'(a)a'\big)(b-a)-\big(f(b)-f(a)\big)(b'-a');\\ \beta =&\big(b'f(a)+bf'(a)a'-a'f(b)-af'(b)b'\big)(b-a)\\ &-\big(bf(a)-af(b)\big)(b'-a'),\end{aligned}$$

and ℓ_{x_2} is given by (5.2). Since $B^{\rm ch}_{x_2}$ is constant along the chords, it does not depend on x_1 if ℓ is fixed. But if the quotient of two linear functions does not depend on the variable, then their coefficients must be proportional, i.e.

$$\alpha(ab'+ba')=-\beta(a'+b').$$

Substituting the corresponding expressions for α and β, we obtain, after elementary calculations, the equivalent identity:

$$a'b'\bigg(\frac{f'(a)+f'(b)}{2}-\frac{f(b)-f(a)}{b-a}\bigg)=0.$$

Dividing by $a'b'$, we have

$$\langle f'\rangle_{[a,b]}=\frac{f'(a)+f'(b)}{2}. \tag{5.4}$$

Thus, under the assumption $a'b'\neq 0$, the derivatives of $B^{\rm ch}$ are constant on the chords $[A,B]$ if and only if their ends satisfy equation (5.4).

Now we turn to the concavity of the function $B^{\rm ch}$ constructed above. We note that at each point of $\Omega_{\rm ch}$, our function is linear in one direction. Therefore, as in the case of extremal tangents discussed in the previous section, it is sufficient to verify the concavity along some other direction. Since the direction x_2 always differs from the direction of chords, it is enough to study the sign of $B^{\rm ch}_{x_2x_2}$. First, using (5.4), we simplify formula (5.3) for $B^{\rm ch}_{x_2}$. Since the expression for $B^{\rm ch}_{x_2}$ does not depend on x_1, we have

$$\begin{aligned}B^{\rm ch}_{x_2}(x_1,x_2)&=\frac{\big(f'(b)b'-f'(a)a'\big)(b-a)-\big(f(b)-f(a)\big)(b'-a')}{(a'+b')(b-a)^2}\\ &=\frac{2f'(b)b'-2f'(a)a'-\big(f'(b)+f'(a)\big)(b'-a')}{2(a'+b')(b-a)}\\ &=\frac{f'(b)-f'(a)}{2(b-a)}.\end{aligned}$$

Since ℓ strictly increases as x_2 grows (this is obvious by the geometric considerations, but the formal proof can be found in the derivation of (5.2)), it is sufficient to study the sign of $B^{\mathrm{ch}}_{x_2\ell}$. By direct calculations, we have

$$
\begin{aligned}
2B^{\mathrm{ch}}_{x_2\ell} &= \frac{f''(b)b' - f''(a)a'}{b-a} - \frac{f'(b)-f'(a)}{(b-a)^2}(b'-a') \\
&= \frac{b'\big(f''(b) - \langle f''\rangle_{[a,b]}\big) - a'\big(f''(a) - \langle f''\rangle_{[a,b]}\big)}{b-a}\,.
\end{aligned}
\tag{5.5}
$$

On the other hand, differentiating equation (5.4) with respect to ℓ, we get

$$
b'\big(f''(b) - \langle f''\rangle_{[a,b]}\big) + a'\big(f''(a) - \langle f''\rangle_{[a,b]}\big) = 0\,. \tag{5.6}
$$

We introduce the following notation:

$$
D_{\mathrm{L}}(a,b) = f''(a) - \langle f''\rangle_{[a,b]} \quad \text{and} \quad D_{\mathrm{R}}(a,b) = f''(b) - \langle f''\rangle_{[a,b]}\,. \tag{5.7}
$$

Equation (5.6), together with the inequalities $b' > 0$ and $a' < 0$, implies that $D_{\mathrm{L}}(a,b)$ and $D_{\mathrm{R}}(a,b)$ have the same sign for every chord $[A,B]$. Thus, by virtue of (5.5), we see that $B^{\mathrm{ch}}_{x_2\ell} \le 0$ if and only if either $D_{\mathrm{L}}(a,b) \le 0$ or $D_{\mathrm{R}}(a,b) \le 0$. What is more, each of these two inequalities implies the other.

We summarize this section in the following proposition.

Proposition 5.1. *Consider a domain $\Omega_{\mathrm{ch}}(a_0,b_0,a_1,b_1)$ foliated entirely by non-intersecting chords $[A,B]$, and parametrize the first coordinates a and b of their endpoints by $\ell = b-a$. Suppose a and b are differentiable functions such that $a' < 0$ and $b' > 0$. Under these assumptions, we can build a function $B^{\mathrm{ch}}(x;\, a_0,b_0,a_1,b_1)$ such that its partial derivatives are constant along the chords $[A,B]$ if and only if all the chords satisfy* (5.4)*. The function B^{ch} can be calculated by* (5.1)*. Also, we have*

$$
B^{\mathrm{ch}}_{x_2}(x) = \frac{f'(b)-f'(a)}{2(b-a)} = \frac{1}{2}\langle f''\rangle_{[a,b]}\,, \tag{5.8}
$$

where a and b are the first coordinates of the endpoints of the chord $[A,B]$ passing through x.

The function B^{ch} is locally concave (*and, therefore, it is a Bellman candidate*) *if and only if for every chord $[A,B]$ one of the following two inequalities is fulfilled*:

$$
D_{\mathrm{L}}(a,b) \le 0 \quad \text{or} \quad D_{\mathrm{R}}(a,b) \le 0. \tag{5.9}
$$

Furthermore, each of these two inequalities implies the other one.

5.2. **Cup.** In the previous section, we dealt with subdomains $\Omega_{\mathrm{ch}}(a_0,b_0,a_1,b_1)$ lying between two chords $[A_0,B_0]$ and $[A_1,B_1]$ in Ω_ε. Now we consider a subdomain arising in the case $a_1 = b_1$.

Definition 5.2. Let $0 \le b_0 - a_0 \le 2\varepsilon$. Consider the subdomain of Ω_ε that lies between $[A_0,B_0]$ and the lower parabola. Suppose there exists a family of non-intersecting chords that foliate this subdomain entirely and have the following properties:

1) if we parametrize the first coordinates a and b of their endpoints by $\ell = b-a$, we obtain the differentiable functions $a(\ell)$ and $b(\ell)$ such that $a' < 0$ and $b' > 0$;
2) each of these chords satisfies equation (5.4);
3) for each chord, one of two inequalities (5.9) is fulfilled.

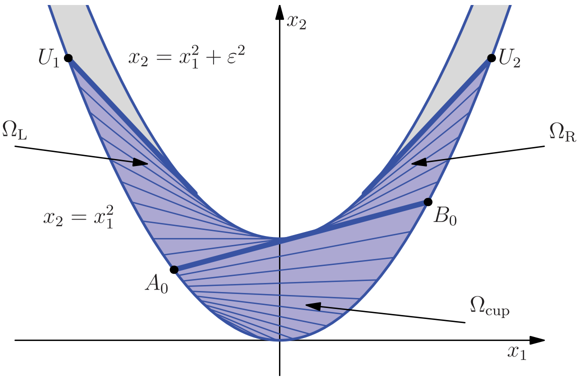

FIGURE 10. A cup Ω_{cup} between Ω_{L} and Ω_{R}.

In such a situation, we call the subdomain being considered *a cup* and denote it by $\Omega_{\mathrm{cup}}(a_0, b_0)$.

The unique point c lying in the intersection of all the intervals $[a, b]$ is called *the origin* of the cup. The points a_0 and b_0 are called *the ends* of the cup, and the value $\ell_0 = b_0 - a_0$ is called *the size* of the cup. Note that if $\ell_0 = 2\varepsilon$, the chord $[A_0, B_0]$ touches the upper parabola. In such a case, we say that the cup $\Omega_{\mathrm{cup}}(a_0, b_0)$ is *full*. Also, the case $\ell_0 = 0$ is not excluded from consideration. In this situation, the cup consists of the single point (c, c^2).

Using (5.1), we construct a function in $\Omega_{\mathrm{cup}}(a_0, b_0)$ that is linear along the chords $[A, B]$. Proposition 5.1 implies that such a function is a Bellman candidate in the cup. We denote it by $B^{\mathrm{cup}}(x; a_0, b_0)$.

Now we assume that $u_1 < a_0 < b_0 < u_2$ and $b_0 - a_0 = 2\varepsilon$. Consider a full cup $\Omega_{\mathrm{cup}}(a_0, b_0)$ together with two domains $\Omega_{\mathrm{L}}(u_1, a_0)$ and $\Omega_{\mathrm{R}}(b_0, u_2)$ adjacent to the cup and foliated by extremals (L) and (R), respectively (see Figure 10).

Consider the union

$$\Omega_{\mathrm{LR}}(u_1, [a_0, b_0], u_2) \stackrel{\mathrm{def}}{=} \Omega_{\mathrm{L}}(u_1, a_0) \cup \Omega_{\mathrm{cup}}(a_0, b_0) \cup \Omega_{\mathrm{R}}(b_0, u_2).$$

In this domain, we are looking for a function such that its partial derivatives are constant along the chords in Ω_{cup} and, respectively, along the corresponding tangents in Ω_{R} and Ω_{L}. Denote the function being sought by $B^{\mathrm{LR}}(x; u_1, [a_0, b_0], u_2)$. In Ω_{cup} it must coincide with B^{cup}. Concerning the subdomains Ω_{L} and Ω_{R}, the corresponding functions B^{L} and B^{R} are calculated by formulas (3.15) and (3.14), where the functions m_{L} and m_{R} are not defined uniquely: we have the freedom to choose the values $m_{\mathrm{L}}(a_0)$ and $m_{\mathrm{R}}(b_0)$ (see (3.12) and (3.8)). But in the situation being considered, there is only one way to choose $m_{\mathrm{L}}(a_0)$ and $m_{\mathrm{R}}(b_0)$ so that the corresponding functions B^{L} and B^{R} glue with B^{cup} continuously. Indeed, on the chord with ends a_0 and b_0, the function B^{cup} can be calculated by the formula

$$B^{\mathrm{cup}}\big(x_1, (a_0 + b_0)x_1 - a_0 b_0\big) = \frac{f(b_0) - f(a_0)}{b_0 - a_0}(x_1 - a_0) + f(a_0).$$

On the other hand, by (3.15) the limit values of B^{L} on this chord are equal to $m_{\mathrm{L}}(a_0)(x_1-a_0)+f(a_0)$. Therefore, the identity

$$m_{\mathrm{L}}(a_0)=\frac{f(b_0)-f(a_0)}{b_0-a_0}=\langle f'\rangle_{[a_0,b_0]}$$

is necessary and sufficient for the concatenation of B^{L} and B^{cup} to be continuous. Using cup equation (5.4), we can rewrite the equation obtained above as

$$m_{\mathrm{L}}(a_0)=\frac{f'(a_0)+f'(b_0)}{2}. \tag{5.10}$$

By $m_{\mathrm{L}}(u;\,a_0)$ denote the coefficient $m_{\mathrm{L}}(u)$ satisfying this condition. Using (3.12), we get

$$m_{\mathrm{L}}(u;\,a_0)=\frac{f'(a_0)+f'(b_0)}{2}e^{(u-a_0)/\varepsilon}+\varepsilon^{-1}e^{u/\varepsilon}\int\limits_u^{a_0} f'(t)e^{-t/\varepsilon}\,dt. \tag{5.11}$$

Thus, in $\Omega_{\mathrm{L}}(u_1,a_0)$, the function $B^{\mathrm{LR}}(x;\,u_1,[a_0,b_0],u_2)$ coincides with the function

$$B^{\mathrm{L}}(x;\,u_1,[a_0,b_0])\stackrel{\mathrm{def}}{=}m_{\mathrm{L}}(u;\,a_0)\,(x_1-u)+f(u), \tag{5.12}$$

where $u=u_{\mathrm{L}}(x_1,x_2)$ can be calculated by (3.3).

Using similar considerations, we see that the concatenation of B^{cup} and B^{R} is continuous if and only if $m_{\mathrm{R}}(u)=m_{\mathrm{R}}(u;\,b_0)$, where

$$m_{\mathrm{R}}(u;\,b_0)=\frac{f'(a_0)+f'(b_0)}{2}e^{(b_0-u)/\varepsilon}+\varepsilon^{-1}e^{-u/\varepsilon}\int\limits_{b_0}^{u} f'(t)e^{t/\varepsilon}\,dt. \tag{5.13}$$

This means that in $\Omega_{\mathrm{R}}(b_0,u_2)$ the function $B^{\mathrm{LR}}(x;\,u_1,[a_0,b_0],u_2)$ being sought must coincide with the function

$$B^{\mathrm{R}}(x;\,[a_0,b_0],u_2)\stackrel{\mathrm{def}}{=}m_{\mathrm{R}}(u;\,b_0)\,(x_1-u)+f(u), \tag{5.14}$$

where $u=u_{\mathrm{R}}(x_1,x_2)$ can be calculated by (3.2).

Before discussing the local concavity of the function $B^{\mathrm{LR}}(x;\,u_1,[a_0,b_0],u_2)$ constructed above, we show that B^{LR} is not only continuous but also C^1-smooth. Let $t_2=B^{\mathrm{LR}}_{x_2}$. We treat t_2 as a function of u in $\Omega_{\mathrm{L}}(u_1,a_0)$ and as a function of a, the left ends of the extremal chords, in $\Omega_{\mathrm{cup}}(a_0,b_0)$. Using (5.8), we obtain

$$t_2(a_0)=\frac{f'(b_0)-f'(a_0)}{2(b_0-a_0)}.$$

On the other hand, by (3.10), (3.11), and (5.10), we have

$$\lim_{u\to a_0-}t_2(u)=\frac{m_{\mathrm{L}}'(a_0;\,a_0)}{2}=\frac{m_{\mathrm{L}}(a_0;\,a_0)-f'(a_0)}{2\varepsilon}=\frac{f'(b_0)-f'(a_0)}{2(b_0-a_0)}.$$

Thus, the function $B^{\mathrm{LR}}_{x_2}$ is continuous at the junction of Ω_{L} and Ω_{cup}. Similarly, we can prove its continuity at the junction of Ω_{cup} and Ω_{R}. But the derivative of B^{LR} in the direction of the chord $[A_0,B_0]$ is also continuous (constant); i.e. on the chord just mentioned, the function B^{LR} has continuous derivatives in two non-collinear directions. Thus, the function B^{LR} turns out to be C^1-smooth. This implies that it is locally concave provided its components $B^{\mathrm{L}}(x;\,u_1,[a_0,b_0])$, $B^{\mathrm{cup}}(x;\,a_0,b_0)$, and $B^{\mathrm{R}}(x;\,[a_0,b_0],u_2)$ are locally concave. As mentioned above, the function B^{cup} is concave by the definition of a cup and Proposition 5.1. Concerning the functions B^{L} and B^{R}, they are locally concave if and only if the following inequalities are fulfilled:

$$\begin{cases} m''_{\mathrm{L}}(u;\, a_0) \geq 0 & \text{for} \quad u \in (u_1, a_0); \\ m''_{\mathrm{R}}(u;\, b_0) \leq 0 & \text{for} \quad u \in (b_0, u_2). \end{cases} \tag{5.15}$$

Now we get expressions for $m''_{\mathrm{L}}(u;\, a_0)$ and $m''_{\mathrm{R}}(u;\, b_0)$. Using equation (3.10) differentiated once, we can express m''_{L} in terms of m'_{L}. After that, using (3.10) one more time, we can express m'_{L} in terms of m_{L}. Applying these considerations to $m''_{\mathrm{L}}(a_0;\, a_0)$, we obtain

$$m''_{\mathrm{L}}(a_0;\, a_0) = \varepsilon^{-1}\big(\varepsilon^{-1} m_{\mathrm{L}}(a_0;\, a_0) - \varepsilon^{-1} f'(a_0) - f''(a_0)\big).$$

Substituting expression (5.10) for $m_{\mathrm{L}}(a_0;\, a_0)$ into this identity, we get

$$m''_{\mathrm{L}}(a_0;\, a_0) = -\varepsilon^{-1}\left[f''(a_0) - \frac{f'(b_0) - f'(a_0)}{2\varepsilon}\right] = -\varepsilon^{-1} D_{\mathrm{L}}(a_0, b_0).$$

Using (3.13), we finally have

$$m''_{\mathrm{L}}(u;\, a_0) = -\varepsilon^{-1} D_{\mathrm{L}}(a_0, b_0) e^{(u-a_0)/\varepsilon} + \varepsilon^{-1} e^{u/\varepsilon} \int_u^{a_0} f'''(t) e^{-t/\varepsilon}\, dt. \tag{5.16}$$

Similar reasoning gives the formula for $m''_{\mathrm{R}}(u;\, b_0)$:

$$m''_{\mathrm{R}}(u;\, b_0) = \varepsilon^{-1} D_{\mathrm{R}}(a_0, b_0) e^{(b_0-u)/\varepsilon} + \varepsilon^{-1} e^{-u/\varepsilon} \int_{b_0}^{u} f'''(t) e^{t/\varepsilon}\, dt. \tag{5.17}$$

As usual, we summarize this section in one proposition.

Proposition 5.3. *Suppose $b_0 - a_0 = 2\varepsilon$ and $\Omega_{\mathrm{cup}}(a_0, b_0)$ is a full cup. Consider domains $\Omega_{\mathrm{L}}(u_1, a_0)$ and $\Omega_{\mathrm{R}}(b_0, u_2)$ adjacent to $\Omega_{\mathrm{cup}}(a_0, b_0)$. A Bellman candidate in the union $\Omega_{\mathrm{LR}}(u_1, [a_0, b_0], u_2)$ has the form*

$$B^{\mathrm{LR}}(x;\, u_1, [a_0, b_0], u_2) = \begin{cases} B^{\mathrm{L}}(x;\, u_1, [a_0, b_0]), & x \in \Omega_{\mathrm{L}}(u_1, a_0); \\ B^{\mathrm{cup}}(x;\, a_0, b_0), & x \in \Omega_{\mathrm{cup}}(a_0, b_0); \\ B^{\mathrm{R}}(x;\, [a_0, b_0], u_2), & x \in \Omega_{\mathrm{R}}(b_0, u_2). \end{cases} \tag{5.18}$$

In addition, inequalities (5.15) *must be fulfilled.*

5.3. **Optimizers on chords.** We consider a domain Ω_{ch} foliated by chords (see Section 5.1). For every point $x \in \Omega_{\mathrm{ch}}$, there is a unique extremal chord $[A, B]$ passing through it. Therefore, a delivery curve coming to x can only start at A or B, because it must run along the extremal. Indeed, in the situation being considered, we have left and right delivery curves: the segments $[A, x]$ and $[x, B]$. Such curves are generated by a step function φ that can take two values: a and b. Namely, if $x = \alpha_- A + \alpha_+ B$, $\alpha_- + \alpha_+ = 1$, we set

$$\varphi(s) = \begin{cases} a, & s \in [0, \alpha_-]; \\ b, & s \in (\alpha_-, 1]. \end{cases}$$

We can see that φ is, indeed, an optimizer for x. Property (1) of Definition 2.7 follows from the fact that all the Bellman points generated by φ lie on the chord $[A, B]$. Property (2) is fulfilled by the construction of φ. Finally, property (3) follows from the linearity of the Bellman candidate along the chord $[A, B]$.

Further, it is easy to see that the curve

$$\gamma_{A}(s) = \big(\langle\varphi\rangle_{[0,s]}, \langle\varphi^2\rangle_{[0,s]}\big), \quad s \in (0,1],$$

is a left delivery curve that starts at A, runs along $[A,B]$, and ends at x. Similarly, we can define the right delivery curve γ_{B} that starts at B and ends, again, at x.

Now we consider the construction $\Omega_{\mathrm{LR}}(u_1,[a_0,b_0],u_2)$ described in Section 5.2. Let W_0 be the tangency point of the chord $[A_0,B_0]$ and the upper parabola. This point is the entry node for both domains $\Omega_{\mathrm{L}}(u_1,a_0)$ and $\Omega_{\mathrm{R}}(b_0,u_2)$. After we connect A_0 and W_0 with the left delivery curve γ_{A_0} generated by the optimizer for W_0, we can continue this curve up to every point in $\Omega_{\mathrm{R}}(b_0,u_2)$ (see Section 3.3). On the other hand, the right delivery curve γ_{B_0} that connects B_0 and W_0 can be continued up to every point in $\Omega_{\mathrm{L}}(u_1,a_0)$.

We conclude that delivery curves can originate not only at $\pm\infty$ but also in cups. Thus, we have all the information required for the construction of delivery curves in domains adjacent to cups.

5.4. **Function f''' changes its sign from plus to minus.** It turns out that the cup, together with two domains Ω_{L} and Ω_{R} adjacent to it, always arises when f''' changes its sign once, from plus to minus. We state and prove the appropriate theorem.

Theorem 5.4. *Suppose $0<\varepsilon<\varepsilon_0$, $f\in\mathfrak{W}^0_{\varepsilon_0}$, and $c_0\neq\pm\infty$. Then we can build a full cup $\Omega_{\mathrm{cup}}(a_0,b_0)$ originating at c_0. We also have*

$$\boldsymbol{B}_\varepsilon(x;\,f) = B^{\mathrm{LR}}(x;\,-\infty,[a_0,b_0],+\infty),$$

where the function B^{LR} is defined by (5.18).

First, we note that a cup is a local construction. Its existence under the conditions of the theorem follows from the general lemma, in which the function f is considered only in some neighborhood of c_0.

Lemma 5.5. *Consider a segment $\Delta=[c-\ell_0,c+\ell_0]$, where $c\in\mathbb{R}$ is its center and the positive number $2\ell_0$ is its length. Consider a function $f\in C^2(\Delta)\cap W^1_3(\Delta)$. Suppose $f'''>0$ a.e. on the left half $[c-\ell_0,c]$ of Δ and $f'''<0$ a.e. on the right half $[c,c+\ell_0]$. Then there exist two functions $a(\ell)$ and $b(\ell)=a(\ell)+\ell$, $\ell\in(0,\ell_0]$, with the following properties*:

1) *$a(\ell)<c<b(\ell)$;*
2) *$a(\ell)$ and $b(\ell)$ solve equation* (5.4);
3) *$D_{\mathrm{L}}\big(a(\ell),b(\ell)\big)<0$ and $D_{\mathrm{R}}\big(a(\ell),b(\ell)\big)<0$;*
4) *a and b are differentiable functions such that $a'<0$ and $b'>0$.*

Setting $\ell_0=2\varepsilon$ and using the lemma just stated, we see that the non-intersecting chords $[A(\ell),B(\ell)]$ form a full cup $\Omega_{\mathrm{cup}}(a_0,b_0)$ with ends $a_0=a(\ell_0)$ and $b_0=b(\ell_0)$.

Further, since $D_{\mathrm{L}}(a_0,b_0)<0$ and $D_{\mathrm{R}}(a_0,b_0)<0$, it follows that conditions (5.15) in Proposition 5.3 are satisfied. Suppose the domains $\Omega_{\mathrm{L}}(-\infty,a_0)$ and $\Omega_{\mathrm{R}}(b_0,+\infty)$ adjoin our cup. Proposition 5.3 tells us that the function $B^{\mathrm{LR}}(x;\,-\infty,[a_0,b_0],+\infty)$ defined by (5.18) is a Bellman candidate in the domain $\Omega_{\mathrm{LR}}(-\infty,[a_0,b_0],+\infty)=\Omega_\varepsilon$. Therefore, Statement 2.2 guarantees that $\boldsymbol{B}_\varepsilon\le B^{\mathrm{LR}}$. The converse estimate $\boldsymbol{B}_\varepsilon\ge B^{\mathrm{LR}}$ follows from the existence of optimizers for each point in Ω_ε (see Section 5.3). It remains to prove Lemma 5.5.

Proof of Lemma 5.5. First, without loss of generality, we can set $c = 0$. This follows from the linear substitution in all the conditions on the required functions a and b.

Now we verify that for any ℓ, $0 < \ell \leq \ell_0$, there exist points a and $b = a + \ell$ solving equation (5.4), and for such points the relation $a < 0 < b$ is always fulfilled. Note that for all the points a and b such that $-\ell_0 \leq a < b \leq 0$, the left part of cup equation (5.4) is strictly smaller than its right part. Indeed, the requirement on the sign of f''' implies that f'' is strictly increasing on $[-\ell_0, 0]$, and so f' is strictly convex on this interval. Thus, on (a, b) the function f' is strictly less than the linear function whose graph contains the points $(a, f'(a))$ and $(b, f'(b))$. This implies that the average of f' over $[a, b]$ is strictly less than the average of this linear function, i.e.

$$\langle f' \rangle_{[a,b]} < \frac{f'(a) + f'(b)}{2}.$$

Similarly, for any points a and b such that $0 \leq a < b \leq \ell_0$, the left part of equation (5.4) is strictly greater than its right part.

If we fix ℓ and set $b = a + \ell$, then we can treat the difference between the left and right parts of (5.4) as a continuous function of $a \in [-\ell_0, 0]$. We see that this function takes both positive and negative values. Therefore, it vanishes at some point a, and the pair a and $b = a + \ell$ solves equation (5.4). Besides, in view of our considerations in the beginning of the proof, we have $a < 0$ and $b > 0$.

Now we prove that $D_{\mathrm{L}}(a, b) < 0$ and $D_{\mathrm{R}}(a, b) < 0$ if a and b solve equation (5.4). Consider the function

$$q(t) = f'(t) + \alpha_1 t + \alpha_2,$$

where the coefficients α_1 and α_2 are chosen so that $q(a) = q(b) = 0$. It is easily shown that such a function has the following properties:

1) $q'' = f'''$;
2) equation (5.4) on the ends of chords is equivalent to the identity $\langle q \rangle_{[a,b]} = 0$;
3) the inequalities $D_{\mathrm{L}}(a, b) < 0$ and $D_{\mathrm{R}}(a, b) < 0$ can be rewritten as $q'(a) < 0$ and $q'(b) < 0$, respectively.

Further, by the condition on the sign of f''', the function q is strictly convex on $[a, 0]$ and strictly concave on $[0, b]$. Thus, by simple geometric considerations, q has at most one root on (a, b). If this root does not exist, then the identity $\langle q \rangle_{[a,b]} = 0$ cannot hold (this identity means precisely that the areas of two hatched domains on Figure 11 are equal). But if $q'(a) \geq 0$ or $q'(b) \geq 0$, the function q has no roots on (a, b) by geometric considerations. Thus, we have proved the estimates $D_{\mathrm{L}}(a, b) < 0$ and $D_{\mathrm{R}}(a, b) < 0$.

Now we find points a_0 and $b_0 = a_0 + \ell_0$ solving equation (5.4). This equation can be written as $\Phi(a_0, \ell_0) = 0$, where

$$\Phi(a, \ell) = \ell\big(f'(a) + f'(a + \ell)\big) - 2\big(f(a + \ell) - f(a)\big).$$

Differentiating Φ with respect to the first variable, we have

$$\begin{aligned}\Phi'_a(a, \ell) &= \ell\big(f''(a) + f''(a + \ell)\big) - 2\big(f'(a + \ell) - f'(a)\big)\\ &= \ell\big(D_{\mathrm{L}}(a, b) + D_{\mathrm{R}}(a, b)\big).\end{aligned}$$

Therefore, $\Phi'_a(a_0, \ell_0) < 0$. Consequently, by the implicit function theorem, there exists an interval $(\tilde{\ell}, \ell_0]$ on which we can define a unique differentiable function $a(\ell)$

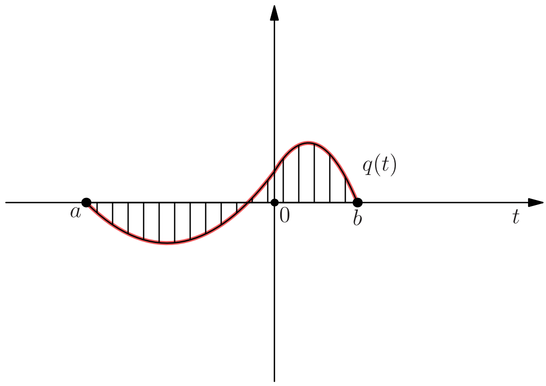

FIGURE 11. A function with zero mean (its convexity changes at $t = 0$).

satisfying the identity $a(\ell_0) = a_0$ and, together with the function $b(\ell) = a(\ell) + \ell$, solving chord equation (5.4). In addition,

$$a'(\ell) = -\frac{\Phi'_\ell\big(a(\ell), \ell\big)}{\Phi'_a\big(a(\ell), \ell\big)}.$$

But

$$\Phi'_\ell\big(a, \ell\big) = \ell f''(a+\ell) + f'(a) - f'(a+\ell) = \ell D_{\mathrm{R}}(a, b),$$

and so $-1 < a'(\ell) < 0$ and $b'(\ell) = a'(\ell) + 1 > 0$ for $\ell \in (\tilde{\ell}, \ell_0]$.

Further, let $(\tilde{\ell}, \ell_0]$ be the union of all the appropriate intervals, i.e. the intervals such that the identity $\Phi(a, \ell) = 0$, together with the requirement $a(\ell_0) = a_0$, defines a unique differentiable function $a(\ell)$ on them. We claim that $\tilde{\ell} = 0$. Indeed, let $\tilde{\ell} > 0$. We choose some decreasing sequence ℓ_n on $(\tilde{\ell}, \ell_0]$ that converges to $\tilde{\ell}$. Then $a_n = a(\ell_n)$ is an increasing sequence and, besides, $a_n < 0$. We denote its limit by $\tilde{a}$. By continuity, we have $\Phi(\tilde{a}, \tilde{\ell}) = 0$. Then, using the implicit function theorem again, we can increase the interval $(\tilde{\ell}, \ell_0]$. But this contradicts the assumption of its maximality.

As a result, we have the functions a and b defined on $(0, \ell_0]$ and satisfying all the conditions required. □

6. General case

In this section we will obtain the function $\boldsymbol{B}_\varepsilon(x;\, f)$ for $f \in \mathfrak{W}^N_{\varepsilon_0}$, $N \in \mathbb{Z}_+$.

6.1. **Trolleybus.** The following considerations, which are not intended to be rigorous, will lead us to a new construction (the last of those that are required for the general case). We have seen in Section 4.3 that in the situation where f''' changes its sign from minus to plus, an angle Ω_{ang} can arise. If f''' changes its sign from plus to minus, then the cup Ω_{cup} arises around the point where the sign changes. Now we assume that f''' changes its sign twice. Then one point where the sign changes generates a cup and the other can generate an angle. It is not difficult to imagine a situation where the angle and the cup stick together. It turns out that they cannot only stick but "mix" with each other and generate one of the constructions shown

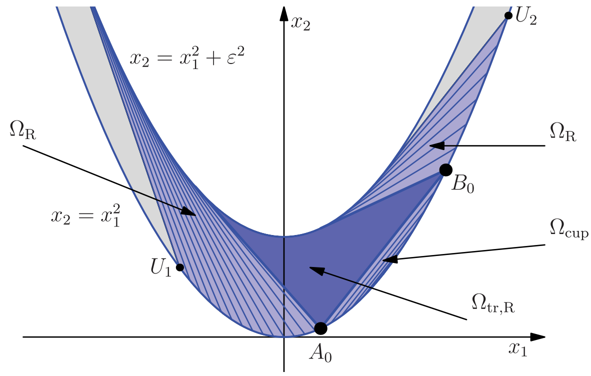

FIGURE 12. A right trolleybus $\Omega_{\mathrm{tr,R}}$.

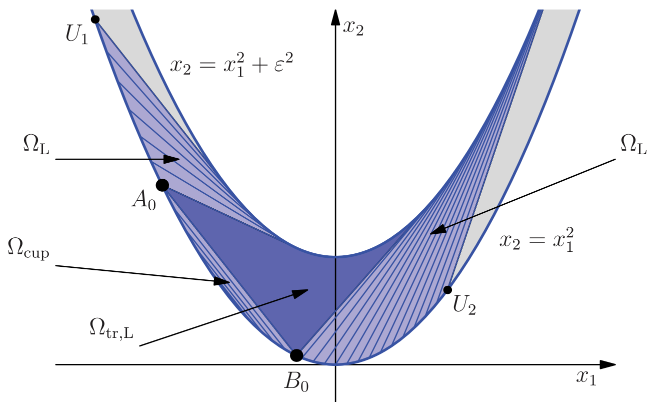

FIGURE 13. A left trolleybus $\Omega_{\mathrm{tr,L}}$.

in Figures 12 and 13 (see [4] for examples). Now we give a rigorous description of such constructions and build corresponding Bellman candidates.

Suppose $u_1 < a_0 < b_0 < u_2$ and $b_0 - a_0 \le 2\varepsilon$. Consider a cup $\Omega_{\mathrm{cup}}(a_0, b_0)$ (it may not be full) and the domains $\Omega_{\mathrm{R}}(u_1, a_0)$ and $\Omega_{\mathrm{R}}(b_0, u_2)$ foliated by the extremal tangents. The quadrangular subdomain of Ω_ε, bounded by the upper chord $[A_0, B_0]$, the right tangents coming from A_0 and B_0, and the arc of the upper parabola is called *the right trolleybus*[3] and is denoted by $\Omega_{\mathrm{tr,R}}(a_0, b_0)$ (see Figure 12). Similarly, we can define the left trolleybus $\Omega_{\mathrm{tr,L}}(a_0, b_0)$ and the corresponding construction shown in Figure 13. Note that for $b_0 - a_0 = 2\varepsilon$ the trolleybus degenerates into an angle adjacent to a cup.

We consider the construction with the right trolleybus. Our goal is to build a Bellman candidate in the domain

$$\Omega_{\mathrm{RR}}(u_1, [a_0, b_0], u_2) = \Omega_{\mathrm{R}}(u_1, a_0) \cup \Omega_{\mathrm{cup}}(a_0, b_0) \cup \Omega_{\mathrm{tr,R}}(a_0, b_0) \cup \Omega_{\mathrm{R}}(b_0, u_2).$$

[3]Glancing at Figure 12, the reader will hardly understand why such a name was chosen. The point is the low artistic skills of the authors. When this construction was drawn on a blackboard for the first time, the one-sided tangents, bounding the subdomain, were almost parallel and looked like trolley poles drawing the electricity from the upper parabola.

We denote the function required by $B^{\mathrm{RR}}(x;\, u_1, [a_0, b_0], u_2)$. In the trolleybus, our candidate is linear by the minimality

$$\begin{aligned} B^{\mathrm{RR}}(x;\, u_1, [a_0, b_0], u_2) &= B^{\mathrm{tr,R}}(x;\, a_0, b_0) \\ &= \beta_1 x_1 + \beta_2 x_2 + \beta_0, \quad x \in \Omega_{\mathrm{tr,R}}(a_0, b_0). \end{aligned}$$

We already know that the Bellman candidate coincides with $B^{\mathrm{cup}}(x;\, a_0, b_0)$ in $\Omega_{\mathrm{cup}}(a_0, b_0)$ and with $B^{\mathrm{R}}(x;\, b_0, u_2)$ in $\Omega_{\mathrm{R}}(b_0, u_2)$. The latter function is not defined uniquely (the value $m_{\mathrm{R}}(b_0)$ must be chosen). The necessary and sufficient conditions for the concatenation of $B^{\mathrm{tr,R}}(x;\, a_0, b_0)$, $B^{\mathrm{cup}}(x;\, a_0, b_0)$, and $B^{\mathrm{R}}(x;\, b_0, u_2)$ to be continuous can be written as

$$\begin{cases} \beta_1 a_0 + \beta_2 a_0^2 + \beta_0 = f(a_0); \\ \beta_1 b_0 + \beta_2 b_0^2 + \beta_0 = f(b_0); \\ m_{\mathrm{R}}(b_0) = \beta_1 + 2(b_0 - \varepsilon)\beta_2. \end{cases} \tag{6.1}$$

Indeed, the first two identities must be fulfilled by the boundary condition, and they imply that $B^{\mathrm{tr,R}}$ is glued to B^{cup} continuously. The last identity guarantees that the concatenation of $B^{\mathrm{tr,R}}(x;\, a_0, b_0)$ and $B^{\mathrm{R}}(x;\, b_0, u_2)$ is continuous. We have obtained this equation by expressing x_2 in terms of x_1 on the right boundary of the trolleybus (see equation (R) in Section 3.1) and then equating the coefficient of x_1 with $m_{\mathrm{R}}(b_0)$.

Now, assume that the functions $B^{\mathrm{cup}}(x;\, a_0, b_0)$ and $B^{\mathrm{R}}(x;\, b_0, u_2)$ are locally concave. In order for their concatenation with $B^{\mathrm{tr,R}}(x;\, a_0, b_0)$ to be locally concave, it is necessary that the jumps of the derivative in x_2 be non-positive on the corresponding boundaries of $\Omega_{\mathrm{tr,R}}$. Using (5.8), we see that on the lower boundary of the trolleybus (i.e. on the chord $[A_0, B_0]$), the jump can be calculated as follows:

$$\delta_1 = \beta_2 - \frac{1}{2}\langle f''\rangle_{[a_0,b_0]}.$$

On the right boundary of the trolleybus (i.e. on the right tangent coming from B_0), the jump can be calculated by the formula

$$\delta_2 = \lim_{u \to b_0+} t_2(u) - \beta_2,$$

where $t_2 = B^{\mathrm{R}}_{x_2}$ in $\Omega_{\mathrm{R}}(b_0, u_2)$. Using (3.7), (3.6), and, after that, the last identity in (6.1), we obtain

$$\lim_{u \to b_0+} t_2(u) = \frac{m_{\mathrm{R}}'(b_0)}{2} = \frac{f'(b_0) - m_{\mathrm{R}}(b_0)}{2\varepsilon} = \frac{f'(b_0) - \beta_1 - 2(b_0 - \varepsilon)\beta_2}{2\varepsilon}.$$

Therefore, we have

$$\delta_2 = \frac{f'(b_0) - \beta_1 - 2\beta_2 b_0}{2\varepsilon}. \tag{6.2}$$

Subtracting the first equation in (6.1) from the second one, we get

$$\beta_1 + \beta_2(a_0 + b_0) = \frac{f(b_0) - f(a_0)}{b_0 - a_0} = \langle f'\rangle_{[a_0,b_0]}. \tag{6.3}$$

Using cup equation (5.4), we obtain

$$\beta_1 + \beta_2(a_0 + b_0) = \frac{f'(a_0) + f'(b_0)}{2}. \tag{6.4}$$

Expressing β_1 in terms of β_2 and substituting the resulting expression into (6.2), we have

$$\begin{aligned}\delta_2 &= \frac{f'(b_0)-f'(a_0)-2\beta_2(b_0-a_0)}{4\varepsilon}\\ &= -\frac{b_0-a_0}{2\varepsilon}\Big(\beta_2-\frac{1}{2}\langle f''\rangle_{[a_0,b_0]}\Big)\\ &= -\frac{b_0-a_0}{2\varepsilon}\delta_1.\end{aligned}$$

But δ_1 and δ_2 must have the same sign and so $\delta_1=\delta_2=0$. In its turn, this condition implies that the concatenation of $B^{\mathrm{cup}}(x;\,a_0,b_0)$, $B^{\mathrm{tr,R}}(x;\,a_0,b_0)$, and $B^{\mathrm{R}}(x;\,b_0,u_2)$ is C^1-smooth (because on the boundaries of the trolleybus the derivatives in two non-collinear directions — along x_2 and along the corresponding boundary — are glued continuously). But if the concatenation is C^1-smooth and its components are locally concave, then it is also locally concave. Therefore, the identity $\delta_1=0$ or

$$\beta_2=\frac{1}{2}\langle f''\rangle_{[a_0,b_0]} \tag{6.5}$$

is a necessary and sufficient condition for the concatenation of the linear function $B^{\mathrm{tr,R}}(x;\,a_0,b_0)$ with the locally concave functions $B^{\mathrm{cup}}(x;\,a_0,b_0)$ and $B^{\mathrm{R}}(x;\,b_0,u_2)$ to be locally concave. Substituting expression (6.5) into (6.3), we get

$$\beta_1=\langle f'\rangle_{[a_0,b_0]}-\frac{1}{2}(b_0+a_0)\langle f''\rangle_{[a_0,b_0]}. \tag{6.6}$$

Also, the expression for β_2 can be substituted in (6.4):

$$\begin{aligned}\beta_1 &= \frac{f'(a_0)+f'(b_0)}{2}-(a_0+b_0)\frac{f'(b_0)-f'(a_0)}{2(b_0-a_0)}\\ &= \frac{b_0f'(a_0)-a_0f'(b_0)}{b_0-a_0}.\end{aligned} \tag{6.7}$$

Summing the first and second equations in (6.1) and, after that, substituting expressions (6.5) and (6.6), we obtain

$$\beta_0=\frac{b_0f(a_0)-a_0f(b_0)}{b_0-a_0}+\frac{1}{2}a_0b_0\langle f''\rangle_{[a_0,b_0]}. \tag{6.8}$$

Finally, substituting expressions (6.5) and (6.7) in the last equation into (6.1), we have

$$\begin{aligned}m_{\mathrm{R}}(b_0) &= \frac{b_0f'(a_0)-a_0f'(b_0)}{b_0-a_0}+(b_0-\varepsilon)\frac{f'(b_0)-f'(a_0)}{b_0-a_0}\\ &= f'(b_0)-\varepsilon\langle f''\rangle_{[a_0,b_0]}.\end{aligned}$$

We note that if $b_0-a_0=2\varepsilon$, then

$$m_{\mathrm{R}}(b_0)=f'(b_0)-\frac{f'(b_0)-f'(a_0)}{2}=m_{\mathrm{R}}(b_0;\,b_0),$$

where $m_{\mathrm{R}}(u;\,b_0)$ is given by (5.13) (that expression was defined only for the case $b_0-a_0=2\varepsilon$). Now we extend the notation $m_{\mathrm{R}}(u;\,b_0)$ to the general case $b_0-a_0\le 2\varepsilon$:

$$m_{\mathrm{R}}(u;\,b_0)=\big(f'(b_0)-\varepsilon\langle f''\rangle_{[a_0,b_0]}\big)e^{(b_0-u)/\varepsilon}+\varepsilon^{-1}e^{-u/\varepsilon}\int_{b_0}^{u}f'(t)e^{t/\varepsilon}\,dt.$$

It is easy to prove that formula (5.17) for $m_{\mathrm{R}}''(u;\,b_0)$ remains true.

Thus, we have

$$m_{\mathrm{R}}(u) = m_{\mathrm{R}}(u;\, b_0), \quad u \in (b_0, u_2),$$

and

$$B^{\mathrm{RR}}(x;\, u_1, [a_0, b_0], u_2) = B^{\mathrm{R}}(x;\, [a_0, b_0], u_2), \quad x \in \Omega_{\mathrm{R}}(b_0, u_2),$$

where the function $B^{\mathrm{R}}(x;\, [a_0, b_0], u_2)$ is still given by (5.14).

Now we consider the concatenation of $B^{\mathrm{tr,R}}$ and $B^{\mathrm{R}}(x;\, u_1, a_0)$. Arguing the same way as for the right boundary of the trolleybus, we get a necessary and sufficient condition for our concatenation to be continuous on the left boundary:

$$m_{\mathrm{R}}(a_0) = \beta_1 + 2(a_0 - \varepsilon)\beta_2.$$

Substituting expressions (6.5) and (6.7) into this formula, we obtain

$$\begin{aligned} m_{\mathrm{R}}(a_0) &= \frac{b_0 f'(a_0) - a_0 f'(b_0)}{b_0 - a_0} + (a_0 - \varepsilon)\frac{f'(b_0) - f'(a_0)}{b_0 - a_0} \\ &= f'(a_0) - \varepsilon \langle f'' \rangle_{[a_0, b_0]}. \end{aligned} \tag{6.9}$$

Using equation (3.6) twice, we have

$$m_{\mathrm{R}}(a_0) = f'(a_0) - \varepsilon m_{\mathrm{R}}'(a_0) = f'(a_0) - \varepsilon\big(f''(a_0) - \varepsilon m_{\mathrm{R}}''(a_0)\big).$$

This allows us to rewrite identity (6.9) as

$$m_{\mathrm{R}}''(a_0) = \varepsilon^{-1}\big(f''(a_0) - \langle f'' \rangle_{[a_0, b_0]}\big) = \varepsilon^{-1} D_{\mathrm{L}}(a_0, b_0), \tag{6.10}$$

where D_{L} is defined by the first relation in (5.7).

Now we verify that the resulting condition implies not only that the concatenation of $B^{\mathrm{R}}(x;\, u_1, a_0)$ and $B^{\mathrm{tr,R}}(x;\, a_0, b_0)$ is continuous but also that it is C^1-smooth. We set $t_2 = B^{\mathrm{R}}_{x_2}$ in $\Omega_{\mathrm{R}}(u_1, a_0)$. Using (3.7), (3.6), and (6.9), we get

$$\lim_{u \to a_0 -} t_2(u) = \frac{m_{\mathrm{R}}'(a_0)}{2} = \frac{f'(a_0) - m_{\mathrm{R}}(a_0)}{2\varepsilon} = \frac{1}{2}\langle f'' \rangle_{[a_0, b_0]} = \beta_2.$$

As usual, this implies the C^1-smoothness of the concatenation on the left boundary of the trolleybus. But since the concatenation is C^1-smooth, it is locally concave provided all its components are locally concave.

Now we discuss the left trolleybus $\Omega_{\mathrm{tr,L}}$ and construct a candidate $B^{\mathrm{LL}}(x;\, u_1, [a_0, b_0], u_2)$ in the union

$$\Omega_{\mathrm{LL}}(u_1, [a_0, b_0], u_2) = \Omega_{\mathrm{L}}(u_1, a_0) \cup \Omega_{\mathrm{cup}}(a_0, b_0) \cup \Omega_{\mathrm{tr,L}}(a_0, b_0) \cup \Omega_{\mathrm{L}}(b_0, u_2).$$

In order to build such a candidate, we can reason in the same way as we did for B^{RR}. We obtain

$$\begin{aligned} B^{\mathrm{LL}}(x;\, u_1, [a_0, b_0], u_2) &= B^{\mathrm{tr,L}}(x;\, a_0, b_0) \\ &= \beta_1 x_1 + \beta_2 x_2 + \beta_0, \quad x \in \Omega_{\mathrm{tr,L}}(a_0, b_0), \end{aligned}$$

where β_1, β_2 and β_0 are the same as for the right trolleybus. Further, defining the function $m_{\mathrm{L}}(u;\, a_0)$, $u \in (u_1, a_0]$, by the formula

$$m_{\mathrm{L}}(u;\, a_0) = \big(f'(a_0) + \varepsilon \langle f'' \rangle_{[a_0, b_0]}\big) e^{(u - a_0)/\varepsilon} + \varepsilon^{-1} e^{u/\varepsilon} \int_u^{a_0} f'(t) e^{-t/\varepsilon}\, dt$$

(clearly, this formula coincides with (5.11) if $b_0 - a_0 = 2\varepsilon$), we obtain

$$B^{\mathrm{LL}}(x;\, u_1, [a_0, b_0], u_2) = B^{\mathrm{L}}(x;\, u_1, [a_0, b_0]), \quad x \in \Omega_{\mathrm{L}}(u_1, a_0),$$

where the function on the right is defined by (5.12). Note that formula (5.16) for $m''_{\mathrm{L}}(u;\, a_0)$ is still correct. Concerning the domain $\Omega_{\mathrm{L}}(b_0, u_2)$, we have

$$B^{\mathrm{LL}}(x;\, u_1, [a_0, b_0], u_2) = B^{\mathrm{L}}(x;\, b_0, u_2), x \in \Omega_{\mathrm{L}}(b_0, u_2),$$

where the coefficient $m_{\mathrm{L}}(u)$, participating in the definition of $B^{\mathrm{L}}(x;\, b_0, u_2)$, satisfies

$$m''_{\mathrm{L}}(b_0) = -\varepsilon^{-1} D_{\mathrm{R}}(a_0, b_0). \tag{6.11}$$

Now we note that by (5.16) identity (6.10) is equivalent to the equation

$$m''_{\mathrm{R}}(a_0) + m''_{\mathrm{L}}(a_0;\, a_0) = 0, \tag{6.12}$$

which has the same form as equation (4.3) for the vertex of an angle. Similarly, from (5.17), it follows that relation (6.11) is equivalent to the equation

$$m''_{\mathrm{L}}(b_0) + m''_{\mathrm{R}}(b_0;\, b_0) = 0. \tag{6.13}$$

Now, we can formulate a proposition in which our construction with a right trolleybus is described.

Proposition 6.1. *Let $u_1 < a_0 < b_0 < u_2$ and $b_0 - a_0 \le 2\varepsilon$. Consider a cup $\Omega_{\mathrm{cup}}(a_0, b_0)$, two domains $\Omega_{\mathrm{R}}(u_1, a_0)$ and $\Omega_{\mathrm{R}}(b_0, u_2)$ foliated by the right extremal tangents, and the linearity domain $\Omega_{\mathrm{tr,R}}(a_0, b_0)$ located between them all. Any Bellman candidate in the union $\Omega_{\mathrm{RR}}(u_1, [a_0, b_0], u_2)$ of these four domains has the form*

$$B^{\mathrm{RR}}(x;\, u_1, [a_0, b_0], u_2) = \begin{cases} B^{\mathrm{R}}(x;\, u_1, a_0), & x \in \Omega_{\mathrm{R}}(u_1, a_0); \\ B^{\mathrm{cup}}(x;\, a_0, b_0), & x \in \Omega_{\mathrm{cup}}(a_0, b_0); \\ B^{\mathrm{tr,R}}(x;\, a_0, b_0), & x \in \Omega_{\mathrm{tr,R}}(a_0, b_0); \\ B^{\mathrm{R}}(x;\, [a_0, b_0], u_2), & x \in \Omega_{\mathrm{R}}(b_0, u_2). \end{cases}$$

Here, $B^{\mathrm{tr,R}}(x;\, a_0, b_0) = \beta_1 x_1 + \beta_2 x_2 + \beta_0$ is the linear function with the coefficients given by (6.5), (6.7), and (6.8). In addition, the inequalities $m''_{\mathrm{R}}(u;\, b_0) \le 0$, $u \in (b_0, u_2)$, and $m''_{\mathrm{R}}(u) \le 0$, $u \in (u_1, a_0)$, together with relation (6.12), must be fulfilled (in order for B^{RR} to be a Bellman candidate indeed).

We also give a symmetric proposition for a left trolleybus.

Proposition 6.2. *Any candidate B^{LL} in Ω_{LL} is the concatenation of $B^{\mathrm{L}}(x;\, u_1, [a_0, b_0])$, $B^{\mathrm{cup}}(x;\, a_0, b_0)$, $B^{\mathrm{tr,L}}(x;\, a_0, b_0)$, and $B^{\mathrm{L}}(x;\, b_0, u_2)$. In addition, the inequalities $m''_{\mathrm{L}}(u;\, a_0) \ge 0$, $u \in (u_1, a_0)$, and $m''_{\mathrm{L}}(u) \ge 0$, $u \in (b_0, u_2)$, together with relation (6.13), must be fulfilled.*

6.2. **Optimizers in trolleybuses.** In this subsection we discuss delivery curves and optimizers in constructions with trolleybuses. We treat in detail only the case of a right trolleybus.

Suppose B is a Bellman candidate in the whole domain Ω_ε and some part of the corresponding foliation forms the construction $\Omega_{\mathrm{RR}}(u_1, [a_0, b_0], u_2)$ described in Proposition 6.1. We already know how to build delivery curves in the domain $\Omega_{\mathrm{R}}(u_1, a_0)$ and in the cup $\Omega_{\mathrm{cup}}(a_0, b_0)$[4] (see Sections 3.3 and 5.3). Suppose A_1 and B_1 are the points where the rear and the front "trolley poles" of $\Omega_{\mathrm{tr,R}}(a_0, b_0)$ touch the upper parabola: $A_1 = \big(a_0 - \varepsilon, (a_0 - \varepsilon)^2 + \varepsilon^2\big)$ and $B_1 = \big(b_0 - \varepsilon, (b_0 - \varepsilon)^2 + \varepsilon^2\big)$. Let γ be the left delivery curve that runs along the upper

[4]It is worth noting that $\Omega_{\mathrm{R}}(u_1, a_0)$ and $\Omega_{\mathrm{cup}}(a_0, b_0)$ are not connected with each other by delivery curves. This situation should not be confused with the case of a full cup and two domains adjacent to it.

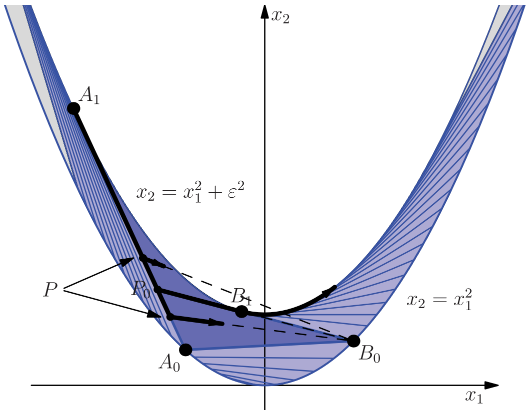

FIGURE 14. $\Omega_{\mathrm{tr,R}}$ and delivery curves.

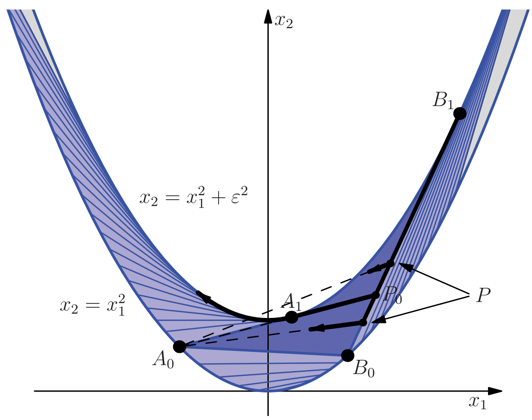

FIGURE 15. $\Omega_{\mathrm{tr,L}}$ and delivery curves.

parabola in $\Omega_{\mathrm{R}}(u_1, a_0)$ and ends at the point A_1. This point is the entry node of $\Omega_{\mathrm{tr,R}}$ and, as we will see later, the curve γ can be continued from A_1 up to each point of the trolleybus. To get an idea of how we are going to do this, the reader can look at Figure 14, which shows various delivery curves in the trolleybus.

Let P_0 be the point where the straight line, containing "the front pole" $[B_1, B_0]$ of the trolleybus intersects "the rear pole" $[A_1, A_0]$. We use Proposition 3.5 from Section 3.3 twice and continue γ with the segment $[A_1, P_0]$ and, after that, with the segment $[P_0, B_1]$. An important feature of the curve just constructed is that it "transits" through the trolleybus and ends at the entry node of $\Omega_{\mathrm{R}}(b_0, u_2)$. Then this curve can be continued up to any point of $\Omega_{\mathrm{R}}(b_0, u_2)$ (see Section 3.3 again). Now we consider the points lying in the triangle with vertices P_0, A_0, and B_0. The curve γ can be continued up to any such point x in the same way as above. First, we find the point P where the straight line containing the segment $[x, B_0]$ intersects the segment $[A_1, A_0]$. After that, we continue γ with the segments $[A_1, P]$ and $[P, x]$.

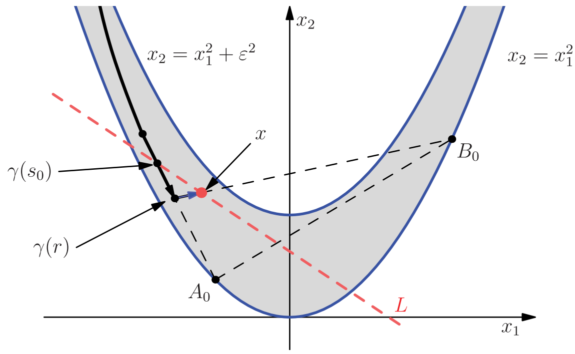

FIGURE 16. Illustration to the proof of Proposition 6.3.

It remains to consider the points of the trolleybus that get into the triangle with vertices P_0, A_1, and B_1. First, we find the point $P \in [A_1, A_0]$ in the same way as described above and continue γ with the segment $[A_1, P]$. By Proposition 3.5, the new curve is still a left delivery curve. We continue it with the segment $[P, x]$. Although the conditions of Proposition 3.5 are not satisfied this time, we still obtain a left delivery curve as a result. The fact is that the conditions of another proposition — some modification of Proposition 3.5 — are fulfilled. This modification allows us to overcome the difficulties appearing from the fact that our curve is continued along the segment intersecting the upper boundary transversally.

Proposition 6.3. *Let γ be a convex left delivery curve that is generated by a test function φ defined on $I = [l, r]$. Suppose γ ends with a straight segment described in Proposition* 3.5*. By A_0 we denote the point where the line containing this segment intersects the lower parabola. On the lower parabola, we also choose a point B_0 such that $0 < b_0 - a_0 \le 2\varepsilon$. Further, let $x \in [\gamma(r), B_0]$ be a point such that the candidate B is linear on $[\gamma(r), x]$. Also, we assume that on the straight segment with which the curve γ ends there exists a point $\gamma(s_0)$, $s_0 \in I$, such that the line L containing the segment $[\gamma(s_0), x]$ does not intersect the upper parabola* (*see Figure* 16)*. If we now continue the curve γ with the segment $[\gamma(r), x]$, then the resulting curve $\widetilde{\gamma}$ remains a left delivery curve. It is generated by the function*

$$\widetilde{\varphi}(s) = \begin{cases} \varphi(s), & s \in I; \\ b_0, & s \in [r, \widetilde{r}]. \end{cases}$$

Proof. The fact that $\widetilde{\varphi}$, $\widetilde{\gamma}$, and B are related by (2.6) and (2.7) can be proved in the same way as in Proposition 3.5. Thus, we only need to verify that $\widetilde{\varphi} \in \mathrm{BMO}_\varepsilon([l, \widetilde{r}])$. We take an arbitrary interval $[c, d] \subset [l, \widetilde{r}]$ and see where the Bellman point $x^{[c,d]} = \big(\langle\varphi\rangle_{[c,d]}, \langle\varphi^2\rangle_{[c,d]}\big)$ is located. If $d \le r$, then the required estimate, as usual, follows from the fact that $\varphi \in \mathrm{BMO}_\varepsilon(I)$. Therefore, it is sufficient to consider the case where $d > r$ and the point $\widetilde{\gamma}(d) = x^{[l,d]}$ lies on the added segment $[\gamma(r), x]$ (in such a case, this point, of course, lies under the line L).

Next, if $c > r$, then $\widetilde{\varphi}$ is identically equal to b_0 on $[c, d]$, and there is nothing to prove. If $s_0 \le c \le r$, then on $[c, d]$ the function $\widetilde{\varphi}$ is a step function with values a_0 and b_0. But then $x^{[c,d]}$ lies on the chord $[A_0, B_0]$, which is contained in Ω_ε entirely, because $|a_0 - b_0| \le 2\varepsilon$.

Now, let $c < s_0$. In this case, the point $\widetilde{\gamma}(c) = x^{[l,c]}$ lies on the initial delivery curve above L. But the points $\widetilde{\gamma}(c)$, $x^{[c,d]}$, and $\widetilde{\gamma}(d)$ lie on one line. The last point is a convex combination of the first two and locates between them. Hence, $x^{[c,d]}$ lies below L and, therefore, under the upper parabola. $\square$

Thus, since certain delivery curves in the trolleybus intersect the upper parabola transversally, it is not always possible to employ Lemma 2.10 and Proposition 3.5 directly. But we can overcome this difficulty using Proposition 6.3.

In left trolleybuses, delivery curves can be constructed exactly the same way. We omit detailed arguments for this case (however, Figure 15 clarifies the matter entirely).

6.3. **Foliation in the general case.** Now, using the components already constructed, we build a global Bellman candidate in the whole domain Ω_ε.

First, we fix some signature Σ consisting of a finite number of symbols R and L that are arranged in an arbitrary order. We associate the pairs RL in this signature with angles, the pairs LR with full cups, and the pairs RR and LL with trolleybuses (right and left, respectively) attached to cups (not necessarily full). We suppose these angles and cups are pairwise disjoint and arranged in the same order as the corresponding pairs of symbols in Σ. We notice that all the domains located between them, together with two domains on the edges, have the form Ω_{R} or Ω_{L}. We assume that these domains are foliated by the suitable tangents. Then, by one of Propositions 3.2, 5.3, 6.1, or 6.2, Bellman candidates are defined uniquely in these domains. If we now assume that near each angle and each cup the conditions of the corresponding proposition — either one of the propositions just listed or Proposition 4.1 about an angle — are satisfied, then we obtain some candidate B^{Σ} in the whole domain Ω_ε. It turns out that the Bellman function we are looking for has precisely such a form.

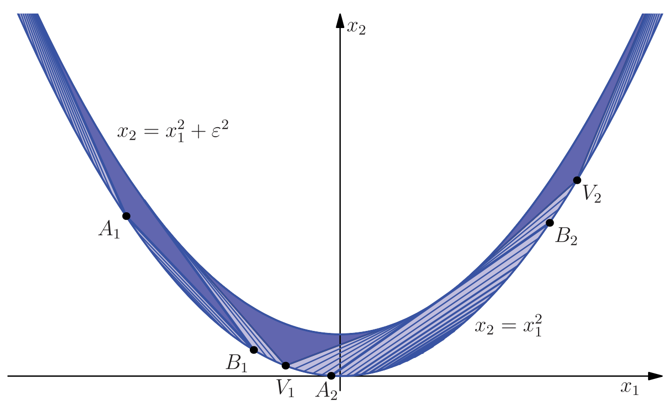

FIGURE 17. A global candidate B^{RRLRL}.

Theorem 6.4. *Suppose $0 < \varepsilon < \varepsilon_0$, $N \in \mathbb{Z}_+$, and $f \in \mathfrak{W}^N_{\varepsilon_0}$. Then we can choose a signature Σ such that a certain Bellman candidate B^{Σ} corresponds to it. In this case, we have $\boldsymbol{B}_\varepsilon(x; f) = B^{\Sigma}(x)$.*

In order to prove this theorem, we need some preparation. First, we present some new definitions.

Let c be a point where the third derivative f''' changes its sign from $+$ to $-$. By Lemma 5.5, there exist continuously differentiable functions $a(\ell)$ and $b(\ell)=a(\ell)+\ell$ on $[0,2\varepsilon]$ that generate a cup originating at c, i.e. $a(0)=b(0)=c$. Together with the pair a and b we will need another pair of mutually inverse functions $\tilde a$ and $\tilde b$. These functions have the same values as a and b, but their arguments are different. They are the first coordinates of the opposite ends of the chord. Namely, each of the pairs $\{\tilde a(u),u\}$ and $\{u,\tilde b(u)\}$ is a pair of points $\{a,b\}$ satisfying the cup equation (5.4).

Fixing the cup size ℓ, we define the following function D on $[a(\ell),b(\ell)]$:

$$D(u)\stackrel{\text{def}}{=}\begin{cases}-D_{\mathrm{L}}(u,\tilde b(u)), & a(\ell)\le u<c;\\ D_{\mathrm{R}}(\tilde a(u),u), & c<u\le b(\ell).\end{cases}$$

Recall that the differentials $D_{\mathrm{L}}(a,b)$ and $D_{\mathrm{R}}(a,b)$ were introduced by (5.7):

$$D_{\mathrm{L}}(a,b)=f''(a)-\langle f''\rangle_{[a,b]},\qquad D_{\mathrm{R}}(a,b)=f''(b)-\langle f''\rangle_{[a,b]}.$$

The function D can be naturally continued to c by zero. Note that the value of ℓ determines the domain of D only; the value of D at any fixed point does not depend on ℓ.

Definition 6.5. The function

$$F(u,\ell)\stackrel{\text{def}}{=}\begin{cases}e^{u/\varepsilon}\Big[D(a(\ell))e^{-a(\ell)/\varepsilon}+\displaystyle\int_u^{a(\ell)}f'''(t)e^{-t/\varepsilon}dt\Big], & u\in(-\infty,a(\ell));\\ D(u), & u\in[a(\ell),b(\ell)]\,;\\ e^{-u/\varepsilon}\Big[D(b(\ell))e^{b(\ell)/\varepsilon}+\displaystyle\int_{b(\ell)}^{u}f'''(t)e^{t/\varepsilon}dt\Big], & u\in(b(\ell),+\infty)\,,\end{cases}$$

defined on the entire real axis, will be called *a force function* or simply *a force.*

It is natural to call the point c *the source* of the force F. We call $[a(\ell),b(\ell)]$ *the screen* of F. We refer to the left and right parts $[a,c]$ and $[c,b]$ as *the left and right screens* respectively.

Some special cases should be mentioned separately. The formula for a force without a screen (zero screen: $\ell=0$) is especially simple:

$$F(u,0)=\begin{cases}e^{u/\varepsilon}\displaystyle\int_u^c f'''(t)e^{-t/\varepsilon}dt, & u\in(-\infty,c]\,;\\ e^{-u/\varepsilon}\displaystyle\int_c^u f'''(t)e^{t/\varepsilon}dt, & u\in[c,+\infty)\,.\end{cases}$$

Forces with sources at infinity provide the simplest cases of the last formula. If $c=+\infty$, then

$$F(u,\ell)=e^{u/\varepsilon}\int_u^{+\infty}f'''(t)e^{-t/\varepsilon}dt,\qquad u\in(-\infty,+\infty)\,,$$

and if $c=-\infty$, then

$$F(u,\ell)=e^{-u/\varepsilon}\int_{-\infty}^{u}f'''(t)e^{t/\varepsilon}dt,\qquad u\in(-\infty,+\infty)\,.$$

These two expressions do not depend on ℓ (a finite screen at infinity cannot touch finite points). Therefore, we can always assume that the forces originating at infinity have zero screens, i.e. $\ell = 0$.

Note that the formula for a force almost coincides with the second derivative of the coefficient m in the expression for the Bellman function. Namely, formula (5.16) for $\varepsilon m''_{\mathrm{L}}(u, a(\ell))$ gives the force on the left of the screen, and formula (5.17) for $\varepsilon m''_{\mathrm{R}}(u, b(\ell))$ coincides with the same force on the right of the screen. Thus, we glue two expressions for $\varepsilon m''_{\mathrm{R}}$ and $\varepsilon m''_{\mathrm{L}}$, extending them continuously to the screen $[a(\ell), b(\ell)]$.

We introduce a few more notions.

Definition 6.6. An interval $[\,c, t^+]$, where

$$t^+ = t^+(\ell) \stackrel{\mathrm{def}}{=} \sup\{t \mid F(s,\ell) \le 0,\ \forall s,\ c \le s \le t\},$$

is called *the right tail* of the force F. *The left tail* is an interval $[t^-, c\,]$, where

$$t^- = t^-(\ell) \stackrel{\mathrm{def}}{=} \inf\{t \mid F(s,\ell) \ge 0,\ \forall s,\ t \le s \le c\}\,.$$

The points $t^-(\ell)$ and $t^+(\ell)$ will be called the ends of the left and right tails correspondingly.

Note that the tails are the maximal intervals where the force has the sign required. They show the size of the maximal region around the cup that the tangents can foliate.

Further, we recall that the equation

$$m''_{\mathrm{R}}(v;\, b_1) + m''_{\mathrm{L}}(v;\, a_2) = 0$$

appears in Proposition 4.1 as an equation for the vertex of the angle $\Omega_{\mathrm{ang}}(v)$ and, also, in Propositions 6.1 and 6.2 as an equation for one of the trolleybus vertices. This inspires the following definition.

Definition 6.7. Two forces are called *balanced* if they satisfy the following two conditions. First, their tails have non-empty intersection. Second, in this intersection we can choose a point v lying strictly between the sources of the forces and such that

$$F_1(v,\ell_1) + F_2(v,\ell_2) = 0\,. \tag{6.14}$$

This point v will be called *the balance point* and the equation above will be called *the balance equation.* A family of forces is called *balanced* if either this family consists of one element or each pair of neighbor forces is balanced.

Suppose we have found a balanced family of forces with 2ε-screens such that the union of the tails covers the entire real axis. Moreover, let no balance point be inside any screen. Then, as was explained in the beginning of this section, we are done. We have the desired foliation consisting of alternating cups and angles with vertices at balance points. The corresponding function B^{Σ} would be the desired Bellman candidate. However, if some points of balance are inside the screens, then such a family does not help to finish the construction so quickly. The following definition helps us to overcome these difficulties.

Definition 6.8. A balanced family of force functions is called *completely balanced* if it is under the following two conditions. First, there are no balance points inside any screen. Second, at least one end of each screen whose size is less than 2ε coincides with a balance point.

We are now ready to state the proposition that immediately implies Theorem 6.4.

Proposition 6.9. *For any $f \in \mathfrak{W}_{\varepsilon_0}^N$, there exists a family of completely balanced forces such that their tails cover the entire axis.*

Now, we explain how to derive our theorem from this proposition. In the simplest case of one force, the Bellman function is already known (see Theorem 3.9 for $c = -\infty$, Theorem 3.10 for $c = +\infty$, and Theorem 5.4 for a finite c). Thus, we consider the situation when we have several completely balanced forces whose tails cover the entire axis. For each force with a source at a finite point, we build a cup whose size is equal to the size of the corresponding screen. If a cup is not full, then we have a balance point at least at one of its ends. We build the right or left trolleybus over such a cup depending on where (at what end of the cup) we have a balance point. After that, we construct the angles with vertices at the remaining balance points. In such a way, we obtain a collection of disjoint constructions, which includes cups, trolleybuses, and angles. We foliate all the remaining subdomains by the left or right tangents. By our definition of balance points and tails, and also by Propositions 3.2, 4.1, 5.3, 6.1, and 6.2, we obtain a Bellman candidate B^{Σ} with a corresponding signature Σ consisting of symbols R and L. As usual, Statement 2.2 implies the estimate $\boldsymbol{B}_\varepsilon \le B^{\Sigma}$, and the reverse inequality $B^{\Sigma} \le \boldsymbol{B}_\varepsilon$ follows from the existence of optimizers in each of the constructions involved (the reader can easily imagine the delivery curves that originate in the full cups or $\pm\infty$ "transit" through the trolleybuses and continue up to the angles).

6.4. **Properties of force functions.** In this subsection, we investigate the properties of force functions needed to prove Proposition 6.9.

Lemma 6.10. *The strict inequality $F < 0$ is fulfilled at all interior points of the right tail, except possibly for the points where f''' changes its sign from $+$ to $-$. With the same possible exception, $F > 0$ at each interior point of the left tail.*

Proof. The fact that the strict inequality is fulfilled in the screen was proved in Lemma 5.5 ($D_{\mathrm{L}} < 0$ and $D_{\mathrm{R}} < 0$). Let $F(u_0, \ell) = 0$ for some $u_0 \in (b, t^+)$. Then

$$F(u, \ell) = \int_{u_0}^{u} f'''(t) e^{-(u-t)/\varepsilon} dt$$

in some neighborhood of u_0. Since $F(u, \ell) \le 0$, the function f''' must be non-positive in some right neighborhood of u_0 and non-negative in some left neighborhood; i.e. u_0 coincides with one of the points c_j. $\square$

We also prove two formulas we use for calculating derivatives of F.

Lemma 6.11. *We have*

$$dD_{\mathrm{L}}(a,b) = \Big(f'''(a) + \frac{2D_{\mathrm{L}}(a,b)}{b-a}\Big) da, \tag{6.15}$$

$$dD_{\mathrm{R}}(a,b) = \Big(f'''(b) - \frac{2D_{\mathrm{R}}(a,b)}{b-a}\Big) db. \tag{6.16}$$

Proof. We begin with writing the derivative of the cup equation (5.6). We use an invariant form not depending on the parametrization:

$$D_{\mathrm{R}}(a,b)db + D_{\mathrm{L}}(a,b)da = 0\,. \tag{6.17}$$

Using this relation, we write the differential of the average $\langle f''\rangle_{[a,b]}$ in two different forms:

$$\begin{aligned} d\langle f''\rangle_{[a,b]} &= \frac{f''(b)db - f''(a)da}{b-a} - \frac{f'(b)-f'(a)}{(b-a)^2}(db-da) \\ &= \frac{D_{\mathrm{R}}db - D_{\mathrm{L}}da}{b-a} = \frac{2D_{\mathrm{R}}db}{b-a} = -\frac{2D_{\mathrm{L}}da}{b-a}\,. \end{aligned}$$

This immediately yields both (6.15) and (6.16). □

After this preparation, it is easy to find the partial derivative of the force with respect to the screen size.

Lemma 6.12. *We have*

$$\frac{\partial}{\partial \ell}F(u,\ell) = \Big(\frac{2}{\ell} - \frac{1}{\varepsilon}\Big)\begin{cases} -D(a)a'e^{-(a-u)/\varepsilon}, & u \in (-\infty, a(\ell))\,; \\ \quad 0, & u \in (a(\ell), b(\ell))\,; \\ -D(b)b'e^{-(u-b)/\varepsilon}, & u \in (b(\ell), +\infty)\,, \end{cases}$$

or

$$\frac{\partial}{\partial \ell}F(u,\ell) = \Big(\frac{2}{\ell} - \frac{1}{\varepsilon}\Big)\frac{D_{\mathrm{L}}D_{\mathrm{R}}}{D_{\mathrm{L}}+D_{\mathrm{R}}}\begin{cases} e^{-(a-u)/\varepsilon}, & u \in (-\infty, a(\ell))\,; \\ 0, & u \in (a(\ell), b(\ell))\,; \\ -e^{-(u-b)/\varepsilon}, & u \in (b(\ell), +\infty)\,. \end{cases}$$

Proof. We can easily check these formulas by direct calculation. Consider, for example, the case $u < a$:

$$\frac{\partial}{\partial \ell}F(u,\ell) = e^{u/\varepsilon}\Big[-\frac{dD_{\mathrm{L}}(a,b)}{d\ell}e^{-a/\varepsilon} + D_{\mathrm{L}}(a,b)\frac{a'}{\varepsilon}e^{-a/\varepsilon} + f'''(a)e^{-a/\varepsilon}a'\Big]\,.$$

Using formula (6.15), we simplify this expression:

$$\frac{\partial}{\partial \ell}F(u,\ell) = D_{\mathrm{L}}(a,b)\Big(\frac{1}{\varepsilon} - \frac{2}{\ell}\Big)e^{-(a-u)/\varepsilon}a'.$$

Thus, we have the first representation of the derivative. To obtain the second one, we must express a' in terms of D_{L} and D_{R}. Taking into account that $db = d\ell + da$ and using (6.17), we have

$$a' = -\frac{D_{\mathrm{R}}(a,b)}{D_{\mathrm{L}}(a,b) + D_{\mathrm{R}}(a,b)}\,.$$

Similarly, we can check the formulas for the case $u > b$. □

Corollary 6.13. *The force is strictly increasing with respect to the screen size on the right of the screen and strictly decreasing on the left. Inside the screen, the force does not depend on this size.*

Proof. In Lemma 5.5, it was proved that $D_{\mathrm{L}}(a,b)<0$, $D_{\mathrm{R}}(a,b)<0$, $a'<0$, and $b'>0$. Therefore, on the whole interval $\ell\in(0,2\varepsilon)$ we have

$$\begin{aligned}\frac{\partial}{\partial\ell}F(u,\ell)&>0 \qquad \text{for}\quad u>b\,;\\ \frac{\partial}{\partial\ell}F(u,\ell)&<0 \qquad \text{for}\quad u<a\,.\end{aligned}$$

Thus we are done. □

Some simple corollaries of this fact are listed below.

Corollary 6.14. *The tails grow as the screen shrinks.*

Corollary 6.15. *If $\ell>0$, then*

$$\begin{aligned}F(u,\ell)&>F(u,0) \qquad \textit{for}\quad u>c\,;\\ F(u,\ell)&<F(u,0) \qquad \textit{for}\quad u<c\,.\end{aligned}$$

The last inequalities will be used together with the following relation between two forces.

Lemma 6.16. *Let F_1 and F_2 be two forces with sources c_1 and c_2, $c_1<c_2$. Then the following two relations between these forces are fulfilled*:

$$\begin{aligned}F_1(u,\ell_1)&=e^{(c_2-u)/\varepsilon}F_1(c_2,\ell_1)+F_2(u,0)\,, \qquad u\ge c_2\,;\\ F_2(u,\ell_2)&=e^{(u-c_1)/\varepsilon}F_2(c_1,\ell_2)+F_1(u,0)\,, \qquad u\le c_1\,.\end{aligned}$$

Proof. The statement of the lemma becomes trivial when rewritten by the definition of forces:

$$\begin{aligned}&e^{-u/\varepsilon}\Big[D_1(b_1)e^{b_1/\varepsilon}+\int_{b_1}^{u}f'''(t)e^{t/\varepsilon}dt\Big]\\ &\quad=e^{(c_2-u)/\varepsilon}e^{-c_2/\varepsilon}\Big[D_1(b_1)e^{b_1/\varepsilon}+\int_{b_1}^{c_2}f'''(t)e^{t/\varepsilon}dt\Big]+e^{-u/\varepsilon}\int_{c_2}^{u}f'''(t)e^{t/\varepsilon}dt\,,\\ &\qquad\qquad u\in[c_2,+\infty)\,.\end{aligned}$$

The second identity is similar. □

We state the following simple corollary.

Corollary 6.17. *Let F_1 and F_2 be two forces with sources c_1 and c_2, $c_1<c_2$. If c_2 gets into the tail of F_1, then $F_1(u)\le F_2(u)$ for $u\ge c_2$. If c_1 gets into the tail of F_2, then the same inequality is true for $u\le c_1$.*

Proof. If $c_2\le t_1^+$, then using Lemma 6.16 and Corollary 6.15 we can write the following inequality:

$$F_1(u,\ell_1)=e^{(c_2-u)/\varepsilon}F_1(c_2,\ell_1)+F_2(u,0)\le F_2(u,0)\le F_2(u,\ell_2)$$

for $u\in[c_2,+\infty)$. In a similar way, if $c_1\ge t_2^-$, then

$$F_2(u,\ell_2)=e^{(u-c_1)/\varepsilon}F_2(c_1,\ell_2)+F_1(u,0)\ge F_1(u,0)\ge F_1(u,\ell_1)$$

for $u\in(-\infty,c_1]$. □

Till now, we were investigating the dependence of a force from the size of its screen. Now we treat the behavior of a force with respect to the first variable.

Lemma 6.18. *We have*

$$\frac{\partial}{\partial u}F(u,\ell)=\begin{cases}-f'''(u)+\varepsilon^{-1}F(u,\ell), & u\in(-\infty,a(\ell))\,;\\[2ex] -f'''(u)+\dfrac{2}{\tilde b(u)-u}F(u,\ell), & u\in(a(\ell),c)\,;\\[2ex] f'''(u)-\dfrac{2}{u-\tilde a(u)}F(u,\ell), & u\in(c,b(\ell))\,;\\[2ex] f'''(u)-\varepsilon^{-1}F(u,\ell), & u\in(b(\ell),+\infty)\,.\end{cases}$$

Proof. The formulas for the derivatives out of the screen are evident. We use Lemma 6.11 to calculate $D'(u)$. On the left screen, we have $a=u$, $b=\tilde b(u)$, and $D(u)=-D_{\mathrm{L}}(u,\tilde b(u))$. Therefore, formula (6.15) yields

$$D'(u)=-f'''(u)+\frac{2D(u)}{\tilde b-u}\,.$$

Similarly, using (6.16), we get

$$D'(u)=f'''(u)-\frac{2D(u)}{u-\tilde a}$$

on the right screen. □

To determine balance points, we need to know the behavior of the sum of two neighbor forces.

Corollary 6.19. *If F_1 and F_2 are two forces with sources c_1 and c_2, $c_1<c_2$, then*

$$\varepsilon\frac{\partial}{\partial u}(F_1(u,\ell_1)+F_2(u,\ell_2))=\begin{cases}F_2(u,\ell_2)-\dfrac{2\varepsilon}{u-\tilde a_1(u)}\,F_1(u,\ell_1), & u\in(c_1,b_1)\,;\\[2ex] F_2(u,\ell_2)-F_1(u,\ell_1), & u\in(b_1,a_2)\,;\\[2ex] \dfrac{2\varepsilon}{\tilde b_2(u)-u}\,F_2(u,\ell_2)-F_1(u,\ell_1), & u\in(a_2,c_2)\,.\end{cases}$$

Corollary 6.20. *If F_1 and F_2 are two forces with sources c_1 and c_2, $c_1<c_2$, then the sum F_1+F_2 is strictly increasing in the intersection of the right tail of F_1 and the left tail of F_2.*

Proof. By the formula from the preceding corollary, we have $\frac{\partial}{\partial u}(F_1+F_2)>0$ for all $u\in(c_1,t_1^+)\cap(t_2^-,c_2)$, except possibly for a finite number of points (see Lemma 6.10). □

Corollary 6.21. *If F_1 and F_2 are two forces with sources c_1 and c_2 such that*

$$c_1<t_2^-\le t_1^+<c_2,$$

then the sum F_1+F_2 has exactly one root in the intersection of the tails, $[t_2^-,t_1^+]$.

Proof. By the preceding corollary, the sum F_1+F_2 is strictly increasing on $[t_2^-,t_1^+]$. Therefore, since the continuous function F_1+F_2 has opposite signs at t_2^- and t_1^+ (because $F_i(t_i^\pm)=0$), it has exactly one root on this interval. □

We conclude our investigation of the force functions with two other important facts.

Lemma 6.22. *If the source of a force function belongs to a tail of another force, then both tails of the first force are included in the tail of the second.*

Proof. First, we note that both sources of two forces cannot be covered by the tails of each other. Indeed, if this occurs, the sum F_1+F_2 would be non-negative at the left end of the segment $[c_1,c_2]$ ($F_1(c_1)=0$, $F_2(c_1)\geq 0$) and non-positive at its right end ($F_1(c_2)\leq 0$, $F_2(c_2)=0$). But since F_1+F_2 is strictly increasing on $[c_1,c_2]$ (see Corollary 6.20), this is impossible.

Assume that c_2 lies in the right tail of F_1. We have to check that both tails of F_2 are in the right tail of F_1, i.e. $[t_2^-,t_2^+]\subset[c_1,t_1^+]$. We have just proved that $c_1<t_2^-$. The second inequality $t_1^+>t_2^+$ is contained in Corollary 6.17. The case where c_1 is in the left tail of F_2 can be treated similarly. □

Lemma 6.23. *If two forces are balanced, then the source of one of them cannot lie in a tail of another one.*

Proof. Let $c_1<c_2$. If we assume that $c_1\geq t_2^-$, then

$$F_1(c_1)+F_2(c_1)=F_2(c_1)\geq 0.$$

If $c_2\leq t_1^+$, then

$$F_1(c_2)+F_2(c_2)=F_1(c_2)\leq 0.$$

In any case, the sum F_1+F_2 cannot have a root on (c_1,c_2); i.e. the forces F_1 and F_2 cannot be balanced. □

6.5. **Algorithm.**

Cleaning. Consider some collection of points $\{c_k\}_{k=0}^N$ and forces $\{F_k\}$ generated by these points. Then we can remove from the collection those points c_k that lie in a tail of some force function F_j, $j\neq k$. We call such an operation *the cleaning.* We denote the set $\{c_{k_j}\}_{j=0}^m$ of points that remain after the cleaning by $\{c_k\}_{k=0}^N$, though the number N may have changed. What is more, the symbol c_k may denote another point after cleaning.

The union of forces' tails cannot become smaller after the cleaning. Indeed, the cleaning removes only those forces whose tails are contained entirely in a tail of some other force.

Compression. Let $\{F_k\}$ be a balanced collection of forces. Suppose some u_{j+1} — the balance point of F_j and F_{j+1} — got into the screen of F_j. We generate a new collection of forces by the following rule. First, we reduce the screen of F_j in such a way that u_{j+1} becomes the right end of this screen. The point u_{j+1} remains to be a balance point of newly defined F_j and old F_{j+1}. The reduction of the screen enlarges the tails of F_j so they could cover some neighbor points c_k. Then we have to make the cleaning. The procedure just described is called *the right compression.* A similar procedure (the decreasing of ℓ_{j+1} and the cleaning), where u_{j+1} gets into the screen of F_{j+1}, is called *the left compression.* We note that the left compression can change the structure of the force collection only on the right of u_{j+1}, and the right compression does not change the structure of the forces on the right of u_{j+1}.

Indeed, consider the right compression. The new tail of the force cannot reach the point c_{j+1} (see Lemma 6.23), because the forces F_j and F_{j+1} are still balanced.

So, all the forces on the right of u_{j+1} remain the same. But what can happen on the left? Nothing can happen provided $j = 0$: either $c_0 = -\infty$ and there is nothing on the left, or the point c_0 is the last point and its left tail still reaches $-\infty$. But if $j > 0$, the numeration of the remaining forces could change. Assume that the former point c_j got a number i, $i \le j$, after the compression. The balance point of the forces F_i and F_{i-1} could move only to the left, because the new force F_i is not less than the old one (either by Lemma 6.13 if there was no cleaning, or by Corollary 6.17 if the cleaning was performed). Consequently, the only new balance point that could get inside a screen is the point in the right screen of F_{i-1}. Thus, the new balance points cannot get into the left screens after the right compression. The only point that can get into the right screen lies on the left of the compressed screen.

The situation is symmetric for the left compression. All the changes occur on the right of the screen being compressed. What is more, the only screen that can get a new balance point is the left screen of the first newly defined force on the right of the screen being compressed.

The whole algorithm. Our algorithm consists of a series of left compressions beginning from F_1 and going to the right, and the right compressions being performed from right to left. Of course, we can change the order of the left and the right compressions. We note that in fact we do not begin from the leftmost and rightmost forces, because there are no balance points both on the left of c_0 and on the right of c_N. Indeed, either $c_0 = -\infty$, or the left tail of F_0 fills the ray $(-\infty, c_0]$. Similarly, either $c_N = +\infty$, or the right tail of F_N fills the ray $[c_N, +\infty)$.

Our algorithm begins with the cleaning of the family $\{F_k(u, 2\varepsilon)\}_{k=0}^{N}$. The tails of neighbor forces have non-empty intersection, because $[c_{j-1}, v_j]$ lies in the tail of F_{j-1}, and $[v_j, c_j]$ lies in the tail of F_j. This property persists after the cleaning, and by Lemma 6.21 we get a balanced family of forces.

Thus, in order to prove that the algorithm provides a system balanced completely, it remains to verify that there are no balance points inside the screens. Indeed, one of the ends of the small screens (those that are smaller than 2ε) coincides with a balance point. Each small screen was compressed, so a balance point arrived at one of its ends. All the points that lay inside the left screens were sent to the boundary of their screens as we performed the left compressions. Hence, we removed all the balance points from the left screens with the left compressions. What is more, this procedure did not send any balance points into the right screens. Similarly, the passage from right to left (execution of the right compressions) removed the balance points from the right screens and did not change the situation inside the left ones. So there are no balance points inside the screens, and we are done.

The only thing we have to mention is that the union of tails of the achieved collection coincides with the whole real line. This is a consequence of the fact that all the tails of the initial family cover the whole line, and both the cleaning and the compression do not reduce this cover.

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