CHARACTERISTIC RANDOM SUBGROUPS OF GEOMETRIC GROUPS AND FREE ABELIAN GROUPS OF INFINITE RANK

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ABSTRACT. We show that if G is a non-elementary word hyperbolic group, mapping class group of a hyperbolic surface or the outer automorphism group of a non-abelian free group, then G has $2^{\aleph_0}$ many non-atomic ergodic invariant random subgroups. If G is a non-abelian free group, then G has $2^{\aleph_0}$ many non-atomic G-ergodic characteristic random subgroups. We also provide a complete classification of characteristic random subgroups of free abelian groups of countably infinite rank and elementary p-groups of countably infinite rank.

CONTENTS

1. INTRODUCTION

The goal of this paper is to classify characteristic random subgroups of the free abelian group of infinite rank and of the infinite dimensional torus and use these random subgroups to construct large families of non-atomic invariant random subgroups in some important non-commutative groups of a geometric nature. The topics of ergodicity and weak-mixing are also included in our research. To achieve this goal we will make use of Pontyagin duality, ergodic theory of group actions, geometric group theory and an important result of Adjan [Ad70] on the existence of independent identities (laws) in a free group of rank ≥ 2 (the latter is based on the solution of Burnside's problem). Let us explain this in more detail.

Let G be a locally compact Hausdorff metrizable topological group with a countable basis of open sets. The space $\mathrm{Sub}(G)$ of all closed subgroups of G admits a natural topology (called the Chabauty topology) (see [BP92, Chapter E.1]). With this topology $\mathrm{Sub}(G)$ is a compact metrizable topological space with a countable basis of open sets. One can characterize convergence in this topology as follows.

$H_n \to H$ if and only if two conditions hold. First, if $g_i \in H_{n_i}$ converges to g, then $g \in H$, and second, if $g \in H$, then there are $g_n \in H_n$ such that $g_n \to g$. It is easy to see that the conjugation action of G on $\mathrm{Sub}(G)$ is continuous.

In general, by a *random subgroup* (RS) of G we mean a random variable which takes values in $\mathrm{Sub}(G)$. We will often identify it with its law, which is a Borel probability measure on $\mathrm{Sub}(G)$.

An *invariant random subgroup* (IRS) of G can be interpreted as a Borel probability measure μ on $\mathrm{Sub}(G)$ that is invariant under the conjugation action of G (i.e. the action of the group of inner automorphisms $\mathrm{Inn}(G)$). We also say that a random subgroup $K \leq G$ is an IRS if its law is conjugation-invariant.

For example, if $N \lhd G$ is normal, then the Dirac mass δ_N is an IRS. Also, if $\Gamma \leq G$ is a lattice in a locally compact group G, then there is an IRS μ_Γ of G supported on the conjugacy class of Γ (μ_Γ is the stabilizer of a Haar-random element of G/Γ [AB+11]). Thus IRS's provide a common generalization of normal subgroups and lattices. More generally, if for some $H \leq G$, the normalizer $N_G(H)$ of H in G is a lattice in G, then there is a naturally associated IRS μ_H supported on the conjugacy class of H. To see this, let $gN_G(H) \in G/N_G(H)$ be a random coset with law equal to the Haar measure. Then gHg^{-1} is an invariant random subgroup. If G is countable, then any IRS of this form is purely atomic. $\mu \in \mathrm{IRS}(G)$ is called *non-atomic* if it has no atoms. It is called *ergodic* if the dynamical system $(G, \mathrm{Sub}(G), \mu)$ is ergodic.

There has been a recent increase in the study of IRS's (see for example [AB+12, BGK13] and the references therein). The name itself was coined in [AGV14]. Observe that groups with $2^{\aleph_0}$ many non-atomic ergodic IRS's include non-abelian free groups [Bo15], classical lamplighter groups [BGK13] and the group of finitely-supported permutations of the natural numbers [Ve11]. It is known that branch and weakly branch groups admit at least one non-atomic ergodic IRS [BG00, Gr11]. There are also results in the opposite direction: groups without any non-atomic ergodic IRS's include lattices in simple higher rank Lie groups [SZ94], Higman-Thompson groups [DM], most special linear groups over countable fields [PT], commensurators of irreducible lattices in most higher rank semisimple Lie groups [CP] and irreducible lattices in semisimple property (T) higher rank Lie groups [Cr].

We will denote by $\mathrm{IRS}(G)$ the set of all IRS's of G. This is a weak*-closed convex subset of the simplex $\mathrm{RS}(G) := \mathcal{P}(\mathrm{Sub}(G))$ of all probability measures on $\mathrm{Sub}(G)$. $\mathrm{IRS}(G)$ consists of all $\mathrm{Inn}(G)$-invariant measures in $\mathrm{RS}(G)$, where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G. Another important notion is that of a characteristic random subgroup (abbreviated as CRS). Recall that a *characteristic* subgroup $K \leq G$ is a subgroup which is $\mathrm{Aut}(G)$-invariant. For example, the torsion part of a countable abelian group is characteristic.

Definition 1. We define $\mathrm{CRS}(G)$ to be the closed convex subset of $\mathrm{IRS}(G)$ consisting of $\mathrm{Aut}(G)$-invariant measures, where $\mathrm{Aut}(G)$ is the group of all continuous automorphisms of G. Elements of $\mathrm{CRS}(G)$ are called characteristic random subgroups of G.

If $\Phi \leq \mathrm{Aut}(G)$ is a subgroup, one can consider a set $\mathrm{RS}_\Phi(G)$ of Φ-invariant probability measures on $\mathrm{Sub}(G)$. Such a situation arises for instance when $N \lhd G$ is a normal subgroup and we are interested in the study of random subgroups in N invariant with respect to conjugation by elements of G. In this case $\Phi \leq \mathrm{Aut}(N)$

equals the image of $\mathrm{Inn}(G)$ in $\mathrm{Aut}(N)$. An example of this sort is considered in [BGK13].

The spaces $\mathrm{IRS}(G)$, $\mathrm{CRS}(G)$, $\mathrm{RS}_\Phi(G)$ are all Choquet simplexes ([Ph01, Section 12]). Their extreme points are indecomposable measures, i.e., measures μ that cannot be presented in the form $\mu = t\mu_1 + (1-t)\mu_2$ for $t \in (0,1)$ and $\mu_1 \neq \mu_2$ belonging to the same simplex. Let $\mathrm{CRS}^e(G) \subset \mathrm{CRS}(G)$ denote the subset of extreme points. In other words, a measure $\mu \in \mathrm{CRS}(G)$ is contained in $\mathrm{CRS}^e(G)$ if there does not exist measures $\mu_1 \neq \mu_2 \in \mathrm{CRS}(G)$ and $t \in (0,1)$ such that $\mu = t\mu_1 + (1-t)\mu_2$.

They can be interpreted as ergodic measures in the sense of ergodic theory of group actions, but we have to issue a warning at this point that the group $\mathrm{Aut}(G)$ (with the compact-open topology) may turn out to be not locally compact second countable or not countable discrete (and this will be the case in the situation related to Theorems 1.3 and 1.4) and for non-locally compact groups the relationship of ergodicity and indecomposability is more complicated as was observed for the first time by Kolmogorov [Fo50]. We will denote by $\mathrm{IRS}^e(G)$, $\mathrm{CRS}^e(G)$, etc., the sets of indecomposable measures.

We recall the definition of ergodic action and of weakly mixing action for a general topological group in §2.5.

Once again, we call a measure $\mu \in \mathrm{RS}$, IRS, CRS, etc. non-atomic if it has no atoms. One of the basic questions in this area is: given an interesting group G and subgroups $\Psi \leq \Phi$ in $\mathrm{Aut}(G)$ does G have any Ψ-ergodic (Ψ-weakly mixing, Ψ-mixing, etc.) Φ-invariant random subgroups?

For instance Theorem 1.2 deals with the case when $G = \mathbb{F}_r$ is a free group, $\Phi = \mathrm{Aut}(\mathbb{F}_r)$ is the group of all automorphisms of $\mathbb{F}_r$, and $\Psi = \mathrm{Inn}(\mathbb{F}_r)$ is the group of all inner automorphisms of $\mathbb{F}_r$ which we identify in a natural way with $\mathbb{F}_r$. It is clear that Ψ-ergodic and Ψ-weakly mixing implies Φ-ergodic and Φ-weakly mixing respectively.

Our first result in this paper is:

Theorem 1.1. *If G is either*

- *a non-elementary Gromov hyperbolic group,*
- *the mapping class group of a (possibly punctured by finitely many points) oriented surface of negative Euler characteristic, or*
- *the outer automorphism group of a non-abelian free group,*

then G has $2^{\aleph_0}$ many non-atomic G-weakly mixing IRS's.

This result we deduce from the next theorem.

Theorem 1.2. *Every non-abelian free group $\mathbb{F}_r$ (of finite or countably infinite rank) has $2^{\aleph_0}$ many non-atomic $\mathrm{Inn}(\mathbb{F}_r)$-weakly mixing CRS's.*

Let us see how Theorem 1.2 implies Theorem 1.1:

Proof of Theorem 1.1 from Theorem 1.2. Let G be a countable group and suppose $N \lhd G$ is a normal non-abelian free subgroup. By Theorem 1.2, N has $2^{\aleph_0}$ many non-atomic N-weakly mixing CRS's. However, each of these CRS's is also a G-weakly mixing IRS of G. This is because N is normal in G, so $\mathrm{Inn}(G)$ naturally maps into $\mathrm{Aut}(N)$ and the image of $\mathrm{Inn}(G)$ in $\mathrm{Aut}(N)$ contains $\mathrm{Inn}(N)$ (also see Lemma 2.12(a) below). So it suffices to show that if G is as in Theorem 1.1, then G has a free non-abelian normal subgroup. If G is a non-elementary hyperbolic

group, then this result is due to Delzant [De96]. Otherwise, it is due to Dahmani-Guirardel-Osin [DGO, Theorems 8.1, 8.5]. □

Remark 1. It follows from the proof above that if a group G has trivial center and possesses a characteristic non-abelian free subgroup N, then G has $2^{\aleph_0}$ many non-atomic G-weakly mixing CRS's.

In order to prove Theorem 1.2, we study CRS's of abelian groups. Although the proof of Theorem 1.2 uses only the existence of a non-atomic ergodic CRS in the free elementary p-group (p-prime) $\bigoplus_{\mathbb{N}}(\mathbb{Z}/p\mathbb{Z})$ of infinite rank, we think it is interesting to obtain a complete classification of indecomposable CRS's of the abelian groups $\mathcal{A} = \bigoplus_{\mathbb{N}} \mathbb{Z}$, $\mathcal{A}_n = \bigoplus_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$, $\hat{\mathcal{A}} = \prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z}$ and $\hat{\mathcal{A}}_n = \prod_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ (Theorems 1.3 and 1.4 below).

To explain, we need more notation. Throughout this paper m, n, i, k will denote non-negative integers, $\mathbb{Z}_+ = \{0, 1, \dots\}$ and $\mathbb{N} = \{1, 2, \dots\}$. For $m, n \in \mathbb{Z}_+$, we will write $m|n$ if $m, n \in \mathbb{N}$ and m divides n. Note $m|0$ for any $m \in \mathbb{Z}_+$, but $0 \nmid n$ for $n \in \mathbb{N}$.

Given locally compact abelian groups G, H let $\text{Hom}(G, H)$ denote the set of continuous homomorphisms from G to H. This is an abelian group under pointwise addition. We consider $\text{Hom}(G, H)$ with the compact-open topology ([HR79, 23.34]). Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus and $\hat{G} = \text{Hom}(G, \mathbb{T})$ denote the *Pontryagin dual* of G.

Given a subgroup $H \leq G$, let $\text{Ann}(H) = \{\alpha \in \hat{G} : \alpha(h) = 0 \ \forall h \in H\}$ be its *annihilator* subgroup. Given a subgroup $S \leq \hat{G}$, let $\text{Ker}(S) = \{g \in G : s(g) = 0 \ \forall s \in S\}$ denote its *kernel*.

Let $\mathcal{A} = \bigoplus_{\mathbb{N}} \mathbb{Z}$ be the free abelian group of countable rank. Its Pontryagin dual group $\hat{\mathcal{A}}$ is naturally isomorphic with $\prod_{\mathbb{N}} \mathbb{T} = \mathbb{T}^{\mathbb{N}}$, the infinite dimensional torus: if $x = \{x_i\} \in \mathcal{A}$, $y = \{y_i\} \in \mathbb{T}^{\mathbb{N}}$, then $y(x) := \sum_{i \in \mathbb{N}} x_i y_i \in \mathbb{T}$. For any $n \in \mathbb{Z}_+$, let $n\mathcal{A} \leq \mathcal{A}$ be the subgroup of elements with all coordinates divisible by n. Let $\mathcal{A}_n = \mathcal{A}/n\mathcal{A} \cong \bigoplus_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$. Then for $n \in \mathbb{N}$, the annihilator subgroup $\text{Ann}(n\mathcal{A}) \leq \hat{\mathcal{A}}$ is naturally isomorphic with $\prod_{\mathbb{N}} \mathbb{Z}[1/n]/\mathbb{Z} = (\mathbb{Z}[1/n]/\mathbb{Z})^{\mathbb{N}}$ which is naturally isomorphic with the dual $\hat{\mathcal{A}}_n$ (Lemma 2.9). Observe that the subgroups $n\mathcal{A} \leq \mathcal{A}$ are *characteristic*: they are invariant under all automorphisms of $\mathcal{A}$. Similarly, $\text{Ann}(n\mathcal{A}) = \hat{\mathcal{A}}_n \leq \hat{\mathcal{A}}$ is characteristic. Indeed, these are all the characteristic subgroups (Lemma 2.10).

In the future we will identify $\hat{\mathcal{A}}_n$ with $\text{Ann}(n\mathcal{A}) \leq \hat{\mathcal{A}}$. If $m, n \in \mathbb{N}$ and $m|n$, then $\hat{\mathcal{A}}_m = \text{Ann}(m\mathcal{A}_n) = (\mathbb{Z}[1/m]/\mathbb{Z})^{\mathbb{N}} \leq (\mathbb{Z}[1/n]/\mathbb{Z})^{\mathbb{N}} = \hat{\mathcal{A}}_n$. In the case $n = 0$, $\text{Ann}(0\mathcal{A}) = \hat{\mathcal{A}}$. Thus $\hat{\mathcal{A}}_0 = \hat{\mathcal{A}}$. Also $\mathcal{A}_0 = \mathcal{A}/0\mathcal{A} = \mathcal{A}$ and $\mathcal{A}_1 = \hat{\mathcal{A}}_1$ is a trivial group.

Definition 2. Let $m \in \mathbb{Z}_+$ and F be a finite abelian group. We say that F *is over* m if either $F = 0$ or each non-trivial summand F_1 of F satisfies $mF_1 \neq 0$. Note that, in particular, any F is over 1 and only the trivial group is over 0.

Note that since F is a finite abelian group, the main structure theorem for finite abelian groups implies that F is isomorphic to $\bigoplus_{i=1}^{s} \mathbb{Z}/p_i^{r_i}\mathbb{Z}$, where p_i are primes. In this case F is over m if and only if for all i we have that $p_i^{r_i}$ does not divide m.

Note that when $F = \bigoplus_{i=1}^{s} \mathbb{Z}/p_i^{r_i}\mathbb{Z}$ and $nF = 0$ it follows that $p_i^{r_i}|n$ and thus we have

$$\mathrm{Hom}(F, \hat{\mathcal{A}}_n) \simeq \bigoplus_{i=1}^{s} \mathrm{Hom}(\mathbb{Z}/p_i^{r_i}\mathbb{Z}, \hat{\mathcal{A}}_n) \simeq \bigoplus_{i=1}^{s} \mathrm{Hom}(\mathbb{Z}/p_i^{r_i}\mathbb{Z}, \mathbb{Z}[1/n]/\mathbb{Z})^{\mathbb{N}}$$

$$\simeq \bigoplus_{i=1}^{s} (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^{\mathbb{N}} \simeq \bigoplus_{i=1}^{s} \hat{\mathcal{A}}_{p_i^{r_i}},$$

$$\mathrm{Hom}(\mathcal{A}_n, F) \simeq \bigoplus_{i=1}^{s} \mathrm{Hom}(\mathcal{A}_n, \mathbb{Z}/p_i^{r_i}\mathbb{Z}) \simeq \bigoplus_{i=1}^{s} \mathrm{Hom}(\mathbb{Z}[1/n]/\mathbb{Z}, \mathbb{Z}/p_i^{r_i}\mathbb{Z})^{\mathbb{N}}$$

$$\simeq \bigoplus_{i=1}^{s} (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^{\mathbb{N}} \simeq \bigoplus_{i=1}^{s} \hat{\mathcal{A}}_{p_i^{r_i}},$$

where the isomorphisms are in fact homeomorphisms ([HR79, 23.34]). The Haar measure on $\mathrm{Hom}(F, \hat{\mathcal{A}}_n)$ and $\mathrm{Hom}(\mathcal{A}_n, F)$ equals the product of Haar measures on $\hat{\mathcal{A}}_{p_i^{r_i}}$.

Given a group H, let $[H]$ denote its isomorphism class.

Theorem 1.3. *Let $\lambda \in \mathrm{CRS}^e(\hat{\mathcal{A}}_n)$. Then there is a unique pair $(m, [F])$ with $m \in \mathbb{Z}_+$, F a finite abelian group over m, such that $m|n$, $nF = 0$ and the random subgroup $\mathrm{Ann}(m\mathcal{A}_n) + h(F)$ has law λ, where $h \in \mathrm{Hom}(F, \hat{\mathcal{A}}_n)$ is a random homomorphism with law equal to the Haar measure on the compact group $\mathrm{Hom}(F, \hat{\mathcal{A}}_n)$.*

In the next theorem we give a characterization of $\mathrm{CRS}^e(\mathcal{A}_n)$.

Theorem 1.4. *Let $\lambda \in \mathrm{CRS}^e(\mathcal{A}_n)$. Then there is a unique pair $(m, [F])$ with $m \geq 0$, F a finite abelian group over m, such that $m|n$, $nF = 0$ and the random subgroup $m\mathcal{A}_n \cap \mathrm{Ker}(h)$ has law λ, where $h \in \mathrm{Hom}(\mathcal{A}_n, F)$ is a random homomorphism with law equal to the Haar measure on the compact group $\mathrm{Hom}(\mathcal{A}_n, F)$.*

Observe that in both Theorems 1.3 and 1.4 the set $\mathrm{CRS}^e(G)$ is infinite countable. In the case of groups $G = \hat{\mathcal{A}}$ or $G = \mathcal{A}$ the sets $\mathrm{CRS}^e(G)$ can be parametrized by $m \in \mathbb{Z}_+$ and a tuple of the form $(p_1^{r_1}, \ldots, p_k^{r_k})$ where p_i are primes, $k_i \in \mathbb{N}$, $p_i \leq p_{i+1}$ and $r_i \leq r_{i+1}$ if $p_i = p_{i+1}$ (such tuples parametrize finite abelian groups). For the case of groups $G = \hat{\mathcal{A}}_n$ or $G = \mathcal{A}_n$, $n \geq 2$, the sets $\mathrm{CRS}^e(G)$ can be parametrized with the same tuples with the additional restriction that both m and all $p_i^{r_i}$ must divide n.

We thus have that for all cases the set $\mathrm{CRS}^e(G)$ is countable. Moreover the set $\mathrm{CRS}^e(G)$ is closed in $\mathrm{CRS}(G)$ (see Corollary 5.5), and hence $\mathrm{CRS}(G)$ is a Bauer simplex in all considered cases (recall that a simplex is called Bauer if the set of extreme points is closed).

Let $n = p \geq 2$ be prime. In this case Theorem 1.4 takes on a particularly simple form. Indeed, if $m|p$, then $m = 1$ or $m = p$. If $m = p$, then F over p and $pF = 0$ implies that $F = 0$. This corresponds to the CRS concentrated on $\{0\} \leq \mathcal{A}_p$. If $m = 1$, then any F is over m. Since $pF = 0$ we have that $F = (\mathbb{Z}/p\mathbb{Z})^k$ for some $k \geq 1$. Thus every non-trivial ergodic CRS of $\mathcal{A}_p$ has the form $\mathrm{Ker}(h)$ where $h \in \mathrm{Hom}(\mathcal{A}_p, (\mathbb{Z}/p\mathbb{Z})^k)$. This implies the random subgroup of $\mathcal{A}_p$ has index p^k almost surely. Moreover, the CRS is completely determined by the exponent k. This special case was obtained earlier by Gnedin-Olshanski [GO09] by completely different methods. Indeed, the idea of [GO09] is to interpret $\mathcal{A}_p$ as a vector space

over the field of order p and obtain a recursive formula for the finite dimensional marginals. Our method is based on the de Finetti-Hewitt-Savage Theorem (e.g. see [Gl03]) and duality and avoids explicit computations.

In [GO09] A. Gnedin and G. Olshanski provided a characterization of random spaces over the Galois field $\mathbb{F}_q$ that is invariant under the natural action of the infinite group of invertible matrices with coefficients from $\mathbb{F}_q$. Our methods allow us to give a different, probabilistic proof of their results. Let us formulate the results.

Let q be a power of a prime and $\mathbb{F}_q$ be the corresponding finite field with q elements. Let V be a locally compact vector space over $\mathbb{F}_q$. Denote by $\mathrm{Sub}_{\mathbb{F}_q}(V)$ the space of closed subspaces of V. Let $\mathrm{Aut}_{\mathbb{F}_q}(V)$ be the group of all homeomorphic invertible linear maps $V \to V$. Denote by $\mathrm{CRS}_{\mathbb{F}_q}(G)$ the Borel probability measures on $\mathrm{Sub}_{\mathbb{F}_q}(V)$ invariant under $\mathrm{Aut}_{\mathbb{F}_q}(V)$ (these are exactly the laws of random spaces of [GO09]). We have

Theorem 1.5. *Let* $\lambda \in \mathrm{CRS}_{\mathbb{F}_q}^e((\mathbb{F}_q)^{\mathbb{N}})$, $\lambda \neq \delta_{(\mathbb{F}_q)^{\mathbb{N}}}$. *Then there is a unique* $\kappa \in \mathbb{Z}_+$ *such that the random subgroup* $h(\mathbb{F}_q^{\kappa})$ *has law* λ, *where a random homomorphism* $h \in \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^{\kappa}, (\mathbb{F}_q)^{\mathbb{N}})$ *is given by the Haar measure on the compact group* $\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^{\kappa}, (\mathbb{F}_q)^{\mathbb{N}})$.

Theorem 1.6. *Let* $\lambda \in \mathrm{CRS}_{\mathbb{F}_q}^e(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$, $\lambda \neq \delta_{\oplus_{\mathbb{N}} \mathbb{F}_q}$. *Then there is a unique* $\kappa \in \mathbb{Z}_+$ *such that the random subgroup* $\mathrm{Ker}(h)$ *has law* λ, *where a random homomorphism* $h \in \mathrm{Hom}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q, \mathbb{F}_q^{\kappa})$ *is given by the Haar measure on the compact group* $\mathrm{Hom}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q, \mathbb{F}_q^{\kappa})$.

Denote by $\mu_{\kappa} \in \mathrm{CRS}_{\mathbb{F}_q}^e(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$ the measure in Theorem 1.6. Define $\tilde{v}_{n,k} = \mu_{\kappa}(\{X \mid \dim_{\mathbb{F}_q}(X \cap V_n) = k\})$. We have

Theorem 1.7. *Suppose that* $n \geq k \geq 0$, *and* $\kappa \geq n - k$. *Then* $\tilde{v}_{n,k}$ *is equal to the number of* $\mathbb{F}_q$-*matrices of size* $\kappa \times n$ *which have rank* $n - k$ *divided by the number of all* $\mathbb{F}_q$-*matrices of size* $\kappa \times n$. *For all other pairs* (n, k), $\tilde{v}_{n,k} = 0$.

We also give an explicit formula for $\tilde{v}_{n,k}$ in Corollary 5.15.

Organization. In §2, we go over background material. A first-time reader may choose to skip this section and refer back to it later as needed. In §3 we prove Theorem 1.2. In §4 - §5 we prove Theorems 1.3-1.4. In §5.2 and §5.3 we sketch the proofs of Theorems 1.5-1.6 and compute the numbers $\tilde{v}_{n,k}$. At the referee's request, the proofs of some easy lemmas are skipped. The interested reader can find them in the arXiv version of this paper which also contains (in the appendix) an alternative proof of Theorem 4.1.

2. Preliminaries

As the scope of this paper is on the border between group theory, probability theory and dynamical systems, we recall some basic definitions related to these fields.

2.1. **Compact-open topology and Braconnier topology.** Let X and Y be topological spaces. Denote by $C(X, Y)$ the set of all continuous maps from X to Y.

Definition 3. *The compact-open topology on* $C(X, Y)$ *is generated by sets* $V(K, U) = \{f \in C(X, Y) \mid f(K) \subseteq U\}$, *where* $K \subseteq X$ *is a compact set and* $U \subseteq Y$ *is an open set.*

Considering this topology on $C(X, Y)$, the following property is well known.

Lemma 2.1. *Suppose X is locally compact Hausdorff. Then the evaluation map $C(X, Y) \times X \to Y$, given by $(f, x) \mapsto f(x)$, is continuous.*

Suppose now that G is a locally compact second countable group. Denote by $\mathrm{Aut}(G)$ the set of all homeomorphic automorphisms of G. Since $\mathrm{Aut}(G) \subset C(G, G)$, we can endow $\mathrm{Aut}(G)$ with topology by restricting the compact-open topology on $C(G, G)$. However, in this topology, $\mathrm{Aut}(G)$ is not necessarily a topological group.

Definition 4. *The Braconnier topology on $\mathrm{Aut}(G)$ is the smallest refinement of the compact-open topology, such that the inverse map $\phi \mapsto \phi^{-1}$ is continuous.*

It is known that for a compact or locally connected (in particular discrete) G, the Braconnier topology coincides with the compact-open topology. That is, the inverse map on $\mathrm{Aut}(G)$ is automatically continuous in the compact-open topology.

The Braconnier topology turns $\mathrm{Aut}(G)$ into a topological group, and from now on we will always consider $\mathrm{Aut}(G)$ with the Braconnier topology.

Lemma 2.2. *The evaluation map $e : \mathrm{Aut}(G) \times G \to G$ is continuous.*

Proof. Note that if we consider the compact-open topology on $\mathrm{Aut}(G)$, then e is continuous by Lemma 2.1 and the assumption that G is a locally compact topological group (hence Hausdorff). It is obvious then that e is continuous for any refinement of the compact-open topology. $\square$

The next proposition shows that the action of $\mathrm{Aut}(G)$ on $\mathrm{Sub}(G)$ is continuous. First we need a lemma.

Lemma 2.3 (Proposition E.1.2 in [BP92])**.** *Let G be a locally compact second countable group. A sequence $\{H_n\} \subset \mathrm{Sub}(G)$ converges to $H \in \mathrm{Sub}(G)$ if and only if two conditions hold:*

   a) *if $g \in G$ and there exist a subsequence n_i and elements $g_i \in H_{n_i}$ such that $g_i \to g$, then $g \in H$,*
   b) *if $g \in H$, then there exist $g_n \in H_n$ such that $g_n \to g$.*

Proposition 2.4. *The map $\mathrm{Aut}(G) \times \mathrm{Sub}(G) \to \mathrm{Sub}(G)$, given by $(\phi, H) \mapsto \phi(H)$, is continuous.*

Proof. Suppose that $\phi_n \to \phi$ in the Braconnier topology and $H_n \to H$ in $\mathrm{Sub}(G)$. We need to show that $\phi_n(H_n) \to \phi(H)$ in $\mathrm{Sub}(G)$.

Note that we have also that $\phi_n^{-1} \to \phi^{-1}$ in the Braconnier topology.

Assume that $g_i = \phi_{n_i}(h_{n_i}) \in \phi_{n_i}(H_{n_i})$ and $g_i \to g$. Then $h_{n_i} = \phi_{n_i}^{-1}(g_i) \to \phi^{-1}(g)$, by Lemma 2.2. It follows that $\phi^{-1}(g) \in H$ and hence $g \in \phi(H)$.

Assume now that $g \in \phi(H)$. Then $g = \phi(h)$ for some $h \in H$. It follows that there exist $h_n \in H_n$ such that $h_n \to h$. We then have that $\phi_n(h_n) \to \phi(h) = g$, by Lemma 2.2. $\square$

2.2. Abelian groups and Pontryagin duality.

We will express the group operations in abelian groups additively. For example, if G is a locally compact second countable abelian group, $n \in \mathbb{N}$ and $x \in G$, then $nx = x + x + \cdots + x$ (n times). Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\hat{G} = \mathrm{Hom}(G, \mathbb{T})$ denote the *Pontryagin dual* of G. Note that

$\mathrm{Hom}(G, \mathbb{T})$ is itself a group under pointwise addition and it is locally compact under the compact-open topology. If G is discrete, then $\hat{G}$ is compact and vice versa. Moreover, $\hat{\hat{G}} = G$.

Whenever we mention the Haar measure on a compact group G, we mean that it is normalized, that is, it is a probability measure.

Proposition 2.5. *The groups* $\mathrm{Aut}(G)$ *and* $\mathrm{Aut}(\hat{G})$ *are naturally isomorphic as topological groups. Indeed the map* $\psi \mapsto \hat{\psi}^{-1}$ *is a continuous isomorphism (with continuous inverse) where*

$$\hat{\psi}(h)(x) = h(\psi(x)).$$

Proof. Note that $\psi_i \to \psi$ in the Braconnier topology if and only if both $\psi_i \to \psi$ and $\psi_i^{-1} \to \psi^{-1}$ in the compact-open topology.

To put this question in a more general context, let G, H be any locally compact abelian group. For $\psi \in \mathrm{Hom}(G, H)$ define $\hat{\psi} \in \mathrm{Hom}(\hat{H}, \hat{G})$ by

$$\hat{\psi}(\phi)(g) = \phi(\psi(g)), \quad \phi \in \hat{H}, g \in G.$$

It now suffices to show that if $\phi_i \to \phi$ in $\mathrm{Hom}(G, H)$, then $\hat{\phi}_i \to \hat{\phi}$ in $\mathrm{Hom}(\hat{H}, \hat{G})$ (in the compact-open topology). Since the map $\psi \mapsto \hat{\psi}$ is linear, this is equivalent to: if $\phi_i \to 0$, then $\hat{\phi}_i \to 0$. Thus the proof will be finished as soon as the next lemma is proved.

Lemma 2.6. $\phi_i \to 0$ *implies that* $\hat{\phi}_i \to 0$.

To prove Lemma 2.6 we need a sequence of lemmas. Let $U = \{z \in \mathbb{T} : |z| < 1/8\}$ be a small neighborhood of $0 \in \mathbb{T}$.

Lemma 2.7. *Let* $K \subseteq G$ *be a compact subset. Define* $K^* = \{h \in \hat{G} \, | \, h(K) \subseteq \bar{U}\}$. *Then* K^* *is a closed compact neighborhood of* $0 \in \hat{G}$.

Proof. The map $G \times \hat{G} \to \mathbb{T}$, given by $(g, h) \mapsto h(g)$ is continuous by Lemma 2.1. We have $0(K) = 0 \in U$, and thus $0 \in K^*$ and by continuity a neighborhood of 0 is in K^*. Again by continuity K^* is closed.

Suppose that there are $h_n \in K^*$ such that $h_n \to \infty$. We have the inclusion $\hat{G} \subset L^\infty(G)$, and the topology on $\hat{G}$ is the restriction of the weak* topology on $L^\infty(G) = (L^1(G))^*$. Moreover, the closure of $\hat{G}$ in $L^\infty(G)$ is the compact set $\hat{G} \cup \{0\}$ (see [F95, proof of Theorem 4.2]). Thus for any $f \in L^1(G)$ we have $\int f(g) \exp(2\pi i h_n(g)) \, dg \to 0$. Take f to be the characteristic function of K. Then $\int f(g) \exp(2\pi i h_n(g)) \, dg = \int_K \exp(2\pi i h_n(g)) \, dg$, and so

$$\left| \int_K \exp(2\pi i h_n(g)) \, dg - \mathrm{Haar}(K) \right| \leq \int_K |\exp(2\pi i h_n(g)) - 1| \, dg$$

$$\leq \int_K |\exp(\pi i/4) - 1| \, dg \leq \frac{\pi}{4} \mathrm{Haar}(K),$$

a contradiction. □

Lemma 2.8. *Suppose* $K_n \subseteq G$, $n \in \mathbb{N}$ *are compact subsets, such that* $K_n \subseteq K_{n+1}$ *for* $n \in \mathbb{N}$ *and* $\bigcup_n K_n = G$. *Then* $K_n^* \supseteq K_{n+1}^*$ *for* $n \in \mathbb{N}$ *and* $\bigcap_n K_n^* = \{0\}$.

Proof of Lemma 2.6. Suppose that $\phi_i : G \to H$ converges to $0 \in \mathrm{Hom}(G, H)$ in the compact-open topology. We need to show that $\hat{\phi}_i : \hat{H} \to \hat{G}$ converges to 0. Let

$C \subset \hat{H}$ be compact and $V \subset \hat{G}$ be an open neighborhood of 0. It suffices to show that $\hat{\phi}_i(C) \subset V$ for all sufficiently large i.

Since G is locally compact second countable, G is σ-compact. So there exists an increasing sequence of compact subsets $K_n \subset G$ satisfying $\bigcup_n K_n = G$. By Lemma 2.8, $\bigcap_n K_n^* = \{0\}$. Since K_n^* are compact, there is an n_0 such that $K_{n_0}^* \subset V$.

In order to show that $\hat{\phi}_i(C) \subset V$ it suffices to check that for each $x \in K_{n_0}$ and $y \in C$, $\hat{\phi}_i(y)(x) \in \bar{U}$. Indeed, this implies $\hat{\phi}_i(C) \subset K_{n_0}^* \subset V$.

We have that $\hat{\phi}_i(y)(x) = y(\phi_i(x))$. Therefore it suffices to check that $\phi_i(K_{n_0}) \subseteq C^*$. By Lemma 2.7, C^* is a neighborhood of 0, and thus, since $\phi_i \to 0$, there exists I such that $i \geq I$ implies that $\phi_i(K_{n_0}) \subseteq C^*$. $\qquad \square$

This finishes the proof of Proposition 2.5. $\qquad \square$

Because of Proposition 2.5, we will not distinguish between $\mathrm{Aut}(G)$ and $\mathrm{Aut}(\hat{G})$. Define $\mathrm{Ann} : \mathrm{Sub}(G) \to \mathrm{Sub}(\hat{G})$ and $\mathrm{Ker} : \mathrm{Sub}(\hat{G}) \to \mathrm{Sub}(G)$ by

$$\mathrm{Ann}(X) = \{h \in \hat{G} : \ h(x) = 0 \ \forall x \in X\},$$
$$\mathrm{Ker}(H) = \{x \in G : \ h(x) = 0 \ \forall h \in H\}.$$

It is easy to check using Lemma 2.3 that both Ann and Ker are continuous maps. Thus they are continuous inverses of each other and they are $\mathrm{Aut}(G)$-equivariant. So they induce homeomorphisms $\mathrm{Ann} : \mathrm{Char}(G) \to \mathrm{Char}(\hat{G})$, $\mathrm{Ker} : \mathrm{Char}(\hat{G}) \to \mathrm{Char}(G)$ and affine homeomorphisms $\mathrm{Ann}_* : \mathrm{CRS}(G) \to \mathrm{CRS}(\hat{G})$ and $\mathrm{Ker}_* : \mathrm{CRS}(\hat{G}) \to \mathrm{CRS}(G)$. Also Ann and Ker are order-reversing in the sense that $H \leq K$ implies $\mathrm{Ann}(H) \geq \mathrm{Ann}(K)$ and similarly with Ker. Lastly, they take addition to intersection and vice versa. To be precise, suppose H, K are subgroups of $\hat{G}$ and H', K' are subgroups of G. Then

(1) $\mathrm{Ker}(H + K) = \mathrm{Ker}(H) \cap \mathrm{Ker}(K)$, $\qquad \mathrm{Ker}(H \cap K) = \mathrm{Ker}(H) + \mathrm{Ker}(K)$,
$\mathrm{Ann}(H' + K') = \mathrm{Ann}(H') \cap \mathrm{Ann}(K')$, $\qquad \mathrm{Ann}(H' \cap K') = \mathrm{Ann}(H') + \mathrm{Ann}(K')$.

Lemma 2.9. *For any $n \geq 2$, $\mathrm{Ann}(n\mathcal{A}) = (\mathbb{Z}[1/n]/\mathbb{Z})^{\mathbb{N}} \leq \mathbb{T}^{\mathbb{N}}$.*

Proof. Recall from the introduction that if $x = \{x_i\} \in \mathcal{A} = \bigoplus_{\mathbb{N}} \mathbb{Z}$ and $y = \{y_i\} \in \hat{\mathcal{A}} = \mathbb{T}^{\mathbb{N}}$, then $y(x) := \sum_{i \in \mathbb{N}} x_i y_i \in \mathbb{T}$. We have that $x \in n\mathcal{A}$ if and only if there exists $x' \in \mathcal{A}$ with $x = nx'$. If $x' = \{x_i'\}$ and $y \in (\mathbb{Z}[1/n]/\mathbb{Z})^{\mathbb{N}}$, then $y(x) = \sum_{i \in \mathbb{N}} nx_i' y_i = \mathbb{Z}$ because $ny_i = \mathbb{Z}$ for all i. This implies $\mathrm{Ann}(n\mathcal{A}) \geq (\mathbb{Z}[1/n]/\mathbb{Z})^{\mathbb{N}}$. To see the other direction, suppose $y \in \mathrm{Ann}(n\mathcal{A})$. Fix $j \in \mathbb{N}$ and let $x = \{x_i\}$ be such that $x_i = n$ if $i = j$ and $x_i = 0$ otherwise. Then $x \in n\mathcal{A}$ so $y(x) = 0 = x_j y_j = ny_j$. Therefore, $y_j \in \mathbb{Z}[1/n]/\mathbb{Z}$. Since j is arbitrary, $y \in (\mathbb{Z}[1/n]/\mathbb{Z})^{\mathbb{N}}$. Since y is arbitrary, $\mathrm{Ann}(n\mathcal{A}) \leq (\mathbb{Z}[1/n]/\mathbb{Z})^{\mathbb{N}}$. $\qquad \square$

2.3. **Characteristic subgroups.** Let $\mathrm{Char}(G)$ denote the set of closed characteristic subgroups of G.

Lemma 2.10. $\mathrm{Char}(\mathcal{A}) = \{r\mathcal{A} : \ r = 0, 1, 2, \dots\}$, $\mathrm{Char}(\hat{\mathcal{A}}) = \{\mathrm{Ann}(r\mathcal{A}) : \ r = 0, 1, 2, \dots\}$, $\mathrm{Char}(\mathcal{A}_n) = \{0\} \cup \{r\mathcal{A}_n : \ r \mid n, r \in \mathbb{N}\}$, $\mathrm{Char}(\hat{\mathcal{A}}_n) = \{\hat{\mathcal{A}}_n\} \cup \{\mathrm{Ann}(r\mathcal{A}_n) : \ r \mid n, r \in \mathbb{N}\}$.

Theorem 2.11. *Every non-abelian free group of finite or countable rank has $2^{\aleph_0}$ many characteristic (in fact, fully invariant) subgroups. Moreover, we can choose these subgroups to lie in the commutator subgroup.*

Proof. Very likely this result is well known to experts, but we did not find a reference. It is a corollary of a famous theorem of Adjan [Ad70]. It can also be derived from [Ol70] although we find it simpler to obtain the result from [Ad70]. We explain the necessary background next.

A subgroup $H \leq G$ is *fully invariant* if $\varphi(H) \leq H$ for every endomorphism $\varphi : G \to G$. This condition is stronger than being characteristic.

An efficient method for contructing fully invariant subgroups is to use laws. Let $X = \{x_1, x_2, \ldots\}$ be a countable alphabet whose elements are considered to be variables taking values in a countable group G and let $\mathbb{F}(X)$ denote the free group with generating set X. For any word

$$w = x_{a_1}^{b_1} \cdots x_{a_n}^{b_n} \in \mathbb{F}(X)$$

let $f_w : G^n \to G$ be the corresponding *word map*:

$$f_w(g_1, \ldots, g_n) = g_{a_1}^{b_1} \cdots g_{a_n}^{b_n}.$$

If $W \subset \mathbb{F}(X)$ is any subset, we let $G(W) \leq G$ be the subgroup generated by the union of the images $\bigcup_{w \in W} f_w(G)$. This subgroup is fully invariant and the quotient group $G/G(W)$ satisfies the laws $w = 1$ ($w \in W$). Every fully invariant subgroup of a free group can be obtained this way [Neu67, Theorem 12.34].

A word $u \in \mathbb{F}(X)$ is a *consequence* of a set $W \subset \mathbb{F}(X)$ if $G(\{u\}) \subset G(W)$ for all G. In other words, any group which satisfies the laws $w = 1$ for $w \in W$ necessarily satisfies the law $u = 1$.

Let $\mathbb{F}_\infty$ denote the free group with countable rank. Suppose $u \in \mathbb{F}(X)$, $W \subset \mathbb{F}(X)$ and $\mathbb{F}_\infty(\{u\}) \subset \mathbb{F}_\infty(W)$. We claim that u is a consequence of W. To see this, let G be any countable group and let $\phi : \mathbb{F}_\infty \to G$ be a homomorphism with the property that for every non-trivial $g \in G$ there is a generator $h \in \mathbb{F}_\infty$ with $\phi(h) = g$. Then $\phi(\mathbb{F}_\infty(W)) = G(W)$ and $\phi(F_\infty(\{u\})) = G(\{u\})$. So $G(\{u\}) \subset G(W)$ which implies u is a consequence of W as claimed.

A subset $I \subset \mathbb{F}(X)$ is called *independent* if there does not exist $u \in I$ such that u is a consequence of $I \setminus \{u\}$. We claim that if I is independent, then the subgroups $\{\mathbb{F}_\infty(A) : A \subset I\}$ are distinct and fully invariant (and therefore, characteristic) subgroups. To see this, suppose $A, B \subset I$ and $\mathbb{F}_\infty(A) = \mathbb{F}_\infty(B)$. It suffices to show that $A = B$. Suppose $b \in B \setminus A$. Then $\mathbb{F}_\infty(\{b\}) \leq \mathbb{F}_\infty(B) = \mathbb{F}_\infty(A) \leq \mathbb{F}_\infty(I \setminus \{b\})$. So b is a consequence of $I \setminus \{b\}$ and therefore I is not independent. This contradiction shows that $B \subset A$. By symmetry $A \subset B$ and so $A = B$ as required.

So for the countable rank case, it suffices to show that there exists an infinite independent subset of $\mathbb{F}(X)$. This problem had been open for a long time before Adjan produced a specific example in [Ad70]. In fact he shows that any odd number $n \geq 4381$, the set

$$I_n = \{(x_1^{np} x_2^{np} x_1^{-np} x_2^{-np})^n : \ p \text{ prime }\}$$

is independent[1] (and in the book [Ad79] Adjan proves the same statement for $n \geq 1001$).

This shows that $\mathbb{F}_\infty$ has $2^{\aleph_0}$ many fully invariant subgroups. Observe that for any natural number $r \geq 2$, $\mathbb{F}_\infty$ is isomorphic to the commutator subgroup of $\mathbb{F}_r$, the rank r free group. Any fully invariant subgroup of the commutator subgroup of $\mathbb{F}_r$ is fully invariant as a subgroup of $\mathbb{F}_r$ because the commutator subgroup is itself fully invariant. □

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[1]Adjan uses the term "irreducible" in place of "independent".

2.4. **The weak* topology.** Let X be a compact metrizable space. Let $\mathcal{P}(X)$ denote the set of all Borel probability measures on X. This set admits the *weak* topology*: a sequence $\{\mu_i\}_{i=1}^{\infty} \subset \mathcal{P}(X)$ converges to $\mu_{\infty} \in \mathcal{P}(X)$ if and only if: for every continuous function $f \in C(X)$

$$\lim_{n\to\infty} \int f \, d\mu_i = \int f \, d\mu_{\infty}.$$

By the Banach-Alaoglu Theorem, $\mathcal{P}(X)$ is compact and metrizable. Moreover, it is a convex subspace of the Banach dual of $C(X)$. Recall that a *Choquet simplex* Δ is a compact convex subset of a locally convex topological vector space with the property that for every $\mu \in \Delta$ there exists a unique Borel probability measure θ on Δ^e, the set of extreme points of Δ, such that $\int \nu \, d\theta(\nu) = \mu$. It is easy to see that $\mathcal{P}(X)$ is a Choquet simplex. Indeed the map $\delta : X \to \mathcal{P}(X)^e$ defined by $\delta(x)$ is the Dirac probability measure concentrated on x, is a homeomorphism. So for any $\mu \in \mathcal{P}(X)$, $\mu = \int \nu \, d\delta_* \mu(\nu)$ is the unique extremal decomposition.

Let G be a topological group. An action of G on X is *continuous* if the map $(g, x) \mapsto gx$ (from $G \times X$ to X) is continuous. A Borel measure $\mu \in \mathcal{P}(X)$ is G-*invariant* if $\mu(gA) = \mu(A)$ for every $g \in G$ and Borel $A \subset X$. Let $\mathcal{P}_G(X) \subset \mathcal{P}(X)$ denote the subspace of G-invariant measures. It is closed in $\mathcal{P}(X)$ and is therefore compact in the weak* topology. It is also convex, so we may let $\mathcal{P}_G^e(X) \subset \mathcal{P}_G(X)$ denote the subspace of extreme points. $\mathcal{P}_G(X)$ is a Choquet simplex (see [Ph01]).

2.5. **Dynamics.** Let G be a topological group and (X, μ) a standard probability space. An action of G on X is *measurable* if the map $(g, x) \mapsto gx$ (from $G \times X$ to X) is measurable. It is *probability-measure-preserving* (pmp) if in addition to being measurable, $\mu(gA) = \mu(A)$ for every $g \in G$ and measurable $A \subset X$. In this case, G acts by unitaries (shifts) on the function space $L^2(X) = L^2(X, \mu)$ via the Koopman representation: $g\phi(x) := \phi(g^{-1}x)$.

Definition 5. A pmp action $G \curvearrowright (X, \mu)$ is *weakly mixing* if $L^2(X)$ has no finite dimensional subspaces which are G-invariant, besides the zero subspace and the subspace of constant functions.

Definition 6. A pmp action $G \curvearrowright (X, \mu)$ is *ergodic* if any G-invariant subset of X is either null or conull.

We can also reformulate this definition in terms of unitary representations. Indeed, $G \curvearrowright (X, \mu)$ is ergodic iff $L^2(X)$ contains no non-constant G-invariant vectors.

We list also the properties of weak-mixing that we will use. First we remind the definition of a factor system.

Definition 7. A $G \curvearrowright (Y, \nu)$ is a *factor* of $G \curvearrowright (X, \mu)$ if there is a G-equivariant measurable map $p : X \to Y$ such that $p_* \mu = \nu$, where $p_* \mu$ is defined by $p_* \mu(B) = \mu(p^{-1}B)$ for every measurable $B \subset Y$. In this case $L^2(Y, \nu)$ isometrically and G-equivariantly embeds in $L^2(X, \mu)$, by the map $\phi \mapsto \phi \circ p$.

Lemma 2.12. *Let $G \curvearrowright (X, \mu)$ be a pmp action.*

    a) *If H is a subgroup of G and the action $H \curvearrowright (X, \mu)$ is weakly mixing, then $G \curvearrowright (X, \mu)$ is weakly mixing.*

    b) *A factor of a weakly mixing system is weakly mixing and a factor of an ergodic system is ergodic.*

3. CHARACTERISTIC RANDOM SUBGROUPS OF FREE GROUPS

Let us recall a few definitions and set notation. We let $\mathbb{F}_r$ denote a free group of rank r with $2 \leq r \leq \aleph_0$. (Recall that the *rank* of a free group is the cardinality of a free generating set.) Keep in mind that every subgroup of a free group is a free group. A subgroup $K \leq \mathbb{F}_r$ is *characteristic* if $\phi(K) = K$ for every $\phi \in \mathrm{Aut}(\mathbb{F}_r)$.

In order to prove Theorem 1.2, we will construct a CRS λ of $\mathbb{F}_r$ for every characteristic subgroup $K \leq \mathbb{F}_r$ which lies in the commutator subgroup (so $K \leq [\mathbb{F}_r, \mathbb{F}_r]$) in such a way that the normal closure $N_\lambda = K$. Theorem 2.11 implies there are $2^{\aleph_0}$ many such characteristic subgroups and so this will imply Theorem 1.2.

The next proposition contains the construction of λ. First we need a definition.

Definition 8 (HT)**.** Let G be a locally compact group and $\lambda \in \mathrm{IRS}(G)$. Recall that the *support* of λ is the smallest closed subset $X \subset \mathrm{Sub}(G)$ such that $\lambda(X) = 1$. The *hull* of λ is the intersection of all subgroups in the support of λ. It is a subgroup of G, denoted by H_λ. The *normal closure* of λ is the smallest closed subgroup of G containing all of the subgroups in the support of λ. It is a subgroup of G denoted by N_λ.

Proposition 3.1. *Let $K \leq \mathbb{F}_r$ be an infinite rank characteristic subgroup and $p \geq 2$ be prime. Let $K_p := [K, K]K^p$ be the subgroup of K generated by the commutator $[K, K]$ and the p-th powers of elements of K. Then there exists a non-atomic CRS λ of $\mathbb{F}_r$ such that $K_p = H_\lambda \leq N_\lambda = K$ where H_λ, N_λ are the hull and normal closure of λ (Definition 8). Moreover, if the action of $\mathbb{F}_r$ on $\widehat{K/K_p} \cong \hat{\mathcal{A}}_p$ is weakly mixing (with respect to Haar measure), then λ is $\mathbb{F}_r$-weakly mixing. The action of $\mathbb{F}_r$ on $\widehat{K/K_p}$ is given by*

$$(g \cdot \phi)(f K_p) = \phi(g^{-1} f g K_p) \quad \forall f \in K, g \in \mathbb{F}_r, \phi \in \widehat{K/K_p}.$$

Proof. K has infinite rank, thus K/K_p is isomorphic with $\mathcal{A}_p$ and therefore $\widehat{K/K_p} \cong \hat{\mathcal{A}}_p \cong (\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}$ is compact. Let X be a Haar-random element of $\widehat{K/K_p}$. Since Haar measure is automorphism-invariant (by Halmos [Ha43]), $\mathrm{Ker}(X) \leq K/K_p$ is a CRS of K/K_p. Recall that X is a homomorphism from K/K_p to $\mathbb{T}$. We let $\mathrm{Ker}(X)$ denote its kernel and let $H \leq K$ be the inverse image of $\mathrm{Ker}(X)$ under the quotient map $q : K \to K/K_p$ (so $H = q^{-1}(\mathrm{Ker}(X))$). Then H is a CRS of K and, since K is characteristic, H is a CRS of $\mathbb{F}_r$. Let λ denote the law of H.

We claim that λ is non-atomic. Every non-trivial element of K/K_p has order p and so X has order p a.s. Therefore, $\mathrm{Ker}(X)$ has index p in K/K_p. The automorphism group of $\mathcal{A}_p = \bigoplus_{\mathbb{N}} (\mathbb{Z}/p\mathbb{Z})$ acts transitively on the set of all index p subgroups. Therefore, the orbit of $\mathrm{Ker}(X)$ under $\mathrm{Aut}(K/K_p)$ is infinite almost surely. This implies that λ is non-atomic.

A subgroup $J \in \mathrm{Sub}(\mathbb{F}_r)$ is in the support of λ if and only if $J \leq K$ and J has index p in K. Since this collection of subgroups generates K, $N_\lambda = K$. The intersection of all such subgroups is $K_p = H_\lambda$.

Now suppose that the action of $\mathbb{F}_r$ on $\widehat{K/K_p}$ is weakly mixing. Let μ denote Haar measure on $\widehat{K/K_p}$ and observe that the map $q^{-1} \mathrm{Ker} : \widehat{K/K_p} \to \mathrm{Sub}(K) \subset \mathrm{Sub}(\mathbb{F}_r)$ is $\mathbb{F}_r$-equivariant (where $\mathbb{F}_r$ acts on $\mathrm{Sub}(K)$ by conjugation) and $q^{-1} \mathrm{Ker}_* \mu = \lambda$. Because weak-mixing is preserved under factors, this implies λ is $\mathbb{F}_r$-weakly mixing. $\square$

Lemma 3.2. *If $K \leq \mathbb{F}_r$ is a characteristic subgroup and K lies inside the commutator subgroup $[\mathbb{F}_r, \mathbb{F}_r]$, then for any prime p, the action of $\mathbb{F}_r$ on $\widehat{K/K_p} \cong \hat{\mathcal{A}}_p$ is weakly mixing with respect to the Haar measure on $\widehat{K/K_p}$.*

We will prove this lemma in the next section. Let us see now how it implies Theorem 1.2:

Proof of Theorem 1.2. By Theorem 2.11 every non-abelian free group admits an uncountable family of characteristic subgroups. The commutator subgroup $[\mathbb{F}_r, \mathbb{F}_r]$ of $\mathbb{F}_r$ is an infinite rank free group. It is also characteristic. A characterisic subgroup of a characteristic subgroup is characteristic in the ambient group, thus Theorem 2.11 implies that the set of all characteristic subgroups of $\mathbb{F}_r$ that lie in $[\mathbb{F}_r, \mathbb{F}_r]$ has cardinality $2^{\aleph_0}$. The theorem now follows immediately from Proposition 3.1 and Lemma 3.2. □

3.1. **Mixing.** In this subsection we prove Lemma 3.2 after the next two lemmas.

Lemma 3.3. *Let G be a countable abelian group and $\hat{G}$ its Pontryagin dual. Let $\Gamma \leq \mathrm{Aut}(\hat{G})$ be a subgroup. Then the action of Γ on $\hat{G}$ is weakly mixing with respect to the Haar measure on $\hat{G}$ if and only if the action of Γ on G has no finite orbits other than the trivial orbit containing the identity. The action of Γ on G is induced from the inclusion $\Gamma \leq \mathrm{Aut}(\hat{G}) = \mathrm{Aut}(G)$ (see Proposition 2.5 for the identification of $\mathrm{Aut}(\hat{G})$ with $\mathrm{Aut}(G)$).*

This lemma is well known. Probably the first time a version of it appeared was in [Ha43, Theorem 1]. There is also a version in [Pe83, Theorem 5.7].

Notation 3.4. For $g \in \mathbb{F}_r$, we let $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathbb{F}_r)$ denote the inner automorphism defined by $\mathrm{Ad}(g)(x) = gxg^{-1}$. More generally, if $C \leq B$ are normal subgroups of $\mathbb{F}_r$, then we also let $\mathrm{Ad}(g) \in \mathrm{Aut}(B/C)$ be the automorphism $\mathrm{Ad}(g)(bC) = gbg^{-1}C$.

Recall that an element $g_0 \in \mathbb{F}_r$ is *primitive* if there exists a free generating set $S \subset \mathbb{F}_r$ that contains g_0.

Lemma 3.5. *Let $K \leq \mathbb{F}_r$ be an infinite rank normal subgroup. Suppose there is a primitive element $g_0 \in \mathbb{F}_r$ such that g_0K has infinite order in $\mathbb{F}_r/K$. Then there exists a free generating set $\mathcal{B}$ for K that is $\mathrm{Ad}(g_0)$-invariant. Moreover, every $\mathrm{Ad}(g_0)$-orbit in $\mathcal{B}$ is infinite.*

Proof. g_0 is primitive, thus there exists a free generating set $S \subset \mathbb{F}_r$ that contains g_0. Let J be the subgroup of $\mathbb{F}_r$ generated by K and g_0. Let $\mathrm{Sch}(J\backslash\mathbb{F}_r, S)$ denote the Schreier right-coset graph of $J\backslash\mathbb{F}_r$: its vertex set is $J\backslash\mathbb{F}_r$ and for every coset $Jh \in J\backslash\mathbb{F}_r$ and every $s \in S$ there is an edge from Jh to Jhs labeled s.

Recall that a subgraph of a graph is *spanning* if it contains all the vertices. It is a *tree* if it is simply connected and it is a *forest* if each of its connected components is a tree. Since $\mathrm{Sch}(J\backslash\mathbb{F}_r, S)$ is a connected graph, there exists a spanning tree $T \subset \mathrm{Sch}(J\backslash\mathbb{F}_r, S)$.

Now consider the Schreier right-coset graph $\mathrm{Sch}(K\backslash\mathbb{F}_r, S)$. The quotient map $\pi : K\backslash\mathbb{F}_r \to J\backslash\mathbb{F}_r$ determines a covering map from $\mathrm{Sch}(K\backslash\mathbb{F}_r, S)$ onto $\mathrm{Sch}(J\backslash\mathbb{F}_r, S)$, also denoted by π. Observe that $\pi^{-1}(T)$ is a spanning forest of $\mathrm{Sch}(K\backslash\mathbb{F}_r, S)$.

Let $T' \subset \text{Sch}(K\backslash\mathbb{F}_r, S)$ be the union of $\pi^{-1}(T)$ with all edges of the form $\{Kg_0^m, Kg_0^{m+1}\}$ for $m \in \mathbb{Z}$. We claim that T' is a spanning tree of $\text{Sch}(K\backslash\mathbb{F}_r, S)$. It is spanning because $\pi^{-1}(T)$ is spanning. To see that T' is connected, we will construct a path in T' from an arbitrary coset $Kx \in K\backslash\mathbb{F}_r$ to the identity coset $K \in K\backslash\mathbb{F}_r$. Since T is a spanning tree there exists a path in T from Jx to J. We observe that $\pi^{-1}(J) = \{Kg_0^m : m \in \mathbb{Z}\}$. Therefore this path lifts to a path in $\pi^{-1}(T)$ from Kx to Kg_0^m for some $m \in \mathbb{Z}$. We may then append to this path all edges of the form (Kg_0^i, Kg_0^{i+1}) for $0 \le i \le m-1$ (if $m \ge 0$) to obtain the required path. The case $m \le 0$ is similar.

To see that T' is simply connected it suffices to show that the connected component of $\pi^{-1}(T)$ intersects the infinite path $p = \{\{Kg_0^m, Kg_0^{m+1}\} : m \in \mathbb{Z}\}$ in exactly one vertex. Indeed suppose that $Kg_0^n \ne Kg_0^m$ are in the same component of $\pi^{-1}(T)$. Let q be a simple path from Kg_0^n to Kg_0^m in $\pi^{-1}(T)$. Since π is a covering map, $\pi(q)$ is a simple path from $J = \pi(Kg_0^m)$ to $J = \pi(Kg_0^n)$. However, this contradicts that π is a covering map and T is a tree. This shows that T' is a spanning tree as required.

Next we observe that T' is invariant under left-multiplication by g_0 (here we are using the fact that K is normal, so multiplication on the left is well defined). This follows from the observation that

$$\pi(g_0 Kx) = \pi(Kg_0 x) = Jx = \pi(Kx)$$

for any x and the path p is also left-g_0-invariant.

Let E be the set of all edges in $\text{Sch}(K\backslash\mathbb{F}_r, S)$ that are not in T'. Since T' is left-g_0-invariant, so is E. Choose an orientation for every edge in E so that the left-action of g_0 on E preserves orientations.

For every vertex v of $\text{Sch}(K\backslash\mathbb{F}_r, S)$, let p_v be the unique oriented path in T' from the identity coset K to v. For every oriented edge $e = (v, w) \in E$ let $p_e = p_v \cdot e \cdot p_w^{-1}$ denote the concatenation of p_v, e and p_w^{-1}. Since p_e is a circuit based at K, reading off its edge labels gives an element $k_e \in K$. Moreover, by a well-known result of Schreier, $\mathcal{B} := \{k_e : e \in E\}$ is a free basis for K. Observe that $k_{g_0 e} = \text{Ad}(g_0)(k_e)$ for any $e \in E$. Since E is left-g_0-invariant, this basis is $\text{Ad}(g_0)$-invariant. Kg_0 has infinite order in $K\backslash\mathbb{F}_r$, thus it follows that for every $e \in E$, $\{g_0^n e\}_{n \in \mathbb{Z}}$ is infinite. Therefore, every $\text{Ad}(g_0)$-orbit in $\mathcal{B} = \{k_e : e \in E\}$ is also infinite. $\square$

Proof of Lemma 3.2. Let $K \le \mathbb{F}_r$ be a characteristic subgroup that lies in the commutator $[\mathbb{F}_r, \mathbb{F}_r]$. Then there exists a primitive element $g_0 \in \mathbb{F}_r$ such that $g_0 K$ has infinite order in $\mathbb{F}_r/K$. For instance, any element of a basis of $\mathbb{F}_r$ satisfies this property. By Lemma 3.5 there exists an $\text{Ad}(g_0)$-invariant free basis $\mathcal{B}$ for K such that every $\text{Ad}(g_0)$-orbit in $\mathcal{B}$ is infinite.

Let $hK_p \in K/K_p$ be a non-identity element. By Lemma 3.3, it suffices to prove that $\{fhf^{-1}K_p : f \in \mathbb{F}_r\}$ is infinite. Since $\mathcal{B}$ is a free basis, $h = b_1^{e_1} \cdots b_n^{e_n}$ for some $b_1, \ldots, b_n \in \mathcal{B}$ and exponents $e_i \in \mathbb{Z} \setminus \{0\}$ such that $b_{i+1} \ne b_i$ for $1 \le i \le n-1$. Then

$$\text{Ad}(g_0)^m(h) = (\text{Ad}(g_0)^m(b_1))^{e_1} \cdots (\text{Ad}(g_0)^m(b_n))^{e_n}.$$

The basis $\mathcal{B}$ is $\text{Ad}(g_0)$-invariant, thus each non-trivial $\text{Ad}(g_0)$-orbit in $\mathcal{B}$ is infinite, and the image $\bar{\mathcal{B}}$ of $\mathcal{B}$ in K/K_p is a basis for the elementary abelian p-group K/K_p, and it follows that $\{\text{Ad}(g_0)^m(h)K_p : m \in \mathbb{Z}\} \subset K/K_p$ is infinite. This proves the lemma. $\square$

4. Random homomorphisms

Suppose G is a countable abelian group and K is a compact abelian group. Let $\text{Hom}(G, K)$ denote the space of all homomorphisms from G to K with the topology of pointwise convergence. This set naturally embeds into the product space K^G as a closed set. By Tychonoff's Theorem, K^G is compact. Therefore, $\text{Hom}(G, K)$ is compact. It is also an abelian group under pointwise addition and the group structure is compatible with its topology. For any subgroup $H \leq G$ and closed subgroup $L \leq K$ there is a natural embedding of $\text{Hom}(G/H, L)$ into $\text{Hom}(G, K)$. Indeed, we may identify $\text{Hom}(G/H, L)$ with the set of homorphisms of $\phi : G \to K$ such that $\phi(G) \leq L$ and $H \leq \text{Ker}(\phi)$. Also $\text{Hom}(G/H, L)$ is a closed subgroup of $\text{Hom}(G, K)$.

The group $\text{Aut}(K)$ acts on $\text{Hom}(G, K)$ by

$$(\psi_* h)(g) = \psi(h(g)) \quad \forall \psi \in \text{Aut}(K), h \in \text{Hom}(G, K), g \in G.$$

A key step in the proof of Theorems 1.3-1.4 is the next result:

Theorem 4.1. *Let G be a countable group and let χ be an $\text{Aut}(\hat{\mathcal{A}})$-invariant and indecomposable Borel probability measure on $\text{Hom}(G, \hat{\mathcal{A}})$. Then there exists a subgroup $H \leq G$ such that χ is the normalized Haar measure on $\text{Hom}(G/H, \hat{\mathcal{A}})$.*

Proof. We identify $\hat{\mathcal{A}}$ with the infinite dimensional torus $\mathbb{T}^{\mathbb{N}}$ and $\text{Hom}(G, \hat{\mathcal{A}})$ with the infinite product $\text{Hom}(G, \mathbb{T})^{\mathbb{N}}$. Let $h = (h_1, h_2, \ldots) \in \text{Hom}(G, \mathbb{T})^{\mathbb{N}}$ be a random homomorphism with law χ. If $\sigma : \mathbb{N} \to \mathbb{N}$ is an arbitrary permutation and $\rho_\sigma \in \text{Aut}(\hat{\mathcal{A}})$ is defined by $\rho_\sigma(x_1, x_2, \ldots) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$, then $(\rho_\sigma)_* h = (h_{\sigma(1)}, h_{\sigma(2)}, \ldots)$. This means that h is an exchangeable sequence of random variables. So the de Finetti-Hewitt-Savage Theorem[2] (see [Gl03] for example) implies that the variables $\{h_i\}_{i \in \mathbb{N}}$ are conditionally iid (independent identically distributed). This means the following. Let $\mathcal{P}(\text{Hom}(G, \mathbb{T}))$ denote the space of all Borel probability measures on $\text{Hom}(G, \mathbb{T})$ with the weak* topology. Then there exists a unique Borel probability measure ω on $\mathcal{P}(\text{Hom}(G, \mathbb{T}))$ such that

$$\chi = \int \mu^{\mathbb{N}} \, d\omega(\mu)$$

where, for any $\mu \in \mathcal{P}(\text{Hom}(G, \mathbb{T}))$, $\mu^{\mathbb{N}}$ denotes the product measure on $\text{Hom}(G, \mathbb{T})^{\mathbb{N}}$.

So there exists a random measure $\mu \in \mathcal{P}(\text{Hom}(G, \mathbb{T}))$ with law ω such that $\{h_i\}$ is an iid sequence of random variables with law μ.

Define $\phi, \psi \in \text{Aut}(\mathbb{T}^{\mathbb{N}})$ by

$$\begin{aligned}
\phi(x_1, x_2, x_3, \ldots) &= (-x_1, x_2, x_3, \ldots), \\
\psi(x_1, x_2, x_3, \ldots) &= (x_1 + x_2, x_2, x_3, \ldots).
\end{aligned}$$

Observe that

$$\begin{aligned}
\phi_*(h_1, h_2, h_3, \ldots) &= (-h_1, h_2, h_3, \ldots), \\
\psi_*(h_1, h_2, h_3, \ldots) &= (h_1 + h_2, h_2, h_3, \ldots).
\end{aligned}$$

χ is invariant under ϕ_*, thus we can repeat the argument above with the automorphisms ρ_σ to conclude that the variables $-h_1, h_2, h_3, \ldots$ are iid with law μ' for some

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[2]This Theorem states that if $X_1, X_2, \ldots$ is a family of random variables each taking values in a compact metrizable space and whose joint law is invariant under all permutations of the indices, then $X_1, X_2, \ldots$ is conditionally iid.

$\mu' \in \mathcal{P}(\mathrm{Hom}(G, \mathbb{T}))$. We know that $h_2, h_3, \ldots$ are iid with law μ, and so it follows that $\mu' = \mu$. Thus if $M : \mathrm{Hom}(G, \mathbb{T}) \to \mathrm{Hom}(G, \mathbb{T})$ is the map $M(k) = -k$, then $M_*\mu = \mu$ for ω-a.e. μ.

Similarly, because χ is invariant under ψ_* the variables $h_1 + h_2, h_2, h_3, \ldots$ are iid with law μ'' for some $\mu'' \in \mathcal{P}(\mathrm{Hom}(G, \mathbb{T}))$. We know that $h_2, h_3, \ldots$ are iid with law μ, thus it follows that $\mu'' = \mu$. For $z \in \mathrm{Hom}(G, \mathbb{T})$, let $A_z : \mathrm{Hom}(G, \mathbb{T}) \to \mathrm{Hom}(G, \mathbb{T})$ be the addition map $A_z(k) = k + z$. Since $h_1 + h_2, h_2$ have law μ and $h_1 + h_2$ is independent of h_2 it follows that $(A_z)_*\mu = \mu$ for μ-a.e. $z \in \mathrm{Hom}(G, \mathbb{T})$ and for ω-a.e. $\mu \in \mathcal{P}(\mathrm{Hom}(G, \mathbb{T}))$.

Let S_μ be the support of μ and S'_μ be the set of all $z \in S_\mu$ such that $(A_z)_*\mu = \mu$. We have already shown that $\mu(S'_\mu) = 1$. So S'_μ is dense in S_μ. Thus for any $z \in S_\mu$ there exists a sequence $\{z_n\}_{n=1}^\infty \subset S'_\mu$ such that $z_n \to z$ as $n \to \infty$. It follows that A_{z_n} converges to A_z uniformly. So $\lim_{n\to\infty}(A_{z_n})_*\mu = (A_z)_*\mu$. Since $(A_{z_n})_*\mu = \mu$ for all n, this implies $(A_z)_*\mu = \mu$. In particular, the support of μ is invariant under the inverses map and addition. So S_μ is a closed subgroup of $\mathrm{Hom}(G, \mathbb{T})$. Moreover, because μ is invariant under addition by elements of S_μ, μ must be the Haar measure of S_μ.

By definition $\mathrm{Hom}(G, \mathbb{T}) = \hat{G}$. Let $H_\mu = \mathrm{Ann}(S_\mu) \leq G$. Then $S_\mu = \mathrm{Hom}(G/H_\mu, \mathbb{T})$ and $\mu^{\mathbb{N}}$ is the Haar measure on $S_\mu^{\mathbb{N}} = \mathrm{Hom}(G/H_\mu, \mathbb{T})^{\mathbb{N}}$ which we identify with $\mathrm{Hom}(G/H_\mu, \mathbb{T}^{\mathbb{N}})$. For any subgroup $H \leq G$, $\mathrm{Hom}(G/H, \mathbb{T}^{\mathbb{N}})$ is $\mathrm{Aut}(\hat{A})$-invariant. The measure ω is uniquely determined by χ and χ is an arbitrary $\mathrm{Aut}(\hat{A})$-invariant probability measure, thus it follows that the Haar measure on $\mathrm{Hom}(G/H, \mathbb{T}^{\mathbb{N}})$ is $\mathrm{Aut}(\hat{A})$-indecomposable for every $H \leq G$. Since χ is $\mathrm{Aut}(\hat{A})$-indecomposable it follows that χ must actually be the Haar measure on a subgroup of the form $\mathrm{Hom}(G/H, \mathbb{T}^{\mathbb{N}})$. $\qquad\square$

5. Characteristic random subgroups of abelian groups

In this section we prove Theorems 1.3-1.4. First we need two lemmas and a proposition. We note that this lemma in other words means that a divisible abelian group is an injective module over $\mathbb{Z}$, which is well known. We include the proof for convenience.

Definition 9. An abelian group X is *divisible* if for every $x \in X$ and integer $n \neq 0$ there exists $y \in X$ such that $ny = x$. For example, $\mathbb{T}$ and $\mathbb{T}^{\mathbb{N}}$ are divisible.

Lemma 5.1. *Let G be a countable abelian group. Also let X be a divisible abelian group. Let $G_0 \leq G$ be a subgroup and $h : G_0 \to X$ a homomorphism. Then there exists a homomorphism $h' : G \to X$ such that $h'(g) = h(g)$ for all $g \in G_0$.*

Definition 10. Define $\overline{\mathrm{Image}} : \mathrm{Hom}(G, \hat{A}) \to \mathrm{Sub}(\hat{A})$ by: $\overline{\mathrm{Image}}(h)$ is the closure of the image of h.

Proposition 5.2. *Let G be a countable abelian group, χ denote the Haar measure on $\mathrm{Hom}(G, \hat{A})$ and $h \in \mathrm{Hom}(G, \hat{A})$ be a random homomorphism with law χ.*

- *If G is finite, then h is injective a.s.*
- *If $G = \mathcal{A}_n$ for some $n \geq 2$, then $\overline{\mathrm{Image}}(h) = \mathrm{Ann}(n\mathcal{A})$ a.s.*
- *If G contains an element of infinite order, then $\overline{\mathrm{Image}}(h) = \hat{A}$ a.s.*
- *If the set of orders of elements of G is unbounded, then $\overline{\mathrm{Image}}(h) = \hat{A}$ a.s.*

Proof. Suppose G is finite. Since χ is $\text{Aut}(\hat{A})$-invariant and ergodic, there exists a subgroup $H \leq G$ such that $\text{Ker}(h) = H$ for χ-a.e. h. Let $h' \in \text{Hom}(G, \hat{A})$ be an injective homomorphism. χ is the Haar measure on $\text{Hom}(G, \hat{A})$, thus χ is invariant under the map $h \mapsto h + h'$. Therefore, $H = \text{Ker}(h) = \text{Ker}(h + h')$ for χ-a.e. h. Since the kernel of h' is trivial, this implies H is trivial. This proves the first item.

Suppose now that $G = A_n = \bigoplus_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$. Then for χ-a.e. $h \in \text{Hom}(G, \hat{A})$, the image of h lies in the subgroup $\text{Ann}(nA)$ (this is because $\text{Ann}(nA)$ contains every element of order n in $\hat{A}$). Let $X_1, X_2, \ldots \in \hat{A}$ be iid random variables each with law equal to the Haar measure on $\text{Ann}(nA)$. Then the law of the subgroup $\overline{\langle X_1, X_2, \ldots \rangle}$ is the same as the law of $\overline{\text{Image}(h)}$ where $h \in \text{Hom}(A_n, \hat{A})$ is chosen uniformly at random. So it suffices to show that $\overline{\langle X_1, X_2, \ldots \rangle} = \text{Ann}(nA)$ a.s. By duality this is equivalent to showing that $\bigcap_{i=1}^{\infty} \text{Ker}(X_i) = nA$ almost surely. We have that $\bigcap_{i=1}^{\infty} \text{Ker}(X_i) \supset nA$, and will now show the inverse inclusion. Note that since X_i are iid, we have that

$$\mathbb{P}\left(v \in \bigcap_i \text{Ker}(X_i) \right) = \lim_i \mathbb{P}(X_1(v) = 0)^i,$$

which is 0 if $\mathbb{P}(X_1(v) = 0) < 1$. Thus it suffices to show that $\mathbb{P}(X_1(v) = 0) = 1$ implies that $v \in nA$. However $\mathbb{P}(X_1(v) = 0) = 1$ says that $X(v) = 0$ for a.a. $X \in \text{Ann}(nA)$ with respect to Haar measure on $\text{Ann}(nA)$. Since the condition $X(v) = 0$ defines a closed set of X's, we have by continuity that $X(v) = 0$ for all $X \in \text{Ann}(nA)$. Thus $v \in \text{Ker}(\text{Ann}(nA)) = nA$ (see §2.2).

To prove the last two items, note first that $\overline{\text{Image}(h)} = \hat{A}$ is equivalent to $\text{Ker}(h(G)) = 0$. So it suffices to show that $\text{Ker}(h(G)) = 0$ almost surely. Since A is countable, this is equivalent to the statement that for any non-zero $v \in A$, $v \notin \text{Ker}(h(G))$ almost surely. Let us identify A with $\bigoplus_{\mathbb{N}} \mathbb{Z}$. Since χ is $\text{Aut}(\hat{A})$-invariant, it suffices to check this condition in the special case $v = ke_1$ where $k \geq 1$ and $e_1 = (1, 0, \ldots)$.

For any $g \in G$, the probability that $ke_1 \in \text{Ker}(h(G))$ is at most the probability that $ke_1 \in \text{Ker}(h(g))$. So it suffices to show there is a sequence $\{g_n\} \subset G$ such that $\mathbb{P}(h(g_n)(ke_1) = 0) \to 0$ as $n \to \infty$ where $\mathbb{P}(\cdot)$ denotes probability.

Fix $g \in G$. Consider the map from $\text{Hom}(G, \hat{A})$ to $\hat{A}$ given by $h \mapsto h(g)$. This map is a continuous homomorphism. So it pushes the Haar measure on $\text{Hom}(G, \hat{A})$ forward to Haar measure, denoted by η, on the image subgroup $\{h(g) : h \in \text{Hom}(G, \hat{A})\}$. By Lemma 5.1, the image subgroup is either $\hat{A}$ (if g has infinite order) or $\text{Ann}(mA)$, if g has order $m < \infty$.

If g has infinite order, then because the closed subgroup $\{x \in \hat{A} : x(ke_1) = 0\}$ has infinite index in $\hat{A}$, $\eta(\{x \in \hat{A} : x(ke_1) = 0\}) = 0$. Equivalently, $\mathbb{P}(h(g)(ke_1) = 0) = 0$. Thus $\overline{\text{Image}(h)} = \hat{A}$.

Suppose g has finite order m. Identify $\text{Ann}(mA)$ with $\prod_{\mathbb{N}} \mathbb{Z}[1/m]/\mathbb{Z}$. We compute that

$$\mathbb{P}(h(g)(ke_1) = 0) = \eta(\{x | x(ke_1) = 0\}) = \frac{\#\{x \in \mathbb{Z}[1/m]/\mathbb{Z} \mid xk = 0\}}{m} = \frac{\gcd(k, m)}{m}.$$

Since this ratio goes to 0 when m goes to infinity, we are done. $\square$

We have the following obvious lemma, which is easily proved using uniqueness in the structure theorem for finite abelian groups. Given a finite abelian group F and integer $n \geq 1$, let $F_{(n)} = \{x \in F \mid nx = 0\}$.

Lemma 5.3. *For any finite abelian group H and any $n \geq 1$ there is a finite abelian group F such that $F/F_{(n)} \cong H$ and F is over n (in the sense of Definition 2). Moreover F is uniquely determined up to isomorphism.*

Proof. By the classification of finite abelian groups, there are primes $p_1, \ldots, p_s$ and positive integers $t_1, \ldots, t_s$ such that $H = \bigoplus_{j=1}^{s} \mathbb{Z}/p_j^{t_j}\mathbb{Z}$.

Let F be any finite abelian group. Without loss of generality, $F = \bigoplus_{i=1}^{r} \mathbb{Z}/q_i^{u_i}\mathbb{Z}$ for some primes q_i and integers $u_i \geq 1$ that are uniquely determined by F. Then

$$F_{(n)} = \bigoplus_{i=1}^{r} \frac{\mathrm{lcm}(n, q_i^{u_i})}{n}\mathbb{Z} \Big/ q_i^{u_i}\mathbb{Z}, \quad F/F_{(n)} \cong \mathbb{Z} \Big/ \frac{\mathrm{lcm}(n, q_i^{u_i})}{n}\mathbb{Z}.$$

Therefore, F is over n if and only if $q_i^{u_i}$ does not divide n for every i. Suppose this is the case. Then $F/F_{(n)} \cong H$ if and only if: after permuting indices if necessary, $r = s$, $p_i = q_i$ for all i and $p_i^{t_i} = \mathrm{lcm}(n, p_i^{u_i})/n$ for all i. This condition uniquely determines u_i. Indeed, $u_i = t_i + k_i$ where $k_i \geq 0$ is determined by: $p_i^{k_i} \mid n$ and $p_i^{k_i+1} \nmid n$. $\qquad\square$

Proof of Theorem 1.3. Observe that the special case $G = \hat{\mathcal{A}}_n$ ($n \geq 2$) follows from the case $G = \hat{\mathcal{A}}$ because $\hat{\mathcal{A}}_n$ is naturally a characteristic subgroup of $\hat{\mathcal{A}}$. Indeed we may regard $\hat{\mathcal{A}}_n \cong (\mathbb{Z}[1/n]/\mathbb{Z})^{\mathbb{N}}$ as a characteristic subgroup of $\hat{\mathcal{A}} \cong \mathbb{T}^{\mathbb{N}}$. Therefore every CRS of $\hat{\mathcal{A}}_n$ is automatically a CRS of $\hat{\mathcal{A}}$. In fact the inclusion map $\hat{\mathcal{A}}_n \to \hat{\mathcal{A}}$ induces an $\mathrm{Aut}(\hat{\mathcal{A}})$-equivariant inclusion map $\mathrm{Sub}(\hat{\mathcal{A}}_n) \to \mathrm{Sub}(\hat{\mathcal{A}})$ which induces an affine embedding $\mathrm{CRS}(\hat{\mathcal{A}}_n) \to \mathrm{CRS}(\hat{\mathcal{A}})$. Therefore it suffices to prove the special case $G = \hat{\mathcal{A}}$.

Let $\lambda \in \mathrm{CRS}^e(\hat{\mathcal{A}})$. Define a measure χ on $\mathrm{Hom}(\mathcal{A}, \hat{\mathcal{A}})$ by

$$\chi = \int \chi_K \, d\lambda(K)$$

where χ_K denotes the Haar measure on the subgroup $\mathrm{Hom}(\mathcal{A}, K) \leq \mathrm{Hom}(\mathcal{A}, \hat{\mathcal{A}})$. Observe that $\overline{\mathrm{Image}}_*\chi = \lambda$. Since λ is $\mathrm{Aut}(\hat{\mathcal{A}})$-invariant, so is χ. By Theorem 4.1, χ is a convex integral of Haar measures on subgroups of $\mathrm{Hom}(\mathcal{A}, \hat{\mathcal{A}})$ of the form $\mathrm{Hom}(\mathcal{A}/H, \hat{\mathcal{A}})$. So λ is a convex integral of measures of the form $\overline{\mathrm{Image}}_*\chi_{\mathcal{A}/H}$ where $\chi_{\mathcal{A}/H}$ denotes Haar measure on $\mathrm{Hom}(\mathcal{A}/H, \hat{\mathcal{A}})$. Since λ is indecomposable and each measure of the form $\overline{\mathrm{Image}}_*\chi_{\mathcal{A}/H}$ is $\mathrm{Aut}(\hat{\mathcal{A}})$-invariant there is a countable abelian group G such that $\lambda = \overline{\mathrm{Image}}_*\chi_G$ where χ_G denotes Haar measure on $\mathrm{Hom}(G, \hat{\mathcal{A}})$.

By Proposition 5.2 if there does not exist a finite bound on the order of elements of G, then the image of h is dense in $\hat{\mathcal{A}}$ for χ_G-a.e. h. Thus $\lambda = \delta_{\hat{\mathcal{A}}}$, the point measure on $\hat{\mathcal{A}}$. This measure corresponds to the pair $(0, [0])$, where 0 is a trivial group (recall the statement of Theorem 1.3).

Suppose now that there does exist a bound on the order of elements of G. Then the first Prüfer Theorem implies

$$G \cong F \oplus \mathcal{A}_{n_1} \oplus \cdots \oplus \mathcal{A}_{n_r}$$

for some finite abelian group F and integers $n_1, \ldots, n_r \geq 1$ (note $\mathcal{A}_1$ is the trivial group). Without loss of generality, we identify G with the direct sum above. So for any $h \in \mathrm{Hom}(G, \hat{\mathcal{A}})$,

$$\overline{h(G)} = \overline{h(F)} + \overline{h(\mathcal{A}_{n_1})} + \cdots + \overline{h(\mathcal{A}_{n_r})}.$$

By Proposition 5.2 we have that $\sum_i \overline{h(\mathcal{A}_{n_i})} = \sum_i \mathrm{Ann}(n_i \mathcal{A}) = \mathrm{Ann}(m\mathcal{A})$ for χ_G-a.e. h where $m = \mathrm{lcm}(n_i)$.

If $F = F_0 \oplus F_1$ for some subgroups F_0, F_1 and $mF_0 = 0$, then $h(F_0) \leq \mathrm{Ann}(m\mathcal{A})$. Therefore

$$\overline{h(G)} = h(F_1) + \mathrm{Ann}(m\mathcal{A}) = \overline{h(F_1 \oplus \mathcal{A}_{n_1} \oplus \cdots \oplus \mathcal{A}_{n_r})}$$

for χ_G-a.e. h. So without loss of generality, we may assume that F does not have any direct summand F_0 with $mF_0 = 0$. In other words, we may assume F is over m (Definition 2). Thus λ corresponds to the pair $(m, [F])$ in the notation of Theorem 1.3. Now we only have to check that the pair $(m, [F])$ is uniquely determined by λ.

$\overline{h(G)}$ is a finite extension of $\mathrm{Ann}(m\mathcal{A})$ for a.e. h, thus the integer m is uniquely determined by λ. By Proposition 5.2, h restricted to F is injective for χ_G-a.e. h. So $F/F_{(m)} \cong \overline{h(G)}/\mathrm{Ann}(m\mathcal{A})$ for χ_G-a.e. h. Lemma 5.3 now implies the isomorphism class of F is uniquely determined by λ and the requirement that F is over m. $\qquad\square$

Proof of Theorem 1.4. Consider first the case $G = \mathcal{A}$. As discussed in §2.2, any indecomposable CRS of $\mathcal{A}$ is of the form $\eta = \mathrm{Ker}_* \lambda$ for some $\lambda \in \mathrm{CRS}^e(\hat{\mathcal{A}})$. So Theorem 1.3 implies λ corresponds to the pair $(m, [F])$, where $m \geq 0$ is an integer and F is a finite abelian group over m. Thus if $h \in \mathrm{Hom}(F, \hat{\mathcal{A}})$ is random with law equal to the Haar measure of $\mathrm{Hom}(F, \hat{\mathcal{A}})$, then η is the law of $\mathrm{Ker}(h(F) + \mathrm{Ann}(m\mathcal{A}))$. By (1) from §2.2, η is the law of

$$\mathrm{Ker}(h(F)) \cap \mathrm{Ker}(\mathrm{Ann}(m\mathcal{A})) = \mathrm{Ker}(h(F)) \cap m\mathcal{A}.$$

Note now that $v \in \mathrm{Ker}(h(F))$ if and only if for every $f \in F$ we have $0 = h(f)(v) = \hat{h}(v)(f)$, where $\hat{h} : \mathcal{A} \to \hat{F}$ is the homomorphism dual to h. Since this is true for every $f \in F$, it follows that $\hat{h}(v) = 0$, or $v \in \mathrm{Ker}(\hat{h})$. Thus $\mathrm{Ker}(h(F)) = \mathrm{Ker}(\hat{h})$. To finish the proof in the case $G = \mathcal{A}$, it is left to note that the duality gives a continuous group isomorphism between $\mathrm{Hom}(F, \hat{\mathcal{A}})$ and $\mathrm{Hom}(\mathcal{A}, \hat{F})$, and that $F \simeq \hat{F}$ since F is finite.

Suppose now that $G = \mathcal{A}_n = \mathcal{A}/n\mathcal{A}$. Since $n\mathcal{A}$ is a characteristic subgroup of $\mathcal{A}$, any CRS on $\mathcal{A}_n$ by taking its preimage under the factor map $\mathcal{A} \to \mathcal{A}/n\mathcal{A} = \mathcal{A}_n$ gives a CRS on $\mathcal{A}$, which will contain $n\mathcal{A}$ almost surely. Thus in order to describe $\mathrm{CRS}^e(\mathcal{A}_n)$ it suffices to describe the set of those elements $\mathrm{CRS}^e(\mathcal{A})$ that contain $n\mathcal{A}$ with probability 1. Clearly, they are those elements of $\mathrm{CRS}^e(\mathcal{A})$ that correspond to the pair $(m, [F])$ with $m | n$ and $nF = 0$.

Note that if $m | n$ and $nF = 0$, then $m\mathcal{A}/n\mathcal{A} = m\mathcal{A}_n$ and $\mathrm{Hom}(\mathcal{A}, F) = \mathrm{Hom}(\mathcal{A}_n, F)$, since for any $h \in \mathrm{Hom}(\mathcal{A}, F)$ we have that $h(n\mathcal{A}) = nh(\mathcal{A}) \subset nF = 0$. Thus an element in $\mathrm{CRS}^e(\mathcal{A}_n)$ corresponding to the pair $(n, [F])$ can be written as $m\mathcal{A}_n \cap \mathrm{Ker}(h)$, where $h \in \mathrm{Hom}(\mathcal{A}_n, F)$ has the Haar law. $\qquad\square$

5.1. **Topology of** $\mathrm{CRS}^e(\hat{\mathcal{A}})$.

Next we describe the topology of the space $\mathrm{CRS}^e(\hat{\mathcal{A}})$, identified with the set of pairs $(n, [F])$ such that $n \geq 0$ and $[F]$ is an isomorphism class of a finite abelian group F over n.

Theorem 5.4. *Let $(n_i, [F_i])$ be a sequence corresponding to $\lambda_i \in \mathrm{CRS}^e(\hat{\mathcal{A}})$. The following statements hold:*

- *If $n_i \to \infty$, then $\lambda_i \to \delta_{\hat{\mathcal{A}}}$.*
- *If n_i stabilizes, but $\mathrm{Maxorder}(F_i)$, the maximal order of elements in F_i, goes to ∞, then $\lambda_i \to \delta_{\hat{\mathcal{A}}}$.*
- *Suppose n_i stabilizes to n. Suppose also that eventually*

$$F_i \cong F \oplus \left(\bigoplus_{j=1}^{s} (\mathbb{Z}/q_j^{t_j}\mathbb{Z})^{m_j(i)} \right)$$

  for some fixed finite abelian group F, integer $s \geq 0$, primes q_j and integers $t_j \geq 0$, and that for all j $\lim_{i \to \infty} m_j(i) = \infty$ (this is the case of bounded $\mathrm{Maxorder}(F_i)$). Then λ_i converges to indecomposable CRS corresponding to the pair $(lcm(n, \{q_j^{t_j}\}_{j=1}^s), [F])$.
- *For any sequence it is possible to choose subsequence which is one of the first three types.*

Corollary 5.5. $\mathrm{CRS}^e(\hat{\mathcal{A}})$ *is closed in* $\mathrm{CRS}(\hat{\mathcal{A}})$ *(i.e.,* $\mathrm{CRS}(\hat{\mathcal{A}})$ *is a Bauer simplex).*

Proof of Theorem 5.4. Recall ([BHK09]) that for a compact group K, given a neighborhood $U \subset K$ of identity, we obtain a neighborhood $N_U(K_1)$ of $K_1 \in \mathrm{Sub}(K)$ in the Chabauty topology by

$$N_U(K_1) := \{H \leq K \mid H + U \supset K_1, \ K_1 + U \supset H\}.$$

Note that for any subgroup $T \in \mathrm{Sub}(K)$, if $H \in N_U(K_1)$, then $H+T \in N_U(K_1+T)$.

Given $\epsilon > 0$ and a finite subset $J \subset \mathbb{N}$ define a neighborhood $U_{J,\varepsilon}$ of the identity in $\hat{\mathcal{A}} = (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ by

$$U_{J,\varepsilon} = \{v = (v_j) \in (\mathbb{R}/\mathbb{Z})^{\mathbb{N}} \mid |v_j| < \varepsilon \text{ for all } j \in J\}$$

where $|x + \mathbb{Z}| := \min_{k \in \mathbb{Z}} |x + k|$. The sets $U_{J,\varepsilon}$ form a neighborhood basis of the identity in $\hat{\mathcal{A}}$.

We will obtain the theorem from a sequence of lemmas.

Lemma 5.6. *For any $m > \varepsilon^{-1}$ and $J \subset \mathbb{N}$, $\mathrm{Ann}(m\mathcal{A}) + U_{J,\varepsilon} = \hat{\mathcal{A}}$.*

Proof. Recall that $\mathrm{Ann}(m\mathcal{A}) = (\mathbb{Z}[1/m]/\mathbb{Z})^{\mathbb{N}}$. In particular, both $\mathrm{Ann}(m\mathcal{A})$ and $U_{J,\varepsilon}$ are product sets. So it suffices to show that for any $j \in J$ the projection of $\mathrm{Ann}(m\mathcal{A}) + U_{J,\varepsilon}$ onto the j-th coordinate is $\mathbb{R}/\mathbb{Z}$. Indeed, this projection is $\{k/m + \varepsilon' + \mathbb{Z} : \ 0 \leq k < m, \ |\varepsilon'| < \varepsilon\}$. If $m > \varepsilon^{-1}$, then this set is all of $\mathbb{R}/\mathbb{Z}$. $\square$

The first item now follows. Indeed, suppose that f is a continuous function on $\mathrm{Sub}(\hat{\mathcal{A}})$. Then, by continuity, for each δ there are J and ε such that $|f(\hat{\mathcal{A}}) - f(H)| < \delta$ for each $H \in N_{U_{J,\varepsilon}}(\hat{\mathcal{A}})$. Note that if $n_i > \varepsilon^{-1}$ and $h \in \mathrm{Hom}(F_i, \hat{\mathcal{A}})$ we have that $\mathrm{Ann}(n_i\mathcal{A}) + h(F_i) \in N_{U_{J,\varepsilon}}(\hat{\mathcal{A}})$ by Lemma 5.6. Thus

$$\left| \lambda_i(f) - f(\hat{\mathcal{A}}) \right| \leq \int_{\mathrm{Hom}(F_i, \hat{\mathcal{A}})} \left| f(\mathrm{Ann}(n_i\mathcal{A}) + h(F_i)) - f(\hat{\mathcal{A}}) \right| d\chi_{F_i}(h) \leq \delta$$

where χ_{F_i} denotes Haar probability measure on $\mathrm{Hom}(F_i, \hat{\mathcal{A}})$. Since δ, f are arbitrary, this implies that if $n_i \to \infty$, then $\lambda_i \to \delta_{\hat{\mathcal{A}}}$ as required. To prove the second item, we need the next lemma.

Lemma 5.7. *Let* $V_{m,J,\epsilon}$ *be the set of all* $h \in \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \hat{\mathcal{A}})$ *such that* $h(\mathbb{Z}/m\mathbb{Z}) + U_{J,\varepsilon} = \hat{\mathcal{A}}$. *Then for any fixed finite set* $J \subset \mathbb{N}$ *and any* ϵ,

$$\lim_{m \to \infty} \chi_{\mathbb{Z}/m\mathbb{Z}}(V_{m,J,\epsilon}) = 1$$

where $\chi_{\mathbb{Z}/m\mathbb{Z}}$ *denotes Haar probability measure on* $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \hat{\mathcal{A}})$.

Proof. Note that the map $h \mapsto h(1)$ identifies $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \hat{\mathcal{A}})$ with $\mathrm{Ann}(m\mathcal{A}) = (\mathbb{Z}[1/m]/\mathbb{Z})^{\mathbb{N}}$. For $j \in J$ define $v_j = h(1)_j \in \mathbb{Z}[1/m]/\mathbb{Z}$. Then $h(\mathbb{Z}/m\mathbb{Z}) + U_{J,\varepsilon} = \hat{\mathcal{A}}$ if and only if for each $j \in J$

$$\mathbb{R}/\mathbb{Z} = \{kv_j + \varepsilon' + \mathbb{Z} : 0 \le k < m, \ |\varepsilon'| < \varepsilon\}.$$

This in turn can happen only if $v_j \in \mathbb{Z}[1/d]/\mathbb{Z}$ for some $d > \varepsilon^{-1}$. To show the statement of the lemma, it suffices to show that the ratio of all such v_j in $\mathbb{Z}[1/m]/\mathbb{Z}$ goes to 1 as $m \to \infty$. Or, what is the same, that the number of $v \in \mathbb{Z}[1/d]/\mathbb{Z}$, such that $d \le \varepsilon^{-1}$, divided by m, goes to 0, but this is obvious. $\square$

From this lemma we have the second item. Indeed, given a continuous function f on $\mathrm{Sub}(\hat{\mathcal{A}})$ and $\delta > 0$ choose J, ε such that $|f(\hat{\mathcal{A}}) - f(H)| < \delta$ for each $H \in N_{U_{J,\varepsilon}}(\hat{\mathcal{A}})$. Let m_i be the maximum order of an element in F_i. Let

$$Q_{i,J,\varepsilon} = \{h \in \mathrm{Hom}(F_i, \hat{\mathcal{A}}) : \mathrm{Ann}(n_i\mathcal{A}) + h(F_i) + U_{J,\varepsilon} = \hat{\mathcal{A}}.$$

Then $\chi_{F_i}(Q_{i,J,\varepsilon}) \ge \chi_{F_i}(V_{m_i,J,\epsilon})$. Thus

$$\left| \lambda_i(f) - f(\hat{\mathcal{A}}) \right| \le \int_{\mathrm{Hom}(F_i, \hat{\mathcal{A}})} \left| f(\mathrm{Ann}(n_i\mathcal{A}) + h(F_i)) - f(\hat{\mathcal{A}}) \right| \, d\chi_{F_i}(h)$$

$$\le \int_{Q_{i,J,\varepsilon}} \left| f(\mathrm{Ann}(n_i\mathcal{A}) + h(F_i)) - f(\hat{\mathcal{A}}) \right| \, d\chi_{F_i}(h)$$

$$+ \int_{\mathrm{Hom}(F_i, \hat{\mathcal{A}}) \setminus Q_{i,J,\varepsilon}} \left| f(\mathrm{Ann}(n_i\mathcal{A}) + h(F_i)) - f(\hat{\mathcal{A}}) \right| \, d\chi_{F_i}(h)$$

$$\le \delta + (1 - \chi_{F_i}(Q_{i,J,\varepsilon}))\|f\|_\infty \le \delta + (1 - \chi_{\mathbb{Z}/m_i\mathbb{Z}}(V_{m_i,J,\varepsilon}))\|f\|_\infty.$$

Lemma 5.7 now implies the second item.

To prove the third item, we need the next two lemmas.

Lemma 5.8. *Fix* $m > 1$. *Consider the set*

$$\{(a_1, \ldots, a_k) \in (\mathbb{Z}[1/m]/\mathbb{Z})^k \,|\, a_1, \ldots, a_k \text{ generate } \mathbb{Z}[1/m]/\mathbb{Z}\}.$$

Then the number of elements in this set divided by m^k *goes to 1 as* k *goes to infinity.*

Proof. It suffices to show that the number of elements in the complement of this set divided by m^k goes to 0. Note that the complement is the union, over $d|m$, $d \ne m$, of sets $(\mathbb{Z}[1/d]/\mathbb{Z})^k$. Clearly $d^k/m^k \to 0$ as $k \to \infty$. $\square$

Lemma 5.9. *Consider the set of those* $h \in \mathrm{Hom}((\mathbb{Z}/m\mathbb{Z})^k, \hat{\mathcal{A}})$ *such that* $h((\mathbb{Z}/m\mathbb{Z})^k) + U_{J,\varepsilon} \supset \mathrm{Ann}(m\mathcal{A})$. *The Haar measure of this set goes to 1 as* k *goes to infinity (for fixed* J, ε, m).

Proof. Denote by e_s, $1 \le s \le k$ the standard generators of $(\mathbb{Z}/m\mathbb{Z})^k$. Since J is finite, and projections onto coordinates of $\hat{\mathcal{A}} = (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ are independent, it suffices to show that for each $j \in J$

$$\chi_{(\mathbb{Z}/m\mathbb{Z})^k} \left(\left\{ h \in \mathrm{Hom}((\mathbb{Z}/m\mathbb{Z})^k, \hat{\mathcal{A}}) : \mathrm{Proj}_j(\mathrm{Image}(h)) = \mathbb{Z}[1/m]/\mathbb{Z} \right\} \right) \to 1$$

as $k \to \infty$ where $\mathrm{Proj}_j : (\mathbb{R}/\mathbb{Z})^{\mathbb{N}} \to \mathbb{R}/\mathbb{Z}$ denotes projection onto the j-th coordinate. The number on the left is equal to the cardinality of the set

$$\{(a_1, \ldots, a_k) \in (\mathbb{Z}[1/m]/\mathbb{Z})^k \, | \, a_1, \ldots, a_k \text{ generate } \mathbb{Z}[1/m]/\mathbb{Z}\}$$

divided by m^k. By Lemma 5.8 we are done. $\square$

We will now prove the third item. Fix a finite set $J \subset \mathbb{N}$ and $\varepsilon > 0$. Recall that $F_i \cong F \oplus (\bigoplus_{j=1}^{s}(\mathbb{Z}/q_j^{t_j}\mathbb{Z})^{m_j(i)})$, and let $k(i) = \min_j\{m_j(i)\}$, $F_i' = \bigoplus_{j=1}^{s}(\mathbb{Z}/q_j^{t_j}\mathbb{Z})^{m_j(i)}$ and $F_i'' = \bigoplus_{j=1}^{s}(\mathbb{Z}/q_j^{t_j}\mathbb{Z})^{k(i)} = (\mathbb{Z}/m\mathbb{Z})^{k(i)}$, where $m = \prod_j q_j^{t_j}$. By assumption, $k(i) \to \infty$ as $i \to \infty$. Note that $F_i'' \subset F_i'$. Thus by Lemma 5.9 we have that the set Q_i of those $h \in \mathrm{Hom}(F_i', \hat{\mathcal{A}})$ such that $h(F_i') + U_{J,\varepsilon} \supset \mathrm{Ann}(m\mathcal{A})$ has Haar measure going to 1 as $i \to \infty$. In other words Q_i is the set of those h such that $h(F_i') \in N_{U_{J,\varepsilon}}(\mathrm{Ann}(m\mathcal{A}))$. We also have that for $h \in Q_i$ and for any $h_1 \in \mathrm{Hom}(F, \hat{\mathcal{A}})$, $\mathrm{Ann}(n_i\mathcal{A}) + h(F_i') + h_1(F) \in N_{U_{J,\varepsilon}}(\mathrm{Ann}(n_i\mathcal{A}) + \mathrm{Ann}(m\mathcal{A}) + h_1(F))$.

We have now, for a continuous function f on $\mathrm{Sub}(\hat{\mathcal{A}})$, using Fubini's Theorem,

$$\lambda_i(f) = \int_{\mathrm{Hom}(F,\hat{\mathcal{A}})} \int_{\mathrm{Hom}(F_i'),\hat{\mathcal{A}}} f(\mathrm{Ann}(n_i\mathcal{A}) + h(F_i') + h_1(F)) \, d\chi_{F_i'}(h) \, d\chi_F(h_1).$$

It now follows, as in the proof of the second item, that the interior integral converges to $f(\mathrm{Ann}(n_i\mathcal{A}) + \mathrm{Ann}(m\mathcal{A}) + h_1(F))$, and so $\lambda_i(f)$ goes to

$$\int_{\mathrm{Hom}(F,\hat{\mathcal{A}})} f(\mathrm{Ann}(n_i\mathcal{A}) + \mathrm{Ann}(m\mathcal{A}) + h_1(F)) \, d\chi_F(h_1)$$

as $i \to \infty$. Since f is arbitrary, this implies the third item.

The last, fourth, item is obvious. $\square$

5.2. Infinite product of Galois fields.
In the last two subsections we derive by different means some results of [GO09] on Grassmanians over a finite field. Let V be a locally compact vector space over $\mathbb{F}_q$. Endow $\mathrm{Aut}_{\mathbb{F}_q}(V)$ with the Braconnier topology. In this setup we use the duality of vector spaces instead of Pontryagin duality: if V is a locally compact vector space over $\mathbb{F}_q$, then $V^* = \mathrm{Hom}_{\mathbb{F}_q}(V, \mathbb{F}_q)$, the set of continuous linear maps, endowed with the compact-open topology.

Since $\mathrm{Aut}_{\mathbb{F}_q}(V) \leq \mathrm{Aut}(V)$, Proposition 2.5 applies to show that $\mathrm{Aut}_{\mathbb{F}_q}(V) \simeq \mathrm{Aut}_{\mathbb{F}_q}(V^*)$. Note that $V^{**} \simeq V$ and $((\mathbb{F}_q)^{\mathbb{N}})^* \simeq \bigoplus_{\mathbb{N}} \mathbb{F}_q$.

Definition 11. We call ϕ in $\mathrm{Aut}_{\mathbb{F}_q}((\mathbb{F}_q)^{\mathbb{N}})$ (respectively in $\mathrm{Aut}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$) *finitary* if it changes only a finite number of coordinates. It is clear that finitary automorphisms form a group. Denote it by $\mathrm{Aut}_{\mathbb{F}_q}^{\mathrm{fin}}((\mathbb{F}_q)^{\mathbb{N}})$ (respectively $\mathrm{Aut}_{\mathbb{F}_q}^{\mathrm{fin}}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$).

It is easy to check that the isomorphism $\phi \mapsto (\phi^*)^{-1}$ between $\mathrm{Aut}_{\mathbb{F}_q}((\mathbb{F}_q)^{\mathbb{N}})$ and $\mathrm{Aut}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$ induces an isomorphism $\mathrm{Aut}_{\mathbb{F}_q}^{\mathrm{fin}}((\mathbb{F}_q)^{\mathbb{N}}) \simeq \mathrm{Aut}_{\mathbb{F}_q}^{\mathrm{fin}}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$.

Proposition 5.10. *The subgroup of finitary automorphisms is dense.*

Corollary 5.11. $\mathrm{CRS}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$ *consists of all measures that are* $\mathrm{Aut}_{\mathbb{F}_q}^{fin}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$-*invariant.*

Proof. Follows from Propositions 5.10 and 2.4. $\square$

This shows that elements of $\mathrm{CRS}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$ are exactly the random spaces defined in [GO09]. The proofs of Theorems 1.5 and 1.6 proceed in the same way as the proof of Theorems 1.3 and 1.4 above, using the following simplified versions of the main auxiliary results.

The next result corresponds with Theorem 4.1:

Theorem 5.12. *Let V be a countable vector space over $\mathbb{F}_q$ and let χ be an $\mathrm{Aut}_{\mathbb{F}_q}((\mathbb{F}_q)^{\mathbb{N}})$-invariant and indecomposable Borel probability measure on $\mathrm{Hom}_{\mathbb{F}_q}(V, (\mathbb{F}_q)^{\mathbb{N}})$. Then there exist a subspace $W \leq V$ such that χ is the normalized Haar measure on $\mathrm{Hom}_{\mathbb{F}_q}(V/W, (\mathbb{F}_q)^{\mathbb{N}})$.*

The next result corresponds with Proposition 5.2:

Proposition 5.13. *Let V be a countable vector space over $\mathbb{F}_q$, χ denote the Haar measure on $\mathrm{Hom}_{\mathbb{F}_q}(V, (\mathbb{F}_q)^{\mathbb{N}})$ and $h \in \mathrm{Hom}_{\mathbb{F}_q}(V, (\mathbb{F}_q)^{\mathbb{N}})$ be a random homomorphism with law χ.*

- *If V is finite, then h is injective a.s.*
- *If $V = \bigoplus_{\mathbb{N}} \mathbb{F}_q$, then $\overline{\mathrm{Image}(h)} = (\mathbb{F}_q)^{\mathbb{N}}$ a.s.*

Note that if V is a countable vector space over $\mathbb{F}_q$, then either V is finite or $V = \bigoplus_{\mathbb{N}} \mathbb{F}_q$.

The proofs of Theorem 5.12 and Proposition 5.13 are similar to the proofs of Theorem 4.1 and Proposition 5.2.

5.3. Finite dimensional approximation.

Let $F_\kappa = (\mathbb{F}_q)^\kappa$, and denote by μ_κ the measure on $\mathrm{Sub}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$ that is the law of $\mathrm{Ker}(h)$, where $h \in \mathrm{Hom}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q, F_\kappa)$ is a random homomorphism with law equal to Haar measure (see Theorem 1.6). Note μ_κ is $\mathrm{Aut}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$-invariant. Let $V_n = \bigoplus_1^n \mathbb{F}_q \subset \bigoplus_{\mathbb{N}} \mathbb{F}_q$.

Consider the map $\mathrm{Sub}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q) \to \mathrm{Sub}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$ given by $X \mapsto X \cap V_n$. Let $\mu_{\kappa,n}$ denote the probability measure on $\mathrm{Sub}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$ which is the pushforward of μ_κ under this map. Note $\mu_{\kappa,n}$ is supported on $\mathrm{Sub}_{\mathbb{F}_q}(V_n)$. Since μ_κ is $\mathrm{Aut}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q)$-invariant, $\mu_{\kappa,n}$ is $\mathrm{Aut}_{\mathbb{F}_q}(V_n)$-invariant.

We are interested in the following numbers:

$$\tilde{v}_{n,k} := \mu_\kappa \left(\left\{ X \in \mathrm{Sub}_{\mathbb{F}_q} \left(\bigoplus_{\mathbb{N}} \mathbb{F}_q \right) \mid \dim_{\mathbb{F}_q}(X \cap V_n) = k \right\} \right),$$

$$v_{n,k} := \mu_{\kappa,n}(\{W\}), \quad W \subseteq V_n, \quad \dim_{\mathbb{F}_q} W = k.$$

Since $\mu_{\kappa,n}$ is $\mathrm{Aut}_{\mathbb{F}_q}(V_n)$-invariant, $v_{n,k}$ does not depend on a specific choice of subspace W. Applying $\mathrm{Aut}_{\mathbb{F}_q}(V_n)$-invariance again, we obtain:

Lemma 5.14. *$\tilde{v}_{n,k} = d_{n,k} v_{n,k}$, where $d_{n,k}$ is the number of k-dimensional subspaces in $\mathbb{F}_q^n$.*

Proof of Theorem 1.7. Let χ denote the Haar probability measure on $\mathrm{Hom}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q, \mathbb{F}_q^\kappa)$. By definition $\mu_\kappa = \mathrm{Ker}_* \chi$. Thus

$$\tilde{v}_{n,k} = \mathrm{Ker}_* \chi \left(\left\{ h \in \mathrm{Hom}_{\mathbb{F}_q} \left(\bigoplus_{\mathbb{N}} \mathbb{F}_q, \mathbb{F}_q^\kappa \right) : \dim(\mathrm{Ker}(h \restriction V_n)) = k \right\} \right).$$

The restriction map $R : \mathrm{Hom}_{\mathbb{F}_q}(\bigoplus_{\mathbb{N}} \mathbb{F}_q, \mathbb{F}_q^\kappa) \to \mathrm{Hom}_{\mathbb{F}_q}(V_n, \mathbb{F}_q^\kappa)$ defined by $R(h) = h \upharpoonright V_n$ is a surjective homomorphism and therefore takes Haar measure to Haar measure. So if χ' is the Haar measure on $\mathrm{Hom}_{\mathbb{F}_q}(V_n, \mathbb{F}_q^\kappa)$, then

$$\tilde{v}_{n,k} = \mathrm{Ker}_* \, \chi'(\{h \in \mathrm{Hom}_{\mathbb{F}_q}(V_n, \mathbb{F}_q^\kappa) : \, \dim(\mathrm{Ker}(h)) = k\}).$$

We observe that $\mathrm{Hom}_{\mathbb{F}_q}(V_n, \mathbb{F}_q^\kappa)$ is a finite set which may be identified with the set of all $\kappa \times n$ matrices with values in $\mathbb{F}_q$. So χ' is the uniform probability measure. The theorem is finished by observing that $h \in \mathrm{Hom}_{\mathbb{F}_q}(V_n, \mathbb{F}_q^\kappa)$ satisfies $\dim(\mathrm{Ker}(h)) = k$ if and only if the matrix representing h has rank $n - k$. $\qquad \square$

We define two quantities.

$$t_n := (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}),$$

$$s_n := (q^n - 1)(q^{n-1} - 1) \cdots (q - 1).$$

Corollary 5.15. *If $n \geq 0$, $n \geq k \geq 0$, and $\kappa \geq n - k$, then*

$$\tilde{v}_{n,k} = q^{(n-k)(n-k-1)/2 - \kappa n} \frac{s_\kappa s_n}{s_{(n-k)} s_{(\kappa-n+k)} s_k}.$$

Proof. Recall that

$$|GL_n(q)| = t_n = q^{\binom{n}{2}} s_n.$$

To see this, observe that $GL_n(q)$ is in 1-1 correspondence with the set of all n-tuples $(v_1, \ldots, v_n)$ such that each $v_i \in \mathbb{F}_q^n$, $v_1 \neq 0$ and for each $i < n$, v_{i+1} is not in the span of $v_1, \ldots, v_i$. Hence there are $q^n - 1$ choices for v_1, $q^n - q$ choices for v_2, $q^n - q^2$ choices for v_3 and so on.

The group $GL_\kappa(q) \times GL_n(q)$ acts on the set $M_{\kappa,n}(\mathbb{F}_q)$ of all matrices of size $\kappa \times n$ by the rule $(A, B)M = AMB^{-1}$. The orbits of this action are exactly the sets of matrices of constant rank. The matrix

$$M_r := \begin{pmatrix} 1_{r \times r} & 0_{r \times (n-r)} \\ 0_{(\kappa-r) \times r} & 0_{(\kappa-r) \times (n-r)} \end{pmatrix}$$

has rank r. The elements of $GL_\kappa(q) \times GL_n(q)$ that stabilize it have the form

$$\left(\begin{pmatrix} a_{r \times r} & b_{r \times (\kappa-r)} \\ 0_{(\kappa-r) \times r} & d_{(\kappa-r) \times (\kappa-r)} \end{pmatrix}, \begin{pmatrix} a_{r \times r} & 0_{r \times (n-r)} \\ c_{(n-r) \times r} & d'_{(n-r) \times (n-r)} \end{pmatrix}^{-1} \right).$$

Thus the number of matrices of rank r in $M_{\kappa \times n}(\mathbb{F}_q)$ is

$$\begin{aligned}
\mathrm{rank}_{\kappa,n,r} &= \frac{|GL_\kappa(q) \times GL_n(q)|}{|\mathrm{Stab}(M_r)|} = \frac{t_\kappa t_n}{t_r t_{\kappa-r} t_{n-r} q^{r(\kappa-r)+r(n-r)}} \\
&= q^{\binom{\kappa}{2} + \binom{n}{2} - \binom{r}{2} - \binom{\kappa-r}{2} - \binom{n-r}{2} - r(\kappa-r) - r(n-r)} \frac{s_\kappa s_n}{s_r s_{(\kappa-r)} s_{(n-r)}} \\
&= q^{r(r-1)/2} \frac{s_\kappa s_n}{s_r s_{(\kappa-r)} s_{(n-r)}}.
\end{aligned}$$

Hence

$$\tilde{v}_{n,k} = \frac{\mathrm{rank}_{\kappa,n,n-k}}{q^{\kappa n}} = q^{(n-k)(n-k-1)/2 - \kappa n} \frac{s_\kappa s_n}{s_{(n-k)} s_{(\kappa-n+k)} s_k}. \qquad \square$$

Recall that $d_{n,k}$ is the number of k-dimensional subspaces in $\mathbb{F}_q^n$. We have

$$
(2) \qquad d_{n,k} = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} = \frac{s_n}{s_{(n-k)} s_k}.
$$

To see this, observe $d_{n,k}$ is the number n-tuples $(v_1, \ldots, v_k)$ with $v_i \in \mathbb{F}_q^n$ such that the span of $v_1, \ldots, v_k$ is k-dimensional divided by $|GL_k(q)|$. Thus

Corollary 5.16. *If $n \geq 0$, $n \geq k \geq 0$, and $\kappa \geq n - k$, then*

$$
v_{n,k} = \frac{\tilde{v}_{n,k}}{d_{n,k}} = q^{(n-k)(n-k-1)/2 - \kappa n} \frac{s_\kappa}{s_{(\kappa - n + k)}}.
$$

Acknowledgements

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