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EXISTENCE OF SURFACE ENERGY MINIMIZING PARTITIONS OF $\mathbb{R}^n$ SATISFYING VOLUME CONSTRAINTS

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This paper is dedicated to the memory of my parents, Dulce M. and Marino Caraballo

Abstract. We give the first proof, with independent smooth norms ϕ_{ij}, of the existence of surface energy minimizing partitions of $\mathbb{R}^n$ into regions having prescribed volumes. Our existence proof significantly extends that of F. Almgren, who in 1976 gave the first such results for the special case in which each ϕ_{ij} is a scalar multiple of a fixed smooth ϕ : $\phi_{ij} = c_{ij}\phi$. Most materials are polycrystalline and do not have surface energy density functions which are scalar multiples of one another, so it is important to extend the theory by removing this restriction, as we have done. We also discuss connections with polycrystalline evolution problems.

1. Introduction

In this paper, we give the first complete proof, with independent norms ϕ_{ij}, of the existence of partitions P of $\mathbb{R}^n$ minimizing the anisotropic surface energy

$$SE(P) = \sum_{1\le i<j\le N} \int_{p\in\Gamma_{ij}(P)} \phi_{ij}\left(n_{P(i)}(p)\right) \, d\mathcal{H}^{n-1}p, \tag{1.1}$$

among all partitions of $\mathbb{R}^n$ into N regions having prescribed volumes. The following is our main theorem. We give the proof in Section 5.

Theorem 1 (Existence of surface energy minimizing partitions). *If $v = (v_2, \ldots, v_N) \in (0,\infty)^{N-1}$ with $N \ge 2$, and $\{\phi_{ij}\}_{1\le i<j\le N}$ is a family of class 1 norms on $\mathbb{R}^n$ ($n \ge 2$) which satisfy BV-ellipticity, then there exists a Caccioppoli partition $Q = (Q(1), Q(2), \ldots, Q(N))$ of $\mathbb{R}^n$ satisfying the volume constraints $\mathcal{L}^n(Q(i)) = v_i$ for each $2 \le i \le N$ and $\mathcal{L}^n(Q(1)) = \infty$, and with $Q(i)$ bounded for each $i \ne 1$, such that $SE(Q)$ equals the infimum of $SE(A)$ among all polycrystals A which satisfy $\mathcal{L}^n(A(i)) = v_i$ for each $2 \le i \le N$ and $\mathcal{L}^n(A(1)) = \infty$.*

The individual regions $A(i)$ in the candidates A for a minimizer may be bounded or unbounded and need not be connected. Since surface energy is within a constant factor of surface area, any partition other than a Caccioppoli partition will

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have infinite surface energy. Caccioppoli partitions with N regions necessarily have precisely one region with infinite volume, which without loss of generality we take to be the first region.

In his landmark monograph [2], F. Almgren established the first-ever existence results (for $n \geq 2$) and regularity results (for $n \geq 3$) for space partitions minimizing (1.1) among all Caccioppoli partitions having $N-1$ bounded regions and one unbounded region and satisfying the volume constraints in our main theorem above, with functions ϕ_{ij} of the form $\phi_{ij} = c_{ij}\phi$ for a fixed class 1 function ϕ, and for coefficients c_{ij} satisfying a condition which he called "partitioning regularity". These hypotheses ensure lower semicontinuity of the surface energy functional (1.1) and existence of surface energy minimizing partitions. In the introduction to his monograph [2], F. Almgren described the historical importance of his work as follows:

> For $n = 2, 3$, such minimal partitioning hypersurfaces have been the subject of numerous papers in mathematics, physics, and especially biology for the past several centuries (see [53], Chap. 4, pp. 88-125, for a discussion and references). This paper presents the first proof of the general mathematical existence of such surfaces. Also, the methods are representative of those required to show the existence and minimality of solutions to a variety of geometric variational problems with constraints.

F. Almgren's work is very impressive in its scope. Existence theory in such a general context is extremely difficult, particularly since there are no a priori restrictions on the geometric and topological complexity of the minimizers. The regularity theory was strengthened for the isotropic case (as in soap films and soap bubble clusters) in $\mathbb{R}^3$ by J. Taylor in the famous paper [49].

Establishing more concrete results, even in the isotropic case, and even for just $N = 3$, can be very difficult. More than a decade after the pioneering work of F. Almgren and J. Taylor, no one had yet shown that the so-called "standard double bubble" was perimeter-minimizing, even in the case of two equal areas in the plane. In 1990, J. Foisy, M. Alfaro, J. Brock, N. Hodges, and J. Zimba showed that the standard double bubble is perimeter-minimizing in $\mathbb{R}^2$ [28]. Five years later, J. Hass, M. Hutchings, and R. Schlafly extended the result to $\mathbb{R}^3$, assuming the regions had equal volumes [34]. Another five years later, M. Hutchings, F. Morgan, M. Ritoré, and A. Ros settled the question in $\mathbb{R}^3$ for arbitrary volumes [36]. The extension to $\mathbb{R}^n$ (for each $n > 3$) was carried out almost a decade after that by B. Reichardt [47].

The anisotropic case, in which surface energy depends on interface orientation (as with crystals and polycrystals), is rather difficult, especially if independent anisotropies are allowed and if N and n can be arbitrary integers greater than one. A short time before J. Foisy et al. proved the double bubble conjecture in the plane, L. Ambrosio and A. Braides ([6], [7]) discovered the first set of conditions (BV-ellipticity) on the ϕ_{ij}'s necessary and sufficient for the lower semicontinuity of the (anisotropic) surface energy functional (1.1). In [54] B. White, also working with the case $\phi_{ij} = c_{ij}\phi$, proved lower semicontinuity under more general conditions on the c_{ij}'s than partitioning regularity. Because he used regions inside a bounded container, existence of a surface energy minimizing configuration follows immediately: any minimizing sequence is automatically uniformly bounded, so standard compactness arguments imply the existence of a convergent subsequence, and then lower semicontinuity of surface energy implies the limit polycrystal is energy minimizing.

In fact, the existence proof from the previous paragraph holds for more general ϕ_{ij}'s than considered in [54]: BV-ellipticity suffices for existence if we restrict the $P(i)$'s to all be inside some fixed, bounded open container $\Omega \subset \mathbb{R}^n$ (see Theorem 5 below). When working in $\mathbb{R}^n$ instead of in a bounded container, however, constructing a bounded minimizing sequence is extremely difficult.

In both F. Almgren's work [2] and B. White's paper [54], because the surface energy density functions are all scalar multiples of a fixed ϕ, the Wulff crystals for the ϕ_{ij}'s are all scaled copies of the Wulff crystal for ϕ. Thus, each interface essentially wants to do the same thing, and that helps with constructions for both existence and regularity. When the ϕ_{ij}'s may have quite different Wulff crystals, as in this paper, the situation is more complicated for both existence and regularity theory. F. Almgren's model applies to physical problems in which surface energy density functions are scalar multiples of one another; such assumptions have been made in the study of soap bubble clusters and incompressible immiscible fluids, for instance.

Typically, polycrystalline surface energy densities do not satisfy $\phi_{ij} = c_{ij}\phi$, so the main existence and regularity results for minimal partitions from F. Almgren's monograph [2] do not apply. Since most materials, naturally occurring or engineered, are polycrystalline, our extension in this paper of F. Almgren's landmark existence result to the setting in which the ϕ_{ij}'s may be independent, as with polycrystalline materials, is of very substantial practical importance as well as mathematical importance.

In the excellent paper [52], J. Taylor described four of the most important challenges of mathematics in materials science. Challenges 1 and 2 are stated as follows:

> Challenge 1. Make mathematical models of grain orientations and boundaries in polycrystals and how they evolve. Then use the models to determine how to control the boundaries in processing.
>
> Challenge 2. Devise a means to prove that a given soap bubble cluster is area-minimizing under the constraint of separating regions of fixed given volumes from each other and from the outside.

Concerning the second challenge, J. Taylor notes that “This challenge for soap bubble clusters is even more challenging for polycrystals with anisotropic surface energies.”

There is a strong, natural connection between the static minimization problem we consider here and crystalline and polycrystalline evolution problems, such as in annealing of metals, in which the desire to minimize surface energy is a key driving force. Consider the case ($N = 2$) of a two-region partition of space, corresponding for instance to a crystal in its melt. There is a single surface energy density function $\phi = \phi_{12}$, and it is well known that its Wulff crystal

$$W_\phi = \{x \in \mathbb{R}^n : x \cdot \xi \le \phi(\xi) \text{ for all } \xi \in \mathbb{R}^n \text{ with } |\xi| = 1\}$$

minimizes surface energy for a given volume (see, for example, [51] for useful history and good references). When surface energies are isotropic (i.e., when the surface energy densities are all multiples of the Euclidean norm ϕ_E, so that surface energy is not direction dependent), the Wulff crystals are simply balls, and spheres provide barriers for mean curvature flow. This phenomenon was first discovered by K. Brakke ([10]), and independently and several years later by R. Hamilton, M. Gage, G. Huisken, and M. Grayson in a series of remarkable papers on mean curvature flow. In [29] and [30], M. Gage and R. Hamilton showed that convex curves in the plane become asymptotically circular and smoothly shrink to a point. G. Huisken [35] then extended these results by showing that convex surfaces tend toward spheres. M. Grayson later showed that any embedded plane curve remains embedded under mean curvature flow and also becomes convex; it follows that embedded plane curves become asymptotically circular and shrink to a point (see [32]). For general surface energy density functions ϕ, as a crystal shrinks by weighted mean curvature flow (essentially a negative gradient flow of surface energy; see [50]), scaled Wulff crystals provide barriers to the evolution from both the inside and the outside (see [5]).

In mean curvature flow (resp., weighted mean curvature flow) of a crystal, the evolving crystal attempts to reduce surface area (resp., surface energy) as fast as possible. So, it is not surprising that it will attempt to become like its Wulff crystal. In this way, Wulff crystals act as barriers in various surface energy reducing evolution problems, even when volume is not conserved. It seems likely that similar knowledge of minimal polycrystals will play a significant role in the study of polycrystalline evolution problems.

In this paper, we require that the collection of norms $\{\phi_{ij}\}$ satisfy BV-ellipticity, since that condition is both necessary and sufficient for lower semicontinuity of surface energy (see [7]). As in [2], we also require that the norms ϕ_{ij} be of class C^1, since we need a technical result of F. Almgren's ([2], Lemma VI.9 (7)) which has not yet been proven to hold without this assumption. F. Almgren's monograph also allowed dependence on position; we will not do so, in order to focus on the dependence on orientation without having to assume the ϕ_{ij}'s are scalar multiples of one another. While the present paper is concerned with existence theory, in [18] we establish partial regularity results in this more general context, including necessary criteria which minimizers must satisfy. Ongoing work is focused on establishing connections between weighted mean curvature flow of polycrystals, as modeled by the author in [13], and the minimal polycrystals from the present paper, which are the polycrystalline analogues of Wulff crystals.

Our existence proof closely follows the intricate plan of F. Almgren [2]. We start with a minimizing sequence (P_k) and capture some of the volume in a limit (§5.2). We use that region for volume adjustments later (to preserve the volume constraints). We cover most of each bounded crystal in each P_k with a finite collection of balls, in such a way that the radii and the number of balls are a priori bounded (§5.4). We show that if the polycrystal extends a distance S or greater from that set (for a well-chosen S), we can truncate it to produce one with less energy and not extending more than a distance S from that set. We do this carefully, so that the surface energy savings from truncation exceeds the surface energy cost required to restore the volumes by using diffeomorphisms (§§5.3 and 5.4). We can thus produce a set (consisting of the union of a finite number of closed balls, where both the radii and the number of balls are a priori bounded) which covers the polycrystal. We can then estimate the diameter of and the number of connected components of that set, and, by translation, put the components inside some a priori fixed, closed ball, centered at the origin. Our estimates are all independent of k (for sufficiently large k), and in this way we construct a minimizing sequence which is bounded. Our estimates must be done carefully, so that the upper bounds can each be estimated a priori, and so that they do not depend on k (for sufficiently large k). Then, using a relevant compactness theorem, we deduce the existence of a limit partition satisfying the volume constraints. BV-ellipticity implies the lower semicontinuity of surface energy, and so this limit partition is surface energy minimizing.

The main new technical difficulties in our setting arise from the fact that the surface energy integrands ϕ_{ij} can be independent of one another, so that integrands for different types of interfaces can behave in radically different ways and are not simply constant multiples of a fixed function ϕ, as in [2]. The primary components of F. Almgren's existence proof in [2] are lower semicontinuity of surface energy, compactness for a certain class of partitions, Proposition VI.12, Lemma VI.13, and Proposition VI.14. Partitioning regularity is enough to guarantee lower semicontinuity of surface energy, in light of [26], Theorem 5.1.5 and its proof. With independent ϕ_{ij}'s, this does not apply, and establishing lower semicontinuity requires a very technical and delicate argument, as well as additional hypotheses to replace partitioning regularity. For our setting, BV-ellipticity is necessary and sufficient for lower semicontinuity (see [7]), so we made this assumption. The compactness argument F. Almgren used is as given in Theorem 4 below. The critically important Propositions VI.12 and VI.14 from [2] are not true as stated for general collections of ϕ_{ij}'s which are not simply scalar multiples of a single, fixed integrand, so it was necessary to prove analogous versions in our setting of these crucial results and to do so in such a way that the many pieces of the complicated proof would combine properly (see Propositions 2 and 4 below). Errors in the statement and proof of the vital Proposition VI.13 from [2] made it necessary to derive a new, corrected result, which we specialized and further modified so as to allow for unbounded regions (see Proposition 3 below). Also, it was important to determine the extent to which other hypotheses assumed in [2] would be necessary. For example, in [2] F. Almgren made use of partitioning regular integrands and partitioning regular coefficient matrices. Our existence proof requires the C^1 smoothness of surface energy integrands (one of several requirements for partitioning regularity of integrands in [2]) because we make use of [2], Lemma VI.9 (7), in an essential way, and we do not know how to prove that result or one sufficiently like it without assuming C^1 smoothness. The greatest challenge, however, is that the proof is quite complex and extremely technical. There were many non-trivial technical challenges in preparing the detailed proof itself which are not apparent from the simple proof outline in the preceding paragraph. Wherever possible we have made use of F. Almgren's results and constructions, citing relevant sections of his work.

In the book [40] and in several papers, F. Morgan (with C. French and S. Greenleaf in [44], with M. Ritoré in [45], and as sole author in [42] and [41]) and others have adapted and made essential use of F. Almgren's constructions, theorems, and general approach from [2] to prove existence theorems for related variational problems involving partitions. In Section 6 we discuss some interesting related work by F. Morgan in more detail. We have all followed the same general strategy of F. Almgren, whose ingenious methods truly "are representative of those required to show the existence and minimality of solutions to a variety of geometric variational problems with constraints".

2. Setting

2.1. **Basic notation.** Throughout this paper, we consider partitions of $\mathbb{R}^n$ into N regions, with $n \geq 2$ and $N \geq 2$. $e_1, e_2, \ldots,$ and e_n denote the standard unit basis vectors for $\mathbb{R}^n$.

Let $O(n)$ denote the group of all orthogonal mappings from $\mathbb{R}^n \to \mathbb{R}^n$, and let $A_*(\mathbb{R}^n, \mathbb{R}^n)$ denote the group of all invertible affine mappings from $\mathbb{R}^n \to \mathbb{R}^n$. We define $\tau : \mathbb{R}^n \to A_*(\mathbb{R}^n, \mathbb{R}^n)$ by setting $(\tau[p])(x) = x - p$ whenever $p, x \in \mathbb{R}^n$ so that, for each $p \in \mathbb{R}^n$, $\tau[p]$ is the translation operator on $\mathbb{R}^n$ which maps p into the origin. When f is a real-valued function and $q \in \mathbb{R}$, we define $\{f < q\} = \{x \in \operatorname{domain}(f) : f(x) < q\}$; the notations $\{f = q\}$, $\{f > q\}$, and so on have obvious meanings.

We will measure volume and surface area in $\mathbb{R}^n$ with n-dimensional Lebesgue measure $\mathcal{L}^n$ and $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$, respectively. We let $B^n(p,r)$ and $U^n(p,r)$ denote, respectively, the closed and open balls with center p and radius r in $\mathbb{R}^n$, and we set $\alpha(n) = \mathcal{L}^n(B^n(\mathbf{0},1))$, where $\mathbf{0}$ denotes the origin in $\mathbb{R}^n$. If $A, B \subset \mathbb{R}^n$, $A \triangle B$ denotes the symmetric difference of A and B: $A \triangle B = (A \setminus B) \cup (B \setminus A)$. If $A, B \subset \mathbb{R}^n$ we write $A \Subset B$ provided $\overline{A}$, the topological closure of A in $\mathbb{R}^n$, is a compact subset of the topological interior of B. If $A, B \subset \mathbb{R}^n$ and $0 < m \leq n$, we write $A \subset_m B$ (i.e., "A is $\mathcal{H}^m$ almost contained in B") when $\mathcal{H}^m(A \setminus B) = 0$, and we write $A =_m B$ (i.e., "A is $\mathcal{H}^m$ almost equal to B") when $\mathcal{H}^m(A \triangle B) = 0$. When $A \subset \mathbb{R}^n$, we let $\partial_{top} A = \overline{A} \cap \overline{\mathbb{R}^n \setminus A}$ denote the topological boundary of A. We note that $x \in \partial_{top} A$ if and only if for each $r > 0$ we have $U^n(x,r) \cap A \neq \emptyset$ and $U^n(x,r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$. Given a point $p \in \mathbb{R}^n$ and a unit vector $u \in \mathbb{R}^n$, we define the open half-spaces $H_+(p,u) = \{x : (x-p) \cdot u > 0\}$ and $H_-(p,u) = \{x : (x-p) \cdot u < 0\}$. $\gamma = \left[n\alpha(n)^{1/n}\right]^{-1}$ is the optimal isoperimetric constant, so that $\mathbf{M}(X)^{(n-1)/n} \leq \gamma \mathbf{M}(\partial X)$ for all $X \in \mathcal{I}_n$, with $n > 1$.

For the remainder of this section, suppose $0 \leq m \leq n$. Here, we follow closely the notation from H. Federer's treatise [26] (see [26], Chapter 1). Let $\Lambda(n,m)$ be the set of all strictly increasing functions from $\{1, 2, \ldots, m\}$ into $\{1, 2, \ldots, n\}$. $\Lambda_m \mathbb{R}^n$ is the vector space of m-vectors of $\mathbb{R}^n$. Whenever $\xi, \eta \in \Lambda_m \mathbb{R}^n$ we let $\xi \cdot \eta$ denote their inner product. $|\cdot|$ and $\|\cdot\|$ are the standard norm and the mass norm, respectively, on $\Lambda_m \mathbb{R}^n$. $\Lambda^m \mathbb{R}^n$ is the dual vector space of *m-covectors* of $\mathbb{R}^n$. We have $\langle \xi, \omega \rangle = \omega(\xi) \in \mathbb{R}$ whenever $\xi \in \Lambda_m \mathbb{R}^n$ and $\omega \in \Lambda^m \mathbb{R}^n$, and $\|\cdot\|$ denotes the comass norm on $\Lambda^m \mathbb{R}^n$. Let $G(n,m)$ denote the Grassmann manifold of all m-dimensional subspaces of $\mathbb{R}^n$.

A function $\phi : \mathbb{R}^n \to [0, \infty)$ is called a *surface energy density function* provided it is a continuous, positive-valued function on unit vectors in $\mathbb{R}^n$, which, when extended by positive homogeneity of degree one to all of $\mathbb{R}^n$, becomes a convex function. That is, $\phi : \mathbb{R}^n \to [0, \infty)$ is a surface energy density function provided ϕ satisfies each of the properties of a norm on $\mathbb{R}^n$ except possibly for the requirement that ϕ be an even function. If it is even, i.e., if $\phi(v) = \phi(-v)$ for all vectors v in $\mathbb{R}^n$, then ϕ is a norm on $\mathbb{R}^n$. A surface energy density function ϕ on $\mathbb{R}^n$ is of *class 1* if it is continuously differentiable at each $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. ϕ_E is the Euclidean norm on $\mathbb{R}^n$, induced by the standard inner product: $\phi_E : v \to \langle v, v \rangle^{1/2}$.

$\{\phi_{ij}\}_{1 \leq i < j \leq N}$ will always denote a family of surface energy density functions, with ϕ_{ij} being the density function for the i-j interface. At times it will be convenient to extend this family to a larger collection $\{\phi_{ij}\}_{1 \leq i,j \leq N}$, in which case we always follow the standard conventions $\phi_{ii} \equiv 0$ and $\phi_{ji}(v) = \phi_{ij}(-v)$ for each $v \in \mathbb{R}^n$. Of course, if the ϕ_{ij}'s in the original collection are norms, then in the extended collection we will have $\phi_{ji} = \phi_{ij}$ for each $i, j \in \{1, 2, \ldots, N\}$.

Whenever $\{\phi_{ij}\}_{1 \le i < j \le N}$ is a collection of surface energy density functions, we define ϕ_0 and ϕ^0 as follows:

$$\phi_0 = \inf_{|v|=1,\ 1 \le i < j \le N} \{\phi_{ij}(v)\} \le \sup_{|v|=1,\ 1 \le i < j \le N} \{\phi_{ij}(v)\} = \phi^0. \tag{2.1}$$

A continuity-compactness argument implies these extrema are attained and that $0 < \phi_0 \le \phi^0 < \infty$, so

$$0 < \phi_0 \le \phi_{ij}(v) \le \phi^0 < \infty, \text{ whenever } |v| = 1 \text{ and } 1 \le i < j \le N, \tag{2.2}$$

and so any piece of interface has surface energy bounded below by ϕ_0 times its surface area and bounded above by ϕ^0 times its surface area. If we have an extended collection $\{\phi_{ij}\}_{1 \le i,j \le N}$, then we simply replace "$1 \le i < j \le N$" with "$1 \le i, j \le N$ and $i \ne j$" in (2.1) and (2.2); of course, ϕ_0 and ϕ^0 do not change when extending $\{\phi_{ij}\}_{1 \le i < j \le N}$ to $\{\phi_{ij}\}_{1 \le i,j \le N}$ as above. If $\tau \in G(n, n-1)$, its orthogonal complement $\tau^\perp$ is the one-dimensional subspace of $\mathbb{R}^n$ orthogonal to the hyperplane τ. Let τ^* denote a unit vector in $\tau^\perp$. Then $-\tau^*$ is likewise a unit vector in $\tau^\perp$, and there are no other such vectors. We can use this elementary construction to extend a norm ϕ on $\mathbb{R}^n$ to a geometric integrand $\widehat{\phi}$ (using F. Almgren's terminology from [2], I.1) as follows: whenever ϕ is a norm on $\mathbb{R}^n$, we define its corresponding *geometric integrand*, $\widehat{\phi} : \mathbb{R}^n \times G(n, n-1) \to [0, \infty)$, by setting $\widehat{\phi}(x, \tau) = \phi(\tau^*)$ whenever $x \in \mathbb{R}^n$ and $\tau \in G(n, n-1)$. Here, τ^* is a unit vector in $\tau^\perp$. Because ϕ is a norm, $\phi(\tau^*) = \phi(-\tau^*)$, and so $\widehat{\phi}$ is well defined.

2.2. **BV functions and sets of finite perimeter.** If $K \subset \mathbb{R}^n$, $p \in \mathbb{R}^n$, and $1 \le m \le n$, the *m-dimensional density of K at p* is

$$\Theta^m(K, p) = \lim_{r \to 0} \left(\mathcal{H}^m(K \cap B^n(p, r)) \;/\; \alpha(m) r^m\right),$$

provided the limit exists. If $K \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$, the vector $u \in \mathbb{R}^n$ is called a *measure-theoretic exterior unit normal to K at p in the sense of Federer* (cf. [27] and [26], 4.5.5) provided $|u| = 1$, $\Theta^n(H_+(p, u) \cap K,\ p) = 0$, and $\Theta^n(H_-(p, u) \setminus K,\ p) = 0$. If no such u exists, we define $n_K(p) = \mathbf{0}$, while if such a u exists it is necessarily unique ([27], Theorem 3.4) and we define $n_K(p) = u$. The function $n_K(\cdot)$ implicitly defines a notion of boundary, the set $\partial_F K$ of points $p \in \mathbb{R}^n$ at which $n_K(p)$ is nonzero: $\partial_F K = \{p \in \mathbb{R}^n : |n_K(p)| = 1\}$. The boundary $\partial_F K$ of K in the sense of Federer relates to the reduced boundary $\partial^* K$ of K introduced by E. De Giorgi (see [19] and [20]) as follows: $\partial_F K \supset \partial^* K$ and $\partial_F K =_{n-1} \partial^* K$; it is a subset of the topological boundary $\partial_F K \subset \partial_{top} K$. R. Caccioppoli had considered a similar boundary as well (see [12]); indeed, in [20] E. De Giorgi remarked that the reduced boundary $\partial^* K$ of a set K consists of the set of centers of the elements of the oriented boundary of K introduced by R. Caccioppoli (see [12]). In this paper, we will work primarily with the boundary in the sense of Federer. If K and L are $\mathcal{L}^n$ measurable subsets of $\mathbb{R}^n$ with $\mathcal{L}^n(K \triangle L) = 0$, then $\partial_F K = \partial_F L$.

Let $u \in L^1_{loc}(\mathbb{R}^n)$. Whenever $U \subset \mathbb{R}^n$ is open, we define the *total variation* of u in U as follows:

$$TV(u,U) = \sup_{\substack{\phi \in C^1_c(U,\mathbb{R}^n) \\ |\phi(x)| \le 1 \text{ for all } x \in U}} \left\{ \int_{x\in U} u(x) \operatorname{div} \phi(x)\, d\mathcal{L}^n x \right\}. \tag{2.3}$$

We can extend $TV(u,\cdot)$ to be a Borel measure on $\mathbb{R}^n$ by setting

$$TV(u,E) = \inf\{TV(u,U) : \ U \text{ is an open set containing } E\}$$

for any Borel set E in $\mathbb{R}^n$. Then $TV(u,\cdot)$ is a Radon measure on $\mathbb{R}^n$ if and only if $TV(u,K) < \infty$ for all compact $K \subset \mathbb{R}^n$. We note that if $u \in C^1(\mathbb{R}^n)$, then integration by parts gives $TV(u,\mathbb{R}^n) = \int_{\mathbb{R}^n} |\nabla u|\, d\mathcal{L}^n$. We let $BV(\mathbb{R}^n) = \{f : f \in L^1(\mathbb{R}^n), \text{ and } TV(f,\mathbb{R}^n) < \infty\}$ denote the space of *functions of bounded variation in* $\mathbb{R}^n$, and we let

$$BV_{loc}(\mathbb{R}^n) = \left\{f : f \in L^1_{loc}(\mathbb{R}^n), \text{ and } TV(f,K) < \infty \text{ for all compact } K \subset \mathbb{R}^n\right\}$$

be the space of functions of *locally bounded variation in* $\mathbb{R}^n$.

When A is an $\mathcal{L}^n$ measurable subset of $\mathbb{R}^n$, we let $P(A) = TV(\chi_A,\mathbb{R}^n)$ be the perimeter of A in $\mathbb{R}^n$, where χ_A denotes the characteristic function of A, taking the value 1 inside A and 0 elsewhere. We say that A has *finite perimeter* in $\mathbb{R}^n$ provided $P(A) < \infty$, and we say that A has *locally finite perimeter* in $\mathbb{R}^n$ provided $\chi_A \in BV_{loc}(\mathbb{R}^n)$. We have $P(A) = \mathcal{H}^{n-1}(\partial_F A) < \infty$ whenever $A \subset \mathbb{R}^n$ is $\mathcal{L}^n$ measurable and has finite perimeter in $\mathbb{R}^n$. $\mathcal{C}$ denotes the collection of all Lebesgue measurable subsets K of $\mathbb{R}^n$ having finite perimeter and finite volume; elements of $\mathcal{C}$ may be unbounded. Some excellent references that treat sets of finite perimeter and functions of bounded variation in detail are [8], [11], [23], [31], [37], [39], and [55].

2.3. **Currents and varifolds.** Let

$$\mathcal{D}^m = \{\varphi : \varphi \text{ is a } C^\infty \text{ differential } m \text{ form on } \mathbb{R}^n \text{ having compact support}\}.$$

An *m-dimensional current T in* $\mathbb{R}^n$ is any element of $\mathcal{D}_m$, the dual space of the real vector space $\mathcal{D}^m$. The *support* of a current $T \in \mathcal{D}_m$ is

$$\operatorname{spt} T = \bigcap \{\Omega \subset \mathbb{R}^n : \Omega \text{ is closed and } \operatorname{spt}(\varphi) \cap \Omega = \emptyset \Rightarrow T(\varphi) = 0\},$$

as with distributions. For each open set W, we define the *mass* of $T \in \mathcal{D}_m$ according to

$$\mathbf{M}_W(T) = \sup\{T(\varphi) : \varphi \in \mathcal{D}^m, \ |\varphi| \le 1, \text{ and } \operatorname{spt}\varphi \subset W\}.$$

When $W = \mathbb{R}^n$, we write $\mathbf{M}(T)$. If $m > 0$ the *boundary* of $T \in \mathcal{D}_m$ is defined so as to satisfy Stokes's theorem; i.e., $\partial T(\varphi) = T(d\varphi)$ whenever $\varphi \in \mathcal{D}^{m-1}$. ∂T is therefore an $m-1$ current in $\mathbb{R}^n$. An m-dimensional current T in $\mathbb{R}^n$ is called *representable by integration in* $\mathbb{R}^n$ provided there exist a Radon measure $\|T\|$ on $\mathbb{R}^n$ (called the *variation measure* of T) and an m-vectorfield $\overrightarrow{T} : \mathbb{R}^n \to \Lambda_m \mathbb{R}^n$ (called the *orientation function* of T) such that $\left\|\overrightarrow{T}(x)\right\| = 1$ for $\|T\|$ almost every $x \in \mathbb{R}^n$, and such that $T = \|T\| \wedge \overrightarrow{T}$ in the sense that

$$T(\varphi) = \int_{x\in\mathbb{R}^n} \left\langle \overrightarrow{T}(x),\ \varphi(x) \right\rangle d\|T\|\, x$$

whenever $\varphi \in \mathcal{D}^m$. If $T \in \mathcal{D}_m$ is such that $\mathbf{M}_W(T) < \infty$ whenever W is bounded, then T is representable by integration in $\mathbb{R}^n$ and $\mathbf{M}_W(T) = \|T\|(W)$ for any open set $W \subset \mathbb{R}^n$.

A current $T \in \mathcal{D}_m$ is an *integer multiplicity locally rectifiable m-dimensional current in* $\mathbb{R}^n$, and we write $T = \mathbf{t}(S, \theta, \overrightarrow{T}) \in \mathcal{R}_m^{\text{loc}}$, provided T is representable by integration as $T = \|T\| \wedge \overrightarrow{T}$, with $\|T\| = \theta\,(\mathcal{H}^m \llcorner S)$, so that

$$T(\varphi) = \int_{x \in S} \left\langle \overrightarrow{T}(x),\ \varphi(x) \right\rangle \theta(x)\, d\mathcal{H}^m x$$

for each $\varphi \in \mathcal{D}^m$, where

1) S is an $\mathcal{H}^m$ measurable and countably m rectifiable subset of $\mathbb{R}^n$,

2) $\theta : S \to \{1, 2, 3, \dots\}$ is $\mathcal{H}^m \llcorner S$ summable (θ is the multiplicity, or density, function for the set S), and

3) $\overrightarrow{T} : S \to \Lambda_m \mathbb{R}^n$ is the orientation function, where $\left\|\overrightarrow{T}(x)\right\| = 1$, $\overrightarrow{T}(x)$ is a simple m-vector, and $\operatorname{Tan}^m(\mathcal{H}^m \llcorner S, x)$ is the m-dimensional linear subspace of $\mathbb{R}^n$ associated with $\overrightarrow{T}(x)$ for $\mathcal{H}^m$ almost every $x \in S$ (see [26], 4.1.28, [4], [5], 3.1.3).

If $T = \mathbf{t}(S, \theta, \overrightarrow{T}) \in \mathcal{R}_m^{loc}$, then $S \subset \operatorname{spt} T$ since $(\operatorname{spt}\varphi) \cap \overline{S} = \emptyset \Rightarrow T(\varphi) = 0$. However, the set $\operatorname{spt} T$ can be much bigger than S in general, unless S is known to be regular. A current $T = \mathbf{t}(S, \theta, \overrightarrow{T}) \in \mathcal{R}_m^{\text{loc}}$ is called an *integer multiplicity rectifiable m-dimensional current in* $\mathbb{R}^n$, and we write $T \in \mathcal{R}_m$, provided S is also bounded and m rectifiable. A current $T \in \mathcal{D}_m$ is an *m-dimensional integral current in* $\mathbb{R}^n$, and we write $T \in \mathcal{I}_m$, if T is a rectifiable m current and (for $m > 0$ only) if ∂T is a rectifiable $(m-1)$ current. It follows from the closure theorem ([26], 4.2.16) that whenever $m > 0$, $\mathcal{I}_m = \{T \in \mathcal{R}_m : \mathbf{M}(\partial T) < \infty\}$. For $T \in \mathcal{D}_m$ we also define the so-called *flat norm* $\mathcal{F}$ (which is a seminorm on $\mathcal{D}_m$) as the infimum of the numbers $\mathbf{M}(Q) + \mathbf{M}(R)$, where the infimum is taken over all currents $R \in \mathcal{R}_m$ and (if $m < n$) $Q \in \mathcal{R}_{m+1}$ for which $T = R + \partial Q$. In particular, rectifiable $(n-1)$ currents S and T are flat close to each other when $S - T$ can be altered slightly (i.e., the piece R has small mass) so as to bound a crystal Q having small mass. $\mathcal{F}$ is useful for determining how close together surfaces are geometrically. The *weak topology* on $\mathcal{D}_m$ is specified by asserting $T_i \to T$ weakly if and only if $T_i(\varphi) \to T(\varphi)$ pointwise for each $\varphi \in \mathcal{D}^m$. Convergence in the mass norm (strong convergence) implies convergence in the flat norm (flat convergence), which implies convergence on fixed m forms (weak convergence).

If $K \in \mathcal{C}$, we let $T = [K]$ be the multiplicity one n current naturally associated with K, i.e., $T = \mathcal{L}^n \llcorner K \wedge (e_1 \wedge \cdots \wedge e_n) = \mathbf{t}(K, 1, e_1 \wedge \cdots \wedge e_n)$, so that

$$[K](w) = \int_{x \in K} \langle e_1 \wedge \cdots \wedge e_n,\ w(x) \rangle\, d\mathcal{L}^n x \tag{2.4}$$

for each $w \in \mathcal{D}^m$. The boundary $(n-1)$ current $\partial[K]$ is given by $\partial[K] = \mathcal{H}^{n-1} \llcorner \partial_F K \wedge *n_K = \mathbf{t}(\partial_F K, 1, *n_K)$, where $*n_K$ is the Hodge dual of n_K. Also, $\mathbf{M}(\partial[K]) = \mathcal{H}^{n-1}(\partial_F K)$. If $K \in \mathcal{C}$, we may also define the complementary current $[\mathbb{R}^n \setminus K] = \mathbf{t}(\mathbb{R}^n \setminus K, 1, e_1 \wedge \cdots \wedge e_n)$.

Suppose $K \in \mathcal{C}$ and let $T = \partial[K]$. Then $\mathbf{M}(T) < \infty$, so T is representable by integration. We may therefore apply [26], 4.5.6, to deduce that $T \in \mathcal{R}_{n-1}^{loc}$, $\|T\| = \mathcal{H}^{n-1} \llcorner \partial_F K$, and for $\mathcal{H}^{n-1}$ almost all $x \in \partial_F K$ we have $\Theta^{n-1}(\partial_F X, x) = 1$,

$\mathrm{Tan}^{n-1}(\partial_F X, x) \in G(n, n-1)$, and $*n_K(x) = \overrightarrow{T}(x)$. This motivates the following definition. We define $\mathrm{Bdy}(K)$ to be the intersection of $\partial_F K$ with the set

$$\left\{x : \Theta^{n-1}(\partial_F X, x) = 1, \ \ \mathrm{Tan}^{n-1}(\partial_F X, x) \in G(n, n-1), \ \ * n_K(x) = \overrightarrow{T}(x)\right\}.$$

We have $\mathrm{Bdy}(K) \subset \partial_F K$ and $\mathrm{Bdy}(K) =_{n-1} \partial_F K$. We also define the *measure-theoretic interior* and *measure-theoretic exterior* of any $K \in \mathcal{C}$ by setting $\mathrm{Int}(K) = \{x : \Theta^n(K, x) = 1\}$ and $\mathrm{Ext}(K) = \{x : \Theta^n(K, x) = 0\}$. When $K \in \mathcal{C}$, the sets $\mathrm{Bdy}(K)$, $\mathrm{Int}(K)$, and $\mathrm{Ext}(K)$ are disjoint, and [26], 4.5.6, implies that their union is $\mathbb{R}^n$ up to a set having $\mathcal{H}^{n-1}$ measure 0. We note that slightly different measure-theoretic notions of boundary, interior, and exterior have been used in the literature. Typically, these definitions agree with ours up to sets having $\mathcal{H}^{n-1}$ measure 0.

The following theorem can be deduced from results on the slicing of locally rectifiable currents by Lipschitz functions in [48], §28.

Theorem 2 (Slicing of crystals)**.** *Suppose* $T = [K]$ *for some* $K \in \mathcal{C}$ *and* $f(x) = \mathrm{dist}(x, B)$ *for some compact* $B \subset \mathbb{R}^n$. *Then* f *is Lipschitz with* $\mathrm{Lip}\, f = 1$, *and for* $\mathcal{L}^1$ *almost every* $q > 0$ *each of the following is true:*

(1) $T \llcorner \{f < q\} \in \mathcal{I}_n$.

(2) *The* $(n-1)$*-dimensional slice current* $\langle T, f, q\rangle$ *satisfies*

$$\begin{aligned} \langle T, f, q\rangle &= \partial\left[T \llcorner \{f < q\}\right] - (\partial T) \llcorner \{f < q\}, \\ \mathrm{spt}\,\langle T, f, q\rangle &\subset (\mathrm{spt}\, T) \cap \{f = q\}, \\ \mathbf{M}(\langle T, f, q\rangle) &\leq m'(q), \ \ \textit{where } m(q) = \|T\|(\{f < q\}). \end{aligned}$$

The standard reference for currents is the treatise [26] by H. Federer. [4], [11], [24], [25], [33], [38], [40], and [48] are also very good references for currents.

An $(n-1)$-*dimensional varifold in* $\mathbb{R}^n$ is simply a Radon measure (i.e., a Borel regular measure which is finite on compact sets) on $\mathbb{R}^n \times G(n, n-1)$. In accordance with the Riesz representation theorem, when V is an $(n-1)$-dimensional varifold in $\mathbb{R}^n$ and $f : \mathbb{R}^n \times G(n, n-1) \to \mathbb{R}$ is a continuous function with compact support, we write $V(f) = \int f \, dV$. We let $\mathcal{V}_{n-1}$ denote the space of all $(n-1)$-dimensional varifolds in $\mathbb{R}^n$, endowed with the weak topology on Radon measures: $V_i \to V$ *weakly* if and only if $V_i(f) \to V(f)$ pointwise for each $f \in C_c(\mathbb{R}^n \times G(n, n-1))$. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, and $V \in \mathcal{V}_m$ with $2 \leq m \leq n$. We define the varifold $f_\# V \in \mathcal{V}_m$ by requiring that

$$(f_\# V)(B) = \int_{\{(x,T):(f(x), Df(x)(T)) \in B\}} |\Lambda_m Df(x) \circ T| \, dV(x, T)$$

whenever B is a Borel subset of $\mathbb{R}^n \times G(n, m)$ (see [1], 3.2, and [2], 1.1 (13)). Suppose S is an $n-1$ rectifiable subset of $\mathbb{R}^n$. We can define the $(n-1)$-dimensional varifold $|S|$ by setting $|S|(A) = \mathcal{H}^{n-1}\left(S \cap \left\{x : \left(x, \ \mathrm{Tan}^{n-1}\left(\mathcal{H}^{n-1} \llcorner S, \ x\right)\right) \in A\right\}\right)$ for each $A \subset \mathbb{R}^n \times G(n, n-1)$. If ϕ is a norm on $\mathbb{R}^n$, and $\widehat{\phi}$ its corresponding geometric integrand, we define $\widehat{\Phi}(|S|) = \int \widehat{\phi} \, d|S|$. We will write $\widehat{\Phi}_{ij}(|S|)$ if $\phi = \phi_{ij}$ for some $1 \leq i < j \leq N$, and we will write $\widehat{\Phi}_E(|S|)$ if $\phi = \phi_E$, the Euclidean norm. The *mass* of the varifold $|S|$ is given by $\mathbf{M}(|S|) = \widehat{\Phi}_E(|S|) = \mathcal{H}^{n-1}(S)$. [1], [2], [24], [33], [40], and [48] are some good references for varifolds.

2.4. **Polycrystals.** By a *partition* or *polycrystal* in $\mathbb{R}^n$ we mean an N-tuple $P = (P(1), P(2), \ldots, P(N))$ such that $P(i)$ is an $\mathcal{L}^n$ measurable subset of $\mathbb{R}^n$ for each

i, $P(i)$ has locally finite perimeter in $\mathbb{R}^n$ for each i, $P(i) \cap P(j) =_n \emptyset$ whenever $i \neq j$, and $\bigcup P(i) =_n \mathbb{R}^n$. We refer to the sets $P(i)$ as *regions* or *crystals.* A partition P is called a *Caccioppoli partition* of $\mathbb{R}^n$ if additionally $P(i)$ has finite perimeter in $\mathbb{R}^n$ for each i. Because of the isoperimetric inequality, any Caccioppoli partition has precisely one region having infinite volume. We let $\mathcal{P}^N$ denote the subcollection of all Caccioppoli partitions of $\mathbb{R}^n$ such that $P(i)$ is bounded for each $i = 2, 3, \ldots, N$.

Definition 1. If (P_k) is a sequence of polycrystals and P is a polycrystal, we say that $P_k \to P$ **in volume** (or **strongly**) provided $\mathcal{L}^n(P_k(i) \triangle P(i)) \to 0$ as $k \to \infty$ for each $1 \leq i \leq N$.

Definition 2. Whenever P is a polycrystal and $1 \leq i < j \leq N$, we define the ***i*-*j* interface** of P as

$$\Gamma_{ij}(P) = \mathrm{Bdy}(P(i)) \cap \mathrm{Bdy}(P(j)),$$

and we define the corresponding varifold $V(i,j) = |\Gamma_{ij}(P)| \in \mathcal{V}_{n-1}$ by setting

$$V(i,j)(A) = \mathcal{H}^{n-1}\left\{x \in \Gamma_{ij}(P) : \left(x, \operatorname{Tan}^{n-1}\left(\mathcal{H}^{n-1} \llcorner \Gamma_{ij}(P), x\right)\right) \in A\right\}$$

for each $A \subset \mathbb{R}^n \times G(n, n-1)$.

Definition 3 (Surface energy and surface area)**.** Suppose P is a polycrystal, $E \subset \mathbb{R}^n$ is an open or closed set, and $\{\phi_{ij}\}_{1 \leq i < j \leq N}$ is a family of surface energy density functions. Whenever $1 \leq i < j \leq N$ we define

$$SE_{ij}(P, E) = \int_{p \in \Gamma_{ij}(P) \cap E} \phi_{ij}\left(n_{P(i)}(p)\right) \, d\mathcal{H}^{n-1}p$$

if $\mathcal{H}^{n-1}(\Gamma_{ij}(P) \cap E) < \infty$, and $SE_{ij}(P, E) = \infty$ otherwise. We then define

$$SE(P, E) = \sum_{1 \leq i < j \leq N} SE_{ij}(P, E). \tag{2.5}$$

If $E = \mathbb{R}^n$ we simply write $SE_{ij}(P)$ and $SE(P)$, respectively. Also, when $\phi_{ij} = \phi_E$ for each i and j with $i \neq j$, surface energy becomes surface area, and we use the notation $SA_{ij}(P, E)$, $SA(P, E)$, $SA_{ij}(P)$, and $SA(P)$ accordingly.

Definition 4 (Volume constraints)**.** Whenever $v = (v_2, \ldots, v_N) \in (0, \infty)^{N-1}$, we let $\mathcal{P}_*^{N,v}$ denote the collection of all Caccioppoli partitions such that $\mathcal{L}^n(P(i)) = v_i$ for each $2 \leq i \leq N$. We let $\mathcal{P}^{N,v} = \left\{P \in \mathcal{P}_*^{N,v} : \mathbb{R}^n \setminus P(1) \text{ is bounded}\right\}$.

Definition 5. A family, $\mathcal{F}$, of polycrystals is called **M**-closed if whenever $P_k \to P$ in volume for some sequence (P_k) of polycrystals in $\mathcal{F}$ and for some polycrystal P we have $P \in \mathcal{F}$.

Proposition 1. *If $N \geq 2$ and $v = (v_2, \ldots, v_N) \in (0, \infty)^{N-1}$, then $\mathcal{P}^{N,v}$ is* **M**-*closed.*

Proof. Suppose $Q \in \mathcal{P}^N$ and (P_k) is any sequence in $\mathcal{P}^{N,v}$ for which $P_k \to Q$ in volume. It follows that $\mathcal{L}^n(Q(i) \triangle P_k(i)) \to 0$ as $k \to \infty$ for each $2 \leq i \leq N$. Then $\mathcal{L}^n(Q(i)) = v_i$ for each $2 \leq i \leq N$, so $Q \in \mathcal{P}^{N,v}$, as desired (also, see [7]). $\square$

Definition 6. Whenever P and Q are polycrystals we define

$$d(P, Q) = \sup_{1 \leq i \leq N} \{\mathcal{L}^n(P(i) \triangle Q(i))\}.$$

3. Lower semicontinuity, compactness, and existence theorems

3.1. **Lower semicontinuity and BV-ellipticity.** L. Ambrosio and A. Braides discovered the first necessary and sufficient condition for lower semicontinuity of the surface energy functional (2.5) with respect to convergence in volume, an integral condition they named BV-ellipticity (cf. [6], [7]). BV-ellipticity ensures that certain perturbations of a planar interface cannot have less surface energy than the original planar interface. It is analogous, for the setting of Caccioppoli partitions of $\mathbb{R}^n$, to C. B. Morrey's quasi-convexity [46].

BV-ellipticity is an integral condition, and, like quasi-convexity, it is not easy to check in practice. Therefore, many other conditions on the ϕ_{ij}'s, sufficient for lower semicontinuity of the surface energy functional (2.5), have been introduced and studied, such as (B)-convexity (introduced in [7]; cf. [8]), joint convexity (see [8] but also [7]), LSC1 and LSC3 (introduced in [13]), B2-convexity (introduced in [43]), and A-convexity, A2-convexity, and directional control (introduced in [14]).

For a collection $\{\phi_{ij}\}_{1\leq i,j\leq N}$ of surface energy density functions, the triangle inequalities $\phi_{ik} \leq \phi_{ij} + \phi_{jk}$ are necessary for lower semicontinuity (or else two parallel planar interfaces can meet and cause surface energy to suddenly increase). In [7], L. Ambrosio and A. Braides showed that the triangle inequalities are sufficient for lower semicontinuity (hence equivalent to BV-ellipticity) when the number of regions, N, equals 3. They also gave an example in the plane demonstrating that lower semicontinuity may fail even if the triangle inequalities hold when $N \geq 6$. In [15], we completed the analysis by showing in $\mathbb{R}^n$ that the triangle inequalities and BV-ellipticity are not equivalent when $N > 3$.

It seemed that B2-convexity might be equivalent to BV-ellipticity. As F. Morgan noted in [43], the example L. Ambrosio and A. Braides used in [7] to show that (B)-convexity is not necessary for lower semicontinuity does not apply to B2-convexity. Also, in the important special case when $\phi_{uv} = c_{uv}\phi$, for a norm ϕ and for positive constants c_{uv} satisfying the triangle inequalities $c_{uv} \leq c_{uw} + c_{wv}$ for each (u, v, w) triple (as with immiscible fluids; see also [2] and [54]), B2-convexity is necessary for lower semicontinuity [43]. Moreover, when $\{\phi_{uv}\}$ is a family of surface energy density functions, each of the conditions above on the ϕ_{uv}'s, except for joint convexity, implies B2-convexity (see [16]). However, in [16] we established that B2-convexity is not necessary for lower semicontinuity; it follows that none of the conditions which implies it can be equivalent to BV-ellipticity, which thus remains the only condition known to be necessary and sufficient for lower semicontinuity (except in very special cases). It is still unknown whether joint convexity is equivalent to BV-ellipticity in general.

Definition 7. A collection $\{\phi_{ij}\}_{1\leq i<j\leq N}$ of surface energy density functions on $\mathbb{R}^n$ satisfies **BV-ellipticity** if $SE\,(P, Q) \geq \phi_{ij}\,(w)$ whenever $1 \leq i < j \leq N$, $w \in \mathbb{R}^n$ is a unit vector, Q is an open unit cube in $\mathbb{R}^n$, centered at the origin and having all faces parallel or perpendicular to w, and P is a Caccioppoli partition in $\mathbb{R}^n$ with $\bigcup_{h\notin\{i,j\}} P\,(h) \Subset Q$, $H_-(\mathbf{0}, w) \setminus Q \subset P\,(i)$, and $H_+(\mathbf{0}, w) \setminus Q \subset P\,(j)$.

Consider a polycrystal P satisfying $\bigcup_{h\notin\{i,j\}} P\,(h) =_n \emptyset$, $P\,(i) =_n H_-(\mathbf{0}, w)$, and $P\,(j) =_n H_+(\mathbf{0}, w)$, where i, j, w, and Q are as in Definition 7. Then $SE\,(P, Q) = \phi_{ij}\,(w)$. Thus we see that, when BV-ellipticity holds, bounded perturbations of a planar interface, possibly involving other regions, are never cheaper than the original planar interface.

The following theorem is due to L. Ambrosio and A. Braides (see [7]). For an alternative proof see [17], in which we also consider lower semicontinuity with respect to flat and weak convergence of integral currents.

Theorem 3 (BV-ellipticity implies strong lower semicontinuity). *Suppose the surface energy density functions* $\{\phi_{ij}\}_{1\leq i<j\leq N}$ *satisfy BV-ellipticity. Suppose* (P_k) *is a sequence of Caccioppoli partitions converging in volume to a Caccioppoli partition* P. *Then* $SE(P) \leq \liminf_{k\to\infty} SE(P_k)$.

3.2. **Compactness and existence theorems.** If we work inside a fixed compact set having finite positive volume, so that all finite-volume regions of all polycrystals are required to be contained in that set, then the existence of a surface energy minimizer satisfying volume constraints is relatively easy to establish, and in this section we give the necessary compactness and existence results. Later, we will prove our main existence result, Theorem 1, by starting with a general minimizing sequence and modifying it to produce a bounded minimizing sequence so that we may apply Theorem 5.

Definition 8. If $X \subset \mathbb{R}^n$ is compact and if $C > 0$, we define $\mathcal{P}^N(X, C)$ to be the collection of all polycrystals P with $\operatorname{spt}([\mathbb{R}^n \setminus P(1)]) \subset X$ and $SA(P) \leq C$.

This compactness result was first used by F. Almgren (see [2], VI.15 (8)). See [14] for a proof.

Theorem 4 (Compactness theorem for polycrystals). *For any positive real numbers* C *and* R, *any sequence in* $\mathcal{P}^N(B^n(\mathbf{0}, R), C)$ *has a strongly convergent subsequence.*

The following existence theorem, established in [14], reduces the question of the existence of surface energy minimizers to the construction of a bounded minimizing sequence, provided the space is **M**-closed.

Theorem 5 (General existence theorem). *Suppose* $R > 0$. *Suppose the surface energy density functions* $\{\phi_{ij}\}_{1\leq i<j\leq N}$ *satisfy BV-ellipticity. Suppose* $\mathcal{F}$ *is any* **M**-*closed, non-empty collection of polycrystals in* $\mathcal{P}^N$ *for which there exists a surface energy minimizing sequence* (P_k) *of polycrystals in* $\mathcal{F}$ *with* $\operatorname{spt}([\mathbb{R}^n \setminus P_k(1)]) \subset B^n(\mathbf{0}, R)$ *for each* $k \geq 1$. *Then there exists a* $P \in \mathcal{F}$ *for which*

$$SE(P) = \inf\{SE(Q) : Q \in \mathcal{F}\}.$$

4. Key estimates and constructions

Definition 9. Whenever $\varepsilon > 0$ and N is a positive integer, let $Z(N, \varepsilon) = \left\{a : a \in (-\varepsilon, \varepsilon)^N \text{ and } \sum_{i=1}^N a_i = 0\right\}$.

We now give a modified form of Proposition VI.12 from [2], with estimate (4) extended so that the bounds hold for each of the ϕ_{ij}'s even though they are independent norms.

Proposition 2. *Suppose* P *is a polycrystal in* $\mathcal{P}^N$, *with* $\mathcal{L}^n(P(i)) > 0$ *for each* $1 \leq i \leq N$. *Suppose* $\{\phi_{ij}\}_{1\leq i<j\leq N}$ *is a family of class* 1 *norms. Then there exist* $\varepsilon > 0$, $M < \infty$, *and class* 1 *functions* $\Psi_1, \Psi_2 : Z(N, \varepsilon) \times \mathbb{R}^n \to \mathbb{R}^n$ *such that each of the following is true:*

(1) *For each* $a \in Z(N, \varepsilon)$, *the functions* $\Psi_1(a, \cdot)$ *and* $\Psi_2(a, \cdot)$ *are diffeomorphisms.*

(2) *The sets* $\{x : \Psi_1(a,x) \neq x$ *for some* $a \in Z(N,\varepsilon)\}$ *and* $\{x : \Psi_2(a,x) \neq x$ *for some* $a \in Z(N,\varepsilon)\}$ *are separated by positive distance.*

(3) *For each sufficiently large* $R \in (0,\infty)$, *and for each* $a \in Z(N,\varepsilon)$,

$$\mathcal{L}^n\left(\Psi_m(a,\cdot)\left[P(i)\cap B^n(\mathbf{0},R)\right]\right) - \mathcal{L}^n\left(P(i)\cap B^n(\mathbf{0},R)\right) = a_i$$

for each $i = 1,\dots,N$ *and for* $m = 1,2$.

(4) *For each* $a \in Z(N,\varepsilon)$, *for each collection of varifolds* $\{V(i,j)\}_{1\le i<j\le N}$ *(with each* $V(i,j) \in \mathcal{V}_{n-1}$*), for each* $1 \le i < j \le N$, *for each* $1 \le u < v \le N$, *and for each* $m = 1,2$,

$$\left|\widehat{\Phi}_{uv}\left(\Psi_m(a,\cdot)_{\#}V(i,j)\right) - \widehat{\Phi}_{uv}\left(V(i,j)\right)\right| \le M \sup_{1\le i<j\le N}\left\{\mathbf{M}\left(V(i,j)\right)\right\}\sum_{i=1}^{N}|a_i|.$$

(5) *For each* $P' \in \mathcal{P}^N$ *with* $d(P,P') < \varepsilon$, *there exist class* 1 *functions* $\alpha_1,\alpha_2 : Z(N,\ \varepsilon/2) \to Z(N,\varepsilon)$ *such that for each* $a \in Z(N,\ \varepsilon/2)$, *for each* $1 \le i \le N$, *and for each* $m = 1,2$, *we have*

$$\mathcal{L}^n\left(\Psi_m(\alpha_m(a),\cdot)\left[P'(i)\cap B^n(0,R)\right]\right) - \mathcal{L}^n\left(P'(i)\cap B^n(0,R)\right) = a_i,$$

for all sufficiently large $R \in (0,\infty)$.

Proof. We use Ψ_1 and Ψ_2 precisely as constructed in [2], VI.8 and VI.11. Conclusions (1), (2), (3), and (5) are the same as [2], VI.12 (1), (2), and (3). (4) above follows from [2], VI.12 (4), as follows. We first define $\sigma(i,j) = 1/2$ if $i \neq j$ and $\phi(i,j) = 0$ if $i = j$. For a fixed u and v with $1 \le u < v \le N$, we may apply [2], VI.12 (4), with $G = F = \widehat{\Phi}_{uv}$. That gives us an $\varepsilon(u,v)$ and an $M(u,v)$ for which (4) above holds with M and ε there replaced by $M(u,v)$ and $\varepsilon(u,v)$, respectively. It follows that (4) holds for each $1 \le u < v \le N$ if we simply define $M = \sup_{1\le u<v\le N}\{M(u,v)\} < \infty$ and $\varepsilon = \inf_{1\le u<v\le N}\{\varepsilon(u,v)\} > 0$. □

The following proposition is a corrected and specialized form of Lemma VI.13 from [2], which we then generalized so as to allow for unbounded regions. We will proceed as in the proof of Lemma VI.13 from [2] to the extent possible. This quite general result allows us to cover most of the underlying set for an integral current with a finite collection of balls, each having radius 2 but not containing each other's centers. Various theorems allow for such a construction; the non-trivial part is that, for our purposes, we will need a uniform bound on the number of such balls, q, that can be applied to all of the currents $[P_k(i)]$, for $i = 2,3,\dots,N$, and for each sufficiently large integer k. Later, in Section 5.4.3, this uniform bound on q will enable us to prove the critical inequality $\operatorname{card}(\Xi_k) \le N_0$ for all sufficiently large k, where N_0 is independent of k. That inequality in turn is instrumental in completing the proof in Section 5.4.3.

Proposition 3. *Suppose* $T = [K]$ *for some* $K \in \mathcal{C}$ *and* $0 < \varepsilon < (2\zeta+1)\mathbf{M}(\partial T)/n$. *Then there exist points* $p_1,p_2,\dots,p_q \in \operatorname{spt} T$ *such that* $|p_i - p_j| > 2$ *whenever* $1 \le i < j \le q$, *and*

$$\mathbf{M}\left(T \llcorner \left(\mathbb{R}^n \setminus \bigcup_{i=1}^{q} B^n(p_i,2)\right)\right) < \varepsilon.$$

If, additionally, we have $\mathbf{M}(T) \ge \varepsilon$, *then we may choose the points* p_i *so that they also satisfy*

$$\mathbf{M}\left(T \llcorner B^n(p_i,1)\right) \ge \frac{1}{\left[2\gamma(2\zeta+1)\right]^n}\left[\frac{\varepsilon}{\mathbf{M}(\partial T)}\right]^n$$

for each $i = 1, \ldots, q$, *and*

$$q \leq \mathbf{M}(T)\left[2\gamma(2\zeta+1)\right]^n \left[\frac{\mathbf{M}(\partial T)}{\varepsilon}\right]^n .$$

If $\mathbf{M}(T) < \varepsilon$, *we may take* $q = 1$. *Here,* $\gamma = \left(n\alpha(n)^{1/n}\right)^{-1}$ *is the optimal isoperimetric constant and* $2\zeta + 1$ *is the covering constant from the Besicovitch Covering Theorem (see* [26], *2.8.14).*

Proof. Define $a = 2\gamma(2\zeta+1)\mathbf{M}(\partial T)/\varepsilon > 2\gamma n$.

Claim 1. If $p \in \operatorname{spt} T$ and $0 < \|T\|(B^n(p,1)) < a^{-n}$, then the set of r values in $(0,1)$ for which $\|\partial T\|(B^n(p,r)) > (a/(2\gamma))\|T\|(B^n(p,r))$ has positive $\mathcal{L}^1$ measure.

Proof of Claim 1. Suppose the claim to be false. Then for $\mathcal{L}^1$ almost every $r \in (0,1)$ the functions $m(r) = \|T\|(B^n(p,r))$ and $n(r) = \|\partial T\|(B^n(p,r))$ satisfy

$$n(r) \leq \frac{a}{2\gamma} m(r) \leq \frac{a}{2\gamma} m(r)^{(n-1)/n} m(1)^{1/n} \leq \frac{1}{2\gamma} m(r)^{(n-1)/n} . \tag{4.1}$$

Let $f(x) = |x - p|$ whenever $x \in \mathbb{R}^n$. For $\mathcal{L}^1$ almost every $r \in (0,1)$, the slice current $\langle T, f, r\rangle$ exists and the conclusions of Theorem 2 hold. For such r, we apply the isoperimetric inequality to the current $T \llcorner B^n(p,r)$ and make use of Theorem 2 and (4.1) to estimate

$$\begin{aligned} m(r)^{(n-1)/n} &\leq \gamma \mathbf{M}(\partial(T \llcorner B^n(p,r))) \leq \gamma\left[\mathbf{M}(\langle T, f, r\rangle) + \mathbf{M}((\partial T) \llcorner B^n(p,r))\right] \\ &\leq \gamma m'(r) + \frac{1}{2} m(r)^{(n-1)/n} , \end{aligned}$$

so that

$$a^{-1} < \frac{1}{2\gamma n} \leq \frac{1}{n} m(r)^{\left(\frac{1}{n}-1\right)} m'(r) = \left[m(r)^{1/n}\right]' .$$

Integrating from 0 to r^* and letting $r^* \uparrow 1$ implies $\|T\|(B^n(p,1)) = m(1) \geq a^{-n}$, a contradiction. ☐

Claim 2. If $B \subset \mathbb{R}^n$ is any non-empty closed set for which

$$\|T\|(\{x \in \mathbb{R}^n : \operatorname{dist}(x,B) > 1\}) \geq \varepsilon,$$

then there exists $p \in \operatorname{spt} T$, with $\operatorname{dist}(p,B) > 1$, such that $\|T\|(B^n(p,1)) \geq a^{-n}$.

Proof of Claim 2. Suppose B is as stated, and let $E = \{x \in \mathbb{R}^n : \operatorname{dist}(x,B) > 1\}$. Then $\|T\|(E) \geq \varepsilon$ by assumption. If no such p exists, then $\|T\|(B^n(p,1)) < a^{-n}$ for all $p \in (\operatorname{spt} T) \cap E$. Claim 1 implies that, for each $p \in (\operatorname{spt} T) \cap E$, there exists a number $r(p) \in (0,1)$ such that

$$\|\partial T\|(B^n(p, r(p))) > \frac{a}{2\gamma}\|T\|(B^n(p, r(p))) . \tag{4.2}$$

The Besicovitch Covering Theorem (see [26], 2.8.14) guarantees that we can cover $(\operatorname{spt} T) \cap E$ with the union of $2\zeta+1$ disjointed subcollections, $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{2\zeta+1}$ of the collection $\mathcal{F} = \{B^n(p, r(p)) : p \in (\operatorname{spt} T) \cap E\}$. Noting that $\|T\|(\mathbb{R}^n \setminus \operatorname{spt} T) = 0$, and proceeding as in [2], VI.13, Part 2, we estimate

$$\begin{aligned}
\varepsilon &\leq \|T\|(E) = \|T\|((\operatorname{spt} T) \cap E) \leq \|T\| \left(\bigcup_{i=1}^{2\zeta+1} \bigcup_{B^n(p,r(p)) \in \mathcal{F}_i} B^n(p, r(p)) \right) \\
&\leq \sum_{i=1}^{2\zeta+1} \sum_{B^n(p,r(p)) \in \mathcal{F}_i} \|T\| (B^n(p, r(p))) \\
&< \frac{2\gamma}{a} \sum_{i=1}^{2\zeta+1} \sum_{B^n(p,r(p)) \in \mathcal{F}_i} \|\partial T\| (B^n(p, r(p))) \ \text{(using (4.2))} \\
&= \frac{2\gamma}{a} \sum_{i=1}^{2\zeta+1} \|\partial T\| \left(\bigcup_{B^n(p,r(p)) \in \mathcal{F}_i} B^n(p, r(p)) \right) \\
&\leq \frac{2\gamma}{a} \sum_{i=1}^{2\zeta+1} \|\partial T\| (\mathbb{R}^n) = \frac{2\gamma}{a} (2\zeta + 1) \mathbf{M}(\partial T),
\end{aligned}$$

which contradicts the definition of a. $\square$

Next, suppose $\mathbf{M}(T) \geq \varepsilon$, and let $B = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \operatorname{spt} T) \geq 2\}$. Then

$$\varepsilon \leq \mathbf{M}(T) = \|T\|(\mathbb{R}^n) = \|T\|(\operatorname{spt} T) \leq \|T\|(\{x \in \mathbb{R}^n : \operatorname{dist}(x, B) > 1\}),$$

and so we may apply Claim 2 to choose $p_1 \in \operatorname{spt} T$ so that $\|T\|(B^n(p_1, 1)) \geq a^{-n}$. If $\|T\|(\mathbb{R}^n \setminus B^n(p_1, 2)) < \varepsilon$, then we're done. If, instead, $\|T\|(\mathbb{R}^n \setminus B^n(p_1, 2)) \geq \varepsilon$, we let $B = B^n(p_1, 1)$ and observe that B satisfies $\|T\|(\{x \in \mathbb{R}^n : \operatorname{dist}(x, B) > 1\}) \geq \varepsilon$, since clearly any point more than 2 units away from p_1 is at least 1 unit away from B. We may then apply Claim 2 to choose $p_2 \in \operatorname{spt} T$ with $\operatorname{dist}(p_2, B) > 1$ (so that $|p_1 - p_2| > 2$) and $\|T\|(B^n(p_2, 1)) \geq a^{-n}$. Continuing in this fashion, we obtain points $p_1, p_2, \ldots, p_k$ such that $|p_i - p_j| > 2$ whenever $i \neq j$ and $\|T\|(B^n(p_i, 1)) \geq a^{-n}$ for each $i = 1, 2, \ldots, k$. If $\|T\| \left(\mathbb{R}^n \setminus \bigcup_{i=1}^k B^n(p_i, 2) \right) < \varepsilon$, then we're done; otherwise, we let $B = \bigcup_{i=1}^k B^n(p_i, 1)$, observe that B satisfies $\|T\|(\{x \in \mathbb{R}^n : \operatorname{dist}(x, B) > 1\}) \geq \varepsilon$, and apply Claim 2 to choose p_{k+1} as before. This process terminates after a finite number, say q, of iterations since $\mathbf{M}(T) < \infty$. Since the $B^n(p_i, 1)$'s are disjoint by construction, we have

$$\mathbf{M}(T) = \|T\|(\mathbb{R}^n) \geq \|T\| \left(\bigcup_{i=1}^q B^n(p_i, 1) \right) = \sum_{i=1}^q \|T\|(B^n(p_i, 1)) \geq q a^{-n},$$

so that $q \leq \mathbf{M}(T) a^n$, as claimed. Finally, suppose $\mathbf{M}(T) < \varepsilon$. Take $q = 1$, and let p be any point in $\operatorname{spt} T$. The proposition follows since $\mathbf{M}(T \llcorner (\mathbb{R}^n \setminus B^n(p, 2))) \leq \mathbf{M}(T) < \varepsilon$. $\square$

The next result is analogous to [2], VI.14, for our setting, in which the surface energy density functions $\{\phi_{ij}\}_{1 \leq i < j \leq N}$ may be independent of one another. For any positive constant $\Gamma > 0$ we construct a constant S, depending only on n, Γ, and the ϕ_{ij}'s, with the property that any Caccioppoli partition P such that $\mathbb{R}^n \setminus P(1)$ is mostly contained in some compact set B can, if necessary, be replaced with another partition P^{r_0}, having no more surface energy than P, and having regions $P^{r_0}(i)$ which, for each $i \geq 2$, are contained in the compact set $B + B^n(\mathbf{0}, S - r_0)$.

Moreover, if P and P^{r_0} are different, the surface energy savings in replacing P with P^{r_0} is at least Γ times the total volume change. The proof requires some care because we want the result to hold for any $\Gamma > 0$.

Proposition 4. *Suppose P is a Caccioppoli partition. Suppose $\Gamma > 0$, and let $S = n\phi^0/(4\Gamma)$. Suppose $B \subset \mathbb{R}^n$ is a non-empty compact set for which*

$$\mathcal{L}^n\left[(\mathbb{R}^n \setminus P(1)) \setminus B\right] < \left(\frac{\phi_0}{16\gamma\Gamma}\right)^n,$$

and let $f(x) = \operatorname{dist}(x, B)$ for $x \in \mathbb{R}^n$. For $r \in [0, S]$, define the polycrystal $P^r \in \mathcal{P}^N$ as follows:

$$P^r(i) = \operatorname{Int}(P(i) \cap \{f \le S - r\}), \text{ for } 2 \le i \le N, \text{ and}$$
$$P^r(1) = \operatorname{Int}\left(\mathbb{R}^n \setminus \bigcup_{i=2}^N P^r(i)\right).$$

Then, if $\mathcal{L}^n((\mathbb{R}^n \setminus P(1)) \cap \{f > S\}) > 0$, there exists an $r_0 \in (0, S)$ such that $SE(P) - SE(P^{r_0}) \ge 4\Gamma(|\beta_1| + \ldots + |\beta_N|)$, where

$$\beta_i = \mathcal{L}^n(P^{r_0}(i)) - \mathcal{L}^n(P(i)) \le 0, \text{ for } 2 \le i \le N, \text{ and } \beta_1 = -\sum_{i=2}^N \beta_i > 0.$$

Proof. Define $X = [(\mathbb{R}^n \setminus P(1)) \cap \{f > 0\}]$. For $\mathcal{L}^1$ almost every $q \in (0, S)$, the slice current $\langle X, f, q\rangle$ exists and the conclusions of Theorem 2 hold. For such values of q, we define

$$\begin{aligned} m(q) &= \|X\|(\{f < q\}) = \mathcal{L}^n((\mathbb{R}^n \setminus P(1)) \cap \{0 < f < q\}), \\ V(q) &= \mathcal{L}^n((\mathbb{R}^n \setminus P(1)) \cap \{f > S - q\}), \\ n(q) &= \mathcal{H}^{n-1}(\partial_F(\mathbb{R}^n \setminus P(1)) \cap \{f > S - q\}), \end{aligned}$$

$$V(q) = V(S) - m(S - q) \le V(S) = \mathbf{M}(X) < \left(\frac{\phi_0}{16\gamma\Gamma}\right)^n. \tag{4.3}$$

Since V is monotonically increasing, $V'(q)$ exists for $\mathcal{L}^1$ almost every $q \in (0, S)$, so we further restrict the q's, if necessary, so that $V'(q)$ is defined. For such q's, we have $V'(q) = m'(S - q) \ge 0$.

Claim 1. The set of r values in $(0, S)$ for which $\phi_0 n(r) \ge \phi^0 V'(r) + 8\Gamma V(r)$ has positive $\mathcal{L}^1$ measure.

Proof of Claim 1. Suppose to the contrary that, for $\mathcal{L}^1$ almost every $r \in (0, S)$, we have $\phi_0 n(r) < \phi^0 V'(r) + 8\Gamma V(r)$. For each such r, we apply the isoperimetric inequality to the n current

$$Y_r = [(\mathbb{R}^n \setminus P(1)) \cap \{x : \operatorname{dist}(x, B) > S - r\}]$$

and make use of Theorem 2 to estimate

$$\begin{aligned} V(r)^{(n-1)/n} &= \mathbf{M}(Y_r)^{(n-1)/n} \le \gamma \mathbf{M}(\partial Y_r) \le \gamma(n(r) + m'(S - r)) \\ &= \gamma(n(r) + V'(r)). \end{aligned}$$

The extreme inequality and the first sentence in the proof of this claim give

$$\gamma^{-1} V(r)^{(n-1)/n} - V'(r) \le n(r) < \frac{\phi^0}{\phi_0} V'(r) + \frac{8\Gamma}{\phi_0} V(r).$$

(4.3) implies $V(r)^{1/n} \leq \phi_0/(16\gamma\Gamma)$, so we have

$$\frac{1}{2}V(r)^{(n-1)/n} \leq \left[1 - \frac{8\gamma\Gamma}{\phi_0}V(r)^{1/n}\right]V(r)^{(n-1)/n} < \gamma\left[\frac{\phi^0}{\phi_0}+1\right]V'(r),$$

which in turn implies $\left[V(r)^{1/n}\right]' > \left[2n\gamma\left(\left(\phi^0/\phi_0\right)+1\right)\right]^{-1} = c$. Integrating from $r=0$ to $r=S$ gives $V(S)^{1/n} - V(0)^{1/n} > cS$, so $V(S) > c^n S^n \geq (\phi_0/(16\gamma\Gamma))^n$, a contradiction. □

It follows that $\phi_0 n(r_0) \geq \phi^0 V'(r_0) + 8\Gamma V(r_0)$ for some $r_0 \in (0,S)$ for which each of the quantities above is defined. Next, we note that the surface energy savings in passing from P to P^{r_0} is at least $\phi_0 n(r_0)$, while the surface energy cost in passing from P to P^{r_0} is at most $\phi^0 V'(r)$. Thus, the net surface energy savings is

$$SE(P) - SE(P^{r_0}) \geq \phi_0 n(r_0) - \phi^0 V'(r_0) \geq 8\Gamma V(r_0), \tag{4.4}$$

and

$$V(r_0) = \sum_{i=2}^{N} \mathcal{L}^n (P(i) \cap \{f > S - r_0\}) = \sum_{i=2}^{N} -\beta_i = \sum_{i=2}^{N} |\beta_i|$$

implies that $|\beta_1| = \beta_1 = -\sum_{i=2}^{N}\beta_i = V(r_0)$, and so $2V(r_0) = \sum_{i=1}^{N}|\beta_i|$. Finally, we use (4.4) to deduce that $SE(P) - SE(P^{r_0}) \geq 4\Gamma\sum_{i=1}^{N}|\beta_i|$, as desired. □

5. Proof of the main existence theorem

We will follow F. Almgren's approach (from [2]) closely, modifying constructions and estimates to make them suitable for our setting as needed.

5.1. **Part 1: Some definitions.** Let $N \geq 2$ be an integer. Suppose $v = (v_2, \ldots, v_N) \in (0,\infty)^{N-1}$. Set $v_{\min} = \inf_{2\leq i\leq N}\{v_i\}$ and $v_{\max} = \sup_{2\leq i\leq N}\{v_i\}$. Clearly, $0 < v_{\min} \leq v_{\max} < \infty$. Suppose $P_1, P_2, P_3, \ldots$ is a sequence of partitions, with $P_k \in \mathcal{P}_*^{N,v}$ for each $k \geq 1$, and satisfying

$$\lim_{k\to\infty} SE(P_k) = \inf\left\{SE(L) : L \in \mathcal{P}_*^{N,v}\right\} = E_{\min}.$$

Later, we will modify this minimizing sequence to produce a bounded minimizing sequence. Let

$$\begin{aligned} a_{\min} &= \inf\left\{\mathcal{H}^{n-1}(\partial_F P_k(i)) : 1 \leq i \leq N, k = 1,2,3,\ldots\right\}, \\ a_{\max} &= \sup\left\{\mathcal{H}^{n-1}(\partial_F P_k(i)) : 1 \leq i \leq N, k = 1,2,3,\ldots\right\}. \end{aligned}$$

The isoperimetric inequality implies that $a_{\min} > 0$, and so $E_{\min} > 0$. Given any v as above, we can form a polycrystal $W \in \mathcal{P}^{N,v}$ by letting $W(i)$ be a scaled Wulff crystal (scaled to have volume v_i) for the 1-i interface for each $i = 2,3,\ldots,N$, where the $W(i)$'s are a positive distance from one another. Since W itself is always a candidate in the minimization, for any $\epsilon > 0$ there exists a k_0 such that $k > k_0$ implies $SE(P_k) < SE(W) + \epsilon$. Since area is within a constant factor of surface energy, we can find a uniform upper bound to the areas $\mathcal{H}^{n-1}(\partial_F P_k(i))$ for all i and for any $k > k_0$. We can trivially bound $\mathcal{H}^{n-1}(\partial_F P_k(i))$ for all i and for any $k \leq k_0$, so it follows that $a_{\max} < \infty$. Thus, $0 < a_{\min} \leq a_{\max} < \infty$.

5.2. **Part 2: Construction of clusters for volume adjustment.** We now proceed as in [2], VI.15. For each pair (k,i), where $k=1,2,3,\ldots$ and $i=2$ to N, we use Proposition 3 with $T=[P_k(i)]$ and with

$$\varepsilon=\frac{1}{2}\inf\left\{\frac{(2\zeta+1)\,a_{\min}}{n},\,v_{\min}\right\}<\inf\left\{\frac{(2\zeta+1)\,\mathbf{M}(\partial T)}{n},\,\mathbf{M}(T)\right\}$$

to choose points $p_k(i)\in\mathbb{R}^n$ for which

$$\mathcal{L}^n\left(P_k(i)\cap B^n\left(p_k(i),1\right)\right)\geq\frac{1}{\left[2\gamma\left(2\zeta+1\right)\right]^n}\left[\frac{\inf\left\{\frac{(2\zeta+1)a_{\min}}{n},\,v_{\min}\right\}}{2a_{\max}}\right]^n>0.$$

Next, fix $R>0$ according to the condition $\alpha(n)\,R^n=2\sum_{i=2}^N v_i$. It follows immediately that $\mathcal{L}^n\left(P_k(1)\cap B^n\left(p_k(i),R\right)\right)\geq(1/2)\,\alpha(n)\,R^n$ for each $k=1,2,3,\ldots$ and $i=2$ to N.

It is not the case that for each i the sequence $(p_k(i))$ must converge as $k\to\infty$. Passing to a subsequence of $1,2,3,\ldots$, if necessary, we define an equivalence relation $\sim$ on $\{2,\ldots,N\}$ as follows: $i\sim j$ provided there exists a sequence $i_1,i_2,\ldots,i_t$ of distinct elements of $\{2,\ldots,N\}$ such that $i_1=i$, $i_t=j$, and $\lim_{k\to\infty}|p_k(i_l)-p_k(i_{l+1})|\leq R$ for each $l=1,2,\ldots,t-1$. Passing to a subsequence, if necessary, we may suppose that $\lim_{k\to\infty}(p_k(i)-p_k(j))$ exists whenever $i\sim j$. We use $\sim$ to partition $\{2,\ldots,N\}$ into equivalence classes $\Lambda_1,\ldots,\Lambda_s$ $(1\leq s\leq N-1)$, and we choose $\lambda_i\in\Lambda_i$ for each $i=1,\ldots,s$. For each $1\leq i\leq s$ and $k\geq 1$, we define $\mathrm{Cluster}_k(i)=\bigcup_{\lambda\in\Lambda_i}B^n\left(p_k(\lambda),R\right)$. We will now define a new sequence (P_k') of polycrystals as follows. For each $k\geq 1$ and for each $2\leq j\leq N$, we set

$$P_k'(j)=\mathrm{Int}\left(\bigcup_{i=1}^{s}\tau\left[p_k(\lambda_i)+4NRi\;e_n\right]\left(P_k(j)\cap\mathrm{Cluster}_k(i)\right)\right).$$

Whenever $k\geq 1$, we set $P_k'(1)=\mathrm{Int}\left(\mathbb{R}^n\setminus\left(P_k'(2)\cup\ldots\cup P_k'(N)\right)\right)$. For all sufficiently large k, the sets $\tau\left[p_k(\lambda_i)+4NRi\;e_n\right]\left(\mathrm{Cluster}_k(i)\right)$, for $i=1,\ldots,s$, are separated by a distance greater than R. Finally, the compactness theorem for polycrystals (Theorem 4) ensures that there exist a polycrystal P' and a subsequence $k_1,k_2,k_3,\ldots$ of $1,2,3,\ldots$ such that the currents $\left[P_{k_j}'(i)\right]$ converge strongly to $[P'(i)]$ as $j\to\infty$. Passing to this subsequence, it follows that

$$\lim_{k\to\infty}\mathcal{L}^n\left(P_k'(i)\,\triangle\,P'(i)\right)=0$$

for each $1\leq i\leq N$.

5.3. **Part 3: Construction of diffeomorphisms for volume adjustment.** The idea here is to effect small volume changes on any subcollection of the N regions by using diffeomorphisms, φ, which act inside small cubes at points z_j on various interfaces. If $p\in\partial_F K_i\cap\partial_F K_j$, then $\Theta^n(K_i,p)=\Theta^n(K_j,p)=1/2$, and so, by using a diffeomorphism, we can essentially increase the volume of region i slightly while decreasing that of region j by the same amount, or we can decrease the volume of region i slightly while increasing that of region j by the same amount. We can record such changes (with 1's and -1's) in a matrix, L, and then we can use a linear combination of many such local perturbations to produce an arbitrary small volume perturbation, including changing the volumes of each of the regions by small, specified amounts, provided $\mathrm{rank}\,(L)=N-1$. We can then combine the diffeomorphisms, via composition, so as to obtain a function Φ which performs all the volume changes in each of the cubes. It is critically important that this can be done with surface energy cost that is no more than a constant times the sum of the volume changes.

5.3.1. *Definition of* $\varphi(\varepsilon, z, \theta)$ ([2], VI.8). The definitions here are from [2], VI.8. We define $h_1 : \mathbb{R} \to \mathbb{R}$ according to $h_1(t) = \left((1/2) - t^2\right)^2$ whenever $|t| \leq \sqrt{2}/2$ and $h_1(t) = 0$ otherwise. Then $h_1 \in C^1(\mathbb{R}, \mathbb{R})$ and $\sup h_1 = 1/4$. We define $h_2 : \mathbb{R}^n \to \mathbb{R}$ by setting $h_2(x) = h_1(x_1) \cdot \ldots \cdot h_1(x_n)$ for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We note that $h_2 \in C^1(\mathbb{R}^n, \mathbb{R})$, with $\operatorname{spt} h_2 = \left\{x : |x_i| \leq \sqrt{2}/2 \text{ for each } i = 1, \ldots, n\right\}$. For each $\varepsilon \in (0,1)$ we define $h_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by setting $h_\varepsilon(x) = c \cdot \left(1/\varepsilon^{n-1}\right) h_2(x/\varepsilon)$ for each $x \in \mathbb{R}^n$. Here, the constant c is chosen so that $h_\varepsilon(x_1, \ldots, x_{n-1}, 0)$ integrates to 1 over all $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. We note that $\sup|\partial h_\varepsilon/\partial x_n| = c\sqrt{6}/\left(9 \cdot 4^{n-1} \cdot \varepsilon^n\right)$. For each $\varepsilon \in (0,1), z \in \mathbb{R}^n, \theta \in O(n)$, we can now define $\varphi(\varepsilon, z, \theta) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by setting, for each $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$\varphi(\varepsilon, z, \theta)(t, x) = x + t h_\varepsilon(\theta(x - z)) \theta^{-1}(e_n).$$

We note that, for small values of t, $\varphi(\varepsilon, z, \theta)(t, \cdot)$ is a diffeomorphism of $\mathbb{R}^n$. Clearly, $\varphi(\varepsilon, z, \theta)(t, \cdot)$ changes points only inside a certain n-cube centered at z; we observe that the ball $B^n\left(z, \varepsilon\frac{\sqrt{2}}{2}\sqrt{n}\right)$ contains this cube, and so $\varphi(\varepsilon, z, \theta)(t, x) = x$ whenever $x \notin B^n\left(z, \varepsilon\frac{\sqrt{2}}{2}\sqrt{n}\right)$.

5.3.2. *Definition of* Φ ([2], VI.11, VI.15 (9),(10)). Lemma VI.10 from [2], together with the remarks from [2], VI.10, Part 2, and also [2], VI.15 (9), applied to P', imply that there exist a positive integer $M \geq N$ and a sequence $z_1, z_2, \ldots, z_M$ of distinct points in $\mathbb{R}^n$ satisfying

$$\operatorname{dist}\left(z_j, \bigcup_{i=1}^{s} \bigcup_{\lambda \in \Lambda_i} \lim_{k \to \infty} \tau\left[p_k(\lambda_i) + 4NRi\, e_n\right](p_k(\lambda))\right) < R$$

for each $1 \leq j \leq M$, and there exist functions $\xi, \eta : \{1, \ldots, M\} \to \{1, \ldots, N\}$, such that $z_j \in \Gamma_{\xi(j), \eta(j)}(P')$ for each $1 \leq j \leq M$, and conditions (2) and (3) of Lemma VI.10 of [2] hold as well. [2], VI.11 (c), with A replaced by P' there, guarantees that we can choose $\theta_1, \theta_2, \ldots, \theta_M \in O(n)$ so that $\theta_j\left(n_{P'(\xi(j))}(z_j)\right) = e_n$ for each $j = 1, \ldots, M$. Let L be the N by M matrix $[L_{ij}]$ with

$$L_{ij} = \begin{cases} 1, & i = \xi(j), \\ -1, & i = \eta(j), \\ 0, & \text{otherwise.} \end{cases}$$

Its rank is $N - 1$ (see [2], VI.11 (d) and VI.10 (2)). [2], VI.11 (d), asserts the existence of a $\delta \in (0,1)$ having the property that if K is an N by M matrix with $|K_{ij} - L_{ij}| < \delta$ for each i, j, then $\operatorname{rank}(K) = N - 1$. We now fix δ as in [2], VI.11 (d).

We next choose $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, as specified in [2], VI.11 (e), with P' replacing A, with the additional requirement that ε_1 be sufficiently small that the closure of the set

$$\bigcup \{x \in \mathbb{R}^n : \varphi(\varepsilon_1, z_j, \theta_j)(t, x) \neq x \text{ for some } t \in \mathbb{R} \text{ and some } 1 \leq j \leq M\}$$

be contained in

$$\left\{ x : \operatorname{dist}\left(x, \bigcup_{i=1}^{s} \bigcup_{\lambda \in \Lambda_i} \lim_{k \to \infty} \tau\left[p_k\left(\lambda_i\right) + 4NRi\ e_n \right] \left(p_k\left(\lambda\right) \right) \right) < R \right\}.$$

Following [2], VI.11 (f), again with P' replacing A there, we define $\Phi : \mathbb{R}^M \times \mathbb{R}^n \to \mathbb{R}^n$ by setting, for each $(t_1, \ldots, t_M) \in \mathbb{R}^M$ and for each $x \in \mathbb{R}^n$,

$$\begin{aligned} \Phi\left(t_1, \ldots, t_M, x\right) = \varphi\left(\varepsilon_1, z_1, \theta_1\right)\left(t_1, \cdot\right) \circ \varphi\left(\varepsilon_1, z_2, \theta_2\right)\left(t_2, \cdot\right) \\ \circ \cdots \circ \varphi\left(\varepsilon_1, z_M, \theta_M\right)\left(t_M, \cdot\right)(x). \end{aligned}$$

[2], VI.9 (1) and VI.11 (e) imply that $\Phi(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism for each $t \in \left(-\varepsilon_2, \varepsilon_2\right)^M$.

5.4. **Part 4: Completing the proof.**

5.4.1. *Estimates on surface energy cost associated with volume corrections.* Proposition 2, applied with P replaced by P', guarantees the existence of numbers $\varepsilon, M' \in (0, \infty)$ and a function Ψ_1 satisfying conclusions (1)-(5) of the proposition. For all sufficiently large integers $k > 1$, we define $\Psi_k : Z(N, \varepsilon) \times \mathbb{R}^n \to \mathbb{R}^n$ by setting, for each $\beta \in Z(N, \varepsilon)$,

$$\Psi_k(\beta, x) = \tau\left[p_k\left(\lambda_i\right) + 4NRi\ e_n\right]^{-1} \circ \Psi_1(\beta, \cdot) \circ \tau\left[p_k\left(\lambda_i\right) + 4NRi\ e_n\right](x)$$

if $1 \leq i \leq s$ and if

$$\operatorname{dist}(x - p_k\left(\lambda_i\right) - 4NRi\ e_n, \bigcup_{\lambda \in \Lambda_i} \lim_{k \to \infty} \tau\left[p_k\left(\lambda_i\right) + 4NRi\ e_n\right]\left(p_k(\lambda)\right) < R,$$

and we set $\Psi_k(\beta, x) = x$ otherwise.

For all sufficiently large k, the hypotheses of Proposition 2(5) are satisfied, so there exist class 1 functions $\alpha_k : Z(N,\ \varepsilon/2) \to Z(N, \varepsilon)$ such that, for all sufficiently large k, for each $\beta \in Z(N,\ \varepsilon/2)$, and for each $j = 1, \ldots, N$, we have

$$\mathcal{L}^n\left(\Psi_k\left(\alpha_k(\beta), \cdot\right)\left[P_k(j) \cap B^n(0, S)\right]\right) - \mathcal{L}^n\left(P_k(j) \cap B^n(0, S)\right) = \beta_j$$

for all sufficiently large $S \in (0, \infty)$, and, in addition

$$\left|SE\left(P_k^*\right) - SE\left(P_k\right)\right| \leq M'' a_{\max} \sum \left|\beta_i\right|, \tag{5.1}$$

where P_k^* is defined according to $P_k^*(i) = \Psi_k\left(\alpha_k(\beta), \cdot\right)\left(P_k(i)\right)$, for each $i = 1, \ldots, N$. The estimate (5.1) is derived by applying Proposition 2(4) with $V(i,j) = \left|\Gamma_{ij}\left(P_k\right)\right|$. Then $\mathbf{M}(V(i,j)) = \mathcal{H}^{n-1}\left(\Gamma_{ij}\left(P_k\right)\right) \leq a_{\max}$, and so

$$\sup_{1 \leq i < j \leq N} \{\mathbf{M}(V(i,j))\} \leq a_{\max}$$

as well. We can apply Proposition 2 (4) for each $1 \leq i < j \leq N$ and add the results to get (5.1), where M'' is easy to compute in terms of M'. We note that M'' is independent of k.

5.4.2. *Covering step and truncation.* Next, define $\Gamma = (1/2)\, M'' a_{\max}$ and

$$\varepsilon_0 = \frac{1}{2} \inf \left\{ \frac{\varepsilon}{2\,(N-1)},\ \frac{v_{\min}}{4},\ \frac{(2\zeta+1)\, a_{\min}}{n},\ \frac{1}{N-1} \left(\frac{\phi_0}{16\gamma\Gamma} \right)^n \right\}.$$

Each of the terms in its definition is independent of k, and so ε_0 is independent of k. For each sufficiently large k and for each $i = 2, \ldots, N$, we apply Proposition 3 with ε replaced by ε_0 there, and with $T = [P_k\,(i)]$. We check that

$$\varepsilon_0 \leq \frac{1}{2} \frac{(2\zeta+1)\, a_{\min}}{n} < \frac{(2\zeta+1)\, \mathcal{H}^{n-1}\,(\partial_F P_k\,(i))}{n} = \frac{(2\zeta+1)\, \mathbf{M}\,(\partial\,[P_k\,(i)])}{n},$$

as required by the proposition. It follows that there exist points $p_k^1\,(i)\,,\, p_k^2\,(i)\,,\, \ldots,$ $p_k^{q(k,i)}\,(i)$ in $\operatorname{spt}\,[P_k\,(i)]$ with

$$\begin{aligned} q\,(k,i) &\leq \mathbf{M}\,([P_k\,(i)])\,[2\gamma\,(2\zeta+1)]^n \left[\frac{\mathbf{M}\,(\partial\,[P_k\,(i)])}{\varepsilon_0} \right]^n \\ &\leq v_{\max}\,[2\gamma\,(2\zeta+1)]^n \left[\frac{a_{\max}}{\varepsilon_0} \right]^n, \end{aligned} \tag{5.2}$$

separated pairwise by distance greater than 2, and having the property that

$$\mathbf{M} \left([P_k\,(i)] \llcorner \left(\mathbb{R}^n \setminus \bigcup_{j=1}^{q(k,i)} B^n \left(p_k^j\,(i)\,, 2 \right) \right) \right) < \varepsilon_0.$$

Thus, we have

$$\mathcal{L}^n \left(P_k\,(i) \setminus \bigcup_{j=1}^{q(k,i)} B^n \left(p_k^j\,(i)\,, 2 \right) \right) < \varepsilon_0. \tag{5.3}$$

For all sufficiently large k, we define the compact sets

$$B_k = \left[\bigcup_{i=2}^{N} B^n\,(p_k\,(i)\,, R) \right] \cup \left[\bigcup_{i=2}^{N} \bigcup_{j=1}^{q(k,i)} B^n \left(p_k^j\,(i) \right) \right].$$

For each sufficiently large k, we apply Proposition 4 with P replaced by P_k, with B replaced by B_k, with Γ as defined above, and with $S = n\phi^0 / (4\Gamma)$ (which we note is also independent of k). We check that the definition of ε_0 and (5.3) imply

$$\mathcal{L}^n \left[\left(\bigcup_{i=2}^{N} P_k\,(i) \right) \setminus B_k \right] \leq \sum_{i=2}^{N} \mathcal{L}^n\,(P_k\,(i) \setminus B_k) < (N-1)\,\varepsilon_0 < \left(\frac{\phi_0}{16\gamma\Gamma} \right)^n, \tag{5.4}$$

as required. It follows from Proposition 4 that for each sufficiently large k there exists $r_k \in (0, S)$ such that

$$SE\,(P_k) - SE\,(P_k^{r_k}) \geq 4\Gamma \sum_{i=1}^{N} |\beta_i\,(k)|\,. \tag{5.5}$$

Here, $P_k^{r_k}$ and $\beta_i(k)$ are defined as follows:

$$\begin{aligned} P_k^{r_k}(i) &= \operatorname{Int}\left(P_k(i) \cap \{x : \operatorname{dist}(x, B_k) \leq S - r_k\}\right), \text{ for } 2 \leq i \leq N, \text{ and} \\ P_k^{r_k}(1) &= \operatorname{Int}\left(\mathbb{R}^n \setminus \bigcup_{i=2}^{N} P_k^{r_k}(i)\right), \\ \beta_i(k) &= \mathcal{L}^n\left(P_k^{r_k}(i)\right) - \mathcal{L}^n\left(P_k(i)\right) \leq 0, \text{ for } 2 \leq i \leq N, \text{ and} \\ \beta_1(k) &= -\sum_{i=2}^{N} \beta_i(k). \end{aligned}$$

5.4.3. *Truncation and volume correction.* We will now show that $\beta(k) = (\beta_1(k), \ldots, \beta_N(k)) \in Z(N, \varepsilon/2)$ for all sufficiently large k. We have

$$\begin{aligned} \sum_{i=2}^{N} |\beta_i(k)| &= \sum_{i=2}^{N} \mathcal{L}^n\left(P_k(i) \cap \{x : \operatorname{dist}(x, B_k) > S - r_k\}\right) \\ &\leq \sum_{i=2}^{N} \mathcal{L}^n\left(P_k(i) \setminus B_k\right) < (N-1)\,\varepsilon_0, \end{aligned} \tag{5.6}$$

because of (5.4). It follows from the definition of ε_0 and from (5.6) that $|\beta_i(k)| < (N-1)\,\varepsilon_0 < \varepsilon/2$ for each $i \geq 2$ and that $|\beta_1(k)| = -\sum_{i=2}^{N} \beta_i(k) < (N-1)\,\varepsilon_0 < \varepsilon/2$, and so $\beta(k) \in Z(N, \varepsilon/2)$ for all sufficiently large k, as claimed. Now, we combine our previous estimates. Using the $\beta_i(k)$'s from above, we define the polycrystals $P_k'' \in \mathcal{P}^N$ by setting $P_k''(i) = \Psi_k(\alpha_k(\beta(k)), \cdot)(P_k^{r_k}(i))$ for each $i \geq 1$ and each sufficiently large k. We note that, by construction and by definition of the $\beta_i(k)$'s, we have $P_k'' \in \mathcal{P}^{N,v}$. The surface energy savings in going from P_k to $P_k^{r_k}$ is

$$SE(P_k) - SE(P_k^{r_k}) \geq 4\Gamma \sum_{i=1}^{N} |\beta_i(k)|, \tag{5.7}$$

by (5.5). The surface energy cost incurred by re-adjusting the volumes, in going from $P_k^{r_k}$ to P_k'', is

$$SE(P_k'') - SE(P_k^{r_k}) \leq M'' a_{\max} \sum |\beta_i| \leq 2\Gamma \sum_{i=1}^{N} |\beta_i(k)|, \tag{5.8}$$

by (5.1) and by the definition of Γ. Subtracting (5.8) from (5.7) gives $SE(P_k) - SE(P_k'') \geq 0$, as desired. Thus, $SE(P_k'') \leq SE(P_k)$, so (P_k'') is a surface energy minimizing sequence; i.e., $\inf\{SE(P_k'') : k = 1, 2, 3, \ldots\} = E_{\min}$.

For all sufficiently large k, we let Ξ_k denote the set of centers of balls in B_k :

$$\Xi_k = \left[\bigcup_{i=2}^{N} \{p_k(i)\}\right] \cup \left[\bigcup_{i=2}^{N} \bigcup_{j=1}^{q(k,i)} \left\{p_k^j(i)\right\}\right], \tag{5.9}$$

$$\begin{aligned} \operatorname{card}(\Xi_k) &\leq (N-1) + (N-1)\sup\{q(k,i) : 2 \leq i \leq N\} \\ &\leq (N-1)\left[1 + v_{\max}\left[2\gamma(2\zeta+1)\right]^n \left[\frac{a_{\max}}{\varepsilon_0}\right]^n\right] = N_0, \end{aligned} \tag{5.10}$$

by (5.2). We note that N_0 is independent of k; without an estimate such as (5.10), this proof would not work. From now on, it will be understood that k is taken to be sufficiently large so that the previous steps hold, and we will not always repeat this assertion.

We note that $B_k \subset \bigcup_{p \in \Xi_k} B^n(p, R+2)$, and so, since we truncate the polycrystals at a distance less than S away from B_k (with a net savings, after restoring the volumes), it follows that

$$\bigcup_{i=2}^{N} \overline{P_k''(i)} \subset \bigcup_{p \in \Xi_k} B^n(p, R+S+2) = F_k.$$

We will now show that F_k is contained in some fixed ball centered at the origin (with radius independent of k) or that we can get it to be contained in such a ball by simply translating its connected components. This is easy to accomplish. We have an a priori upper bound, N_0, on the cardinality of Ξ_k, and we can easily estimate the diameter of any connected component of F_k: since the number of closed balls (and hence of connected components) in F_k does not exceed N_0, and since each ball has radius no more than $R+S+2$, any connected component of F_k has diameter no more than $2N_0(R+S+2)$, which we note is independent of k.

One way to proceed at this point would be to use Jung's theorem ([26], 2.10.41) to put each connected component of F_k inside a closed ball having diameter not exceeding $\sqrt{2n/(n+1)}\,(2N_0(R+S+2)) = N_1$. We could then translate each of these balls so that they remain separated by positive distance (not exceeding $1/N_0$) and have centers on the x_n axis, with one ball having its center at the origin. In this way, we can translate all of these balls in such a way that they are disjoint and are all contained in $B^n(0, N_0N_1+2)$, which is what we need. We would then have to explicitly define the translation functions that do this, and then we would replace P_k'' with a new polycrystal, P_k''', which is formed from P_k'' by simply translating components of F_k. Both volume and surface energy are preserved by translation of separate components, and so (P_k''') is an a priori bounded, surface energy minimizing sequence, as desired.

Alternatively, we could proceed as in [2], VI.15 (16), by setting $r = 2N_0(2N_0(R+S+2))$, and choosing $s = 2r$. F. Almgren defined functions

$$f^k : \left(\bigcup_{p \in \Xi_k} B^n(p, R+S+2) \right) \to B^n(0, r+2N_0(R+S+2))$$

by showing how to translate each connected component C_m of F_k in such a way that the components remain separated and inside the ball $B^n(0, r+2N_0(R+S+2))$. We define $P_k'''(i) = f^k(P_k''(i))$ for each $2 \le i \le N$, and set

$$P_k'''(1) = \operatorname{Int}(\mathbb{R}^n \setminus (P_k'''(2) \cup \ldots \cup P_k'''(N))).$$

It follows from the discussion above that

$$P_k'''(2) \cup \ldots \cup P_k'''(N) \subset B^n(0, s+2N_0(R+S+2))$$

for all sufficiently large k and that (P_k''') is a surface energy minimizing sequence, so that

$$\inf\{SE(P_k''') : k = 1, 2, 3, \ldots\} = E_{\min}.$$

Now that we have produced a bounded minimizing sequence, since $\mathcal{P}^{N,v}$ is **M**-closed we may use Theorems 4 and 5 to deduce that there exist a $Q \in \mathcal{P}^{N,v}$ and a subsequence $\left(P_{k(j)}'''\right)$ of (P_k''') for which $P_{k(j)}''' \to Q$ in volume as $j \to \infty$, with

$SE(Q) = \inf \left\{ SE(L) : L \in \mathcal{P}_*^{N,v} \right\} = E_{\min}$. Any polycrystal satisfying the volume constraints but which is not in $\mathcal{P}_*^{N,v}$ must have infinite total boundary surface area, and hence infinite surface energy, and so $SE(Q)$ equals the infimum of $SE(L)$ over all polycrystals L for which $\mathcal{L}^n(L(i)) = v_i$ for each $2 \leq i \leq N$. □

6. Other related work

F. Almgren's methods are quite adaptable to other settings. In [41], Section 4.3, F. Morgan uses several key constructions from F. Almgren's argument to study the existence of a partition of $\mathbb{R}^n$ into regions $R_0, R_1, \ldots, R_m$ (with R_0 having infinite volume in his formulation) having prescribed volumes and a prescribed "index set" $\mathcal{T}$ (a collection of subsets of $\{ij : 0 \leq i, j \leq m\}$ "prescribing how the S_{ij} can come together", where S_{ij} denotes the $(n-1)$-dimensional interface between regions R_i and R_j), and minimizing

$$\sum \mathbf{M}(S_{ij}) + \lambda \sum_{I \in \mathcal{T}} \mathbf{M}(C_I), \tag{6.1}$$

where $\lambda > 0$ and each C_I is an $(n-2)$-dimensional "singular set" where the S_{ij}'s (for each ij in I) meet, and $\partial S_{ij} = \Sigma_{ij \in I} C_I$. As set up in [41], each interface and singular set is counted multiple times in the energy term (6.1). In Section 4.6 of [41], F. Morgan remarks,

> The existence Theorem 4.3 extends to produce a bounded minimizer of
>
> $$\sum \Phi_{ij}(S_{ij}) + \sum \lambda_I \mathbf{M}(C_I)$$
>
> for general norms Φ_{ij} and positive constants λ_I. (Here $\Phi_{ij}(S_{ij}) = \int_{S_{ij}} \Phi_{ij}(\mathbf{n})$, where $\mathbf{n}$ is the oriented unit normal to S_{ij}.) Without the curvature bounds on the S_{ij}'s of the regularity Theorem 3.2, another argument, similar to [A2, VI.14], is needed to ensure compact support, in turn needed in the existence proof. To outline the arguments for the harder case $n \geqq 3$, suppose a minimizer fails to have compact support. ... a contradiction.

(Here, A2 is [2].) F. Morgan uses proof by contradiction to outline a proof, for the case $n \geq 3$, that if a minimizer exists then it must have just one unbounded region. We note that his energy term is always different from ours since $\lambda_I > 0$; the main existence theorem in our paper does not involve the C_I's in any way, and our main results are distinct from those in [41].

Also, our setting is a bit different from that of [41] in another fundamental way. In the caption to Figure 2a and 2b there, F. Morgan writes, "The regions R_i do not completely determine the surfaces S_{ij}." His Figures 2a and 2b of [41] depict two partitions having nearly the same R_1, R_2, and R_3 but different S_{ij}'s: in his Figure 2a the regions R_1 and R_2 meet along a flat interface S_{12}, while in his Figure 2b the interface S_{12} is replaced by flat S_{13} and S_{23} interfaces separated by an "infinitesimal layer of R_3 through the middle". His R_i's in the two figures are the same except for that infinitesimal layer of R_3 having zero volume. In our formulation, our interfaces are defined using reduced boundaries, which are unaltered by zero-volume changes in the regions. In particular, if we were to put an infinitesimal layer of region 3 in between regions 1 and 2, which meet along a planar interface, there would be only a 1-2 interface in our formulation. No interface would form with this infinitesimally thin part of region 3 since the reduced boundary of a set having zero volume is empty.

A proof of the existence result stated in [41], §4.6, which is based on the standard compactness/lower semicontinuity argument would not work without severely restricting the index set T and also restricting the norms Φ_{ij} and the constants λ_I. One issue is that new interfaces and new "edges" can be created in the limit, causing total energy to go up suddenly in the limit without creating any new types of interfaces or edges. For example, consider a sequence of clusters C_α such that for each positive integer α the cluster C_α consists of two congruent and parallel solid, closed right circular cylinders, the first being region 1 and the second being region 2 (with region 0 being the complement of the union of regions 1 and 2), each separated by a positive distance, and such that the distance between the cylinders becomes arbitrarily close to 0 as α increases, so that C_α converges in volume to a limit configuration C involving the two cylinders intersecting along a boundary edge. We can form another sequence C'_α by adding to each C_α a translated copy of C so that the C'_α sequence converges in volume to two copies of C. Then for any norms Φ_{ij} and positive constants λ_I the energy of the limit will exceed the limit inferior of the energies of the C'_α clusters, violating lower semicontinuity, even though no new types of interfaces or edges were created in the limit. Unless T is made so restrictive as to rule out all examples of this sort (and several other families of sequences of clusters where new edges form in the limit as surfaces meet), the energy (6.1) will not be lower semicontinuous with respect to convergence in volume.

Even with T severely restricted, the Φ_{ij}'s in the more general energy term that F. Morgan remarked on would have to satisfy BV-ellipticity, which is necessary for the lower semicontinuity of the surface energy component of the energy with respect to convergence in volume. In addition to that, the λ_I's in the more general energy term F. Morgan discussed would have to be restricted so as to satisfy triangle-type inequalities, chosen to preclude the possibility of several low-energy edges merging in the limit to produce a costly, previously non-existent, higher-energy edge. It is not clear that even all of those conditions together would imply lower semicontinuity of the energy with respect to convergence in volume. The same general issues would be present when trying to use a compactness/lower semicontinuity argument with lower-dimensional energies of all orders.

The existence results proven and discussed in [41] are for the case where the coefficients of the lower-dimensional energies of the singular sets C_I are strictly positive. Although those existence results do not apply to our setting, as those coefficients are zero in our surface energy and so our energies are always different in a significant way from those in [41], it is worth considering how some of the results and constructions from [41] could be modified in order to produce an alternative existence proof in our setting.

First, several changes (such as those noted above for the Φ_{ij}'s, for the λ_I's, and for T) would need to be made in order to ensure lower semicontinuity of the energy being minimized.

Second, Lemma 3.1 from [41], used in the proof outline for the existence result discussed in [41], §4.6, is stated for the special case where the Φ_{ij}'s are each equal to Φ_E, the Euclidean norm. This is a special case of a result of F. Almgren ([2], Theorem VI.2 (3), which is proved in Lemma VI.12). However, F. Almgren himself established it only for the case where the Φ_{ij}'s are scalar multiples of a fixed, smooth Φ, with the additional hypotheses of "partitioning regularity". It would be necessary to extend F. Almgren's result to the case of independent Φ_{ij}'s, as we did in Proposition 2.

There is another technical issue related to Lemma 3.1 from [41]. This result is critical for the main existence proof in [41]. Even our own result, Proposition 2, which is much more general, would not be enough to carry out the proof of the existence result mentioned in [41], §4.6, which is stated for general, not necessarily smooth, norms Φ_{ij}. We are not aware of any extensions of our Proposition 2 in the literature which do not require smoothness of the norms. Proving such a result seems to be quite difficult.

Next, Lemma 4.1 of [41], stated for the isotropic case, would have to be stated and proved for the case of independent Φ_{ij}'s, as in our paper. Additionally, Lemma 4.2 of [41] (a modification of a special case of F. Almgren's Lemma VI.13) is stated and proven for integral n currents, which necessarily have compact support. Yet, in order to allow for regions R_i which may be unbounded, as in our setting, the proofs in [41] would need to make use of such a result on locally integral currents of the form $[K]$, where K is a possibly unbounded Lebesgue measurable subset of $\mathbb{R}^n$ having finite perimeter and finite volume. Our Proposition 3, itself modeled after [2], Lemma VI.13, is the first proven result which allows this. A similar result would be needed to extend the existence results from [41] to our setting. Also, the boundedness proof sketch in [41], §4.6, introduces the ratio $(n-1)\,/\,(n-2)$, which restricts the proof to $n \geq 3$.

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