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SINGULAR TWISTED SUMS GENERATED BY COMPLEX INTERPOLATION

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Abstract. We present new methods to obtain singular twisted sums $X \oplus_\Omega X$ (i.e., exact sequences $0 \to X \to X \oplus_\Omega X \to X \to 0$ in which the quotient map is strictly singular) when X is an interpolation space arising from a complex interpolation scheme and Ω is the induced centralizer.

Although our methods are quite general, we are mainly concerned with the choice of X as either a Hilbert space or Ferenczi's uniformly convex Hereditarily Indecomposable space. In the first case, we construct new singular twisted Hilbert spaces (which includes the only known example so far: the Kalton-Peck space Z_2). In the second case we obtain the first example of an H.I. twisted sum of an H.I. space.

During our study of singularity we introduce the notion of a disjointly singular twisted sum of Köthe function spaces and construct several examples involving reflexive p-convex Köthe function spaces (which includes the function space version of the Kalton-Peck space Z_2).

We then use Rochberg's description of iterated twisted sums to show that there is a sequence $\mathcal{F}_n$ of H.I. spaces so that $\mathcal{F}_{m+n}$ is a singular twisted sum of $\mathcal{F}_m$ and $\mathcal{F}_n$, while for $l > n$ the direct sum $\mathcal{F}_n \oplus \mathcal{F}_{l+m}$ is a nontrivial twisted sum of $\mathcal{F}_l$ and $\mathcal{F}_{m+n}$.

1. Introduction

For all unexplained notation see Sections 2 (background on exact sequences and quasi-linear maps) and 3 (background and preliminary results on complex interpolation and centralizers).

This paper focuses on the study of the existence and properties of exact sequences

$$0 \longrightarrow X \xrightarrow{\;j\;} \Diamond \xrightarrow{\;q\;} X \longrightarrow 0, \tag{1.1}$$

in which the Banach space X has been obtained by complex interpolation. The exact sequence is called *nontrivial* when $j(X)$ is not complemented in the middle space $\Diamond$, which is then called a (nontrivial) twisted sum of X (or a twisting of X or even a twisted X). The exact sequence is called *singular* (and $\Diamond$ is called a *singular twisted sum*) when the operator q is strictly singular. The key example on which all the theory is modeled is the Kalton-Peck twisted Hilbert space Z_2 obtained in [36], which provides the first and only known singular sequence

$$0 \longrightarrow \ell_2 \xrightarrow{\;j\;} Z_2 \xrightarrow{\;q\;} \ell_2 \longrightarrow 0.$$

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Singular sequences correspond to twisted sums which are, in some sense, as far as possible from being direct sums. For example, in Kalton-Peck example Z_2, the natural copy of ℓ_2 does not even admit a "relative" summand, i.e. there is no infinite-dimensional subspace Z of Z_2 forming a topological direct sum $j(\ell_2) \oplus Z$ inside Z_2.

In [30] Kalton showed that exact sequences (1.1) are in correspondence with certain nonlinear maps $F : X \to X$, called quasi-linear maps. So, twisted sum spaces, and in particular exact sequences, can be written in the form

$$0 \longrightarrow X \longrightarrow X \oplus_F X \longrightarrow X \longrightarrow 0. \tag{1.2}$$

Following [13, 18], we say that a quasi-linear map F is *singular* if the associated exact sequence (1.2) is singular. In [36] Kalton and Peck presented a method to show an explicit construction of quasi-linear maps on Banach spaces with unconditional basis. This method was refined by Kalton [32] and extended to Köthe function spaces. The special type of quasi-linear maps obtained by this method were called *centralizers*. The main examples are the so-called *Kalton-Peck maps:*

$$\mathcal{K}_\phi(x) = x\phi\left(-\log \frac{|x|}{\|x\|}\right),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz map . The choice of the function $\phi_r(t) = t^r$ (when $t \geq 1$) and $\phi_r(t) = t$ (when $0 \leq t \leq 1$) with $0 < r \leq 1$ will have special interest for us, and we will call $\mathcal{K}_r$ the associated Kalton-Peck map.

Definition 1.1. We will simply write $\mathcal{K}$ for the Kalton-Peck map $\mathcal{K}_1$; i.e.,

$$\mathcal{K}(x) = x\left(-\log \frac{|x|}{\|x\|}\right).$$

In [36] it is shown that $\mathcal{K}$ is singular on ℓ_p-spaces for $1 < p < \infty$, in [18] for $p = 1$, and in [13] for the whole range $0 < p < \infty$. Kalton soon observed that $\mathcal{K}$ on ℓ_2 could be generated from the interpolation scale of ℓ_p-spaces. Taking this as a starting point, Kalton unfolds in [32, 33] the existence of a correspondence between centralizers defined on Köthe function spaces and interpolation scales of Köthe function spaces. This opens the door to the possibility of obtaining nontrivial quasi-linear maps in Banach spaces generated by an interpolation scale, even when no unconditional structure is present.

Such is the point of view we adopt in this paper to tackle the study of singular centralizers and singular quasi-linear maps on Banach spaces obtained by complex interpolation. In the case of centralizers this leads us to obtain new singularity results for Kalton-Peck sums of sequence spaces as well as of function spaces and, in particular, new singular twisted Hilbert spaces. We introduce a new concept of singularity, that we call *disjoint singularity*, which is relevant to the study of interpolation schemes of function spaces. In the case of general quasi-linear maps, not just centralizers, we "localize" the techniques developed and apply them to spaces with monotone basis and obtain the first H.I. twisted sum of an H.I. space.

A description of the contents of the paper is in order: after this introduction and a preliminary Section 2 on basic facts about exact sequences and quasi-linear maps, Section 3 takes root in Kalton's work and so it contains an analysis of centralizers arising from an interpolation scheme; the analysis is centered on an interpolation couple (X_0, X_1) and the centralizer Ω_θ obtained at the interpolation space $X_\theta = (X_0, X_1)_\theta$; although the results extend (see subsection 5.4) to cover the case of a measurable family of spaces. We observe that such centralizers admit an overall form as $\Omega_\theta(x) = x \log \frac{a_0(x)}{a_1(x)}$, where $a_0(x)^{1-\theta}a_1(x)^\theta$ is a Lozanovskii factorization of $|x|$ with respect to the couple (X_0, X_1); from this we derive a few consequences. Section 4 contains the two fundamental estimates we use throughout the paper: Lemma 4.5 (estimate for nonsingular maps) and Lemma 4.8 (general estimate for centralizers arising from an interpolation scheme). Section 5 contains several criteria for singularity based on the previous two lemmata. The first two subsections treat the Banach lattice case: we recover the singularity of Kalton-Peck maps associated to the interpolation scale of ℓ_p-spaces as a particular case of a general criterion for singularity, and we prove the disjoint singularity of Kalton-Peck maps associated to the interpolation scale of L_p-spaces, for which it was known that they were not singular. We also prove the disjoint singularity of Kalton-Peck maps on more general p-convex Köthe spaces by means of the interpolation formula $X = (L_\infty, X^{(p)})_{1/p}$. In the third subsection, we give conditions implying the singularity in the conditional case (spaces admitting a basis not necessarily unconditional) which will be needed to cover the case of H.I. spaces. In Section 6 we obtain new singular twisted Hilbert spaces; we also complete previous results by showing that centralizers $\mathcal{K}_\phi$ are singular under rather mild conditions on ϕ, satisfied in particular by the complex versions [34] of $\mathcal{K}$. In Section 7 we connect the results about singular sequences with the twisting of H.I. spaces: a twisted sum of an H.I. space and an arbitrary space is H.I. if and only if it is singular. One of the difficulties for such construction is, as we show, that a nontrivial twisted sum of two H.I. spaces can be decomposable; note that it was known [27, Theorem 1] that such twisted sums should be at most 2-decomposable. Section 8 applies the previous techniques to the quasi-linear map associated to the construction of Ferenczi's H.I. space $\mathcal{F}$ [25] by complex interpolation of a suitable family of Banach spaces. In Section 9 we apply the results of [12] to complete and improve the results in Sections 7 and 8 by showing new natural H.I. and decomposable twistings; precisely, that there is a sequence $(\mathcal{F}_n)$ of H.I. spaces so that:

(i) For each $m, n \geq 1$ there is a singular exact sequence
$$0 \longrightarrow \mathcal{F}_m \longrightarrow \mathcal{F}_{m+n} \longrightarrow \mathcal{F}_n \longrightarrow 0.$$

(ii) For each $l, m, n \geq 1$ with $l > n$ there is a nontrivial exact sequence
$$0 \longrightarrow \mathcal{F}_l \longrightarrow \mathcal{F}_n \oplus \mathcal{F}_{l+m} \longrightarrow \mathcal{F}_{m+n} \longrightarrow 0.$$

2. Exact sequences, twisted sums and quasi-linear maps

A *twisted sum* of two Banach spaces Y and Z is a space X which has a subspace isomorphic to Y with the quotient X/Y isomorphic to Z. The space X is a quasi-Banach space in general [36]. Recall that a Banach space is *B-convex* when it does not contain ℓ_1^n uniformly. Theorem 2.6 of [30] implies that a twisted sum of two B-convex Banach spaces is isomorphic to a Banach space.

An exact sequence $0 \to Y \to X \to Z \to 0$, where Y, Z are Banach spaces and the arrows are (bounded) operators, is a diagram in which the kernel of each arrow coincides with the image of the preceding one. By the open mapping theorem this means that the middle space X is a twisted sum of Y and Z.

Two exact sequences $0 \to Y \to X_1 \to Z \to 0$ and $0 \to Y \to X_2 \to Z \to 0$ are *equivalent* if there exists an operator $T : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{i} & X_1 & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow[j]{} & X_2 & \xrightarrow[p]{} & Z & \longrightarrow & 0 \end{array}$$

The classical 3-lemma (see [17, p. 3]) shows that T must be an isomorphism. An exact sequence is trivial if and only if it is equivalent to $0 \to Y \to Y \times Z \to Z \to 0$, where $Y \times Z$ is endowed with the product norm. In this case we say that the exact sequence *splits.*

A map $F : Z \to Y$ is called *quasi-linear* if it is homogeneous and there is a constant M such that $\|F(u+v) - F(u) - F(v)\| \le M(\|u\| + \|v\|)$ for all $u, v \in Z$. There is a correspondence (see [17, Theorem 1.5.c, Section 1.6]) between exact sequences $0 \to Y \to X \to Z \to 0$ of Banach spaces and a special kind of quasi-linear maps called z-linear maps, which are those satisfying $\|F(\sum_{i=1}^n u_i) - \sum_{i=1}^n F(u_i)\| \le M \sum_{i=1}^n \|u_i\|$ for all finite sets $u_1, \dots, u_n \in Z$. A quasi-linear map $F : Z \to Y$ induces the exact sequence $0 \to Y \xrightarrow{j} Y \oplus_F Z \xrightarrow{p} Z \to 0$ in which $Y \oplus_F Z$ denotes the vector space $Y \times Z$ endowed with the quasi-norm $\|(y,z)\|_F = \|y - F(z)\| + \|z\|$. The embedding is $j(y) = (y, 0)$ while the quotient map is $p(y,z) = z$. When F is z-linear, this quasi-norm is equivalent to a norm [17, Chapter 1]. On the other hand, the process to obtain a z-linear map from an exact sequence $0 \to Y \xrightarrow{i} X \xrightarrow{q} Z \to 0$ of Banach spaces is the following one: get a homogeneous bounded selection $b : Z \to X$ for the quotient map q, and then a linear selection $\ell : Z \to X$ for the quotient map. Then $\omega = i^{-1}(b - \ell)$ is a z-linear map from Z to Y. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{i} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow[j]{} & Y \oplus_\omega Z & \xrightarrow[p]{} & Z & \longrightarrow & 0 \end{array}$$

obtained by taking as $T : X \to Y \oplus_\omega Z$ the operator $T(x) = (x - \ell q x, qx)$ shows that the upper and lower exact sequences are equivalent. Two quasi-linear maps $F, F' : Z \to Y$ are said to be equivalent, denoted $F \equiv G$, if the difference $F - F'$ can be written as $B + L$, where $B : Z \to Y$ is a homogeneous bounded map (not necessarily linear) and $L : Z \to Y$ is a linear map (not necessarily bounded). Of course two quasi-linear maps are equivalent if and only if the associated exact sequences are equivalent. Thus, two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Omega Z & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Psi Z & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

(or two quasi-linear maps Ω, Ψ) are equivalent (i.e., $\Omega \equiv \Psi$) if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Omega Z & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Psi Z & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

with $\alpha = id_Y$ and $\gamma = id_Z$. Imposing other conditions on the maps α, β, γ yields other notions of equivalence appearing in the literature. From the most restrictive to the more general they are:

(1) Bounded equivalence [32,33] (see also Section 3 below): asking that $\Omega - \Psi$ is bounded.
(2) Equivalence, as defined above.
(3) Projective equivalence [36]: asking α, γ to be scalar multiples of the identity. Equivalently, $\Omega \equiv \mu\Psi$ for some scalar μ.
(4) Isomorphic equivalence [10,19]: asking α, β, γ to be isomorphisms. In quasi-linear terms, this means that $\alpha\Omega \equiv \Psi\gamma$.

Obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Each of these notions yields a "natural" isomorphism β between $Y \oplus_\Omega Z$ and $Y \oplus_\Psi Z$ of a specific form prescribed by the forms of the maps α and γ. Moreover, we will also consider:

(5) Permutative projective equivalence: when Y and Z have unconditional bases (e_n), asking $T_\sigma\Omega \equiv \mu\Psi T_\sigma$ for some scalar μ and some operator $T_\sigma(\sum_i x_i e_i) = \sum_i x_i e_{\sigma(i)}$ induced by a permutation σ of the integers. When $\mu = 1$ we will just say that Ω and Λ are permutatively equivalent.
(6) More generally, module equivalence: asking α, γ to have the form $x \to \lambda x$ for some bounded λ.

A few facts about the connections between quasi-linear maps and the associated exact sequences will be needed in this paper, and can be explicitly found in [20, Section 1]. Given an exact sequence $0 \to Y \to X \to Z \to 0$ with associated quasi-linear map F and an operator $\alpha : Y \to Y'$, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{i} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle T}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & Y' & \xrightarrow{i'} & \mathrm{PO} & \xrightarrow{q'} & Z & \longrightarrow & 0. \end{array} \tag{2.1}$$

The lower sequence is called the *push-out sequence,* its associated quasi-linear map is equivalent to $\alpha \circ F$, and the space PO is called the *push-out space.* When F is z-linear, so is $\alpha \circ F$. Given a commutative diagram such as (2.1) the *diagonal push-out sequence* is the exact sequence generated by the quasi-linear map $F \circ q'$ and is equivalent to the exact sequence

$$0 \longrightarrow Y \xrightarrow{d} Y' \oplus X \xrightarrow{m} \mathrm{PO} \longrightarrow 0$$

where $d(y) = (-\alpha y, iy)$ and $m(y', x) = i'y' + Tx$.

3. Complex interpolation and centralizers

Here we explain the connections between complex interpolation, twisted sums and quasi-linear maps that we use throughout the paper.

3.1. **Complex interpolation and twisted sums.** We describe the complex interpolation method for a pair of spaces following [5]. Other general references are [21,33,35,41].

Let $\mathbb{S}$ denote the closed strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$ in the complex plane, and let $\mathbb{S}^\circ$ be its interior and $\partial\mathbb{S}$ be its boundary. Given an admissible pair (X_0, X_1) of complex Banach spaces, we denote by $\mathcal{H} = \mathcal{H}(X_0, X_1)$ the space of functions

$g : \mathbb{S} \to \Sigma := X_0 + X_1$ satisfying the following conditions:

(1) g is $\|\cdot\|_\Sigma$-bounded and $\|\cdot\|_\Sigma$-continuous on $\mathbb{S}$, and $\|\cdot\|_\Sigma$-analytic on $\mathbb{S}^\circ$;
(2) $g(it) \in X_0$ for each $t \in \mathbb{R}$, and the map $t \in \mathbb{R} \mapsto g(it) \in X_0$ is bounded and continuous;
(3) $g(it+1) \in X_1$ for each $t \in \mathbb{R}$, and the map $t \in \mathbb{R} \mapsto g(it+1) \in X_1$ is bounded and continuous.

The space $\mathcal{H}$ is a Banach space under the norm

$$\|g\|_{\mathcal{H}} = \sup\{\|g(j+it)\|_j : j = 0,1; t \in \mathbb{R}\}.$$

For $0 < \theta < 1$, define the interpolation space

$$X_\theta = (X_0, X_1)_\theta = \{x \in \Sigma : x = g(\theta) \text{ for some } g \in \mathcal{H}\}$$

with the norm $\|x\|_\theta = \inf\{\|g\|_H : x = g(\theta)\}$. So $(X_0, X_1)_\theta$ is the quotient of $\mathcal{H}$ by $\ker \delta_\theta$, and thus it is a Banach space.

For $0 < \theta < 1$, we will consider the maps $\delta_\theta^n : \mathcal{H} \to \Sigma$ (the evaluation of the n^{th}-derivative at θ) that appear in Schechter's version of the complex method of interpolation [42]. Note that $\delta_\theta \equiv \delta_\theta^0$ is bounded by the definition of $\mathcal{H}$, and this fact and the Cauchy integral formula imply the boundedness of δ_θ^n for $n \geq 1$ (see also [12]). We will also need the following result (see [15, Theorem 4.1]):

Lemma 3.1. $\delta'_\theta : \ker \delta_\theta \to X_\theta$ *is bounded and onto for* $0 < \theta < 1$.

For future use, note that given $G \in \ker\, \delta_\theta$, the function H defined by $H(z) = G(z)/(z-\theta)$ belongs to $\mathcal{H}$ and satisfies $\delta'_\theta(G) = H(\theta)$, which implies the estimate

$$\|\delta'_{\theta|\ker\ \delta_\theta}\| \leq \operatorname{dist}(\theta, \partial\mathbb{S})^{-1}.$$

Lemma 3.1 provides the connection with exact sequences and twisted sums through the following push-out diagram:

$$\begin{tikzcd} 0 \arrow[r] & \ker\delta_\theta \arrow[r, "i_\theta"] \arrow[d, "\delta'_\theta"'] & \mathcal{H} \arrow[r, "\delta_\theta"] \arrow[d] & X_\theta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X_\theta \arrow[r] & \mathrm{PO} \arrow[r] & X_\theta \arrow[r] & 0 \end{tikzcd} \tag{3.1}$$

whose lower row is obviously a twisted sum of X_θ.

Apart from the obvious description as a push-out space, PO can be represented as:

(1) *A twisted sum space.* Let $B_\theta : X_\theta \to \mathcal{H}$ be a bounded homogeneous selection for δ_θ, and let $L_\theta : X_\theta \to \mathcal{H}$ be a linear selection. The map $\omega_\theta = B_\theta - L_\theta : X_\theta \to \ker\delta_\theta$ is an associated quasi-linear map for the upper sequence in diagram (3.1). The lower push-out sequence then comes defined by the quasi-linear map $\delta'_\theta\omega_\theta$. Hence, $\mathrm{PO} \simeq X_\theta \oplus_{\delta'_\theta\omega_\theta} X_\theta$.

(2) *A derived space.* With the same notation as above, set

$$d_{\delta'_\theta B_\theta}(X_\theta) = \{(y,z) \in \Sigma \times X_\theta \ : \ y - \delta'_\theta B_\theta z \in X_\theta\}$$

endowed with the quasi-norm $\|(y,z)\|_d = \|y - \delta'_\theta B_\theta z\|_{X_\theta} + \|z\|_{X_\theta}$. This is a twisted sum of X_θ since the embedding $y \to (y,0)$ and quotient map $(y,z) \to z$ yield an exact sequence

$$0 \longrightarrow X_\theta \longrightarrow d_{\delta'_\theta B_\theta}(X_\theta) \longrightarrow X_\theta \longrightarrow 0.$$

Moreover, the two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_\theta & \longrightarrow & X_\theta \oplus_{\delta'_\theta \omega_\theta} X_\theta & \longrightarrow & X_\theta & \longrightarrow & 0 \\ & & \| & & \big\downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & X_\theta & \longrightarrow & d_{\delta'_\theta B_\theta}(X_\theta) & \longrightarrow & X_\theta & \longrightarrow & 0 \end{array}$$

are isometrically equivalent via the isometry $T(y,z) = (y + \delta'_\theta L_\theta z, z)$.

Thus, we can pretend that the quasi-linear map associated to the push-out sequence is $\delta'_\theta B_\theta$, usually more intuitive than the true quasi-linear map $\delta'_\theta(B_\theta - L_\theta)$. Such a map has been sometimes called "the Ω-operator". Needless to say, the Ω-operator depends on the choice of B_θ. However the difference between two associated Ω-operators must be bounded:

$$\|\delta'_\theta(\tilde{B}_\theta - B_\theta)x\|_{X_\theta} \leq \|\delta'_{\theta|\ker \delta_\theta}\|(\|\tilde{B}_\theta\| + \|B_\theta\|)\|x\|_{X_\theta}.$$

The derived space admits a third useful representation; see [41, p. 323] for an embryonic finite-dimensional version, also quoted in [22, p. 218]; see [15, Prop. 7.1] for a general version involving two compatible interpolators, and [12] for a rather complete exposition, variations and applications of that representation.

Proposition 3.2. *$d_{\delta'_\theta B_\theta}(X_\theta) = \{(f'(\theta), f(\theta)) : f \in \mathcal{H}\}$, and the quotient norm of $\mathcal{H}/(\ker \delta_\theta \cap \ker \delta'_\theta)$ is equivalent to the quasi-norm $\|(\cdot,\cdot)\|_d$.*

Proof. Given $f \in \mathcal{H}$, since $f - B_\theta(f(\theta)) \in \ker \delta_\theta$, by Lemma 3.1 one has

$$f'(\theta) - \delta'_\theta B_\theta(f(\theta)) = \delta'_\theta(f - B_\theta(f(\theta))) \in X_\theta;$$

hence $(f'(\theta), f(\theta)) \in d_{\delta'_\theta B_\theta}(X_\theta)$. Conversely, let $(y,z) \in d_{\delta'_\theta B_\theta}(X)$. We have $z \in X_\theta$, so $B_\theta z \in \mathcal{H}$. Since $y - \delta'_\theta B_\theta z \in X_\theta$, there exists $g \in \ker \delta_\theta$ such that $y - \delta'_\theta B_\theta z = g'(\theta)$. Thus taking $f = B_\theta z + g$ we have $f(\theta) = z$ and $f'(\theta) = y$, and the equality is proved.

To prove the equivalence of norms, given $(y,z) \in d_{\delta'_\theta B_\theta}(X)$, take $f \in \mathcal{H}$ with $\|f\| \leq 2\,\mathrm{dist}(f, \ker \delta_\theta \cap \ker \delta'_\theta)$ such that $y = f'(\theta)$ and $z = f(\theta)$. Then $\|z\|_{X_\theta} = \mathrm{dist}(f, \ker \delta_\theta)$ and

$$\|y - \delta'_\theta B_\theta z\|_{X_\theta} = \|\delta'_\theta(f - B_\theta z)\|_{X_\theta}.$$

Since $f - B_\theta z \in \ker \delta_\theta$, we get

$$\begin{aligned} \|(y,z)\|_d &\leq \|\delta'_{\theta|\ker \delta_\theta}\|(1 + \|B_\theta\|)\|f\| + \|f\| \\ &\leq 2(\|\delta'_{\theta|\ker \delta_\theta}\|(1 + \|B_\theta\|) + 1)\,\mathrm{dist}(f, \ker \delta_\theta \cap \delta'_\theta), \end{aligned}$$

and thus there must exist a constant C so that $\mathrm{dist}(f, \ker \delta_\theta \cap \delta'_\theta) \leq C\|(y,z)\|_d$ by the open mapping theorem. $\square$

The results mentioned so far remain valid in the wider context of the general method of interpolation considered in [35, Section 10]. In Section 8 we will need to work with the complex interpolation method associated to a family $(X_{(0,t)}, X_{(1,t)})_{t\in\mathbb{R}}$ of complex Banach spaces as described in [21], which is a special case of the general method mentioned above.

3.2. **Centralizers.** Given a measure space (Σ, μ), we will consider the ambient space L_0 of all μ-measurable functions $\Sigma \to \mathbb{R}$. A Köthe function space X over Σ is a Banach space of (equivalence classes of) locally integrable functions of L_0 such that

(1) $1_A \in X$ provided A is a measurable set of finite measure.
(2) If $f, g \in L_0$ with $|g| \leq |f|$ and $f \in X$, then $g \in X$ and $\|g\| \leq \|f\|$.

If L_∞ denotes the space of μ-measurable essentially bounded functions in their sup norm, Köthe spaces are L_∞-modules under the natural action $L_\infty \times X \to X$ given by $(f, x) \longrightarrow fx$ (pointwise product).

In what follows the measure μ will be countably additive and Σ a finite or σ-finite base space.

A particularly interesting case for us is when $\Sigma = \mathbb{N}$ and μ is the counting measure; here we encounter, among other, Banach spaces with a 1-unconditional basis with their associated ℓ_∞-structure.

Definition 3.3. A centralizer on a Köthe function space X is a homogeneous map $\Omega : X \longrightarrow L_0$ such that $\|\Omega(ax) - a\Omega(x)\|_X \leq C\|x\|_X\|a\|_\infty$ for all $a \in L_\infty$ and $x \in X$.

A centralizer on X will be denoted by $\Omega : X \curvearrowright X$. We use this notation to stress the fact that a centralizer on X is not a map $X \to X$, but only a map $X \to L_0$, so that the differences $\Omega(ax) - a\Omega(x)$ belong to X. This notion corresponds to Kalton's "homogeneous centralizer" introduced in [32, p. 12]. Centralizers arise naturally in a complex interpolation scheme in which the interpolation scale of spaces share a common L_∞-module structure: in such a case, the space $\mathcal{H}$ also enjoys the same L_∞-module structure in the form $(u \cdot f)(z) = u \cdot f(z)$. In this way, the fundamental sequence of the interpolation scheme $0 \to \ker \delta_\theta \to \mathcal{H} \to X_\theta \to 0$ is an exact sequence in the category of L_∞-modules. In an interpolation scheme starting with a couple (X_0, X_1) of Köthe function spaces, the map $\Omega_\theta = {\delta_\theta}' B_\theta$ is a centralizer on X_θ.

For a centralizer $\Omega : X \curvearrowright X$ on a Köthe function space X, it was proved in [32, Lemma 4.2] that there exists $M > 0$ such that $\|\Omega(u+v) - \Omega(u) - \Omega(v)\|_X \leq M(\|u\|_X + \|v\|_X)$. So we can assume that a centralizer is a quasi-linear map. The smallest of the constants M satisfying the above inequality is denoted $\rho(\Omega)$. Note that $\Omega : X \curvearrowright X$ induces an exact sequence in the category of (quasi-)Banach L_∞-modules $0 \to X \to d_\Omega(X) \to X \to 0$, where

$$d_\Omega(X) = \{(w, z) \in L_0 \times X : w - \Omega z \in X\}$$

endowed with the quasi-norm $\|(w, z)\|_\Omega = \|w - \Omega z\|_X + \|z\|_X$ and with embedding $y \to (y, 0)$ and quotient map $(w, z) \to z$. The derived space $d_\Omega(X)$ admits an L_∞-module structure defined by $a(w, z) = (aw, az)$. Kalton proved in [32, Section 4] that every self-extension of a Köthe function space X over a σ-finite measure space is (equivalent to) the extension induced by a centralizer on X.

A centralizer $\Omega : X \curvearrowright X$ is said to be *bounded* when there exists a constant $C > 0$ so that $\|\Omega(x)\|_X \leq C\|x\|_X$ for all $x \in X$, which in particular means that $\Omega(x) \in X$ for all $x \in X$. Two centralizers $\Omega_1 : X \curvearrowright X$ and $\Omega_2 : X \curvearrowright X$ are *equivalent* if and only if the induced exact sequences are equivalent, which happens if and only if there exists a linear map $L : X \to L_0$ so that $\Omega_1 - \Omega_2 - L$ is bounded. Two centralizers $\Omega_1 : X \curvearrowright X$ and $\Omega_2 : X \curvearrowright X$ are said to be *boundedly equivalent* when $\Omega_1 - \Omega_2$ is bounded. The interest in this notion (which, to some extent, plays for centralizers the role triviality plays for quasi-linear maps) stems from the following outstanding result of Kalton [33, Theorem 7.6]. A centralizer will be called real if $\Omega(f)$ is a real function whenever f is a real function. The centralizers obtained from a complex interpolation schema as before are real.

Theorem 3.4. *Let X be a separable superreflexive Köthe function space. Then there exists a constant c (depending on the concavity of a q-concave renorming of X) such that if $\Omega : X \curvearrowright X$ is a real centralizer on X with $\rho(\Omega) \leq c$, then*

(1) *There is a pair of Köthe function spaces X_0, X_1 such that $X = (X_0, X_1)_{1/2}$ and $\Omega - \Omega_{1/2}$ is bounded.*
(2) *The spaces X_0, X_1 are uniquely determined up to equivalent renorming.*

An example could help to understand the situation: taking the couple (ℓ_1, ℓ_∞), the map $B(x) = x^{2(1-z)}$ is a homogeneous bounded selection for the evaluation map $\delta_{1/2} : \mathcal{H} \to \ell_2$; hence the interpolation procedure yields the centralizer $-2\mathcal{K}$, while the couple (ℓ_p, ℓ_{p^*}) yields $-2(\frac{1}{p} - \frac{1}{p^*})\mathcal{K}$. As we see, the centralizer is the same in both cases, up to a scalar factor. This cannot be overlooked since, as Theorem 3.4 explains, it actually determines the end points of the interpolation scale. Proposition 3.7 clarifies this point.

The following result is a digression from the topic of the paper (which is why we will omit explanations about the required background, which can be found in [33, Theorem 7.6]). It is presented here to remark for future use that the condition on $\rho(\Omega)$, which is necessary for existence, is not necessary for uniqueness:

Proposition 3.5. *Let X be a separable superreflexive Köthe function space. Assume that $X = (X_0, X_1)_\theta = (Y_0, Y_1)_\theta$, where $0 < \theta < 1$ and X_i, Y_i are Köthe function spaces. If the induced centralizers Ω_X and Ω_Y are boundedly equivalent, then $X_0 = Y_0$ and $X_1 = Y_1$.*

Proof. We follow Kalton's notation and the first steps of the proof of uniqueness in Kalton's theorem [33, Theorem 7.6], which is written in the case $\theta = 1/2$. Since Ω_X and Ω_Y are boundedly equivalent, $\Omega_X^{[1]}$ and $\Omega_Y^{[1]}$ are boundedly equivalent. Hence on a suitable strict semi-ideal, Φ^{Ω_X} is equivalent to $\Phi_{Y_1} - \Phi_{Y_0}$, while $(1-\theta)\Phi_{Y_0} + \theta\Phi_{Y_1}$ is equivalent to Φ_X. Thus, up to equivalence, Φ_{Y_0} and Φ_{Y_1} are uniquely determined. [33, Proposition 4.5] shows then that the spaces Y_0 and Y_1 are unique up to equivalence of norm. □

3.3. **Centralizers and Lozanovskii's decomposition.** Here we follow Kalton (see [33, formula (3.2)]) to obtain a formula for the centralizer associated to an interpolation scale of Köthe function spaces with endpoints (X_0, X_1). Let $0 < \theta < 1$, and suppose that one of the spaces X_0, X_1 has the Radon-Nikodym property. Recall that given X, Y, two Köthe function spaces on the same measure space, one defines

$$XY = \{u \in L_0 : u = fg, \quad f \in X, g \in Y\},$$

endowed with the norm

$$\|u\|_{XY} = \inf\{\|f\|\|g\| : u = fg, f \in X, g \in Y\}.$$

Also, given X and $r > 0$, then one writes $X^r = \{f \in L_0 : |f|^r \in X\}$. The space X is said to be r-convex if X^r is isomorphic to a Banach space when endowed with the quasi-norm $\rho(f) = \||f|^{1/r}\|_X^r$.

The Lozanovskii decomposition formula allows us to show (see [35, Theorem 4.6]) that $X_\theta = X_0^{1-\theta} X_1^\theta$, with

$$\|x\|_\theta = \inf\{\|y\|_0^{1-\theta}\|z\|_1^\theta : y \in X_0, z \in X_1, |x| = |y|^{1-\theta}|z|^\theta\}.$$

By homogeneity we may always assume that $\|y\|_0 = \|z\|_1$ for y, z in this infimum. When $\|y\|_0, \|z\|_1 \leq K\|x\|_\theta$ we shall say that $|x| = |y|^{1-\theta}|z|^\theta$ is a *K-optimal decomposition* for x. When x is finitely supported or X is uniformly convex, a 1-optimal (or simply, optimal) decomposition may be achieved. A simple choice of $B_\theta(x)$ can be made for positive x as follows: Let $a_0(x), a_1(x)$ be a $(1+\epsilon)$-optimal (or optimal when possible) Lozanovskii decomposition for x. Since $\|x\|_\theta = \|a_0(x)\|_0 = \|a_1(x)\|_1$, set $B_\theta(x) \in \mathcal{H}$ given by $B_\theta(x)(z) = |a_0(x)|^{1-z}|a_1(x)|^z$. One thus gets for positive x the formula

$$\Omega_\theta(x) = \delta'_\theta B_\theta(x) = |a_0(x)|^{1-\theta}|a_1(x)|^\theta \log\frac{|a_1(x)|}{|a_0(x)|} = x \log\frac{|a_1(x)|}{|a_0(x)|}.$$

Using $B_\theta(x) = (\text{sgn } x)B_\theta(|x|)$ for general x one still gets

$$\Omega_\theta(x) = x \log\frac{|a_1(x)|}{|a_0(x)|}. \tag{3.2}$$

The Lozanovskii approach can be used to make explicit the Kalton correspondence between centralizers and interpolation scales in some special cases. Recall that the p-convexification of a Köthe function space X is defined by the norm $|||x||| = \||x|^p\|^{1/p}$. Conversely, when X is p-convex, the p-concavification of X is given by $|||x||| = \||x|^{1/p}\|^p$. Modulo the fact that every superreflexive space may be renormed to be p-convex for some $p > 1$, the following proposition interprets Kalton-Peck maps defined on uniformly convex spaces as induced by specific interpolation schemes.

Proposition 3.6. *Let $0 < \theta < 1 < p < \infty$, and let X be a Banach space with 1-unconditional basis (respectively a Köthe function space). Then $X_\theta = (\ell_\infty, X)_\theta$ (respectively $(L_\infty(\mu), X)_\theta$) is the θ^{-1}-convexification of X, and the induced centralizer on X_θ is*

$$\Omega(x) = \theta^{-1}\, x \log(|x|/||x||_\theta).$$

Conversely if X is p-convex and X^p is the p-concavification of X, then $X = (\ell_\infty, X^p)_{1/p}$ (respectively $X = (L_\infty(\mu), X^p)_{1/p}$), and the induced centralizer is defined on X by

$$\Omega(x) = p\, x \log(|x|/||x||).$$

Proof. We write the proof for the unconditional basis, the other being analogous. For normalized positive x in X_θ, write $x = a_0(x)^{1-\theta}a_1(x)^\theta$ and look for such a (normalized) decomposition which is optimal. Since $a_0(x) \in \ell_\infty$, we may assume that $a_0(x)$ has constant coefficients equal to 1 on the support of x. Otherwise, we may increase the coordinates of $a_0(x)$ that are not 1 to 1, therefore diminishing the corresponding coordinates of $a_1(x)$ and nonincreasing the norm of $a_1(x)$ by 1-unconditionality, and still get something optimal. So $a_0(x) = 1_{\text{supp}}(x)$ and $x = a_1(x)^\theta$. Therefore $\|x\|_\theta = \|a_1(x)\|^\theta = \|x^{1/\theta}\|^\theta$. So X_θ is the θ^{-1}-convexification of X and

$$\Omega_\theta(x) = x \log(a_1(x)/a_0(x)) = \frac{1}{\theta}\, x \log(x).$$

As for the converse, note that when we interpolate ℓ_∞ and some Y we have $|a_1(x)| = |x|^p$ for x normalized in Y_θ, so if we interpolate ℓ_∞ and $Y = X^{(p)}$ we obtain for such x

$$\|x\|_{Y_\theta} = 1 = \|a_1(x)\|_Y = \||x|^p\|_Y = \|(|x|^p)^\theta\|_X^p = \|x\|_X^p,$$

and therefore $X = Y_\theta = (\ell_\infty, X^{(p)})_\theta$.

The remaining part of the converse is an immediate consequence of the first part of the proposition. □

As we announced before Theorem 3.4, we now show the dependence of the scalar factor with respect to different choices of endpoints in a given interpolation scale:

Proposition 3.7. *Let (X_0, X_1) be an admissible pair of Köthe function spaces and, for some $0 < \alpha < \beta < 1$, also consider the admissible pair (X_α, X_β). Let $\alpha < \theta < \beta$ so that one has $(X_0, X_1)_\theta = (X_\alpha, X_\beta)_\rho$ for some $0 < \rho < 1$. Let Ω (resp. Ω') denote the centralizers generated by the couple (X_0, X_1) (resp. (X_α, X_β)). Then $\Omega'_\rho = (\beta - \alpha)\Omega_\theta$.*

Proof. It is easy to check (see [35, Theorem 4.5]) that ρ is given by $\alpha(1-\rho)+\beta\rho = \theta$. Let us consider the centralizers

$$\Omega_\theta(x) = x \log \frac{|a_1(x)|}{|a_0(x)|} \quad \text{and} \quad \Omega'_\rho(x) = x \log \frac{|a_\beta(x)|}{|a_\alpha(x)|}.$$

Since $x = a_0(x)^{1-\theta}a_1(x)^\theta$, $1-\theta = (1-\alpha)(1-\rho) + (1-\beta)\rho$ and $\theta = \alpha(1-\rho) + \beta\rho$ we get

$$x = \big(a_0(x)^{1-\alpha}a_1(x)^\alpha\big)^{1-\rho}\big(a_0(x)^{1-\beta}a_1(x)^\beta\big)^\rho.$$

Thus taking $a_\alpha(x) = a_0(x)^{1-\alpha}a_1(x)^\alpha$ and $a_\beta(x) = a_0(x)^{1-\beta}a_1(x)^\beta$ it is not difficult to check that the minimality of $x = a_0(x)^{1-\theta}a_1(x)^\theta$ implies the minimality of $x = a_\alpha(x)^{1-\rho}a_\beta(x)^\rho$, and the equality $\Omega'_\rho(x) = (\beta - \alpha)\Omega_\theta(x)$ follows from the properties of the logarithm function. □

3.4. **The case of Orlicz function spaces.** As an example and for further application, we now describe the centralizers associated to Orlicz function spaces over a measure space (Σ, μ). Recall that an *N-function* is a map $\varphi : [0,\infty) \to [0,\infty)$ which is strictly increasing, continuous, $\varphi(0) = 0$, $\varphi(t)/t \to 0$ as $t \to 0$, and $\varphi(t)/t \to \infty$ as $t \to \infty$. An N-function φ satisfies the *Δ_2-property* if there exists a number $C > 0$ such that $\varphi(2t) \le C\varphi(t)$ for all $t \ge 0$. For $1 < p < \infty$, $\varphi(t) = t^p$ is an N-function satisfying the Δ_2-property.

When an N-function φ satisfies the Δ_2-property, the *Orlicz space* $L_\varphi(\mu)$ is given by

$$L_\varphi(\mu) = \{f \in L_0(\mu) : \varphi(|f|) \in L_1(\mu)\}$$

with the norm

$$\|f\| = \inf\{r > 0 : \int \varphi(|f|/r)d\mu \le 1\}.$$

The following result was proved in [29], and a clear exposition can be found in [14].

Proposition 3.8. *Let φ_0 and φ_1 be two N-functions satisfying the Δ_2-property, and let $0 < \theta < 1$. Then the formula $\varphi^{-1} = \big(\varphi_0^{-1}\big)^{1-\theta}\big(\varphi_1^{-1}\big)^\theta$ defines an N-function φ satisfying the Δ_2-property, and $\big(L_{\varphi_0}(\mu), L_{\varphi_1}(\mu)\big)_\theta = L_\varphi(\mu)$.*

Next we give a formula for the centralizer associated to a Hilbert space obtained by complex interpolation of Orlicz spaces. Note that once we have defined a centralizer Ω for normalized $0 \leq f \in X$, we can define $\Omega(0) = 0$ and $\Omega(g) = g \cdot \Omega(|g|/\|g\|)$ for $0 \neq g \in X$.

Proposition 3.9. *Let φ_0 and φ_1 be two N-functions satisfying the Δ_2-property and such that $t = \varphi_0^{-1}(t) \cdot \varphi_1^{-1}(t)$. Then $\big(L_{\varphi_0}(\mu), L_{\varphi_1}(\mu)\big)_{1/2} = L_2(\mu)$ and the induced centralizer is*

$$\Omega_{1/2}(f) = f \,\log \frac{\varphi_1^{-1}(f^2)}{\varphi_0^{-1}(f^2)} = 2f \,\log \frac{\varphi_1^{-1}(f^2)}{f} \quad (0 \leq f \in L_2(\mu), \|f\|_2 = 1).$$

Proof. First we consider the general case $\varphi^{-1} = \big(\varphi_0^{-1}\big)^{1-\theta}\big(\varphi_1^{-1}\big)^{\theta}$, as in Proposition 3.8. For $0 \leq f$ normalized in $L_\varphi(\mu)$ we can write $f = \big(\varphi_0^{-1}\varphi(f)\big)^{1-\theta}\big(\varphi_1^{-1}\varphi(f)\big)^{\theta}$.

Thus, a homogeneous bounded selection for the quotient map $\mathcal{H} \to L_\varphi(\mu)$ is given by

$$B_\theta(f)(z) = \big(\varphi_0^{-1}\varphi(f)\big)^{1-z}\big(\varphi_1^{-1}\varphi(f)\big)^{z}.$$

By differentiation one gets $B_\theta(f)'(z) = B_\theta(f)(z) \,\log \frac{|\varphi_1^{-1}(\varphi(f))|}{|\varphi_0^{-1}\varphi(f)|}$; hence

$$\Omega_{1/2}(f) = B_{1/2}(f)'(1/2) = f \,\log \frac{|\varphi_1^{-1}(\varphi(f))|}{|\varphi_0^{-1}\varphi(f)|},$$

which gives the desired result when $\varphi(t) = t^2$. □

3.5. **Additional properties.** A *unit* in L_∞ is an element which only takes the values ± 1.

Definition 3.10. Let X be an L_∞-module. A map $F : X \to L_0$ is called *exact* if for each $x \in X$ and every unit u one has $F(ux) = uF(x)$. It is called *contractive* if $\text{supp}\ F(x) \subset \text{supp}\ x$ for every $x \in X$.

Every exact map is contractive: If the measure is finite, pick $u \in L_\infty$ as the function with value 1 on the support of x and -1 elsewhere, and then $ux = x$; therefore $u\Omega(x) = \Omega(ux) = \Omega(x)$, which means that $\text{supp}\,\Omega(x)$ is included in the support of x. A simple decomposition argument yields the σ-finite case. If the exactness condition $F(ux) = uF(x)$ for all units is forced to $F(ax) = aF(x)$ for all $a \in L_\infty$, then one gets trivial mappings. It is shown in [7, Lemma 1] and [8, Lemma 2.a] that mappings $\Phi : X \to L_0$ such that $\Phi(ax) = a\Phi(x)$ for all $a \in L_\infty$ have the form $\Phi(x) = fx$ for some $f \in L_0$.

It is not hard to pick an exact centralizer associated to an interpolation scale: proceed as in Section 3.3 starting with a contractive bounded selection B_θ on positive normalized elements to then set $B_\theta(x) = (\text{sgn}\ x)B_\theta(|x|)$.

Lemma 3.11. *Let (X_0, X_1) be an admissible pair of Köthe function spaces. The centralizer $\Omega_\theta = \delta'_\theta B_\theta$ on $X_\theta = (X_0, X_1)_\theta$ is exact, contractive and, whenever X_0 and X_1 are spaces with a normalized 1-unconditional basis (e_n), it can be chosen so that $\Omega_\theta(e_n) = 0$ for all n.*

Lemma 3.11 and an appeal to Kalton's theorem 3.4 immediately yield (1) and (2) in the following result. We present straightforward proofs that can also be used in the quasi-linear context.

Lemma 3.12.

(1) *Every centralizer Ω on a Köthe function space admits a exact centralizer ω such that $\Omega - \omega$ is bounded.*
(2) *Every exact centralizer (resp. quasi-linear map) Ω between Banach spaces with unconditional basis admits an exact centralizer (resp. quasi-linear map) ω such that $\omega(e_n) = 0$ and $\Omega - \omega$ is linear and exact.*
(3) *Every contractive centralizer (resp. quasi-linear map) Ω between Köthe function spaces admits, for every sequence (f_n) of disjointly supported vectors, a contractive centralizer (resp. quasi-linear map) ω such that $\omega(f_n) = 0$ and $\Omega - \omega$ is linear and contractive.*

Proof. Assertion (1) is in [32, Prop. 4.1]: set $\omega(x) = \|x\| \operatorname{sgn}(x)\Omega(|x|/\|x\|)$ for $x \neq 0$. To prove (2), note that since Ω is contractive, $\Omega(e_n) = \mu_n e_n$, and we may define the multiplication linear map $\ell(x) = \mu x$, where $\mu = (\mu_n)_n$. Thus $\omega = \Omega - \ell$ is the desired map. To prove (3), define as above a linear map by $\ell(f_n) = \Omega(f_n)$. If Ω is contractive, so is ℓ, and thus $\omega = \Omega - \ell$ is the desired map. □

We will need a more general result. Given a Köthe function space X, its Köthe dual X' is defined as the space of all measurable functions g for which the integral $\int fgd\mu$ defines a continuous functional on X. A Köthe function space X is said to be minimal if the characteristic functions of finite measure sets generate a dense subspace. The following result is due to Félix Cabello and we reproduce it here with his permission:

Lemma 3.13. *Let X be a minimal Köthe space on a σ-finite measure. Then X' is complemented in X^* by an L_∞-module projection.*

Proof. It is enough to treat the case of a finite measure space. Consider $L_\infty(\Sigma)$ as a subspace of X via the natural inclusion $\kappa : L_\infty(\Sigma) \to X$. Thus, $\kappa^*(x^*) \in L_\infty(\Sigma)^*$. Let $R : L_\infty(\Sigma)^* \to L_1(\Sigma)$ be the Radon-Nikodým projection, and define $P : X^* \to X'$ as

$$P(x^*) = R(\kappa^*(x^*)).$$

P is a morphism of L_∞-modules, and we show that P is a continuous projection of X^* onto its submodule X'. Obviously, $P(g) = g$ for all $g \in X'$. We see that $P(x^*)$ belongs to X' for all $x^* \in X^*$. It clearly suffices to work with nonnegative elements x^*. Let $g = P(x^*)$. Then, for every nonnegative $f \in L_\infty(\Sigma)$ one has

$$\int_\Sigma fgd\mu \leq x^*(f) \leq \|x^*\|_{X^*}\|f\|_X.$$

Hence, $g \in X'$ and $\|g\|_{X^*} \leq \|x^*\|_{X^*}$. The continuity of P is thus clear and the proof follows from this. □

A Köthe function space X is said to be maximal if whenever (f_n) is an increasing sequence in X converging almost everywhere to f, with $f_n \geq 0$ and $\sup_n \|f_n\|_X < \infty$, then $f \in X$ and $\|f\|_X = \sup_n \|f_n\|_X$. For every Köthe space X, its Köthe dual X' is a maximal Köthe space equipped with the dual norm. If X is maximal, then $X = X'_{oo}$, where X_{oo} is the subspace spanned by the characteristic functions of sets of finite measure in X'. If a Köthe space X is separable, then it is maximal if and only if it does not contain c_0; and, in any case, it is maximal if and only if $X = X''$. In particular, reflexive Köthe spaces are maximal. One has:

Corollary 3.14. *A maximal Köthe function space is complemented in its bidual.*

Proof. The Köthe space X_{oo} is minimal, and thus Lemma 3.13 immediately yields that $X = X_{oo}{}'$ is complemented in $X_{oo}{}^*$. $\square$

We are ready to:

Lemma 3.15. *Let X be a maximal Köthe function space.*

1. *Every exact map $B + L : X \to L_0$ with $B : X \to X$ homogeneous bounded and $L : X \to L_0$ linear can be written as $B + L = \mathcal{B} + \mathcal{L}$ with $\mathcal{B} : X \to X$ homogeneous bounded, $\mathcal{L} : X \to L_0$ linear and both exact.*
2. *If, moreover, X has an unconditional basis (e_n) and $(B + L)(e_n) = 0$ for all n, then $B + L$ is bounded.*

Proof. (1) Let U denote the abelian group of units in in L_∞. Then U is amenable, so there exists a left invariant finitely additive mean m on U allowing us to define for any bounded $f : U \to \mathbb{R}$ an integral $\int_U f(u)dm$. Since X is complemented in its bidual we may then define (see [6]) for any bounded $f : U \to X$ an element $x = \int_U f(u)dm \in X$ in the natural way, i.e.

$$\phi(x) = \int_U \phi(f(u))dm$$

for every $\phi \in X^*$. One can therefore define the bounded homogeneous map $\mathcal{B} : X \to X$ as follows:

$$\mathcal{B}(x) = \int_U uB(ux)dm.$$

By the invariance of m, the map $\mathcal{B}$ is exact: let v be a unit,

$$\mathcal{B}(vx) = \int_U uB(uvx)dm = \int_U vuvB(uvx)dm = v\mathcal{B}(x).$$

Let us show that $B - \mathcal{B}$ is linear, for which it is enough to show that $\mathcal{L} = B + L - \mathcal{B}$ is linear. By the exactness of $B + L$ we have

$$\begin{aligned} \int_U (B+L)(x+y) &- uB(ux+uy) \\ &- (B+L)(x) + uB(ux) \\ &- (B+L)(y) + uB(uy)\ dm \\ &= \int_U u(B+L)(u(x+y)) - uB(u(x+y)) \\ &- u(B+L)(ux) + uB(ux) - u(B+L)(uy) + uB(uy)\ dm \\ &= 0. \end{aligned}$$

Thus $\mathcal{L}$ is linear and exact and $B + L = \mathcal{B} + \mathcal{L}$.

(2) Since $\mathcal{L}$ is exact, it is contractive; hence $\mathcal{L}(e_n) = \lambda_n e_n$. If $(\mathcal{B} + \mathcal{L})(e_n) = (B + L)(e_n) = 0$, then $\mathcal{L}(e_n) = -\mathcal{B}(e_n)$, and thus the sequence (λ_n) must be bounded. So unconditionality applies to make $\mathcal{L}$ bounded, and thus also $\mathcal{B} + \mathcal{L} = B + L$ is bounded. $\square$

Corollary 3.16. *Let Ω and Ψ be exact centralizers on a maximal Köthe space with 1-unconditional basis (e_n) and such that $\Omega(e_n) = \Psi(e_n) = 0$ for all $n \in \mathbb{N}$. If Ω and Λ are equivalent, then they are boundedly equivalent.*

Proof. $\Omega - \Lambda = B + L$ is still exact and vanishes on the e_n, and thus it is bounded. $\square$

This, after Kalton's theorem 3.4, yields that given two interpolation scales of spaces $X = (X_0, X_1)$ and $Y = (Y_0, Y_1)$ with common 1-unconditional normalized basis, satisfying $E = (X_0, X_1)_\theta = (Y_0, Y_1)_\theta$, the exact sequences they induce at E are not equivalent unless X_0 and Y_0 (resp. X_1 and Y_1) are equal up to equivalence of norms.

To obtain our basic estimate (Lemma 4.5) we need to generalize Lemma 3.15 to cover the case of maps between two different modules. We are interested in the particular case in which one has to combine two related actions: let X be an L_∞-Banach module and let $W \subset X$ be a subspace generated by disjointly supported elements $W = [u_n]$. Consider in this case the subspace $L_\infty^W \subset L_\infty$ formed by the elements which are constant on the supports of all the u_n. Let U_W be its group of units. We say that a map $F : W \to X$ is *relatively exact* if $F(ux) = uF(x)$ for all $u \in U_W$ and $x \in W$, and we say that F is *relatively contractive* if $\mathrm{supp}_X F(x) \subset \mathrm{supp}_X x$ for all $x \in W$. One has:

Lemma 3.17. *Let X be a Köthe function space, and let W be a subspace of X generated by disjointly supported elements. Let $F : X \to L_\infty$ be a map.*

(1) *If F is exact, then the restriction $F_{|W}$ is relatively exact.*
(2) *Every relatively exact map $W \to X$ is relatively contractive.*
(3) *Assume X is a maximal Köthe function space. A relatively exact map $B + L : W \to L_0$ with $B : W \to X$ homogeneous bounded and $L : W \to L_0$ linear can be written as $B + L = \mathcal{B} + \mathcal{L}$ with $\mathcal{B} : W \to X$ homogeneous bounded, $L : W \to L_0$ linear and both relatively exact.*

Proof. Assertion (1) is clear and (2) has the same proof as before. To get (3), work as before defining for $x \in W$,

$$\mathcal{B}(x) = \int_{U_W} uB(ux)dm,$$

where m is a left invariant finitely additive mean on U_W. □

4. Singularity and estimates for exact centralizers

Recall that an operator between Banach spaces is said to be *strictly singular* if no restriction to an infinite-dimensional closed subspace is an isomorphism.

Definition 4.1. A quasi-linear map (in particular, a centralizer) is said to be *singular* if its restriction to every infinite-dimensional closed subspace is never trivial. An exact sequence induced by a singular quasi-linear map is called a *singular sequence.*

It is well known [18] that a quasi-linear map is singular if and only if the associated exact sequence has a strictly singular quotient map. So singular quasi-linear maps induce twisted sums which are, in some sense, as nontrivial as is possible. The following notion is more suitable to work with Köthe function spaces.

Definition 4.2. A quasi-linear map on a Köthe function space is called *disjointly singular* if its restriction to every subspace generated by a disjoint sequence is never trivial.

It is not hard to check that a quasi-linear map $F : Z \to Y$ is disjointly singular if and only if the quotient map of the induced exact sequence $0 \to Y \to Y \oplus_F Z \stackrel{q}{\to} Z \to 0$ is never an isomorphism on a subspace X of $Y \oplus_F Z$ such that $q(X)$ is generated by disjoint vectors in Z. Observe that when Z has an unconditional basis and the lattice structure one considers is the one induced by the basis, then the two notions coincide since saying that q is an isomorphism on some subspace is the same as saying that it is an isomorphism on some subspace whose image is generated by blocks of the basis. Thus:

Lemma 4.3. *A quasi-linear map $F : Z \to Y$ on a Banach space Z with unconditional basis is singular if and only if it is disjointly singular with respect to the induced lattice structure.*

Singularity implies disjoint singularity but, as we shall see, the reverse implication does not hold in general. Of course, a disjointly singular quasi-linear map is nontrivial. The following "transfer principle" ([18], [13]) will be essential for us.

Lemma 4.4. *If a quasi-linear map defined on a Banach space with basis is trivial on some infinite-dimensional subspace, then it is also trivial on some subspace spanned by normalized blocks of the basis.*

Observe that if $F : X \to Y$ is a quasi-linear map on a Köthe space X, and if for some sequence (u_n) of disjointly supported vectors and some constant K one has

$$\left\| F(\sum \lambda_j u_j) - \sum \lambda_j F(u_j) \right\| \leq K \| \sum \lambda_j u_j \|$$

for all choices of scalars (λ_j), then F is not singular. Indeed, the estimate above means that the linear selection for the quotient map $Y \oplus_F [u_j] \to [u_j]$ given by $u_j \to (F u_j, u_j)$ is continuous. Under exactness conditions we can get a partial converse.

Lemma 4.5. *Let $\Omega : X \curvearrowright X$ be an exact centralizer on a maximal Köthe function space. If Ω is not disjointly singular, then there exists a subspace W of X generated by a disjoint sequence and a constant K such that given vectors $u_1, \ldots, u_n$ in W there are vectors $z_1, \ldots, z_n$ in X with* supp $z_i \subset$ supp u_i *and $\|z_i\| \leq K \|u_i\|$ such that for all scalars $\lambda_1, \ldots, \lambda_n$ one has*

$$\left\| \Omega(\sum_{i=1}^n \lambda_i u_i) - \sum_{i=1}^n \lambda_i \Omega(u_i) \right\| \leq K \left(\| \sum_{i=1}^n \lambda_i u_i \| + \| \sum_{i=1}^n \lambda_i z_i \| \right). \tag{4.1}$$

Proof. Since Ω is not disjointly singular, it is trivial on some subspace $W = [u_n]$ spanned by disjointly supported vectors. Then by Lemma 3.17 there exists a linear relatively exact map $\Lambda : W \to X$ so that $\Omega_{|W} - \Lambda$ is bounded. Since both Ω and Λ must be relatively contractive (Lemma 3.17 (2)), so is $\Omega - \Lambda$. Set $z_i = (\Omega - \Lambda)(u_i)$ and $K = \|\Omega_{|W} - \Lambda\|$. □

The preceding estimate can be considered as a subtler version of the "upper p-estimates" argument for nonsplitting, which can be quickly described as: if the space X verifies some type of upper p-estimate and the twisted sum $X \oplus_\Omega X$ splits, then the space $X \oplus_\Omega X$ must also verify the upper p-estimate (in general, if X has type p, then $X \oplus_\Omega X$ has only type $p + \varepsilon$ for every ε; see [31]). Therefore, given a sequence of independent vectors (u_n) in the unit ball of X the elements $(0, u_n)$ in $X \oplus_\Omega X$ should verify an upper p-estimate, which amounts to

$$\|\Omega(\sum_{i=1}^n u_i) - \sum_{i=1}^n \Omega(u_i)\| \leq C \sqrt[p]{n}.$$

We now introduce the notion of the standard class of finite families of elements of Köthe spaces to simplify the exposition.

Definition 4.6. A *standard class* $\mathcal{S}$ is a correspondence that assigns to each Köthe function space X (resp. spaces with 1-unconditional bases) a class $\mathcal{S}(X)$ of finite sequences of elements of the unit ball of X satisfying

(i) whenever $(x_1, \ldots, x_n) \in \mathcal{S}(X)$ and supp $z_i \subset$ supp x_i for all i, then $(z_1, \ldots, z_n) \in \mathcal{S}(X)$;
(ii) whenever $W \subset X$ is a subspace generated by disjoint vectors (resp. generated by successive vectors) then $S(W) \subset \mathcal{S}(X)$.

The two standard classes we shall use in this paper are the class $\mathcal{D}$ of disjointly supported vectors in Köthe spaces and the class $\mathcal{S}$ of "Schreier" successive vectors on 1-unconditional bases, i.e. families $(x_1, \ldots, x_n)$ such that $n <$ supp $x_1 < \cdots <$ supp x_n. Other examples such as successive vectors on 1-unconditional bases could also be of interest for other applications. Given a standard class $\mathcal{S}$ and a space X, we consider the following indicator function $M_{X,\mathcal{S}}(\cdot) : \mathbb{N} \to \mathbb{R}^+$ defined by

$$M_{X,\mathcal{S}}(n) = \sup\{\|x_1 + \ldots + x_n\| : \ (x_1, \ldots, x_n) \in \mathcal{S}(X)\}.$$

Lemma 4.5 can be rewritten as:

Lemma 4.7. *Let $\mathcal{S}$ be a standard class, and let $\Omega : X \curvearrowright X$ be an exact centralizer on a maximal Köthe function space. If Ω is not disjointly singular, then there exists a subspace W of X generated by a disjointly supported sequence of vectors and a constant K such that given $(u_1, \ldots, u_n) \in \mathcal{S}(W)$ one has*

$$\Big\|\Omega(\sum_{i=1}^n u_i) - \sum_{i=1}^n \Omega(u_i)\Big\| \leq K M_{X,\mathcal{S}}(n).$$

We now arrive at the core of our method:

Lemma 4.8. *Let (X_0, X_1) be an admissible couple of Köthe function spaces, fix $0 < \theta < 1$, and let Ω_θ be the induced centralizer on X_θ. If $(x_1, \ldots, x_n) \in \mathcal{S}(X_\theta)$, then*

$$\Big\|\Omega_\theta\big(\sum_{i=1}^n x_i\big) - \sum_{i=1}^n \Omega_\theta(x_i) - \log\frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}\Big(\sum_{i=1}^n x_i\Big)\Big\|_\theta \leq 3\frac{M_{X_0,\mathcal{S}}(n)^{1-\theta} M_{X_1,\mathcal{S}}(n)^\theta}{\operatorname{dist}(\theta, \partial\mathbb{S})}.$$

Proof. To simplify, let us write $M(n, z) = M_{X_0,\mathcal{S}}(n)^{1-z} M_{X_1,\mathcal{S}}(n)^z$. Given $0 < \epsilon < 1/4$, let $(x_i) \in \mathcal{S}$ be an n-tuple in the unit ball of X_θ. Let B_θ be a $(1+\epsilon)$-bounded selection $X_\theta \to \mathcal{H}$ such that supp $B_\theta(x) \subset$ supp x for all x. Let $F_i = B_\theta(x_i)$ for each i. Note that $\big(F_i(z)\big)$ is an n-tuple in $\mathcal{S}$ for any z in the strip. Let F be the function

$$F(z) = \frac{F_1(z) + \cdots + F_n(z)}{M(n, z)}$$

for $z \in \mathbb{S}$. We know that $\|F\| \leq 1 + \epsilon$ and

$$F(\theta) = \frac{1}{M(n, \theta)}(x_1 + \ldots + x_n).$$

Set $k = \|F(\theta)\|^{-1}$. The map $\Phi : F(\theta) \mapsto F$ defines a linear bounded selection on the one-dimensional subspace $[F(\theta)]$ having norm at most k. Therefore

$$\|B_{\theta|[F(\theta)]} - \Phi\| \leq 1 + \epsilon + k \leq k(1 + \epsilon) + \epsilon + k.$$

Thus, if $x \in [F(\theta)]$, and denoting $\delta' = \delta'_\theta$,

$$\|(\delta' B_\theta - \delta'\Phi)(x)\|_\theta \leq (2k + k\epsilon + \epsilon)\|\delta'_{|\ker\ \delta_\theta}\|\|x\|_\theta.$$

In particular

$$\left\|(\delta' B_\theta - \delta'\Phi)(\sum_{i=1}^n x_i)\right\|_\theta \leq (2k + k\epsilon + \epsilon)\,\mathrm{dist}(\theta, \partial\mathbb{S})^{-1} \left\|\sum_{i=1}^n x_i\right\|_\theta,$$

or equivalently

$$\left\|(\delta' B_\theta - \delta'\Phi)(\sum_{i=1}^n x_i)\right\|_\theta \leq \mathrm{dist}(\theta, \partial\mathbb{S})^{-1}(2+\epsilon+\frac{\epsilon}{k})M(n,\theta) \leq 3\,\mathrm{dist}(\theta, \partial\mathbb{S})^{-1}M(n,\theta).$$

On the other hand,

$$F'(\theta) = F(\theta)\log\frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)} + \frac{1}{M(n,\theta)}\sum_i B_\theta(x_i)'(\theta),$$

which means

$$\delta'\Phi(\sum_i x_i) = \log\frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}(\sum_i x_i) + \sum_i \delta' B_\theta(x_i).$$

Therefore

$$\delta'\Phi(\sum_i x_i) - \delta' B_\theta(\sum_i x_i) = \sum_i \delta' B_\theta(x_i) - \delta' B_\theta(\sum_i x_i) + \log\frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}(\sum_i x_i),$$

which yields

$$\Big\|\sum_i \delta' B_\theta(x_i) - \delta' B_\theta(\sum_i x_i) + \log\frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}(\sum_i x_i)\Big\|_\theta \leq 3\,\mathrm{dist}(\theta, \partial\mathbb{S})^{-1}M(n,\theta);$$

hence

$$\Big\|\Omega_\theta(\sum_{i=1}^n x_i) - \sum_{i=1}^n \Omega_\theta(x_i) - \log\frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}(\sum_{i=1}^n x_i)\Big\|_\theta \leq 3\,\mathrm{dist}(\theta, \partial\mathbb{S})^{-1}M(n,\theta), \tag{4.2}$$

as desired. □

Observe that the estimate above applies (after suitable normalization) to all real centralizers; the form that such an estimate should adopt for complex centralizers or for centralizers generated by arbitrary families is not equally clear. We now show that the function $\theta \mapsto M_{X_\theta,\mathcal{S}}(n)$ is log-convex:

Lemma 4.9. *Given an interpolation scale of Köthe function spaces associated to a pair* (X_0, X_1) *one has*

$$M_{X_\theta,\mathcal{S}}(n) \leq M_{X_0,\mathcal{S}}(n)^{1-\theta} M_{X_1,\mathcal{S}}(n)^\theta.$$

Proof. Let $F(z) = (F_1(z) + \cdots + F_n(z))/M(n,z)$ be the function in the proof of Lemma 4.8. The inequalities $\|F(\theta)\|_\theta \leq \|F\| \leq 1+\epsilon$ imply $\|x_1 + \cdots + x_n\|_\theta \leq (1+\epsilon)M(n,\theta)$, from which the conclusion follows. □

5. Criteria for singularity

Here we give some results that will allow us to recognize singular or disjointly singular exact sequences.

5.1. **A general criterion in Köthe function spaces.** Our setting in this section is the following: $\mathcal{S}$ is a standard class; (X_0, X_1) is an interpolation couple of Köthe function spaces generating the interpolation scale (X_θ) for which we assume that X_θ is maximal for $0 < \theta < 1$; and Ω_θ is the induced centralizer on X_θ. A combination of Lemma 4.7, Lemma 4.8 and Lemma 4.9 yields the following result.

Proposition 5.1. *If Ω_θ is not disjointly singular, then there exists a subspace $W \subset X_\theta$ spanned by disjointly supported vectors and a constant K such that*

$$\left|\log \frac{M_{X_0,\mathcal{S}}(n)}{M_{X_1,\mathcal{S}}(n)}\right| M_{W,\mathcal{S}}(n) \leq K M_{X_0,\mathcal{S}}(n)^{1-\theta} M_{X_1,\mathcal{S}}(n)^{\theta}. \tag{5.1}$$

More general criteria could be obtained with different variations of M_X (say, using sequences of vectors whose norms are at most some prescribed varying values instead of vectors of norm at most 1). We shall not pursue this line since nothing of this will be needed for the applications in which we are interested. Recall that two functions $f, g : \mathbb{N} \to \mathbb{R}$ are called equivalent, something we denote $f \sim g$, if $0 < \liminf f(n)/g(n) \leq \limsup f(n)/g(n) < +\infty$.

Definition 5.2. Given a standard class $\mathcal{S}$ and its associated indicator function $M_{\mathcal{S}}(\cdot)$ we will say that a space X is self-similar with respect to $\mathcal{S}$ (or simply self-similar if the relative class $\mathcal{S}$ is clear) if $M_{W,\mathcal{S}} \sim M_{X,\mathcal{S}}$ for every infinite-dimensional subspace $W \subset X$ generated by a sequence of disjointly supported vectors.

For our first application we pick as $\mathcal{S}$ the standard class $\mathcal{D}$ of all disjointly supported sequences in a Köthe function space X, and simplify the notation to $M_X(n) = M_{X,\mathcal{D}}(n)$. As a direct application of the criterion in Proposition 5.1 we have:

Proposition 5.3. *Let (X_0, X_1) be an interpolation couple of two Köthe function spaces so that M_{X_0} and M_{X_1} are not equivalent. If X_θ is self-similar (with respect to $\mathcal{D}$) and $M_{X_\theta} \sim M_{X_0}^{1-\theta} M_{X_1}^{\theta}$, then Ω_θ is disjointly singular.*

Proof. Otherwise, the estimate (5.1) yields that for some subspace $W \subset X$ generated by a sequence of disjointly supported vectors and some constants K, K' one has

$$\left|\log \frac{M_{X_0}(n)}{M_{X_1}(n)}\right| M_W(n) \leq K M_{X_\theta}(n) \leq K' M_W(n),$$

which is impossible unless M_{X_0} and M_{X_1} are equivalent. □

Let us see these criteria at work.

The scale of ℓ_p-spaces. The simplest case of course concerns the scale of ℓ_p-spaces, $1 < p < +\infty$. These spaces are self-similar with $M_{\ell_p}(n) = n^{1/p}$, while reiteration theorems allow one to fix X_0 and X_1 at any two different values p, q so that $\lim |\log \frac{M_{X_0}(n)}{M_{X_1}(n)}| = \lim |\log n^{1/p-1/q}| = +\infty$. Thus, the induced centralizer, which is actually (projectively equivalent to) the Kalton-Peck ℓ_∞-centralizer $\mathcal{K}$, is disjointly singular, hence singular on ℓ_p. A different proof that the Kalton-Peck centralizer on ℓ_p is singular for $0 < p < \infty$ can be found in [13, Theorem 2(b)].

The scale of L_p-spaces. The case of L_p-spaces, $1 < p < +\infty$, is also simple: since disjointly supported sequences span ℓ_p in L_p, Proposition 5.1 yields that if the twisted sum fails to be disjointly singular, then

$$\left|\log \frac{M_{L_\infty}(n)}{M_{L_1}(n)}\right| M_{\ell_p}(n) \leq K M_{L_\infty}^{1-\frac{1}{p}}(n) M_{L_1}^{\frac{1}{p}}(n).$$

Therefore $(\log n)n^{1/p} \leq K n^{1/p}$, which is impossible. Thus, since the induced centralizer on L_p is actually (projectively equivalent to) the Kalton-Peck L_∞-centralizer $\mathcal{K}$, one has:

Proposition 5.4. *For $1 < p < +\infty$, the Kalton-Peck L_∞-centralizer $\mathcal{K}$ on L_p is disjointly singular.*

In [9] it was shown that no L_∞-centralizer on L_p is singular for $0 < p < \infty$; it had been previously shown in [43] that the Kalton-Peck L_∞-centralizer $\mathcal{K}$ on L_p is not singular since it becomes trivial on the Rademacher copy of ℓ_2. Proposition 5.4 tells us that it is not trivial on any subspace generated by disjointly supported vectors. Cabello [9] remarks that it would be interesting to know whether there exist singular quasi-linear maps $L_p \to L_p$ for $p < 2$.

The scale of L_p-spaces in their associated unconditional structure. Consider now the scale of L_p-spaces in their ℓ_∞-module structure generated by the Haar basis. In [13] it is shown that the associated Kalton-Peck centralizer $\mathcal{K}$ is singular for $2 \leq p < \infty$. However, Khintchine's inequality makes it possible to define $B_\theta(r) = f_r$ (the constant function $f_r(z) = r$) on the subspace ℓ_2^R generated by the Rademacher functions, so $\Omega_\theta(r) = \delta'_\theta B_\theta(r) = 0$ on ℓ_2^R and thus Ω_θ is not singular. Since the Haar basis is unconditional, this means that it is not disjointly singular either. This means, in particular, that the Kalton-Peck ℓ_∞-centralizer relative to the Haar basis is not the ℓ_∞-centralizer induced by the interpolation scale of L_p-spaces in their ℓ_∞-module structure.

Kalton-Peck maps on more general scales. We may however obtain more information regarding $\mathcal{K}$. Combining Proposition 5.3 together with Proposition 3.6 we get:

Theorem 5.5. *For $p > 1$, let X be a maximal p-convex Köthe function space that is self-similar with respect to $\mathcal{D}$. Then, the Kalton-Peck map $\mathcal{K}$ is disjointly singular on X.*

Proof. Since X is p-convex we have $X = (L_\infty, X^p)_{1/p}$ by Proposition 3.6. Furthermore the centralizer $\Omega_{1/p}$ induced by this interpolation scheme is a multiple of $\mathcal{K}$, and thus one is disjointly singular if and only if so is the other. Since the norm on X^p is defined as $\|x\| = \||x|^{1/p}\|_X^p$, we have immediately that $M_{X^p}(n) = M_X(n)^p$. Since X is p-convex, M_X is not bounded, and so M_X^p is not equivalent to M_{L_∞} (which is constantly 1). Furthermore

$$M_{L_\infty}(n)^{1-\frac{1}{p}} M_{X^p}(n)^{\frac{1}{p}} = (M_X(n)^p)^{1/p} = M_X(n).$$

By Proposition 5.3 the centralizer $\Omega_{1/p}$, hence $\mathcal{K}$, is disjointly singular. $\square$

5.2. **The criterion in spaces with unconditional bases.** We now consider an asymptotic variation of M_X more suitable to work with sequence spaces. Let $\mathcal{S}$ be the standard class of Schreier sequences; i.e., sequences of finitely supported elements $x_1, \dots, x_n$ of norm at most one such that $n < x_1 < \cdots < x_n$. On spaces X with unconditional basis we define $A_X = M_{X,\mathcal{S}}$ as

$$A_X(n) = \sup\{\|x_1 + \ldots + x_n\| : \|x_i\| \le 1,\ n < x_1 < \ldots < x_n\}.$$

Proposition 5.1 becomes:

Proposition 5.6. *Let (X_0, X_1) be an interpolation couple of Banach spaces with a common 1-unconditional basis, and let $0 < \theta < 1$. If Ω_θ is not singular, then there exists a block subspace $W \subset X_\theta$ and a constant K such that*

$$\left|\log \frac{A_{X_0}(n)}{A_{X_1}(n)}\right| A_W(n) \le K A_{X_0}^{1-\theta}(n) A_{X_1}^{\theta}(n).$$

Proposition 5.3 can be reformulated as follows.

Proposition 5.7. *Let (X_0, X_1) be an interpolation couple of Banach spaces with a common 1-unconditional basis so that A_{X_0} and A_{X_1} are not equivalent. If X_θ is self-similar with respect to $\mathcal{S}$ and $A_{X_\theta} \sim A_{X_0}^{1-\theta} A_{X_1}^{\theta}$, then Ω_θ is singular.*

Let us see these criteria at work:

Asymptotically ℓ_p-spaces. Recall that a Banach space with a basis is said to be *asymptotically ℓ_p* if there exists $C \ge 1$ such that for all n and normalized $n < x_1 < \ldots < x_n$ in X, the sequence $(x_i)_{i=1}^n$ is C-equivalent to the basis of ℓ_p^n. The ℓ_p-spaces are asymptotically ℓ_p and Tsirelson's space is asymptotically ℓ_1. One has

Proposition 5.8. *Let (X_0, X_1) be an interpolation pair of Banach spaces with a common 1-unconditional basis. Let $p_0 \neq p_1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Assume one of the following conditions holds:*

1. *The spaces X_j, $j = 0, 1$, are reflexive asymptotically ℓ_{p_j}.*
2. *Successive vectors in X_j, $j = 0, 1$, satisfy an asymptotic upper ℓ_{p_j}-estimate; and for every block subspace $W \subset X_\theta$, there exist a constant C and, for each n, a finite block-sequence $n < y_1 < \ldots < y_n$ in B_W such that $\|y_1 + \cdots + y_n\| \ge C^{-1} n^{1/p}$.*

Then the induced centralizer Ω_θ is singular.

Kalton-Peck maps on more general scales. With the same method as in Theorem 5.5 we get:

Proposition 5.9. *Let X be a p-convex reflexive space with 1-unconditional basis and self-similar with respect to $\mathcal{S}$. Then the Kalton-Peck map $\mathcal{K}$ on X is singular.*

Proposition 5.9 applies to spaces such as the p-convexified Tsirelson spaces $T^{(p)}$, $p > 1$, since $A_W(n) \sim n^{1/p}$ for any block subspace $W \subset T^{(p)}$.

5.3. **The criterion in spaces with monotone bases.** The lattice structure was not used in the proof of Lemma 4.8, apart from the use of supports, which can be replaced by ranges (ran x is the minimal interval of integers containing its support). This can be done as follows: Let $\Omega : X \to X$ be a quasi-linear map acting on a space with 1-monotone basis. Since this case does not fit under the umbrella of Kalton's theorem, it could well occur that Ω could not be recovered from an interpolation scheme. In fact, the maps Ω_θ appearing in an interpolation process are not ℓ_∞-centralizers or necessarily contractive. However, they can be chosen to be "range contractive", in the sense of verifying ran $\Omega_\theta(x) \subset$ ran x. Indeed if for $x \in c_{00}$, $b_\theta(x)$ is an almost optimal selection, then $B_\theta(x) = 1_{\mathrm{ran}x} b_\theta(x)$ will also be almost optimal and range contractive, so $\delta'_\theta B_\theta$ will be the required map. The transfer principle still works, and thus a nonsingular $\Omega : X \to X$ must be trivial on some subspace W generated by blocks of the basis.

Successive vectors and asymptoticity still make sense, so that the function A_X still may be defined. Thus, a proof entirely similar to that of Lemma 4.8 using instead the function

$$F(z) = \frac{1}{A_{X_0}(n)^{1-z} A_{X_1}(n)^z}(B_\theta(y_1) + \cdots + B_\theta(y_n))(z)$$

yields the estimate

$$\left\| \Omega_\theta(\sum_{i=1}^n y_i) - \sum_{i=1}^n \Omega_\theta(y_i) - \log \frac{A_{X_0}(n)}{A_{X_1}(n)} \sum_i y_i \right\|_\theta \le \frac{3}{\mathrm{dist}(\theta, \partial \mathbb{S})} A_{X_0}^{1-\theta} A_{X_1}^{\theta}(n), \tag{5.2}$$

for all $n < y_1 < \cdots < y_n$ in the unit ball of X_θ, in an interpolation scale (X_0, X_1) of spaces with common 1-monotone basis. One can also prove that the function $\theta \mapsto A_{X_\theta}(n)$ is log-convex working as in Lemma 4.9.

In this case, however, the analogue of Proposition 5.1 is not automatic because the estimate in Lemma 4.5 requires lattice structure in a deep way, and thus something new is needed to cover the conditional case: We now show how the lattice structure may be replaced by hypotheses of local unconditionality and complementation.

Proposition 5.10. *Assume we have an interpolation couple (X_0, X_1) formed with spaces having a common 1-monotone basis. Given $0 < \theta < 1$, assume that for every block subspace $W \subset X_\theta$ there exists for every n a finite sequence of successive vectors $n < y_1 < \cdots < y_n$ with $\|y_i\| \le 1$ $\forall i = 1, \ldots, n$, and constants $\varepsilon_n, \lambda_n, M_n$ satisfying*

(i) *The block-sequence is ε_n-optimal, in the sense that*

$$\left\| \sum_{i=1}^n y_i \right\| \ge \varepsilon_n A_{X_0}(n)^{1-\theta} A_{X_1}(n)^\theta.$$

(ii) *The block-sequence $\{y_1, \ldots, y_n\}$ is λ_n-unconditional.*

(iii) *The space $[y_1, \ldots, y_n]$ is M_n-complemented in X_θ and thus*

$$\liminf_{n \to +\infty} \frac{\lambda_n^3 M_n}{\varepsilon_n \left| \log \frac{A_{X_0}(n)}{A_{X_1}(n)} \right|} = 0.$$

Then Ω_θ is singular.

Proof. Suppose that the restriction of Ω_θ to some subspace of X is trivial. By the hypothesis Ω_θ is trivial on some block subspace Y_θ of X_θ, and we can pick for any n a λ_n-unconditional, ε_n-optimal, finite sequence $[y_i]_{i=1}^n$ of blocks in B_{Y_θ} that is M_n-complemented in X_θ by a projection P_n.

Then a local version of the proof of Lemma 3.17 (3) can be made. Let $\ell : Y_\theta \to L_0$ be a linear map so that $\|\Omega_{|Y_\theta} - \ell\| \le K$. Then let $G_n \simeq \{-1, 1\}^n$ be the group of units of ℓ_∞^n acting on $Y_n = [y_1, \dots, y_n]$ in the natural way by change of signs of the coordinates on the y_i's, and let, for $y \in Y_n$, $\psi_n(y)$ be the finite average

$$\psi_n(y) = \mathrm{Ave}_{u\in G_n} uP_n(\Omega_{|Y_\theta} - \ell)(uy).$$

Note that ψ_n takes values in Y_n and that this homogeneous map is bounded by $KM_n\lambda_n^2$. It is also an exact ℓ_∞^n-centralizer in the sense that $\psi_n(uy) = u\psi_n(y)$ for $u \in G_n$, so supp $\psi_n(y) \subset$ supp y for $y \in Y_n$. This implies that $\psi_n(y_i) = \mu_i y_i$ for some scalars μ_i with $|\mu_i| \le KM_n\lambda_n^2$. Thus

$$\begin{aligned}(5.3)\qquad \|\psi_n(\textstyle\sum_{i=1}^n y_i) - \sum_{i=1}^n \psi_n(y_i)\| &\le KM_n\lambda_n^2\|\textstyle\sum_{i=1}^n y_i\| + \|\sum_{i=1}^n \mu_i y_i\| \\ &\le KM_n\lambda_n^2(1+\lambda_n)\|\textstyle\sum_{i=1}^n y_i\|.\end{aligned}$$

Consider the estimate (5.2), and observe that replacing Ω_θ by $\Omega_\theta - \ell$ with ℓ linear changes nothing, and projecting and averaging on $\pm$ signs as in the definition of ψ_n only changes the estimate by $\lambda_n\|P_n\| \le \lambda_n M_n$; so one gets

$$\Big\|\psi_n(\sum_{i=1}^n y_i) - \sum_{i=1}^n \psi_n(y_i) - \log\frac{A_{X_0}(n)}{A_{X_1}(n)}\sum_{i=1}^n y_i\Big\| \le k_\theta M_n\lambda_n A_{X_0}(n)^{1-\theta}A_{X_1}(n)^\theta.$$

On the other hand by log-convexity of A_{X_θ} we can rewrite (5.3) as

$$(5.4)\qquad \Big\|\psi_n(\sum_i y_i) - \sum_i \psi_n(y_i)\Big\| \le KM_n\lambda_n^2(1+\lambda_n)A_{X_0}^{1-\theta}(n)A_{X_1}^\theta(n).$$

Putting both estimates together we get

$$\Big|\log\frac{A_{X_0}(n)}{A_{X_1}(n)}\Big| \cdot \Big\|\sum_{i=1}^n y_i\Big\| \le \big(K\lambda_n(1+\lambda_n) + k_\theta\big)M_n\lambda_n A_{X_0}^{1-\theta}(n)A_{X_1}^\theta(n).$$

Condition (i) yields that

$$\varepsilon_n\left|\log\frac{A_{X_0}(n)}{A_{X_1}(n)}\right| \le \big(K\lambda_n(1+\lambda_n) + k_\theta\big)M_n\lambda_n,$$

in contradiction with the hypothesis. □

To see this criterion at work recall that the notion of an asymptotically ℓ_p-space is meaningful in the conditional case: There are even H.I. spaces asymptotically ℓ_1, such as those defined by Argyros and Deliyanii [2].

Corollary 5.11. *Assume we have an interpolation couple (X_0, X_1) formed with spaces having a common 1-monotone basis. Let $1 \le p_0 \ne p_1 \le +\infty$, $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Assume that:*

(1) *The spaces X_j, $j = 0, 1$, satisfy an asymptotic upper ℓ_{p_j}-estimate.*
(2) *For every block subspace $W \subset X_\theta$ there exists a constant C so that for each n there is a C-unconditional finite block-sequence $n < y_1 < \ldots < y_n$ in the unit ball of W such that:*
   (a) *$\|y_1 + \cdots + y_n\| \ge C^{-1}n^{1/p}$;*
   (b) *$[y_1, \cdots, y_n]$ is C-complemented in X_θ.*

Then Ω_θ is singular.

5.4. **Interpolation of families of spaces.** The preceding criteria also work for centralizers induced by complex interpolation of a family of spaces (see [21]). Indeed, take a family of compatible Banach spaces $\{X_{(j,t)} : j = 0, 1; t \in \mathbb{R}\}$ with index in the boundary of $\mathbb{S}$, and denote by $\Sigma(X_{j,t})$ the algebraic sum of these spaces with the norm

$$\|x\|_\Sigma = \inf\{\|x_1\|_{(j_1,t_1)} + \cdots + \|x_n\|_{(j_n,t_n)} : x = x_1 + \cdots + x_n\}.$$

Let $\mathcal{H}(X_{j,t})$ denote the space of functions $g : \mathbb{S} \to \Sigma := \Sigma(X_{j,t})$ which are $\|\cdot\|_\Sigma$-bounded, $\|\cdot\|_\Sigma$-continuous on $\mathbb{S}$ and $\|\cdot\|_\Sigma$-analytic on $\mathbb{S}^\circ$; and satisfy $g(it) \in X_{(0,t)}$ and $g(it+1) \in X_{(1,t)}$ for each $t \in \mathbb{R}$. Note that $\mathcal{H}(X_{j,t})$ is a Banach space under the norm

$$\|g\|_\mathcal{H} = \sup\{\|g(j+it)\|_{(j,t)} : j = 0, 1; t \in \mathbb{R}\}.$$

For each $\theta \in (0, 1)$, or even $\theta \in \mathbb{S}$, we define

$$X_\theta := \{x \in \Sigma(X_{j,t}) : x = g(\theta) \text{ for some } g \in \mathcal{H}(X_{j,t})\}$$

with the norm $\|x\|_\theta = \inf\{\|g\|_\mathcal{H} : x = g(\theta)\}$. Clearly X_θ is the quotient of $\mathcal{H}(X_{j,t})$ by the kernel of the evaluation map $\ker \delta_\theta$, and thus it is a Banach space.

All the ingredients of our constructions straightforwardly adapt to this context, and the only relevant modification is to set $A_j(n) = \operatorname{ess\,sup}_{t\in\mathbb{R}} A_{X_{j+it}}(n)$ instead of $A_{X_j}(n)$, $j = 0, 1$. One therefore has the following proposition, which will be necessary in Section 8, and whose proof is similar to those of Proposition 5.10 and Corollary 5.11.

Proposition 5.12. *Consider an interpolation scheme given by a family $\{X_{(j,t)} : j = 0, 1; t \in \mathbb{R}\}$ of spaces with a common 1-monotone basis. Let $1 \le p_0 \ne p_1 \le +\infty$, $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Assume that:*

1. *All the spaces $X_{j,t}$ satisfy an asymptotic upper ℓ_{p_j}-estimate with uniform constant.*
2. *For every block subspace $W \subset X_\theta$ there exist a constant C so that for each n there is a C-unconditional finite block-sequence $n < y_1 < \ldots < y_n$ in the unit ball of W such that*
   (a) *$\|y_1 + \cdots + y_n\| \ge C^{-1} n^{1/p}$;*
   (b) *$[y_1, \cdots, y_n]$ is C-complemented in X_θ.*

Then Ω_θ is singular.

6. Singular twisted Hilbert spaces

In many cases, complex interpolation between a Banach space and its dual gives $(X, X^*)_{1/2} = \ell_2$. See e.g., the comments in [40, around Theorem 3.1]. Also Watbled [44] claims that her results cover the case of spaces with a 1-unconditional basis X. We do not know whether there could be counterexamples with a monotone basis. So, for the sake of clarity, let us briefly explain the situation.

Given a Banach space X with a normalized basis (e_n), we denote by (e_n^*) the corresponding sequence of biorthogonal functionals. We identify X with $\{\big(e_n^*(x)\big) : x \in X)\}$, and its antidual space $\hat{X}^*$ with $\{\overline{\big(x^*(e_n)\big)} : x^* \in X)\}$, both linear subspaces of ℓ_∞, in such a way that $X \cap \hat{X}^*$ is continuously embedded in ℓ_2. Indeed, $x = (a_n) \in X \cap \hat{X}^*$ implies $x(x) = \sum |a_n|^2 \le \|x\|_X \cdot \|x\|_{\hat{X}^*}$.

Proposition 6.1. *Let X be a Banach space with a monotone shrinking basis. Then $(X, \hat{X}^*)_{1/2} = \ell_2$ with equality of norms.*

Proof. It is enough to show that ℓ_2 is continuously embedded in $X + \hat{X}^*$ and apply [44, Corollary 4]. Let $T : X \cap \hat{X}^* \to \ell_2$ be the embedding. Since the basis is shrinking, $X \cap \hat{X}^*$ is dense in both X and $\hat{X}^*$. Thus the dual of $X \cap \hat{X}^*$ is $X^* + (\hat{X}^*)^* = X^{**} + \hat{X}^*$ [5, 2.7.1 Theorem], and the conjugate operator T^* embeds ℓ_2 into $X + \hat{X}^*$, which is a closed subspace of $X^{**} + \hat{X}^*$ by the arguments in [44, p. 204]. □

We have a similar result for Köthe function spaces X. Observe that in this case X^* and $\hat{X}^*$ coincide as sets.

Proposition 6.2 ([44, Corollary 5])**.** *Let X be a Köthe function space on a σ-finite measure space (Σ, μ). Suppose that $X \cap X^*$ is dense in X and*

$$L_1(\Sigma) \cap L_\infty(\Sigma) \subset X \cap X^* \subset L_2(\Sigma) \subset X + X^* \subset L_1(\Sigma) + L_\infty(\Sigma).$$

Then $(X, X^*)_{1/2} = L_2(\Sigma)$.

Arguing as in Proposition 6.1, we can show that the conditions X and X^* intermediate spaces between $L_1(\Sigma)$ and $L_\infty(\Sigma)$, and $X \cap X^*$ dense in both X and X^* imply the hypothesis of Proposition 6.2. Thus, in all the previous situations the twisted sum space induced by the interpolation of a space and its antidual is a twisted Hilbert space. In [12, 6.2] the authors conjecture that it is a Hilbert space if and only if $X_0 = \ell_2$. Kalton's theorem 3.4 only yields that the associated centralizer is unbounded unless $X_0 = \ell_2$. When the endpoints are sequence spaces, a direct application of either Proposition 5.3 or 5.7 fits appropriately in this situation since Hilbert spaces are self-similar (with respect to both $\mathcal{D}$ and $\mathcal{S}$).

Proposition 6.3. *The interpolation of a reflexive asymptotically ℓ_p-space, $p \neq 2$, with its antidual induces a singular twisted Hilbert space.*

Thus, interpolation of Tsirelson's space T with its dual T^* produces a singular sequence that by Corollary 3.16 cannot be equivalent to the sequence $0 \longrightarrow \ell_2 \longrightarrow Z_2 \longrightarrow \ell_2 \longrightarrow 0$. Regarding distinguishing sequences one should get at least that they are not projectively equivalent. When the endpoints have symmetric bases one can obtain that if the sequences are not projectively equivalent, then they are not permutatively projectively equivalent. Indeed, given a reflexive Banach space X with normalized subsymmetric basis (e_n), we denote as usual [37]

$$\lambda_X(n) = \Big\| \sum_{i=1}^n e_i \Big\|_X.$$

Then $\lambda_{X^*}(n) \sim n/\lambda_X(n)$ (see [37, Proposition 3.a.6]).

Lemma 6.4. *Let X, Y be reflexive spaces with normalized* 1*-unconditional and* 1*-subsymmetric bases. Let Ω^X and Ω^Y be the induced centralizers at ℓ_2 corresponding to the scales (X, X^*) and (Y, Y^*). If Ω^X and Ω^Y are c-projectively equivalent, then*

$$n\lambda_X(n)^{-2} \sim (n\lambda_Y(n)^{-2})^c.$$

If, moreover, the spaces have symmetric bases, then the same conclusion follows when Ω^X and Ω^Y are permutatively c-projectively equivalent.

Proof. The induced centralizers at $(X, X^*)_{1/2}$ (resp. $(Y, Y^*)_{1/2}$) defined on terms of the Lozanovskii decompositions yield

$$\big(\Omega^X - c\Omega^Y\big)(x) = \Big(\log \frac{|a_0(x)|}{|a_1(x)|} - c \log \frac{|a_0'(x)|}{|a_1'(x)|}\Big) x.$$

Pick $x = \sum_{i=1}^{n} x_i e_i$ with $x_i = 1/\sqrt{n}$ and apply the above formula with

$$|a_0(x)| = \lambda_X(n)^{-1} 1_{[1,n]}, \quad |a_1(x)| = \frac{\lambda_X(n)}{n} 1_{[1,n]}, \quad \text{and}$$
$$|a_0'(x)| = \lambda_Y(n)^{-1} 1_{[1,n]}, \quad |a_1'(x)| = \frac{\lambda_Y(n)}{n} 1_{[1,n]}.$$

If $\Omega^X - c\Omega^Y$ is trivial, then it is bounded by Corollary 3.16, so the function $\log(n\lambda_X(n)^{-2}) - c\log(n\lambda_Y(n)^{-2})$ on $\mathbb{N}$ is bounded, which implies that the functions $n\lambda_X(n)^{-2}$ and $(n\lambda_Y(n)^{-2})^c$ are equivalent.

When the bases are symmetric, permutations are isometries, and thus $\lambda(n)$ is invariant under permutations. □

Symmetric Orlicz spaces have symmetric bases, and thus they can be covered by the previous result:

Proposition 6.5. *Let ℓ_{M_α} be the symmetric Orlicz space with function $M_\alpha(t) = e^{-t^{-\alpha}}, \alpha > 0$. The induced centralizers at $\ell_2 = (\ell_{M_\alpha}, \ell_{M_\alpha}^*)_{1/2}$ for different values of α are not permutatively projectively equivalent.*

Proof. The choice of M_α in the statement yields $\lambda_{\ell_{M_\alpha}}(n) \sim (\log n)^{1/\alpha}$. It is thus impossible for different $\alpha, \beta > 0$ and any c to have

$$n\lambda_{\ell_{M_\alpha}}(n)^{-2} \sim (n\lambda_{\ell_{M_\beta}}(n)^{-2})^c.$$

□

Even without symmetric bases, asymptotically ℓ_p-spaces are, in a sense, "almost" invariant under permutations, thus:

Proposition 6.6. *Let X, Y be spaces with asymptotically ℓ_p normalized* 1*-unconditional bases, $1 \le p \le +\infty$. Then the singular twisted Hilbert sums induced by the interpolation couples (X, X^*) and (Y, Y^*) at $1/2$ are (permutatively) projectively equivalent if and only if the bases of X and Y are (permutatively) equivalent.*

Proof. The key is to show that projective equivalence actually implies equivalence, hence bounded equivalence, which implies, by Kalton's result (Proposition 3.5), that the bases of X and Y are equivalent.

Assume thus that the induced centralizers are λ-projectively equivalent. By Lemma 3.12 (3) and Corollary 3.16

$$\sum_i a_i^2 \Big(\log \frac{\mu_i}{\nu_i} - \lambda \log \frac{\mu_i'}{\nu_i'} \Big)^2 \le K,$$

whenever $x = \sum_i a_i e_i$ in ℓ_2 is normalized, and $a_i^2 = \nu_i \mu_i = \nu_i' \mu_i'$ with

$$1 \le \|\sum_i \nu_i e_i\|_X, \|\sum_i \mu_i e_i\|_{X^*}, \|\sum_i \nu_i' e_i\|_Y, \|\sum_i \mu_i' e_i\|_{Y^*} \le c.$$

Taking x with support far enough on the basis, we may choose $a_i = n^{-1/2}$ and $\nu_i = \nu_i' \simeq n^{-1/p}$, $\mu_i = \mu_i' \simeq n^{-1/p'}$. Then $|(1-\lambda)\log n|^2 \le K'$, which means that $\lambda = 1$. Therefore we have equivalence, and even bounded equivalence, by Corollary 3.16.

To deduce the permutative projective equivalence case from the projective equivalence case just note that if a basis (e_n) is asymptotically ℓ_p, then any permutation of (e_n) is again asymptotically ℓ_p "in the long distance", in the sense that there exists $C \geq 1$ and a function $f : \mathbb{N} \to \mathbb{N}$ such that for all n and normalized $f(n) < x_1 < \ldots < x_n$ in X, the sequence $(x_i)_{i=1}^n$ is C-equivalent to the basis of ℓ_p^n. $\square$

In Banach space theory it is interesting to decide whether the twisted Hilbert spaces one obtains are simply isomorphic as Banach spaces. Although the techniques in this paper cannot in principle distinguish sequences beyond module equivalence, there are two situations in which one can obtain the following.

Kalton obtained in [34] a family $Z_2(\alpha)$ of complex twisted Hilbert spaces induced by the centralizers

$$\mathcal{K}_{i\alpha}(x) = x\left(-\log\frac{|x|}{\|x\|}\right)^{1+i\alpha}$$

for $-\infty < \alpha < \infty$ (see also [32]). These are not real centralizers, and thus [33] only guarantees they can be obtained as induced by interpolation of three Köthe spaces. Kalton shows in [34] that the spaces $Z_2(\alpha)$ and $Z_2(\beta)$ are not isomorphic for $\alpha \neq \beta$. Let us show that the induced exact sequences are singular:

Definition 6.7. A Lipschitz function $\phi : [0+\infty) \to \mathbb{C}$ with $\phi(0) = 0$ is called *expansive* if for every M there exists N such that $|s-t| \geq N \Rightarrow |\phi(s) - \phi(t)| \geq M$.

Observe that Lipschitz functions for which $\lim_{t\to\infty} \phi'(t) = 0$ are not expansive. In particular the functions ϕ_r for $0 < r < 1$ are not expansive, while ϕ_1 is expansive. Moreover,

Lemma 6.8. *The Lispchitz function $\phi(t) = t^{1+i\alpha}$ is expansive.*

Proof. $|\phi(s) - \phi(t)| = |se^{i\alpha\log(s)} - te^{i\alpha\log(t)}| \geq ||s| - |t|| = |s-t|$. $\square$

We finally show:

Proposition 6.9. *Let X be a space with a normalized* 1*-unconditional basis that is self-similar with respect to $\mathcal{D}$ and such that $\lim_{n\to\infty} M_X(n) = \infty$. If ϕ is an expansive Lipschitz function, then the Kalton-Peck map $\mathcal{K}_\phi$ is singular.*

Proof. To simplify notation we write $\Omega = \mathcal{K}_\phi$. Observe that Ω is a contractive centralizer. Assume that Y is a sublattice of X such that $\Omega_{|Y}$ is trivial. Let M be arbitrary positive, N be such that $|s-t| \geq N \Rightarrow |\phi(s)-\phi(t)| \geq M$, and n be such that $M_Y(n) \geq 2e^N$. We may consider disjoint vectors $y_1, \ldots, y_n$ in Y of norm at most 1 such that $\|y_1 + \cdots + y_n\| \geq M_Y(n)/2$. An easy calculation shows that

$$\Omega(\sum_i y_i) - \sum_i \Omega(y_i) = \sum_i y_i\big(\phi(-\log(\sum_i y_i/K)) - \phi(-\log(\sum_i y_i))\big),$$

where $K = \|\sum_{i=1}^n y_i\|$. Each coordinate of the vector $\log(\sum_i y_i) - \log(\sum_i y_i/K)$ is $\log K$, which is larger than $\log(M_Y(n)/2) \geq N$. Therefore each coordinate of the vector $\phi(-\log(\sum_i y_i)) - \phi(-\log(\sum_i y_i/K))$ is larger than M in modulus. We deduce that

$$\|\Omega(\sum_i y_i) - \sum_i \Omega(y_i)\| \geq M\|\sum_i y_i\| \geq MM_Y(n)/2.$$

By Lemma 4.7, this implies for some fixed constant k that $kM_X(n) \geq MM_Y(n)/2$; therefore $M_X \not\sim M_Y$, a contradiction which proves that Ω is singular. $\square$

Observe that the condition $\lim_{n\to\infty} M_X(n) = \infty$ can be obtained assuming that X is self-similar with respect to $\mathcal{D}$ and does not contain c_0. Summing up the situation, one has:

Proposition 6.10. *Given $\alpha \in \mathbb{R}$, the exact sequences*

$$0 \longrightarrow \ell_2 \longrightarrow Z_2(\alpha) \longrightarrow \ell_2 \longrightarrow 0$$

are singular, and for $\alpha \neq \beta$ the spaces $Z_2(\alpha)$ and $Z_2(\beta)$ are not isomorphic.

Our second example is provided by the Kalton-Peck centralizers $\mathcal{K}_r$ (which are those induced by the Lipschitz functions $\phi_r(t) = t$ for $0 \leq t \leq 1$, and $\phi_r(t) = t^r$ for $1 < t < \infty$). We will call $\ell_2(\phi_r) = \ell_2 \oplus_{\mathcal{K}_r} \ell_2$ the twisted sum spaces they generate. Note that $\ell_2(\phi_1) = Z_2$. It was shown in [36, Corollary 5.5] that the spaces $\ell_2(\phi_r)$ are mutually nonisomorphic for different values of $0 < r \leq 1$. We already know [36, Corollary 5.5] that $\mathcal{K} = \mathcal{K}_{\phi_1}$ is singular but, since the function ϕ_r is not expansive for $r < 1$, we do not know if also $\mathcal{K}_r$ is singular for $0 < r < 1$. As every centralizer does, $\mathcal{K}_r$ comes generated by some interpolation scale. We now show that it is a scale of Orlicz spaces.

Proposition 6.11. *Let $0 < r < 1$ and $\varphi_0,\ \varphi_1$ be the maps $[0,\infty) \to [0,\infty)$ defined by*

$$\varphi_0^{-1}(t) = t^{\frac{1}{2}+\frac{1}{4}(-\log t)^{r-1}}, \quad \varphi_1^{-1}(t) = t^{\frac{1}{2}-\frac{1}{4}(-\log t)^{r-1}},$$

on a neighborhood of 0, and extended to $[0,\infty)$ to be N-functions with the Δ_2-property. Then $\Omega_{(\ell_{\varphi_0},\ell_{\varphi_1})_{1/2}}$ is boundedly equivalent to $\mathcal{K}_r$.

Proof. We note that everything here is well defined since by the choice of r and after an easy calculation, $t^{3/4} \leq \varphi_0^{-1}(t) \leq t^{1/4}$, $t^{3/4} \leq \varphi_1^{-1}(t) \leq t^{1/4}$ and $\varphi_1^{-1}(t)$ and $\varphi_0^{-1}(t)$ are increasing, for t in some neighborhood of 0. This is enough to make sure that φ_1 and φ_0 define N-function Orlicz spaces. The Δ_2-property is also satisfied on a neighborhood of 0. Indeed

$$\begin{aligned}
\varphi_0^{-1}(9t) &= 3t^{\frac{1}{2}+\frac{1}{4}(-\log 9t)^{r-1}} \\
&= 3\varphi_0^{-1}(t)t^{\frac{1}{4}[(-\log 9-\log t)^{r-1}-(-\log t)^{r-1}]} \\
&= 3\varphi_0^{-1}(t)\exp\big(-\tfrac{1}{4}(-\log t)^r[(1+\tfrac{\log 9}{\log t})^{r-1}-1]\big).
\end{aligned}$$

The exponential in this expression is easily seen to tend to 1 when t tends to 0, so close enough to 0, $\varphi_0^{-1}(9t) \geq 2\varphi_0^{-1}(t)$, and φ_0 satisfies the Δ_2 condition $\varphi_0(2s) \leq 9\varphi(s)$ for s in a neighborhood of 0. The same holds for φ_1. Since $\varphi_0^{-1}(t)\varphi_1^{-1}(t) = t$ on a neighborhood of 0, the equality $(\ell_{\varphi_0},\ell_{\varphi_1})_{1/2} = \ell_2$ holds up to equivalence of bases.

Let ψ be the map so that

$$\varphi_1^{-1}(t) = t^{\frac{1}{2}-\frac{1}{4}\psi(-\log(t))}.$$

Note that ψ is continuous, $\psi(s) = s^{r-1}$ for s on a neighborhood V of $+\infty$, and only the value of $\psi(s)$ for $s \geq 0$ is relevant here. Suppose that $\|x\|_2 = 1$. Then the centralizer Ω associated to $(\ell_{\varphi_0},\ell_{\varphi_1})_{1/2} = \ell_2$ (see Proposition 3.9) is given by

$$\Omega(x) = 2x\log\frac{\varphi_1^{-1}(|x|^2)}{|x|} = 2x\log|x|^{-\frac{1}{2}\psi(-\log|x|)} = x\psi(-\log|x|)(-\log|x|),$$

while $\mathcal{K}_r(x)_n = x_n \cdot (-\log|x_n|)^r$ whenever $|x_n|$ is less than some constant c depending on V. So we deduce that

$$\begin{aligned}\|\Omega(x) - \mathcal{K}_r(x)\|^2 &\leq \sum_{|x_n|\geq c} 2(\Omega(x))_n^2 + (\mathcal{K}_r(x))_n^2 \\ &\leq 2\big((-\log c)^2 \sup_{[0,-\log c]} |\psi| + (-\log c)^{2r}\big).\end{aligned}$$

Since Ω and $\mathcal{K}_r$ are homogeneous, they are boundedly equivalent. In particular, $\ell_2 \oplus_\Omega \ell_2$ and $\ell_2(\phi_r)$ are isomorphic. □

7. The twisting of H.I. spaces

A Banach space X is said to be *indecomposable* if it cannot be decomposed as $A \oplus B$ for two infinite-dimensional subspaces A, B. An infinite-dimensional space X is said to be *hereditarily indecomposable* (H.I., for short) if all subspaces are indecomposable [28]. It is said to be *quotient hereditarily indecomposable* (Q.H.I., for short) if all quotients of all its subspaces are indecomposable [26]. In particular, Q.H.I. spaces are H.I. The existence of Q.H.I. Banach spaces was proved in [26]. The simplest connection between H.I. spaces and the theory of singular exact sequences is described in the following folklore proposition; we present its proof for the sake of completeness.

Lemma 7.1. *Given an exact sequence of Banach spaces*

$$0 \longrightarrow Y \longrightarrow X \xrightarrow{q} Z \longrightarrow 0,$$

the space X is H.I. if and only if Y is H.I. and q is strictly singular.

Proof. Suppose X is H.I. Then clearly Y is H.I., and if q is not strictly singular, $q_{|V}$ is an isomorphism for some (infinite-dimensional) subspace V of X; hence $Y \oplus V$ is a subspace of X and thus X cannot be H.I. Conversely, suppose that q is strictly singular. If X is not H.I. we can find a decomposable subspace $X_1 \oplus X_2$ of X, and q has compact (even nuclear) restrictions on some subspaces $Y_1 \subset X_1$ and $Y_2 \subset X_2$. Thus we can assume that there exists a bijective isomorphism $\tau : X \to X$ such that $\tau(Y_1)$ and $\tau(Y_2)$ are contained in Y. Since $\tau(Y_1) \oplus \tau(Y_2)$ is closed, we conclude that Y is not H.I. □

The basic question we tackle in this section is whether it is possible to obtain nontrivial twisted sums of H.I. spaces.

We will use the homological language to denote the existence of a nontrivial twisted sum of A and B as $\operatorname{Ext}(B, A) \neq 0$. On one hand, of course, is that there exist H.I. twisted sums of H.I. spaces: if X is a Q.H.I. space and Y is a subspace of X with $\dim Y = \dim X/Y = \infty$, then X is a nontrivial twisted sum of the two H.I. spaces Y and X/Y. However, what one is looking for is to obtain methods to twist two specified H.I. spaces. Recall that the Kalton-Peck method [36] to twist spaces only works, in principle, under unconditionality assumptions. A second method is to use the local theory of exact sequences as developed in [11]. The following result is a good example; we could not find it explicitly in the literature, but it is certainly known:

Proposition 7.2. *If X is a B-convex Banach space, then* $\operatorname{Ext}(X, X) \neq 0$.

Proof. If X contains ℓ_2^n uniformly complemented, as it is in the case of B-convex Banach spaces, then $\mathrm{Ext}(X, \ell_2) \neq 0$ [11]. And if $\mathrm{Ext}(X, X) = 0$, then $\mathrm{Ext}(X, \ell_2) = 0$ [11]. □

The only currently known B-convex H.I. space is the one constructed by the second author in [25]. Calling this space $\mathcal{F}$ one gets $\mathrm{Ext}(\mathcal{F}, \mathcal{F}) \neq 0$. However this is not entirely satisfactory since this twisting does not provide any information about the twisted sum space, apart from its existence. So we formulate the following question:

Problem 7.3. Let X be an H.I. space. Does there exist an H.I. twisted sum of X?

Focusing again on Ferenczi's space $\mathcal{F}$, which is obtained via an interpolation scheme, i.e., $\mathcal{F} = X_\theta$ for a certain configuration of spaces, the induced centralizer Ω_θ provides a natural twisted sum of $\mathcal{F}$ with itself that we call $\mathcal{F}_2$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F} \longrightarrow 0.$$

We will show in Section 8 that this sequence is singular, which implies that $\mathcal{F}_2$ is H.I.

By the characterization in Lemma 7.1 it is tempting to believe that a twisted sum of two H.I. spaces is H.I. whenever it is not trivial. However, let us show that this is not the case. Recall that two Banach spaces A, B are said to be totally incomparable if no infinite-dimensional subspace of A is isomorphic to a subspace of B. The following two results are revisitations of [26, Proposition 23].

Proposition 7.4. *In a commutative diagram*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & Y \arrow[r, "f"] \arrow[d, "j"'] & X_1 \arrow[r, "q_f"] \arrow[d, "i"] & Z_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X_2 \arrow[r, "g"] \arrow[d, "q_j"'] & \mathrm{PO} \arrow[r, "q_g"'] \arrow[d, "q_i"] & Z_1 \arrow[r] & 0 \\ & Z_2 \arrow[r, equal] \arrow[d] & Z_2 \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd} \tag{7.1}$$

if q_f and q_j are strictly singular and Z_1 and Z_2 are totally incomparable, then also q_g and q_i are strictly singular.

Proof. Let V be a subspace of PO. If it contains a subspace V_1 where both q_g and q_i are isomorphisms, then Z_1, Z_2 are not incomparable. If on every subspace of V both q_g, q_i are singular, then they are singular. So, assume there is a subspace V_1 of V where q_g is singular and q_i is an isomorphism. Then there is a further subspace V_2 of V_1 that is the isomorphic image $(g + k_1)(W_2)$ of some $W_2 \subset X_2$ with k_1 compact; and since q_j is strictly singular, some subspace $V_3 \subset V_2$ is the isomorphic image $(j + k_2)(W_3)$ of some $W_3 \subset Y$ with k_2 compact.

This makes $q_i(g+k_1)(j+k_2)(W_2) = q_i(if+k')(W_2) = q_ik'(W_2)$ an isomorphism, which is impossible since k' is compact. $\square$

Proposition 7.5. *There exists a nontrivial twisted sum of two H.I. spaces which is indecomposable but not H.I.*

Proof. It was proved in [26, Prop. 25] that there exist two reflexive Q.H.I. spaces X_1, X_2 admitting infinite-dimensional subspaces $Y_1 \subset X_1$ and $Y_2 \subset X_2$ such that Y_1 is isometric to Y_2 and X_1/Y_1 and X_2/Y_2 are infinite dimensional and totally incomparable. To simplify notation let us say $Y_1 = f(Y)$ and $Y_2 = j(Y)$. Finally set $Z_1 = X_1/f(Y)$ and $Z_2 = X_2/j(Y)$. Note that X_1^* and X_2^* are both Q.H.I.

Form the push-out diagram (7.1) and observe that PO must be H.I. since, say, g_g is strictly singular. Being reflexive (by a 3-space argument), its dual PO* must be indecomposable. So the dual sequence of, say, the middle horizontal

$$0 \longrightarrow Z_1^* \longrightarrow \mathrm{PO}^* \longrightarrow X_2^* \longrightarrow 0$$

cannot split. And since X_1 and X_2 are Q.H.I. both Z_1^* and X_2^* are H.I. It remains to show that PO* is not H.I.; the reason is that the following sequence is exact:

$$0 \longrightarrow Y \xrightarrow{gj=if} \mathrm{PO} \longrightarrow Z_1 \oplus Z_2 \longrightarrow 0$$

and therefore the dual sequence

$$0 \longrightarrow Z_1^* \oplus Z_2^* \longrightarrow \mathrm{PO}^* \longrightarrow Y^* \longrightarrow 0$$

is exact and PO* cannot be H.I. $\square$

We can present an alternative construction of nontrivial and non-H.I. twisted sums of H.I. spaces. Let us say that a Banach space X *admits a singular extension* if there exists a singular exact sequence

$$0 \longrightarrow X \longrightarrow \diamondsuit \xrightarrow{q} Z \longrightarrow 0;$$

i.e., an exact sequence with q strictly singular. By Lemma 7.1, an H.I. space admits a singular extension if and only if it admits a non trivial extension which is an H.I. space.

Proposition 7.6. *Every separable H.I. space X which admits a singular extension is a complemented subspace of a nontrivial twisted sum of two H.I. spaces.*

Proof. Let $0 \to X \xrightarrow{i} \diamondsuit \xrightarrow{q} Z \to 0$ be a singular extension of X with $\diamondsuit$ separable. It follows from Lemma 7.1 that $\diamondsuit$ is H.I. By [3, Theorems 14.5 and 14.8] there exists a separable H.I. space W and a surjective operator $p : W \to \diamondsuit$ with infinite-dimensional kernel. Note that p is strictly singular by Lemma 7.1. We consider the closed subspace $\mathrm{PB} = \{(w, x) \in W \oplus X : p(w) = i(x)\}$ of $W \oplus X$ and the projection operators $\alpha : \mathrm{PB} \to W$ and $\beta : \mathrm{PB} \to X$. Note that β is strictly singular because $i\beta = q\alpha$, and that β is surjective with $\ker \beta = \ker p$ an H.I. space. Hence PB is H.I.

Since the operator $\tau : Z \oplus X \longrightarrow \diamondsuit$ defined as $\tau(w, x) = i(x) - p(w)$ is surjective, we have a twisted sum of two H.I. spaces

$$0 \longrightarrow \mathrm{PB} \longrightarrow W \oplus X \xrightarrow{\tau} Y \longrightarrow 0. \tag{7.2}$$

To finish the proof it is enough to show that this twisted sum is nontrivial. Indeed, otherwise τ would be in the class Φ of operators with complemented kernel and finite-codimensional closed range. The class Φ is stable under strictly singular perturbations [1, Theorem 7.23], and thus the operator $T : Z \oplus W \longrightarrow \diamondsuit$ defined as $T(w, x) = i(x)$ would define an isomorphism of X onto a finite-codimensional subspace of $\diamondsuit$, which is not possible. $\square$

We do not know if every separable H.I. space admits a singular extension. On the other hand, the exact sequence (7.2) also shows that there are nontrivial twisted sums of H.I. spaces which are decomposable ("two" is the maximum number of summands by [27, Theorem 1]). In Section 9 we will give other examples of this kind. To conclude this section, we formulate another general problem about twisting H.I:

Problem 7.7. Does there exist an H.I. space X so that $\mathrm{Ext}(X, X) = 0$?

Note (see [4]) that there are only three types of known spaces X for which $\mathrm{Ext}(X, X) = 0$: injective spaces, the spaces $L_1(\mu)$ and c_0.

8. An H.I. twisted sum of $\mathcal{F}$

Ferenczi's H.I. uniformly convex space $\mathcal{F}$ [25] comes induced by a complex interpolation scheme associated to a family of Banach spaces (briefly described in Subsection 5.4) setting $X_{(1,t)} = \ell_q$, $q > 1, t \in \mathbb{R}$, and as $X_{(0,t)}$ certain Gowers-Maurey-like spaces with 1-monotone basis. We fix $\theta \in (0, 1)$, and define $\mathcal{F} = \{x \in \Sigma(X_{j,t}) : x = g(\theta)$ for some $g \in \mathcal{H}(X_{j,t})\}$ with the quotient norm of $\mathcal{H}(X_{j,t})/\ker\delta_\theta$, given by $\|x\|_\theta = \inf\{\|g\|_{\mathcal{H}} : x = g(\theta)\}$. In this section we will show that the space $\mathcal{F}$ satisfies the hypotheses of Proposition 5.12 with $C = 1 + \epsilon$ for any $\epsilon > 0$ and thus:

Theorem 8.1. *The induced exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F} \longrightarrow 0.$$

is singular. Therefore $\mathcal{F}_2$ is H.I.

We have trivial upper ℓ_1-estimates in spaces $X_{(0,t)}$ and upper ℓ_q-estimates in spaces $X_{(1,t)}$. So we only need to check the ℓ_p-condition of Proposition 5.12 in the middle space X_θ, for $\frac{1}{p} = 1-\theta+\frac{\theta}{q}$. Let $f(x) := \log_2(1+x)$. We first state estimates relative to successive vectors in the space $\mathcal{F}$ [25, Proposition 1], as well as estimates for successive functionals in $\mathcal{F}^*$ obtained by standard duality arguments:

Lemma 8.2. *For all successive vectors $x_1 < \cdots < x_n$ in $\mathcal{F}$,*

$$\frac{1}{f(n)^{1-\theta}}\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p} \leq \Big\|\sum_{i=1}^n x_i\Big\| \leq \Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p},$$

and for all successive functionals $\phi_1 < \cdots < \phi_n$ in $\mathcal{F}^*$,

$$\Big(\sum_{i=1}^n \|\phi_i\|^{p'}\Big)^{1/p'} \leq \Big\|\sum_{i=1}^n \phi_i\Big\| \leq f(n)^{1-\theta}\Big(\sum_{i=1}^n \|\phi_i\|^{p'}\Big)^{1/p'}.$$

In [25], ℓ_{p+}^n-averages are defined as normalized vectors of the form $\sum_{i=1}^n x_i$, where the x_i's are successive of norm at most $(1+\epsilon)n^{-1/p}$, and may be found in any block subspace of $\mathcal{F}$ (see [25, Lemma 2]). However here we need to control not only the norm of $\sum_{i=1}^n x_i$ but also of $\sum_{i=1}^n \pm x_i$ for any choice of signs $\pm$, so [25, Lemma 2] is not quite enough. To this end we shall use RIS sequences as defined in [25, Definition 3].

RIS sequences with constant $C > 1$ are successive sequences of $\ell_{p+}^{n_k}$-averages with a technical "rapidly" increasing condition on the n_k's and therefore are also present in every block subspace of $\mathcal{F}$. Every subsequence of an RIS sequence is again an RIS sequence. In what follows L is some lacunary infinite subset of $\mathbb{N}$ whose exact definition may be found in [25]. As a consequence of Lemma 8.2, [25, Lemma 10] and standard duality arguments we have:

Lemma 8.3. *Let $y_1 < \cdots < y_n$ be an RIS sequence in $\mathcal{F}$, with constant $1+\epsilon^2/100$, where $n \in [\log N, \exp N]$ for some N in L, and $0 < \epsilon < 1/16$. Then*

$$\frac{n^{1/p}}{f(n)^{1-\theta}} \leq \| \sum_{i=1}^{n} y_i \| \leq (1+\epsilon) \frac{n^{1/p}}{f(n)^{1-\theta}}.$$

Furthermore if for all i, $\phi_i \in \mathcal{F}^*$ satisfies $\|\phi_i\| = \phi_i(y_i) = 1$ and $\operatorname{ran} \phi_i \subset \operatorname{ran} y_i$, *then*

$$(1+\epsilon)^{-1} f(n)^{1-\theta} n^{1/p'} \leq \| \sum_{i=1}^{n} \phi_i \| \leq f(n)^{1-\theta} n^{1/p'}.$$

We deduce the existence of sequences satisfying the condition of Proposition 5.12 in any block subspace of $\mathcal{F}$:

Proposition 8.4. *Let Y be a block-sequence of $\mathcal{F}$, $n \in \mathbb{N}$, and $\epsilon > 0$. Then there exists a block-sequence $y_1 < \cdots < y_n$ in Y and a block-sequence $\psi_1 < \cdots < \psi_n$ in $\mathcal{F}^*$ such that:*

(1) $(1+\epsilon)^{-1} \leq \|\psi_i\| \leq 1 \leq \|y_j\| \leq 1+\epsilon$ *and* $\psi_i(y_j) = \delta_{ij}$ *for* $i, j = 1, \ldots, n$,
(2) *for any complex* $\alpha_1, \ldots, \alpha_n$, $\| \sum_{i=1}^{n} \alpha_i y_i \| \geq (1+\epsilon)^{-1} (\sum_{i=1}^{n} |\alpha_i|^p)^{1/p}$,
(3) *for any complex* $\alpha_1, \ldots, \alpha_n$, $\| \sum_{i=1}^{n} \alpha_i \psi_i \| \leq (1+\epsilon) (\sum_{i=1}^{n} |\alpha_i|^{p'})^{1/p'}$.

In particular the block-sequence $y_1 < \cdots < y_n$ of Y is $(1+\epsilon)$-equivalent to the unit vector basis of ℓ_p^n and $[y_1, \ldots, y_n]$ is $(1+\epsilon)$-complemented in Y.

Proof. Assuming $\epsilon \leq 1/16$, pick m such that $\operatorname{dist}(mn, N) < n$ for some $N \in L$ and big enough to ensure that m and mn belong to $[\log N, \exp N]$, and that $f(mn)/f(m) < 1+\epsilon$. Denote $M = mn$. Let $x_1, \ldots, x_M$ be an RIS in Y with constant $1+\epsilon^2/100$ and $\phi_1, \ldots, \phi_M$ be a sequence of successive norming functionals in X^* for $x_1, \ldots, x_M$.

Now for $j = 1, \ldots, n$, let

$$y_j = \frac{f(m)^{1-\theta}}{m^{1/p}} \sum_{i=(j-1)m+1}^{jm} x_i \text{ and } \psi_j = \frac{1}{f(m)^{1-\theta} m^{1/p'}} \sum_{i=(j-1)m+1}^{jm} \phi_j.$$

Since $x_{(j-1)m+1}, \ldots, x_{jm}$ is an RIS with constant $1+\epsilon^2/100$, we have by Lemma 8.3 that for $j = 1, \ldots, n$,

$$1 \leq \|y_j\| \leq (1+\epsilon), \quad (1+\epsilon)^{-1} \leq \|\psi_j\| \leq 1,$$

and clearly $\psi_j(y_k) = \delta_{j,k}$. For any complex $\alpha_1, \ldots, \alpha_n$, Lemma 8.2 implies

$$\frac{m^{1/p}}{f(m)^{1-\theta}} \| \sum_{j=1}^{n} \alpha_j y_j \| \geq \frac{(\sum_{j=1}^{n} m |\alpha_j|^p)^{1/p}}{f(M)^{1-\theta}},$$

so

$$\| \sum_{j=1}^{n} \alpha_j y_j \| \geq (\sum_{j=1}^{n} |\alpha_j|^p)^{1/p} (\frac{f(m)}{f(M)})^{1-\theta} \geq (\sum_{j=1}^{n} |\alpha_j|^p)^{1/p} (1+\epsilon)^{-1}.$$

Lemma 8.2 also implies

$$f(m)^{1-\theta} m^{1/p'} \|\sum_{j=1}^{n} \alpha_j \psi_j\| \leq f(M)^{1-\theta} (\sum_{j=1}^{n} m|\alpha_j|^{p'})^{1/p'},$$

and thus $\|\sum_{j=1}^{n} \alpha_j \psi_j\| \leq (1+\epsilon)(\sum_{j=1}^{n} |\alpha_j|^{p'})^{1/p'}$.

Clearly $(y_i)_{i=1}^{n}$ is $(1+\epsilon)$-equivalent to the unit basis of ℓ_p^n. We claim that $Px = \sum_{i=1}^{n} \psi_i(x) y_i$ defines a projection from $\mathcal{F}$ onto $[y_1, \ldots, y_n]$ of norm at most $(1+\epsilon)^{2p}$. Indeed, for $x \in \mathcal{F}$,

$$\|Px\|^p \leq (1+\epsilon)^p (\sum_{i=1}^{n} |\psi_i(x)|^p) = (1+\epsilon)^p (\sum_{i=1}^{n} \alpha_i |\psi_i(x)|^{p-1} \psi_i(x))$$

for some $\alpha_1, \ldots, \alpha_n$ of modulus 1. So

$$\|Px\|^p \leq (1+\epsilon)^p \|x\| \|\sum_{i=1}^{n} \alpha_i |\psi_i(x)|^{p-1} \psi_i\| \leq (1+\epsilon)^{p+1} \|x\| (\sum_{i=1}^{n} |\psi_i(x)^{p-1}|^{p'})^{1/p'}.$$

Since

$$\sum_{i=1}^{n} |\psi_i(x)^{p-1}|^{p'} = \sum_{i=1}^{n} |\psi_i(x)|^p \leq (1+\epsilon)^p \|Px\|^p,$$

we deduce $\|Px\|^p \leq (1+\epsilon)^{p+1+p/p'} \|x\| \|Px\|^{p/p'}$; therefore $\|Px\| \leq (1+\epsilon)^{2p} \|x\|$. This concludes the proof of the claim, and up to appropriate choice of ϵ, that of the proposition. □

9. Iterated twisting of $\mathcal{F}$

The results in this section are the particular cases of [12, Cor. 2 and Prop. 3] for the admissible families yielding Ferenczi's space. For the sake of completeness we include a rather complete sketch with somewhat different proofs. To unify the notation, let us set $\mathcal{F}_1 = \mathcal{F}$. As above, let $\mathcal{F}_2$ denote the self-extension of $\mathcal{F}_1$ obtained in Section 8. As it is showed in Proposition 3.2,

$$\mathcal{F}_2 = \{(g'(\theta), g(\theta)) \,:\, g \in \mathcal{H}(X_{j,t})\},$$

endowed with the quotient norm of $\mathcal{H}(X_{j,t})/(\ker \delta_\theta \cap \ker \delta'_\theta)$. Let us show that the twisting process can be iterated obtaining a sequence $(\mathcal{F}_n)$ of H.I. spaces such that $\mathcal{F}_{n+m}$ is a twisted sum of $\mathcal{F}_n$ and $\mathcal{F}_m$.

Given a function $g \in \mathcal{H}(X_{j,t})$ and an integer $k \in \mathbb{N}$, we denote $\hat{g}[k] := g^{(k-1)}(\theta)/(k-1)!$, the $(k)^{th}$ coefficient of the Taylor series of g at θ. Following the constructions in [12], we define for $n \geq 3$:

$$\mathcal{F}_n := \{(\hat{g}[n], \ldots, \hat{g}[2], \hat{g}[1]) \,:\, g \in \mathcal{H}(X_{j,t})\}$$

endowed with the quotient norm of $\mathcal{H}(X_{j,t})/\bigcap_{k=0}^{n-1} \ker \delta_\theta^{(k)}$.

Proposition 9.1. *Let $m, n \in \mathbb{N}$ with $m > n$.*

1. *The expression $\pi_{m,n}(x_m, \ldots, x_n, \ldots, x_1) = (x_n, \ldots, x_1)$ defines a surjective operator $\pi_{m,n} : \mathcal{F}_m \to \mathcal{F}_n$.*
2. *The expression $i_{n,m}(x_n, \ldots, x_1) := (x_n, \ldots, x_1, 0, \ldots, 0)$ defines an isomorphic embedding $i_{n,m} : \mathcal{F}_n \to \mathcal{F}_m$ with $\operatorname{ran}(i_{n,m}) = \ker(\pi_{m,m-n})$.*
3. *The operator $\pi_{m,n}$ is strictly singular.*

Proof. (1) Since $\mathrm{dist}(g, \bigcap_{k=0}^{n-1} \ker \delta_\theta^{(k)}) \le \mathrm{dist}(g, \bigcap_{k=0}^{m-1} \ker \delta_\theta^{(k)})$, we have $\|\pi_{m,n}\| \le 1$. And it is obvious that $\pi_{m,n}$ is surjective.

(2) Let $\phi \in H^\infty(\mathbb{S})$ be a scalar function such that $\hat{\phi}[k] = \delta_{k,m-n}$ for $1 \le k \le m$. For the existence of ϕ, we consider a conformal equivalence $\varphi : \mathbb{S} \to \mathbb{D}$ satisfying $\varphi(\theta) = 0$, and the polynomial $p(z) := (z-\theta)^{m-n}$. The function $p \circ \varphi^{-1} \in H(\mathbb{D})$ admits a representation $p\circ\varphi^{-1}(\omega) = \sum_{l=0}^\infty a_l \omega^l$, and it is not difficult to check that $\phi(z) := \sum_{l=0}^m a_l \varphi(z)^l$ defines a function that satisfies the required conditions.

Given $(x_n, \dots, x_1) \in \mathcal{F}_n$, we take $g \in \mathcal{H}(X_{j,t})$ such that $\hat{g}[k] = x_k$ for $k = 1, \dots, n$. Then $f := \phi \cdot g \in \mathcal{H}(X_{j,t})$ with $\|f\| \le \|\phi\|_\infty \cdot \|g\|$ and, by the Leibnitz rule,

$$\hat{f}[k] = \sum_{l=1}^k \hat{\phi}[l]\hat{g}[k-l].$$

Thus $\hat{f}[k] = 0$ for $1 \le k \le m-n$ and $\hat{f}[k] = \hat{g}[k-m+n]$ for $m-n < k \le m$; i.e., $(\hat{f}[m], \dots, \hat{f}[1]) = (x_n, \dots, x_1, 0, \dots, 0)$. Hence $i_{n,m}$ is well defined and $\|i_{n,m}\| \le \|\phi\|_\infty$.

Clearly $i_{n,m}$ is injective and $\mathrm{ran}(i_{n,m}) \subset \ker(\pi_{m,m-n})$. Let $(y_n, \dots, y_1, 0, \dots, 0)$ in $\ker(\pi_{m,m-n})$. Then there exists $g \in \mathcal{H}(X_{j,t})$ such that $\hat{g}[k] = 0$ for $1 \le k \le m-n$ and $\hat{g}[k] = y_{k-m+n}$ for $m-n < k \le m$. Since g has a zero of order $m-n$ at θ, there exists $f \in \mathcal{H}(X_{j,t})$ such that $g(z) = f(z)(z-\theta)^{m-n}$, and it is not difficult to check that $i_{n,m}(\hat{f}[n], \dots, \hat{f}[1]) = (y_n, \dots, y_1, 0, \dots, 0)$.

(3) Since $\pi_{m,n} = \pi_{m-1,n}\pi_{m,m-1}$ for $m > n+1$, it is enough to prove that $\pi_{m,m-1}$ is strictly singular. We will do it by induction.

We proved in Theorem 8.1 that $\pi_{2,1}$ is strictly singular. Let $m > 2$ and assume that $\pi_{m-1,m-2}$ is strictly singular. Note that $\pi_{m,1} = \pi_{m,2}\pi_{2,1}$; hence $\pi_{m,1}$ is also strictly singular.

We consider the following commuting diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_{m-1} & \xrightarrow{i_{m-1,m}} & \mathcal{F}_m & \xrightarrow{\pi_{m,1}} & \mathcal{F}_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_{m-1,m-2}} & & \downarrow{\scriptstyle \pi_{m,m-1}} & & \| & & \\ 0 & \longrightarrow & \mathcal{F}_{m-2} & \xrightarrow[i_{m-2,m-1}]{} & \mathcal{F}_{m-1} & \xrightarrow[\pi_{m-1,1}]{} & \mathcal{F}_1 & \longrightarrow & 0. \end{array} \tag{9.1}$$

By (1) and (2), the two rows are exact. Suppose that M is an infinite-dimensional closed subspace of $\mathcal{F}_m$ such that $\pi_{m,m-1}|_M$ is an isomorphism. Since $\pi_{m,m-1}i_{m-1,m}$ is strictly singular and $\mathrm{ran}(i_{m-1,m}) = \ker(\pi_{m,1})$, $M \cap \ker(\pi_{m,1})$ is finite dimensional and $M + \ker(\pi_{m,1})$ is closed. But this is impossible, because $\pi_{m,1}$ is strictly singular. □

Since superreflexivity is a 3-space property [17], as an immediate consequence we get:

Corollary 9.2. *Let $m, n \in \mathbb{N}$. Then the sequence*

$$0 \longrightarrow \mathcal{F}_m \xrightarrow{i_{m,m+n}} \mathcal{F}_{m+n} \xrightarrow{\pi_{m+n,n}} \mathcal{F}_n \longrightarrow 0$$

is exact and singular. Therefore, all the spaces $\mathcal{F}_n$ are super-reflexive H.I.

Next we show that there are natural nontrivial twisted sums of spaces $\mathcal{F}_n$ which are not H.I. Let $l, m, n \in \mathbb{N}$ with $l > n$. We consider the following push-out diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_l \arrow[r, "i_{l,l+m}"] \arrow[d, "\pi_{l,n}"'] & \mathcal{F}_{l+m} \arrow[r, "\pi_{l+m,m}"] \arrow[d, "\pi_{l+m,n+m}"] & \mathcal{F}_m \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F}_n \arrow[r, "i_{n,n+m}"'] & \mathcal{F}_{n+m} \arrow[r, "\pi_{n+m,m}"'] & \mathcal{F}_m \arrow[r] & 0. \end{tikzcd} \tag{9.2}$$

Proposition 9.3. *Let $l, m, n \in \mathbb{N}$ with $l > n$. Then the diagonal push-out sequence*

$$0 \longrightarrow \mathcal{F}_l \xrightarrow{\ i\ } \mathcal{F}_n \oplus \mathcal{F}_{l+m} \xrightarrow{\ \pi\ } \mathcal{F}_{m+n} \longrightarrow 0 \tag{9.3}$$

obtained from diagram (9.2) *is a nontrivial exact sequence.*

Proof. As we saw in Section 2, the maps i and π are given by

$$i(x) = (-\pi_{l,n}\, x,\, i_{l,l+m}\, x) \quad \text{and} \quad \pi(y, z) = i_{n,n+m}\, y + \pi_{l+m,n+m}\, z,$$

and it is easy to check that the sequence (9.3) is exact. Since $l > n$, every operator from $\mathcal{F}_l$ or $\mathcal{F}_{m+n}$ into $\mathcal{F}_n$ is strictly singular. Thus $\mathcal{F}_l \oplus \mathcal{F}_{m+n}$ is not isomorphic to $\mathcal{F}_n \oplus \mathcal{F}_{l+m}$, and the exact sequence (9.3) is nontrivial. □

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