

THE AFFINE AND PROJECTIVE GROUPS ARE MAXIMAL

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ABSTRACT. We show that the groups $AGL_n(\mathbb{Q})$ (for $n \geq 2$) and $PGL_n(\mathbb{Q})$ (for $n \geq 3$), seen as closed subgroups of S_ω , are maximal-closed.

Subgroups of the infinite symmetric group S_ω (or more generally, S_κ for any cardinal) have been studied extensively, and much work has been done concerning maximal subgroups. For instance, in [MP90], the authors proved that if a subgroup $G \leq S_\omega$ is not highly transitive, then it is contained in a maximal subgroup. In particular, closed (in the usual product topology) proper subgroups of S_ω are contained in maximal subgroups. Macpherson and Neumann [MN90, Observation 6.1] proved that if $H \leq S_\omega$ is maximal and non-open, H must be highly transitive. In particular, closed non-open subgroups cannot be maximal.

Later, in [BST93], the authors proved that it is consistent with ZFC that for every $\kappa \geq \omega$, there is a subgroup $G \leq S_\kappa$ which is not contained in a maximal subgroup. As far as we know, it is an open question whether this can be proved in ZFC. However, not much is known about maximal-closed subgroups (i.e., groups that are maximal in the family of closed subgroups of S_ω). In [MN90, Example 7.10] several examples of such groups are given, including $AGL_\omega(p)$ for a prime p (meaning the affine group acting on the infinite dimensional vector space over \mathbb{F}_p) and $PGL_\omega(q)$ for a prime power q .

If $G \leq S_\omega$ is the automorphism group of an ω -categorical structure M , then closed supergroups of M in S_ω are in one-to-one correspondence with reducts of M . The full classification of reducts is known for a number of ω -categorical structures ([Cam76, Tho91, Tho96, BP11]). Nevertheless, the main question asked by Thomas over 20 years ago remains unresolved: we do not know if every homogeneous structure on a finite relational language has only finitely many reducts.

In another direction, Junker and Ziegler [JZ08] asked for a converse to that question: if a structure is not ω -categorical, does it necessarily have infinitely many reducts? Let M be a countable structure in a language L and let $\text{Aut}(M) \leq S_\omega$ be its automorphism group. A (*proper*) *group reduct* of M is a structure M' with the same domain as M whose automorphism group $\text{Aut}(M')$ properly contains $\text{Aut}(M)$. It is non-trivial if it is not the whole of S_ω . On the other hand, a (*proper*) *definable reduct* of M is a proper reduct of the structure M in the usual model theoretic sense (see Definition 1.38). It is non-trivial if it is not interdefinable

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with equality. In general, classifying group reducts and definable reducts of a given structure are two independent questions: a group reduct need not be definable, and a definable reduct need not admit additional automorphisms. Thus Junker and Ziegler's question breaks into two sub-questions.

In a recent paper [BM13], Bodirsky and Macpherson have answered the two versions of Junker and Ziegler's question negatively: there are non- ω -categorical structures which admit no reducts in either sense. The automorphism groups involved in their work are of a different flavor than those that appear in the study of ω -categorical structures. Whereas the techniques used in the study of reducts of ω -categorical structures involve mainly Ramsey properties, Bodirsky and Macpherson make use of the theory of Jordan groups. Jordan groups can be classified according to the type of structure that they preserve. Bodirsky and Macpherson focus on automorphism groups of D -relations.

In this paper, we study some members of another family of Jordan groups: that of automorphism groups of Steiner systems. The most natural Steiner system is the family of lines in an affine or projective space, giving rise to the groups $AGL_n(\mathbb{Q})$ and $PGL_n(\mathbb{Q})$. We show, making an essential use of Adeleke and Macpherson's classification theorem of Jordan groups, that those groups are maximal-closed in S_ω (in dimensions larger than 1). This answers a question of Macpherson and Bodirsky from [BM13] about examples of *countable* maximal-closed subgroups of S_ω . We also deduce from these results that the structures (\mathbb{Q}^n, f) for $n \geq 1$, where $f(x, y, z) = x + y - z$, admit no definable reduct, which answers another question from [BM13].

After our paper was submitted it came to our attention that in a recent work, Bogomolov and Rovinsky [BR13] proved that for $n \geq 3$, $PGL_n(F)$ is a maximal-closed subgroup of the group of all permutations of F for any infinite field F , using completely different methods. In particular their result implies ours on $PGL_n(\mathbb{Q})$. However, our result about $AGL_n(\mathbb{Q})$ does not extend to all infinite fields (see an example after Definition 1.1).

A few words about the proof. The classification theorem for Jordan groups (Fact 1.11) says that any 3-transitive Jordan group must preserve a structure of one of the following types: a cyclic separation relation, a D -relation, a k -Steiner system, or a limit of Steiner systems (all of which will be defined later). Thus to prove that $AGL_n(\mathbb{Q})$, say, is maximal-closed, we will show that any group properly containing it does not preserve any one of these structures. The separation relation, D -relation and Steiner system are treated with combinatorial *ad-hoc* constructions. To rule out limits of Steiner systems, we use a slightly more general (or generalizable) construction which shows that if a (sufficiently transitive) group containing $AGL_n(\mathbb{Q})$ preserves a limit of Jordan sets (see Definition 1.28), then it actually preserves a Steiner system. We believe that this argument could apply to other groups as well.

1. $AGL_n(\mathbb{Q})$ IS MAXIMAL

Definition 1.1. For a vector space V over a field F , $AGL(V)$ is the group of permutations of V consisting of maps of the form $x \mapsto Tx + b$ where $T \in GL(V)$ (i.e., an invertible linear map) and $b \in V$. For $n < \omega$, let $AGL_n(\mathbb{Q})$ be $AGL(\mathbb{Q}^n)$ and also let $AGL_\omega(\mathbb{Q})$ be $AGL(\mathbb{Q}^{<\omega})$ (where $\mathbb{Q}^{<\omega}$ is the vector space of infinite sequences with finite support). Similarly we define $GL_n(\mathbb{Q})$.

Note that for a vector space V over \mathbb{Q} , $AGL(V)$ is the group of automorphisms of the structure (V, f) where $f(x, y, z) = x + y - z$.

The group S_ω of all permutations of \mathbb{N} is a topological group where the topology is the product topology (i.e., the one induced by $\mathbb{N}^{\mathbb{N}}$). In fact for a natural metric it is a Polish group — $d(\sigma_1, \sigma_2) = 1/(\min\{n \mid \sigma_1(n) \neq \sigma_2(n) \vee \sigma_1^{-1}(n) \neq \sigma_2^{-1}(n)\} + 1)$. For $n \leq \omega$, we can embed $AGL_n(\mathbb{Q})$ naturally (after choosing a bijection from \mathbb{Q}^n , or $\mathbb{Q}^{<\omega}$, to \mathbb{N}) as a subgroup of S_ω . It is then a closed subgroup, since it is the automorphism group of a structure. Note also that for $n < \omega$ it is countable, while for $n = \omega$ it is uncountable.

Examples.

- [BM13, Proposition 5.7] Take the structure (\mathbb{Q}, f) , where $f(x, y, z) = x + y - z$. Then its automorphism group is $AGL_1(\mathbb{Q})$. It is also countable and closed, but it is not maximal. For a prime p , let $C_p(x, y, z)$ be $v_p(y - x) < v_p(z - y)$, where v_p is the p -adic valuation. This is a C -relation on \mathbb{Q} , and its automorphism group is uncountable:

For every $\eta : \omega \rightarrow 2$, let $\sigma_\eta : \mathbb{Q} \rightarrow \mathbb{Q}$ be the following map: for each $x \in \mathbb{Q}$, let $\sum_{i=-n}^\infty a_i p^i$ be its (unique) p -adic expansion (where $n \in \mathbb{N}$, $a_i < p$ and $a_{-n} \neq 0$), and let $\sigma_\eta(x) = a_{-n} p^{-n} + \sum_{i=-(n+1)}^0 a_i^{\eta(-i)} p^i + \sum_{i=1}^\infty a_i p^i$ where $a_i^0 = a_i$ and $a_i^1 = a_i + 1 \pmod p$. Then $\{\sigma_\eta \mid \eta : \omega \rightarrow 2\}$ is a set of distinct automorphisms of (\mathbb{Q}, C) .

- Now take the group $AGL_2(R)$ where R is any field extension of \mathbb{Q} . The plane R^2 can be seen as a \mathbb{Q} -vector space, and so the group of affine transformation over \mathbb{Q} properly contains $AGL_2(R)$; hence that group is not maximal.

In this section we will prove:

Theorem 1.2. *For $2 \leq n \leq \omega$, $AGL_n(\mathbb{Q})$ is a maximal-closed subgroup of S_ω . So for $2 \leq n < \omega$, $AGL_n(\mathbb{Q})$ is a countable maximal-closed group.*

Assumption 1.3. *Throughout this section, we fix $2 \leq n \leq \omega$. Let $\Omega = \mathbb{Q}^n$ in case $n < \omega$ and $\mathbb{Q}^{<\omega}$ in case $n = \omega$.*

As we noted, $AGL_n(\mathbb{Q})$ is closed. In order to prove that it is maximal, it is enough to show that any group G containing it is *highly transitive*, i.e., k -transitive for every $k < \omega$. So this is what we will do.

Definition 1.4. For 3 collinear points $(a, b, c) \in \Omega^3$ we will say that they have *ratio* $r \in \mathbb{Q}$ if $c = ra + (1 - r)b$.

The following lemma (whose proof we leave to the reader) describes the orbits of triples in Ω under the action of $AGL_n(\mathbb{Q})$.

Lemma 1.5. *$AGL_n(\mathbb{Q})$ is 2-transitive. Moreover, the orbits of its action on triples of distinct elements from Ω are:*

- *For each $r \in \mathbb{Q} \setminus \{0, 1\}$ and $a \neq b$, $\{(a, b, c) \mid c = ra + (1 - r)b\}$ — all triples of ratio r .*
- *$\{(a, b, c) \mid (a, b, c) \text{ are not collinear}\}$.*

Fact 1.6 ([Sch62, BR98]—The fundamental theorem of affine geometry). For a permutation σ of Ω , $\sigma \in AGL_n(\mathbb{Q})$ iff σ preserves lines iff σ takes triples of collinear points to collinear points.

Remark 1.7. In the proof of Fact 1.6, one first proves that if σ maps lines to lines (equivalently 3 collinear points to 3 collinear points), then it must take affine planes to affine planes. This is done by noting that if two lines intersect, then they lie on the same plane. Then, the main point is to prove additivity of σ , which is done geometrically; i.e., assuming that $\sigma(0) = 0$, for any $u, v \neq 0$ in Ω , $u + v$ is the intersection of the lines L_1 — the unique line in the plane containing u, v which is parallel to v containing u — and L_2 , which is defined similarly. Since $\sigma(u + v)$ is in the same plane spanned by $\sigma(0), \sigma(u)$ and $\sigma(v)$, the same geometrical property holds for $\sigma(u + v)$.

Lemma 1.8. *Let σ be any permutation of Ω . Suppose that L is a line containing 0 which σ does not map to a line. Then for any $r \in \mathbb{Q} \setminus \{0, 1\}$ there are 3 collinear points of ratio r on L which are sent by σ to 3 non-collinear points.*

Proof. For $a \neq b \in \Omega$, denote by $L(a, b) = \{c \in \Omega \mid \exists r (c = ra + (1 - r)b)\}$ the unique line that contains both a and b . Let E be the equivalence relation defined on $L \setminus \{0\}$ by: a and b are equivalent if $L(\sigma(0), \sigma(a)) = L(\sigma(0), \sigma(b))$. Equivalently, a and b are equivalent if $\sigma(b) \in L(\sigma(0), \sigma(a))$.

Suppose for contradiction that for some $r \in \mathbb{Q} \setminus \{0, 1\}$, all 3 collinear points of ratio r are sent by σ to collinear points. This means that for any $a \in L \setminus \{0\}$, $[a]_E$ contains ra (as $(a, 0, ra)$ has ratio r), $(1 - r)a$ (as $(0, a, (1 - r)a)$ has ratio r), $\frac{1}{r}a$ (as $(\frac{1}{r}a, 0, a)$ has ratio r) and $\frac{1}{1-r}a$ (as $(0, \frac{1}{1-r}a, a)$ has ratio r).

Next, we note that $a E -a$ for $a \in L \setminus \{0\}$. Since $0 = a - a = r(\frac{1}{r}a) + (1 - r)(-\frac{1}{1-r}a)$, $(\frac{1}{r}a, -\frac{1}{1-r}a, 0)$ has ratio r , so $\sigma(0) \in L(\sigma(\frac{1}{r}a), \sigma(-\frac{1}{1-r}a))$ (note that $\frac{1}{r} \neq -\frac{1}{1-r}$). This implies that $\frac{1}{r}a E -\frac{1}{1-r}a$. But we already know that $\frac{1}{r}a E a$ and $\frac{1}{1-r}(-a) E -a$. Together we are done.

Now we will show that if $b E a$, then $a + b E a$ for $a, b \in L \setminus \{0\}$. There are two cases to consider. First, assume that $\frac{1}{r}a = \frac{1}{1-r}b$; this implies that $a + b = \frac{1}{r}a$, which we already know is E -equivalent to a . If not, then as $(\frac{1}{r}a, \frac{1}{1-r}b, a + b)$ is a tuple of 3 collinear points of ratio r , our assumption implies that $\sigma(a + b)$ is in $L(\sigma(\frac{1}{r}a), \sigma(\frac{1}{1-r}b))$. Since $\frac{1}{1-r}b E b E a E \frac{1}{r}a$, this line equals $L(\sigma(0), \sigma(a))$ and we are done.

We conclude that for all $a \in L \setminus \{0\}$ and $n \in \mathbb{Z}$, $na E a$. But then $\frac{1}{n}a E a$ as well, so $[a]_E = L \setminus \{0\}$. This contradicts our assumption. \square

Corollary 1.9. *If G is a permutation group of Ω properly containing $AGL_n(\mathbb{Q})$, then G is 3-transitive.*

Proof. By Fact 1.6, there is some $\sigma \in G$ which does not preserve lines. By pre-composing with an element from $AGL_n(\mathbb{Q})$, we may assume that there is a line L containing 0 which σ does not preserve. The corollary follows from Lemma 1.5 and Lemma 1.8. \square

Now we are going to use a theorem of Adeleke and Macpherson about the classification of Jordan groups.

Definition 1.10. Suppose G is a group acting on a set X . A set $\emptyset \neq A \subseteq X$ of size at least 2 is a *Jordan set* if the pointwise stabilizer $G_{X \setminus A}$ acts transitively on A .

If there is a Jordan set $A \subseteq X$ such that for all $k \in \mathbb{N}$ for which G is k -transitive, $|X \setminus A| \geq k$, then (G, X) (or just G) is called a *Jordan group*.

Let A be an affine subspace of Ω of dimension $< n$. Then $\Omega \setminus A$ is a Jordan set. To see this, assume without loss that A is a linear subspace (i.e., $0 \in A$). Then, let B be a basis of A , and let $x, y \notin A$. Then $B \cup \{x\}$ and $B \cup \{y\}$ are both independent, so there is some $\sigma \in GL_n(\mathbb{Q})$ taking x to y while fixing B (so also A). Hence $\Omega \setminus A$ witnesses that $AGL_n(\mathbb{Q})$ is a Jordan group. In fact, any group G containing $AGL_n(\mathbb{Q})$ is a Jordan group by the same argument.

Let us now turn to the classification theorem of Adeleke and Macpherson:

Fact 1.11 ([AM96, Theorem 1.0.2] Adeleke and Macpherson, 1996). Suppose G is an infinite 3-transitive Jordan group acting on a space X which is not highly transitive. Then G must preserve on X one of the following structures:

- (1) a cyclic separation relation;
- (2) a D relation;
- (3) a Steiner k system for $k \geq 2$;
- (4) a limit of Steiner systems.

For the reader who looks at the reference in [AM96], the relevant clauses in Theorem 1.0.2 there are (v) and (vi). Also note that G is automatically primitive (i.e., does not preserve any non-trivial equivalence relation on Ω), since it is 2-transitive.

We will show that in fact $AGL_n(\mathbb{Q})$ does not preserve any of the structures in (1)–(3) (except the case $k = 2$ in (3)) and that any group properly containing it does not preserve a structure of the form (3) or (4). Using Corollary 1.9 (3-transitivity) we conclude that any such group is highly transitive.

So in each following subsection we will rule out one of these structures. Our definitions are all taken from [BMMN97] except for that of a limit of Steiner systems, which is taken from [AM96].

1.1. A cyclic separation relation.

Definition 1.12. A quaternary relation S defined on a set X is a *cyclic separation relation* if it satisfies the following conditions for all $\alpha, \beta, \gamma, \delta, \varepsilon \in X$:

- (s1) $S(\alpha, \beta; \gamma, \delta) \Rightarrow S(\beta, \alpha; \gamma, \delta) \wedge S(\gamma, \delta; \alpha, \beta)$;
- (s2) $S(\alpha, \beta; \gamma, \delta) \wedge S(\alpha, \gamma; \beta, \delta) \Leftrightarrow \beta = \gamma \vee \alpha = \delta$;
- (s3) $S(\alpha, \beta; \gamma, \delta) \Rightarrow S(\alpha, \beta; \gamma, \varepsilon) \vee S(\alpha, \beta; \delta, \varepsilon)$;
- (s4) $S(\alpha, \beta; \gamma, \delta) \vee S(\alpha, \gamma; \delta, \beta) \vee S(\alpha, \delta; \beta, \gamma)$.

The idea is that there is, in the background, a circle C , and $S(\alpha, \beta, \gamma, \delta)$ holds iff γ, δ are on different “components” of $C \setminus \{\alpha, \beta\}$. See Figure 1.1.

Lemma 1.13. *Suppose that S is a cyclic separation relation on a set X . Then the following holds for all distinct $\alpha, \beta, \gamma, \delta, \delta'$ in X :*

- (s5) $S(\alpha, \beta; \gamma, \delta) \wedge S(\alpha, \beta; \gamma, \delta') \Rightarrow S(\alpha, \delta; \beta, \delta') \vee S(\alpha, \delta'; \beta, \delta)$.

Proof. One easily sees that this property holds by observing Figure 1.1. However, we will give a formal proof. We first claim that a variant of (s3) holds:

- (s3') For all distinct $\alpha, \beta, \gamma, \delta, \varepsilon \in X$, if $S(\alpha, \beta; \gamma, \delta)$ holds, then exactly one of $S(\alpha, \beta; \gamma, \varepsilon)$ or $S(\alpha, \beta; \delta, \varepsilon)$ holds.

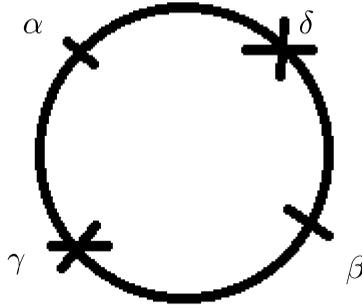


FIGURE 1.1. A cyclic separation relation

Indeed, suppose for contradiction that both $S(\alpha, \beta; \gamma, \varepsilon)$ and $S(\alpha, \beta; \delta, \varepsilon)$ hold. By (s3) and (s1) applied to $S(\alpha, \beta; \gamma, \varepsilon)$, either $S(\delta, \beta; \gamma, \varepsilon)$ or $S(\alpha, \delta; \gamma, \varepsilon)$ holds. Suppose the former occurs. Then it cannot be that $S(\gamma, \beta; \varepsilon, \delta)$ holds (or else $S(\beta, \delta; \gamma, \varepsilon)$ and $S(\beta, \gamma; \delta, \varepsilon)$ by (s1), contradicting (s2)). So applying (s3) (and (s1)) to $S(\alpha, \beta; \delta, \varepsilon)$ we get $S(\alpha, \gamma; \delta, \varepsilon)$. Since $S(\alpha, \gamma; \beta, \delta)$ is impossible by (s2), by applying (s3) to $S(\alpha, \gamma; \delta, \varepsilon)$ we get $S(\alpha, \gamma; \beta, \varepsilon)$, which contradicts $S(\alpha, \beta; \gamma, \varepsilon)$ by (s2). The other possibility is that $S(\alpha, \delta; \gamma, \varepsilon)$ holds, and it leads to a similar contradiction by replacing α and β in the argument.

Now suppose $\alpha, \beta, \gamma, \delta, \delta' \in X$, and that $S(\alpha, \beta; \gamma, \delta)$ and $S(\alpha, \beta; \gamma, \delta')$ hold. If the conclusion does not hold, then by (s4), $S(\alpha, \beta; \delta, \delta')$ holds. But then this contradicts (s3'), since now we can replace both δ and δ' with γ . \square

Proposition 1.14. *$AGL_n(\mathbb{Q})$ does not preserve a cyclic separation relation on Ω .*

Proof. Suppose it does.

Let L be a line, and choose 3 points on it, $a, b, c \in L$. Let d be any point which is not on L . By (s4), we may assume that $S(a, b; c, d)$ holds. Since $\Omega \setminus L$ is a Jordan set, the same is true for any $d' \notin L$. Finally, by (s5) we get that $S(a, d; b, d')$ or $S(a, d'; b, d)$ holds for any $d, d' \notin L$. By translation, we may assume that $0 \in L$ and that $b = -a$. By applying $GL_n(\mathbb{Q})$, we may assume that $a = e_1$, where $\{e_i \mid i < n\}$ form the standard basis for Ω . Now choose $d = e_2, d' = -e_2$.

So we have $S(e_1, e_2; -e_1, -e_2)$ or $S(e_1, -e_2; -e_1, e_2)$. There is some $\sigma \in GL_n(\mathbb{Q})$ that maps e_2 to $-e_2$, fixing e_1 . Then $\sigma(-e_2) = e_2$. So in any case we contradict (s2) by applying σ . \square

1.2. A D -relation.

Definition 1.15. A quaternary relation D defined on a set X is a D -relation if it satisfies the following conditions for all $\alpha, \beta, \gamma, \delta, \varepsilon \in X$:

- (d1) $D(\alpha, \beta; \gamma, \delta) \Rightarrow D(\beta, \alpha; \gamma, \delta) \wedge D(\gamma, \delta; \alpha, \beta)$;
- (d2) $D(\alpha, \beta; \gamma, \delta) \Rightarrow \neg D(\alpha, \gamma; \beta, \delta)$;
- (d3) $D(\alpha, \beta; \gamma, \delta) \Rightarrow D(\varepsilon, \beta; \gamma, \delta) \vee D(\alpha, \beta; \gamma, \varepsilon)$;
- (d4) $(\alpha \neq \gamma \wedge \beta \neq \gamma) \Rightarrow D(\alpha, \beta; \gamma, \gamma)$;
- (d5) α, β, γ distinct $\Rightarrow \exists \delta (\gamma \neq \delta \wedge D(\alpha, \beta; \gamma, \delta))$.

The idea behind this relation is that there is some tree in the background, and D is applied to “directed branches” where for such branches $\alpha, \beta, \gamma, \delta, D(\alpha, \beta; \gamma, \delta)$

holds if the shortest “path” between α and β does not intersect the shortest path between γ and δ . See Figure 1.2.

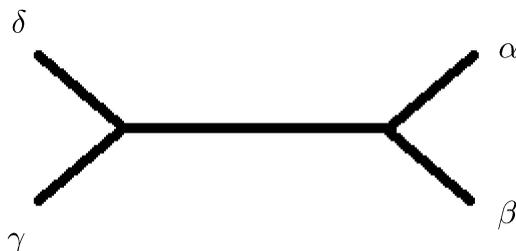


FIGURE 1.2. D relation

As in Section 1.1, we have:

Lemma 1.16. *Suppose that D is a D -relation on a set X . Then the following axiom holds:*

(d6) *If $D(\alpha, \beta; \gamma, \delta)$ and $D(\alpha, \beta; \gamma, \delta')$, then $D(\alpha, \beta; \delta, \delta')$.*

Proof. Use (d3) and (d1) to try to replace γ by δ' in $D(\alpha, \beta; \gamma, \delta)$ and then to try to replace γ by δ in $D(\alpha, \beta; \gamma, \delta')$. If both fail, then it must be that $D(\delta', \beta; \gamma, \delta)$ and $D(\delta, \beta; \gamma, \delta')$, which together contradict (d2) and (d1). \square

Proposition 1.17. *$AGL_n(\mathbb{Q})$ does not preserve a D -relation on Ω .*

Proof. Suppose it does.

Let L be a line in Ω . Let $a, b, c \in L$ be distinct. By (d5) for some $d \neq c$, $D(a, b; c, d)$. Note that by (d2) and (d1), it must be that $d \neq a, b$. If $d \in L$, then by (d3) we can replace either d or a by some $d' \notin L$. Using (d1) in the latter case, we may then assume that $D(a, b; c, d)$ holds with $d \notin L$. Since $\Omega \setminus L$ is a Jordan set, by (d6) we have that $D(a, b; d, d')$ for any $d, d' \notin L$.

By applying $AGL_n(\mathbb{Q})$, it follows that for any a, b, d and d' , if d and d' are not on the line determined by a, b , then $D(a, b; d, d')$.

So, letting $\{e_i \mid i < n\}$ be the standard basis of Ω , let $a = -e_1$, $b = -e_2$, $d = e_1$ and $d' = e_2$. But then let $\sigma \in GL_n(\mathbb{Q})$ replace e_2 by $-e_2$ while fixing e_1 . Then by applying σ it follows that $D(-e_1, -e_2; e_1, e_2)$ and $D(-e_1, e_2; e_1, -e_2)$. By (d1) and (d2) this is a contradiction. \square

1.3. Steiner systems.

Definition 1.18. Let $k \in \mathbb{N}$ be such that $k \geq 2$. A *Steiner k -system* (X, \mathcal{B}) consists of a set X of *points* and set \mathcal{B} of *blocks* where $\mathcal{B} \subseteq \mathcal{P}(X)$, for all $\mathbf{b}_1, \mathbf{b}_2 \in \mathcal{B}$, $|\mathbf{b}_1| = |\mathbf{b}_2| > k$ and:

- (1) There is more than one block.
- (2) If $\alpha_1, \dots, \alpha_k$ are distinct points in X , then there is a unique block $\mathbf{b} \in \mathcal{B}$ containing them.

Example 1.19. Let $X = \Omega$ and \mathcal{B} be the set of lines in X . Then (X, \mathcal{B}) forms a 2-Steiner system. So $AGL_n(\mathbb{Q})$ preserves a 2-Steiner system on Ω .

Remark 1.20. If (X, \mathcal{B}) is a k -Steiner system, then for any permutation σ of X , σ preserves \mathcal{B} iff σ preserves the $k + 1$ -ary relation R defined as $R(x_1, \dots, x_{k+1})$ iff x_1, \dots, x_{k+1} lie in the same block.

Lemma 1.21. *If (Ω, \mathcal{B}) is a k -Steiner system such that $AGL_n(\mathbb{Q})$ preserves and $a_1, \dots, a_k \in \Omega$ are distinct points contained in some affine subspace A of dimension $< n$, then the block $\mathfrak{b} \in \mathcal{B}$ they determine is contained in A .*

Proof. Suppose that for some $y \notin A$, $y \in \mathfrak{b}$. Then since $\Omega \setminus A$ is a Jordan set, it follows by (2) in Definition 1.18 that \mathfrak{b} contains $\Omega \setminus A$. Let L be a line disjoint from A in Ω . So \mathfrak{b} contains L (in particular, k points from L), but also some points outside of L . By the same argument, \mathfrak{b} contains $\Omega \setminus L$. Together $\mathfrak{b} = \Omega$, contradicting (1) in Definition 1.18. □

Lemma 1.22. *If G is any group of permutations of Ω properly containing $AGL_n(\mathbb{Q})$ which preserves a k -Steiner system on Ω , then $k > 2$.*

Proof. By Lemma 1.21, if $k = 2$, then blocks are contained in lines. However, by Corollary 1.9, the action of G on Ω is 3-transitive, so it takes 3 points on the same block (so collinear) to 3 non-collinear points (so not on the same block). Contradiction. □

Lemma 1.23. *If (Ω, \mathcal{B}) is a k -Steiner system that $AGL_n(\mathbb{Q})$ preserves, then all blocks $\mathfrak{b} \in \mathcal{B}$ have size at least $k + 2$.*

Note that by Definition 1.18, blocks must contain at least $k + 1$ points, but this lemma asks for one more.

Proof. We divide the proof into two cases:

Case 1. $k = 2l + 1$ is odd.

Let $\mathfrak{b} \in \mathcal{B}$ be the unique block containing

$$s = \{-le_1, -(l - 1)e_1, \dots, -e_1, 0, e_1, \dots, (l - 1)e_1, le_1\},$$

where as usual $\{e_i \mid i < n\}$ is a standard basis for Ω . Then this block contains at least one more point x , which we already know is on the line $L = \mathbb{Q}e_1$ by Lemma 1.21. Then any map $\sigma \in GL_n(\mathbb{Q})$ taking e_1 to $-e_1$ will take x to $-x$ while fixing s , so $-x \in \sigma(\mathfrak{b})$ but $\sigma(\mathfrak{b}) = \mathfrak{b}$ since they both contain s . So \mathfrak{b} contains at least $k + 2$ points.

Case 2. $k = 2l$ is even.

Let $\mathfrak{b} \in \mathcal{B}$ be the unique block containing

$$s' = \{-le_1, -(l - 1)e_1, \dots, -e_1, e_1, \dots, (l - 1)e_1, le_1\}.$$

Again, \mathfrak{b} contains at least one more point x on this line. If the argument for the odd case fails, it means that $x = 0$. So now we may assume that \mathfrak{b} contains $s = s' \cup \{0\}$. Let $\sigma \in AGL_n(\mathbb{Q})$ be the map $x \mapsto x + e_1$. Then $|\sigma(s) \cap s| \geq k$ (namely $\sigma(s) \cap s = \{-(l - 1)e_1, \dots, le_1\}$), so $\sigma(\mathfrak{b}) = \mathfrak{b}$. But then \mathfrak{b} contains $(l + 1)e_1$ and we are done. □

Remark 1.24. Suppose that $L \subseteq \Omega$ is a line and that $\sigma \in AGL_n(\mathbb{Q})$ with $\sigma(L) = L$. If for some $a \neq b, x \in L$, $\sigma(a) = b$ and $\sigma(x) = x$, then unless $x = (a + b)/2$, $\langle \sigma^m(a) \mid m < \omega \rangle$ is without repetition.

Indeed, after conjugating by an element from $AGL_n(\mathbb{Q})$ we may assume that $L = \mathbb{Q}e_1$ and that $x = 0$. Thus it is enough to note that if $\tau \in GL_1(\mathbb{Q}) = \mathbb{Q}^\times$, $r \in \mathbb{Q} \setminus \{0\}$, then for all m , $\tau^m(r) = r$ iff $\tau^m = \text{id}$, which implies that $\tau = \text{id}$ or $\tau = -\text{id}$.

Proposition 1.25. $AGL_n(\mathbb{Q})$ does not preserve a k -Steiner system on Ω for $k > 2$.

Proof. Suppose $AGL_n(\mathbb{Q})$ preserves the k -Steiner system (Ω, \mathcal{B}) and $k > 2$. Let \mathfrak{b} be the block determined by $\{me_1 \mid 0 \leq m < k - 1\} \cup \{e_2\}$. By Lemma 1.23, this block contains at least two more points, x and y , and neither of them is in the line $L' = \mathbb{Q}e_1$ (by Lemma 1.21). By Lemma 1.21, x and y are in the plane spanned by $\{e_1, e_2\}$. Note that in general, for distinct $a, b, c \in \Omega$, if $(a + b)/2, (b + c)/2, (a + c)/2$ are collinear, then a, b, c are collinear and contained in the same line. This line contains b as $b = \frac{a+b}{2} + \frac{b+c}{2} - \frac{a+c}{2}$, and similarly a and c . It follows that at least for one pair from $\{e_2, x, y\}$, say x and y , $(x + y)/2 \notin L'$.

Case 1. The line L determined by y and x intersects the line L' .

Call the intersection point x_0 . Let $\sigma \in AGL_n(\mathbb{Q})$ fix L' and map x to y . Then σ must fix setwise the line L . By Remark 1.24, $\langle \sigma^m(x_0) \mid m < \omega \rangle$ is infinite and contained in L . But this contradicts Lemma 1.21.

Case 2. L does not intersect L' .

Since x, y are in the plane spanned by $\{e_1, e_2\}$, this means that $y - x \in L'$ (i.e., the line determined by x and y is parallel to L'). So if $\sigma \in AGL_n(\mathbb{Q})$ fixes L' and takes x to y , then easily $\langle \sigma^m(x) \mid m < \omega \rangle$ is infinite and contained in the unique line containing x and parallel to L' , and we reach the same contradiction. □

Corollary 1.26. If G is a group of permutations of Ω which properly contains $AGL_n(\mathbb{Q})$, then G does not preserve a k -Steiner system for $k \geq 2$.

Proof. Follows from Lemma 1.22 and Proposition 1.25. □

1.4. A limit of Steiner systems. The following definition is taken from [AM96, Theorem 5.8.4, Theorem 5.8.2].

Definition 1.27. Let (G, X) be a Jordan group. Then G is said to preserve on X a *limit of Steiner systems* if there is some $3 \leq m \in \mathbb{N}$, some totally ordered index set (J, \leq) with no greatest element, and a strictly increasing chain $\langle \Pi_i \mid i \in J \rangle$ of subsets of X such that:

- (1) $\bigcup \{\Pi_i \mid i \in J\} = X$;
- (2) for each $i \in J$, $G_{\{\Pi_i\}}$ is $(m - 1)$ -transitive on Π_i (where $G_{\{\Pi_i\}}$ is the setwise stabilizer of Π_i) and preserves a non-trivial Steiner $(m - 1)$ -system on Π_i ;
- (3) if $i < j$, then Π_i is a subset of a block of the $G_{\{\Pi_j\}}$ -invariant Steiner $(m - 1)$ -system on Π_j ;
- (4) for all $g \in G$ there is $i_0 \in J$, dependent on g , such that for every $i > i_0$ there is $j \in J$ such that $g(\Pi_i) = \Pi_j$ and the image under g of every $(m - 1)$ -Steiner block on Π_i is an $(m - 1)$ -block on Π_j ;
- (5) for each $i \in J$, the set $X \setminus \Pi_i$ is a Jordan set for (G, X) .

We will show that if G is a group of permutations of Ω properly containing $AGL_n(\mathbb{Q})$, and G preserves a limit of Steiner systems, then if G is not highly

transitive, it must already preserve a k -Steiner system for some $k \in \mathbb{N}$, contradicting Corollary 1.26.

In fact, we will not use the full definition of a limit of Steiner system. Instead, we will use the following definition:

Definition 1.28. Let (G, X) be a Jordan group. Then G is said to preserve on X a *limit of Jordan sets* if there is some totally ordered index set (J, \leq) with no greatest element, and a strictly increasing chain $\langle \Pi_i \mid i \in J \rangle$ of subsets of X such that:

- (1) $\bigcup \{\Pi_i \mid i \in J\} = X$;
- (2) for all $g \in G$ there is $i_0 \in J$, dependent on g , such that for every $i > i_0$ there is $j \in J$ such that $g(\Pi_i) = \Pi_j$;
- (3) for each $i \in J$, the set $X \setminus \Pi_i$ is a Jordan set for (G, X) .

Lemma 1.29. *Suppose that G preserves a limit of of Jordan sets on X as witnessed by $\langle \Pi_i \mid i \in J \rangle$. Then for every $g \in G$, for all i large enough either $g^{-1}(\Pi_i) \subseteq \Pi_i$ or $g(\Pi_i) \subseteq \Pi_i$.*

Proof. By (2) of Definition 1.28, for all i large enough, $g(\Pi_i) = \Pi_j$ for some $j \in J$. Fix such i , and suppose that $j \geq i$. Then as $\Pi_i \subseteq \Pi_j$, $\Pi_i \subseteq g(\Pi_i) \Rightarrow g^{-1}(\Pi_i) \subseteq \Pi_i$. If $j < i$, we get that $g(\Pi_i) \subseteq \Pi_i$. □

Assumption 1.30. *Suppose that G is a group of permutation of Ω , properly containing $AGL_n(\mathbb{Q})$ and preserving a limit of Jordan sets as witnessed by (J, \leq) and $\langle \Pi_i \mid i \in J \rangle$. Also, fix some $m \in \mathbb{N}$ such that G is m -transitive but not $(m + 1)$ -transitive.*

Definition 1.31. Say that a tuple of distinct elements $\bar{a} = (a_0, \dots, a_m) \in (\Omega)^{m+1}$ is *very large* if for some $i \in J$, $a_0, \dots, a_{m-1} \in \Pi_i$ and $a_m \notin \Pi_i$. Say that an $(m + 1)$ -tuple \bar{a} is *large* if its orbit contains a very large $m + 1$ -tuple.

Lemma 1.32. *The large $(m + 1)$ -tuples consist of one orbit.*

Proof. We need to show that if \bar{a} and \bar{b} are large, then for some $\sigma \in G$, $\sigma(\bar{a}) = \bar{b}$. We may assume that both \bar{a} and \bar{b} are very large. Let $i \in J$ be such that $a_0, \dots, a_{m-1} \in \Pi_i$ and $a_m \notin \Pi_i$. By m -transitivity, for some $\sigma \in G$, $\sigma(\bar{a} \upharpoonright m) = \bar{b} \upharpoonright m$. Let $i_0 \in J$ correspond to (2) of Definition 1.28 applied to σ , and let $i' > i_0, i$. Since $\Omega \setminus \Pi_{i'}$ is a Jordan set and $\Pi_{i'} \neq \Omega$, we can fix Π_i and move a_m out of $\Pi_{i'}$. This allows us to assume that $i > i_0$. So $\sigma(\Pi_i) = \Pi_j$ for some $j \in J$, and $\bar{b} \upharpoonright m \subseteq \Pi_j$. By moving b_m , fixing some $\Pi_{j'}$ containing $\bar{b} \upharpoonright m$, we may assume that $b_m \notin \Pi_j$.

But since $\Omega \setminus \Pi_j$ is a Jordan set and $\sigma(a_m) \notin \Pi_j$, we can map $\sigma(a_m)$ to b_m fixing Π_j via some $\tau \in G$. Then $\tau \circ \sigma(\bar{a}) = \bar{b}$. □

Lemma 1.33. *If $\bar{a} = (a_1, \dots, a_{m+1}) \in \Omega^{m+1}$ is such that $\bar{a} \upharpoonright m \subseteq L$ for some line L and a_{m+1} is not in L , then \bar{a} is large.*

Proof. Let Π_i be such that Π_i contains $\bar{a} \upharpoonright m$. We claim that $\Omega \setminus \Pi_i \not\subseteq L$. Suppose not, i.e., that $\Omega \setminus L \subseteq \Pi_i$. Let $L' \neq L$ and let $\sigma \in AGL_n(\mathbb{Q})$ take L to L' . We may assume (perhaps increasing i) that $\sigma(\Pi_i) = \Pi_j$ for some $j \in J$, and so $\Omega \setminus L' \subseteq \Pi_j$. Let $j' > i, j$; then $\Pi_{j'} \supseteq \Omega \setminus A$ where $|A| \leq 1$. But then J must have a last element.

Since we can map a_m to any point outside of L , we can map it to a point outside of Π_i . □

Lemma 1.34. *If $\bar{a} \in \Omega^{m+1}$ is large and $\pi \in S_{m+1}$ is any permutation, then $\bar{a}^\pi = \langle a_{\pi(i)} \mid i < m \rangle$ is also large.*

Proof. It is enough to prove it for π of the form $(k \ m)$ for some $k < m$. Since large tuples form one orbit (Lemma 1.32), it is enough to show the lemma for *one* large tuple.

We will find a line L and some $i \in J$ such that $L \cap \Pi_i$ contains at least $m - 1$ points: $a_0, \dots, a_{k-1}, a_{k+1}, \dots, a_{m-1}$, and both $L \setminus \Pi_i$ and $\Pi_i \setminus L$ are non-empty, say containing x and a_k respectively. Then $\bar{a} = (a_0, \dots, a_{m-1}, x)$ is very large by definition, but also \bar{a}^π is large by Lemma 1.33. Together we are done.

First assume that $n = \omega$, and let $\{e_j \mid j \in \mathbb{Z}\}$ be the standard basis for Ω enumerated by \mathbb{Z} . Let $\sigma_0 \in GL_n(\mathbb{Q})$ take e_j to e_{-j} (so $\sigma^2 = \text{id}$). By Lemma 1.29, for all i large enough, Π_i contains 0 and it is closed under $-\text{id}$ and under σ_0 .

By applying Lemma 1.29 with σ being translation by e_1 , $\sigma(x) = x + e_1$, for all large enough i , Π_i is closed under translation by e_1 and by $-e_1$. Indeed, if for instance Π_i is closed under σ , then $(-\text{id})\sigma(-\text{id})(\Pi_i) \subseteq \Pi_i$ but $(-\text{id})\sigma(-\text{id}) = \sigma^{-1}$.

By applying Lemma 1.29 with σ being base shift, i.e., $\sigma \in GL_n(\mathbb{Q})$ and $\sigma(e_j) = e_{j+1}$, for all i large enough Π_i is closed under base shift and its inverse. This follows again from the fact that $\sigma^{-1} = \sigma_0\sigma\sigma_0$.

Now it follows that for i large enough, Π_i is closed under translation by $\pm e_j$ for all $j \in \mathbb{Z}$. Indeed, suppose σ_j is translation by e_j , and $\tau_{j \rightarrow 1} \in GL_n(\mathbb{Q})$ takes e_k to e_{k-j+1} . Then $\sigma_j = \tau_{j \rightarrow 1}^{-1}\sigma_1\tau_{j \rightarrow 1}$.

But now, for all i large enough, Π_i contains all the integer valued linear combinations of $\{e_j \mid j \in \mathbb{Z}\}$, which is just $\mathbb{Z}^{<\omega}$.

Let $x \in \Omega \setminus \Pi_i$. Then, if $p \in \mathbb{N}$ is the product of the denominators of the rationals appearing in x , then the line $L = \mathbb{Z}px$ intersects Π_i in infinitely many points. In particular, we can find $m - 1$ points on $L \cap \Pi_i$ and find some $a_k \in \mathbb{Z}^{<\omega} \setminus L$.

The proof for finite n is similar but simpler, since we do not need to use σ_0 , as the base shift modulo n map $\sigma \in GL_n(\mathbb{Q})$ is of finite order; hence $\sigma(\Pi_i) = \Pi_i$ for all large enough $i \in J$, so automatically Π_i is closed under σ^{-1} . □

In light of Lemma 1.34 we can extend the definition of large tuples to $(m + 1)$ -sets. We will define an m -Steiner system on Ω . For a subset of $s \subseteq \Omega$ of size m , let

$$\mathfrak{b}_s = \{x \in \Omega \mid s \cup \{x\} \text{ is not large}\}.$$

Equivalently, \mathfrak{b}_s is the set of points $x \in \Omega$ such that whenever s is sent by G to some line, x is sent to the same line (this follows from Lemma 1.32 and Lemma 1.33). It follows that $s \subseteq \mathfrak{b}_s$.

Lemma 1.35. *Assume $s \subseteq \Omega$ is of size m , and let $a \in s$. Then if $b \in \mathfrak{b}_s$, then $\mathfrak{b}_s = \mathfrak{b}_{(s \setminus \{a\}) \cup \{b\}}$.*

Proof. Let $t = (s \setminus \{a\}) \cup \{b\}$. To see that $\mathfrak{b}_t \subseteq \mathfrak{b}_s$, take some $x \in \mathfrak{b}_t$ and some $\sigma \in G$ that sends s to some line L . Then $\sigma(b) \in L$, so $\sigma(t) \subseteq L$, so $\sigma(x) \in L$.

For the other direction, assume that $x \in \mathfrak{b}_s$ and that $\sigma \in G$ maps t to a line L . Since $t \cup \{a\} = s \cup \{b\}$ is not large (since $b \in \mathfrak{b}_s$), it follows that $\sigma(a) \in L$, and thus $\sigma(x) \in L$. □

Let

$$\mathcal{B} = \{\mathfrak{b}_s \mid s \subseteq \Omega, |s| = m\}.$$

Corollary 1.36. (Ω, \mathcal{B}) is an m -Steiner system on Ω preserved by G .

Proof. Note that by definition, G preserves \mathcal{B} and $\sigma(\mathfrak{b}_s) = \mathfrak{b}_{\sigma(s)}$ for all $\sigma \in G$ and $s \subseteq \Omega$ of size m . By m -transitivity, it follows that $|\mathfrak{b}_1| = |\mathfrak{b}_2|$ for any $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{B}$. We already noted that $|\mathfrak{b}| \geq m$ for all $\mathfrak{b} \in \mathcal{B}$. If $|\mathfrak{b}| = m$ for all $\mathfrak{b} \in \mathcal{B}$, then all $(m+1)$ -sets are large, and by Lemma 1.32, this would mean that G is $(m+1)$ -transitive. This shows that $|\mathfrak{b}| > m$ for all $\mathfrak{b} \in \mathcal{B}$.

Since there is a large tuple, there is more than one block.

Finally we must check that if $s \subseteq \Omega$ is of size m and $\mathfrak{b} \in \mathcal{B}$ contains s , then $\mathfrak{b} = \mathfrak{b}_s$. Suppose $\mathfrak{b} = \mathfrak{b}_t$ for some t . Then $s \subseteq \mathfrak{b}_t$, so by Lemma 1.35 we can replace every element of t by an element of s until we get $\mathfrak{b}_s = \mathfrak{b}_t$. \square

1.5. Conclusion.

Corollary 1.37. *Theorem 1.2 holds: $AGL_n(\mathbb{Q})$ is a maximal-closed subgroup of S_ω for $\omega \geq n \geq 2$.*

Proof. Suppose that G is some group of permutations of Ω strictly containing $AGL_n(\mathbb{Q})$.

By Corollary 1.9, G is 3-transitive. Since G is a Jordan group as witnessed by e.g., lines, we may apply Fact 1.11. By Proposition 1.14, Proposition 1.17 and Corollary 1.26, G must preserve a limit of Steiner systems. But by Corollary 1.36, which assumes even the weaker hypothesis of Assumption 1.30, unless G is highly transitive, G preserves an m -Steiner system where G is m -transitive but not $(m+1)$ -transitive (so $m \geq 3$), and this contradicts Corollary 1.26. \square

Recall from the introduction the question of Junker and Ziegler [JZ08]: does every non- ω -categorical theory admit infinitely many reducts? Bodirsky and Macpherson [BM13] have answered this question negatively. Our next corollary (which was noticed by David Evans) gives yet another counterexample. This also answers a question from [BM13].

Definition 1.38. A structure N is a (*definable*) *reduct* of a structure M if they share the same universe, and the basic relations of N are \emptyset -definable subsets of M . The structure N is a *proper reduct* if there are \emptyset -definable subsets of M which are not \emptyset -definable in N . For a complete first order theory T in the language L , we say that a (complete first order) theory T' in the language L' is a (proper) *reduct* of T if there is a model M of T and a model N of T' such that N is a (proper) reduct of M . Equivalently, every model of T has a (proper) reduct that is a model of T' . The theory T' is a *trivial* reduct when T' is the theory of an infinite set with no structure.

Corollary 1.39. *The theory $T = Th(\mathbb{Q}, f)$ where $f(x, y, z) = x + y - z$ has no non-trivial proper definable reduct.*

Proof. Let $L = \{f\}$. Then T is a reduct of T_0 , the theory of a divisible torsion free abelian group (or a vector space over \mathbb{Q}) in the language $\{+, 0\}$. The vector space $\mathbb{Q}^{<\omega}$ is a countable saturated model of T_0 (since T_0 is strongly minimal, or by quantifier elimination). Thus its reduct M to L (so $M \models T$) is also saturated. Suppose that M has a proper reduct N with language L' . Since N is a proper reduct, by saturation there are two tuples \bar{a}, \bar{b} in N such that $\bar{a} \equiv_{L'} \bar{b}$ but $\bar{a} \not\equiv_L \bar{b}$. By saturation of N , there is an automorphism of N that maps \bar{a} to \bar{b} , thus the automorphism group of N is strictly larger than $\text{Aut}(M) = AGL_\omega(\mathbb{Q})$. By

Theorem 1.2, $\text{Aut}(N)$ is the full permutation group of $\mathbb{Q}^{<\omega}$, thus N is the trivial structure. \square

2. $PGL_n(\mathbb{Q})$ IS MAXIMAL

In this section we will show using the same techniques as in Section 1 that $PGL_n(\mathbb{Q})$ is maximal for all $n \leq \omega$. Recall:

Definition 2.1. For a vector space V over a field F , let $P(V)$ be the set of one dimensional subspaces of V . Let $PGL(V)$ be the group $GL(V)/Z(GL(V))$ (where $Z(GL(V))$ is just the group $\{\alpha \text{id} \mid \alpha \in F^\times\}$). Elements of $P(V)$ are called *points*, while two-dimensional subspaces of V are called *lines*. A point x lies on a line L if $x \subseteq L$. For $n < \omega$, let $PGL_n(\mathbb{Q}) = PGL(\mathbb{Q}^n)$, and let $PGL_\omega(\mathbb{Q}) = PGL(\mathbb{Q}^{<\omega})$.

With this definition of lines, points and incidence, $P(V)$ satisfies all the axioms of a projective space [BR98], and $PGL(V)$ is a group of automorphism of the projective space structure. In fact, by Fact 2.3 below, this is the group of all automorphisms of the projective space in case the field is \mathbb{Q} (or in general when $\text{Aut}(F)$ is trivial).

As in Section 1, we assume:

Assumption 2.2. *Throughout this section, we fix $3 \leq n \leq \omega$. Let $V = \mathbb{Q}^n$ as a vector space in case $n < \omega$ and $\mathbb{Q}^{<\omega}$ in case $n = \omega$. Let $\Omega = P(V)$.*

Fact 2.3 ([BR98, Corollary 3.5.9]—The fundamental theorem of projective geometry). For a permutation σ of Ω , $\sigma \in PGL_n(\mathbb{Q})$ iff σ preserves lines iff σ takes triples of collinear points to collinear points.

It follows from Fact 2.3 that $PGL_n(\mathbb{Q})$ is a closed subgroup of the group of permutations of Ω (which is countable) — it is the automorphism group of the structure (Ω, R) where $R(x, y, z)$ holds iff x, y, z are collinear.

Remark 2.4. Suppose that U is a hyperplane in V (a subspace of co-dimension 1). So U can be seen as an affine space with its structure of lines and points.

Let $G = G_U = GL_n(\mathbb{Q})_{\{U\}}$ be the setwise stabilizer of U . Let $PG = G/Z(G)$ (where $Z(G) = \{\alpha \text{id} \mid \alpha \in \mathbb{Q}^\times\}$). Let $X = X_U = \{x \in \Omega \mid x \subseteq U\}$. Then PG acts on $\Omega \setminus X$. This action is equivalent to the action of $AGL_{n-1}(\mathbb{Q})$ on U (if $n = \omega$, $n - 1 = \omega$). To see this, choose a basis B of U and $b \in V$ such that $B \cup \{b\}$ is a basis of V . Present each point x in $\Omega \setminus X$ uniquely as $\langle b + u_x \rangle$ — the span of $b + u_x$ — where $u_x \in U$. Similarly, present each $\sigma \in PG$ uniquely as $T \cdot Z(G)$, where $T \in G$ with $T(b) = b + u_\sigma$ for $u_\sigma \in U$. Then $\sigma(x) = \langle T(b + u_x) \rangle = \langle Tb + Tu_x \rangle = \langle b + u_\sigma + Tu_x \rangle$. Thus, by identifying x with u_x and σ with the map $u \mapsto u_\sigma + Tu$, we get the desired equivalence.

Moreover, the identification of $\Omega \setminus X$ with the affine space U (via $x \mapsto u_x$) preserves lines: collinear points in Ω map to collinear points in U . In fact, it preserves affine/projective subspaces as well, meaning that if $W \subseteq V$ is a subspace, then $P(W) \cap (\Omega \setminus X)$ is an affine subspace of $\Omega \setminus X$ (or \emptyset), and for any affine subspace $A \subseteq \Omega \setminus X$, the projective subspace $P(W)$ generated by A in Ω intersects $\Omega \setminus X$ in A .

Easily, Ω is covered by affine spaces, and if B is a basis of V , then $\Omega = \bigcup_{b \in B} (\Omega \setminus X_{U_b})$, where U_b is the hyperplane spanned by $B \setminus \{b\}$. After choosing U , we call $\Omega \setminus X$ the corresponding *affine space* and X its *hyperplane at infinity*.

If $\Omega \setminus X$ is an affine space and L is a line in it, then (the prolongation of) L intersects X in exactly one point, which we call *the point at infinity* of L . It follows that if σ is an affine map of $\Omega \setminus X$ that preserves $L \cap (\Omega \setminus X)$, then the unique extension of σ to $PGL_n(\mathbb{Q})$ preserves the point at infinity of L .

Note that since V cannot be covered by finitely many hyperplanes, it follows that for any finite set $s \subseteq V \setminus \{0\}$ there is some hyperplane U such that $s \cap U = \emptyset$. This means that given finitely many points from Ω , there is some hyperplane U such that all these points are in $\Omega \setminus X_U$.

Theorem 2.5. *Under Assumption 2.2, $PGL_n(\mathbb{Q})$ is a maximal-closed permutation group of Ω .*

Proof. The proof follows the same lines as the proof of Theorem 1.2. We will go over the steps, using the notation of Remark 2.4 and the notation of the proof of Theorem 1.2.

Step 1. Analyze the orbits of $PGL_n(\mathbb{Q})$ on triples of distinct elements from Ω . Here, as opposed to the situation in Lemma 1.5, there are only two orbits — the set of collinear triples and the set of non-collinear triples. Hence Corollary 1.9 follows at once from Fact 2.3: if G is a group of permutations of Ω properly containing $PGL_n(\mathbb{Q})$, then it is 3-transitive.

Step 2. Observe that $PGL_n(\mathbb{Q})$ is a Jordan group. Indeed the complement of a line or of any proper projective subspace is a Jordan set. Now we may apply Fact 1.11.

Step 3. We deal with the S - and D -relations exactly as we did in Sections 1.1 and 1.2, by working within an affine space. In the S -relation case, we can first fix any hyperplane U and choose our line and points in $\Omega \setminus X_U$. Since the action of $PG_U \leq PGL_n(\mathbb{Q})$ on $\Omega \setminus X_U$ is equivalent to the action of $AGL_n(\mathbb{Q})$ on U , we get that PG_U cannot preserve an S -relation on $\Omega \setminus X_U$, so the same follows for $PGL_n(\mathbb{Q})$ and Ω (because the axioms for the S -relation are universal). In the D -relation case, we first choose a line L and we get that $D(a, b; c, d)$ holds for $a, b, c \in L$ and $d \notin L$. Then we can choose a hyperplane U such that $a, b, c, d \in \Omega \setminus X_U$, and we work within $\Omega \setminus X_U$ with PG_U and reach a contradiction as in Section 1.2.

Step 4. We deal with the Steiner system case.

Lemma 1.21 remains true by replacing the affine subspace by projective subspace A with a similar proof: instead of choosing a line L disjoint of A , choose a line L such that $|A \cap L| \leq 1$. But if \mathfrak{b} is a block which contains $\Omega \setminus \{x\}$ for some x , then by transitivity, $\mathfrak{b} = \Omega$. Lemma 1.22 follows.

For Lemma 1.23, we have to reverse the even and odd cases. Start by working within some affine space $\Omega \setminus X_U$ as in Step 3, which we identify with \mathbb{Q}^{n-1} (or $\mathbb{Q}^{<\omega}$). For even $k = 2l \geq 4$, choose $k - 1$ points in $\Omega \setminus X_U$, $-(l - 1)e_1, \dots, 0, \dots, (l - 1)e_1$ and then add the point at infinity of the line containing these points. Now there is one more point, x , which must be on the line L and hence must be in $\Omega \setminus X_U$. The affine map $-\text{id}$ then preserves L , and hence also the point at infinity, but takes x to $-x$, hence it adds one more point. For odd $k = 2l + 1 \geq 3$, choose $k - 1$ points in $\Omega \setminus X_U$, $-le_1, \dots, -e_1, e_1, \dots, le_1$ and the point at infinity of this line. Now there is one more point x which again must be in the line but also in $\Omega \setminus X_U$, so either this point is $\neq 0$, in which case we can proceed as in the odd case, or this point is 0,

in which case we can translate by e_1 , fixing the point at infinity, and get one more point.

Proposition 1.25 now follows with the same proof. First we choose any $k - 1$ points on some line, and we add one more point out of it. We know that there are two more points which belong to the projective space (in fact a plane) generated by the k chosen points. Choose some hyperplane U such that $\Omega \setminus X_U$ contains all of these points. So now we work within the affine space $\Omega \setminus X_U$, and produce infinitely many points in a block, all contained in the same line. Note that in the proof there, the choice of points on the line was not important.

This shows that any group $G \supseteq PGL_n(\mathbb{Q})$ cannot preserve a k -Steiner system on Ω for $k \geq 2$.

Step 5. The limit of the Steiner system case.

The proof as in Section 1.4 works with some minor modifications. So again we show that if $PGL_n(\mathbb{Q})$ preserves a limit of Jordan sets on Ω and it is m -transitive but not $(m + 1)$ -transitive, then it must preserve an m -Steiner system on Ω .

Lemmas 1.29 and 1.32 hold with exactly the same proofs, as well as Lemma 1.33 (which only used the fact the two lines intersect in at most one point).

Lemma 1.34 requires some small modification. The proof uses exactly the same technique, but now we first choose a basis B of V , and we show that for some $i \in J$ large enough, for all $b \in B$, Π_i contains all the points with integer coefficients in the affine space which corresponds to b (i.e., which corresponds to the hyperplane spanned by $B \setminus \{b\}$) under the identification described in Remark 2.4. These points correspond to points which admit a tuple of homogeneous coordinates consisting of integers, at least one of which is 1. Since any point of Ω belongs to at least one of these affine spaces, the same proof will work.

Suppose first that $n = \omega$, and let $\{e_i \mid i \in \mathbb{Z}\}$ be a basis for V . Denote by A_i the affine space corresponding to e_i . Using the same technique as the proof of Lemma 1.34, we find $i \in J$ large enough so that Π_i is preserved under the projective maps induced by the linear maps σ_1 , which maps $e_i \mapsto e_{i+1}$, its inverse, σ_2 which fixes e_0 , maps $e_i \mapsto e_{i+1}$ for $i \neq -1$, and maps $e_{-1} \mapsto e_1$, and its inverse. We also want Π_i to be preserved under the affine map of translating by e_1 in the affine space that corresponds to e_0 , and its inverse. It follows that with the map σ_1 we cover all affine spaces and with the map σ_2 we cover all the coordinates within one affine space. Now if i is large enough so that Π_i contains the 0 point of the affine space corresponding to e_0 , it will follow that Π_i is as required.

For n finite the proof is the same (but simpler).

The rest of the proof is exactly the same, so the theorem follows. □

Remark 2.6. The same proof shows that $PGL_n(K)$ is maximal for any field K of characteristic 0 in which the only roots of unity are $\{1, -1\}$ (though Lemma 1.34 needs a slightly different argument). However, recall from the introduction that Bogomolov and Rovinsky [BR13] proved that for $n \geq 3$, $PGL_n(F)$ is a maximal-closed subgroup of the group of all permutations of F for any infinite field F .

3. OPEN QUESTIONS

Question 3.1. Is $PGL_2(F)$ maximal for an algebraically closed field F of transcendence degree ≥ 1 ?

Note that the action of $P\Gamma L_2(F)$ on $P(F^2)$ preserves the 3-Steiner system whose blocks are conjugates of $\text{acl}(\mathbb{Q}) \cup \{\infty\}$.

Question 3.2. Is the automorphism group of the geometry of a homogeneous structure constructed using the Hrushovski construction maximal-closed?

This question might be a bit too vague, so here is a precise special case. Consider the Hrushovski construction giving a homogeneous 2-Steiner system described in [BMMN97, Chapter 15] or [Hru93, Section 5]. Namely, let R be a ternary relation symbol, and define a pre-dimension δ on the class of 3-hypergraphs (R -structures in which R is symmetric and holds only for tuples of distinct points) by $\delta(A) = |A| - |R^A|$. Consider the family of 3-hypergraphs A for which $\delta(A_0) \geq \min\{|A_0|, 2\}$ for any $A_0 \subseteq A$. The usual Fraïssé-Hrushovski amalgamation associated with this class and pre-dimension gives a countable structure M equipped with a dimension function d . Consider the permutations of M which preserve the dimension. Is this group maximal?

Question 3.3. Is there a k -Steiner system on a countable set whose automorphism group G is k -transitive and also preserves an l -Steiner system for $l \neq k$? One may also add the condition that the k -blocks are Jordan complements for G .

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