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INTERPOLATION SETS AND THE SIZE OF QUOTIENTS OF FUNCTION SPACES ON A LOCALLY COMPACT GROUP

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Abstract. We devise a fairly general method for estimating the size of quotients between algebras of functions on a locally compact group. This method is based on the concept of interpolation set we introduced and studied recently and unifies the approaches followed by many authors to obtain particular cases.

We find in this way that there is a linear isometric copy of $\ell_\infty(\kappa)$ in each of the following quotient spaces:

- $\mathcal{WAP}_0(G)/C_0(G)$ whenever G contains a subset X that is an E-set (see the definition in the paper) and $\kappa = \kappa(X)$ is the minimal number of compact sets required to cover X. In particular, $\kappa = \kappa(G)$ when G is an SIN-group.
- $\mathcal{WAP}(G)/\mathcal{B}(G)$, when G is any locally compact group and $\kappa = \kappa(Z(G))$ and $Z(G)$ is the centre of G, or when G is either an IN-group or a nilpotent group and $\kappa = \kappa(G)$.
- $\mathcal{WAP}_0(G)/\mathcal{B}_0(G)$, when G and κ are as in the foregoing item.
- $\mathcal{CB}(G)/\mathcal{LUC}(G)$, when G is any locally compact group that is neither compact nor discrete and $\kappa = \kappa(G)$.

1. Introduction

The main focus throughout the paper will be on C^*-algebras of functions on a locally compact group G with identity e. If $\ell_\infty(G)$ denotes the C^*-algebra of bounded, scalar-valued functions on G with the supremum norm, our concern will be with the following subalgebras of $\ell_\infty(G)$: the algebra $\mathcal{CB}(G)$ of continuous bounded functions, the algebra $\mathcal{LUC}(G)$ of bounded left uniformly continuous functions, the algebra $\mathcal{WAP}(G)$ of weakly almost periodic functions, the Fourier-Stieltjes algebra $B(G)$, the uniform closure of $B(G)$ denoted by $\mathcal{B}(G)$ and best known as the Eberlein algebra, the algebra $\mathcal{AP}(G)$ of almost periodic functions, and the algebra $C_0(G) \oplus \mathbb{C}1$, where $C_0(G)$ consists of the functions in $\mathcal{CB}(G)$ vanishing at infinity.

The spectra of these algebras $\mathcal{A}(G)$ define some of the best-known semigroup compactifications in the sense of [5]. These are compact right (or left) topological semigroups $G^{\mathcal{A}}$ having a dense, continuous, homomorphic copy of G contained in their topological centres (i.e., the map $x \mapsto sx \quad (x \mapsto xs) : G^{\mathcal{A}} \to G^{\mathcal{A}}$ is continuous for each $s \in G$). For instance, the compactification $G^{\mathcal{LUC}}$ is the spectrum of $\mathcal{LUC}(G)$, and is usually referred to as the $\mathcal{LUC}$- or $\mathcal{LC}$-compactification of G. It

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is the largest semigroup compactification in the sense that any other semigroup compactification is a quotient of $G^{\mathcal{LUC}}$. When G is discrete, $G^{\mathcal{LUC}}$ and the Stone-Čech compactification βG are the same. The $\mathcal{WAP}$-compactification $G^{\mathcal{WAP}}$ is the spectrum of $\mathcal{WAP}(G)$; it is the largest semitopological semigroup compactification. The Bohr or $\mathcal{AP}$-compactifiaction is the spectrum of $\mathcal{AP}(G)$ and is the largest topological (semi)group compactification.

The Banach duals of these C^*-algebras can also be made into Banach algebras with a convolution type product extending in most cases that of the group algebra $L^1(G)$. We may recall that $L^\infty(G)$ is the Banach dual of the group algebra $L^1(G)$ and consists of all scalar-valued functions which are measurable and essentially bounded with respect to the Haar measure; two functions are identified if they coincide on a locally null set, and the norm is given by the essential supremum norm. We may also recall that the product making $L^\infty(G)^*$ into a Banach algebra is the first (or the second) Arens product on the second dual space $L^1(G)^{**}$ of the group algebra, and that $\mathcal{LUC}(G)^*$ may be seen as a quotient Banach algebra of $L^1(G)^{**}$. For more details, see for instance [17]. These two Banach algebras have been studied extensively in recent years. Particular attention has been given to properties related to Arens regularity of the group algebra $L^1(G)$ and to the topological centres of $G^{\mathcal{LUC}}$, $\mathcal{LUC}(G)^*$ and $L^1(G)^{**}$. For the latest, see [8] and the references therein.

The definitions of all these function algebras will be given in the next section. But for the moment the following diagram summarizes already the inclusion relationships known to hold among these algebras. See [13, page 143] for the first inclusion; [13, Lemma 2.1] for the first equality; [45] or [5, Theorem 4.3.13] for the second equality; [5, Corollary 4.4.11] or [9] for the third inclusion; the rest is easy to check.

$$\begin{aligned} C_0(G)\oplus \mathcal{AP}(G) &\subseteq \mathcal{B}(G)=\mathcal{AP}(G)\oplus\mathcal{B}_0(G)\subseteq \mathcal{WAP}(G)=\mathcal{AP}(G)\oplus\mathcal{WAP}_0(G)\\ &\subseteq \mathcal{LUC}(G)\cap\mathcal{RUC}(G)\subseteq\mathcal{LUC}(G)\subseteq\mathcal{CB}(G)\\ &\subseteq L^\infty(G). \end{aligned}$$

When G is finite, the diagram is trivial. When G is infinite and compact, the diagram reduces to $\mathcal{CB}(G)\subseteq L^\infty(G)$.

The task of comparing these algebras and estimating the sizes of the quotients formed among them has already been taken by many authors. We now give a brief review of what is known in this respect.

In the review below as well as in our study of quotients between the above algebras the compact covering number will appear at several points. We recall that the *compact covering number* of a topological space X is the smallest cardinal number $\kappa(X)$ of compact subsets of X required to cover X.

Comparing $L^\infty(G)$ with its subspaces. Already in 1961, Civin and Yood proved in their seminal paper [15] that the quotient space $L^\infty(G)/\mathcal{CB}(G)$ is infinite-dimensional for any non-discrete locally compact Abelian group and deduced that the radical of the Banach algebra $L^\infty(G)^*$ (with one of the Arens products as a product) is also infinite-dimensional.

This idea was pushed further by Gulick in [35, Lemma 5.2] when G is Abelian, and proved that the quotient $L^\infty(G)/\mathcal{CB}(G)$ is even non-separable and so is the radical of $L^\infty(G)^*$. Then Granirer proved in [33] the same results for any non-discrete locally compact group.

A decade later, Young produced, for any infinite locally compact group G, a function in $L^\infty(G)$ which is not in $\mathcal{WAP}(G)$, proving the non-Arens regularity of the group algebra $L^1(G)$ for any such a group; see [56].

There was also [6, Theorem 4.2] where the quotient $\mathcal{LUC}(G)/\mathcal{WAP}(G)$ was seen to contain a linear isometric copy of $\ell_\infty(\kappa)$, where κ is the compact covering of G. A fortiori, the quotient $L^\infty(G)/\mathcal{WAP}(G)$ contains the same copy, a fact that was used in [6, Theorem 4.4] to deduce that the group algebra is even extremely non-Arens regular in the sense of Granirer, whenever κ is larger than or equal to the minimal cardinal $\chi(G)$ of a basis of neighbourhoods at the identity.

It was also proved in [6, Section 4] that the quotient $L^\infty(G)/\mathcal{CB}(G)$ always contains a linear isometric copy of ℓ_∞, yielding extreme non-Arens regularity for the group algebra of compact metrizable groups. Due to a result by Rosenthal proved in [50, Proposition 4.7, Theorem 4.8], larger copies of ℓ_∞ cannot be expected in $L^\infty(G)$ when G is compact. The question on extreme non-Arens regularity of the group algebra was recently settled by the authors of the present paper using a technique inspired by Theorem 2.11. We actually find in [26] that, for any compact group G, $L^\infty(G)/\mathcal{CB}(G)$ contains a copy of $L^\infty(G)$. This fact together with [6, Theorem 4.4] gives that $L^1(G)$ is extremely non-Arens regular for any infinite locally compact group.

Comparing $\mathcal{CB}(G)$ with its subspaces. In 1966, Comfort and Ross [16, Theorem 4.1] compared the spaces $\mathcal{CB}(G)$ and $\mathcal{AP}(G)$ for an arbitrary topological group, and proved that they are equal if and only if G is pseudocompact (i.e., every continuous scalar-valued function on G is bounded). In 1970, Burckel showed in [9] that $\mathcal{CB}(G)$ and $\mathcal{WAP}(G)$ are equal if and only if G is compact. In [3], Baker and Butcher compared $\mathcal{CB}(G)$ and $\mathcal{LUC}(G)$ for locally compact groups, and proved that these two spaces are equal if and only if G is either discrete or compact. This result was extended recently by Filali and Vedenjuoksu in [28, Theorem 4.3] to all topological groups which are not P-groups. The authors of [28] proved that if G is a topological group which is not a P-group, then $\mathcal{CB}(G) = \mathcal{LUC}(G)$ if and only if G is pseudocompact. In [20], Dzinotyiweyi showed that the quotient $\mathcal{CB}(G)/\mathcal{LUC}(G)$ is non-separable if G is a non-compact, non-discrete, locally compact group. This theorem was generalized in [6, Theorem 3.1] and [7, Theorem 4.1], where $\mathcal{CB}(G)/\mathcal{LUC}(G)$ was seen to contain in fact a linear isometric copy of ℓ_∞ whenever G is a non-precompact topological group which is not a P-group. So this theorem also actually improved Dzinotyiweyi's result for locally compact groups. For non-discrete, P-groups, the quotient $\mathcal{CB}(G)/\mathcal{LUC}(G)$ was seen to be trivial in the case when for instance G is a Lindelöf P-group (see [28, Theorem 5.1]), but may also contain a linear isometric copy of ℓ_∞ for some other P-groups (see [6, Theorem 3.3]). In [7, Theorem 3.1], using a technique due to Alas (see [1]), the quotient space $\mathcal{CB}(G)/\mathcal{LUC}(G)$ was also seen to contain a linear isometric copy of ℓ_∞ whenever G is a non-SIN topological group.

In the locally compact situation, our answer in the present paper is precise and definite. We prove, in Section 5, that there is a linear isometric copy of $\ell_\infty(\kappa)$ in $\mathcal{CB}(G)/\mathcal{LUC}(G)$, where as before κ is the compact covering G, if and only if G is neither compact nor discrete. This leads again to a linear isometric copy of $\ell_\infty(\kappa)$ into the quotient $L^\infty(G)/\mathcal{WAP}(G)$, and of course may be used to deduce again the extreme non-Arens regularity of $L^1(G)$ when $\kappa \geq \chi(G) \geq \omega$ as in [6, Theorem 4.4].

Comparing $\mathcal{LUC}(G)$ with $\mathcal{WAP}(G)$. In 1972, Granirer showed that $\mathcal{LUC}(G) = \mathcal{WAP}(G)$ if and only if G is compact [32].

It is not difficult to check that $G^{\mathcal{LUC}}$ is a semitopological semigroup (i.e., the topological centre of $G^{\mathcal{LUC}}$ is the whole of $G^{\mathcal{LUC}}$) if and only if $\mathcal{LUC}(G) = \mathcal{WAP}(G)$. The same observation can also be made for $\mathcal{LUC}(G)^*$. This means that $G^{\mathcal{LUC}}$ or $\mathcal{LUC}(G)^*$ is a semitopological semigroup if and only if G is compact, i.e., $G^{\mathcal{LUC}} = G$ is a compact group and $\mathcal{LUC}(G)^*$ coincides with the measure algebra $M(G)$.

More recently, Granirer's result was deduced by Lau and Pym in [43, Proposition 3.6] as a corollary of their main theorem on the topological centre of $G^{\mathcal{LUC}}$ being G, and again by Lau and Ülger in [44, Corollary 3.8] as a corollary of the topological centre of $L^1(G)^{**}$ being $L^1(G)$ [42].

Moreover, Granirer showed in the same paper that if G is non-compact and amenable, then the quotient $\mathcal{LUC}(G)/\mathcal{WAP}(G)$ contains a linear isometric copy of ℓ_∞, and so it is not separable. This result was extended by Chou in [10] to E-groups (see below for the definition), then by Dzinotyiweyi in [20] to all non-compact locally compact groups, and generalized by Bouziad and Filali in [6, Theorem 2.2] to all non-precompact topological groups. Moreover, as already mentioned above, this result was improved in [6, Theorem 4.2] when G is a non-compact locally compact group, by having a copy of $\ell_\infty(\kappa)$ in the quotient $\mathcal{LUC}(G)/\mathcal{WAP}(G)$.

Comparing $\mathcal{WAP}(G)$ with its subspaces. In the "regular" side of the inclusion diagram, when we compare $\mathcal{WAP}(G)$ with its subspaces, the situation is not simpler. It is true that the Fourier-Stieltjes algebra $B(G)$ may be dense in $\mathcal{WAP}(G)$ (i.e., $\mathcal{WAP}(G) = \mathcal{B}(G)$), as in the case of minimally weakly almost periodic groups studied by Veech, Chou and Ruppert; see [55], [12] and [53]. For these groups, $\mathcal{WAP}(G) = \mathcal{AP}(G) \oplus C_0(G)$. However, if G is a non-compact group, then $B(G)$ is far from being dense in $\mathcal{WAP}(G)$ in general as it shall soon be explained.

When comparing $\mathcal{WAP}(G)$ with $\mathcal{AP}(G)$ and $C_0(G)$, we may recall first that $\mathcal{WAP}(G) = \mathcal{AP}(G) \oplus \mathcal{WAP}_0(G)$. Burckel proved in [9] that $C_0(G) \subsetneq \mathcal{WAP}_0(G)$ when G is an Abelian, non-compact, locally compact group. In [10], Chou considered E-groups and proved that the quotient $\mathcal{WAP}_0(G)/C_0(G)$ contains a linear isometric copy of ℓ_∞. In Section 4, we improve this result by showing that ℓ_∞ may be replaced by an isometric copy of the larger space $\ell_\infty(\kappa(X))$ in the quotient space $\mathcal{WAP}_0(G)/C_0(G)$, where $\kappa(X)$ is the compact covering of any E-set X contained in G. So when G is an SIN-group, these quotients contain a copy of $\ell_\infty(\kappa(G))$. For the same class of groups, we prove also that the quotient $\mathcal{WAP}(G)/(\mathcal{AP}(G) \oplus C_0(G))$ is non-separable.

In Section 5, we deal with the non-compact, IN-groups, and with non-compact nilpotent groups. In this class of groups, the results of the previous section shall be considerably improved. Rudin proved in [51] that $\mathcal{B}(G) \subsetneq \mathcal{WAP}(G)$ if G is a locally compact Abelian group and contains a closed discrete subgroup which is not of bounded order. This was followed by [49], where Ramirez extended Rudin's result to any non-compact, locally compact, Abelian group. Then in [13], Chou extended and strengthened the theorem to all non-compact IN-groups and nilpotent groups by showing that the quotient $\mathcal{WAP}(G)/\mathcal{B}(G)$ contains a linear isometric copy of ℓ_∞.

We shall strengthen Chou's result in Section 5 by showing that, in these cases, there is in fact a linear isometric copy of $\ell_\infty(\kappa)$ in the quotient spaces $\mathcal{WAP}(G)/\mathcal{B}(G)$, $\mathcal{WAP}(G)/(\mathcal{AP}(G) \oplus C_0(G))$ and $\mathcal{WAP}_0(G)/\mathcal{B}_0(G)$, where κ is as before the compact covering of G. Our method of proof also shows that $\mathcal{WAP}(G)/\mathcal{B}(G)$ always contains a copy of $\ell_\infty(\kappa(Z(G)))$.

It is worthwhile to note that all this confirms an observation made in [5, page 216], and gives indeed an indication on the size and complexity of the $\mathcal{WAP}$-compactification $G^{\mathcal{WAP}}$ and the Banach algebra $\mathcal{WAP}(G)^*$.

OUTLINE. The underlying structure in many of the proofs that estimate the size of $\mathcal{A}_2(G)/\mathcal{A}_1(G)$ for C^*-subalgebras $\mathcal{A}_1(G) \subseteq \mathcal{A}_2(G)$ of $\ell_\infty(G)$, depends on the existence of sets of interpolation for $\mathcal{A}_2(G)$ that are not sets of interpolation for $\mathcal{A}_1(G)$ (see for instance [6], [7], [10], [13] or [20]). One of the main objectives of the present paper is to make that structure emerge in a clear fashion. A first, but essential, step towards this objective is to work with the right concept of interpolation sets. We will use here the general concept of interpolation set introduced in [24] that extends several related classical ones and show how to apply it in this setting. The resulting interpolation sets are characterized in [24] in term of topological group properties, thereby making them easier to manipulate. We finally illustrate the scope of our approach by studying some concrete cases. We shall in particular study under this light the following quotients: $\mathcal{WAP}_0(G)$ by $C_0(G)$ and $\mathcal{WAP}(G)$ by $\mathcal{AP}(G)\oplus C_0(G)$ for E-groups, $\mathcal{WAP}(G)$ by $\mathcal{B}(G)$, $\mathcal{WAP}(G)$ by $\mathcal{AP}(G) \oplus C_0(G)$ and $\mathcal{WAP}_0(G)$ by $\mathcal{B}_0(G)$ for IN-groups and nilpotent groups, $\mathcal{CB}(G)$ by $\mathcal{LUC}(G)$ for locally compact groups.

1.1. **The function algebras.** We start by recalling the definitions of the function algebras we are interested in; for more details the reader is directed for example to [5].

Let G be a topological group. For each function f defined on G, the left translate f_s of f by $s \in G$ is defined on G by $f_s(t) = f(st)$. For each $s \in G$, the left translation operator $L_s\colon \ell_\infty(G) \to \ell_\infty(G)$ is defined as $L_s(f) = f_s$. The supremum norm of an element $f \in \ell_\infty(G)$ will be denoted as $\|f\|_\infty$.

A function $f \in \ell_\infty(G)$ is *left uniformly continuous* when, if for every $\epsilon > 0$, there exists a neighbourhood U of e such that

$$|f(s) - f(t)| < \epsilon \quad \text{whenever} \quad st^{-1} \in U.$$

The algebra of left uniformly continuous functions on G is denoted by $\mathcal{LUC}(G)$. We warn the reader that some authors refer to these functions as right uniformly continuous because they are uniformly continuous for the *right* uniform structure on G. We however follow [5], where left uniformly continuous means left norm continuous in the sense that the map $s \mapsto f_s : G \to \mathcal{CB}(G)$ is norm-continuous.

A function $f \in \mathcal{CB}(G)$ is *almost periodic* when the set of all its left (equivalently, right) translates is a relatively norm compact subset in $\mathcal{CB}(G)$. The algebra of almost periodic functions on G is denoted by $\mathcal{AP}(G)$.

A function $f \in \mathcal{CB}(G)$ is *weakly almost periodic* when the set of all its left (equivalently, right) translates makes a relatively weakly compact subset in $\mathcal{CB}(G)$. The algebra of weakly almost periodic functions on G is denoted by $\mathcal{WAP}(G)$.

The *Fourier-Stieltjes algebra* $B(G)$ is the linear span of the set of all continuous positive definite functions on G. Equivalently, $B(G)$ is the space of coefficients of unitary representations of G when G is locally compact. As the Fourier-Stieltjes algebra is not uniformly closed we will work with the *Eberlein algebra* $\mathcal{B}(G)$, which is the uniform closure of $B(G)$, in symbols $\mathcal{B}(G) = \overline{B(G)}^{\|\cdot\|_\infty}$.

Let μ be the unique invariant mean on $\mathcal{WAP}(G)$ (see [5], or [9]). Put

$$\mathcal{WAP}_0(G) = \{f \in \mathcal{WAP}(G) : \mu(|f|) = 0\},$$
$$\mathcal{B}_0(G) = \{f \in \mathcal{B}(G) : \mu(|f|) = 0\}.$$

In [13, page 143], Chou denoted $B(G) \cap \mathcal{WAP}_0(G)$ by $B_c(G)$, and observed that $\mathcal{B}_0(G) = \overline{B_c(G)}$ when G is a locally compact group.

1.2. **The spectrum as a compactification.** Let G be a topological group, $\mathcal{A}(G) \subseteq \ell_\infty(G)$ be a unital C^*-subalgebra and denote by $G^{\mathcal{A}}$ the spectrum (the set of non-zero multiplicative linear functionals) of $\mathcal{A}(G)$. Equipped with the topology of pointwise convergence, $G^{\mathcal{A}}$ becomes a compact Hausdorff topological space. There is a canonical morphism $\epsilon_{\mathcal{A}} \colon G \to G^{\mathcal{A}}$ given by evaluations

$$\epsilon_{\mathcal{A}}(s)(f) = f(s) \text{ for every } f \in \mathcal{A}(G) \text{ and } s \in G.$$

This map is continuous if and only if $\mathcal{A}(G) \subseteq \mathcal{CB}(G)$, and injective on G if and only if $\mathcal{A}(G)$ separates the points of G. We may recall, for example, that the map $\epsilon_{\mathcal{A}}$ is injective on G (and in fact a homeomorphism onto its image in $G^{\mathcal{A}}$) whenever $C_0(G) \subseteq \mathcal{A}(G)$. This is not a necessary condition since it may also happen that $\epsilon_{\mathcal{A}}$ is injective when $C_0(G) \cap \mathcal{A}(G) = \{0\}$ as it is the case when G is a locally compact, maximally almost periodic and $\mathcal{A}(G) = \mathcal{AP}(G)$. It may also happen that $\epsilon_{\mathcal{A}}$ is injective on a given subset T of G. We will then identify T as a subset of $G^{\mathcal{A}}$. This situation occurs when for example T is an $\mathcal{A}(G)$-interpolation set.

The C^*-algebra $\mathcal{A}(G)$ is *left translation invariant* when $f_s \in \mathcal{A}(G)$ for every $f \in \mathcal{A}(G)$ and $s \in G$. In this case for every $x \in G^{\mathcal{A}}$ and $f \in \mathcal{A}(G)$ we may define a function xf on G by $xf(s) = x(f_s)$. A translation invariant C^*-subalgebra $\mathcal{A}(G)$ of $\mathcal{CB}(G)$ containing 1, and xf for every $f \in \mathcal{A}(G)$ and $x \in G^{\mathcal{A}}$ is called m-admissible in [5, Definition 2.10]. For simplicity we will use the term *admissible algebra* instead.

When $\mathcal{A}(G)$ is an admissible C^*-subalgebra of $\mathcal{CB}(G)$, $G^{\mathcal{A}}$ can be equipped with the product $G^{\mathcal{A}}$ given by

$$xy(f) = x(yf) \quad \text{for every} \quad x, y \in G^{\mathcal{A}} \text{ and } f \in \mathcal{A}(G).$$

$G^{\mathcal{A}}$ then becomes a semigroup compactification of the topological group G in the sense of [5]. This means that $G^{\mathcal{A}}$ is a compact semigroup having a continuous, dense, homomorphic, image of G such that the mappings

$$x \mapsto xy \colon G^{\mathcal{A}} \to G^{\mathcal{A}} \quad \text{and} \quad x \mapsto \epsilon_{\mathcal{A}}(s)x \colon G^{\mathcal{A}} \to G^{\mathcal{A}}$$

are continuous for every $y \in G^{\mathcal{A}}$ and $s \in G$.

The algebras $C_0(G) \oplus \mathbb{C}$, $\mathcal{AP}(G)$, $C_0(G) \oplus \mathcal{AP}(G)$, $\mathcal{B}(G)$, $\mathcal{WAP}_0(G) \oplus \mathbb{C}$, $\mathcal{WAP}(G)$ and $\mathcal{LUC}(G)$ are all known to be admissible; see for example [5]. But when G is locally compact, $\mathcal{CB}(G)$ is not admissible unless G is either discrete or compact; see [3] or [28] for more.

When G is a locally compact group and $\mathcal{A}$ is a unital left translation invariant C^*-subalgebra of $\mathcal{LUC}(G)$, the spectrum $G^{\mathcal{A}}$ has *the joint continuity property*, that is, the map

$$(s, x) \mapsto \epsilon_{\mathcal{A}}(s)x \colon G \times G^{\mathcal{A}} \to G^{\mathcal{A}}$$

is continuous; see [5, Theorem 1.4.2].

A recent account on semigroup compactifications is given in [29].

1.3. **A few words on notation.** All our groups will be multiplicative and their identity element will be denoted as e. The characteristic function of a set T will be denoted as 1_T. If X is a set and $T \subseteq X$, given $f \in \ell_\infty(X)$, we define $\|f\|_T = \sup\{|f(x)| \colon x \in T\}$ so that $\|f\|_\infty = \|f\|_X$. The morphism $\epsilon_\mathcal{A}$ maps G into $G^\mathcal{A}$ faithfully if $\mathcal{A}(G)$ separates points. If $X \subseteq G$, we will denote the closure of $\epsilon_\mathcal{A}(X)$ simply as $\overline{X}^\mathcal{A}$, while the closure of X in G will be denoted as $\overline{X}$. The reason for this is that in most of our applications the algebra $\mathcal{A}(G)$ separates points of G and therefore $\epsilon_\mathcal{A}$ may be used to identify G with a subset of $G^\mathcal{A}$.

A standard application of Gelfand duality identifies $\mathcal{A}(G)$ with $\mathcal{CB}(G^\mathcal{A})$. Under this identification, to every $f \in \mathcal{A}(G)$ there corresponds $f^\mathcal{A} \in \mathcal{CB}(G^\mathcal{A})$ in such a way that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\epsilon_\mathcal{A}} & G^\mathcal{A} \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle f^\mathcal{A}} \\ & & \mathbb{C} \end{array} \tag{1}$$

When $\epsilon_\mathcal{A}$ is injective $f^\mathcal{A}$ can be seen as an *extension* of f to $G^\mathcal{A}$.

2. Interpolation sets and quotients of function spaces

We begin our work by introducing in precise terms the sets we will be using, and then we prove the impact they have in measuring the size of our quotient spaces $\mathcal{A}_2(G)/\mathcal{A}_1(G)$. This is achieved in Theorem 2.11.

It is worthwhile to note that this theorem may also be applied to obtain most (if not all) of the results concerning the quotient spaces of the various function algebras mentioned in the introduction; it is of course necessary at each time to construct the required interpolation sets.

Our final main results in this section and in the rest of the paper concern C^*-algebras of bounded functions on a locally compact group, but definitions and properties shall also be proved for a general Hausdorff topological group whenever this makes sense.

Definition 2.1. Let G be a topological group and $\mathcal{A}(G) \subseteq \ell_\infty(G)$. A subset $T \subseteq G$ is said to be

(i) an $\mathcal{A}(G)$-*interpolation set* if every bounded function $f \colon T \to \mathbb{C}$ can be extended to a function $\overline{f} \colon G \to \mathbb{C}$ such that $\overline{f} \in \mathcal{A}(G)$,

(ii) an *approximable* $\mathcal{A}(G)$-*interpolation set* if it is an $\mathcal{A}(G)$-interpolation set and for every neighbourhood U of e, there is an open neighbourhood V of e with $\overline{V} \subseteq \operatorname{int}(U)$ such that, for each $T_1 \subseteq T$ there is $h \in \mathcal{A}(G)$ with $h(VT_1) = \{1\}$ and $h(G \setminus (UT_1)) = \{0\}$.

Remark 2.2. $\mathcal{A}(G)$-interpolation sets for some concrete algebras $\mathcal{A}(G) \subseteq \ell_\infty(G)$ have been a frequent object of study; see [29] and [24] for more details and references. See also [31] for the most recent account on the subject.

Approximable interpolation sets appeared in the early 1970s as a crucial step in Drury's proof of the union theorem of Sidon sets; see [18]. Other well-known interpolation sets are also approximable as for instance *translation-finite sets* considered by Ruppert in [52] (and called R_W-sets by Chou in [14]) that turn out to be the approximable $\mathcal{WAP}(G)$-interpolation sets of discrete groups; see [24] for more in this respect.

When G is discrete, the definition of approximable $\mathcal{A}(G)$-interpolation set is much simpler. In that case $T \subset G$ is an approximable $\mathcal{A}(G)$-interpolation set if and only if T is an $\mathcal{A}(G)$-interpolation such that $1_T \in \mathcal{A}(G)$, or equivalently, if every bounded function supported on T is in $\mathcal{A}(G)$.

It should however be recalled that approximable $\mathcal{A}(G)$-interpolation sets do not make sense for every C^*-subalgebra $\mathcal{A}(G)$ of $\ell_\infty(G)$. For example, no subset in a non-compact locally compact group can be an approximable $\mathcal{AP}(G)$-interpolation set; see [24, Section 3 and Corollary 4.24].

2.1. **The quotients.** The following lemma contains some elementary consequences of the definitions of interpolation and approximable interpolation sets. The identification of $\overline{T}^{\mathcal{A}}$ with the Stone-Čech compactification of T (with the discrete topology) allows us to use the powerful property of extreme disconnectedness of the latter compactification. As the reader will quickly notice this is the key in the arguments leading to the main results in this section. The main results start with a generalization of a theorem proved by Chou [13] for $B(G)$ (Theorem 2.5) to arbitrary C^*-subalgebras of $\ell_\infty(G)$. Along with some rather technical lemmas, this provides us with the conditions stated in Theorem 2.11 and Corollary 2.12 under which the quotient $\mathcal{A}_2(G)/\mathcal{A}_1(G)$ ($\mathcal{A}_1(G) \subset \mathcal{A}_2(G)$ being admissible C^*-subalgebras of $\mathcal{CB}(G)$) contains a linear isometric copy of $\ell_\infty(\kappa)$ for some cardinal κ.

Lemma 2.3. *Let G be a topological group. Let $\mathcal{A}(G)$ be a C^*-subalgebra of $\ell_\infty(G)$ with $1 \in \mathcal{A}(G)$ and $T \subseteq G$.*

(i) *T is an $\mathcal{A}(G)$-interpolation set if and only if $\epsilon_{\mathcal{A}}$ is injective on T and the canonical embedding of T extends to a homeomorphism between $\overline{T}^{\mathcal{A}}$ and βT_d, the Stone-Čech-compactification of T equipped with the discrete topology, that leaves the points of T fixed.*

(ii) *T is an $\mathcal{A}(G)$-interpolation set if and only if for every pair of subsets $T_1, T_2 \subset T$, $T_1 \cap T_2 = \emptyset$ implies $\overline{T_1}^{\mathcal{A}} \cap \overline{T_2}^{\mathcal{A}} = \emptyset$.*

(iii) *If T is an $\mathcal{A}(G)$-interpolation set and $f\colon T \to \mathbb{C}$ is a bounded function, then f has an extension $\overline{f} \in \mathcal{A}(G)$ with $\|\overline{f}\|_\infty = \|f\|_T$.*

(iv) *If T is an approximable $\mathcal{A}(G)$-interpolation set, then for every bounded function $h\colon T \to \mathbb{C}$ and every neighbourhood U of the identity, there is $f \in \mathcal{A}(G)$ such that*

$$f\big|_T = h, \quad f(G \setminus UT) = \{0\} \quad \text{and} \quad \|f\|_\infty = \|h\|_T.$$

Proof. First observe that $\epsilon_{\mathcal{A}}$ is injective on every $\mathcal{A}(G)$-interpolation set T: if $t_1 \neq t_2 \in T$, there is $f \in \ell_\infty(T)$ with $f(t_1) \neq f(t_2)$. Take $\bar{f} \in \mathcal{A}(G)$ extending f. By (1) $\bar{f} = \bar{f}^{\mathcal{A}} \circ \epsilon_{\mathcal{A}}$, hence $\epsilon_A(t_1) \neq \epsilon_A(t_2)$.

Assertion (i) follows then from the universal property defining the Stone-Čech compactification of a discrete space. In fact, the restriction of the evaluation map $\epsilon_{\mathcal{A}}$ to T gives a homeomorphism of the discrete set T_d onto its image in $G^{\mathcal{A}}$. So $\overline{T}^{\mathcal{A}}$ is a (topological) compactification of T_d, and we may apply [21, Corollary 3.6.3].

Assertion (ii) also follows directly from a well-known characterization of the Stone-Čech compactification of a discrete space; see for instance [21, Corollary 3.6.2].

To prove (iii), let $f\colon T \to \mathbb{C}$ with $\|f\|_T = M$ be given. If B_M is the closed disc of radius M centred at 0 (in $\mathbb{C}$), we can use (i) and the universal property of βT_d to find a continuous function $f^\beta\colon \overline{T}^{\mathcal{A}} \to B_M$ with $f^\beta\big|_T = f$. Then, by Tietze's extension theorem, f^β can be extended to a continuous function $f^{\mathcal{A}}\colon G^{\mathcal{A}} \to B_M$, the restriction $f^{\mathcal{A}}\big|_G$ is then the desired extension.

To prove (iv), let T be an approximable $\mathcal{A}(G)$-interpolation set. First, we find, using (iii), $f_1 \in \mathcal{A}$ with $f_1\big|_T = h$ and $\|f_1\|_\infty = \|h\|_T$. The definition of approximable $\mathcal{A}(G)$-interpolation sets provides a neighbourhood V with $\overline{V} \subseteq \operatorname{int}(U)$ and $f_2 \in \mathcal{A}$ such that

$$f_2(VT) = \{1\} \quad \text{and} \quad f_2(G \setminus UT) = \{0\}.$$

Using [21, 3.2.20], we can assume (taking the minimum of f_2 and the function that is constant and equal to 1) that $\|f_2\|_\infty = 1$. The product $f_1 \cdot f_2$ then coincides with h on T and vanishes off UT. □

Remark 2.4. Note that if in the lemma above $\mathcal{A}(G) \subseteq \mathcal{CB}(G)$, then T is necessarily discrete since *every* bounded function on T must be continuous.

Observe as well that the sole existence of an infinite $\mathcal{A}(G)$-interpolation set T in G, implies that $G^{\mathcal{A}}$ contains a copy of βT_d, where T_d is the discrete set T. The compactification $G^{\mathcal{A}}$ is therefore large and topologically involved.

The following theorem, due to Chou [13], has its roots in a result of Ramirez (see Theorem 2.3 of [19]) in the Abelian setting. This theorem is used by Chou, loc. cit., to find an isometric copy of ℓ_∞ inside $\mathcal{WAP}(G)/\mathcal{B}(G)$ for a discrete group G. This was originally the departing point of our paper.

Theorem 2.5 (Chou, [13, Lemma 3.11])**.** *Let G be a discrete group. A subset $T \subseteq G$ fails to be a $B(G)$-interpolation set if and only if there is a bounded function $f \in \ell_\infty(G)$, with $\|f\|_\infty = 1$ such that*

$$\|\phi - f\|_T \geq 1 \quad \text{for all} \quad \phi \in B(G).$$

Remark 2.6. It is an immediate consequence of the previous theorem that $\mathcal{B}(G)$-interpolation sets are also $B(G)$-interpolation sets (i.e., Sidon sets). We do not know whether Theorem 2.5 remains valid for all locally compact groups.

The result in Theorem 2.5 is more natural when the function algebra is a C^*-subalgebra. It is not surprising therefore that it holds for *any* C^*-subalgebra. The next lemma proves even more.

Lemma 2.7. *Let G be a topological group, $\mathcal{A}_1(G) \subseteq \mathcal{A}_2(G) \subseteq \ell_\infty(G)$ be two C^*-subalgebras with $1 \in \mathcal{A}_1(G)$, and let $(T_\eta)_{\eta<\kappa}$ be a family of disjoint subsets of G such that*

(i) *each T_η fails to be an $\mathcal{A}_1(G)$-interpolation set,*
(ii) *$T = \bigcup_{\eta<\kappa} T_\eta$ is an approximable $\mathcal{A}_2(G)$-interpolation set.*

Then for each open neighbourhood U of e, there is a function $f \in \mathcal{A}_2(G)$ with $\|f\|_\infty = 1$ such that

$$f(G \setminus UT) = \{0\} \quad \text{and} \quad \|f - \phi\|_{T_\eta} \geq 1 \ \text{for every } \eta < \kappa \text{ and every } \phi \in \mathcal{A}_1(G).$$

Proof. Let $T = \bigcup_{\eta<\kappa} T_\eta$ be an approximable $\mathcal{A}_2(G)$-interpolation set as stated in the lemma. Let U be an open neighbourhood of e.

To avoid cumbersomeness, we abuse our notation and use the same letters to denote subsets of T and their images in $G^{\mathcal{A}_1}$.

Then, by Statement (ii) of Lemma 2.3, each T_η must contain two disjoint subsets $T_{1,\eta}, T_{2,\eta}$ such that $\overline{T_{1,\eta}}^{\mathcal{A}_1} \cap \overline{T_{2,\eta}}^{\mathcal{A}_1} \neq \emptyset$. Define for each $\eta < \kappa$, a function $h_\eta \colon G \to [-1,1]$ with $h_\eta(T \setminus T_\eta) = \{0\}$,

$$h_\eta(T_{1,\eta}) = \{1\} \quad \text{and} \quad h_\eta(T_{2,\eta}) = \{-1\}.$$

Then consider the function $h \colon G \to [-1,1]$ given by

$$h(t) = h_\eta(t) \ \text{ if } t \in T_\eta \text{ for some } \eta < \kappa \ \text{ and } h(G \setminus T) = \{0\}.$$

By Statement (iv) of Lemma 2.3, there is a a function $f \in \mathcal{A}_2(G)$ such that

$$f(G \setminus UT) = 0, \ \ f\big|_T = h \ \text{ and } \|f\|_\infty = \|h\|_T = 1.$$

Now let ϕ be any function in $\mathcal{A}_1(G)$, and take $\varepsilon > 0$. Given $\eta < \kappa$, we are going to prove that $\|f - \phi\|_{T_\eta} \geq 1 - \varepsilon$.

Take $p_\eta \in \overline{T_{1,\eta}}^{\mathcal{A}_1} \cap \overline{T_{2,\eta}}^{\mathcal{A}_1}$ and pick $t_{1,\eta} \in T_{1,\eta}$ and $t_{2,\eta} \in T_{2,\eta}$ with

$$|\phi(t_{1,\eta}) - \phi^{\mathcal{A}_1}(p_\eta)| < \varepsilon \quad \text{and} \quad |\phi(t_{2,\eta}) - \phi^{\mathcal{A}_1}(p_\eta)| < \varepsilon,$$

where $\phi^{\mathcal{A}_1}$ denotes the extension of ϕ to $G^{\mathcal{A}_1}$. Then

$$\begin{aligned}(2) \qquad 2 &= |h_\eta(t_{1,\eta}) - h_\eta(t_{2,\eta})| = |h(t_{1,\eta}) - h(t_{2,\eta})| = |f(t_{1,\eta}) - f(t_{2,\eta})| \\ &\leq |f(t_{1,\eta}) - \phi(t_{1,\eta})| \\ &\quad + |\phi(t_{1,\eta}) - \phi^{\mathcal{A}_1}(p_\eta)| + |\phi^{\mathcal{A}_1}(p_\eta) - \phi(t_{2,\eta})| + |\phi(t_{2,\eta}) - f(t_{2,\eta})|.\end{aligned}$$

It follows that either $|f(t_{1,\eta}) - \phi(t_{1,\eta})| \geq 1 - \varepsilon$ or $|f(t_{2,\eta}) - \phi(t_{2,\eta})| \geq 1 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we find that $\|f - \phi\|_{T_\eta} \geq 1$. Since $\|f\|_\infty = 1$ and $f(G \setminus UT) = \{0\}$, we see that f is the required function. □

Lemma 2.8. *Let G be a locally compact group, $\mathcal{A}(G)$ be a unital left invariant C^*-subalgebra of $l_\infty(G)$, U be a neighbourhood of e, and $T \subseteq G$ be an approximable $\mathcal{A}(G)$-interpolation set, which may be partitioned as $T = \bigcup_{\eta<\kappa} T_\eta$ with $UT_\eta \cap UT_{\eta'} = \emptyset$ whenever $\eta \neq \eta'$. Then there is a neighbourhood V of e such that for each $f \in \mathcal{A}(G)$ with $f(G \setminus VT) = \{0\}$ and each $\mathbf{c} = (c_\eta)_{\eta<\kappa} \in \ell_\infty(\kappa)$, the function given by*

$$g(G \setminus VT) = \{0\} \text{ and } g\big|_{VT_\eta} = c(\eta) f\big|_{VT_\eta} \ \text{ for each } \eta < \kappa,$$

is in $\mathcal{A}(G)$.

Proof. Consider the neighbourhood V provided by the definition of approximable $\mathcal{A}(G)$-interpolation set for the neighbourhood U.

For a non-empty subset M of κ, put $T_M = \bigcup_{\eta \in M} T_\eta$. For disjoint subsets M_1 and M_2 of κ, the assumption of the lemma implies $UT_{M_1} \subseteq G \setminus UT_{M_2}$ and so, by Definition 2.1(ii), $\overline{VT_{M_1}}^{\mathcal{A}} \cap \overline{UT_{M_2}}^{\mathcal{A}} = \emptyset$, and in particular, $\overline{VT_{M_1}}^{\mathcal{A}} \cap \overline{VT_{M_2}}^{\mathcal{A}} = \emptyset$. For arbitrary $M_1, M_2 \subset \kappa$, it follows that

$$\overline{VT_{M_1 \cap M_2}}^{\mathcal{A}} = \overline{VT_{M_1} \cap VT_{M_2}}^{\mathcal{A}} = \overline{VT_{M_1}}^{\mathcal{A}} \cap \overline{VT_{M_2}}^{\mathcal{A}}.$$

This implies that, for $x \in \overline{VT}^{\mathcal{A}}$, $\mathcal{U}_x = \{M \subseteq \kappa : x \in \overline{VT_M}^{\mathcal{A}}\}$ is a filter on κ. It is actually an ultrafilter on κ, since for any $M \subseteq \kappa$, $x \notin \overline{VT_M}^{\mathcal{A}}$ implies that $x \in \overline{VT_{\kappa\setminus M}}^{\mathcal{A}}$.

Now it is clear that the mapping

$$x \mapsto \mathcal{U}_x : \overline{VT}^{\mathcal{A}} \to \beta\kappa$$

is continuous since $x \in \overline{VT_M}^{\mathcal{A}}$ if and only if $M \in \mathcal{U}_x$ (and, thus, the inverse image of every closed set is closed).

Let c^β and $f^{\mathcal{A}}$ denote the respective continuous extensions of c and f to $\beta\kappa$ and $G^{\mathcal{A}}$. Then the function g^* given by $g^*(x) = c^\beta(\mathcal{U}_x) f^{\mathcal{A}}(x)$ is a continuous extension of g to $\overline{VT}^{\mathcal{A}}$, and so g extends continuously to $G^{\mathcal{A}}$, proving that $g \in \mathcal{A}(G)$. $\square$

For algebras $\mathcal{A}(G)$ containing $\mathcal{LUC}(G)$ it is easy to see that a weaker assumption on T is enough.

Lemma 2.9. *Let G be a locally compact group, $\mathcal{A}(G)$ be a unital left invariant C^*-subalgebra of $l_\infty(G)$ with $\mathcal{LUC}(G) \subseteq \mathcal{A}(G)$, U be a neighbourhood of e, and let $T \subseteq G$ be a set which may be partitioned as $T = \bigcup_{\eta<\kappa} T_\eta$ with $UT_\eta \cap UT_{\eta'} = \emptyset$ whenever $\eta \neq \eta'$. Then there is a neighbourhood V of e such that for each $f \in \mathcal{A}(G)$ with $f(G \setminus VT) = \{0\}$ and each $\mathbf{c} = (c_\eta)_{\eta<\kappa} \in \ell_\infty(\kappa)$, the function given by*

$$g(G \setminus VT) = \{0\} \text{ and } g\big|_{VT_\eta} = c(\eta) f\big|_{VT_\eta} \text{ for each } \eta < \kappa,$$

is in $\mathcal{A}(G)$.

Proof. Let V be any symmetric neighbourhood of the identity with $V^2 \subseteq U$. Then the assumption $UT_\eta \cap UT_{\eta'} = \emptyset$, for $\eta \neq \eta'$, implies that $V^2 T_{M_1} \cap V^2 T_{M_2} = \emptyset$ for every pair of disjoint subsets M_1 and M_2 of κ. By Katětov's Theorem (see [40, 41] or [30]), there is a function $f \in \mathcal{LUC}(G)$ such that $h(VT_{M_1}) = 0$ and $h(VT_{M_2}) = 1$, and so $\overline{VT_{M_1}}^{\mathcal{A}} \cap \overline{VT_{M_2}}^{\mathcal{A}} = \emptyset$. The proof of Lemma 2.8 can then be repeated *verbatim* in this case. $\square$

Remarks 2.10. (i) We would like to thank the referee for proposing the present proof of Lemma 2.9 which is much shorter and easier to read than our original one.

(ii) The reader might have noticed that for the proof of Lemma 2.8 the full force of the approximable $\mathcal{A}(G)$-interpolation property of T is not needed. First, the fact that T is an $\mathcal{A}(G)$-interpolation set is not used and, secondly, the sets T_η need not be approximable interpolation sets, it is enough to separate by functions of the algebra each pair of *blocks* T_η and $T_{\eta'}$, as emphasized in the proof of Lemma 2.9.

(iii) A known theorem due to Veech asserts that the left action of a locally compact group G on $G^{\mathcal{LUC}}$ is free, i.e., $sx \neq x$ for every $x \in G^{\mathcal{LUC}}$ and $s \in G$, $s \neq e$; see [54], or [48] for a shorter proof. The original proof of the second statement of the previous lemma, which was rather involved, revealed that Veech's property in fact holds in $G^{\mathcal{A}}$ at any point in the closure of the approximable $\mathcal{A}(G)$-interpolation sets with $\mathcal{A} \subset \mathcal{LUC}(G)$. That is, if T is any such a set, $x \in \overline{T}^{\mathcal{A}}$ and $g \neq e$ in G, then $gx \neq x$ and $xg \neq x$ in $G^{\mathcal{A}}$. This property was proved for $G^{\mathcal{WAP}}$ in [4] and [23] using t-sets. These sets are approximable $\mathcal{WAP}(G)$-interpolation sets and have also been used to study the algebra in the Stone-Čech compactification βS when S is a discrete semigroup, see [22] and [38], and in $G^{\mathcal{LUC}}$, see [27]. These matters are dealt with in our recent work [25].

Theorem 2.11. *Let G be a locally compact group and let $\mathcal{A}_1(G) \subset \mathcal{A}_2(G) \subseteq \mathcal{LUC}(G)$ be two unital C^*-subalgebras of $\ell_\infty(G)$ with $\mathcal{A}_2(G)$ left invariant. Suppose that G contains a neighbourhood U of the identity and a family of sets $\{T_\eta \colon \eta < \kappa\}$ such that*

(i) *$UT_\eta \cap UT_{\eta'} = \emptyset$ for every $\eta \neq \eta' < \kappa$,*
(ii) *T_η fails to be an $\mathcal{A}_1(G)$-interpolation set for every $\eta < \kappa$, and*
(iii) *$T = \bigcup_{\eta<\kappa} T_\eta$ is an approximable $\mathcal{A}_2(G)$-interpolation set.*

Then there is a linear isometry $\Psi \colon \ell_\infty(\kappa) \to \mathcal{A}_2(G)/\mathcal{A}_1(G)$.

Proof. Let V be the neighbourhood of the identity provided by Lemma 2.8.

Since $T = \bigcup_{\eta<\kappa} T_\eta$ is an approximable $\mathcal{A}_2(G)$-interpolation set and each T_η fails to be an $\mathcal{A}_1(G)$-interpolation set, we take from Lemma 2.7 a function $f \in \mathcal{A}_2(G)$ with $\|f\|_\infty = 1$ such that

$$f(G \setminus VT) = \{0\} \quad \text{and} \quad \|f - \phi\|_{T_\eta} \geq 1 \quad \text{for all} \quad \phi \in \mathcal{A}_1(G) \quad \text{and} \quad \eta < \kappa.$$

For each $\mathbf{c} = (c_\eta)_{\eta<\kappa} \in \ell_\infty(\kappa)$, we define a function $f_\mathbf{c} : G \to \mathbb{C}$ with $f_\mathbf{c}(G \setminus VT) = \{0\}$ and

$$f_\mathbf{c}(vt) = c_\eta f(vt) \quad \text{if} \quad t \in T_\eta \quad \text{and} \quad \eta < \kappa,$$

i.e., with the notation of Lemma 2.8, $f_\mathbf{c}\big|_{VT_\eta} = \mathbf{c}(\eta) f\big|_{VT_\eta}$.

Then $f_\mathbf{c} \in \mathcal{A}_2(G)$ by Lemma 2.8. Obviously, the map $\Psi \colon \ell_\infty(\kappa) \to \mathcal{A}_2(G)/\mathcal{A}_1(G)$ given by

$$\Psi(\mathbf{c}) = f_\mathbf{c} + \mathcal{A}_1(G) \quad \text{for every} \quad \mathbf{c} \in \ell_\infty(\kappa)$$

is linear. We next check that it is isometric.

The same argument of [13, Theorem 3.12] shows now that, for every $\eta < \kappa$,

$$\begin{aligned} \|\Psi(\mathbf{c})\|_{\mathcal{A}_2(G)/\mathcal{A}_1(G)} &= \inf\{\|f_\mathbf{c} - \phi\|_\infty : \phi \in \mathcal{A}_1(G)\} \\ &\geq \inf\{\|f_\mathbf{c} - \phi\|_{T_\eta} : \phi \in \mathcal{A}_1(G)\} \\ &= \inf\{\|c_\eta f - \phi\|_{T_\eta} : \phi \in \mathcal{A}_1(G)\} \\ &= |c_\eta| \inf\{\|f - \phi\|_{T_\eta} : \phi \in \mathcal{A}_1(G)\} \\ &\geq |c_\eta|, \end{aligned}$$

where the last inequality follows from the choice of f. Since, obviously,

$$\|\Psi(\mathbf{c})\|_{\mathcal{A}_2(G)/\mathcal{A}_1(G)} \leq \|f_\mathbf{c}\|_\infty = \|\mathbf{c}\|, \quad \text{for every} \quad \mathbf{c} = (c_\eta)_{\eta<\kappa} \in \ell_\infty(\kappa),$$

we see that Ψ is the required isometry. □

Corollary 2.12. *If in the above theorem $\mathcal{A}_2(G) = \mathcal{CB}(G)$, then the quotient $\mathcal{CB}(G)/\mathcal{A}_1(G)$ contains a linearly isometric copy of $\ell_\infty(\kappa)$.*

Proof. The proof of Theorem 2.11 remains valid in this case applying Lemma 2.9 instead of Lemma 2.8. □

Remark 2.13. Two C^*-subalgebras of $\ell_\infty(G)$ may be different, and yet produce a small quotient (i.e., separable), for example if G is a minimally weakly almost periodic group ([14], [51], [55]), then $\mathcal{WAP}(G)/\mathcal{AP}(G) = C_0(G)$. If $G = SL(2,\mathbb{R})$, then $\mathcal{WAP}(G) = C_0(G) \oplus \mathbb{C}1$, and so $\mathcal{WAP}(G)/C_0(G) = \mathbb{C}$. In the theorem and corollary above, we have just met conditions under which this is not so.

Corollary 2.14. *Under the hypotheses of Theorem* 2.11 *or Corollary* 2.12*, the quotient space $\mathcal{A}_2(G)/\mathcal{A}_1(G)$ is non-separable.*

3. Interpolation sets

The definitions in this section gather the topological group-theoretic properties that will correspond to the interpolation sets needed in the three sections that follow. Once these interpolation sets are at hand, an application of Theorem 2.11 and Corollary 2.12 will lead immediately to the desired conclusion on the quotients.

Definition 3.1. Let G be a non-compact topological group. We say that a subset T of G is

(i) *right translation-finite* if every infinite subset $L \subseteq G$ contains a finite subset F such that $\bigcap\{b^{-1}T \colon b \in F\}$ is finite; *left translation-finite* if every infinite subset $L \subseteq G$ contains a finite subset F such that $\bigcap\{Tb^{-1} \colon b \in F\}$ is finite; and *translation-finite* when it is both right and left translation-finite;

(ii) *right translation-compact* if every non-relatively compact subset $L \subseteq G$ contains a finite subset F such that $\bigcap\{b^{-1}T \colon b \in F\}$ is relatively compact; *left translation-compact* if every non-relatively compact subset $L \subseteq G$ contains a finite subset F such that $\bigcap\{Tb^{-1} \colon b \in F\}$ is relatively compact; and *translation-compact* when it is both left and right translation-compact;

(iii) a *right t-set* (*left t-set*) if there exists a compact subset K of G containing e such that $gT \cap T$ (respectively, $Tg \cap T$) is relatively compact for every $g \notin K$; and a *t-set* when it is both a right and a left t-set.

In addition to these sets, we also need to recall the known notions related to uniform discreteness. Along with translation-finite and translation-compact sets, these sets are essential for the rest of the paper.

Definition 3.2. Let G be a topological group, T be a subset of G and U be a neighbourhood of e. We say that T is *right U-uniformly discrete* if

$$Us \cap Us' = \emptyset \quad \text{for every} \quad s \neq s' \in T.$$

The set T being *left U-uniformly discrete* is defined analogously. We say that T is *right uniformly discrete* (resp. *left uniformly discrete*) when it is right U-uniformly discrete (resp. left U-uniformly discrete) for some neighbourhood U of e. If T is both left and right uniformly discrete, we say that T is *uniformly discrete.*

We also need to establish the range of locally compact groups to which our methods apply in the next two sections; these are those locally compact groups for which the existence of a good supply of $\mathcal{WAP}$-functions is guaranteed.

Recall that a locally compact group G is an *$IN-$group* if it has a compact invariant neighbourhood of the identity. We recall also from [10], that a locally compact group G is an *E-group* if it contains a non-relatively compact set X such that for each neighbourhood U of e, the set

$$\bigcap\{x^{-1}Ux : x \in X \cup X^{-1}\}$$

is again a neighbourhood of e. The set X is called an *E-set.* This is a large class of locally compact groups. This includes of course all non-compact $SIN-$groups, the groups with a non-compact centre such as the matrix group $GL(n, \mathbb{R})$, and the direct product of any E-group with any locally compact group.

A detailed study of approximable $\mathcal{LUC}(G)$- and $\mathcal{WAP}(G)$-interpolation sets, with some precise characterizations, is carried out in the recent paper [24]. We summarize in Lemma 3.3 the results that will be needed in the present paper.

Lemma 3.3 ([24, Lemma 4.8 and Proposition 3.3 (iii)]). *Let G be a topological group and let $T \subset G$.*

(i) *If the underlying topological space of G is normal, then all discrete closed subsets of G are approximable $\mathcal{CB}(G)$-interpolation sets.*

(ii) *If T is right (resp. left) uniformly discrete, then T is an approximable $\mathcal{LUC}(G)$-interpolation set (resp. $\mathcal{RUC}(G)$-interpolation set).*

(iii) *If G is assumed to be metrizable, then every $\mathcal{LUC}(G)$-interpolation set (resp. $\mathcal{RUC}(G)$-interpolation set) is right (left) uniformly discrete.*

(iv) *If G is an E-group and T is an E-set in G which is right (or left) uniformly discrete with respect to U^2 for some neighbourhood U of the identity such that UT is translation-compact, then T is an approximable $\mathcal{WAP}_0(G)$-interpolation set.*

(v) *If G is a metrizable E-group, an E-set $T \subset G$ is a $\mathcal{WAP}(G)$-interpolation set if and only if UT is translation-compact for some compact neighbourhood U of the identity such that T is right (or left) uniformly discrete with respect to U^2.*

The following lemma will be needed later on in Section 5.

Lemma 3.4. *Let G be a locally compact group, let H be a closed subgroup of G and let $T \subset H$.*

(i) *If UT is a right t-set in H for some compact neighbourhood U of the identity e in H, then VT is a right t-set in G for every compact neighbourhood V of e in G.*

(ii) *If in addition T is central, then the left analogue of Statement (i) holds also.*

Proof. Let U be a compact neighbourhood of e in H, and suppose that UT is a right t-set in H. By definition there is a compact subset $K \subseteq H$ such that $gUT \cap UT$ is relatively compact whenever $g \notin K$.

Let V be a compact neighbourhood of e and let $K' = VKV^{-1}$.

Take $g \in G$ but $g \notin K'$. Suppose $gVT \cap VT$ is not relatively compact. There is then a net (g_α) in $gVT \cap VT$ with no convergent subnet. For each α, there are $v_\alpha, w_\alpha \in V$ and $t_\alpha, s_\alpha \in T$ such that $g_\alpha = v_\alpha t_\alpha = g w_\alpha s_\alpha$, and so

$$t_\alpha s_\alpha^{-1} = v_\alpha^{-1} g w_\alpha \quad \text{for every} \quad \alpha.$$

We can assume, by considering subnets if necessary, that (v_α) and (w_α) converge to $v, w \in V$, respectively. Therefore $(t_\alpha s_\alpha^{-1})$ is a net in H which converges to $h = v^{-1} g w$. Since H is closed, $h \in H$. Thus, the net $(t_\alpha s_\alpha^{-1})$ is eventually in hU. This means that the net (t_α) belongs to $hUT \cap UT$. Since (g_α) has no convergent subnet but (v_α) has, we know that (t_α) cannot have convergent subnets either. We deduce therefore that $hUT \cap UT$ is not relatively compact. That however would imply that $h \in K$, and so that $g \in K'$, a contradiction. Hence, VT is a right t-set in G.

To prove the analogous statement for left t-sets, we suppose in addition that T is central and put again $K' = VKV^{-1}$. Using the fact that T is central, the same argument leads to

$$s_\alpha^{-1} t_\alpha = w_\alpha g v_\alpha^{-1} \quad \text{for every} \quad \alpha.$$

By taking subnets, we see that (t_α) is eventually in $TUh \cap TU = UTh \cap UT$ with $h \in VgV^{-1}$. Thus, h must be in K, and so g is in K'. ∎

Remark 3.5. Notice that the above argument is also meaningful when $G = H$. In that case it shows the following: if $T \subset G$ is such that UT is a right t-set for *some* compact neighbourhood U of e in G, then VT is a right t-set for *every* compact neighbourhood V of e in G. We thank the referee for pointing out to us that the proof holds for any compact neighbourhood of e.

To state a well-known necessary condition for a subset $T \subseteq G$ to be a $\mathcal{B}(G)$-interpolation set we need the concept of large squares that we recall from [13, Definition 3.3]:

A finite subset F of G is an *n-square* if $F = AB$ where $|A| = |B| = n$ and $|F| = n^2$. A subset of a group is then said to *contain large squares* if it contains an n-square for every $n \in \mathbb{N}$.

Large squares are incompatible with Sidon sets, as proved in [13, Proposition 3.4]. We restate this theorem here, stressing $\mathcal{B}(G)$.

Theorem 3.6 (Chou, [13])**.** *Let G be a topological group. A $\mathcal{B}(G)$-interpolation set cannot contain large squares.*

Proof. Let G_d be the group G with the discrete topology. If T is a $\mathcal{B}(G)$-interpolation set, then T is also a $\mathcal{B}(G_d)$-interpolation set. By Remark 2.6, T is then a $B(G_d)$-interpolation set and, by [13, Proposition 3.4], T cannot contain large squares. □

4. The quotients of $\mathcal{WAP}_0(G)$ by $C_0(G)$ and $\mathcal{WAP}(G)$ by $\mathcal{AP}(G) \oplus C_0(G)$

In [10], Chou considered E-groups and proved that the quotient space $\mathcal{WAP}_0(G)/C_0(G)$ contains a linear isometric copy of ℓ_∞. In this section, we strengthen this result and prove that if G is an E-group, then there is a linear isometric copy of $\ell_\infty(\kappa)$ in the quotient $\mathcal{WAP}_0(G)/C_0(G)$ where κ is the compact covering number of an E-set contained in G. In particular, $\kappa = \kappa(G)$ when G is an SIN-group.

Our method applies further to show that the quotient $\mathcal{WAP}(G)/(\mathcal{AP}(G) \oplus C_0(G))$ is non-separable.

Theorem 4.1. *Let G be a non-compact locally compact E-group having an E-set X with a compact covering number κ. Then the quotient space $\mathcal{WAP}_0(G)/C_0(G)$ contains a linear isometric copy of $\ell_\infty(\kappa)$.*

Proof. Let V be a fixed compact symmetric neighbourhood of e. Then we consider a set $T \subset X$ as that constructed in Section 2 of [23]. This set has the following properties:

(i) $\kappa(T) = |T| = \kappa(X)$.
(ii) T is right V^2-uniformly discrete.
(iii) VT is a t-set.

For completeness, we recall from [23] the construction of the set T. We may assume that $e \in X$ and start with $x_0 = e$.. Suppose that the elements x_β have been selected for all $\beta < \alpha$ with $\alpha < \kappa$. Set

$$X_\alpha = \bigcup_{\beta_1,\beta_2,\beta_3<\alpha} V^2 x_{\beta_1}^{\epsilon_1} x_{\beta_2}^{\epsilon_2} V^2 x_{\beta_3}^{\epsilon_3} V,$$

where each $\epsilon_i = \pm 1$. Since $\kappa(X_\alpha) < \kappa$, we pick x_α in $X \setminus X_\alpha$ for our set T. In this way, we form a set $T = \{x_\alpha : \alpha < \kappa\}$.

We obtain from Lemma 3.3 that T is an approximable $\mathcal{WAP}_0(G)$-interpolation set. Since every infinite subset of T is uniformly discrete and $C_0(G)$-interpolation sets must be relatively compact, and so finite (see the proof of Proposition 3.3 of [24]) any decomposition $T = \bigcup_{\eta<\kappa} T_\eta$ as a disjoint union of κ-many infinite subsets leaves us in position to apply Theorem 2.11 and finish the proof. □

We deduce first that the quotient $\mathcal{WAP}(G)/\left(\mathcal{AP}(G) \oplus C_0(G)\right)$ contains an isomorphic copy of $\ell_\infty(\kappa)$.

Corollary 4.2. *Let G be a non-compact locally compact E-group having an E-set X with a compact covering number κ. The quotient space $\mathcal{WAP}(G)/\left(\mathcal{AP}(G) \oplus C_0(G)\right)$ then contains an* isomorphic *copy of $\ell_\infty(\kappa)$.*

Proof. We only have to recall that $\mathcal{WAP}(G) = \mathcal{AP}(G) \oplus \mathcal{WAP}_0(G)$, [45]. Therefore $\mathcal{WAP}(G)/(\mathcal{AP}(G) \oplus C_0(G))$ is isomorphic to $\mathcal{WAP}_0(G)/C_0(G)$, and then apply Theorem 4.1. □

If we want to use our Theorem 2.11 to obtain a linear *isometric* copy of $\ell_\infty(\kappa)$ in the quotient $\mathcal{WAP}(G)/\mathcal{AP}(G)$ we need first an approximable $\mathcal{WAP}(G)$-interpolation set which is not an $\mathcal{AP}(G)$-interpolation set. If G, for instance, is discrete this means we need a translation-finite set that is not an I_0-set. Such sets can be easily found in $\mathbb{Z}$, the additive group of integers: $T = \{3^n + n \colon n \in \mathbb{N}\} \cup \{3^n \colon n \in \mathbb{N}\}$ is such an example; see [31, Example 1.5.2] for a (simple) proof. For arbitrary discrete groups, an example as simple as that has escaped us but see the recent paper [36] by Hare and Ramsey for a generic construction in discrete Abelian groups. A considerably more complicated construction can be used to obtain an approximable $\mathcal{WAP}(G)$-interpolation set that is not a $\mathcal{B}(G)$-interpolation set, *a fortiori* not an $\mathcal{AP}(G)$-interpolation set when G is an *IN*-group, a nilpotent group or a group with large enough centre; see Section 5 (note that the set T above *is* a Sidon set, i.e., a $B(G)$-interpolation set).

Corollary 4.3. *Let G be a non-compact, locally compact E-group having an E-set X with a compact covering number κ. Then the Banach space $\mathcal{WAP}(G)$ contains a linear isometric copy L of $\ell_\infty(\kappa)$ such that*

$$\frac{\|f\|}{2} \leq \|f + \mathcal{AP}(G) \oplus C_0(G)\|_q \leq \|f\| \quad \textit{for every} \quad f \in L.$$

In particular, the quotient space $\mathcal{WAP}(G)/(\mathcal{AP}(G) \oplus C_0(G))$ is non-separable.

Proof. Since $\mathcal{WAP}_0(G)$ is an ideal in the C^*-algebra $\mathcal{WAP}(G)$, we see that every element in $G^{\mathcal{AP}}$ can be extended trivially on $\mathcal{WAP}_0(G)$. It follows that $\|f+g\| \geq \|g\|$ for every $f \in \mathcal{WAP}_0(G)$ and $g \in \mathcal{AP}(G)$. Therefore,

$$\|f + g\| \geq \frac{1}{2}\|f\| \quad \text{for every} \quad f \in \mathcal{WAP}_0(G) \quad \text{and} \quad g \in \mathcal{AP}(G)$$

(use the previous inequality when $\frac{1}{2}\|f\| \leq \|g\|$, and the triangle inequality when $\frac{1}{2}\|f\| \geq \|g\|$). This gives for the quotient norms

$$\|f + \mathcal{AP}(G) \oplus C_0(G)\|_q \geq \frac{1}{2}\|f + C_0(G)\|_q \quad \text{for every} \quad f \in \mathcal{WAP}_0(G).$$

Now Theorem 4.1 completes the proof since it gives an isometric copy L of $\ell_\infty(\kappa)$ in $\mathcal{WAP}_0(G)$ such that $\|f\| = \|f + C_0(G)\|_q$ for every $f \in L$. □

The results in this section will be considerably improved in the next section when G is an IN-group or a nilpotent group.

5. The quotient of $\mathcal{WAP}(G)$ by $\mathcal{B}(G)$

The situation is much more delicate with $\mathcal{WAP}(G)/\mathcal{B}(G)$. Already in the cases dealt with by Rudin in [51] and by Ramirez in [49] proving that $\mathcal{B}(G) \subsetneq \mathcal{WAP}(G)$ the arguments were quite involved. Elaborating on the work by Rudin and Ramirez, Chou proved in [13] that the quotient space $\mathcal{WAP}(G)/\mathcal{B}(G)$ contains a linear isometric copy of ℓ_∞ whenever G is a non-compact, locally compact, IN-group or a nilpotent group. In all these papers the key argument consists of constructing a t-set that contains large squares. Here we follow that thread and find copies of $\ell_\infty(\kappa)$ for κ as large as possible in $\mathcal{WAP}(G)/\mathcal{B}(G)$ by applying Theorem 2.11.

More precisely, we shall strengthen Chou's theorems by showing that there is a copy of $\ell_\infty(\kappa)$ in the quotient $\mathcal{WAP}(G)/\mathcal{B}(G)$ when G is either an IN-group or a nilpotent group and $\kappa = \kappa(G)$, and that, in general, for every locally compact group a copy of $\ell_\infty(\kappa(Z(G)))$ can be found in the quotient $\mathcal{WAP}(G)/\mathcal{B}(G)$, where $Z(G)$ is the algebraic centre of G.

The following technical lemma establishes that a group cannot be covered by β-cosets of finitely many different subgroups of index larger than β. This is similar to a theorem, known at least from the times of [47], in which only finitely many cosets are allowed.

Lemma 5.1. *Let G be any group with $|G| = \kappa$. Suppose that there is a finite collection $\{H_1, \dots, H_n\}$ of subgroups of G such that G can be covered by $\beta < \kappa$ right-cosets of them, i.e., such that*

$$G = \bigcup_{j=1}^{n} \bigcup_{i \in I_j} H_j x_{i,j}, \qquad \textit{with } |I_1| + \cdots + |I_n| = \beta < \kappa. \tag{3}$$

Then some of the subgroups H_j have index at most β.

Proof. This is proved by induction on n. The theorem is obvious if $n = 1$. Assume the theorem has been proved for unions of cosets of $n-1$ different subgroups and suppose

$$G = \bigcup_{j=1}^{n} \bigcup_{i \in I_j} H_j x_{i,j}, \qquad \text{with } |I_1| + \cdots + |I_n| = \beta < \kappa. \tag{$*$}$$

If $|G : H_n| > \beta$, there is $x \in G$ such that $x \notin \bigcup_{i \in I_n} H_n x_{i,n}$. Since

$$H_n x \cap H_n x_{i,n} = \emptyset \quad \text{for every} \quad i \in I_n,$$

we obtain

$$H_n x \subseteq \bigcup_{j=1}^{n-1} \bigcup_{i \in I_j} H_j x_{i,j},$$

and so

$$H_n = \bigcup_{j=1}^{n-1} \bigcup_{i \in I_j} H_j x_{i,j} x^{-1} \cap H_n = \bigcup_{j=1}^{n-1} \bigcup_{i \in I_j} (H_j \cap H_n) y_{i,j}, \tag{$**$}$$

where the $y_{i,j}$'s have been suitably chosen in H_n (if $h_j x_{i,j} x^{-1} = y_{i,j}$ is in $H_j x_{i,j} x^{-1} \cap H_n$, then $x_{i,j} x^{-1} = h_j^{-1} y_{i,j}$). Applying our inductive hypothesis, we deduce that there is j_0 with $|H_n\colon (H_n \cap H_{j_0})| \leq \beta$. (One may also proceed directly and replace H_n from $(**)$ in $(*)$, then apply the inductive hypothesis.)

There is therefore a family $\{z_s\colon s \in S\} \subset H_n$ with $|S| \leq \beta$ such that $H_n = \bigcup_{s\in S}(H_n \cap H_{j_0})z_s$ and we may replace (3) by

$$G = \left(\bigcup_{j=1}^{n-1} \bigcup_{i\in I_j} H_j x_{i,j}\right) \cup \left(\bigcup_{i\in I_n} \bigcup_{s\in S} H_{j_0} z_s x_{i,n}\right).$$

Since this is a cover of G by cosets of at most $n-1$ different subgroups of G we deduce from our inductive hypothesis that some of the subgroups H_j, $1 \leq j \leq n-1$, have index at most β. □

Lemma 5.2. *Let G be a locally compact group containing a normal subgroup $H \subset G$. If $|G\colon H| = \kappa \geq \omega$, then G contains a family $\{T_\eta\colon \eta < \kappa\}$ of subsets such that, putting $T = \bigcup_{\eta<\kappa} T_\eta$,*

- (i) *$Ht \cap Ht' = \emptyset$ for every $t \neq t' \in T$.*
- (ii) *T_η contains large squares for every $\eta < \kappa$.*
- (iii) *If $U \subset H$ is symmetric and compact, then $UTg \cap UT$ and $gUT \cap UT$ are relatively compact for every $g \notin \bigcup_{t\in T} t^{-1}U^2 t$.*

Proof. For each $\eta < \kappa$, in this proof let $\mathfrak{C}_\eta(H)$ denote the set

$$\mathfrak{C}_\eta(H) = \{gH \in G/H\colon\ |\mathfrak{Cl}(gH)| \leq \eta\},$$

where $\mathfrak{Cl}(gH)$ denotes the conjugacy class of gH in G/H. If $A \subseteq G/H$, $\mathfrak{Cl}(A)$ will stand for the set $\{(g^{-1}ag)H\colon gH \in G/H, aH \in A\}$. $\pi\colon G \to G/H$ will denote the canonical quotient mapping. For $n < \omega$, we define $\langle A\rangle_n$ to be the set of all products of at most n elements in $A \cup A^{-1}$.

We will define for each $\eta < \kappa$ and $n < \omega$ two collections of finite sets

$$C_{\eta,n} = \{x_{\eta,n,k}\colon 0 \leq k \leq n\} \quad \text{and} \quad D_{\eta,n} = \{y_{\eta,n,k}\colon 0 \leq k \leq n\}.$$

These sets are defined recursively. First we order the set $\kappa \times \omega$ in the canonical way ([39, 3.12]): For $\eta, \eta' < \kappa$ and $n, n' < \omega$, we define $(\eta, n) < (\eta', n')$ if either $\max(\eta, n) < \max(\eta', n')$, or $\max(\eta, n) = \max(\eta', n')$ and $\eta < \eta'$, or $\max(\eta, n) = \max(\eta', n')$, $\eta = \eta'$ and $n < n'$. This way, the cardinal of the set $\{(\eta, n)\colon (\eta, n) < (\eta_0, n_0)\}$ is less than κ for every $(\eta_0, n_0) \in \kappa \times \omega$.

We now define $x_{0,0,0} = y_{0,0,0} = e$ and set $C_{0,0} = \{x_{0,0,0}\}$, $D_{0,0} = \{y_{0,0,0}\}$.

Assume that the sets $C_{\gamma,j}$ and $D_{\gamma,j}$ have been defined for $(\gamma, j) < (\eta, n)$. We now define the set $C_{\eta,n}$.

To that end we need to define $R_{\eta,n,0} = \bigcup_{(\gamma,j)<(\eta,n)} (\pi(C_{\gamma,j}) \cup \pi(D_{\gamma,j}))$ and consider a cardinal number $f(\eta, n, 0)$ such that

$$\kappa > f(\eta, n, 0) > (n+1)^2 \left|R_{\eta,n,0}\right|^4 + \left(|R_{\eta,n,0}| + n + 1\right)^8. \tag{4}$$

Note that, in particular, $f(\eta, n, 0)$ is finite when $\kappa = \omega$.

We then choose $x_{\eta,n,0}$ such that

$$\pi(x_{\eta,n,0}) \notin \left\langle R_{\eta,n,0}, \mathfrak{Cl}\left(R_{\eta,n,0} \cap \mathfrak{C}_{f(\eta,n,0)}\right) \right\rangle_8. \tag{5}$$

The choice of this element will be possible because $|\langle R_{\eta,n,0}\rangle_8| \leq f(\eta,n,0) < \kappa = |G\colon H|$ and conjugacy classes of elements of $\mathfrak{C}_{f(\eta,n,0)}$ do not have, by definition, more than $f(\eta,n,0)$-elements.

Suppose now that $x_{\eta,n,l}$ has been defined for $0 \leq l < k$. The element $x_{\eta,n,k}$ is then defined similarly. We first define

$$R_{\eta,n,k} = \bigcup_{(\gamma,j)<(\eta,n)} \Big(\pi(C_{\gamma,j}) \cup \pi(D_{\gamma,j})\Big) \cup \Big\{\pi(x_{\eta,n,0}),\ldots,\pi(x_{\eta,n,k-1})\Big\}$$

and $f(\eta,n,k)$ such that

$$\kappa > f(\eta,n,k) > (n+1)^2\,|R_{\eta,n,k}|^4 + (|R_{\eta,n,k}|+2n+1)^8\,. \tag{6}$$

Then $x_{\eta,n,k}$ is obtained as above:

$$\pi(x_{\eta,n,k}) \notin \Big\langle R_{\eta,n,k}, \mathfrak{Cl}\left(R_{\eta,n,k} \cap \mathfrak{C}_{f(\eta,n,0)}\right)\Big\rangle_8. \tag{7}$$

Once the sets $C_{\gamma,j}$ have been defined for $(\gamma,j) \leq (\eta,n)$ and the sets $D_{\gamma,j}$ have been defined for $(\gamma,j) < (\eta,n)$, we define the set $D_{\eta,n}$.

The process starts as in the case of $C_{\eta,n}$; in this case the elements of this latter set must also be avoided. We thus start by defining the sets:

$$S_{\eta,n,0} = \left(\bigcup_{(\gamma,j)\leq(\eta,n)} \pi(C_{\gamma,j})\right) \cup \left(\bigcup_{(\gamma,j)<(\eta,n)} \pi(D_{\gamma,j})\right) \text{ and, for } 1 \leq k \leq n,$$

$$S_{\eta,n,k} = \left(\bigcup_{(\gamma,j)\leq(\eta,n)} \pi(C_{\gamma,j})\right) \cup \left(\bigcup_{(\gamma,j)<(\eta,n)} \pi(D_{\beta,i})\right) \cup \Big\{\pi(y_{\eta,n,0}),\ldots,\pi(y_{\eta,n,k-1})\Big\}.$$

We use the following claim to define $y_{\eta,n,k}$.

Claim 1. There is $y_{\eta,n,k} \in G$ such that

$$\pi(y_{\eta,n,k}) \notin \langle S_{\eta,n,k}\rangle_8 \tag{8}$$

and

$$C_{\eta,n}C_{\eta,n}^{-1} \cap y_{\eta,n,k}\pi^{-1}\left(\langle R_{\eta,n,k}\rangle_4\right) y_{\eta,n,k}^{-1} = \{e\}. \tag{9}$$

We first enumerate $(C_{\eta,n}C_{\eta,n}^{-1} \setminus \{e\}) \cap \pi^{-1}\left(\langle \mathfrak{Cl}(R_{\eta,n,k})\rangle_4\right)$ as $\{a_1,\ldots,a_l\}$. Then let

$$R_j = \pi^{-1}\left(\langle R_{\eta,n,k}\rangle_4 \cap \mathfrak{Cl}(\pi(a_j))\right)$$

and choose, for each j, $1 \leq j \leq l$, and each $r \in R_j$ an element $y_{j,r} \in G$ and an element $h_{j,r} \in H$ with

$$r = y_{j,r}^{-1} a_j h_{j,r} y_{j,r}.$$

Suppose now that no $y \in G$ can be found so that conditions (8) and (9) are satisfied. In that case some R_j must be non-empty and, indeed,

$$G = \langle S_{\eta,n,k}\rangle_8 \cup \left(\bigcup_{j=1}^{l} \bigcup_{r\in R_j} L_{j,r}\right),$$

where

$$L_{j,r} = \left\{g \in G\colon r = g^{-1}a_j g\right\}.$$

Observe now that $\pi(L_{j,r}) \subseteq C_{G/H}\big(\pi(a_j)\big)\pi(y_{j,r})$, where

$$C_{G/H}\big(\pi(a_j)\big) = \{\pi(g) \in G/H \colon \pi(ga_j) = \pi(a_j g)\}$$

is the centralizer of $\pi(a_j)$. Therefore,

$$G/H = \langle S_{\eta,n,k}\rangle_8 \cup \left(\bigcup_{j=1}^{l} \bigcup_{r\in R_j} C_{G/H}\big(\pi(a_j)\big)\pi(y_{j,r}) \right). \tag{10}$$

Now, $|S_{\eta,n,0}| \leq |R_{\eta,n,0}| + n + 1$ and $|S_{\eta,n,k}| \leq |R_{\eta,n,0}| + 2n + 1$, for $1 \leq k \leq n$. We therefore deduce from (4) and (6) that $(n+1)^2 \left|\langle R_{\eta,n,k}\rangle_4\right| + |\langle S_{\eta,n,k}\rangle_8| < f(\eta,n,k)$ for $0 \leq k \leq n$.

If the elements of the set $\langle S_{\eta,n,k}\rangle_8$ are viewed as cosets of the trivial subgroup $\{e_{G/H}\}$, we see in (10) that G/H can be decomposed as a union of less than $(n+1)^2|\langle R_{\eta,n,k}\rangle_4| + |\langle S_{\eta,n,k}\rangle_8|$ cosets. We deduce from Lemma 5.1 that some of them must correspond to a subgroup of index at most $f(\eta,n,k)$. Thus, there is j, $1 \leq j \leq l$ such that $|G/H \colon C_{G/H}(\pi(a_j))| = |\mathfrak{Cl}(\pi(a_j))| < f(\eta,n,k)$. We conclude that $\pi(a_j) \in \mathfrak{C}_{f(\eta,n,k)}$.

Since $\pi(a_j) \cap \mathfrak{Cl}\left(R_{\eta,n,k}\right) \neq \emptyset$, we find that $a_j \in \pi^{-1}\left(\mathfrak{Cl}(R_{n,\eta,k} \cap \mathfrak{C}_{f(\eta,n,k)})\right)$. If $a_j = x^{-1}_{\eta,n,k_1}x_{\eta,n,k_2}$ and $k_2 > k_1$, this goes against condition (7) in the choice of x_{η,n,k_2} and finishes the proof of the claim.

We have therefore constructed two families

$$C_{\eta,n} = \{x_{\eta,n,k} \colon 0 \leq k \leq n\} \quad \text{and } D_{\eta,n} = \{y_{\eta,n,k} \colon 0 \leq k \leq n\}, \quad (\eta,n) \in \kappa\times\omega,$$

with properties (7), (8) and (9). We now check that the sets

$$T_\eta = \bigcup_{n<\omega} (D_{\eta,n}C_{\eta,n}), \quad \eta < \kappa,$$

satisfy the desired properties.

First of all we see that $|D_{\eta,n}C_{\eta,n}| = (n+1)^2$. If this were not the case, there would be $1 \leq k_1, k_2, k_3, k_4 \leq n$ with $k_1 \neq k_3$ and $k_2 < k_4$ such that $y_{\eta,n,k_1}x_{\eta,n,k_2} = y_{\eta,n,k_3}x_{\eta,n,k_4}$. But then $y_{\eta,n,k_4} \in \langle C_{\eta,n}, y_{\eta,n,k_2}\rangle_2$ which goes against our choice of the elements in $D_{\eta,n}$. Therefore, T_η contains n-squares for every n.

In order to prove the last statement, we take U a compact symmetric subset of H and $g \notin U^2$. Choose $t_1 = y_{\eta_1,n_1,k_1}x_{\eta_1,n_1,k_1'}$ and $t_2 = y_{\eta_2,n_2,k_2}x_{\eta_2,n_2,k_2'}$, $u_1, u_2 \in U$, with

$$gu_1t_1 = u_2t_2 \in gUT \cap UT.$$

We order the 3-tuples (η_i, n_i, k_i) lexicographically with respect to the last entry, that is, $(\eta_i,n_i,k_i) > (\eta_i', n_i', k_i')$ if either $(\eta_i,n_i) > (\eta_i',n_i')$ or $(\eta_i,n_i) = (\eta_i',n_i')$ and $k_i > k_i'$.

Assume that $(\eta_2,n_2) \geq (\eta_1,n_1)$.

Now let $gu_3t_3 = u_4t_4$ with $t_3 = y_{\eta_3,n_3,k_3}x_{\eta_3,n_3,k_3'}$ and $t_4 = y_{\eta_4,n_4,k_4}x_{\eta_4,n_4,k_4'}$, $u_3, u_4 \in U$ be any other element of $gUT \cap UT$.

Then

$$\begin{aligned} g &= u_4 y_{\eta_4,n_4,k_4}x_{\eta_4,n_4,k_4'}x^{-1}_{\eta_3,n_3,k_3'}y^{-1}_{\eta_3,n_3,k_3}u_3^{-1} \\ &= u_2 y_{\eta_2,n_2,k_2}x_{\eta_2,n_2,k_2'}x^{-1}_{\eta_1,n_1,k_1'}y^{-1}_{\eta_1,n_1,k_1}u_1^{-1}. \end{aligned}$$

Let $(\eta_{i_1}, n_{i_1}, k_{i_1}) = \max\{(\eta_i,n_i,k_i) \colon 1 \leq i \leq 4\}$. There must then be i_2, $1 \leq i_2 \leq 4$, $i_2 \neq i_1$, with $\eta_{i_1} = \eta_{i_2}$, $n_{i_1} = n_{i_2}$ and $y_{\eta_{i_1},n_{i_1},k_{i_1}} = y_{\eta_{i_2},n_{i_2},k_{i_2}}$, for otherwise $y_{\eta_{i_1},n_{i_1},k_{i_1}} \in \langle S_{\eta_{i_1},n_{i_1},k_{i_1}}\rangle_7$.

Claim 2. It is not possible that $(\eta_4, n_4) = (\eta_3, n_3) > (\eta_2, n_2)$. Should this be the case, then $y_{\eta_4,n_4,k_4} = y_{\eta_3,n_3,k_3}$ and

$$x_{\eta_3,n_3,k'_4} x^{-1}_{\eta_3,n_3,k'_3} \in y^{-1}_{\eta_3,n_3,k_3} \left(y_{\eta_2,n_2,k_2} x_{\eta_2,n_2,k'_2} x^{-1}_{\eta_1,n_1,k'_1} y^{-1}_{\eta_1,n_1,k_1} \right) y_{\eta_3,n_3,k_3} H.$$

It follows from our condition (9) in the choice of y_{η_3,n_3,k_3} (Claim 1, page 593) that $x_{\eta_3,n_3,k'_4} = x^{-1}_{\eta_3,n_3,k'_3}$ but this is only possible if $g \in U^2$ (take into account that $gu_3t_3 = u_4t_4$ and that $y_{\eta_4,n_4,k_4} = y_{\eta_3,n_3,k_3}$), and the claim is proved.

The same argument shows that it is not possible that $(\eta_1, n_1) = (\eta_2, n_2) > (\eta_i, n_i)$, with $i = 3, 4$.

We deduce that either $(\eta_4, n_4) = (\eta_2, n_2)$ or $(\eta_3, n_3) = (\eta_2, n_2)$. But, since the element $gu_3t_3 = u_4t_4$ was chosen arbitrarily in $gUT \cap UT$, it follows that

$$gUT \cap UT \subseteq (gUD_{\eta_2,n_2} C_{\eta_2,n_2}) \cup (UD_{\eta_2,n_2} C_{\eta_2,n_2}),$$

and this is a relatively compact set.

We now check that $UTg \cap UT$ is compact. Choose $t_1 = y_{\eta_1,n_1,k_1} x_{\eta_1,n_1,k'_1}$ and $t_2 = y_{\eta_2,n_2,k_2} x_{\eta_2,n_2,k'_2}$, $u_1, u_2 \in U$ with

$$u_1t_1g = u_2t_2 \in UTg \cap UT.$$

Let

$$u_3 y_{\eta_3,n_3,k_3} x_{\eta_3,n_3,k'_3} g = u_4 y_{\eta_4,n_4,k_4} x_{\eta_4,n_4,k'_4}, \quad u_3, u_4 \in U,$$

be any other element of $UTg \cap UT$ with $(\eta_3, n_3) \geq (\eta_4, n_4)$. We have that

$$\begin{aligned} g &= t_1^{-1} u_1^{-1} u_2 t_2 \\ &= x^{-1}_{\eta_3,n_3,k'_3} y^{-1}_{\eta_3,n_3,k_3} u_3^{-1} u_4 y_{\eta_4,n_4,k_4} x_{\eta_4,n_4,k'_4} \in x^{-1}_{\eta_3,n_3,k'_3} y^{-1}_{\eta_3,n_3,k_3} y_{\eta_4,n_4,k_4} x_{\eta_4,n_4,k'_4} H. \end{aligned}$$

As in the preceding case we assume that $(\eta_2, n_2) \geq (\eta_1, n_1)$ and it is enough to see that neither $(\eta_4, n_4) = (\eta_3, n_3) > (\eta_2, n_2)$, nor $(\eta_2, n_2) = (\eta_1, n_1) > (\eta_3, n_3)$.

If $(\eta_4, n_4) = (\eta_3, n_3) > (\eta_2, n_2)$, then necessarily $(\eta_4, n_4, k_4) = (\eta_3, n_3, k_3)$ and $t_1^{-1} u_1^{-1} u_2 t_2 \in x^{-1}_{\eta_3,n_3,k'_3} x_{\eta_3,n_3,k'_4} H$. If $k'_3 > k'_4$, then $x_{\eta_3,n_3,k'_3} \in \langle R_{\eta_3,n_3,k'_3} \rangle_5$, against the election of x_{η_3,n_3,k'_3}. But $k'_3 = k'_4$, implies that $t_3 = t_4$ (recall that $(\eta_4, n_4, k_4) = (\eta_3, n_3, k_3)$) and $g \in t_3^{-1} U^2 t_3$. We rule out analogously the possibility $(\eta_1, n_1) = (\eta_2, n_2)$ and argue as above to prove that

$$UTg \cap UT \subseteq (UD_{\eta_2,n_2} C_{\eta_2,n_2}) g \cup (UD_{\eta_2,n_2} C_{\eta_2,n_2}),$$

and conclude that $UTg \cap UT$ is relatively compact. □

Corollary 5.3. *Let G be a locally compact group, H a subgroup of G and $N_G(H) = \{g \in G \colon gH = Hg\}$ be the normalizer of H in G. If $|N_G(H) \colon H| = \kappa \geq \omega$, then $N_G(H)$ contains a family $\{T_\eta \colon \eta < \kappa\}$ of subsets such that*

(i) *if $T = \bigcup_{\eta<\kappa} T_\eta$, then $Ht \cap Ht' = \emptyset$ for every $t \neq t' \in T$;*
(ii) *T_η contains large squares for every $\eta < \kappa$;*
(iii) *if $T = \bigcup_{\eta<\kappa} T_\eta$ and $U \subset H$ is compact, then $UTg \cap UT$ and $gUT \cap UT$ are relatively compact for every $g \notin \bigcup_{t\in T} t^{-1} U^2 t$.*

Proof. Since H is a normal subgroup of $N_G(H)$, we can apply Lemma 5.2 to $N_G(H)$. Statements (i) and (ii) remain the same if $N_G(H)$ is replaced by G. As for Statement (iii), one notices that $gUT \cap UT \neq \emptyset$ and $UTg \cap UT \neq \emptyset$ both imply that $g \in N_G(H)$, and so this statement also follows from Lemma 5.2. □

We obtain a first consequence for groups with large centre.

Theorem 5.4. *Let G be a locally compact group, $Z(G)$ be the algebraic centre of G and put $\kappa = \kappa(Z(G))$. If $\kappa \neq 1$, then there is always a linear isometry $\Psi\colon \ell_\infty(\kappa) \to \mathcal{WAP}(G)/\mathcal{B}(G)$.*

Proof. The centre $Z(G)$, as every locally compact Abelian group, always contains an open subgroup G_0 topologically isomorphic to $\mathbb{R}^n \times K$, with K compact. Therefore, G must contain two subgroups $H_1 \subset H_2 \subseteq Z(G)$ with H_1 open in H_2 and $|H_2\colon H_1| = \kappa$. Indeed, if $|Z(G)\colon G_0| \geq \omega$, then we take $H_2 = Z(G)$ and $H_1 = G_0$. If G_0 has finite index and $Z(G)$ is not compact, then $\kappa = \omega$ and so we may take $H_2 = \mathbb{Z}$ and $H_1 = \{e\}$.

Let $\{T_\eta : \eta < \kappa\}$ be the family of subsets of H_2 provided by Lemma 5.2. By Theorem 3.6, none of them is a $\mathcal{B}(G)$-interpolation set. If $T = \bigcup_{\eta<\kappa} T_\eta$ and U is a compact neighbourhood of the identity in H_2 with $U \subset H_1$, then, since H_2 is commutative, UT is a t-set in H_2. By Lemma 3.4, we can find a compact neighbourhood V of the identity in G, such that VT is a t-set in G. If V is chosen so that $V^4 \cap H_2 \subset H_1$ (remember H_1 is open in H_2), then T is V^2-uniformly discrete, and by Lemma 3.3, T is an approximable $\mathcal{WAP}(G)$-interpolation set.

It suffices now to apply Theorem 2.11. □

Lemma 5.2 can be readily applied to discrete groups. To further expand its applicability we follow the usual path applying well-known structure theorems. The following lemma for instance is the analogue of Lemma 4.4 of [13].

Lemma 5.5. *Let G be a locally compact group and let N be a closed subgroup of G.*

(i) *If N is normal, the quotient map $\pi\colon G \to G/N$ induces linear isometries*

$$\Pi\colon \mathcal{WAP}(G/N)/\mathcal{B}(G/N) \to \mathcal{WAP}(G)/\mathcal{B}(G) \quad \text{and}$$
$$\Pi_0\colon \mathcal{WAP}_0(G/N)/\mathcal{B}_0(G/N) \to \mathcal{WAP}_0(G)/\mathcal{B}_0(G).$$

(ii) *If N is open, there are linear isometries*

$$\Psi\colon \mathcal{WAP}(N)/\mathcal{B}(N) \to \mathcal{WAP}(G)/\mathcal{B}(G) \quad \text{and}$$
$$\Psi_0\colon \mathcal{WAP}_0(N)/\mathcal{B}_0(N) \to \mathcal{WAP}_0(G)/\mathcal{B}_0(G).$$

Proof. We first prove (i). The map $\phi \mapsto \phi \circ \pi$ clearly defines a linear isometry $\tilde{\pi}\colon \mathcal{WAP}(G/N) \to \mathcal{WAP}(G)$. By [11, Theorem], we have

$$\tilde{\pi}(\mathcal{B}(G/N)) = \tilde{\pi}(\mathcal{CB}(G/N)) \cap \mathcal{B}(G).$$

By [9], we have

$$\tilde{\pi}(\mathcal{WAP}(G/N)) = \tilde{\pi}(\mathcal{CB}(G/N)) \cap \mathcal{WAP}(G).$$

Since $\mathcal{B}(G) \subseteq \mathcal{WAP}(G)$, we see that

$$\tilde{\pi}(\mathcal{B}(G/N)) = \tilde{\pi}(\mathcal{CB}(G/N)) \cap \mathcal{WAP}(G) \cap \mathcal{B}(G) = \tilde{\pi}(\mathcal{WAP}(G/N)) \cap \mathcal{B}(G) \tag{11}$$

so that $\tilde{\pi}$ induces a linear isomorphism

$$\Pi\colon \mathcal{WAP}(G/N)/\mathcal{B}(G/N) \to \mathcal{WAP}(G)/\mathcal{B}(G),$$

given by

$$\Pi(\phi + \mathcal{B}(G/N)) = \tilde{\pi}(\phi) + \mathcal{B}(G).$$

We check that Π is an isometry. If $\phi \in \mathcal{WAP}(G/N)$,

$$\begin{aligned}\|\Pi(\phi+\mathcal{B}(G/N))\| &= \|\tilde{\pi}(\phi)+\mathcal{B}(G)\| \\ &= \inf\{\|\tilde{\pi}(\phi)+\psi\|\colon \psi\in\mathcal{B}(G)\} \\ &\le \inf\{\|\tilde{\pi}(\phi+\psi)\|\colon \psi\in\mathcal{B}(G/N)\} \\ &= \inf\{\|\phi+\psi\|\colon \psi\in\mathcal{B}(G/N)\} = \|\phi+\mathcal{B}(G/N)\|.\end{aligned}$$

For the reverse inequality, we follow the path of Lemma 2.3 of [12] and consider the invariant mean μ_N on $\mathcal{WAP}(N)$. For $\phi \in \mathcal{WAP}(G)$, we define the function $\phi^N\colon G\to\mathbb{C}$ by

$$\phi^N(g)=\mu_N(\phi_g), \quad \text{where } \phi_g(h)=\phi(gh).$$

By invariance of μ_N, the function ϕ^N is constant on the cosets of N and therefore induces a continuous function on G/N. Clearly, $\|\phi^N\|_{G/N} \le \|\phi\|_G$.

Now Lemma 2.3 of [12] proves in fact that $\phi^N \in \mathcal{WAP}(G/N)$. Moreover, by first considering positive-definite functions, it is also easily checked that $\psi^N \in \mathcal{B}(G/N)$ for every $\psi \in \mathcal{B}(G)$.

Note as well that for $\phi \in \mathcal{WAP}(G/N)$, we have $\tilde{\pi}(\phi)^N=\phi$.

Now if $\phi \in \mathcal{WAP}(G/N)$ and $\psi \in \mathcal{B}(G)$,

$$\begin{aligned}\|\tilde{\pi}(\phi)+\psi\| &\ge \|(\tilde{\pi}(\phi)+\psi)^N\| \\ &= \|\tilde{\pi}(\phi)^N+\psi^N\| \\ &= \|\phi+\psi^N\| \\ &\ge \|\phi+\mathcal{B}(G/N)\|.\end{aligned}$$

And the remaining inequality

$$\|\Pi(\phi+\mathcal{B}(G/N))\| = \|\tilde{\pi}(\phi)+\mathcal{B}(G)\| \ge \|\phi+\mathcal{B}(G/N)\|$$

follows.

We prove now the analogue statements for $\mathcal{WAP}_0(G)$ and $\mathcal{B}_0(G)$. We check first that $\tilde{\pi}$ maps $\mathcal{WAP}_0(G/N)$ into $\mathcal{WAP}_0(G)$ and $\mathcal{B}_0(G/N)$ into $\mathcal{B}_0(G)$. Consider the adjoint of $\tilde{\pi}$, this is the map given by

$$\tilde{\pi}^* : \mathcal{WAP}(G)^* \to \mathcal{WAP}(G/N)^*, \quad \tilde{\pi}^*(\nu)=\nu\circ\tilde{\pi}.$$

Note that if $\mu \in \mathcal{WAP}(G)^*$ is invariant, then $\tilde{\pi}^*(\mu) \in \mathcal{WAP}(G/N)^*$ is invariant. To see this, let $\bar{s}=\pi(s)\in G/N$ and $f \in \mathcal{WAP}(G/N)$ and note that $\tilde{\pi}(f_{\bar{s}})=(\tilde{\pi}(f))_s$, and so

$$\tilde{\pi}^*(\mu)(f_{\bar{s}})=\mu(\tilde{\pi}(f_{\bar{s}}))=\mu((\tilde{\pi}(f))_s)=\mu(\tilde{\pi}(f))=\tilde{\pi}^*(\mu)(f).$$

Thus, $\tilde{\pi}^*(\mu)$ is invariant on $\mathcal{WAP}(G/N)$.

Now let $f \in \mathcal{WAP}_0(G/N)$ and μ be the invariant mean on $\mathcal{WAP}(G)$. Then $\tilde{\pi}(f) \in \mathcal{WAP}(G)$ and

$$\mu(|\tilde{\pi}(f)|)=\mu(|f\circ\pi|)=\mu(|f|\circ\pi)=\tilde{\pi}^*(\mu)(|f|)=0.$$

Thus, $\tilde{\pi}(f)\in\mathcal{WAP}_0(G)$. To see that $\tilde{\pi}(f)\in\mathcal{B}_0(G)$ when $f\in\mathcal{B}_0(G/N)$, we argue in a similar way using the fact that $\tilde{\pi}(f)\in\mathcal{B}(G)$ by [12, Theorem]. Accordingly,

$$\tilde{\pi}(\mathcal{B}_0(G/N)) \subseteq \tilde{\pi}(\mathcal{WAP}_0(G/N))\cap\mathcal{B}_0(G).$$

The reverse inclusion is checked as follows. If $f\in\tilde{\pi}(\mathcal{WAP}_0(G/N))\cap\mathcal{B}_0(G)$, then by (11) f is clearly in $\tilde{\pi}(\mathcal{B}(G/N))$. So let $g\in\mathcal{B}(G/N)$ with $f=\tilde{\pi}(g)$. We only need to make sure that $\tilde{\pi}^*(\mu)(|g|) = 0$. But this is also clear from the following identity:

$$\tilde{\pi}^*(\mu)(|g|) = \mu(\tilde{\pi}(|g|)) = \mu(|g| \circ \pi) = \mu(|g \circ \pi|) = \mu(|\tilde{\pi}(g)|) = \mu(|f|).$$

Thus, we obtain the analogue of (11)

$$\tilde{\pi}(\mathcal{B}_0(G/N)) = \tilde{\pi}(\mathcal{WAP}_0(G/N)) \cap \mathcal{B}_0(G) \tag{12}$$

so that $\tilde{\pi}$ induces a linear isomorphism

$$\Pi_0 \colon \mathcal{WAP}_0(G/N)/\mathcal{B}_0(G/N) \to \mathcal{WAP}_0(G)/\mathcal{B}_0(G),$$

given by

$$\Pi_0(\phi + \mathcal{B}_0(G/N)) = \tilde{\pi}(\phi) + \mathcal{B}_0(G).$$

To check that Π_0 is an isometry, we proceed precisely as for Π.

For the proof of (ii), we associate to each $\phi \in \mathcal{WAP}(N)$ the function

$$\phi_N(g) = \phi(g) \quad \text{if} \quad g \in N \quad \text{and} \quad \phi_N(g) = 0 \quad \text{if} \quad g \notin N. \tag{13}$$

Then ϕ_N is in $\mathcal{WAP}(G)$ by [46, Lemma 5.4], [9, Theorem 3.14] or [10, Lemma 2.4]. If ϕ happens to be in $B(N)$, then $\phi_N \in B(G)$ by [37, page 280] or [13, Lemma 4.1] and this obviously extends to $\mathcal{B}(N)$ and $\mathcal{B}(G)$.

Define then $\Psi \colon \mathcal{WAP}(N)/\mathcal{B}(N) \to \mathcal{WAP}(G)/\mathcal{B}(G)$ by

$$\Psi(\phi + \mathcal{B}(N)) = \phi_N + \mathcal{B}(G).$$

It is easy to check that Ψ is a linear isometry.

To prove the second statement of (ii), note that the extension of ϕ_N defined in (13) is clearly in $\mathcal{WAP}_0(G)$ if $\phi \in \mathcal{WAP}_0(N)$, and in $\mathcal{B}_0(G)$ if $\phi \in \mathcal{B}_0(N)$. It is again straightforward to verify that

$$\Psi_0 \colon \mathcal{WAP}_0(N)/\mathcal{B}_0(N) \to \mathcal{WAP}_0(G)/\mathcal{B}_0(G), \quad \Psi_0(\phi + \mathcal{B}_0(N)) = \phi_N + \mathcal{B}_0(G)$$

is the required linear isometry. □

We finally reach our main results.

Theorem 5.6. *Let G be a a non-compact, locally compact, IN-group and put $\kappa = \kappa(G)$. Then there is a linear isometry $\Psi \colon \ell_\infty(\kappa) \to \mathcal{WAP}(G)/\mathcal{B}(G)$.*

Proof. By Theorem 2.13 of [34], there is an open normal subgroup N of G that contains a compact normal subgroup K with N/K Abelian.

Suppose first that $\kappa(N) = \kappa$. Then $\kappa = \kappa(N/K)$. By Theorem 5.4, there is a linear isometric copy of $\ell_\infty(\kappa)$ in $\mathcal{WAP}(N/K)/\mathcal{B}(N/K)$. We then apply (i) and (ii) of Lemma 5.5 to obtain a linear isometric copy of $\ell_\infty(\kappa)$ in $\mathcal{WAP}(G)/\mathcal{B}(G)$.

If $\kappa(N) < \kappa$, it follows that $\kappa = |G : N|$. We apply Lemma 5.2 to the discrete group G/N. Let $\{T_\eta : \eta < \kappa\}$ be the collection of subsets obtained in that lemma (in this case the subgroup H of that lemma is trivial) and let $T = \bigcup_{\eta<\kappa} T_\eta$. By (iii) in that lemma, the set T is a t-set (note that $U = \{e\}$ in this case, hence $Tg \cap T$ and $gT \cap T$ are finite if $g \neq e$) while each of the sets T_η contains large squares. Therefore, T is a $\mathcal{WAP}(G/N)$-interpolation set by Lemma 3.3, while none of the sets T_η is a $\mathcal{B}(G/N)(G)$-interpolation set by Theorem 3.6.

By Theorem 2.11 there is an isometric embedding

$$\ell_\infty(\kappa) \to \mathcal{WAP}(G/N)/\mathcal{B}(G/N).$$

Lemma 5.5 then provides the desired copy of $\ell_\infty(\kappa)$ in $\mathcal{WAP}(G)/\mathcal{B}(G)$. □

Theorem 5.6 also leads naturally to an improvement of [13, Theorem 4.6].

Theorem 5.7. *Let G be a a non-compact, locally compact, nilpotent group and put $\kappa = \kappa(G)$. Then the quotient $\mathcal{WAP}(G)/\mathcal{B}(G)$ contains a linear isometric copy of $\ell_\infty(\kappa)$.*

Proof. The case $\kappa(Z(G)) = \kappa(G)$ is already proved in Theorem 5.4. So we may assume that $\kappa(Z(G)) < \kappa(G)$. We argue by induction on the length n of the upper central series of G (the nilpotency length of G)

$$\{e\} = G_0 \subset G_1 \subset \ldots \subset G_{n-1} \subset G_n = G \text{ with } Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)).$$

If $n = 1$, then G is Abelian and so Theorem 5.4 or Theorem 5.6 applies.

Assume as inductive hypothesis that the claim holds for groups of nilpotency length at most $n-1$ and suppose G has nilpotency length n. Since $\kappa(G) = \kappa(Z(G)) + \kappa(G/Z(G))$ and the case $\kappa(Z(G)) = \kappa(G)$ has already been ruled out, we can assume that $\kappa(G/Z(G)) = \kappa(G)$. Our inductive hypothesis ($\kappa(G/Z(G))$ has nilpotency length $n-1$) and Lemma 5.5 then provide the desired isometry. □

When G is an IN-group or a nilpotent group, we recover and further improve the results obtained in Section 4.

Corollary 5.8. *Let G be a non-compact IN-group or a nilpotent group and let κ be the compact covering of G. Then each of the quotient spaces $\mathcal{WAP}(G)/(\mathcal{AP}(G) \oplus C_0(G))$ and $\mathcal{WAP}_0(G)/\mathcal{B}_0(G)$ contains a linear isometric copy of $\ell_\infty(\kappa)$.*

Proof. That the first quotient contains a copy of $\ell_\infty(\kappa)$ follows directly from Theorem 5.6 and Theorem 5.7 if we recall the inclusion $\mathcal{AP}(G) \oplus C_0(G) \subseteq \mathcal{B}(G)$ (see [13, page 143]).

For the second quotient, we argue as in Theorem 5.6. None of the sets T_η, $\eta < \kappa$, constructed in all cases needed in the proof of Theorem 5.6, is a $\mathcal{B}_0(G)$-interpolation set. On the other hand, proceeding precisely as in Theorems 5.6 and 5.7 (and using the right statements of Lemma 5.5), we see that $T = \bigcup_{\eta<\kappa} T_\eta$ is an approximable $\mathcal{WAP}_0(G)$-interpolation set. □

6. On the quotient of $\mathcal{CB}(G)$ by $\mathcal{LUC}(G)$

When G is a non-compact, non-discrete, locally compact group, Dzinotyiweyi showed in [20] that the quotient $\mathcal{CB}(G)/\mathcal{LUC}(G)$ is non-separable. When G is a non-precompact, topological group which is not a P-group, this theorem was generalized and improved in [6, Theorem 3.1] and [7, Theorem 4.1], where a linear isometric copy of ℓ_∞ was proved to be contained in $\mathcal{CB}(G)/\mathcal{LUC}(G)$. This section is concerned again with locally compact groups. Our theorem is then more precise and definite. We prove there is a linear isometric copy of $\ell_\infty(\kappa)$ in $\mathcal{CB}(G)/\mathcal{LUC}(G)$, where as before κ is the compact covering G, if and only if G is neither compact nor discrete.

Lemma 6.1. *Every non-discrete locally compact group contains a faithfully indexed sequence $\{x_n : n \in \mathbb{N}\}$ that converges to the identity.*

Proof. A locally compact group always contains a compact subgroup K such that G/K is a metrizable topological space (see [2, Theorem 4.3.29], for instance). Infinite compact groups on the other hand always contain non-trivial convergent sequences ([2, Theorem 4.1.7 and Exercise 4.1.f]). If K is infinite we are done. If K is finite, G is non-discrete and metrizable; it therefore contains non-trivial convergent sequences. □

Theorem 6.2. *Let G be a locally compact group. Then $\mathcal{CB}(G)/\mathcal{LUC}(G)$ contains a linear isometric copy of $\ell_\infty(\kappa(G))$ if and only if G is neither compact nor discrete.*

Proof. The necessity is clear since $\mathcal{CB}(G) = \mathcal{LUC}(G)$ if G is either compact or discrete.

If G is not compact we can find a compact neighbourhood of the identity U and a U^2-right uniformly discrete subset $X = \{x_\alpha \colon \alpha < \kappa\} \subseteq G$ with $\kappa = \kappa(G)$. This is clear if G is σ-compact. If $\kappa > \omega$, we consider $H = \langle U \rangle$, the subgroup generated by U. Then $\kappa = |G : H|$ and any system of representatives of right cosets of H constitutes an H-right uniformly discrete set of cardinality κ.

Partition X in κ-many countable subsets $X = \bigcup_{\alpha<\kappa} X_\alpha$. Enumerate, for each $\alpha < \kappa$, $X_\alpha = \{x_{\alpha,n} \colon n < \omega\}$. Since G is not discrete, U contains (by Lemma 6.1) a faithfully indexed sequence $S = \{s_n \colon n < \omega\}$ converging to the identity. With these ingredients, we define

$$T_{\alpha,n} = \{s_j x_{\alpha,n} \colon 1 \leq j \leq n\},\ T_\alpha = \bigcup_n T_{\alpha,n} \ \text{ and } \ T = \bigcup_\alpha T_\alpha.$$

Obviously, $UT_\alpha \cap UT_{\alpha'} = \emptyset$ for every $\alpha \neq \alpha' < \kappa$.

Each set T_α fails to be an $\mathcal{LUC}(G)$-interpolation set. Indeed, the function $f \colon T_\alpha \to \mathbb{C}$ such that

$$f(s_{2j}x_{\alpha,n}) = 1 \quad \text{for every} \quad j,n \in \mathbb{N} \quad \text{with} \quad 1 \leq 2j \leq n \text{ and}$$
$$f(s_{2j+1}x_{\alpha,n}) = -1 \quad \text{for every} \quad j,n \in \mathbb{N} \quad \text{with} \quad 1 \leq 2j+1 \leq n$$

cannot coincide on T_α with any $\phi \in \mathcal{LUC}(G)$, since given $\varepsilon > 0$, we can choose j large enough and $n \geq 2j+1$, so that

$$|\phi(s_{2j}x_{\alpha,n}) - \phi(s_{2j+1}x_{\alpha,n})| < \varepsilon \quad \text{while} \quad f(s_{2j}x_{\alpha,n}) - f(s_{2j+1}x_{\alpha,n}) = 2.$$

We now prove that T is an approximable $\mathcal{CB}(G)$-interpolation set. Since the sequence (s_j) is taken in U and X is right U^2-uniformly discrete, we readily see that the open set $Ux_{\alpha,n} \cap T$ is contained in $\{s_j x_{\alpha,n} \colon 1 \leq j \leq n\}$. T is therefore discrete.

Next we check that T is closed. Since $\lim_{j\to\infty} s_j = e$ and $s_j \in U$ for all j, there exists a neighbourhood W of the identity such that $Ws_j \subset U$ for all j. Now take a neighbourhood W_1 of the identity such that $W_1 W_1^{-1} \subseteq W$. Let $x \in G$ be arbitrarily chosen. If, for some $\alpha < \kappa$ and $n < \omega$, $s_{j,n}x_{\alpha,n} \in W_1 x$, then $W_1 x \subseteq Ux_{\alpha,n}$. It follows that $W_1 x \cap T \subseteq Ux_{\alpha,n} \cap T$. Since the latter set (see the preceding paragraph) is finite, we conclude that for every $x \in G$, $W_1 x \cap T$ is either empty or finite. So T is closed. Since the topological space underlying G is normal, T is an approximable $\mathcal{CB}(G)$-interpolation set by Lemma 3.3.

Corollary 2.12 now implies that $\mathcal{CB}(G)/\mathcal{LUC}(G)$ contains a linear isometric copy of $\ell_\infty(\kappa)$ with $\kappa = |X| = \kappa(G)$. □

The equivalence of the first two statements of the following corollary was proved by Baker and Butcher in [3]; see also [28] for a different proof.

Corollary 6.3. *Let G be a locally compact group with a compact covering number κ. Then the following statements are equivalent:*

(1) *G is neither compact nor discrete.*
(2) *$\mathcal{CB}(G) \neq \mathcal{LUC}(G)$.*
(3) *$\mathcal{CB}(G)/\mathcal{LUC}(G)$ contains a linear isometric copy of $\ell_\infty(\kappa(G))$.*

Proof. (1) $\Longrightarrow$ (3) is proved in the theorem above. (3) $\Longrightarrow$ (2) is obvious and (2) $\Longrightarrow$ (1) is clear. □

Remark 6.4. Theorem 6.2 implies a fortiori that the space $L^\infty(G)/\mathcal{LUC}(G)$ as well as $L^\infty(G)/\mathcal{WAP}(G)$ contains a linear isometric copy of $\ell_\infty(\kappa)$. The arguments used in [6, Section 4], may be applied again to deduce that the group algebra $L^1(G)$ is extremely non-Arens regular whenever κ is greater than or equal to the local character $\chi(G)$ of G (this is the least cardinality of an open base at the identity of G). To obtain the full result, however, harder work is necessary. This is achieved in our recent article [26].

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