TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 368, Number 11, November 2016, Pages 8029–8046
http://dx.doi.org/10.1090/tran6699
Article electronically published on January 27, 2016

ON THE $BP\langle n\rangle$-COHOMOLOGY OF ELEMENTARY ABELIAN p-GROUPS

GEOFFREY POWELL

Abstract. The structure of the $BP\langle n\rangle$-cohomology of elementary abelian p-groups is studied, obtaining a presentation expressed in terms of BP-cohomology and mod-p singular cohomology, using the Milnor derivations.

The arguments are based on a result on multi-Koszul complexes which is related to Margolis's criterion for freeness of a graded module over an exterior algebra.

1. Introduction

Understanding the generalized group cohomology of elementary abelian p-groups for a cohomology theory $E^*(-)$ is of interest both as a first step towards the study of generalized group cohomology, inspired in part by the results of Quillen for singular cohomology, and also since Lannes' theory [Lan92] implies that it yields information on the p-local homotopy type of the spaces of the Ω-spectrum representing E.

In studying the spectra of interest in chromatic homotopy theory, it is natural to commence by the complex oriented theories. Here the state of knowledge is incomplete once one moves outside the cases admitting descriptions as formal schemes (see [HKR00]) or the classical cases corresponding to singular cohomology or the periodic Morava K-theories.

The universal example, complex cobordism MU, is of interest. For elementary abelian p-groups, one can reduce to Brown-Peterson theory, BP; this corresponds to working p-locally, hence restricting to p-typical formal group laws. Landweber showed that $BP^*(BV)$, for V an elementary abelian p-group, can be described in terms of the formal group structure (the situation for Brown-Peterson homology is much more complicated [JW85, JWY94]).

Wilson [Wil73, Wil75] introduced and studied the theories $BP\langle n\rangle$, for $n \in \mathbb{N}$, which interpolate between $BP = BP\langle\infty\rangle$ and the mod-p Eilenberg-MacLane spectrum $H\mathbb{F}_p = BP\langle -1\rangle$. These provide a first step towards other theories of significant interest in chromatic homotopy theory; moreover, they are important in understanding the BP-cohomology of Eilenberg-MacLane spaces (cf. [RWY98]).

The cases $BP\langle -1\rangle = H\mathbb{F}_p$, $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$ and $BP\langle 1\rangle$ are understood ($BP\langle 1\rangle$ identifies with the Adams summand of p-local connective complex K-theory). Hitherto, for $n > 1$, results on $BP\langle n\rangle^*(BV)$ have concentrated on low degree or small rank behaviour; for example, Strickland [Str00] gave an analysis of the first (in

Received by the editors November 20, 2013 and, in revised form, January 13, 2015 and March 13, 2015.

2010 *Mathematics Subject Classification.* Primary 55N20, 55N22, 20J06.

Key words and phrases. Brown-Peterson theory, Johnson-Wilson theories, Milnor primitive, elementary abelian group, formal group.

terms of the rank) occurrence of v_n-torsion in $BP\langle n\rangle^*(BV)$, exhibiting a relationship between formal group theory and the action of the Milnor derivations on $H\mathbb{F}_p^*(BV)$.

This paper shows that this is the tip of the iceberg: the Milnor derivations explain all the v_n-torsion, without restriction on the rank of V. The structure of $BP\langle n\rangle^*(BV)$ is determined in terms of the contribution from formal groups obtained from $BP^*(BV)$ by base change, and from mod-p cohomology $H\mathbb{F}_p^*(BV)$, considered as a module over $\Lambda(Q_0,\ldots,Q_n)$. Namely, there is a short exact sequence

$$0 \to L_n \hookrightarrow \big(BP\langle n\rangle^* \otimes_{BP^*} BP^*(BV)\big) \oplus \mathbf{tors}_{v_n} \twoheadrightarrow BP\langle n\rangle^*(BV) \to 0,$$

where the v_n-torsion $\mathbf{tors}_{v_n} \subset BP\langle n\rangle^*(BV)$ is a trivial $BP\langle n\rangle^*$-module, which is isomorphic to the image $\mathrm{Im}(Q_0\ldots Q_n) \subset H\mathbb{F}_p^*(BV)$ of the iterated Milnor operation, and the kernel L_n is identified explicitly (see Theorem 7.8).

This result is derived from the general result, Theorem 6.1, for which the key input is the behaviour of the quotients (for $n \in \mathbb{N}$):

$$\mathscr{H}^*(X,n) := \Big\{\bigcap_{i=0}^{n} \mathrm{Ker}(Q_i)\Big\}/\mathrm{Im}(Q_0\ldots Q_n)$$

associated to the mod-p cohomology of a space X. The fundamental property is that the Thom reduction from BP to mod-p cohomology induces a surjection onto $\mathscr{H}^*(X,n)$.

The proof of this for the case $X = BV$ is a modification of Margolis's criterion [Mar83] for a module over the exterior algebra $\Lambda(Q_0,\ldots,Q_n)$ on the Milnor derivations Q_i to be free; this establishes a fundamental property of the structure of $H\mathbb{F}_p^*(BV)$ (see Theorem 7.2).

The argument can be generalized to the study of any MU-module spectrum which is constructed from BP by forming the quotient by a cofinite subset of a suitable set of algebra generators $\{v_i | i \in \mathbb{N}\}$ for BP_* (where $v_0 = p$). For instance, the methods recover the author's results on connective complex K-theory [Pow14]; moreover, they also apply to connective Morava K-theories, adding a useful perspective on existing results, such as Kuhn's study of the periodic theory [Kuh87] and the results of Wilson on the Hopf ring of periodic Morava K-theory [Wil84], and Hara, on the Hopf ring of the connective theory [Har91]. Similarly, the methods extend to the study of integral versions of connective Morava K-theory, generalizing the results for connective complex K-theory. For simplicity of exposition, these applications are not treated in the current paper; however, the main input is provided by Proposition 7.4, which is proved in full generality.

Organization of the paper. Section 2 provides background and Section 3 introduces the subquotient which bounds the indeterminacy of the Thom reduction map in terms of the action of the Milnor primitives. Section 4 proves technical results which control injectivity and surjectivity of certain reduction maps. The fulcrum is Section 5, which shows how the v_n-torsion can be controlled in odd degrees under appropriate hypotheses; Section 6 exhibits the ramifications to the full $BP\langle n\rangle$-cohomology. Finally, in Section 7, these techniques are applied to the case of elementary abelian p-groups, proving the Margolis-type vanishing result, which provides the necessary input.

2. Preliminaries

2.1. **Torsion theories.** This section fixes notation and recalls a standard result on the relation between torsion submodules and annihilator submodules.

Let R be a commutative ring and $R[v]$ the polynomial algebra on v. For M an $R[v]$-module, the v-torsion submodule $\mathbf{tors}_v M$ is the set of v-torsion elements $\{m \in M | \exists t\ v^t m = 0\}$ and $\mathrm{Ker}_v M$ is the kernel of multiplication by v, $M \stackrel{v}{\to} M$, so that $\mathrm{Ker}_v M \cong \mathrm{Tor}_1^{R[v]}(R, M)$ and $\mathrm{Ker}_v M \subset \mathbf{tors}_v M$. The v-cotorsion $\mathbf{cotors}_v M$ is the quotient $M/\mathbf{tors}_v M$, so that there is a natural short exact sequence

$$0 \to \mathbf{tors}_v M \to M \to \mathbf{cotors}_v M \to 0.$$

This is a standard example of a hereditary torsion theory.

The proof of the following is straightforward.

Lemma 2.1. *For M an $R[v]$-module, the following conditions are equivalent:*

(1) $\mathrm{Ker}_v M = \mathbf{tors}_v M$*;*
(2) $vM \cap \mathrm{Ker}_v M = 0$*;*
(3) *the projection $M \twoheadrightarrow M/vM$ induces a monomorphism $\mathrm{Ker}_v M \hookrightarrow M/vM$.*

If these conditions are satisfied, there is a short exact sequence

$$0 \to \mathrm{Ker}_v M \to M/vM \to (\mathbf{cotors}_v M)/v \to 0.$$

Remark 2.2. In the application, rings and modules are graded; as usual, the appropriate commutativity condition is graded commutativity (with Koszul signs). However, where this intervenes, the rings are concentrated in even degrees, so signs do not appear.

2.2. **The Wilson theories $BP\langle n\rangle$.** Fix a prime p and consider the Brown-Peterson spectrum BP and the associated Wilson spectra $BP\langle n\rangle$ (cf. [Wil75, Str00, Tam00]), equipped with the reduction maps

$$BP \stackrel{\rho_n}{\to} BP\langle n\rangle \stackrel{\rho^n_{n-1}}{\to} BP\langle n-1\rangle,$$

which can be constructed in the category of MU-modules. The $BP\langle n\rangle$ can be taken to be commutative MU-ring spectra so that the reduction maps are morphisms of ring spectra [Str00, Section 3]. The coefficient rings are $BP_* \cong \mathbb{Z}_{(p)}[v_i | i \geq 0]$, $BP\langle n\rangle_* \cong BP_*/(v_i | i > n)$, where $|v_n| = 2(p^n - 1)$ and $v_0 = p$, by convention; thus $BP\langle n\rangle_* \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$ for $n \geq 1$. In particular $BP\langle -1\rangle = H\mathbb{F}_p$ and $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$ are Eilenberg-MacLane spectra.

Multiplication by v_n fits into the cofibre sequence which defines q_n:

$$\Sigma^{|v_n|} BP\langle n\rangle \stackrel{v_n}{\to} BP\langle n\rangle \stackrel{\rho^n_{n-1}}{\to} BP\langle n-1\rangle \stackrel{q_n}{\to} \Sigma^{|v_n|+1} BP\langle n\rangle. \tag{1}$$

The following is clear:

Lemma 2.3. *For X a spectrum and $n \in \mathbb{N}$, q_n induces a map*

$$BP\langle n-1\rangle^*(X) \xrightarrow{q_n} \mathrm{Ker}(v_n)^{*+|v_n|+1} \subset BP\langle n\rangle^{*+|v_n|+1}(X).$$

The composite $\rho_n q_n : BP\langle n-1\rangle \to \Sigma^{|v_n|+1} BP\langle n-1\rangle$ is a derivation (cf. [Str00, Section 3]). More generally, as in *loc. cit.*, one considers the derivation induced for MU-modules by the derivation $MU/v_n \to \Sigma^{|v_n|+1} MU/v_n$, which provides compatibility; the operation on $H\mathbb{F}_p$ coincides with the Milnor derivation Q_n (up to sign), by [Str00, Proposition 3.1].

This compatibility implies the following (cf. [Tam00, Proposition 4-4]).

Lemma 2.4. *For $n \in \mathbb{N}$, the following diagram commutes:*

$$\begin{tikzcd} BP\langle n\rangle \arrow[r, "q_{n+1}"] \arrow[d, "\rho^n_{-1}"'] & \Sigma^{|Q_{n+1}|}BP\langle n+1\rangle \arrow[d, "\rho^{n+1}_{-1}"] \\ H\mathbb{F}_p \arrow[r, "\pm Q_{n+1}"'] & \Sigma^{|Q_{n+1}|}H\mathbb{F}_p. \end{tikzcd}$$

Hence (up to possible sign), the composite $Q_n \dots Q_0 : H\mathbb{F}_p \to \Sigma^{\sum |Q_i|}H\mathbb{F}_p$ factors across ρ^n_{-1} as

$$H\mathbb{F}_p \xrightarrow{q_n \dots q_0} \Sigma^{\sum |Q_i|}BP\langle n\rangle \xrightarrow{\rho^n_{-1}} \Sigma^{\sum |Q_i|}H\mathbb{F}_p.$$

When considering the $BP\langle n\rangle$-cohomology of a space, the following can be applied.

Proposition 2.5. *For X a space such that $BP^{\mathrm{odd}}(X)=0$ and $n \in \mathbb{N}$, $BP\langle n\rangle^{\mathrm{odd}}(X)$ is v_i-torsion for $0 \leq i \leq n$.*

Proof. [Wil75, Corollary 5.6] shows that the reduction map $(\rho_n)^t : BP^t(X) \to BP\langle n\rangle^t(X)$ is surjective for $t \leq 2\big(\frac{p^n-1}{p-1}\big)$, hence $BP\langle n\rangle^t(X)$ is zero in this range. The result is a straightforward consequence. □

The condition $BP\langle n-1\rangle^{\mathrm{odd}}(X) = 0$ arises naturally at the start of the inductive arguments; the following observation records its immediate ramifications.

Proposition 2.6. *For X a spectrum and $0 < n \in \mathbb{N}$ such that $BP\langle n-1\rangle^{\mathrm{odd}}(X) = 0$, the following properties hold:*

(1) $\mathbf{tors}^{\mathrm{even}}_{v_n} = 0$, *where* $\mathbf{tors}_{v_n} \subset BP\langle n\rangle^*(X)$*;*
(2) *if $BP\langle n\rangle^{\mathrm{odd}}(X)$ contains no v_n-divisible elements, then*

$$BP\langle n\rangle^{\mathrm{odd}}(X) = 0.$$

Proof. The result follows from the long exact sequence associated to the cofibre sequence (1). For example, the hypothesis $BP\langle n-1\rangle^{\mathrm{odd}}(X) = 0$ implies that any element of $BP\langle n\rangle^{\mathrm{odd}}(X)$ is the image of an odd degree element under multiplication by v_n; repeating the argument, any such element is (infinitely) v_n-divisible. □

3. The image of the Thom reduction

The image in cohomology of the Thom reduction map $BP \to H\mathbb{Z}_{(p)}$ is of significant interest in general (see [Tam97], for example); here we consider the image of $\rho^n_{-1} : BP\langle n\rangle \to H\mathbb{F}_p$ and its relation with the action of the Milnor derivations Q_i on mod-p cohomology.

The following is well known; a proof is included for the convenience of the reader.

Proposition 3.1. *For X a spectrum and $n \in \mathbb{N}$, the reduction map ρ^n_{-1} induces a map of $BP\langle n\rangle^*$-modules: $\rho^n_{-1} : BP\langle n\rangle^*(X) \to H\mathbb{F}^*_p(X)$ such that*

$$\mathrm{Im}(Q_0 \dots Q_n) \subset \mathrm{Image}(\rho^n_{-1}) \subset \bigcap_{i=0}^{n} \mathrm{Ker}(Q_i).$$

Proof. The inclusion $\mathrm{Im}(Q_0 \dots Q_n) \subset \mathrm{Image}(\rho^n_{-1})$ is a consequence of the factorization of $Q_0 \dots Q_n$ across ρ^n_{-1}, given by Lemma 2.4.

For the upper bound, since $\rho^n_{-1} = \rho^{n-1}_{-1}\rho^n_{n-1}$, it suffices to show that $\mathrm{Image}(\rho^n_{-1}) \subset \mathrm{Ker}(Q_n)$; this follows from the commutative diagram

$$\begin{tikzcd}
BP\langle n\rangle^*(X) \arrow[r, "\rho^n_{n-1}"'] \arrow[rr, bend left, "0"] \arrow[dr, "\rho^n_{-1}"'] & BP\langle n-1\rangle^*(X) \arrow[r, "q_n"'] \arrow[d, "\rho^{n-1}_{-1}"] & BP\langle n\rangle^{*+|Q_n|}(X) \arrow[d, "\rho^n_{-1}"] \\
& H\mathbb{F}_p^*(X) \arrow[r, "\pm Q_n"'] & H\mathbb{F}_p^{*+|Q_n|}(X),
\end{tikzcd}$$

where the commutative square is provided by Lemma 2.4. □

Notation 3.2. For X a spectrum and $n \in \mathbb{N}$, let $\mathscr{H}^*(X,n)$ denote the graded subquotient of $H\mathbb{F}_p^*(X)$

$$\mathscr{H}^*(X,n) := \big\{\bigcap_{i=0}^{n} \mathrm{Ker}(Q_i)\big\}/\mathrm{Im}(Q_0 \dots Q_n).$$

Remark 3.3. Proposition 3.1 shows that $\mathscr{H}^*(X,n)$ bounds the indeterminacy of the image of ρ^n_{-1}. In particular, if $\mathscr{H}^t(X,n) = 0$, then $\mathrm{Image}(\rho^n_{-1})^t = \mathrm{Im}(Q_0 \dots Q_n)^t$.

Remark 3.4. For M a graded module over the exterior algebra $\Lambda(Q_0, \dots, Q_n)$, $\bigcap_{i=0}^n \mathrm{Ker}(Q_i) \subset M$ identifies with the socle $\mathrm{soc}(M)$ of M. If M is bounded below and of finite type, it can be written as $M \cong F \oplus \overline{M}$, where F is a free $\Lambda(Q_i|0 \leq i \leq n)$-module and $\overline{M}$ contains no free sub-module (see [Mar83], for example). The inclusion $\mathrm{Im}(Q_0 \dots Q_n) \subset \bigcap_{i=0}^n \mathrm{Ker}(Q_i)$ corresponds to the inclusion $\mathrm{soc}(F) \hookrightarrow \mathrm{soc}(M)$ and the quotient identifies with $\mathrm{soc}(\overline{M})$.

Hence, $\mathscr{H}^*(X,n)$ gives a measure of the failure of $H\mathbb{F}_p^*(X)$ to be free as an $\Lambda(Q_0, \dots, Q_n)$-module, when X is a connective spectrum with cohomology of finite type.

At the opposite extreme, if $Q_0, \dots, Q_n$ act trivially upon $H\mathbb{F}_p^*(X)$ (for example, if the latter is concentrated in even degrees), then there is an identification $\mathscr{H}^*(X,n) \cong H\mathbb{F}_p^*(X)$.

By Proposition 3.1, ρ^n_{-1} maps to $\bigcap_{i=0}^n \mathrm{Ker}(Q_i) \subset H\mathbb{F}_p^*(X)$, hence induces a map to $\mathscr{H}^*(X,n)$.

Corollary 3.5. *For X a spectrum and $n \in \mathbb{N}$, $\rho^n_{-1} : BP\langle n\rangle^*(X) \to H\mathbb{F}_p^*(X)$ surjects to $\bigcap_{i=0}^n \mathrm{Ker}(Q_i)$ if and only if the induced map $BP\langle n\rangle^*(X) \to \mathscr{H}^*(X,n)$ is surjective.*

Proof. A straightforward consequence of Proposition 3.1. □

Remark 3.6. Surjectivity to $\bigcap_{i=0}^n \mathrm{Ker}(Q_i)$ is a natural condition to consider; for $n = 1$ it arises in the work of Kane [Kan82] on finite H-spaces via connective K-theory.

When X is a space, further information can be obtained by exploiting multiplicative structure. (Henceforth, cohomology is taken to be reduced, so a disjoint basepoint is required.)

Proposition 3.7. *For X a space and $n \in \mathbb{N}$,*

(1) *the cup product on $H\mathbb{F}_p^*(X_+)$ induces a graded commutative algebra structure on $\mathscr{H}^*(X_+, n)$;*
(2) *the reduction map $BP\langle n\rangle^*(X_+) \to \mathscr{H}^*(X_+, n)$ is a morphism of $BP\langle n\rangle^*$-algebras.*

Proof. The first statement is an immediate consequence of the fact that the operations Q_i are derivations and the second is a formal consequence of the construction of the reduction. □

4. Injectivity and surjectivity for generalized reduction maps

Fix $n \in \mathbb{N}$; for a spectrum X, $\rho_n^{n+1} : BP\langle n+1\rangle \to BP\langle n\rangle$ induces a morphism of $BP\langle n+1\rangle_*$-modules

$$BP\langle n+1\rangle^*(X) \stackrel{\rho_n^{n+1}}{\to} BP\langle n\rangle^*(X),$$

which is not surjective in general. Similarly one can consider the reduction

$$BP^*(X) \stackrel{\rho_n}{\to} BP\langle n\rangle^*(X).$$

Wilson's result, [Wil75, Corollary 5.6], gives surjectivity in low degrees, for X a suspension spectrum.

General criteria for injectivity and surjectivity are introduced in this section.

4.1. Surjecting to $BP\langle n\rangle$-cohomology.

The short exact sequence

$$0 \to BP\langle n+1\rangle^*(X)/v_{n+1} \to BP\langle n\rangle^*(X) \to \mathrm{Ker}(v_{n+1})^{*+|Q_{n+1}|} \to 0 \tag{2}$$

is induced by the cofibre sequence $BP\langle n+1\rangle \stackrel{\rho_n^{n+1}}{\to} BP\langle n\rangle \stackrel{q_{n+1}}{\to} \Sigma^{|Q_{n+1}|}BP\langle n+1\rangle$.

Remark 4.1. Identifying $\mathrm{Ker}(v_{n+1})$ as $\mathrm{Tor}_1^{\mathbb{Z}_{(p)}[v_{n+1}]}(\mathbb{Z}_{(p)}, BP\langle n+1\rangle^*(X))$, the sequence (2) can be viewed as a universal coefficient short exact sequence; cf. [JW73, Proposition 5.7], where homology is considered.

By restriction to $\mathbf{tors}_{v_n} \subset BP\langle n\rangle^*(X)$, q_{n+1} gives a natural map

$$\kappa_n : \mathbf{tors}_{v_n} \to \Sigma^{|Q_{n+1}|}\mathrm{Ker}(v_{n+1});$$

and the inclusion $\mathbf{tors}_{v_n} \subset BP\langle n\rangle^*(X)$ together with $BP\langle n+1\rangle^*(X) \stackrel{\rho_n^{n+1}}{\to} BP\langle n\rangle^*(X)$ induce

$$\sigma_n : BP\langle n+1\rangle^*(X) \oplus \mathbf{tors}_{v_n} \to BP\langle n\rangle^*(X).$$

Similarly, write

$$\tilde{\sigma}_n : BP^*(X) \oplus \mathbf{tors}_{v_n} \to BP\langle n\rangle^*(X)$$

for the map obtained by replacing ρ_n^{n+1} with ρ_n.

Lemma 4.2. *For X a spectrum, the following conditions are equivalent:*

(1) *$\sigma_n : BP\langle n+1\rangle^*(X) \oplus \mathbf{tors}_{v_n} \to BP\langle n\rangle^*(X)$ is surjective;*
(2) *$\kappa_n : \mathbf{tors}_{v_n} \to \Sigma^{|Q_{n+1}|}\mathrm{Ker}(v_{n+1})$ is surjective;*
(3) *$BP\langle n+1\rangle^*(X) \to \mathbf{cotors}_{v_n} BP\langle n\rangle^*(X)$, induced by ρ_n^{n+1}, is surjective.*

Proof. Straightforward. □

The following result illustrates how the identification of $BP\langle n+1\rangle^{\mathrm{odd}}(X)$ leads to a criterion for the surjectivity of σ_n; this is a warm-up for the proof of Theorem 6.1.

Proposition 4.3. *Let X be a spectrum such that $BP\langle n\rangle^{\mathrm{odd}}(X) = \mathbf{tors}^{\mathrm{odd}}_{v_n}$ and ρ^{n+1}_{-1} induces an isomorphism*

$$BP\langle n+1\rangle^{\mathrm{odd}}(X) \xrightarrow{\cong} \mathrm{Im}(Q_0 \dots Q_{n+1})^{\mathrm{odd}} \subset H\mathbb{F}_p^{\mathrm{odd}}(X).$$

Then $\sigma_n : BP\langle n+1\rangle^*(X) \oplus \mathbf{tors}_{v_n} \twoheadrightarrow BP\langle n\rangle^*(X)$ is surjective.

Proof. Since $\mathbf{tors}^{\mathrm{odd}}_{v_n} \cong BP\langle n\rangle^{\mathrm{odd}}(X)$ by hypothesis, it suffices to show that σ_n surjects in even degree. By Lemma 4.2, it suffices to show that

$$\kappa_n : \mathbf{tors}^{2s}_{v_n} \to \mathrm{Ker}(v_{n+1})^{2s+|Q_{n+1}|}$$

is surjective, for all $s \in \mathbb{Z}$. The hypothesis on $BP\langle n+1\rangle^{\mathrm{odd}}(X)$ implies that $\mathrm{Ker}(v_{n+1})^{\mathrm{odd}} = BP\langle n+1\rangle^{\mathrm{odd}}(X)$, which embeds as $\mathrm{Im}(Q_0 \dots Q_{n+1})^{\mathrm{odd}}$ in $H\mathbb{F}_p^{\mathrm{odd}}(X)$.

Lemma 2.4 shows that $Q_0 \dots Q_{n+1}$ factors across $q_{n+1} \dots q_0$ and, hence, across $q_n \dots q_0 : H\mathbb{F}_p \to \Sigma^{\sum |Q_i|} BP\langle n\rangle$, which maps to $\mathrm{Ker}(v_n) \subset \mathbf{tors}_{v_n} \subset BP\langle n\rangle^*(X)$ in cohomology, by Lemma 2.3; since κ_n is induced by q_{n+1}, Lemma 2.4 implies surjectivity to $\mathrm{Ker}(v_{n+1})$ in odd degrees, as required. □

For X a spectrum and $n \in \mathbb{N}$, the reduction map ρ^n_{n-1} fits into a diagram

$$\begin{tikzcd} \mathbf{tors}_{v_n} \arrow[r, hook] & BP\langle n\rangle^*(X) \arrow[d, "\rho^n_{n-1}"] \\ \mathbf{tors}_{v_{n-1}} \arrow[r, hook] & BP\langle n-1\rangle^*(X). \end{tikzcd}$$

It is tempting to assert that the diagram can be completed to a commutative square, using the structure theory of BP_*BP-comodules [JY80, Theorem 0.1] and the stable comodule structure on $BP^*(X)$ provided (after suitable completion) by [Boa95, Sections 11, 15]. However, the passage to the Wilson theories $BP\langle n\rangle$ is delicate.

For this reason, the hypothesis that ρ^n_{n-1} sends $\mathbf{tors}_{v_n}$ to $\mathbf{tors}_{v_{n-1}}$ is included in the following result.

Proposition 4.4. *Let X be a spectrum and $n \in \mathbb{N}$ such that*

(1) *$\tilde{\sigma}_n : BP^*(X) \oplus \mathbf{tors}_{v_n} \twoheadrightarrow BP\langle n\rangle^*(X)$ is surjective;*
(2) *for $0 \leq j \leq n$,*
    (a) *$\mathbf{tors}_{v_j} \hookrightarrow BP\langle j\rangle^*(X) \to BP\langle j-1\rangle^*(X)$ factors across $\mathbf{tors}_{v_{j-1}} \subset BP\langle j-1\rangle^*(X)$;*
    (b) *$\sigma_j : BP\langle j\rangle^*(X) \oplus \mathbf{tors}_{v_{j-1}} \twoheadrightarrow BP\langle j-1\rangle^*(X)$ is surjective.*

Then, for $0 \leq j \leq n$, $\tilde{\sigma}_j : BP^*(X) \oplus \mathbf{tors}_{v_j} \twoheadrightarrow BP\langle j\rangle^*(X)$ is surjective;

Proof. The result is proved by a straightforward downward induction on j. □

Remark 4.5. The result will be applied in the case where $\rho_n : BP^*(X) \to BP\langle n\rangle^*(X)$ is itself surjective, hence establishing the first point of the hypotheses.

4.2. **Injectivity and base change.** The reduction map ρ^{n+1}_n induces a morphism of $BP\langle n\rangle^*$-modules:

$$BP\langle n\rangle^* \otimes_{BP\langle n+1\rangle^*} BP\langle n+1\rangle^*(X) \to BP\langle n\rangle^*(X)$$

and, by base change,

$$\mathbb{F}_p \otimes_{BP\langle n+1\rangle^*} BP\langle n+1\rangle^*(X) \to \mathbb{F}_p \otimes_{BP\langle n\rangle^*} BP\langle n\rangle^*(X)$$

which need not *a priori* be injective. Criteria for the injectivity of this and related morphisms are considered in this section.

The following terminology is used:

Definition 4.6. A $BP\langle n\rangle^*$-module M is trivial if it is given by restriction of a $\mathbb{F}_p$-vector space structure along $BP\langle n\rangle^* \twoheadrightarrow \mathbb{F}_p$.

The following basic lemma extracts the formal part of the argument employed in Propositions 4.8 and 4.9 below.

Lemma 4.7. *Let $\mathscr{C}$, $\mathscr{D}$ be abelian categories and*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "\gamma"'] & D \end{tikzcd}$$

be a cartesian square in $\mathscr{C}$. Then

(1) *α is injective if and only if γ is injective;*
(2) *the square is also cocartesian if and only if the associated total complex*

$$A \to B \oplus C \to D$$

is a short exact sequence.

Suppose that the square is both cartesian and cocartesian and $F : \mathscr{C} \to \mathscr{D}$ is a right exact functor. If $F(\alpha)$ is a monomorphism, then

$$\begin{tikzcd} F(A) \arrow[r, "F(\alpha)"] \arrow[d] & F(B) \arrow[d] \\ F(C) \arrow[r, "F(\gamma)"'] & F(D) \end{tikzcd}$$

is both cartesian and cocartesian and $F(\gamma)$ is injective.

Proposition 4.8. *For X a spectrum and $n \in \mathbb{N}$ such that*

(1) *$\mathbf{tors}_{v_n}$ is trivial as a $BP\langle n\rangle^*$-module;*
(2) *$\sigma_n : BP\langle n+1\rangle^*(X) \oplus \mathbf{tors}_{v_n} \twoheadrightarrow BP\langle n\rangle^*(X)$ is surjective;*

the morphism induced by ρ_n^{n+1}:

$$\mathbb{F}_p \otimes_{BP\langle n+1\rangle^*} BP\langle n+1\rangle^*(X) \to \mathbb{F}_p \otimes_{BP\langle n\rangle^*} BP\langle n\rangle^*(X)$$

is injective.

Proof. The surjection σ_n induces a short exact sequence

$$0 \to K_n \to (BP\langle n+1\rangle^*(X)/v_{n+1}) \oplus \mathbf{tors}_{v_n} \twoheadrightarrow BP\langle n\rangle^*(X) \to 0$$

of $BP\langle n\rangle^*$-modules, corresponding to a cartesian and cocartesian diagram (of monomorphisms)

$$\begin{tikzcd} K_n \arrow[r, hook] \arrow[d, hook] & \mathbf{tors}_{v_n} \arrow[d, hook] \\ BP\langle n+1\rangle^*(X)/v_{n+1} \arrow[r, hook] & BP\langle n\rangle^*(X). \end{tikzcd}$$

Lemma 4.7 applied to the right exact functor $\mathbb{F}_p \otimes_{BP\langle n\rangle^*} -$ implies that

$$\mathbb{F}_p \otimes_{BP\langle n+1\rangle^*} BP\langle n+1\rangle^*(X) \to \mathbb{F}_p \otimes_{BP\langle n\rangle^*} BP\langle n\rangle^*(X)$$

is injective, since $\mathbf{tors}_{v_n}$ is a trivial $BP\langle n\rangle^*$-module, so that $\mathbb{F}_p \otimes_{BP\langle n\rangle^*} (K_n \hookrightarrow \mathbf{tors}_{v_n})$ identifies with the monomorphism $K_n \to \mathbf{tors}_{v_n}$. $\square$

The method of proof can also be applied to consider the morphism

$$BP\langle n\rangle^* \otimes_{BP^*} BP^*(X) \hookrightarrow BP\langle n\rangle^*(X)$$

induced by $\rho_n : BP \to BP\langle n\rangle$.

Proposition 4.9. *For X a spectrum and $n \in \mathbb{N}$ such that*

1. $BP\langle n\rangle^* \otimes_{BP^*} BP^*(X) \hookrightarrow BP\langle n\rangle^*(X)$ *is injective;*
2. $\mathbf{tors}_{v_n} = \mathrm{Ker}(v_n)$*;*
3. $\tilde{\sigma}_n : BP^*(X) \oplus \mathbf{tors}_{v_n} \twoheadrightarrow BP\langle n\rangle^*(X)$ *is surjective;*

the morphism induced by ρ_{n-1}:

$$BP\langle n-1\rangle^* \otimes_{BP^*} BP^*(X) \to BP\langle n-1\rangle^*(X)$$

is injective.

Proof. The hypotheses provide a short exact sequence

$$0 \to L_n \to (BP\langle n\rangle^* \otimes_{BP^*} BP^*(X)) \oplus \mathbf{tors}_{v_n} \to BP\langle n\rangle^*(X) \to 0$$

such that the components $L_n \to \mathbf{tors}_{v_n}$ and $L_n \to BP\langle n\rangle^* \otimes_{BP^*} BP^*(X)$ are injective. As in the proof of Proposition 4.8, applying $BP\langle n-1\rangle^* \otimes_{BP\langle n\rangle^*} -$ (which identifies with $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)}[v_n]} -$) yields the horizontal short exact sequence below:

(3)

$$\begin{tikzcd} L_n \arrow[r] & BP\langle n-1\rangle^* \otimes_{BP^*} BP^*(X) \oplus \mathbf{tors}_{v_n} \arrow[r] & BP\langle n-1\rangle^* \otimes_{BP\langle n\rangle^*} BP\langle n\rangle^*(X) \arrow[d, hook, dotted] \\ & & BP\langle n-1\rangle^*(X), \end{tikzcd}$$

where the additional vertical inclusion is induced by ρ^n_{n-1}. The injectivity of the left hand horizontal morphism follows from the fact that multiplication by v_n acts trivially on $\mathbf{tors}_{v_n}$, so that $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)}[v_n]} (L_n \hookrightarrow \mathbf{tors}_{v_n})$ identifies with the inclusion $L_n \hookrightarrow \mathbf{tors}_{v_n}$.

By Lemma 4.7, it follows that

$$BP\langle n-1\rangle^* \otimes_{BP^*} BP^*(X) \to BP\langle n-1\rangle^* \otimes_{BP\langle n\rangle^*} BP\langle n\rangle^*(X)$$

is injective. Composing with the vertical monomorphism completes the proof. $\square$

5. Controlling the v_n-torsion in odd degrees

5.1. **The bounded torsion case.** The following result shows how the v_n-torsion can be understood in odd degrees under suitable hypotheses. This will be applied in Section 6 to deduce the main general result of the paper, Theorem 6.1.

Proposition 5.1. *Let X be a spectrum and $n \in \mathbb{N}$ be such that $H\mathbb{Z}^{\mathrm{odd}}_{(p)}(X) = BP\langle 0\rangle^{\mathrm{odd}}(X) \hookrightarrow H\mathbb{F}_p^{\mathrm{odd}}(X)$ and there exists $N \in \mathbb{N}$ such that, for $0 \leq j \leq n$:*

1. $v_j^N BP\langle j\rangle^{\mathrm{odd}}(X) = 0$*;*
2. $\mathrm{image}(\rho^j_{-1})^{\mathrm{odd}} = \mathrm{Im}(Q_0 \dots Q_j)^{\mathrm{odd}} \subset H\mathbb{F}_p^{\mathrm{odd}}(X)$*;*

then, for $0 \leq j \leq n$, ρ^j_{-1} *induces an isomorphism:*

$$BP\langle j\rangle^{\mathrm{odd}}(X) \cong \mathrm{Im}(Q_0 \dots Q_j)^{\mathrm{odd}}.$$

In particular, $BP\langle j\rangle^{\mathrm{odd}}(X)$ *is a trivial* $BP\langle n\rangle^*$*-module.*

Proof. The result is proved by upward induction upon j, starting with $j = 0$, which forms part of the hypothesis. For the inductive step, suppose the result established for smaller values of j. We need to prove that the reduction $BP\langle j\rangle^*(X) \to H\mathbb{F}^*_p(X)$ is injective in odd degrees.

By the inductive hypothesis, the kernel of $\rho^j_{-1} : BP\langle j\rangle^{\mathrm{odd}}(X) \to H\mathbb{F}^{\mathrm{odd}}_p(X)$ coincides with the kernel of $\rho^j_{j-1} : BP\langle j\rangle^{\mathrm{odd}}(X) \to BP\langle j-1\rangle^{\mathrm{odd}}(X)$, which is the image of multiplication by v_j (restricted to odd degrees). Hence, if $x = x_0$ (of odd degree) is in the kernel of ρ^j_{-1}, there is an odd degree element x'_1 such that $v_j x'_1 = x_0$. Now $\rho^j_{-1}(x'_1) = Q_0 \dots Q_j y_1$ for some $y_1 \in H\mathbb{F}^*_p(X)$, by the hypothesis on the image of $(\rho^j_{-1})^{\mathrm{odd}}$. Thus, consider the element $x_1 := x'_1 - (\pm)(q_j \dots q_0)y_1$, where the sign is taken so that $\rho^j_{-1}(\pm(q_j \dots q_0)y_1) = Q_0 \dots Q_j y_1$ (using Lemma 2.4); by construction $\rho^j_{-1}(x_1) = 0$ and $v_j x_1 = v_j x'_1 = x_0$.

Suppose $x_0 \neq 0$; then $x_1 \neq 0$ and the argument can be repeated to form a sequence of non-trivial elements $x_s \in BP\langle j\rangle^{\mathrm{odd}}(X)$ such that $v^s_j x_s = x_0 \neq 0$, $s \in \mathbb{N}$. This contradicts the hypothesis that the $BP\langle j\rangle^{\mathrm{odd}}(X)$ is bounded v_j-torsion; hence $(\rho^j_{-1})^{\mathrm{odd}}$ is injective. ☐

Remark 5.2.

(1) The hypothesis on the image of $(\rho^j_{-1})^{\mathrm{odd}}$ is implied, for example, by the condition $\mathscr{H}(X, j)^{\mathrm{odd}} = 0$.
(2) If $H\mathbb{F}^*_p(X)$ is Q_0-acyclic, then $BP\langle 0\rangle^*(X) = H\mathbb{Z}^*_{(p)}(X)$ embeds in $H\mathbb{F}^*_p(X)$; in particular, the required embedding hypothesis holds in odd degrees.

5.2. **The Noetherian case.** When X is a space, the bounded torsion hypothesis required in Proposition 5.1 can sometimes be provided by exploiting the algebra structure of $BP\langle n\rangle^*(X_+)$, in particular in the presence of a finiteness hypothesis.

Proposition 5.3. *Let* X *be a space and* $n \in \mathbb{N}$ *such that* $BP^{\mathrm{odd}}(X) = 0$, $BP\langle n\rangle^*(X_+)$ *is a Noetherian algebra and* $BP\langle n\rangle^{\mathrm{odd}}(X) = 0$.

Then, for all $j \leq n$, $BP\langle j\rangle^{\mathrm{odd}}(X)$ *is a Noetherian* $BP\langle n\rangle^*(X_+)$*-module and there exists* $N \in \mathbb{N}$ *such that*

$$v^N_i BP\langle j\rangle^{\mathrm{odd}}(X) = 0$$

for all $0 \leq i \leq j$.

Proof. The fact that $BP\langle j\rangle^{\mathrm{odd}}(X)$ is a Noetherian $BP\langle n\rangle^*(X_+)$-module is proved by a standard downward induction upon j. Proposition 2.5 implies that, for $0 \leq i \leq j$, $BP\langle j\rangle^{\mathrm{odd}}(X)$ is v_i-torsion. Since, for each j, $BP\langle j\rangle^{\mathrm{odd}}(X)$ is finitely-generated over $BP\langle n\rangle^*(X_+)$, there is a uniform bound on the torsion. ☐

6. Criteria for trivial torsion

The following is the main general result of the paper; it is applied in the following section to the case $X = BV$, for V an elementary abelian p-group.

Theorem 6.1. *Let X be a space and $n \in \mathbb{N}$ for which the following hypotheses are satisfied:*

(1) $BP^{\mathrm{odd}}(X) = BP\langle n\rangle^{\mathrm{odd}}(X) = 0$;
(2) $BP\langle n\rangle^*(X_+)$ *is Noetherian;*
(3) $BP\langle 0\rangle^*(X) \hookrightarrow H\mathbb{F}_p^*(X)$ *is a monomorphism with image* $\mathrm{Im}(Q_0)$;
(4) $\mathscr{H}(X,j)^{\mathrm{odd}} = 0$ *for* $0 \le j \le n$.

Then, for $0 \le j \le n$, $\mathbf{tors}_{v_j}$ is a trivial $BP\langle j\rangle^*$-module which identifies as:

$$\begin{aligned}\mathbf{tors}_{v_j} &\cong \mathrm{Im}(q_j \dots q_0) \subset BP\langle j\rangle^*(X)\\ &\cong \mathrm{Im}(Q_0 \dots Q_j) \subset H\mathbb{F}_p^*(X);\end{aligned}$$

in particular $BP\langle j\rangle^{\mathrm{odd}}(X) \cong \mathrm{Im}(Q_0 \dots Q_j)^{\mathrm{odd}}$.

Moreover:

(1) *the reduction map ρ^j_{j-1} induces a monomorphism $\mathbf{tors}_{v_j} \hookrightarrow \mathbf{tors}_{v_{j-1}}$, which corresponds to the natural inclusion $\mathrm{Im}(Q_0 \dots Q_j) \hookrightarrow \mathrm{Im}(Q_0 \dots Q_{j-1})$;*
(2) $\sigma_j : BP\langle j+1\rangle^*(X) \oplus \mathbf{tors}_{v_j} \to BP\langle j\rangle^*(X)$ *is surjective;*
(3) *the reduction map ρ^j_{-1} induces a monomorphism*

$$\mathbb{F}_p \otimes_{BP\langle j\rangle^*} BP\langle j\rangle^*(X) \hookrightarrow H\mathbb{F}_p^*(X).$$

If, furthermore, $BP\langle j\rangle^*(X) \twoheadrightarrow \mathscr{H}^*(X,j)$ is surjective for $0 \le j \le n$, then the reduction map ρ^j_{-1} induces an isomorphism

$$\mathbb{F}_p \otimes_{BP\langle j\rangle^*} BP\langle j\rangle^*(X) \cong \bigcap_{i=0}^{j} \mathrm{Ker}(Q_i) \subset H\mathbb{F}_p^*(X).$$

Proof. Under the given hypotheses, by Remark 5.2, Propositions 5.1 and 5.3 together apply to determine $BP\langle j\rangle^{\mathrm{odd}}(X)$ for $0 \le j \le n$.

Consider $\mathrm{Ker}(v_j)$ in degree $2t + |v_j|$, for $j \ge 1$; this fits into a commutative diagram:

$$\begin{tikzcd} BP\langle j\rangle^{2t-1}(X) \arrow[r, hook] & BP\langle j-1\rangle^{2t-1}(X) \arrow[r, two heads] \arrow[d, hook] \arrow[dr, dotted, "\alpha"] & \mathrm{Ker}(v_j)^{2t+|v_j|} \arrow[d, "\nu"] \arrow[dr, hook] & \\ & H\mathbb{F}_p^{2t-1}(X) \arrow[r, "\pm Q_j"'] & H\mathbb{F}_p^{2t+|v_j|}(X) & BP\langle j\rangle^{2t+|v_j|}(X), \arrow[l, "\rho^j_{-1}"] \end{tikzcd}$$

where the top row is the short exact sequence (2) and the commutative square is provided by Lemma 2.4. Here, by the odd degree case, the morphism $BP\langle j\rangle^{2t-1}(X) \to BP\langle j-1\rangle^{2t-1}(X)$ identifies as the monomorphism

$$\mathrm{Im}(Q_0 \dots Q_j)^{2t-1} \hookrightarrow \mathrm{Im}(Q_0 \dots Q_{j-1})^{2t-1}.$$

The morphism α indicated by the dotted arrow factors as

$$\mathrm{Im}(Q_0 \dots Q_{j-1})^{2t-1} \overset{\pm Q_j}{\twoheadrightarrow} \mathrm{Im}(Q_0 \dots Q_j)^{2t+|v_j|} \subset H\mathbb{F}_p^{2t+|v_j|}(X),$$

hence has kernel $\big(\mathrm{Ker}(Q_j) \cap \mathrm{Im}(Q_0 \dots Q_{j-1})\big)^{2t-1}$, which contains $\mathrm{Im}(Q_0 \dots Q_j)^{2t-1}$. The quotient

$$\big(\mathrm{Ker}(Q_j) \cap \mathrm{Im}(Q_0 \dots Q_{j-1})/\{\mathrm{Im}(Q_0 \dots Q_j)\}\big)^{2t-1}$$

embeds in $\mathscr{H}(X,j)^{2t-1}$ and hence is trivial, by hypothesis. Thus the kernel of α coincides with the image of $BP\langle j\rangle^{2t-1}(X)$ in $BP\langle j-1\rangle^{2t-1}(X)$. It follows that the vertical morphism ν induces an isomorphism:

$$\mathrm{Ker}(v_j)^{2t+|v_j|} \xrightarrow{\cong} \mathrm{Im}(Q_0 \ldots Q_j)^{2t+|v_j|} \subset H\mathbb{F}_p^{2t+|v_j|}(X).$$

In particular, the composite $\mathrm{Ker}(v_j)^{2t+|v_j|} \subset BP\langle j\rangle^{2t+|v_j|}(X) \to H\mathbb{F}_p^{2t+|v_j|}(X)$ is a monomorphism. Hence, by Lemma 2.1, in even degrees

$$\mathrm{Ker}(v_j)^{2*} = \mathbf{tors}_{v_j}^{2*} \cong \mathrm{Im}(Q_0 \ldots Q_j)^{2*}.$$

This completes the proof of the main statement.

If $j \geq 0$, since $\mathbf{tors}_{v_j}$ maps injectively to $H\mathbb{F}_p^*(X)$ by ρ_{-1}^j, which factorizes as $\rho_{-1}^{j-1}\rho_{j-1}^j$, it is clear that $\mathbf{tors}_{v_j} \hookrightarrow \mathbf{tors}_{v_{j-1}}$ is injective and is as stated. Moreover, the morphism $\kappa_j : \mathbf{tors}_{v_j}^* \to \mathrm{Ker}(v_{j+1})^{*+|Q_{j+1}|}$ is the surjection

$$\mathrm{Im}(Q_0 \ldots Q_j)^* \twoheadrightarrow \mathrm{Im}(Q_0 \ldots Q_{j+1})^{*+|Q_{j+1}|}$$

induced by $\pm Q_{j+1}$. Thus, by Lemma 4.2, the morphism σ_j is surjective.

This allows Proposition 4.8 to be applied for $0 \leq j \leq n$ to deduce, by increasing induction on j, that the reduction morphism ρ_{-1}^j induces a monomorphism

$$\mathbb{F}_p \otimes_{BP\langle j\rangle^*} BP\langle j\rangle^*(X) \hookrightarrow H\mathbb{F}_p^*(X).$$

Finally, under the additional hypothesis of surjectivity to $\mathscr{H}^*(X,j)$, the image is identified by Corollary 3.5. ☐

Corollary 6.2. *Under the hypotheses of Theorem* 6.1*, if, in addition, the reduction morphism*

$$\rho_n : BP^*(X) \twoheadrightarrow BP\langle n\rangle^*(X)$$

is surjective, then, for each $0 \leq j \leq n$*, the morphisms* $\rho_j : BP \to BP\langle j\rangle$ *and* $q_j \ldots q_0 : H\mathbb{F}_p \to \Sigma^{\sum |Q_i|}BP\langle j\rangle$ *induce a surjection*

$$BP^*(X) \oplus H\mathbb{F}_p^{*-\sum |Q_i|}(X) \twoheadrightarrow BP\langle j\rangle^*(X).$$

Proof. The result follows by combining the conclusions of Theorem 6.1 with Proposition 4.4. ☐

This corollary can be strengthened under an additional hypothesis. The statement of the following result uses the conclusions of Proposition 4.4, Theorem 6.1 and the notation introduced in Proposition 4.9.

Proposition 6.3. *Under the hypotheses of Theorem* 6.1 *if, in addition, the reduction morphism* ρ_n *induces an isomorphism*

$$BP\langle n\rangle^* \otimes_{BP^*} BP^*(X) \xrightarrow{\cong} BP\langle n\rangle^*(X),$$

then, for $0 \leq j \leq n$*, the morphism* ρ_j *induces a monomorphism*

$$BP\langle j\rangle^* \otimes_{BP^*} BP^*(X) \hookrightarrow BP\langle j\rangle^*(X)$$

and, if L_j *denotes the kernel of the surjection*

$$(BP\langle j\rangle^* \otimes_{BP^*} BP^*(X)) \oplus \mathbf{tors}_{v_j} \twoheadrightarrow BP\langle j\rangle^*(X),$$

then $L_n = \mathbf{tors}_{v_n}$ *and, for* $n \geq j \geq 0$*, the inclusion* $\mathbf{tors}_{v_j} \hookrightarrow \mathbf{tors}_{v_{j-1}}$ *induces a short exact sequence*

$$0 \to L_j \to L_{j-1} \to \big(\mathrm{Ker}(Q_j) \cap \mathrm{Im}(Q_0 \ldots Q_{j-1})\big)/\mathrm{Im}(Q_0 \ldots Q_j) \to 0.$$

Proof. The injectivity of $BP\langle j\rangle^* \otimes_{BP^*} BP^*(X) \hookrightarrow BP\langle j\rangle^*(X)$ follows by applying Proposition 4.9, using the conclusions of Theorem 6.1. The proof of the remaining statements extends the methods of the proof of Proposition 4.9, using the fact that the reduction ρ^j_{j-1} induces a monomorphism $\iota_j : \mathbf{tors}_{v_j} \hookrightarrow \mathbf{tors}_{v_{j-1}}$.

Since $BP\langle j\rangle^* \otimes_{BP^*} BP^*(X)$ is concentrated in even degrees and $\mathbf{tors}^{\mathrm{odd}}_{v_j}$ coincides with $BP\langle j\rangle^{\mathrm{odd}}(X)$, L_j is concentrated in even degrees. Moreover, since L_j injects to $\mathbf{tors}_{v_j}$, L_j is a trivial $BP\langle j\rangle^*$-module.

It is straightforward to show that $L_n = \mathbf{tors}_{v_n}$. Then, for $n \geq j > 0$, the diagram (3) of the proof of Proposition 4.9 extends to a commutative diagram in which the rows and columns are short exact sequences:

$$\begin{tikzcd}
L_j \arrow[r] \arrow[d] & BP\langle j-1\rangle^*\otimes_{BP^*}BP^*(X) \oplus \mathbf{tors}_{v_j} \arrow[r] \arrow[d, "1\oplus\iota_j"'] & BP\langle j-1\rangle^*\otimes_{BP\langle j\rangle^*}BP\langle j\rangle^*(X) \arrow[d, hook] \\
L_{j-1} \arrow[r] \arrow[d] & BP\langle j-1\rangle^*\otimes_{BP^*}BP^*(X) \oplus \mathbf{tors}_{v_{j-1}} \arrow[r] \arrow[d] & BP\langle j-1\rangle^*(X) \arrow[d] \\
L_{j-1}/L_j \arrow[r] & \mathbf{tors}_{v_{j-1}}/\mathbf{tors}_{v_j} \arrow[r] & \mathrm{Ker}(v_j)^{*+|Q_j|}
\end{tikzcd}$$

and the right hand column is the universal coefficient short exact sequence. This diagram provides the natural inclusion of L_j to L_{j-1}. (As remarked above, the case of interest is where the degree of the middle column is even; the odd degree case has already been used in the proof of Theorem 6.1.)

Theorem 6.1 identifies the quotient $\mathbf{tors}_{v_{j-1}}/\mathbf{tors}_{v_j}$ as

$$\mathrm{Im}(Q_0 \ldots Q_{j-1})/\mathrm{Im}(Q_0 \ldots Q_j)$$

and $\mathrm{Ker}(v_j)^{*+|Q_j|}$ as $\mathrm{Im}(Q_0 \ldots Q_j)$, in appropriately shifted degree; the surjection is induced by the Milnor derivation Q_j. The identification of the subquotient L_{j-1}/L_j follows. □

7. The case of elementary abelian p-groups

7.1. **Generalized Margolis vanishing.** The structure of $H\mathbb{F}_p^*(BV_+)$ is well known; by the Künneth theorem, it suffices to describe the rank one case. For p odd, $H\mathbb{F}_p^*(B\mathbb{Z}/p_+) \cong \Lambda(u) \otimes \mathbb{F}_p[v]$, with $|u| = 1$ and $|v| = 2$ with Bockstein $\beta u = v$; for $p = 2$, $H\mathbb{F}_2^*(B\mathbb{Z}/2_+) \cong \mathbb{F}_2[u]$, where $|u| = 1$. The action of the Milnor primitives is determined as follows: for p odd Q_i acts trivially on v and $Q_i u = v^{p^i}$; for $p = 2$, $Q_i u = u^{2^{i+1}}$.

For p odd and V an elementary abelian p-group of finite rank, the above gives the isomorphism

$$H\mathbb{F}_p^*(BV_+) \cong \Lambda^*(V^\sharp) \otimes S^*(V^\sharp),$$

where $V^\sharp$ denotes the linear dual, Λ^* the exterior algebra and S^* the symmetric algebra. This provides a bigrading which is related to the standard grading by $H\mathbb{F}_p^n(BV_+) \cong \bigoplus_{a+2b=n} \Lambda^a(V^\sharp) \otimes S^b(V^\sharp)$. The Milnor primitives respect the decomposition, in the sense that

$$Q_i : \Lambda^a(V^\sharp) \otimes S^b(V^\sharp) \to \Lambda^{a-1}(V^\sharp) \otimes S^{b+p^i}(V^\sharp). \tag{4}$$

This is a Koszul-complex type differential.

Remark 7.1. Similar statements are obtained for $p=2$ by filtering, based on the isomorphism of $\mathbb{F}_2[u^2]$-modules: $\mathbb{F}_2[u] \cong \Lambda(u) \otimes \mathbb{F}_2[u^2]$.

The map $B\mathbb{Z}/p \to \mathbb{C}P^\infty$ induced by the inclusion $\mathbb{Z}/p \subset S^1$ of pth roots of unity, induces a morphism of unstable algebras $H\mathbb{F}_p^*(\mathbb{C}P^\infty_+) \cong \mathbb{F}_p[x] \hookrightarrow H\mathbb{F}_p^*(B\mathbb{Z}/p_+)$, with $|x| = 2$, determined by $x \mapsto v$ (respectively $x \mapsto u^2$ for $p=2$). Since $H\mathbb{F}_p^*((\mathbb{C}P^\infty)^{\times d}_+)$ is concentrated in even degrees, as observed in Remark 3.4, it can be identified with $\mathscr{H}((\mathbb{C}P^\infty)^{\times d}_+)$.

Theorem 7.2. *Let V be an elementary abelian p-group of rank d and $n \in \mathbb{N}$. Then the map $B\mathbb{Z}/p \to \mathbb{C}P^\infty$ induces a surjection*

$$H\mathbb{F}_p^*((\mathbb{C}P^\infty)^{\times d}_+) = \mathscr{H}((\mathbb{C}P^\infty)^{\times d}_+) \twoheadrightarrow \mathscr{H}^*(BV_+, n).$$

In particular $\mathscr{H}^*(BV_+, n)^{\mathrm{odd}} = 0$ and the Thom reduction $BP \to H\mathbb{F}_p$ induces a surjection

$$BP^*(BV_+) \twoheadrightarrow \mathscr{H}^*(BV_+, n).$$

Proof. The case p odd is treated below; the argument adapts to the case $p=2$ by filtering using the number of terms of odd degree in monomials (cf. Remark 7.1).

Since $H\mathbb{F}_p^*(\mathbb{C}P^\infty_+)$ is concentrated in even degrees, the Milnor operations act trivially, hence the Thom reduction maps to $\bigcap_{i=0}^\infty \mathrm{Ker}(Q_i)$ and the morphism to $\mathscr{H}^*(BV_+, n)$ is defined. The first statement is proved using a refinement of Margolis' criterion for the freeness of modules over exterior algebras [Mar83, Theorem 8(a), Section 18.3], exploiting the filtration induced by the number of exterior generators.

Namely, a straightforward reduction (using the behaviour (4) of the Milnor operations with respect to the bigrading) implies that it is sufficient to show that an element $x \in \Lambda^a(W) \otimes S^b(W)$ which lies in $\bigcap_{i=0}^n \mathrm{Ker}(Q_i)$ is in the image of

$$(Q_0 \dots Q_n) : \Lambda^{a+n+1}(W) \otimes S^{b-\sum |Q_i|}(W) \to \Lambda^a(W) \otimes S^b(W),$$

where W is written for $V^\sharp$, for notational simplicity. This is a case of Proposition 7.4 below.

The Thom reduction $BP^*((\mathbb{C}P^\infty)^{\times d}_+) \twoheadrightarrow H\mathbb{F}_p^*((\mathbb{C}P^\infty)^{\times d}_+)$ is surjective; moreover, Landweber showed that $BP^*((\mathbb{C}P^\infty)^{\times d}_+) \twoheadrightarrow BP^*(BV_+)$ is surjective (this is included in [Str00, Proposition 2.3]). The final statement follows. $\square$

The acyclicity of the Koszul complex is restated below; it is valid for all primes (with the appropriate interpretation of the operation Q_i at $p=2$).

Lemma 7.3. *For $i, a, n \in \mathbb{N}$, the Koszul complex yields an acyclic complex:*

$$\dots \to \Lambda^n \otimes S^{a-np^i} \xrightarrow{Q_i} \Lambda^{n-1} \otimes S^{a-(n-1)p^i} \xrightarrow{Q_i} \dots \to S^a \to \overline{S^a_i} \to 0,$$

where $\overline{S^a_i}$ is the truncated symmetric power, imposing the relation $w^{p^i} = 0$.

The following result can be deduced from [Mar83, Theorem 8(a), Section 18.3]; a direct proof is given here, since this indicates the very general nature of the result.

Proposition 7.4. *Let W be an elementary abelian p-group of finite rank, $\emptyset \neq \mathscr{I} \subset \mathbb{N}$ be a non-empty, finite indexing set and $0 < a \in \mathbb{N}$. If $x \in \Lambda^a(W) \otimes S^b(W)$ is an element such that $Q_i x = 0$, $\forall i \in \mathscr{I}$, then there exists an element $y \in \Lambda^{a+|\mathscr{I}|}(W) \otimes S^{b-\sum_{i\in\mathscr{I}}|Q_i|}(W)$ such that*

$$x = (\prod_{i\in\mathscr{I}} Q_i)y.$$

In particular, $x = 0$ if either $a + |\mathscr{I}| > \dim W$ or $b < \sum_{i\in\mathscr{I}} |Q_i|$.

Proof. The proof is by induction on $|\mathscr{I}|$, with an internal induction on $|x|$, where the degree of an element of $\Lambda^a(W) \otimes S^b(W)$ is $a+2b$. For $|\mathscr{I}| = 1$, the result holds by Lemma 7.3; the initial step of the $|x|$ induction is a straightforward consequence of connectivity, since the degree of elements is non-negative.

For the inductive step of the degree induction, consider $x \in \Lambda^a(W) \otimes S^b(W)$ and $\mathscr{I} = \{i_1 < i_2 < \ldots < i_t\}$ as in the statement, supposing that the result holds for all indexing sets $\mathscr{J}$ with $|\mathscr{J}| < |\mathscr{I}| = t$ and for such elements of degree $< |x|$.

To prove the result, it is sufficient to construct elements $\alpha_k \in \Lambda^{a+t-1}(W) \otimes S^{b-\sum_{i\in\mathscr{I}\setminus\{i_1\}}|Q_i|}(W)$, for $t \geq k \geq 1$ which satisfy the following properties:

$$(\prod_{i\in\mathscr{I}\setminus\{i_1\}} Q_i)\alpha_k = x, \tag{5}$$

$$(\prod_{1\leq s\leq k} Q_{i_s})\alpha_k = 0. \tag{6}$$

Indeed, the element α_1 then satisfies $Q_{i_1}\alpha_1 = 0$, so that acyclicity of the complex of Lemma 7.3 implies the existence of y such that $Q_{i_1} y = \alpha_1$. By condition (5) for α_1, this satisfies $x = (\prod_{i\in\mathscr{I}} Q_i)y$, as required.

The construction of the α_k is by descending induction on k; by induction upon $|\mathscr{I}|$, there exists an element α_t such that $x = (\prod_{i\in\mathscr{I}\setminus\{i_1\}} Q_i)\alpha_t$. Since $Q_{i_1}x = 0$, by hypothesis, the second condition required of α_t also holds, so this forms the initial step of the descending induction.

For the inductive step ($t \geq k > 1$), consider α_k and form the element $\beta_k := (\prod_{1\leq s<k} Q_{i_s})\alpha_k$, which is the obstruction to α_k being taken for α_{k-1}. In the case $k = t$, $|x| - |\beta_t| = |Q_{i_t}| - |Q_{i_1}| > 0$.

Condition (6) for α_k implies that $Q_{i_s}\beta_k = 0$, for $1 \leq s \leq k$. Hence, the global inductive hypothesis in the proof of the theorem yields an element γ_k such that $(\prod_{1\leq s\leq k} Q_{i_s})\gamma_k = \beta_k$ (for $k = t$, this is induction on the degree, using the fact that $|\beta_t| < |x|$, and, for $k < t$, induction on $|\mathscr{I}|$).

Taking $\alpha_{k-1} := \alpha_k - Q_{i_k}\gamma_k$, the required conditions are satisfied, completing the inductive step. □

Remark 7.5. As suggested by the referee, it is interesting no observe the following consequence: $H\mathbb{F}_p^*(BV_+) \cong \Lambda^*(V^\sharp) \otimes S^*(V^\sharp)$ contains no free $\Lambda(Q_i|i \in \mathscr{I})$-submodule if $|\mathscr{I}| > \dim V$. In the case $|\mathscr{I}| = \dim V$, $\Lambda^{|\mathscr{I}|}(V^\sharp)$ is one dimensional and the free $\Lambda(Q_i|i \in \mathscr{I})$-module summand of $H\mathbb{F}_p^*(BV_+)$ is generated by $\Lambda^{|\mathscr{I}|}(V^\sharp) \otimes S^*(V^\sharp)$.

The reader may wish to compare Proposition 7.4 with the analysis of the stable summands of $\Sigma^\infty BV_+$ which have cohomology that is free over $\Lambda(Q_i|i \in \mathscr{I})$; here Margolis's criterion can be applied directly, as in [CK89].

7.2. **The structure of the $BP\langle n\rangle$-cohomology of elementary abelian p-groups.** The description of $BP\langle n\rangle^*(BV)$ is obtained by applying Theorem 6.1. The required property of the torsion of $BP\langle n\rangle^{\mathrm{odd}}(BV)$ is provided by the following:

Proposition 7.6 ([Str00, Proposition 2.3])**.** *For V an elementary abelian p-group of rank $d \leq n+1$, $BP\langle n\rangle^*(BV_+)$ is a Noetherian algebra concentrated in even degrees, which has no p-torsion if $d < n+1$.*

Notation 7.7. Write $\mathbf{\Psi} H\mathbb{F}_p^*(BV) \subset H\mathbb{F}_p^*(BV)$ for the augmentation ideal of the polynomial subalgebra if p is odd and for the double $\Phi H\mathbb{F}_2^*(BV)$ if $p=2$. Thus, $\mathbf{\Psi} H\mathbb{F}_p^*(BV)$ coincides with the image of ρ_{-1}.

Theorem 7.8. *For V an elementary abelian p-group of finite rank and $j \in \mathbb{N}$, the following statements hold:*

(1) $\mathbf{tors}_{v_j}$ *is a trivial $BP\langle j\rangle^*$-module which identifies as:*

$$\begin{aligned} \mathbf{tors}_{v_j} &\cong \mathrm{Im}(q_j \ldots q_0) \subset BP\langle j\rangle^*(BV) \\ &\cong \mathrm{Im}(Q_0 \ldots Q_j) \subset H\mathbb{F}_p^*(BV) \end{aligned}$$

and, in particular, $BP\langle j\rangle^{\mathrm{odd}}(BV) \cong \mathrm{Im}(Q_0 \ldots Q_j)^{\mathrm{odd}}$.

(2) *The reduction map ρ^j_{-1} induces an isomorphism*

$$\mathbb{F}_p \otimes_{BP\langle j\rangle^*} BP\langle j\rangle^*(BV) \cong \bigcap_{i=0}^{j} \mathrm{Ker}(Q_i) \subset H\mathbb{F}_p^*(BV).$$

(3) *The reduction map ρ_j induces a monomorphism*

$$BP\langle j\rangle^* \otimes_{BP^*} BP^*(BV) \hookrightarrow BP\langle j\rangle^*(BV)$$

which is an isomorphism modulo v_j-torsion and, is an isomorphism for $j > \mathrm{rank}(V)$; in particular,

$$BP\langle j\rangle^*(BV)\Big[\frac{1}{v_j}\Big] \cong BP\langle j\rangle^*\Big[\frac{1}{v_j}\Big] \otimes_{BP^*} BP^*(BV).$$

(4) *The reduction map ρ_j and localization induces a monomorphism*

$$BP\langle j\rangle^*(BV) \hookrightarrow H\mathbb{F}_p^*(BV) \oplus \Big(BP\langle j\rangle^*\Big[\frac{1}{v_j}\Big] \otimes_{BP^*} BP^*(BV)\Big).$$

(5) *The morphism $\tilde{\sigma}_j$ induces a short exact sequence*

$$0 \to L_j \hookrightarrow \big(BP\langle j\rangle^* \otimes_{BP^*} BP^*(BV)\big) \oplus \mathbf{tors}_{v_j} \twoheadrightarrow BP\langle j\rangle^*(BV) \to 0$$

where L_j is isomorphic to $\mathbf{\Psi} H\mathbb{F}_p^*(BV) \cap \mathrm{Im}(Q_0 \ldots Q_j) \subset H\mathbb{F}_p^*(BV)$.

Proof. The first two statements follow from Theorem 6.1, using Theorem 7.2 and Proposition 7.6 to show that the hypotheses are satisfied. Statements (3) and (4) follow from Theorem 6.1 and Proposition 6.3.

Finally, the identification of L_j follows by analysing the information furnished in Proposition 6.3, using the fact that the image of L_j in $\mathbf{tors}_{v_j} \cong \mathrm{Im}(Q_0 \ldots Q_j) \subset H\mathbb{F}_p^*(BV)$ also lies in $\mathbf{\Psi} := \mathbf{\Psi} H\mathbb{F}_p^*(BV)$. Namely, the proof is by downward induction on j, starting from $j > \mathrm{rank}(V)$, for which the result is clear. Theorem 7.2 implies that $\mathbf{\Psi} \hookrightarrow H\mathbb{F}_p^*(BV)$ induces a surjection $\mathbf{\Psi} \twoheadrightarrow \mathscr{H}^*(BV, j)$, with kernel

$\mathbf{\Psi} \cap \mathrm{Im}(Q_0 \dots Q_j)$. The inductive step follows from the observation that the square below is cartesian:

$$\begin{tikzcd} & \mathbf{\Psi} \cap \mathrm{Im}(Q_0 \dots Q_{j-1}) \arrow[r, hook] \arrow[d, two heads] & \mathbf{\Psi} \arrow[d, two heads] \\ L_{j-1}/L_j \cong & \big(\mathrm{Ker}(Q_j) \cap \mathrm{Im}(Q_0 \dots Q_{j-1})\big)/\mathrm{Im}(Q_0 \dots Q_j) \arrow[r, hook] & \mathscr{H}^*(BV, j), \end{tikzcd}$$

where the isomorphism is given by Proposition 6.3. ☐

Remark 7.9. The method of proof applies *mutatis mutandis* to any spectrum constructed from BP by forming the quotient by a cofinite subset of a suitable set of generators $\{v_i | i \geq 0\}$ of BP_*.

Theorem 7.8 yields the following precise description of the failure of surjectivity of the reduction map ρ_j for the cohomology of BV, a far-reaching generalization of the result of Strickland [Str00].

Corollary 7.10. *For V an elementary abelian p-group of finite rank and $j \in \mathbb{N}$, the morphism* $BP \amalg \Sigma^{\sum |Q_i|} H\mathbb{F}_p \xrightarrow{(\rho_j, q_j \dots q_0)} BP\langle j\rangle$ *induces a surjection*

$$BP^*(BV) \oplus H\mathbb{F}_p^{* - \sum |Q_i|}(BV) \twoheadrightarrow BP\langle j\rangle^*(BV).$$

Proof. Follows from Corollary 6.2, using [Str00, Proposition 2.3] to treat the cases $n \gg 0$. ☐

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Laboratoire Angevin de Recherche en Mathématiques, UMR 6093, Faculté des Sciences, Université d'Angers, 2 Boulevard Lavoisier, 49045 Angers, France

E-mail address: `Geoffrey.Powell@math.cnrs.fr`