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PROPER HOLOMORPHIC MAPPINGS BETWEEN INVARIANT DOMAINS IN $\mathbb{C}^n$

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Abstract. In the present paper, we prove the following result generalizing some well-known related results about biholomorphic or proper holomorphic mappings between some special domains in $\mathbb{C}^n$. Let G_1 and G_2 be two compact Lie groups, which act linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^{G_j} = \mathbb{C}$ for $j = 1, 2$. Let $0 \in \Omega_j$ be bounded G_j-invariant domains in $\mathbb{C}^n$ for $j = 1, 2$. If $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping, then f extends holomorphically to an open neighborhood of $\overline{\Omega}_1$, and in addition if $f^{-1}(0) = \{0\}$, then f is a polynomial mapping. We also prove that if $0 \in \Omega$ is a G_1-invariant pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^{G_1} = \mathbb{C}$, then Ω is orbit convex. The second result is used to prove the first one.

1. Introduction

Special domains such as circular domains, Reinhardt domains, etc., could be regarded in a unified way as examples of invariant domains with respect to some compact Lie group actions. In the present paper, we'll discuss certain structures and properties of biholomorphic or proper holomorphic mappings between some invariant domains in a setting of any compact Lie group actions.

1.1. **Some well-known theorems.**

In this subsection, we recall some well-known results by Cartan, Kaup, Heinzner, Bell, et al., about biholomorphic mappings or proper holomorphic mappings between some special domains in $\mathbb{C}^n$. The goal of the present paper is to give a unified result from the point view of the group actions.

A classical theorem of H. Cartan [11] asserts that if $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded circular domains in $\mathbb{C}^n$ which contain 0, and if $f(0) = 0$, then f is a linear mapping.

Cartan's theorem was later generalized by W. Kaup [26] to biholomorphic mappings between some special invariant domains with particular S^1 actions.

A domain Ω in $\mathbb{C}^n$ is called a $(p_1, \cdots, p_n)$-domain if it is stable under the transformations

$$(z_1, \cdots, z_n) \mapsto (t^{p_1} z_1, \cdots, t^{p_n} z_n)$$

where $t \in S^1 := \{s \in \mathbb{C} : |s| = 1\}$ and $p_1, \cdots, p_n$ are integers. When all p_i are equal, the $(p_1, \cdots, p_n)$-domain is just the circular domain.

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If $0 \in \Omega$ is a bounded $(p_1, \cdots, p_n)$-domain and $p_i > 0$ for all i, then W. Kaup [26] proved that every automorphism of Ω extends holomorphically to an open neighborhood of the topological closure $\overline{\Omega}$ of Ω.

Later on, P. Heinzner extended Kaup's result to the domains invariant with respect to some actions of any compact Lie groups in [22].

Let G be a compact Lie group and let G act linearly on $\mathbb{C}^n$, i.e., there is a continuous representation $\rho : G \to GL(\mathbb{C}^n)$.

Denote by $\mathcal{O}(\mathbb{C}^n)^G$ the set of G-invariant entire functions, i.e.,

$$\mathcal{O}(\mathbb{C}^n)^G := \{f \in \mathcal{O}(\mathbb{C}^n) : f \circ \rho(g) = f \text{ for all } g \in G\}.$$

A domain Ω is called G-invariant if $\rho(g) \cdot \Omega = \Omega$ for any $g \in G$.

P. Heinzner [22] proved that if $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$, then for any bounded G-invariant domain Ω in $\mathbb{C}^n$ which contains 0, any automorphism f of Ω can be extended holomorphically to an open neighborhood of $\overline{\Omega}$, and in addition if $f(0) = 0$, then f is a polynomial mapping.

The above three authors all dealt with the automorphisms or biholomorphic mappings on some group invariant domains.

S. Bell [6] considered a kind of more general mappings than biholomorphic mappings, the so-called proper holomorphic mappings, but between more special domains (i.e., bounded circular domains containing 0) than those Kaup and Heinzner dealt with.

He proved the following:

Consider any proper holomorphic mapping $f : \Omega_1 \to \Omega_2$, where Ω_1 and Ω_2 are bounded circular domains containing 0 in $\mathbb{C}^n$.

(1) Let $K(z, w)$ be the Bergman kernel function of Ω_1. If for each compact subset D of Ω_1 there is an open set $G = G(D)$ containing $\overline{\Omega}_1$ such that $K(z, w)$ extends holomorphically with respect to $z \in G$ for all $w \in D$, then f can be extended holomorphically to an open neighborhood of $\overline{\Omega}_1$.

(2) If $f^{-1}(0) = \{0\}$, then f is a polynomial mapping.

1.2. **A generalized result.** In the present paper, we extend the above results due to Cartan, Kaup, Heinzner and Bell, et al., to proper holomorphic mappings between some invariant domains w.r.t. arbitrary compact Lie groups. We find that the assumption (1) of S. Bell on the Bergman kernel functions in his above theorem is redundant (see Theorem 3.3).

We will prove the following theorem:

Theorem 1.1. *Let G_j be compact Lie groups, acting linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^{G_j} = \mathbb{C}$, and let Ω_j be G_j-invariant bounded domains in $\mathbb{C}^n$ which contain 0 for $j = 1, 2$. Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping. Then*

(1) *f can extend holomorphically to an open neighborhood of $\overline{\Omega}_1$.*

(2) *If in addition $f^{-1}(0) = \{0\}$, then f is a polynomial mapping.*

C. Fefferman [20] proved that any biholomorphic mapping between two strictly bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary can extend smoothly to the boundary.

After that, S. Bell and D. Catlin [9], and K. Diederich and J. Fornaess [18] proved that: let Ω_1 and Ω_2 be two bounded pseudoconvex domains with smooth (or analytic) boundary. If Ω_1 satisfies condition R, then any proper holomorphic mapping from Ω_1 to Ω_2 can extend smoothly (or holomorphically) to $\overline{\Omega_1}$.

It should be noted that we need neither the boundary to be smooth nor the domain to be pseudoconvex in the above Theorem 1.1.

In the proof of Theorem 1.1, we extend the idea of S. Bell [6] in a setting of group actions by using S. Bell's transformation formula for the Bergman projections, and by combining it with some results about compact Lie group actions.

1.3. **A conjecture about orbit convexity.** In this paper, we'll present a result (Theorem 1.2) to confirm a conjecture in the paper [29] under an additional assumption, which will be used to prove Theorem 1.1.

For a Lie group G which acts linearly on $\mathbb{C}^n$ by ρ, we always write gz or $g \cdot z$ for $\rho(g)z$ where $g \in G$ and $z = (z_1, \cdots, z_n)' \in \mathbb{C}^n$ as a column vector.

For a compact Lie group G, denote by $G^{\mathbb{C}}$ the complexification of G. If G acts linearly on $\mathbb{C}^n$, then $G^{\mathbb{C}}$ acts linearly on $\mathbb{C}^n$.

Fact 1.1. If $\mathfrak{g}$ is the Lie algebra of G, and $exp : \mathfrak{g} \to G$ is the exponential mapping, then the first Cartan decomposition theorem says that $G^{\mathbb{C}} = G \cdot exp(i\mathfrak{g})$ and $G \cap exp(i\mathfrak{g}) = \{e\}$, where e is the identity of G.

We first recall the definitions of orbit connected and orbit convex.

Definition 1.1. Let X be a $G^{\mathbb{C}}$-space. Denote by $b_z : G^{\mathbb{C}} \to X$ the mapping $g \mapsto g \cdot z$ for a given $z \in X$. A G-set D in X is called orbit connected if $b_z^{-1}(D)$ is G-connected for all $z \in D$, i.e., the set $b_z^{-1}(D)/G = \{G \cdot x : x \in b_z^{-1}(D)\}$ endowed with the quotient topology is connected. When G is connected, this just means that $b_z^{-1}(D)$ is connected.

It's known that G is a totally real submanifold of maximal dimension in $G^{\mathbb{C}}$.

Fact 1.2. Let G be a connected Lie group and let D be a domain containing G in $G^{\mathbb{C}}$. Then if a holomorphic function on D is zero on G, then it is identically zero on D. This is the so-called uniqueness theorem.

Definition 1.2. Let X be a $G^{\mathbb{C}}$-space. A G-invariant subset U of X is said to be orbit convex if for each $z \in U$ and $v \in i\mathfrak{g}$ such that $exp(v) \cdot z \in U$, it follows that $exp(tv) \cdot z \in U$ for all $t \in [0, 1]$.

It's clear that an orbit convex set is orbit connected.

X. Y. Zhou [36] proved the following theorem:

Let $(S^1)^k$ act linearly on $\mathbb{C}^n$, and $\mathcal{O}(\mathbb{C}^n)^{(S^1)^k} = \mathbb{C}$. Let Ω be an $(S^1)^k$-invariant domain of holomorphy in $\mathbb{C}^n$ which contains 0. Then Ω is orbit convex.

According to the above result and other results, X. Y. Zhou (see [29]) conjectured that:

Conjecture 1.1. *Let G be a compact Lie group and act linearly on $\mathbb{C}^n$. Then any G-invariant domain of holomorphy Ω in $\mathbb{C}^n$ which contains* 0 *is orbit convex.*

In the present paper, we obtain the following:

Theorem 1.2. *The above conjecture is true if we assume that $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$.*

As a consequence, we get an important property (Theorem 2.6) about such invariant domains which is used to prove the extension property of the Bergman kernel functions, and is also one of the keys to the proof of Theorem 1.1 (1).

In the present paper, the considered invariant domains refer to the G-invariant domains with $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$, where G is a compact Lie group and acts linearly on

$\mathbb{C}^n$. It's easy to see that the domains such as circular and quasi-circular domains are special cases of our considered invariant domains.

The organization of the present paper is as follows. We'll prove Theorem 1.2 and its consequence in section 2. In section 3, we'll prove the extension property of the Bergman kernel functions and give some other results needed for proving Theorem 1.1. After that, we'll give a proof of Theorem 1.1 in section 4. In section 5, we'll also extend some theorems proved by S. Bell in [5, 7, 8].

2. Some results about the invariant domains

2.1. **Some definitions and related results about Lie group action.** In the present subsection, we will recall some results about the invariant domains and prove a result on the Runge property of the invariant domains with trivial invariant entire holomorphic functions.

First, we recall some definitions about group actions and the Fourier theorem for compact group actions.

Definition 2.1. Let G be a Lie group. A complex space X together with a group homomorphism ρ from G into the group $Aut(X)$ is called a complex G-space if the action $G \times X \to X, (g, x) \mapsto \rho(g)(x)$ is continuous.

We write $g \cdot x$ or gx for $\rho(g)(x)$.

If G is a complex Lie group, and the action $G \times X \to X, (g, x) \mapsto \rho(g)(x)$ is holomorphic, we call X a holomorphic G-space.

For a subset $A \subseteq X$, A is called G-invariant if $G \cdot A := \{g \cdot x : x \in A, g \in G\} = A$.

A holomorphic mapping $\psi : X \to Y$ between two complex G-spaces X and Y is called equivariant if $\psi(g \cdot x) = g \cdot \psi(x)$ for all $g \in G, x \in X$.

We denote by $Hol_G(X, Y)$ the set of all holomorphic equivariant mappings from X to Y.

Let the compact Lie group G act linearly on $\mathbb{C}^n$ and G act trivially on $\mathbb{C}$; then $\mathcal{O}(\mathbb{C}^n)^G = Hol_G(\mathbb{C}^n, \mathbb{C})$.

Fourier Theorem 2.1. *Let G be a compact Lie group, and let X be a complex G-space. Let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of G. Suppose that $\rho : G \to GL(V)$ is an irreducible unitary representation in a complex vector space V of dimension $d(\rho)$, and f is a holomorphic function on X. Denote*

$$f_\rho := d(\rho) \int_G (f \circ g^{-1})\rho(g^{-1}) d\mu,$$

where μ is the normalized Haar measure on G. Then $f_\rho \in Hol_G(X, Hom_{\mathbb{C}}(V, V))$, and $Trf_\rho \in \mathcal{O}(X)$ depends only on the equivalence classes of irreducible unitary representation ρ, where Tr stands for the trace of $Hom_{\mathbb{C}}(V, V)$.

The Fourier theorem says that $\sum_{\rho \in \hat{G}} Trf_\rho$ converges uniformly on all compact subsets of X to f.

The above theorem is an important corollary of a theorem of Harish-Chandra on infinite dimensional representation theory of a compact Lie group.

Second, we recall the definition of the complexification of the complex G-space with real Lie group G action.

Definition 2.2 (see [23])**.** Let G be a Lie group and let X be a complex G-space. Let $G^{\mathbb{C}}$ be a complexification of G. A holomorphic $G^{\mathbb{C}}$-space $X^{\mathbb{C}}$ together with a

G-map ι from X into $X^{\mathbb{C}}$ is called a G-complexification of the G-space X if for every holomorphic G-map ϕ from X into a holomorphic $G^{\mathbb{C}}$-space Y there exists one and only one holomorphic $G^{\mathbb{C}}$-map $\phi^{\mathbb{C}}$ from $X^{\mathbb{C}}$ into Y, such that

$$\phi^{\mathbb{C}} \circ \iota = \phi.$$

Lemma 2.1 (Extension Lemma; [32], see also [23])**.** *Let X and Y be holomorphic $G^{\mathbb{C}}$-spaces, and let $U \subset X$ be an orbit connected open G-set. Then any $\varphi \in Hol_G(U, Y)$ has a holomorphic extension $\tilde{\varphi} \in Hol_{G^{\mathbb{C}}}(G^{\mathbb{C}} \cdot U, Y)$.*

Proposition 2.2 (see [23])**.** *Let X be a holomorphic $G^{\mathbb{C}}$-space, let $D \subset X$ be a $G^{\mathbb{C}}$-domain, and let φ be a G-invariant plurisubharmonic function on D. Then the G-set $D_0 = \{x \in D : \varphi(x) < 1\}$ is orbit convex.*

Now we present the following fact which will be used frequently.

Fact 2.1. For any $w \in \mathbb{C}^n$, $\overline{G^{\mathbb{C}} \cdot w}$ is a closed G-invariant complex subvariety of $\mathbb{C}^n$ (see [30]). When $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$, any closed G-invariant complex subvariety of $\mathbb{C}^n$ contains 0 (see [22]). Furthermore, we have $G^{\mathbb{C}} \cdot B_r = \mathbb{C}^n$ for any $r > 0$, where B_r is the ball with radius r centered at the origin.

Finally, we prove the following proposition which will be used.

Proposition 2.3. *Let G be a compact Lie group which acts linearly on $\mathbb{C}^n$ and $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$, and let Ω be a G-invariant domain in $\mathbb{C}^n$ which contains 0. Then Ω is a Runge domain, $G^{\mathbb{C}} \cdot \Omega = \mathbb{C}^n$, and the envelope of holomorphy $\tilde{\Omega}$ of Ω is schlicht and G-invariant.*

Proof. Take $r > 0$ small enough, such that $B_r \subset \Omega$.

Without loss of generality, one may assume that G is a closed subgroup of the unitary group $U(n)$. Therefore, B_r is G-invariant.

By the above fact, we get $G^{\mathbb{C}} \cdot \Omega = \mathbb{C}^n$.

For any $f \in \mathcal{O}(\Omega)$, by the Fourier theorem, we have $f = \sum_{\rho \in \hat{G}} Trf_\rho$.

B_r is orbit convex, hence is orbit connected. Since

$$f_\rho|_{B_r} \in Hol_G(B_r, Hom_{\mathbb{C}}(V, V)),$$

then we can extend $f_\rho|_{B_r}$ holomorphically to $\mathbb{C}^n$, by using Extension Lemma 2.1.

As Ω is connected, it follows from the uniqueness theorem that Trf_ρ can extend holomorphically to $\mathbb{C}^n$. Hence, f can be approximated by entire functions uniformly on compact subsets of Ω.

Thus, $(\Omega, G^{\mathbb{C}} \cdot \Omega) = (\Omega, \mathbb{C}^n)$ is a Runge pair.

Then the envelope of holomorphy $\tilde{\Omega}$ of Ω which is certainly still G-invariant is schlicht (see [22] or [37]). $\square$

2.2. **Some new results about the invariant domains.** We now prove Theorem 1.2 stated as follows again.

Theorem 2.4. *Let the compact Lie group G act linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. If $0 \in \Omega$ is a G-invariant pseudoconvex domain in $\mathbb{C}^n$, then Ω is orbit convex.*

Proof. We may assume $G \subset U(n)$, since G is compact and acts linearly on $\mathbb{C}^n$.

By Proposition 2.3, Ω is a Runge domain. For each $z_0 \in \Omega$ and $v \in i\mathfrak{g}$ with $exp(v) \cdot z_0 \in \Omega$, we need to prove $exp(tv) \cdot z_0 \in \Omega$ for all $t \in [0, 1]$.

Denote $D := G \cdot \{z_0, exp(v) \cdot z_0\}$ and

$$\widehat{D} := \{z \in \Omega : |g(z)| \leq \sup_{w \in D} |g(w)|, \forall g \in \mathcal{O}(\Omega)\}.$$

Since Ω is a domain of holomorphy, it follows that $\widehat{D}$ is compact.

We can find an open subset $U \Subset \Omega$ such that $\widehat{D} \subset U$.

For any $z \in bU$, there exists a $g \in \mathcal{O}(\Omega)$, such that

$$\sup_{w \in \widehat{D}} |g(w)| < 1$$

and $|g(z)| = 4$. We can approximate g by an entire function $\tilde{g}$ to get

$$\sup_{w \in \widehat{D}} |\tilde{g}(w)| < 2$$

and $|g(z)| > 3$. By the compactness of bU, there exist

$$g_1, \cdots, g_m \in \mathcal{O}(\mathbb{C}^n)$$

such that

$$\sup_{w \in \widehat{D}, 1 \leq j \leq m} |g_j(w)| < 2$$

and

$$\max_{1 \leq j \leq m} |g_j(z)| > 3$$

for any $z \in bU$.

Let $\varphi(z) = \max_{1 \leq j \leq m} |g_j(z)|$ and $W = \{z \in U : \varphi(z) < 5/2\}$. Denote

$$\psi(z) = \begin{cases} \varphi(z) - 2, & z \in W, \\ max\{\varphi(z) - 2, 1/2\}, & z \in \mathbb{C}^n \backslash W. \end{cases}$$

It's easy to see that ψ is a continuous plurisubharmonic function on $\mathbb{C}^n$, and $\widehat{A} \subset \{z \in \mathbb{C}^n : \psi(z) < 0\} \subset \Omega$.

Define $\tilde{\psi}(z) := \int_G \psi(k \cdot z) d\mu$. Then $\tilde{\psi}$ is a G-invariant continuous plurisubharmonic function on $\mathbb{C}^n$, and $\widehat{A} \subset \{z \in \mathbb{C}^n : \tilde{\psi}(z) < 0\} \subset \Omega$.

By Proposition 2.2, $\{z \in \mathbb{C}^n : \tilde{\psi}(z) < 0\}$ is orbit convex. Therefore $exp(tv) \cdot z_0 \in \{z \in \mathbb{C}^n : \tilde{\psi}(z) < 0\} \subset \Omega$ for all $t \in [0, 1]$. □

Lemma 2.5 (see [37]). *Let S^1 act linearly on $\mathbb{C}^n$. Let $D \subset \mathbb{C}^n$ be an S^1-invariant orbit connected pseudoconvex domain. Then*

$$D = \{z \in \mathbb{C}^* \cdot D : s(z) < 1, t(z) < 1\},$$

where

$$s(z) = \inf_{t \cdot z \in D} |t|, \qquad t(z) = \inf_{t \cdot z \in D} \frac{1}{|t|}$$

are plurisubharmonic on $\mathbb{C}^* \cdot D$.

Now we prove the following consequence (Theorem 2.6) of Theorem 2.4, which will be needed for the proof of the extension property of the Bergman kernel functions.

Theorem 2.6. *Let the compact Lie group G act linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. If $0 \in \Omega$ is a G-invariant bounded pseudoconvex domain in $\mathbb{C}^n$, then $U \cdot \Omega \Supset \Omega$ for any open subset U in $G^{\mathbb{C}}$ which contains G.*

Proof. We may assume $U = U^{-1}$, where $U^{-1} = \{g^{-1} : g \in U\}$. Otherwise, we may take $U \cap U^{-1}$ instead of U.

We only need to show $b\Omega \subset U \cdot \Omega$. If the conclusion is not true, then there exists $z_0 \in b\Omega$ such that $U \cdot z_0 \cap \Omega = \emptyset$.

Since $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$, it follows from Fact 2.1 that there exists $v \in \mathfrak{g}$ such that $exp(iv) \cdot z_0 \in \Omega$, where $\mathfrak{g}$ is the Lie algebra of G.

Since $\mathfrak{g} \cong \mathbb{R}^{dim\ K}$, there exists a Euclidean metric on it.

As the set $A = \{u \in \mathfrak{g} : \overline{exp(\mathbb{R} \cdot u)} = exp(\mathbb{R} \cdot u) \cong S^1\}$ is dense in $\mathfrak{g}$, we may assume $v \in A$. Therefore $G_1 = exp(\mathbb{R} \cdot v) \cong S^1$.

Let $\rho : S^1 \to G_1$ be a Lie group isomorphism; then it induces an S^1 action on $\mathbb{C}^n$ from the action of G_1. Furthermore, one has $G_1^{\mathbb{C}} \cdot \Omega = \mathbb{C}^* \cdot \Omega \ni z_0$. It follows from Theorem 2.4 that Ω is orbit convex with respect to $G^{\mathbb{C}}$. Hence Ω is orbit convex with respect to $\mathbb{C}^*$ and it is also orbit connected.

By Lemma 2.5, we have

$$\Omega = \{z \in \mathbb{C}^* \cdot \Omega : s(z) < 1, t(z) < 1\},$$

where

$$s(z) = \inf_{t \cdot z \in \Omega} |t|, \qquad t(z) = \inf_{t \cdot z \in \Omega} \frac{1}{|t|}$$

are plurisubharmonic on $\mathbb{C}^* \cdot \Omega$.

Given an S^1-invariant plurisubharmonic function φ on $\mathbb{C}^* \cdot \Omega$, denote $\psi(x, y) = \varphi(e^{x+iy} \cdot z_0)$. Then ψ is subharmonic on $\mathbb{C}$, and $\psi(x, y)$ is independent of y, hence $\psi(x, 0)$ is convex for $x \in \mathbb{R}$.

Notice that $s(z)$ and $t(z)$ are S^1-invariant plurisubharmonic, so $h_1(x) = s(e^x \cdot z_0)$ and $h_2(x) = t(e^x \cdot z_0)$ are convex for $x \in \mathbb{R}$.

Since $exp(iv) \cdot z_0 \in \Omega$, there exists a $t_0 \in \mathbb{R}$ (we may assume $t_0 > 0$), such that $exp(iv) \cdot z_0 = e^{t_0} \cdot z_0$.

For $x \in [0, t_0]$, one may write $x = (1 - \frac{x}{t_0}) \times 0 + \frac{x}{t_0} \times t_0$; then

$$h_1(x) \le (1 - \frac{x}{t_0}) h_1(0) + \frac{x}{t_0} h_1(t_0)$$

and

$$h_2(x) \le (1 - \frac{x}{t_0}) h_2(0) + \frac{x}{t_0} h_2(t_0).$$

As $h_1(t_0) < 1$, $h_2(t_0) < 1$, $h_1(0) \le 1$ and $h_2(0) \le 1$, we obtain that both $h_1(x)$ and $h_2(x) < 1$ for $x \in (0, t_0]$. Therefore, $exp(ixv) \cdot z_0 \in \Omega$ for any $x \in (0, 1]$.

However, it's clear that $\{exp(ixv) : x \in (0, 1]\} \cap U \neq \emptyset$. This contradicts the assumption $U \cdot z_0 \cap \Omega = \emptyset$. The theorem thus follows. □

2.3. **A result about transversely acting domains.** We adopt the notion introduced in [1] here. Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary.

Definition 2.3 (see [1]). Let $G \subseteq Aut(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$ be a Lie subgroup of $Aut(\Omega)$ with respect to the compact-open topology. For each $z_0 \in b\Omega$, denote $T^h_{z_0} b\Omega := T_{z_0} b\Omega \cap J T_{z_0} b\Omega$, where J is the multiplication by $\sqrt{-1}$.

We will say that G acts transversely on Ω if for each $z_0 \in b\Omega$ the image of the tangent map $(\Psi_{z_0})_* : T_e G \to T_{z_0} b\Omega$ associated to the map $\Psi_{z_0} : G \to b\Omega, g \mapsto g(z_0)$ is not contained in $T^h_{z_0} b\Omega$.

Ω is said to have transverse symmetries if Ω admits a Lie group of automorphisms acting transversely on Ω.

Theorem 2.7. *Let G be a compact Lie group, which acts linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. If $0 \in \Omega$ is a G-invariant bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, then G acts transversely on Ω.*

Proof. Let $\Phi_{z_0} : G^{\mathbb{C}} \to \mathbb{C}^n, g \mapsto g(z_0)$. We need to prove that for any $z_0 \in b\Omega$, there is $u = u(z_0) \in \mathfrak{g}$, such that $(\Phi_{z_0})_*(iu) = J\ (\Phi_{z_0})_*(u)$ is not contained in $T_{z_0}b\Omega$.

Given $\eta \in (\frac{1}{2}, 1)$, for any $z_0 \in b\Omega$, there is a neighborhood U of z_0 and a smooth defining function ρ of Ω, such that $-(-\rho)^\eta$ is a strictly plurisubharmonic function on $\Omega \cap U$ (see [17]).

Since $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$, it follows from Fact 2.1 that there exists a $v \in i\mathfrak{g}$ such that $exp(v) \cdot z_0 \in \Omega$. We may suppose $v \in iA$, where A is the same as that in the proof of Theorem 2.6. Then we have $exp(tv) \cdot z_0 \in \Omega$ for any $t \in (0, 1]$.

There is a positive constant $r \leq 1$ sufficiently small such that $exp(\xi v) \cdot z_0 \in U$ for any $\xi \in \Delta_r := \{\zeta \in \mathbb{C} : |\zeta| < r\}$.

Since $exp(\xi v) \cdot z_0 \in \Omega$ for any $\xi \in \{\zeta \in \Delta_r : Re\ \zeta > 0\}$, then $f(\xi) := -(-\rho(exp(\xi v) \cdot z_0))^\eta$ is a plurisubharmonic function on $\{\zeta \in \Delta_r : Re\ \zeta > 0\}$.

As $f(\xi) < 0$ on $\{\zeta \in \Delta_r : Re\ \zeta > 0\}$, it follows from the Hopf Lemma that there exists a positive constant c, such that

$$\overline{lim}_{t \to 0^+} \frac{f(t)}{t} \leq -c, \quad \text{i.e. } f(t) \leq -ct \quad \text{for } 0 < t \ll 1. \tag{1}$$

Note that $f(t) = -(-\rho(exp(tv) \cdot z_0))^\eta$. Combining the above inequality, we have

$$g(t) := -\rho(exp(tv) \cdot z_0) \geq c^{\frac{1}{\eta}} t^{\frac{1}{\eta}}.$$

Note that g is smooth and $g(0) = 0$. If $g'(0) = 0$, then $g(t) = O(|t|^2)$. This is impossible, since it is easy to see that $\lim_{t \to 0^+} \frac{|t|^2}{t^{\frac{1}{\eta}}} = 0$ for $\eta \in (\frac{1}{2}, 1)$.

Therefore, $g'(0) \neq 0$, i.e., $(\Phi_{z_0})_*(v)$ is not contained in $T_{z_0}b\Omega$. Hence, we have thus proved the theorem. □

Remark 2.1. It is proved by D. Barrett [1] that if Ω is a smooth bounded domain in $\mathbb{C}^n$ which has transverse symmetries, then Ω satisfies condition R, i.e., the Bergman projection P maps $C^\infty(\overline{\Omega})$ continuously into $C^\infty(\overline{\Omega})$.

The above theorem thus gives another example of domains satisfying condition R, although we will not use it in the rest of the paper.

2.4. **Lemmas about the topology of invariant domains.** Denote by $X^{\mathbb{C}}$ the G-complexification of the complex G-space X. There is a topological relationship between X and $X^{\mathbb{C}}$ as follows.

Lemma 2.8 (see [24])**.** *Let G be a compact Lie group and let X be a Stein G-manifold. Then the manifolds X and $X^{\mathbb{C}}$ are G-homotopically equivalent.*

Lemma 2.9. *Let the compact Lie group G act linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. If $0 \in \Omega$ is a G-invariant pseudoconvex domain in $\mathbb{C}^n$, then Ω is simply connected.*

Proof. It follows from Theorem 2.4 that Ω is orbit convex.

By Extension Lemma 2.1, we know that $G^{\mathbb{C}} \cdot \Omega$ is the G-complexification of the G-space Ω. By Fact 2.1, we have $G^{\mathbb{C}} \cdot \Omega = \mathbb{C}^n$. It follows from Lemma 2.8 that Ω and $\mathbb{C}^n$ are G-homotopically equivalent. Thus Ω is simply connected. □

By the proof of the above lemma, we know that Ω is actually a cell.

3. Some preliminary results for the proof of Theorem 1.1

We give some preliminary results for the proof of Theorem 1.1 in this section.

3.1. **A useful lemma.** For $z \in \mathbb{C}^n$, set $|z| = (\sum_{i=1}^n |z_i|^2)^{\frac{1}{2}}$. Set $||k|| = \underset{|z|=1}{sup}|kz|$ for any $k \in M(n,n)$ which is the set of complex $n \times n$ matrices. Denote by k' the transpose matrix of k and by $\overline{k}$ the complex conjugation of k.

It is easy to see that $||k|| = ||k'||$, $max\{|k_{ij}|\} \leq ||k||$, and $||k_1k_2|| \leq ||k_1|| \cdot ||k_2||$.

Lemma 3.1. *Let G be a closed subgroup of the unitary group $U(n)$ and $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. Then there is a constant $a > 0$ such that for any z with $|z| = 1$ and $r \gg 1$, there is $k \in G^{\mathbb{C}}$ which depends on z and r, satisfying $|k \cdot z| < 1/r$, $||k^{-1}|| \leq r^a$ and $||\bar{k}'^{-1}|| \leq r^a$.*

Proof. Since $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$, then it follows from Fact 2.1. that there exists $k \in G^{\mathbb{C}}$ satisfying $|kz| < e^{-1}$ for any z with $|z| = 1$.

By the continuity, there is a neighborhood U_z of z in $S^{2n-1} = \{w \in \mathbb{C}^n; |w| = 1\}$, such that for any $w \in U_z$, $|kw| < e^{-1}$.

By the compactness of S^{2n-1}, there is a constant $c > 0$ such that for any $z \in S^{2n-1}$, there is $k \in G^{\mathbb{C}}$ satisfying $|k \cdot z| < e^{-1}$, $||k^{-1}|| < c$ and $||\bar{k}'^{-1}|| < c$.

For any $r \gg 1$, let $s = [log\, r] + 1$. For any $z \in S^{2n-1}$, there are $k_1, \cdots, k_s \in G^{\mathbb{C}}$, such that $|k_1k_2 \cdots k_s \cdot z| = |k_1 \cdot \frac{k_2 \cdots k_s \cdot z}{|k_2 \cdots k_s \cdot z|}||k_2 \cdots k_s \cdot z| < e^{-1}|k_2 \cdots k_s \cdot z| < e^{-s} < 1/r$, $||k_i^{-1}|| < c$ and $||\bar{k}_i'^{-1}|| < c$ for $i = 1, \cdots, s$.

Let $k = k_1k_2 \cdots k_s$; then $|kz| < 1/r$, $||k^{-1}|| < c^s \leq cc^{log\, r} = cr^{log\, c}$ and $||\bar{k}'^{-1}|| < c^s \leq cr^{log\, c}$.

Denote $a = log\, c + 1$. If $r > c$, then $||k^{-1}|| \leq r^a$ and $||\bar{k}'^{-1}|| \leq r^a$. □

3.2. **Some results about the Bergman kernels of invariant domains.** We recall some basic definitions and facts about the Bergman projections and the Bergman kennel functions.

The Bergman projection P associated to a bounded domain Ω contained in $\mathbb{C}^n$ is the orthogonal projection of $L^2(\Omega)$ onto its closed subspace $H(\Omega)$ consisting of L^2 holomorphic functions.

The Bergman kernel is given by $K(z,w) = \sum_{i=1}^{\infty} f_i(z)\overline{f_i(w)}$, where $\{f_1, f_2, f_3, \ldots\}$ is a complete orthonormal basis of $H(\Omega)$.

The connection between the Bergman projection P and the Bergman kernel function $K(z,w)$ is

$$P(\phi)(z) = \int_{\Omega} K(z,w)\phi(w)dV_w$$

for all ϕ in $L^2(\Omega)$.

The Bergman kernel $K(z,w)$ is holomorphic in z and antiholomorphic in w, and $K(z,w) = \overline{K(w,z)}$.

In the proof of the next lemma, we modify the idea of the proof of Lemma B in [6] so that it suits for a more general case of group actions in the present paper.

Lemma 3.2. *Let G be a compact Lie group which acts linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. Suppose that $0 \in \Omega$ is a bounded G-invariant domain in $\mathbb{C}^n$. Then for any $z \in \Omega, w \in \Omega$ and $k \in G^{\mathbb{C}}$, such that $kz \in \Omega, \bar{k}'^{-1}w \in \Omega$, we have $K(z,w) = K(kz, \bar{k}'^{-1}w)$.*

Proof. For any fixed $k \in G$, the mapping Φ defined via $\Phi(z) = kz$ is an automorphism of the domain Ω.

By the transformation formula, the Bergman kernel function $K(z, w)$ satisfies the identity

$$Det[\Phi'(z)]K(\Phi(z), \Phi(w))Det[\overline{\Phi'(w)}] = K(z, w).$$

Since $k = \bar{k}'^{-1}$ for $k \in G \subset U(n)$, then the above formula yields

$$K(z, w) = K(kz, \bar{k}'^{-1}w).$$

By Proposition 2.3, the envelope of holomorphy $\tilde{\Omega}$ of Ω is schlicht and G-invariant.

The Bergman kernel function $K(z, w)$ can be extended holomorphically with respect to $z \in \tilde{\Omega}$ and antiholomorphically with respect to $w \in \tilde{\Omega}$.

We still write $K(z, w)$ for the extension in the following. It's easy to see that

$$K(z, w) = K(kz, \bar{k}'^{-1}w) \tag{2}$$

holds for any $z \in \tilde{\Omega}, w \in \tilde{\Omega}$ and $k \in G$.

Given any $z \in \Omega, w \in \Omega$ and $k_0 \in G^{\mathbb{C}}$, such that $k_0 z \in \Omega, \bar{k}_0'^{-1}w \in \Omega$. By the first Cartan decomposition theorem, we may write $k_0 = k_1 exp(v)$, where $k_1 \in G$ and $v \in i\mathfrak{g}$.

By Theorem 2.4, we have $\tilde{\Omega}$ is orbit convex. Then there is $\varepsilon > 0$ such that $k_1 exp(\xi v)z \in \tilde{\Omega}$ and $\overline{k_1 exp(\xi v)}'^{-1} w \in \tilde{\Omega}$ for all $\xi = x + iy \in E := (-\varepsilon, 1 + \varepsilon) \times \mathbb{R}$.

Denote $g(\xi) = K(k_1 exp(\xi v)z, \overline{k_1 exp(\xi v)}'^{-1} w)$; it's clear that g is holomorphic on E. Note that $k_1 exp(iyv) \in G$.

It follows from (2) that $g(iy) = K(z, w)$, therefore $g(\xi) \equiv K(z, w)$ for $\xi \in E$.

Choosing $\xi = 1$, we have $g(1) = K(k_0 z, \bar{k}_0'^{-1}w) = K(z, w)$. The lemma thus follows. $\square$

Now we prove the following extension property of the Bergman kernel functions.

Theorem 3.3. *Let G be a compact Lie group which acts linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. Suppose that $0 \in \Omega$ is a bounded G-invariant domain in $\mathbb{C}^n$. Let $K(z, w)$ be the Bergman kernel function of Ω. For each compact subset D of Ω, there is an open set $G = G(D)$ depending on D and containing $\overline{\Omega}$ such that $K(z, w)$ extends holomorphically with respect to z in G for all $w \in D$.*

Proof. For any compact subset $D \subset \Omega$, we can find a small open neighborhood U of G in $G^{\mathbb{C}}$, such that $\bar{k}'^{-1} \cdot D \subset \Omega$ for all $k \in U$.

By Proposition 2.3, the envelope of holomorphy $\tilde{\Omega}$ of Ω is schlicht and G-invariant.

The Bergman kernel function $K(z, w)$ can be extended holomorphically with respect to $z \in \tilde{\Omega}$ and antiholomorphically with respect to $w \in \tilde{\Omega}$.

In this paragraph, we fix $w \in D$. It follows from Lemma 3.2 that $K(z, w) = K(kz, \bar{k}'^{-1}w)$ for $z \in \Omega$ and $k \in U$ such that $kz \in \Omega$. The right hand side of the above equality is well defined on $k^{-1} \cdot \tilde{\Omega}$, therefore we may extend $K(z, w)$ to $k^{-1} \cdot \tilde{\Omega}$ via $K(z, w) = K(kz, \bar{k}'^{-1}w)$.

We can extend $K(z, w)$ holomorphically with respect to z for $z \in U^{-1} \cdot \tilde{\Omega}$ and $w \in D$.

If $k_1 \in U,\ k_2 \in U,\ z_1 \in \tilde{\Omega},\ z_2 \in \tilde{\Omega}$, such that $z = k_1^{-1}z_1 = k_2^{-1}z_2$, then $z_1 = k_1 k_2^{-1} z_2$ and $K(k_1 z, \bar{k}_1'^{-1}w) = K(k_1 k_2^{-1} z_2, \bar{k}_1'^{-1}w) = K(z_2, \bar{k}_2'^{-1}w) = K(k_2 z, \bar{k}_2'^{-1}w)$. The second identity follows from Lemma 3.2.

It follows from Theorem 2.6 that $U^{-1} \cdot \tilde{\Omega} \Supset \tilde{\Omega} \supset \Omega$. Therefore, we can extend $K(z, w)$ holomorphically with respect to z to a neighborhood $G := U^{-1} \cdot \tilde{\Omega}$ of $\overline{\Omega}$ for all $w \in D$.

The theorem thus follows. $\square$

For multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, where α_j is a nonnegative integer, we set $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We shall use the shorthand notation, $K^{\bar{\alpha}}(z, w) := \frac{\partial^\alpha K}{\partial \bar{w}^\alpha}(z, w)$ for multi-indices α.

Let ϕ belong to $\mathcal{C}_0^\infty(B_1)$ which satisfies $\phi(z) = \phi(|z|)$ and $\int_{B_1} \phi = \int_{B_1} \phi_\epsilon = 1$, where $\phi_\epsilon(w) := \epsilon^{-2n}\phi(w/\epsilon)$.

Theorem 3.4. *Let G be a closed subgroup of $U(n)$ with $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. Let $0 \in \Omega$ be a bounded G-invariant domain in $\mathbb{C}^n$. Suppose $K(z, w)$ is the Bergman kernel function associated to Ω. Then $K^{\bar{\alpha}}(z, 0)$ is a polynomial with $\deg K^{\bar{\alpha}}(z, 0) \leq a|\alpha|$, where a is the same as in Lemma* 3.1.

For any multi-index β, there are constants $c_{\beta,\alpha}$ such that

$$z^\beta = \sum_{|\alpha| \leq a|\beta|} c_{\beta,\alpha} K^{\bar{\alpha}}(z, 0).$$

Proof. Without loss of generality, we may assume that $\bar{B}_1 \subset \Omega$.

Since $G \subset U(n)$, then $K(kz, kw) = K(z, w)$ for any $k \in G$ and $z, w \in \Omega$. Therefore we have $K(kz, w) = K(z, k^{-1}w) = K(z, \bar{k}'w)$.

Set $k = (k_{ij})$. Hence

$$\frac{\partial K}{\partial \bar{w}_{i_1}}(kz, w) = \sum_{j_1=1}^{n} k_{i_1 j_1} \frac{\partial K}{\partial \bar{w}_{j_1}}(z, \bar{k}'w),$$

and

$$\frac{\partial^2 K}{\partial \bar{w}_{i_1} \partial \bar{w}_{i_2}}(kz, w) = \sum_{j_1, j_2=1}^{n} k_{i_1 j_1} k_{i_2 j_2} \frac{\partial^2 K}{\partial \bar{w}_{j_1} \partial \bar{w}_{j_2}}(z, \bar{k}'w).$$

Inductively, we have

$$\begin{aligned}&\frac{\partial^m K}{\partial \bar{w}_{i_1} \partial \bar{w}_{i_2} \cdots \partial \bar{w}_{i_m}}(kz, w) \\ &\quad = \sum_{j_1, j_2, \cdots, j_m=1}^{n} k_{i_1 j_1} k_{i_2 j_2} \cdots k_{i_m j_m} \frac{\partial^m K}{\partial \bar{w}_{j_1} \partial \bar{w}_{j_2} \cdots \partial \bar{w}_{j_m}}(z, \bar{k}'w).\end{aligned}$$

In particular, we have

$$\begin{aligned}&\frac{\partial^m K}{\partial \bar{w}_{i_1} \partial \bar{w}_{i_2} \cdots \partial \bar{w}_{i_m}}(kz, 0) \\ &\quad = \sum_{j_1, j_2, \cdots, j_m=1}^{n} k_{i_1 j_1} k_{i_2 j_2} \cdots k_{i_m j_m} \frac{\partial^m K}{\partial \bar{w}_{j_1} \partial \bar{w}_{j_2} \cdots \partial \bar{w}_{j_m}}(z, 0).\end{aligned} \tag{3}$$

We claim that $\frac{\partial^m K}{\partial \bar{w}_{i_1} \partial \bar{w}_{i_2} \cdots \partial \bar{w}_{i_m}}(z, 0)$ can extend holomorphically to $\mathbb{C}^n$, and the equation (3) is hold for any $z \in \mathbb{C}^n$ and $k \in G^{\mathbb{C}}$.

We first restrict $\frac{\partial^m K}{\partial \bar{w}_{i_1} \partial \bar{w}_{i_2} \cdots \partial \bar{w}_{i_m}}(z, 0)$ to B_1.

Since $G \subset U(n)$ and B_1 is orbit convex, then $b_z(B_1) := \{k \in G^{\mathbb{C}} : kz \in B_1\}$ is G-connected.

For fixed $z \in B_1$, both sides of equation (3) are well defined on $b_z(B_1)$, and holomorphic with respect to $k \in b_z(B_1)$.

As they are equal on G and $b_z^{-1}(B_1)$ is G-connected, it follows from the uniqueness theorem that equation (3) also holds for $k \in b_z^{-1}(B_1)$.

For any $z_1, z_2 \in B_1$ and $k_1, k_2 \in G^{\mathbb{C}}$ such that $k_1 z_1 = k_2 z_2$, we set $k_1 = (k_{ij}^1)$, $k_2 = (k_{ij}^2)$, $k_1^{-1} = (k_{ij}^{-1})$. Then we have $z_1 = (k_1^{-1}k_2)z_2$, $k_1^{-1}k_2 \in b_{z_2}(B_1)$, and

$$
\begin{aligned}
&\sum_{j_1,\cdots,j_m} k^1_{i_1j_1}\cdots k^1_{i_mj_m}\frac{\partial^m K}{\partial \bar{w}_{j_1}\cdots\partial \bar{w}_{j_m}}(z_1,0)\\
&=\sum_{j_1,\cdots,j_m} k^1_{i_1j_1}\cdots k^1_{i_mj_m}\frac{\partial^m K}{\partial \bar{w}_{j_1}\cdots\partial \bar{w}_{j_m}}((k_1^{-1}k_2)z_2,0)\\
&=\sum_{\substack{j_1,\cdots,j_m\\ l_1,\cdots,l_m}} k^1_{i_1j_1}\cdots k^1_{i_mj_m}(k_1^{-1}k_2)_{j_1l_1}\cdots(k_1^{-1}k_2)_{j_ml_m}\frac{\partial^m K}{\partial \bar{w}_{l_1}\cdots\partial \bar{w}_{l_m}}(z_2,0)\\
&=\sum_{\substack{j_1,\cdots,j_m\\ l_1,\cdots,l_m\\ s_1,\cdots,s_m}} k^1_{i_1j_1}\cdots k^1_{i_mj_m}k^{-1}_{j_1s_1}k^2_{s_1l_1}\cdots k^{-1}_{j_ms_m}k^2_{s_ml_m}\frac{\partial^m K}{\partial \bar{w}_{l_1}\cdots\partial \bar{w}_{l_m}}(z_2,0)\\
&=\sum_{l_1,\cdots,l_m} k^2_{i_1l_1}\cdots k^2_{i_ml_m}\frac{\partial^m K}{\partial \bar{w}_{l_1}\cdots\partial \bar{w}_{l_m}}(z_2,0).
\end{aligned}
\tag{4}
$$

Therefore, we can extend $\frac{\partial^m K}{\partial \bar{w}_{i_1}\partial \bar{w}_{i_2}\cdots\partial \bar{w}_{i_m}}(z,0)|_{B_1}$ holomorphically to $G^{\mathbb{C}} \cdot B_1 = \mathbb{C}^n$ via equation (3) by defining

$$
\begin{aligned}
&\frac{\partial^m K}{\partial \bar{w}_{i_1}\partial \bar{w}_{i_2}\cdots\partial \bar{w}_{i_m}}(kz,0)\\
&:=\sum_{j_1,j_2,\cdots,j_m} k_{i_1j_1}k_{i_2j_2}\cdots k_{i_mj_m}\frac{\partial^m K}{\partial \bar{w}_{j_1}\partial \bar{w}_{j_2}\cdots\partial \bar{w}_{j_m}}(z,0).
\end{aligned}
$$

It follows from (4) that this is well defined. We get that $\frac{\partial^m K}{\partial \bar{w}_{i_1}\partial \bar{w}_{i_2}\cdots\partial \bar{w}_{i_m}}(z,0)$ can extend holomorphically to $\mathbb{C}^n$. We use the same notation for the extension.

Since Ω is connected, it follows that the extension function is equal to the original function on Ω. We have thus finished the proof of the claim.

Hence

$$
\begin{aligned}
&\frac{\partial^m K}{\partial \bar{w}_{i_1}\partial \bar{w}_{i_2}\cdots\partial \bar{w}_{i_m}}(z,0)\\
&=\sum_{j_1,j_2,\cdots,j_m} k_{i_1j_1}k_{i_2j_2}\cdots k_{i_mj_m}\frac{\partial^m K}{\partial \bar{w}_{j_1}\partial \bar{w}_{j_2}\cdots\partial \bar{w}_{j_m}}(k^{-1}z,0),
\end{aligned}
\tag{5}
$$

for any $z \in \mathbb{C}^n, k \in G^{\mathbb{C}}$.

It follows from Lemma 3.1 that there is a constant a such that for any z with $|z| = r$ and any $r \gg 1$, there exists $k \in G^{\mathbb{C}}$ satisfying $|k^{-1}z| < 1$ and $||k|| \leq r^a$.

Denote

$$
c_m = sup\{|\frac{\partial^m K}{\partial \bar{w}_{j_1}\partial \bar{w}_{j_2}\cdots\partial \bar{w}_{j_m}}(z,0)|; |z| \leq 1, 1 \leq j_1,\cdots,j_m \leq n\}.
$$

It follows from (5) that

$$\begin{aligned}
&|\frac{\partial^m K}{\partial \bar{w}_{i_1}\partial \bar{w}_{i_2}\cdots\partial \bar{w}_{i_m}}(z,0)| \\
&= |\sum_{j_1,j_2,\cdots,j_m} k_{i_1j_1}k_{i_2j_2}\cdots k_{i_mj_m}\frac{\partial^m K}{\partial \bar{w}_{j_1}\partial \bar{w}_{j_2}\cdots\partial \bar{w}_{j_m}}(k^{-1}z,0)| \\
&\le c_m \sum_{j_1,j_2,\cdots,j_m} |k_{i_1j_1}k_{i_2j_2}\cdots k_{i_mj_m}| \\
&\le c_m n^m r^{am} \\
&= c_m n^m |z|^{am}.
\end{aligned}$$

Since $\frac{\partial^m K}{\partial \bar{w}_{i_1}\partial \bar{w}_{i_2}\cdots\partial \bar{w}_{i_m}}(z,0)$ is an entire function, then $\frac{\partial^m K}{\partial \bar{w}_{i_1}\partial \bar{w}_{i_2}\cdots\partial \bar{w}_{i_m}}(z,0)$ is a polynomial with degree $\le am$.

As $K^{\bar{\alpha}}(z,w)$ is antiholomorphic with respect to w, it follows from the mean value property that

$$K^{\bar{\alpha}}(z,0) = \int_\Omega K^{\bar{\alpha}}(z,w)\phi_\epsilon(w)dV_w = \int_\Omega K(z,w)(-1)^{|\alpha|}\frac{\partial^\alpha}{\partial \bar{w}^\alpha}\phi_\epsilon(w)dV_w, \tag{6}$$

and for any $h \in H(\Omega)$, one has

$$\begin{aligned}
\frac{\partial^\alpha h}{\partial w^\alpha}(0) &= \int_\Omega \frac{\partial^\alpha h}{\partial w^\alpha}(w)\phi_\epsilon(w)dV_w \\
&= \int_\Omega h(w)(-1)^{|\alpha|}\frac{\partial^\alpha}{\partial w^\alpha}\phi_\epsilon(w)dV_w \\
&= \int_\Omega (-1)^{|\alpha|}\frac{\partial^\alpha}{\partial w^\alpha}\phi_\epsilon(w)dV_w \int_\Omega K(w,z)h(z)dV_z \qquad (7)\\
&= \int_\Omega h(z)dV_z \overline{\int_\Omega K(z,w)(-1)^{|\alpha|}\frac{\partial^\alpha}{\partial \bar{w}^\alpha}\phi_\epsilon(w)dV_w} \\
&= \int_\Omega h(w)\overline{K^{\bar{\alpha}}(w,0)}dV_w.
\end{aligned}$$

The last equality follows from (6).

If

$$\sum c_\alpha K^{\bar{\alpha}}(z,0) = 0$$

for finitely many $c_\alpha \neq 0$, then it follows from (7) that $\sum \bar{c}_\alpha \frac{\partial^\alpha h}{\partial z^\alpha}(0) = 0$ for any $h \in H(\Omega)$. Hence $c_\alpha = 0$ for any α, so $\{K^{\bar{\alpha}}(z,0); \alpha \in \mathbb{N}^n\}$ is linearly independent.

Given $h \in H(\Omega)$. If $\int_\Omega K^{\bar{\alpha}}(z,0)\overline{h(z)}dV_z = 0$ for any indices α, then it follows from (7) that $\frac{\partial^\alpha h}{\partial z^\alpha}(0) = 0$ for any indices α. Therefore $h = 0$.

Therefore, the linear combinations of $\{K^{\bar{\alpha}}(z,0); \alpha \in \mathbb{N}^n\}$ are dense in $H(\Omega)$.

Because $deg\, K^{\bar{\alpha}}(z,0) \le a|\alpha|$, it follows from (7) that

$$\int_\Omega K^{\bar{\alpha}}(z,0)\overline{K^{\bar{\beta}}(z,0)}dV_z = 0, \quad \text{if} \quad |\beta| > a|\alpha|. \tag{8}$$

By the Gram-Schmidt process and (8), one can find a complete orthonormal basis $\{f_\alpha : \alpha \in \mathbb{N}^n\}$ with

$$f_\alpha(z) = \sum_{|\alpha|/a \le |\beta| \le |\alpha|} b_{\alpha,\beta} K^{\bar{\beta}}(z,0). \tag{9}$$

Since $z^\beta \in H(\Omega)$, then $z^\beta = \sum_{\alpha\in\mathbb{N}^n} d_{\beta,\alpha} f_\alpha$, with $d_{\beta,\alpha} = \int_\Omega z^\beta \overline{f_\alpha(z)} dV_z$. By (7) and (9), it follows that $d_{\beta,\alpha} = 0$ for $|\alpha| > a|\beta|$. Therefore

$$z^\beta = \sum_{|\alpha|\le a|\beta|} d_{\beta,\alpha} f_\alpha = \sum_{|\alpha|\le a|\beta|} c_{\beta,\alpha} K^{\bar\alpha}(z,0). \qquad \square$$

Corollary 3.5. *For each $0 < \epsilon < 1$ and each multi-index β, there are complex numbers $c_{\beta,\alpha}$ such that*

$$\phi_{\beta,\epsilon}(w) := \sum_{|\alpha|\le a|\beta|} (-1)^{|\alpha|} c_{\beta,\alpha} \frac{\partial^{|\alpha|}\phi_\epsilon}{\partial \bar w^\alpha}(w), P\phi_{\beta,\epsilon}(z) = z^\beta,$$

where a is the same constant as in Lemma 3.1.

Proof. It follows from Theorem 3.4 that

$$z^\beta = \sum_{|\alpha|\le a|\beta|} c_{\beta,\alpha} K^{\bar\alpha}(z,0).$$

Set

$$\phi_{\beta,\epsilon}(z) = \sum_{|\alpha|\le a|\beta|} c_{\beta,\alpha} (-1)^{|\alpha|} \frac{\partial^\alpha}{\partial \bar w^\alpha} \phi_\epsilon(w).$$

It follows from the equality (6) and the above equality that $z^\beta = P(\phi_{\beta,\epsilon})(z)$. $\square$

Remark 3.1. If Ω is a bounded circular domain which contains 0, it is not hard to see that $K^{\bar\alpha}(z,0)$ is an $|\alpha|$-homogenous polynomial and the above theorem and corollary are relatively easy to prove; see S. Bell [6].

The above theorem and corollary will be used to prove Theorem 1.1 (2).

We also need a lemma which is proved in [5] by S. Bell.

Lemma 3.6 (see [5])**.** *Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded domains Ω_1 and Ω_2 contained in $\mathbb{C}^n$. Let P_i denote the Bergman projections associated to Ω_i for $i = 1, 2$, and let $u = Det[f']$. Then the Bergman projection transform formula*

$$P_1(u \cdot \phi \circ f) = u \cdot (P_2\phi) \circ f$$

holds for all ϕ in $L^2(\Omega_2)$.

4. Proof of Theorem 1.1

In this section, we'll give the proof of Theorem 1.1 from the point of view of group actions, combining some new ideas about the group actions and modifying some ideas of S. Bell [6] in order to suit our case. We will give the details of the proof for the sake of completeness.

Proof of Theorem 1.1. We may suppose, without loss of generality, that Ω_1 and Ω_2 both contain $\bar B_1$.

Let $K(z, w)$ be the Bergman kernel function associated to Ω_1.

Now let $\phi_{\alpha,\epsilon}$ be the function associated to Ω_2 as in Corollary 3.5 and let $u = Det[f']$. Then it follows from Lemma 3.6 that

$$u \cdot f^\alpha = u \cdot (z^\alpha \circ f) = P_1(u \cdot (\phi_{\alpha,\epsilon} \circ f)) \tag{10}$$

where P_1 is the Bergman projection associated to Ω_1.

We may rewrite (10) in integral form:

$$u(z)f(z)^\alpha = \int_{\Omega_1} K(z,w)u(w)\phi_{\alpha,\epsilon}(f(w))dV_w \tag{11}$$
$$= \int_{f^{-1}(\overline{B_\epsilon})} K(z,w)u(w)\phi_{\alpha,\epsilon}(f(w))dV_w.$$

Fix $\epsilon \in (0,1)$. Since $Supp(\phi_{\alpha,\epsilon} \circ f) \subset \{z : |f(z)| \leq \epsilon\} = f^{-1}(\overline{B_\epsilon})$ and f is proper, we know that $f^{-1}(\overline{B_\epsilon})$ is compact in Ω_1.

By Theorem 3.3, we can extend $K(z,w)$ holomorphically to an open neighborhood of $\overline{\Omega_1}$ with respect to z for all $w \in f^{-1}(\overline{B_\epsilon})$.

Hence, by (11), we can extend uf^α holomorphically to an open neighborhood of $\overline{\Omega_1}$, which is independent of α.

Because the ring of germs of holomorphic functions is a unique factorization domain, f can be extended holomorphically to an open neighborhood of $\overline{\Omega}_1$.

Actually, when one proves that f extends to a neighborhood of $\overline{\Omega}_1$, one should first take $\alpha = (0, \ldots, 0)$. This gives an extension of u. Hence, for every component f_j of f we have $f_j = \varphi_j/u$, where each of u and φ_j extends.

One can now use the fact that the ring of germs of holomorphic functions at a boundary point is a UFD, and represent each of u and φ_j as a product of irreducible factors. Now, taking any power of f_j one observes that f_j extends near every boundary point.

We have thus finished the proof of Theorem 1.1 (1).

Now we turn to prove Theorem 1.1 (2). In the following, we suppose furthermore that $f^{-1}(0) = \{0\}$.

Let r be a large positive number, it follows from Lemma 3.1 that for any $z_0 \in B_r$, there is $k \in G_1^{\mathbb{C}}$, such that $k \cdot z_0 \in B_1$ and $||\bar{k}'^{-1}|| \leq r^{a_1}$, where a_1 is the constant number associated to G_1 as in Lemma 3.1.

By Lemma 3.2, it is obvious that $K(z,w) = K(kz, \bar{k}'^{-1}w)$ holds for any $z \in \Omega_1$, $w \in \Omega_1$ and $k \in G_1^{\mathbb{C}}$ satisfying $kz \in \Omega_1$ and $\bar{k}'^{-1}w \in \Omega_1$.

Similarly to the proof of Theorem 3.3, we can extend $K(z,w)$ holomorphically with respect to z to $\bigcup_{\{k\in G^{\mathbb{C}}:||\bar{k}'^{-1}||\leq r^{a_1}\}} k^{-1} \cdot B_1$ for all $w \in B_{1/r^{a_1}}$.

As $\bigcup_{\{k\in G^{\mathbb{C}}:||\bar{k}'^{-1}||\leq r^{a_1}\}} k^{-1} \cdot B_1 \supset B_r$, then $K(z,w)$ extends holomorphically as a function of z to B_r whenever $w \in B_{1/r^{a_1}}$.

We shall now prove that $u \cdot f^\alpha$ is a polynomial for each α, including $\alpha = (0, \cdots, 0)$, by showing that the functions $u \cdot f^\alpha$ are entire functions which satisfy an estimate of the form $|u(z)f(z)^\alpha| \leq C|z|^p$ for $|z|$ large enough.

Now, since u is a polynomial and $u \cdot f^\alpha$ is a polynomial for each α, and the ring of polynomial is a unique factorization domain, we conclude that f must be a polynomial mapping following the same argument above for proving the extension of f across the boundary.

Since $f^{-1}(0) = \{0\}$, the nullstellensatz implies that there are holomorphic functions $a_{ij}(z)$ and positive integers k_i such that $z_i^{k_i} = \sum_{j=1}^n a_{ij}(z)f_j(z)$ near $z = 0$.

There are positive constants m and c' such that $|z|^m \leq c'|f(z)|$ for all $z \in U$, where U is a small neighborhood of 0.

Furthermore, since f is proper, it follows that $inf_{z\in\Omega_1\setminus U}|f(z)| > 0$. Therefore, there is a positive constant c, such that $|z|^m \leq c|f(z)|$ for all $z \in \Omega_1$.

Then,

$$Supp(\phi_{\alpha,\epsilon} \circ f) \subset \{z : |f(z)| \leq \epsilon\} \subset \{z : |z| \leq (c\epsilon)^{1/m}\}.$$

Note that $K(z,w)$ extends to $B_r \times B_{1/r^{a_1}}$. If r is a large positive number and ϵ is chosen so that $(c\epsilon)^{1/m} < 1/r^{a_1}$, then it follows from equation (11) that $u \cdot f^\alpha$ extends holomorphically to B_r. Therefore, we conclude that $u \cdot f^\alpha$ is an entire function for each α.

Notice that

$$|\phi_{\alpha,\epsilon}(w)| = |\sum_{|\beta|\le a_2|\alpha|} (-1)^{|\beta|} c_{\alpha,\beta} \frac{\partial^{|\beta|}}{\partial \bar{w}^\beta} \phi_\epsilon(w)| \le A_\alpha \frac{1}{\epsilon^{2n+a_2|\alpha|}}, \tag{12}$$

where a_2 is the constant number associated to G_2 as in Lemma 3.1.

We're left to show that $|u(z)f(z)^\alpha| \le C|z|^p$. Fix a point $z \in \mathbb{C}^n$, with $|z| \gg 1$. Pick ϵ so that $(c\epsilon)^{1/m} = \frac{1}{|z|^{a_1}}$, i.e., $\epsilon = \frac{1}{c|z|^{a_1 m}}$. Note that $Supp(\phi_{\alpha,\epsilon} \circ f) \subset B_{\frac{1}{|z|^{a_1}}}$. Choose $k \in G_1^{\mathbb{C}}$ such that $|kz| < 1$ and $||\bar{k}'^{-1}|| \le |z|^{a_1}$. It follows from identities (11), (12) and Lemma 3.2 that

$$\begin{aligned}
&|u(z)f(z)^\alpha| \\
&= |\int_{\Omega_1} K(kz, \bar{k}'^{-1} w) u(w) (\phi_{\alpha,\epsilon} \circ f)(w) dV_w| \\
&\le C_1 (\underset{\bar{B}_1 \times \bar{B}_1}{Sup} |K(\zeta,\xi)|) ||u \cdot \phi_{\alpha,\epsilon} \circ f||_{L^2(\Omega_1)} \\
&\le C_2 ||\phi_{\alpha,\epsilon}||_{L^2(\Omega_2)} \\
&\le C_3 \frac{1}{\epsilon^{n+a_2|\alpha|}} \\
&\le C_4 |z|^{a_1 m (n + a_2|\alpha|)},
\end{aligned}$$

where $C_1, \cdots, C_4$ are independent of z. We have thus proved the theorem. □

5. Some further results and remarks

Remark 5.1. For any bounded circular domain $0 \in \Omega \subset \mathbb{C}^n$, one may ask whether any proper holomorphic mapping from Ω to Ω, with $f^{-1}(0) = \{0\}$, is a homogenous polynomial. This is not true in general.

For example, let $F(z_1,z_2) = (z_1, z_1 + z_2)$, $\varphi(z_1,z_2) = (z_1, z_2^2)$ and $\Omega = F(\Delta \times \Delta)$; then it is easy to check that Ω is bounded circular. Let $f(z_1,z_2) := F \circ \varphi \circ F^{-1}(z_1,z_2) = (z_1, z_1 + (z_2 - z_1)^2)$, then $f : \Omega \to \Omega$ is a proper holomorphic mapping, and $f^{-1}(0) = \{0\}$. However f is not homogenous.

For two smooth domains in $\mathbb{C}^n$ with $n > 1$, it is interesting to know whether all proper holomorphic mappings are biholomorphic. We extend a theorem proved by S. Bell [5].

Proposition 5.1. *Let G_j be a compact Lie group, acting linearly on $\mathbb{C}^n$ with $n > 1$, and $\mathcal{O}(\mathbb{C}^n)^{G_j} = \mathbb{C}$, for $j = 1, 2$. Let Ω_1 be a G_1-invariant bounded strictly pseudoconvex domain containing 0 with $\mathcal{C}^2$-boundary, and let Ω_2 be a G_2-invariant bounded domain containing 0 with $\mathcal{C}^2$-boundary. Suppose that $f : \Omega_1 \to \Omega_2$ is proper holomorphic. Then f is biholomorphic and extends to be a biholomorphism between larger domains D_1 and D_2 which contain $\overline{\Omega_1}$ and $\overline{\Omega_2}$, respectively. Hence Ω_2 must also be strictly pseudoconvex.*

Proof. The proof is similar to [5]. For the sake of completeness, we give details of the proof.

Let r_2 be a defining function for Ω_2. By Theorem 1.1, f extends holomorphically to a neighborhood of $\overline{\Omega}_1$. Since the normal derivative of f is nonzero by the Hopf lemma, then $r_1 := r_2 \circ f$ is a defining function for Ω_1.

Since Ω_1 is a strictly pseudoconvex domain, then there is $C \gg 1$ such that $\rho = exp(Cr_1) - 1$ is a strictly plurisubharmonic definition function for Ω_1. Hence

$$[\frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}] = J(f)'[(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} exp(Cr_2)) \circ f]\overline{J(f)}$$

is a strictly positive definite Hermitian matrix at $z \in b\Omega_1$, where $J(f)$ is the complex Jacobian of f.

So $Det(Jf)$ cannot vanish on $b\Omega_1$ and $[\frac{\partial^2}{\partial z_i \partial \bar{z}_j} exp(Cr_2)]$ is a strictly positive definite Hermitian matrix for $z \in b\Omega_2$. As $dim\, \Omega_1 > 1$, $Det(J(f))$ cannot vanish on Ω_1.

Hence f is an unbranched covering map, and therefore Ω_2 is also strictly pseudoconvex. By Lemma 2.9, we get Ω_2 is simply connected. Therefore, f is biholomorphic. □

Combining Bell's works [7] with the results in the present paper, we can obtain the following proposition:

Proposition 5.2. *Let G be a compact Lie group, which acts linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. Suppose Ω_1 is a bounded domain in $\mathbb{C}^n$ whose associated Bergman kernel function is a rational function, and suppose Ω_2 is a bounded G-invariant domain in $\mathbb{C}^n$ that contains the origin. Then any proper holomorphic mapping $f : \Omega_1 \to \Omega_2$ must be rational.*

Finally, we give a remark about proper holomorphic correspondences between Ω_1 and Ω_2, where Ω_1, Ω_2 are the same as in Theorem 1.1.

We now recall the definition of holomorphic correspondences (see [31], [3] or [8]).

Definition 5.1 (see [8])**.** Let D_1 and D_2 be two domains in $\mathbb{C}^n$, and let Γ be a subvariety of $D_1 \times D_2$. Let $\pi_j : \Gamma \to D_j$ be the projections for $j = 1, 2$. A set-valued mapping which the variety Γ gives rise to via $f = \pi_2 \pi_1^{-1}$ is called a holomorphic correspondence.

If both π_1 and π_2 are proper, then f is said to be proper. If Γ is an algebraic variety, then f is said to be algebraic.

Using Bell's proof [8], and the above Theorem 3.4 and Corollary 3.5, one may obtain

Proposition 5.3. *Let Ω_1 and Ω_2 be the same as in Theorem 1.1. If $f : \Omega_1 \multimap \Omega_2$ is a proper holomorphic correspondence, and $f^{-1}(0) = \{0\}$, then f is algebraic.*

6. Note added under revision

When the paper was under revision, we noted two recent papers by F. Rong [28] and A. Yamamori [35] posted on ArXiv. They studied the degree of the automorphisms of a bounded quasi-circular domain Ω which preserves 0. F. Rong's paper [28] contains the theorems in A. Yamamori's papers [34, 35]. Here, a quasi-circular domain is just a $(p_1, \cdots, p_n)$-domain with $p_i > 0$ for all i.

One can use the main results in the present paper to give an alternative proof of the main theorem in F. Rong's paper. Furthermore, we can actually obtain a more general theorem on the degree of automorphisms of a G-invariant domain which preserves 0, where the domain has trivial invariant holomorphic functions.

We give the proof of F. Rong's main result by using the method of the present paper.

Theorem 6.1. *There exists a bounded $(m_1, m_2, \cdots, m_n)$-domain $\Omega \subset \mathbb{C}^n$ which contains 0, where $m_i > 0$ for any $i \geq 1$. If $f \in Aut(\Omega)$, $f(0) = 0$, then the degree of f is not bigger than $max\{|\delta| : \delta m^t = \gamma m^t, |\gamma| = |\beta|, \beta m^t = m_i,\ for\ i = 1, \cdots, n\}$.*

Proof. As before, let $K(z, w)$ be the the Bergman kernel function on the domain Ω. Since Ω is an $(m_1, m_2, \cdots, m_n)$-domain, we have

$$K((t^{m_1}z_1, \cdot\cdot\cdot, t^{m_n}z_n), (t^{m_1}w_1, \cdot\cdot\cdot, w^{m_n}z_n)) = K(z, w),$$

for any $z, w \in \Omega$ and $t \in S^1$.

Therefore we can get

$$K((t^{m_1}z_1, \cdot\cdot\cdot, t^{m_n}z_n), w) = K(z, (\overline{t^{m_1}}w_1, \cdot\cdot\cdot, \overline{t^{m_n}}w_n)).$$

Differentiating the above formula with respect to $\bar{w}$, we obtain that

$$K^{\bar{\alpha}}((t^{m_1}z_1, \cdot\cdot\cdot, t^{m_n}z_n), w) = t^{\alpha m^t} K^{\bar{\alpha}}(z, (\overline{t^{m_1}}w_1, \cdot\cdot\cdot, \overline{t^{m_n}}w_n)).$$

Taking $w = 0$ in the above identity, we see that

$$K^{\bar{\alpha}}((t^{m_1}z_1, \cdot\cdot\cdot, t^{m_n}z_n), 0) = t^{\alpha m^t} K^{\bar{\alpha}}(z, 0).$$

For z near 0, we can write $K^{\bar{\alpha}}(z, 0) = \sum_\beta c_\beta z^\beta$.

It follows that $K^{\bar{\alpha}}(z, 0) = \sum_{\beta m^t = \alpha m^t} c_\beta z^\beta$ and $degK^{\bar{\alpha}}(z, 0) \leq max\{|\beta| : \beta m^t = \alpha m^t\}$.

We claim that the set of polynomials $P^N = \{K^{\bar{\alpha}}(z, 0) : \alpha m^t = N\}$ forms a basis for the $\mathbb{C}$-linear space $\langle z^\beta : \beta m^t = N\rangle$.

We have proved that the functions in P^N are linearly independent. Furthermore, the cardinality of P^N is equal to the dimension of $\langle z^\beta : \beta m^t = \alpha m^t\rangle$. Hence, each monomial z^α can be written in the form

$$z^\alpha = \sum_{\beta m^t = \alpha m^t} c_{\alpha,\beta} K^{\bar{\beta}}(z, 0).$$

As $u(z) = detJf(z)$ is a polynomial without zeros, then $u(z)$ is actually a nonzero constant c. Therefore $K(f(z), f(w))|c|^2 = K(z, w)$. It follows that $K(f(z), w)|c|^2 = K(z, f^{-1}(w))$ for $z, w \in \Omega$.

So

$$f_i(z) = \sum_{\beta m^t = m_i} c_{i,\beta} K^{\bar{\beta}}(f(z), 0) = \sum_{\beta m^t = m_i} c_{i,\beta} |c|^{-2} \frac{\partial^\beta}{\partial \bar{w}^\beta} K(z, f^{-1}(w))|_{w=0}.$$

As $\frac{\partial^\beta}{\partial \bar{w}^\beta} K_1(z, f^{-1}(w))|_{w=0} = \sum_{|\gamma| \leq |\beta|} d_\gamma K_1^{\bar{\gamma}}(z, 0)$, where d_γ are constants, we get that $deg\frac{\partial^\beta}{\partial \bar{w}^\beta} K_1(z, f^{-1}(w))|_{w=0} \leq max\{|\delta| : \delta m^t = \gamma m^t, |\gamma| = |\beta|, \beta m^t = m_i$, for $i = 1, \cdots, n\}$. We have thus proved the theorem. □

Actually the same proof yields a more general theorem in the setting of the present paper than the above theorem.

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