

COERCIVITY OF WEIGHTED KOHN LAPLACIANS: THE CASE OF MODEL MONOMIAL WEIGHTS IN \mathbb{C}^2

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ABSTRACT. The *weighted Kohn Laplacian* \square_φ is a natural second order elliptic operator associated to a weight $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$ and acting on $(0, 1)$ -forms, which plays a key role in several questions of complex analysis.

We consider here the case of *model monomial weights* in \mathbb{C}^2 , i.e.,

$$\varphi(z, w) := \sum_{(\alpha, \beta) \in \Gamma} |z^\alpha w^\beta|^2,$$

where $\Gamma \subseteq \mathbb{N}^2$ is finite. Our goal is to prove coercivity estimates of the form

$$(*) \quad \square_\varphi \geq \mu^2,$$

where $\mu : \mathbb{C}^n \rightarrow \mathbb{R}$ acts by pointwise multiplication on $(0, 1)$ -forms, and the inequality is in the sense of self-adjoint operators. We proved in 2015 how to derive from $(*)$ new pointwise bounds for the weighted Bergman kernel associated to φ . Here we introduce a technique to establish $(*)$ with

$$\mu(z, w) = c(1 + |z|^a + |w|^b) \quad (a, b \geq 0),$$

where $a, b \geq 0$ depend on (and are easily computable from) Γ . As a corollary we also prove that, for a wide class of model monomial weights, the spectrum of \square_φ is discrete if and only if the weight is *not decoupled*, i.e., Γ contains at least a point (α, β) with $\alpha \neq 0 \neq \beta$.

Our methods comprise a new *holomorphic uncertainty principle* and linear optimization arguments.

1. INTRODUCTION

1.1. Motivations and goal of the paper. Since the work of Hörmander [Hör65], many results in several complex variables have been established where a key role is played by a (typically, plurisubharmonic) weight $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$. In particular, an effective way of estimating the Bergman kernel of a weakly pseudoconvex domain in \mathbb{C}^{n+1} is to consider first the same problem on the model domain associated to a plurisubharmonic function $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$:

$$\Omega_\varphi := \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : \Im(z_{n+1}) > \varphi(z)\},$$

which, after a reduction to the boundary and a Fourier transform in the $\Re(z_{n+1})$ -variable, leads to the consideration of a *weighted Bergman kernel* in \mathbb{C}^n (see [Has98]). The adjective “weighted” refers to the fact that the underlying measure is $e^{-2\varphi}$ times Lebesgue measure (see Section 2.4). Typically one works under some finite-type assumption on domains, and hence on weights, the prototypical case being when φ is a plurisubharmonic non-harmonic polynomial.

Many papers are dedicated to Bergman kernels on domains and weighted Bergman kernels on \mathbb{C}^n : the situation is well-understood when $n = 1$ (see, e.g.,

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[NRSW89], [Chr91], [Rai06], and [Rai07]), and there are satisfactory results when $n \geq 2$ if the domain or the weight satisfies some auxiliary assumption (e.g., it is convex [MS94], [MS97], its complex Hessian has comparable eigenvalues [Koe02], it is decoupled [NS06], or geometrically separated [CD14]).

The papers just cited cover many interesting special cases, but the case of a generic plurisubharmonic non-harmonic polynomial weight $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$ is far from being understood, and it is expected to raise delicate algebraic and geometric questions. A significant step in this direction has been taken recently by Nagel and Pramanik [NP], who obtained pointwise bounds for the diagonal values of the Bergman kernel of Ω_φ when φ is, in the author's terminology, a *model monomial weight*, i.e.,

$$\varphi(z_1, \dots, z_n) := \sum_{\alpha=(\alpha_1, \dots, \alpha_n) \in \Gamma} |z_1^{\alpha_1} \cdots z_n^{\alpha_n}|^2,$$

where $\Gamma \subseteq \mathbb{N}^n \setminus \{0\}$ is finite and non-empty.

We recently proved in [Dal15] that pointwise bounds for the weighted Bergman kernel relative to $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$ can be obtained whenever the *weighted Kohn Laplacian* \square_φ , a second order elliptic operator naturally associated to φ and acting on $(0, 1)$ -forms, is known to be μ -coercive, i.e.,

$$(1) \quad \square_\varphi \geq \mu^2,$$

where $\mu : \mathbb{C}^n \rightarrow \mathbb{R}$ acts by pointwise multiplication on $(0, 1)$ -forms, and the inequality is in the sense of self-adjoint operators (see Section 2.1 for the precise definitions). The estimate we get for the weighted Bergman kernel is characterized by an exponential decay which depends quantitatively on μ (see Section 2.4 for details). We highlight the fact that coercivity conditions like (1) are often useful in the study of elliptic operators and have other interesting consequences (see, e.g., Theorem 6 in the following).

The goal of this paper is to establish (1) when φ is a model monomial weight in \mathbb{C}^2 , and with

$$\mu(z, w) = c(1 + |z|^a + |w|^b) \quad (a, b \geq 0),$$

where $a, b \geq 0$ depend on (and are easily computable from) Γ . This is somehow the simplest class of weights that falls outside the scope of the existing literature.

1.2. Structure of the paper. After defining rigorously weighted Kohn Laplacians, μ -coercivity (Section 2.1), and model monomial weights (Section 2.2), we state our theorems in Section 2.3. Section 2.4 relies on [Dal15] to deduce estimates on the weighted Bergman kernel associated to model monomial weights in \mathbb{C}^2 .

The proofs of our theorems are outlined in Section 3 and consist of two main ingredients: a linear optimization argument which exploits the specific algebraic nature of our weights (Section 4) and a more general *holomorphic uncertainty principle*, which we introduce in Section 5. The two ingredients are put together in Section 6, where the proofs are concluded.

2. DEFINITIONS AND STATEMENT OF THE RESULTS

2.1. Weighted Kohn Laplacians and μ -coercivity. Let $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$ be a C^2 plurisubharmonic weight. This means that the *complex Hessian* $H_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \right)_{j,k=1}^n$ is positive semidefinite, i.e.,

$$(H_\varphi(z)v, v) \geq 0 \quad \forall z \in \mathbb{C}^n, \quad v \in \mathbb{C}^n.$$

We begin by introducing the weighted L^2 space

$$L^2(\mathbb{C}^n, \varphi) := \left\{ f : \mathbb{C}^n \rightarrow \mathbb{C} : \int_{\mathbb{C}^n} |f|^2 e^{-2\varphi} < +\infty \right\}.$$

We denote by $L_{(0,q)}^2(\mathbb{C}^n, \varphi)$ the Hilbert space of $(0, q)$ -forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$. Since we will be working only with forms of degree less than or equal to 2, we confine our discussion to these cases. Adopting the standard notation for differential forms, we have that $L_{(0,0)}^2(\mathbb{C}^n, \varphi) = L^2(\mathbb{C}^n, \varphi)$,

$$L_{(0,1)}^2(\mathbb{C}^n, \varphi) := \left\{ u = \sum_{1 \leq j \leq n} u_j d\bar{z}_j : u_j \in L^2(\mathbb{C}^n, \varphi) \quad \forall j \right\},$$

and

$$L_{(0,2)}^2(\mathbb{C}^n, \varphi) := \left\{ w = \sum_{1 \leq j < k \leq n} w_{jk} d\bar{z}_j \wedge d\bar{z}_k : w_{jk} \in L^2(\mathbb{C}^n, \varphi) \quad \forall j, k \right\}.$$

For the norms and the scalar products in these Hilbert spaces of forms we use the same symbols $\|\cdot\|_\varphi$ and $(\cdot, \cdot)_\varphi$; i.e., if $u, \tilde{u} \in L_{(0,1)}^2(\mathbb{C}^n, \varphi)$, we have

$$\|u\|_\varphi^2 = \sum_{1 \leq j \leq n} \|u_j\|_\varphi^2, \quad (u, \tilde{u})_\varphi = \sum_{1 \leq j \leq n} (u_j, \tilde{u}_j)_\varphi,$$

while if $w, \tilde{w} \in L_{(0,2)}^2(\mathbb{C}^n, \varphi)$, we have

$$\|w\|_\varphi^2 = \sum_{1 \leq j < k \leq n} \|w_{jk}\|_\varphi^2, \quad (w, \tilde{w})_\varphi = \sum_{1 \leq j < k \leq n} (w_{jk}, \tilde{w}_{jk})_\varphi.$$

This ambiguity should not be a source of confusion.

We now introduce the initial fragment of the *weighted $\bar{\partial}$ -complex*:

$$(2) \quad L^2(\mathbb{C}^n, \varphi) \xrightarrow{\bar{\partial}} L_{(0,1)}^2(\mathbb{C}^n, \varphi) \xrightarrow{\bar{\partial}} L_{(0,2)}^2(\mathbb{C}^n, \varphi).$$

The symbol $\bar{\partial}$ denotes as usual both the operator $\bar{\partial} : L^2(\mathbb{C}^n, \varphi) \rightarrow L_{(0,1)}^2(\mathbb{C}^n, \varphi)$ defined on the domain

$$\mathcal{D}_0(\bar{\partial}) := \left\{ f \in L^2(\mathbb{C}^n, \varphi) : \frac{\partial f}{\partial \bar{z}_j} \in L^2(\mathbb{C}^n, \varphi) \quad \forall j \right\}$$

by the formula $\bar{\partial}f = \sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$, and the operator $\bar{\partial} : L_{(0,1)}^2(\mathbb{C}^n, \varphi) \rightarrow L_{(0,2)}^2(\mathbb{C}^n, \varphi)$ defined on the domain

$$\mathcal{D}_1(\bar{\partial}) := \left\{ u = \sum_j u_j d\bar{z}_j \in L_{(0,1)}^2(\mathbb{C}^n, \varphi) : \frac{\partial u_k}{\partial \bar{z}_j} - \frac{\partial u_j}{\partial \bar{z}_k} \in L^2(\mathbb{C}^n, \varphi) \quad \forall j, k \right\}$$

by the formula $\bar{\partial}u = \sum_{j < k} \left(\frac{\partial u_k}{\partial \bar{z}_j} - \frac{\partial u_j}{\partial \bar{z}_k} \right) d\bar{z}_j \wedge d\bar{z}_k$.

The weighted $\bar{\partial}$ -complex (2) is a complex, i.e.,

$$(3) \quad \bar{\partial}f \in \mathcal{D}_1(\bar{\partial}) \quad \text{and} \quad \bar{\partial}\bar{\partial}f = 0 \quad \forall f \in \mathcal{D}_0(\bar{\partial}).$$

Taking the Hilbert space adjoints of the operators in (2) (as we can, since the operators are closed and densely defined), we have the dual complex:

$$L^2(\mathbb{C}^n, \varphi) \xleftarrow{\bar{\partial}_\varphi^*} L_{(0,1)}^2(\mathbb{C}^n, \varphi) \xleftarrow{\bar{\partial}_\varphi^*} L_{(0,2)}^2(\mathbb{C}^n, \varphi).$$

We use the index φ in the symbols for these operators to stress the fact that not only the domains $\mathcal{D}_1(\bar{\partial}_\varphi^*) \subseteq L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ and $\mathcal{D}_2(\bar{\partial}_\varphi^*) \subseteq L^2_{(0,2)}(\mathbb{C}^n, \varphi)$, but also the formal expressions of $\bar{\partial}_\varphi^*$ depend on the weight φ . We omit these formulas, since they will play no direct role in what follows.

Definition 1. The *weighted Kohn Laplacian* is defined by the formula

$$\square_\varphi := \bar{\partial}_\varphi^* \bar{\partial} + \bar{\partial} \bar{\partial}_\varphi^*$$

on the domain

$$\mathcal{D}(\square_\varphi) := \{u \in L^2_{(0,1)}(\mathbb{C}^n, \varphi) : u \in \mathcal{D}_1(\bar{\partial}) \cap \mathcal{D}_1(\bar{\partial}_\varphi^*), \bar{\partial}u \in \mathcal{D}_2(\bar{\partial}_\varphi^*) \text{ and } \bar{\partial}_\varphi^* u \in \mathcal{D}_0(\bar{\partial})\}.$$

The weighted Kohn Laplacian is a densely defined, closed, self-adjoint and non-negative operator on $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. The details of the routine arguments proving this fact can be found in [Has14].

Finally, let us introduce the quadratic form

$$(4) \quad \mathcal{E}_\varphi(u, v) := (\bar{\partial}u, \bar{\partial}v)_\varphi + (\bar{\partial}_\varphi^* u, \bar{\partial}_\varphi^* v)_\varphi,$$

defined for $u, v \in \mathcal{D}(\mathcal{E}_\varphi) := \mathcal{D}_1(\bar{\partial}) \cap \mathcal{D}_1(\bar{\partial}_\varphi^*)$. Notice that, by definition of Hilbert space adjoints,

$$(\square_\varphi u, v)_\varphi = \mathcal{E}_\varphi(u, v) \quad \forall u \in \mathcal{D}(\square_\varphi), \quad \forall v \in \mathcal{D}(\mathcal{E}_\varphi).$$

We will simply write $\mathcal{E}_\varphi(u)$ for $\mathcal{E}_\varphi(u, u)$.

The well-known Morrey-Kohn-Hörmander formula gives an alternative expression for $\mathcal{E}_\varphi(u)$. In order to state it, we identify the $(0, 1)$ -form $u = \sum_{j=1}^n u_j d\bar{z}_j$ with the vector field $u = (u_1, \dots, u_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$, so that $(H_\varphi u, u) = \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k$. The Morrey-Kohn-Hörmander formula is the following identity:

$$(5) \quad \mathcal{E}_\varphi(u) = \sum_{j,k} \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial \bar{z}_k} \right|^2 e^{-2\varphi} + 2 \int_{\mathbb{C}^n} (H_\varphi u, u) e^{-2\varphi} \quad \forall u \in \mathcal{D}(\mathcal{E}_\varphi).$$

A proof may be found in [Has14] (or [CS01], for the similar unweighted case). Identity (5) reveals the fundamental role played by H_φ in the analysis of \square_φ .

We conclude this section with the key notion of μ -coercivity that already appeared in our previous paper [Dal15].

Definition 2. Given a measurable function $\mu : \mathbb{C}^n \rightarrow [0, +\infty)$, we say that \square_φ is μ -coercive if the following inequality holds:

$$(6) \quad \mathcal{E}_\varphi(u) \geq \|\mu u\|_\varphi^2 \quad \forall u \in \mathcal{D}(\mathcal{E}_\varphi).$$

In view of (5), μ -coercivity is equivalent to the estimate

$$(7) \quad \sum_{j,k} \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial \bar{z}_k} \right|^2 e^{-2\varphi} + 2 \int_{\mathbb{C}^n} (H_\varphi u, u) e^{-2\varphi} \geq \int_{\mathbb{C}^n} \mu^2 |u|^2 e^{-2\varphi} \quad \forall u \in \mathcal{D}(\mathcal{E}_\varphi).$$

The concept of μ -coercivity is a very natural one in the theory of elliptic operators. One can think of it as a spatially localized spectral gap condition (the usual spectral gap condition corresponds to the case when μ equals a positive constant). The following lemma shows how qualitative information on the spectrum of \square_φ may be deduced from μ -coercivity.

Lemma 3. Assume that \square_φ is μ -coercive for some μ such that

$$\lim_{z \rightarrow \infty} \mu(z) = +\infty.$$

Then the operator \square_φ has discrete spectrum.

We recall that we say that a self-adjoint operator has discrete spectrum if its spectrum is a discrete subset of \mathbb{R} consisting of eigenvalues of finite multiplicity.

Proof. This is essentially contained in [Has11], where the author proves that \square_φ admits a compact inverse N_φ if and only if for every $\varepsilon > 0$ there exists $R < +\infty$ such that if

$$u \in \mathcal{D}(\mathcal{E}_\varphi) \quad \text{is such that} \quad \mathcal{E}_\varphi(u) \leq 1,$$

then

$$\int_{|z| \geq R} |u|^2 e^{-2\varphi} \leq \varepsilon.$$

This condition is clearly equivalent to μ -coercivity for some μ diverging at infinity. To conclude the proof, recall that a compact operator has discrete spectrum and that if the inverse of a self-adjoint operator has discrete spectrum, the same is true of the operator itself. \square

2.2. Model monomial weights. Let us specialize to $n = 2$. We use z and w as coordinates on \mathbb{C}^2 , and we adopt the convention $\mathbb{N} := \{0, 1, \dots\}$.

Definition 4. If $\Gamma \subseteq \mathbb{N}^2$ is finite, we define the *model monomial weight* associated to Γ as follows:

$$\varphi_\Gamma(z, w) := \sum_{(\alpha, \beta) \in \Gamma} |z^\alpha w^\beta|^2 \quad \forall (z, w) \in \mathbb{C}^2.$$

A model monomial weight is said to be *decoupled* if $\Gamma \subseteq \mathbb{N} \times \{0\} \cup \{0\} \times \mathbb{N}$.

A model monomial weight is said to be *homogeneous* if there are $m, n \geq 1$ for which the following two properties hold:

- (1) $\{(m, 0), (0, n)\} \subseteq \Gamma$,
- (2) every $(\alpha, \beta) \in \Gamma$ lies on the line segment connecting $(m, 0)$ and $(0, n)$, i.e.,

$$(8) \quad n\alpha + m\beta = nm \quad \forall (\alpha, \beta) \in \Gamma.$$

Model monomial weights are sums of moduli squared of holomorphic functions, and thus they are plurisubharmonic.

Of course one could consider the analogous definitions in \mathbb{C}^n , associating a model monomial weight to any finite $\Gamma \subseteq \mathbb{N}^n$, but here we shall only treat the two-dimensional case.

Any homogeneous model monomial weight is homogeneous with respect to a system of not necessarily isotropic dilations, i.e.,

$$\varphi_\Gamma(t^{\frac{1}{m}} z, t^{\frac{1}{n}} w) = t^2 \varphi_\Gamma(z, w) \quad \forall t > 0 \text{ and } (z, w) \in \mathbb{C}^2.$$

In our analysis a key role will be played by the two quantities σ and τ (depending on Γ) defined as the smallest non-negative real numbers such that

$$(9) \quad \frac{1}{\sigma} \leq \frac{\beta}{\alpha} \leq \tau \quad \forall (\alpha, \beta) \in \Gamma \text{ such that } \alpha, \beta \neq 0.$$

One can choose two points (α_1, β_1) and (α_2, β_2) of Γ such that

$$\alpha_1 = \sigma\beta_1 \quad \text{and} \quad \beta_2 = \tau\alpha_2.$$

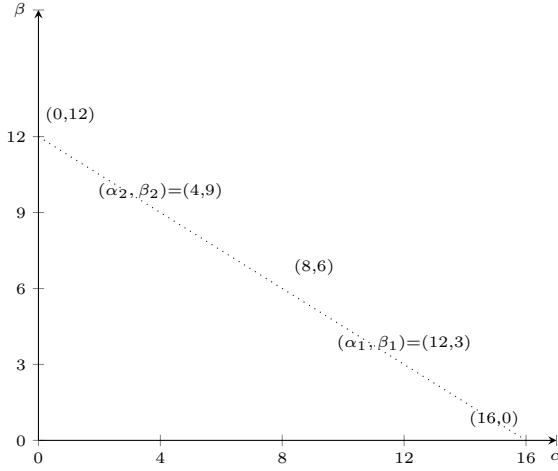


FIGURE 1. The plot of the set Γ corresponding to the homogeneous model weight $\varphi_\Gamma(z, w) = |z|^{32} + |z|^{24}|w|^6 + |z|^{16}|w|^{12} + |z|^8|w|^{18} + |w|^{24}$. In this case $\sigma = 4$ and $\tau = \frac{9}{4}$.

Notice that $\sigma, \tau < +\infty$. If φ_Γ is decoupled, then σ and τ are set to be equal to 0, while if φ_Γ is not decoupled both σ and τ are always positive. See Figure 1 for an example.

2.3. Our results. We can finally state our two main results. The first states that if a model monomial weight is homogenous, the associated Kohn Laplacian is μ -coercive for a μ which can be easily computed from Γ .

Theorem 5. *Let φ_Γ be a homogeneous model monomial weight. Then \square_{φ_Γ} is μ -coercive, where*

$$\mu(z, w) = c(1 + |z|^\sigma + |w|^\tau).$$

Here $c > 0$ is a constant that depends on Γ .

The second one is of a qualitative nature and concerns discreteness of the spectrum for an almost arbitrary model monomial weight.

Theorem 6. *If φ_Γ is a model monomial weight for which there are $m, n \geq 2$ such that $(m, 0), (0, n) \in \Gamma$, then the Kohn Laplacian \square_φ has discrete spectrum if and only if φ_Γ is not decoupled.*

This is, to the author's knowledge, an interesting new phenomenon: as soon as a mixed monomial is "added" to a two-dimensional decoupled model monomial weight, the spectrum becomes discrete.

We plan to study the same questions in \mathbb{C}^n , when $n \geq 3$. We expect that our arguments should require a significant effort to be generalized to more variables. Notice that it is not clear how a generalization of Theorem 6 should look in \mathbb{C}^3 : we know by work of Haslinger and Helffer (Theorem 6.1 of [HH07]) that \square_φ does not have discrete spectrum when $\varphi(z) = |z_1|^{2m_1} + |z_2|^{2m_2} + |z_3|^{2m_3}$. Is it enough to "add" a mixed term of the form $|z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}|^2$ ($\alpha_1, \alpha_2, \alpha_3 \neq 0$) to the weight to make the spectrum discrete?

2.4. Pointwise estimates of weighted Bergman kernels. In [Dal15] (see also [Dal14]) we proved that information about μ -coercivity of \square_φ can be converted in a pointwise estimate of the weighted Bergman kernel with respect to φ . In order to state our result more precisely, we need to recall a few definitions.

The *weighted Bergman space* with respect to the C^2 plurisubharmonic weight $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$ is defined as follows (notice that here we are not assuming that φ is a model monomial):

$$A^2(\mathbb{C}^n, \varphi) := \{h : \mathbb{C}^n \rightarrow \mathbb{C} : h \text{ is holomorphic and } h \in L^2(\mathbb{C}^n, \varphi)\}.$$

If $h \in A^2(\mathbb{C}^n, \varphi)$, then in particular it is harmonic and satisfies the mean value property $h(z) = \frac{1}{|B(z, r)|} \int_{B(z, r)} h$. The Cauchy-Schwarz inequality yields

$$(10) \quad |h(z)| \leq \frac{1}{|B(z, r)|} \sqrt{\int_{B(z, r)} e^{2\varphi} \|h\|_\varphi} \quad \forall h \in A^2(\mathbb{C}^n, \varphi),$$

for any $z \in \mathbb{C}^n$ and $r > 0$. This estimate has two elementary consequences:

- (a) $A^2(\mathbb{C}^n, \varphi)$ is a closed subspace of $L^2(\mathbb{C}^n, \varphi)$ (by (10) convergence of a sequence of $A^2(\mathbb{C}^n, \varphi)$ in the $\|\cdot\|_\varphi$ -norm implies uniform convergence, which preserves holomorphicity). We denote by B_φ the orthogonal projector of $L^2(\mathbb{C}^n, \varphi)$ onto $A^2(\mathbb{C}^n, \varphi)$.
- (b) The evaluation mappings $h \mapsto h(z)$ are continuous linear functionals of $A^2(\mathbb{C}^n, \varphi)$, and the Riesz Lemma yields a function $K_\varphi : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ such that

$$h(z) = \int_{\mathbb{C}^n} K_\varphi(z, w) h(w) e^{-2\varphi(w)} d\mathcal{L}(w) \quad \forall z \in \mathbb{C}^n,$$

and $\overline{K_\varphi(z, \cdot)} \in A^2(\mathbb{C}^n, \varphi)$ for every $z \in \mathbb{C}^n$.

The operator B_φ is called the *weighted Bergman projector* and the function K_φ the *weighted Bergman kernel* associated to the weight φ . It is immediate to see that

$$B_\varphi(f)(z) = \int_{\mathbb{C}^n} K_\varphi(z, w) f(w) e^{-2\varphi(w)} d\mathcal{L}(w) \quad \forall z \in \mathbb{C}^n,$$

for every $f \in L^2(\mathbb{C}^n, \varphi)$; i.e., K_φ is the integral kernel of B_φ . Since B_φ is self-adjoint, $K_\varphi(z, w) = \overline{K_\varphi(w, z)}$.

In [Dal14], we defined the class of *admissible weights*. Those are the C^2 plurisubharmonic weights φ such that:

- (1) the following L^∞ doubling condition holds ($B(z, r) := \{z' : |z' - z| < r\}$):

$$\sup_{B(z, 2r)} \Delta\varphi \leq D \sup_{B(z, r)} \Delta\varphi \quad \forall z \in \mathbb{C}^n, r > 0,$$

for some finite constant D which is independent of z and r ;

- (2) there exists $c > 0$ such that

$$(11) \quad \inf_{z \in \mathbb{C}^n} \sup_{B(z, c)} \Delta\varphi > 0.$$

If φ is an admissible weight, the function

$$(12) \quad \rho(z) := \sup \left\{ r > 0 : \sup_{B(z, r)} \Delta\varphi \leq r^{-2} \right\}$$

is a *radius function*; i.e., it is Borel measurable and there exists a constant $C < +\infty$ such that for every $z \in \mathbb{C}^n$ we have

$$(13) \quad C^{-1}\rho(z) \leq \rho(z') \leq C\rho(z) \quad \forall z' \in B(z, \rho(z)).$$

See Section 4 of [Dal15] for details. We now state the main result of [Dal15].

Theorem 7. *Let φ be an admissible weight and assume that there exists*

$$\kappa : \mathbb{C}^n \rightarrow (0, +\infty)$$

such that:

- (i) κ is a bounded radius function,
- (ii) \square_φ is κ^{-1} -coercive.

Then there exists $\varepsilon > 0$ such that the pointwise bound

$$|K_\varphi(z, w)| \lesssim e^{\varphi(z)+\varphi(w)} \frac{\max\{\kappa(z), \rho(z)\}}{\rho(z)^n \rho(w)^n} \frac{e^{-\varepsilon d(z, w)}}{\rho(z)^n \rho(w)^n}$$

holds for every $z, w \in \mathbb{C}^n$, where $d(z, w)$ is the distance associated to the Riemannian metric

$$\frac{\sum_{j=1}^n dx_j^2 + dy_j^2}{\max\{\kappa(z), \rho(z)\}^2} \quad (z_j = x_j + iy_j).$$

We refer to the paper for the proof and a deeper discussion of this result.

To see that the information contained in Theorem 5 can be plugged into Theorem 7, we need to see that model monomial weights fit into the general picture of [Dal15], which we just recalled. We prove the following two claims:

- (a) model monomial weights are admissible,
- (b) $(1 + |z|^a + |w|^b)^{-1}$ is a bounded radius function for every $a, b \geq 0$.

To verify claim (a), we begin by noticing that a model monomial weight φ is a sum of squares of holomorphic functions, and hence it is C^2 and plurisubharmonic (alternatively, this follows from Proposition 8 in Section 4).

Conditions (1) and (2) of the definition of admissibility only depend on the fact that $\Delta\varphi$ is a non-negative polynomial on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ (n not necessarily equal to 2). Let $d \in \mathbb{N}$ be the degree of this polynomial. The mappings

$$p \mapsto \sup_{|u| \leq 1} |p(u)| \quad \text{and} \quad p \mapsto \sup_{|u| \leq 2} |p(u)|$$

are norms on the finite-dimensional vector space of real polynomials in $2n$ real variables of degree $\leq d$ on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, and therefore they are equivalent. In particular ($z \in \mathbb{C}^n$),

$$\begin{aligned} \sup_{|u-z| \leq 2r} \Delta\varphi(u) &= \sup_{|u| \leq 2} \Delta\varphi(z + ru) \\ &\leq D \sup_{|u| \leq 1} \Delta\varphi(z + ru) = D \sup_{|u-z| \leq r} \Delta\varphi(u). \end{aligned}$$

This proves condition (1).

As z varies in \mathbb{C}^n , the polynomial $\Delta\varphi(z + \cdot)$ varies on a hyperplane not containing the origin of the vector space of real polynomials in $2n$ real variables of degree $\leq d$. To see this, just notice that any of the coefficients of a monomial of highest degree

of $\Delta\varphi$ is not affected by translations. Since $p \mapsto \sup_{|u| \leq 1} |p(u)|$ is a norm, we have

$$\inf_{z \in \mathbb{C}^n} \sup_{|u-z| \leq 1} \Delta\varphi(u) = \inf_{z \in \mathbb{C}^n} \sup_{|u| \leq 1} \Delta\varphi(z+u) > 0,$$

that is, condition (2). This concludes the proof of claim (a).

To prove claim (b), just notice that the following stronger statement holds (if $\kappa(z, w) := (1 + |z|^a + |w|^b)^{-1}$):

$$C^{-1}\kappa(z_0, w_0) \leq \kappa(z, w) \leq C\kappa(z_0, w_0) \quad \forall (z, w) \in B((z_0, w_0), 1).$$

In fact, it is equivalent to the elementary estimate

$$(1 + |z|^a + |w|^b) \leq C(1 + |z_0|^a + |w_0|^b) \quad \forall (z, w) \in B((z_0, w_0), 1).$$

Thus, one can apply Theorem 7 to deduce new pointwise estimates for K_φ when φ is a homogeneous model monomial weight.

It may be interesting to point out that the function ρ defined by (12) satisfies the following approximate identity when φ is a model monomial weight:

$$(14) \quad \rho(z) \approx \min \left\{ \left| \frac{\partial^{\alpha+\beta} \Delta\varphi(z)}{\partial z^\alpha \partial \bar{z}^\beta} \right|^{-\frac{1}{|\alpha|+|\beta|+2}} : \alpha, \beta \in \mathbb{N}^n \right\},$$

where the implicit constant depends on the degree of φ and n . To prove (14), one can observe that $\sup_{B(z, r)} \Delta\varphi \leq r^{-2}$ is equivalent to $\sum_{\alpha, \beta} r^{2+|\alpha|+|\beta|} \left| \frac{\partial^{\alpha+\beta} \Delta\varphi(z)}{\partial z^\alpha \partial \bar{z}^\beta} \right| \lesssim 1$, as a simple Taylor expansion reveals.

See also the proof of Theorem 6, where we establish a weaker μ -coercivity bound with μ of the form $1 + |z|^\delta + |w|^\delta$. By the considerations above, this gives pointwise bounds for weighted Bergman kernels associated to more general model monomial weights.

3. OUTLINE OF THE PROOFS

Proving Theorem 5 and Theorem 6 boils down to establishing μ -coercivity of certain weighted Kohn Laplacians for an appropriate μ . This is clear for Theorem 5, while it requires a little discussion in the case of Theorem 6.

First of all, the “only if” part of Theorem 6 follows from work of Haslinger and Helffer: Theorem 6.1 of [HH07] states that if the weight is decoupled, then the spectrum of the weighted Kohn Laplacian is not discrete (this works for a wide class of weights including polynomial ones). Thus we are reduced to proving the “if” part. In view of Lemma 3, it is enough to show that if φ is a non-decoupled model monomial weight with $(m, 0), (0, n) \in \Gamma$ for some $m, n \geq 2$, then \square_φ is μ -coercive for some μ diverging at infinity.

Thus, recalling (7), our goal is the estimate

$$(15) \quad \sum_{j=1}^2 \left(\int_{\mathbb{C}^2} \left| \frac{\partial u_j}{\partial \bar{z}} \right|^2 e^{-2\varphi} + \int_{\mathbb{C}^2} \left| \frac{\partial u_j}{\partial \bar{w}} \right|^2 e^{-2\varphi} \right) + 2 \int_{\mathbb{C}^2} (H_\varphi u, u) e^{-2\varphi} \gtrsim \int_{\mathbb{C}^2} \mu^2 |u|^2 e^{-2\varphi}$$

for every $u \in \mathcal{D}(\mathcal{E}_\varphi)$, in two cases:

- (a) when φ is a homogeneous model monomial weight and $\mu = 1 + |z|^\sigma + |w|^\tau$ (with σ and τ as in Section 2.2),
- (b) when φ is a non-decoupled model monomial weight with $(m, 0), (0, n) \in \Gamma$ for some $m, n \geq 2$, and μ diverging at infinity (dependent on φ).

Notice that we use the notation $A \gtrsim B$ to denote $A \geq cB$ for a constant c , which is allowed to depend only on the weight.

We now introduce the function

$$\lambda_\Gamma(z, w) := \min_{v \in \mathbb{C}^2 \setminus \{0\}} \frac{(H_{\varphi_\Gamma}(z, w)v, v)}{|v|^2},$$

which equals the minimal eigenvalue of the complex Hessian of φ_Γ , and set as our goal the estimate for scalar-valued functions:

$$(16) \quad \begin{aligned} & \int_{\mathbb{C}^2} \left| \frac{\partial u}{\partial \bar{z}} \right|^2 e^{-2\varphi} + \int_{\mathbb{C}^2} \left| \frac{\partial u}{\partial \bar{w}} \right|^2 e^{-2\varphi} + 2 \int_{\mathbb{C}^2} \lambda_\Gamma |u|^2 e^{-2\varphi} \\ & \gtrsim \int_{\mathbb{C}^2} \mu^2 |u|^2 e^{-2\varphi} \quad \forall u \in C_c^\infty(\mathbb{C}^2), \end{aligned}$$

with φ and μ as described above. The deduction of (15) from (16) is a simple approximation argument, which we omit.

The first consequence of (16) is that \square_{φ_Γ} is μ -coercive for $\mu \approx \sqrt{\lambda_\Gamma}$. Unfortunately, this is not enough to establish our theorems: for example, it will be clear later that λ_Γ never diverges at infinity. This situation has to be compared to that in the theory of Schrödinger operators where the operator $-\Delta + V$ has discrete spectrum even if the potential $V : \mathbb{R}^d \rightarrow [0, +\infty)$ does not diverge at infinity. Discreteness of the spectrum of $-\Delta + V$ is in fact well-known (see, e.g., [Iwa86]) to be equivalent to the energy estimate

$$(17) \quad \int_{\mathbb{R}^d} |\nabla u|^2 + \int_{\mathbb{R}^d} V|u|^2 \gtrsim \int_{\mathbb{R}^d} \mu^2 |u|^2,$$

for some μ diverging at infinity. This fact has to be compared with Lemma 3. In this context, it is in virtue of the uncertainty principle that we expect (17) to hold for some μ^2 larger than V .

Inspired by this similarity, we look for a *holomorphic uncertainty principle* that may serve an analogous purpose for our problem. Notice that the left hand of (16) differs from that of (17) in two relevant aspects: the presence of the weight and the nature of the “kinetic term”, which contains only the barred derivatives and is therefore weaker.

To turn these ideas into actual proofs, we proceed as follows:

- (1) In Section 4 we begin by showing that λ_Γ is comparable to a rational function of $|z|^2$ and $|w|^2$ that can be computed from Γ (Section 4.1). This is the part where we use the specific nature of model monomial weights. Then, thanks to this approximate formula and a linear optimization argument (Section 4.2), we split \mathbb{C}^2 in regions where λ_Γ is bounded from below by different functions of the form $|z|^{2a}|w|^{2b}$, where $a, b \in \mathbb{Q}$. If the weight is homogeneous we obtain sharp estimates (and as a consequence the statement of Theorem 5 is quantitative in nature).
- (2) In Section 5 we introduce our *holomorphic uncertainty principle* to take care of the regions where λ_Γ is too small.
- (3) Finally, in Section 6 we prove (16): outside of a hyperbolic neighborhood of the complex coordinate axes whose shape is dictated by the weight φ , we use the estimates of Section 4, while on this neighborhood we exploit the holomorphic uncertainty principle.

We would like to highlight the fact that the holomorphic uncertainty principle works for weights that are not necessarily polynomial, and in fact we believe that some more general formulation of it may hold and be useful for other problems as well.

4. ESTIMATING λ_Γ

4.1. Approximate formula for λ_Γ . Since $\varphi_\Gamma(z, w) = \sum_{(\alpha, \beta) \in \Gamma} |z|^{2\alpha} |w|^{2\beta}$, model weights only depend on the squared moduli of the coordinates. In view of this, we introduce the polynomial

$$(18) \quad p_\Gamma(x, y) := \sum_{(\alpha, \beta) \in \Gamma} x^\alpha y^\beta \quad (x, y) \in \mathbb{R}_{\geq 0}^2,$$

and in what follows we think of x and y both as independent non-negative variables and as denoting $|z|^2$ and $|w|^2$ respectively, so that

$$\varphi_\Gamma(z, w) = p_\Gamma(|z|^2, |w|^2) = p_\Gamma(x, y).$$

This ambiguity will not be a source of confusion.

We now prove a very useful formula for the determinant and the trace of the complex Hessian H_{φ_Γ} of a model monomial weight. In order to state it, we associate to any $\Gamma \subseteq \mathbb{N}^2$ four further subsets of \mathbb{N}^2 :

$$\begin{aligned} \Gamma_r &:= \{(\alpha, \beta) \in \Gamma : \alpha \neq 0\} \quad (r \text{ stands for "right"},) \\ \Gamma_u &:= \{(\alpha, \beta) \in \Gamma : \beta \neq 0\} \quad (u \text{ stands for "upper"},) \\ \Gamma^{(1)} &:= \{(\alpha, \beta) + (\gamma, \delta) : (\alpha, \beta), (\gamma, \delta) \in \Gamma \text{ linearly independent}\} - (1, 1), \\ \Gamma^{(2)} &:= [\Gamma_r - (1, 0)] \cup [\Gamma_u - (0, 1)]. \end{aligned}$$

Here $\Gamma_r - (1, 0)$ denotes the collection $\{(\alpha - 1, \beta) : (\alpha, \beta) \in \Gamma_r\}$, and the other symbols have analogous meanings. Observe that if (α, β) and (γ, δ) are linearly independent elements of \mathbb{N}^2 , then $(\alpha + \gamma - 1, \beta + \delta - 1) \in \mathbb{N}^2$, and hence $\Gamma^{(1)} \subseteq \mathbb{N}^2$.

Proposition 8. *If $\Gamma \subseteq \mathbb{N}^2$ is finite, then*

$$\begin{aligned} \det(H_{\varphi_\Gamma}(z, w)) &\approx \varphi_{\Gamma^{(1)}}(z, w), \\ \operatorname{tr}(H_{\varphi_\Gamma}(z, w)) &\approx \varphi_{\Gamma^{(2)}}(z, w), \end{aligned}$$

where the implicit constants depend only on Γ .

In particular, this proposition shows that the model monomial weight φ_Γ is weakly plurisubharmonic (i.e., λ_Γ vanishes) on the set where $\varphi_{\Gamma^{(1)}}$ vanishes. Since $\varphi_{\Gamma^{(1)}}$ is itself a monomial model weight, this set may be easily determined from $\Gamma^{(1)}$ and may be empty, the origin, a complex coordinate axes ($\{z = 0\}$ or $\{w = 0\}$), or $\{z = 0\} \cup \{w = 0\}$. We omit the elementary details.

Proof. Let $h_1, \dots, h_N : \mathbb{C}^2 \rightarrow \mathbb{C}$ be holomorphic functions and consider the weight

$$\varphi := \sum_{j=1}^N |h_j|^2.$$

We have (for the sake of simplicity, we denote $\frac{\partial}{\partial z}$ (resp. $\frac{\partial}{\partial \bar{z}}$) by ∂_z (resp. $\bar{\partial}_z$))

$$\partial_z \bar{\partial}_z \varphi = \sum_j |\partial_z h_j|^2, \quad \partial_w \bar{\partial}_w \varphi = \sum_j |\partial_w h_j|^2$$

and

$$\partial_z \bar{\partial}_w \varphi = \sum_j \partial_z h_j \overline{\partial_w h_j}, \quad \partial_w \bar{\partial}_z \varphi = \sum_j \partial_w h_j \overline{\partial_z h_j}.$$

Hence

$$\begin{aligned} \det(H_\varphi) &= \partial_z \bar{\partial}_z \varphi \cdot \partial_w \bar{\partial}_w \varphi - \partial_z \bar{\partial}_w \varphi \cdot \partial_w \bar{\partial}_z \varphi \\ &= \sum_{j,k} |\partial_z h_j|^2 |\partial_w h_k|^2 - \sum_{j,k} \partial_z h_j \overline{\partial_w h_j} \partial_w h_k \overline{\partial_z h_k} \\ &= \frac{1}{2} \left(\sum_{j,k} |\partial_z h_j|^2 |\partial_w h_k|^2 + |\partial_w h_j|^2 |\partial_z h_k|^2 - 2\Re(\partial_z h_j \overline{\partial_w h_j} \partial_w h_k \overline{\partial_z h_k}) \right) \\ &= \frac{1}{2} \sum_{j,k} |\partial_z h_j \partial_w h_k - \partial_w h_j \partial_z h_k|^2. \end{aligned}$$

We also have

$$\text{tr}(H_\varphi) = \partial_z \bar{\partial}_z \varphi + \partial_w \bar{\partial}_w \varphi = \sum_j |\partial_z h_j|^2 + |\partial_w h_j|^2.$$

Specializing to $\varphi_\Gamma(z, w) := \sum_{(\alpha, \beta) \in \Gamma} |z^\alpha w^\beta|^2$, we obtain (here l.i. stands for “linearly independent”)

$$\begin{aligned} \det(H_{\varphi_\Gamma}(z, w)) &= \frac{1}{2} \sum_{(\alpha, \beta), (\gamma, \delta) \in \Gamma} (\alpha\delta - \beta\gamma)^2 |z^{\alpha+\gamma-1} w^{\beta+\delta-1}|^2 \\ &\approx \sum_{\substack{(\alpha, \beta), (\gamma, \delta) \in \Gamma \\ \text{l.i.}}} |z^{\alpha+\gamma-1} w^{\beta+\delta-1}|^2 \\ &\approx \varphi_{\Gamma^{(1)}}(z, w) \end{aligned}$$

and

$$\begin{aligned} \text{tr}(H_{\varphi_\Gamma}(z, w)) &= \sum_{(\alpha, \beta) \in \Gamma} \alpha^2 |z^{\alpha-1} w^\beta|^2 + \beta^2 |z^\alpha w^{\beta-1}|^2 \\ &\approx \sum_{(\alpha, \beta) \in \Gamma: \alpha \neq 0} |z^{\alpha-1} w^\beta|^2 + \sum_{(\alpha, \beta) \in \Gamma: \beta \neq 0} |z^\alpha w^{\beta-1}|^2 \\ &\approx \varphi_{\Gamma^{(2)}}(z, w). \end{aligned}$$

It is easy to see that the implicit constants in the approximate equalities above depend only on Γ . \square

Since $\det(H_{\varphi_\Gamma})$ equals the product of the eigenvalues of H_{φ_Γ} , and $\text{tr}(H_{\varphi_\Gamma})$ equals their sum, by Proposition 8 we have

$$(19) \quad \varphi_{\Gamma^{(1)}}(z, w) \approx \lambda_\Gamma(z, w) \cdot \varphi_{\Gamma^{(2)}}(z, w),$$

for any finite $\Gamma \subseteq \mathbb{N}^2$.

4.2. A linear optimization argument to estimate λ_Γ . If $(u, v) \in \mathbb{R}^2$, we consider the curve in the non-negative quadrant $\mathbb{R}_{\geq 0}^2$:

$$C_{u,v}: \quad t \mapsto (t^u, t^v) \quad (t \geq 1).$$

Notice that if (u', v') is proportional to (u, v) , $C_{u,v}$ and $C_{u',v'}$ have the same range. If $A \subseteq \mathbb{N}^2$ is finite, using the notation (18) we have

$$p_A(C_{u,v}(t)) = \sum_{(\alpha,\beta) \in A} (t^u)^\alpha (t^v)^\beta = \sum_{(\alpha,\beta) \in A} t^{u\alpha+v\beta} \approx t^{m_{u,v}(A)} \quad (t \geq 1),$$

where $m_{u,v}(A)$ is the maximum of the linear functional $(\xi, \eta) \mapsto u\xi + v\eta$ on the set $A \subseteq \mathbb{R}^2$, and the implicit constant depends on A and is independent of u, v .

Since as (u, v) varies in \mathbb{R}^2 , the ranges of the curves $C_{u,v}$ cover $(0, +\infty)^2$, studying the family of optimization problems

$$\max_{(\xi,\eta) \in A} u\xi + v\eta \quad (u, v) \in \mathbb{R}^2,$$

one can estimate $p_A(x, y)$ from below and from above (up to multiplicative constants depending on A) on the whole positive quadrant. Notice that since A is finite, we have a partition of \mathbb{R}^2 depending on which point of A the maximum of $(\xi, \eta) \mapsto u\xi + v\eta$ is attained, and as a consequence, we get a partition of the positive quadrant on each part of which the polynomial p_A is comparable to a different monomial.

We are interested in estimating $\lambda_\Gamma(z, w)$ when $(x, y) = (|z|^2, |w|^2)$ lies on the curve $C_{u,v}$. By formula (19),

$$p_{\Gamma(1)}(|z|^2, |w|^2) \approx \lambda_\Gamma(z, w) \cdot p_{\Gamma(2)}(|z|^2, |w|^2).$$

If $(|z|^2, |w|^2) = (t^u, t^v) = C_{u,v}(t)$ ($t \geq 1$), the discussion above gives

$$(20) \quad \lambda_\Gamma(z, w) \approx t^{m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)})}.$$

We now present our analysis of the optimization problems of determining

$$m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) \quad \forall (u, v) \in \mathbb{R}^2,$$

first in the homogeneous case (Proposition 9), where we obtain more precise results, and then in the more general case of the weights appearing in Theorem 6 (Proposition 11).

Proposition 9. *Let φ_Γ be a homogeneous model monomial weight. Let*

$$m, n, \sigma, \tau, \alpha_1, \beta_1, \alpha_2, \beta_2$$

be as in Section 2.2. We define the three regions of \mathbb{C}^2 :

$$\begin{aligned} E_1 &:= \{|z| \geq 1, |w| \leq |z|^{\frac{m}{n}}\}, \\ E_2 &:= \{|w| \geq 1, |w|^\nu \leq |z| \leq |w|^{\frac{n}{m}}\}, \end{aligned}$$

and

$$E_3 := \{|w| \geq 1, |z| \leq |w|^\nu\},$$

where $\nu := \frac{n-1-\beta_2}{\alpha_2-1}$. The following approximate identities hold:

$$\begin{aligned} \lambda_\Gamma(z, w) &\approx |z|^{2\alpha_1} |w|^{2(\beta_1-1)} \quad \forall (z, w) \in E_1, \\ \lambda_\Gamma(z, w) &\approx |w|^{2(n-1)} \quad \forall (z, w) \in E_2, \\ \lambda_\Gamma(z, w) &\approx |z|^{2(\alpha_2-1)} |w|^{2\beta_2} \quad \forall (z, w) \in E_3. \end{aligned}$$

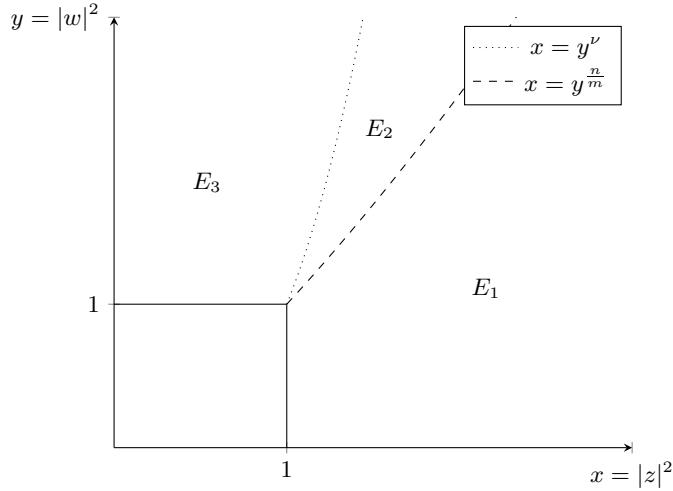


FIGURE 2

Figure 2 depicts the regions appearing in Proposition 9.

Proof. Observe that for homogeneous model monomial weights the definitions of $\Gamma^{(1)}$ and $\Gamma^{(2)}$ take the slightly simpler forms:

$$\begin{aligned}\Gamma^{(1)} &:= \{(\alpha + \gamma, \beta + \delta) : (\alpha, \beta) \neq (\gamma, \delta) \in \Gamma\} - (1, 1), \\ \Gamma^{(2)} &:= (\Gamma \setminus \{(0, n)\} - (1, 0)) \cup (\Gamma \setminus \{(m, 0)\} - (0, 1)).\end{aligned}$$

Fix $(u, v) \in \mathbb{R}^2$ such that either u or v is positive: the union of the corresponding family of curves $C_{u,v}$ is $E_1 \cup E_2 \cup E_3$. By convexity considerations it is clear that the maximum of $u\xi + v\eta$ on Γ is attained at $(m, 0)$ if $um \geq vn$, while it is attained at $(0, n)$ if $um \leq vn$. We separately analyze the two cases, assuming without loss of generality that $m \geq n$ (if needed, one may exchange the symbols “ z ” and “ w ”).

Case I: $v \leq \frac{m}{n}u$. We have

$$m_{u,v}(\Gamma^{(1)}) = um + m_{u,v}(\Gamma \setminus \{(m, 0)\}) - u - v$$

and

$$\begin{aligned}m_{u,v}(\Gamma^{(2)}) &= \max\{m_{u,v}(\Gamma \setminus \{(0, n)\}) - u, m_{u,v}(\Gamma \setminus \{(m, 0)\}) - v\} \\ &= \max\{um - u, m_{u,v}(\Gamma \setminus \{(m, 0)\}) - v\}.\end{aligned}$$

Hence

$$(21) \quad m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) = \min\{m_{u,v}(\Gamma \setminus \{(m, 0)\}) - v, u(m - 1)\}.$$

It is almost immediate to see that the maximum of $u\xi + v\eta$ on $\Gamma \setminus \{(m, 0)\}$ is attained at the point (α_1, β_1) satisfying $\frac{\alpha_1}{\beta_1} = \sigma$, which we introduced above. Identity (21) becomes

$$m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) = \min\{u\alpha_1 + v\beta_1 - v, u(m - 1)\}.$$

The inequality $u\alpha_1 + v\beta_1 - v \leq u(m - 1)$ holds if and only if

$$(22) \quad v \leq \frac{m - 1 - \alpha_1}{\beta_1 - 1}u.$$

This condition depends only on the ratio of u and v , as it should. Observe that

$$\frac{m}{n} \leq \frac{m-1-\alpha_1}{\beta_1-1}.$$

In fact, the inequality above is obviously equivalent to $mn-n \geq \alpha_1 n + m\beta_1 - m$, and recalling the homogeneity condition (8) we see that this is the same as $m \geq n$, which we assumed before. This shows that condition (22) is a consequence of $v \leq \frac{m}{n}u$ (since $u > 0$ in this case) and thus

$$(23) \quad m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) = u\alpha_1 + v\beta_1 - v.$$

Case II: $u \leq \frac{n}{m}v$. Proceeding analogously to the case $um \geq vn$, this time formula (21) is replaced by

$$m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) = \min\{m_{u,v}(\Gamma \setminus \{(0, n)\}) - u, v(n-1)\},$$

and the maximum of $ux+vy$ on $\Gamma \setminus \{(0, n)\}$ is attained at the point (α_2, β_2) satisfying $\frac{\beta_2}{\alpha_2} = \tau$. Hence

$$m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) = \min\{u\alpha_2 + v\beta_2 - u, v(n-1)\}.$$

Here is the difference from Case I: the minimum above equals $u\alpha_2 + v\beta_2 - u$ if and only if

$$u \leq \frac{n-1-\beta_2}{\alpha_2-1}v,$$

but this condition is not automatically implied by the inequality $u \leq \frac{n}{m}v$. In fact,

$$\frac{n-1-\beta_2}{\alpha_2-1} \leq \frac{n}{m},$$

as may be easily verified using (8) and the fact that $n \leq m$. Thus there are two further subcases: if

$$u \leq \frac{n-1-\beta_2}{\alpha_2-1}v,$$

then

$$(24) \quad m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) = u\alpha_2 + v\beta_2 - u,$$

while if

$$\frac{n-1-\beta_2}{\alpha_2-1}v \leq u \leq \frac{n}{m}v,$$

then

$$(25) \quad m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) = v(n-1).$$

Putting (20), (23), (24), and (25) together, we conclude the proof. \square

We state as a separate corollary the consequence of Proposition 9 that will be used in the proof of Theorem 5.

Corollary 10. *Let φ_Γ be a homogeneous model monomial weight. Let σ and τ be as in Section 2.2. If we define the region of \mathbb{C}^2 :*

$$E := \{|z| \geq 1, |w| \geq |z|^{-\sigma}\} \cup \{|w| \geq 1, |z| \geq |w|^{-\tau}\},$$

we have

$$\lambda_\Gamma(z, w) \gtrsim |z|^{2\sigma} + |w|^{2\tau} \quad \forall (z, w) \in E.$$

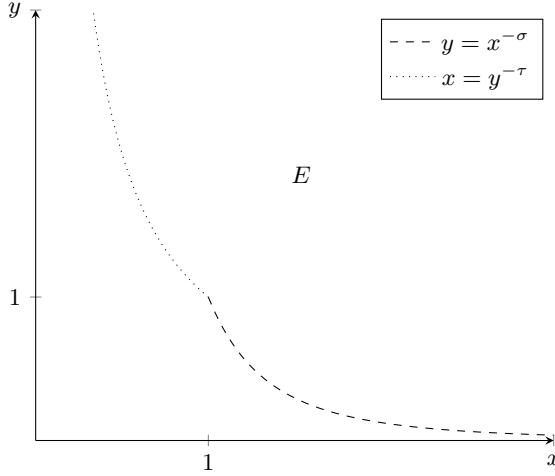


FIGURE 3

Figure 3 depicts the region E appearing in Proposition 11. Notice that its complement contains two hyperbolic neighborhoods of the coordinate axes.

Proof. Let $E_1, E_2, E_3, \sigma, \tau, \alpha_1, \beta_1, \alpha_2, \beta_2, m, n, \nu$ be as in Proposition 9.

Observe that $E_2 \subseteq E \subseteq E_1 \cup E_2 \cup E_3$ (this is apparent from the pictures, and comes from the fact that ν, m, n, σ and τ are all non-negative).

If $(z, w) \in E \cap E_1$, then $|z|^{\frac{m}{n}} \geq |w| \geq |z|^{-\sigma}$ and, by Proposition 9,

$$\lambda_\Gamma(z, w) \approx |z|^{2\alpha_1} |w|^{2(\beta_1-1)} \geq |z|^{2(\alpha_1-\sigma\beta_1+\sigma)} = |z|^{2\sigma}$$

and

$$\lambda_\Gamma(z, w) \approx |z|^{2\alpha_1} |w|^{2(\beta_1-1)} \geq |w|^{2(\frac{m}{n}\alpha_1+\beta_1-1)} = |w|^{2(n-1)}.$$

In the first identity we used the definition of (α_1, β_1) , while in the second one we used (8). Notice that $\tau = \frac{\beta_2}{\alpha_2} \leq \beta_2 \leq n-1$, and hence

$$(26) \quad \lambda_\Gamma(z, w) \gtrsim |z|^{2\sigma} + |w|^{2\tau} \quad \forall (z, w) \in E \cap E_1.$$

If $(z, w) \in E_2$, in particular $|z| \leq |w|^{\frac{n}{m}}$ and $|w|^{2(n-1)} \geq |z|^{2\frac{m}{n}(n-1)}$. Notice that $|w| \geq 1$ on E_2 and $\nu \geq 0$ and thus $|z|$ is also ≥ 1 . If we show that $\frac{m}{n}(n-1) \geq \sigma$, we can then deduce that $|w|^{2(n-1)} \geq |z|^{2\sigma}$. To prove the inequality above one can plug in the identity $\sigma = \frac{\alpha_1}{\beta_1}$ and use (8). This, together with Proposition 9 and the already observed fact that $\tau \leq n-1$, allows us to write that

$$(27) \quad \lambda_\Gamma(z, w) \gtrsim |w|^{2(n-1)} \gtrsim |z|^{2\sigma} + |w|^{2\tau} \quad \forall (z, w) \in E_2.$$

Finally, if $(z, w) \in E \cap E_3$, then in particular $|w|^{\frac{n}{m}} \geq |z| \geq |w|^{-\tau}$, and Proposition 9 yields

$$\lambda_\Gamma(z, w) \gtrsim |z|^{2(\alpha_2-1)} |w|^{2\beta_2} \geq |w|^{2(-\tau\alpha_2+\tau+\beta_2)} = |w|^{2\tau}$$

and

$$\lambda_\Gamma(z, w) \gtrsim |z|^{2(\alpha_2-1)} |w|^{2\beta_2} \geq |z|^{2(\alpha_2-1+\frac{m}{n}\beta_2)} = |z|^{2(m-1)}.$$

The last identity follows from (8). Since $\sigma = \frac{\alpha_1}{\beta_1} \leq \alpha_1 \leq (m-1)$, we have

$$(28) \quad \lambda_\Gamma(z, w) \gtrsim |z|^{2\sigma} + |w|^{2\tau} \quad \forall (z, w) \in E \cap E_3.$$

Putting (26), (27), and (28) together we obtain the thesis. \square

Let us proceed with the analysis of the weights appearing in Theorem 6.

Proposition 11. *Let φ_Γ be a non-decoupled model monomial weight such that $(m, 0), (0, n) \in \Gamma$ for some $m, n \geq 2$. Let σ and τ be as in Section 2.2. If we define the region of \mathbb{C}^2 :*

$$E := \{|z| \geq 1, |w| \geq |z|^{-\sigma}\} \cup \{|w| \geq 1, |z| \geq |w|^{-\tau}\},$$

we have

$$\lambda_\Gamma(z, w) \gtrsim |z|^{2\delta} + |w|^{2\delta} \quad \forall (z, w) \in E,$$

for some $\delta > 0$ depending on Γ .

As anticipated in Section 3, the bounds of Proposition 11 are not sharp in general, but they are sufficient for our purposes.

Proof. We are going to show that if $(u, v) \in \mathbb{R}^2$ is such that $u \geq 0$ and $v \geq -\sigma u$, then

$$(29) \quad m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) \geq \delta \max\{u, v\}.$$

Recalling (20), this proves that

$$\lambda_\Gamma(z, w) \gtrsim |z|^{2\delta} + |w|^{2\delta}$$

in the region $\{|z| \geq 1, |w| \geq |z|^{-\sigma}\}$. By symmetry, this also implies the same bound (29) in the region $\{|w| \geq 1, |z| \geq |w|^{-\tau}\}$, and hence the statement of the proposition.

To prove the claimed inequality, fix (u, v) satisfying the assumptions above. We distinguish three cases depending on whether the maximum of $u\xi + v\eta$ on Γ is attained on the x -axis, on the y -axis, or on $(\mathbb{N} \setminus \{0\})^2$. In the analysis of the first two cases it will be useful to denote by m and n the largest natural numbers such that $(m, 0) \in \Gamma$ and $(0, n) \in \Gamma$. By assumption, $m, n \geq 2$.

We denote by $\Gamma_{u,v}$ the subset of Γ whose elements are not multiples of a fixed maximizer of $u\xi + v\eta$ on Γ . It is easy to see that

$$(30) \quad m_{u,v}(\Gamma^{(1)}) = m_{u,v}(\Gamma) + m_{u,v}(\Gamma_{u,v}) - u - v.$$

Case I: $m_{u,v}(\Gamma)$ is attained on $\mathbb{N} \times \{0\}$. In this case $m_{u,v}(\Gamma) = mu$ and $\Gamma_{u,v} = \Gamma_u$ (recall the definition of Γ_u in Section 4.1). Moreover $m_{u,v}(\Gamma_r) = m_{u,v}(\Gamma) = mu$. By (30), we have

$$\begin{aligned} & m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) \\ &= mu + m_{u,v}(\Gamma_u) - u - v - \max\{mu - u, m_{u,v}(\Gamma_u) - v\} \\ &= \min\{m_{u,v}(\Gamma_u) - v, (m-1)u\}. \end{aligned}$$

Inequality (29) is equivalent to the following four inequalities:

$$(m-1)u \geq \delta u, \quad (m-1)u \geq \delta v, \quad m_{u,v}(\Gamma_u) - v \geq \delta v, \quad m_{u,v}(\Gamma_u) - v \geq \delta u.$$

Since $m \geq 2$, if we choose $\delta \leq 1$ the first inequality holds. Since the $m_{u,v}(\Gamma)$ is attained on the x -axis, we also have $mu \geq nv$, which implies the second one, if we choose $\delta \leq \frac{m-1}{m}n$.

To prove the third inequality, we distinguish two cases: $v \geq 0$ and $v < 0$.

If $v \geq 0$, we observe that $m_{u,v}(\Gamma_u) \geq nv$ (since $\{0\} \times \mathbb{N} \subseteq \Gamma_u$), and $nv \geq (1+\delta)v$, if we choose $\delta \leq n - 1$ (which is a positive quantity, by the assumption $n \geq 2$).

If $v < 0$, the inequality follows trivially from $m_{u,v}(\Gamma_u) \geq 0$. To see this, recall the assumption $v \geq -\sigma u$. Since φ_Γ is not decoupled, there is an element $(\alpha_1, \beta_1) \in \Gamma_u$ such that $\frac{\alpha_1}{\beta_1} = \sigma$. In particular, $m_{u,v}(\Gamma_u) \geq u\alpha_1 + v\beta_1 \geq 0$.

We are left with the fourth inequality. We observe that

$$m_{u,v}(\Gamma_u) = \max_{(\alpha,\beta) \in \Gamma_u} u\alpha + \beta v$$

is a continuous function of v (for u fixed) which is differentiable with derivative ≥ 1 outside a finite set. Since $\delta u + v$ has derivative 1, it is enough to prove the fourth inequality when $v = -\sigma u$. By definition of σ , $m_{u,-\sigma u}(\Gamma_u) = 0$, and our inequality becomes $\sigma u \geq \delta u$. Choosing $\delta \leq \sigma$, we are done.

This completes the analysis of the first case.

Case II: $m_{u,v}(\Gamma)$ is attained on $\{0\} \times \mathbb{N}$. In this case $v \geq 0$, and necessarily $u \geq -\tau v$ (recall that $u \geq 0$). Thus we can repeat the argument of Case I exchanging the role of the two variables u and v .

Case III: $m_{u,v}(\Gamma)$ is attained on $(\mathbb{N} \setminus \{0\})^2$. In this case $m_{u,v}(\Gamma) = m_{u,v}(\Gamma_r) = m_{u,v}(\Gamma_u)$, and

$$\begin{aligned} & m_{u,v}(\Gamma^{(1)}) - m_{u,v}(\Gamma^{(2)}) \\ &= m_{u,v}(\Gamma) + m_{u,v}(\Gamma_{u,v}) - u - v - \max\{m_{u,v}(\Gamma) - u, m_{u,v}(\Gamma) - v\} \\ &= m_{u,v}(\Gamma_{u,v}) - \max\{u, v\}. \end{aligned}$$

Since $\Gamma_{u,v}$ contains the coordinate axis, $m_{u,v}(\Gamma_{u,v}) \geq \max\{mu, nv\}$, and we can bound the expression above by $(\min\{m, n\} - 1) \max\{u, v\}$. This completes the analysis of Case III, and hence the proof of (29). \square

5. A HOLOMORPHIC UNCERTAINTY PRINCIPLE

Corollary 10 and Proposition 11 allow us to prove the μ -coercivity estimate (16) for every test function u supported on the set

$$E := \{|z| \geq 1, |w| \geq |z|^{-\sigma}\} \cup \{|w| \geq 1, |z| \geq |w|^{-\tau}\}$$

whenever either φ is a homogeneous model monomial weight and $\mu = 1 + |z|^\sigma + |w|^\tau$ or φ is a non-decoupled model monomial weight such that $(m, 0), (0, n) \in \Gamma$ (for some $m, n \geq 2$) and $\mu = 1 + |z|^\delta + |w|^\delta$ ($\delta > 0$).

To take care of the complement of this region, and thus to complete the proof of Theorem 5 and Theorem 6, we prove in this section the following lemma.

We denote by $D(z, r)$ the disc of center $z \in \mathbb{C}$ and radius r .

Lemma 12. *Let $V : D(z, r) \rightarrow [0, +\infty)$ be a measurable function and define*

$$c := \inf_{z' \in D(z, r) \setminus D(z, \frac{r}{2})} V(z').$$

If $f \in L^2(D(z, r))$ is such that $\frac{\partial f}{\partial \bar{z}} \in L^2(D(z, r))$, then

$$(31) \quad \int_{D(z, r)} \left| \frac{\partial f}{\partial \bar{z}} \right|^2 + \int_{D(z, r)} V|f|^2 \gtrsim \min \left\{ c, \frac{1}{r^2} \right\} \int_{D(z, r)} |f|^2.$$

The proof is based on a Poincaré-type inequality related to the $\frac{\partial}{\partial \bar{z}}$ operator and an elementary consequence of the Cauchy formula, which we now discuss.

Put $D := D(0, 1)$. It is well-known (cf., e.g., Theorem 4.3.4 of [CS01]) that $\frac{\partial}{\partial \bar{z}}$ is solvable in $L^2(D)$, i.e., that if $g \in L^2(D)$, then there exists $f \in L^2(D)$ such that $\frac{\partial f}{\partial \bar{z}} = g$ and

$$\int_D |f|^2 \lesssim \int_D |g|^2.$$

If $f \in L^2(D)$ is such that $\frac{\partial f}{\partial \bar{z}} \in L^2(D)$, the above solvability result yields $\tilde{f} \in L^2(D)$ such that $f - \tilde{f}$ is holomorphic and $\int_D |\tilde{f}|^2 \lesssim \int_D \left| \frac{\partial f}{\partial \bar{z}} \right|^2$. In particular, denoting by $B : L^2(D) \rightarrow L^2(D)$ the orthogonal projection onto the space of L^2 holomorphic functions (i.e., the unweighted Bergman projector of the unit disc), we have

$$(32) \quad \int_D |f - B(f)|^2 \leq \int_D |f - (\tilde{f})|^2 \lesssim \int_D \left| \frac{\partial f}{\partial \bar{z}} \right|^2.$$

This is the inequality we need. One should compare it with the usual Poincaré inequality in which $\frac{\partial}{\partial \bar{z}}$ is replaced by ∇ , and B by $\frac{1}{|D|} \int_D$. Of course one could rescale the estimate to apply it to an arbitrary disc.

The second ingredient is the following inequality, which holds for every holomorphic function $h : D \rightarrow \mathbb{C}$:

$$(33) \quad \int_D |h|^2 \lesssim \int_{D \setminus \frac{1}{2}D} |h|^2,$$

and which follows easily from the Cauchy integral formula.

Proof of Lemma 12. By a trivial rescaling it is enough to prove the lemma for $z = 0$ and $r = 1$. Then let $V : D \rightarrow [0, +\infty)$ be such that $V \geq c$ on $D \setminus \frac{1}{2}D$, and $f \in L^2(D)$ be such that $\frac{\partial f}{\partial \bar{z}} \in L^2(D)$. If $\varepsilon > 0$ is a small parameter to be fixed later and we write $f = f_i + f_e$, where f_e is zero on $\frac{1}{2}D$ and f_i is zero on $D \setminus \frac{1}{2}D$, we have the following two possibilities:

- (1) either $\int_D |f_e|^2 \geq \varepsilon \int_D |f_i|^2$,
- (2) or $\int_D |f_e|^2 < \varepsilon \int_D |f_i|^2$.

If (1) happens, a significant portion of the L^2 mass of f is contained in the corona $D \setminus \frac{1}{2}D$ and

$$\int_D |f_e|^2 \geq \frac{\varepsilon}{2} \int_D |f_i|^2 + \frac{1}{2} \int_D |f_e|^2 \geq \frac{\varepsilon}{2} \int_D |f|^2.$$

Therefore

$$\int_D \left| \frac{\partial f}{\partial \bar{z}} \right|^2 + \int_D V|f|^2 \geq \int_{D \setminus \frac{1}{2}D} V|f|^2 \geq c \int_D |f_e|^2 \geq c \frac{\varepsilon}{2} \int_D |f|^2,$$

and (31) holds.

If (2) happens, we use (32):

$$\int_D \left| \frac{\partial f}{\partial \bar{z}} \right|^2 + \int_D V|f|^2 \gtrsim \int_D |f - B(f)|^2.$$

By the linearity of B and condition (2), we have

$$\begin{aligned} \int_D |f - B(f)|^2 &\geq \frac{1}{2} \int_D |f_i - B(f_i)|^2 - \int_D |f_e - B(f_e)|^2 \\ &\geq \frac{1}{2} \int_D |f_i - B(f_i)|^2 - \int_D |f_e|^2 \\ &\geq \frac{1}{2} \int_D |f_i - B(f_i)|^2 - \varepsilon \int_D |f_i|^2. \end{aligned}$$

In the second line we used the fact that $1 - B$ is an orthogonal projection.

We claim that

$$(34) \quad \int_D |f_i - B(f_i)|^2 \geq a \int_D |f_i|^2,$$

where a is some small absolute constant.

Inequality (34) immediately implies, choosing $\varepsilon = \frac{a}{4}$, that

$$\int_D \left| \frac{\partial f}{\partial \bar{z}} \right|^2 + \int_D V|f|^2 \gtrsim \int_D |f_i|^2 \gtrsim \int_D |f|^2.$$

We are reduced to proving (34). In order to do this, we separate the two cases (for a new parameter δ):

- (1) $\int_{D \setminus \frac{1}{2}D} |B(f_i)|^2 \geq \delta \int_D |f_i|^2$,
- (2) $\int_{D \setminus \frac{1}{2}D} |B(f_i)|^2 < \delta \int_D |f_i|^2$.

If (1) holds, then

$$\int_D |f_i - B(f_i)|^2 \geq \int_{D \setminus \frac{1}{2}D} |f_i - B(f_i)|^2 = \int_{D \setminus \frac{1}{2}D} |B(f_i)|^2 \geq \delta \int_D |f_i|^2.$$

The equality above holds, because f_i vanishes on the region of integration by definition.

If (2) holds instead, we apply (33) to the holomorphic function $B(f_i)$ to deduce that $\int_D |B(f_i)|^2 \lesssim \delta \int_D |f_i|^2$. If we choose δ small enough we can write

$$\int_D |f_i - B(f_i)|^2 \geq \frac{1}{2} \int_D |f_i|^2 - \int_D |B(f_i)|^2 \geq \frac{1}{4} \int_D |f_i|^2.$$

This concludes the proof of (34). \square

Notice how the nature of the uncertainty principle of the previous result is revealed by its proof: it shows that a function f defined on a disc cannot be concentrated on a strictly smaller disc without having a large “holomorphic kinetic energy” $\int_{D(z,r)} \left| \frac{\partial f}{\partial \bar{z}} \right|^2$.

One should also compare Lemma 12 with the Fefferman-Phong inequalities (see, e.g., [Fef83] or [She99]), where c is replaced by some kind of average of V on the disc. Notice that one cannot hope for an improvement of Lemma 12 of the form:

$$(35) \quad \int_{D(z,r)} \left| \frac{\partial f}{\partial \bar{z}} \right|^2 + \int_{D(z,r)} V|f|^2 \gtrsim \frac{1}{r^2} \min \left\{ \int_{D(z,r)} V, 1 \right\} \int_{D(z,r)} |f|^2.$$

In fact, if $V \equiv 1$ on $D(0, \frac{1}{2})$ and $V \equiv 0$ on $D(0, 1) \setminus D(0, \frac{1}{2})$, we can test the hypothetical inequality (35) on $f(z) = z^m$ and obtain

$$\frac{4^{-m}}{m} \approx \int_{D(0, \frac{1}{2})} |z|^{2m} \gtrsim \int_{D(0, 1)} |z|^{2m} \approx \frac{1}{m},$$

which is a contradiction when m tends to $+\infty$.

6. ENERGY ESTIMATES

Let $\varphi = \varphi_\Gamma$ be a model monomial weight. As in Section 4, we put

$$E := \{|z| \geq 1, |w| \geq |z|^{-\sigma}\} \cup \{|w| \geq 1, |z| \geq |w|^{-\tau}\},$$

where $\sigma = \max_{(\alpha, \beta) \in \Gamma_u} \frac{\alpha}{\beta}$ and $\tau = \max_{(\alpha, \beta) \in \Gamma_r} \frac{\beta}{\alpha}$.

Proposition 13. *Assume that there are $a, b \geq 0$ such that*

$$(36) \quad \lambda_\Gamma(z, w) \gtrsim |z|^{2a} + |w|^{2b} \quad \forall (z, w) \in E.$$

Then

$$\int_{\mathbb{C}^2} \left| \frac{\partial u}{\partial \bar{z}} \right|^2 e^{-2\varphi} + \int_{\mathbb{C}^2} \left| \frac{\partial u}{\partial \bar{w}} \right|^2 e^{-2\varphi} + 2 \int_{\mathbb{C}^2} \lambda_\Gamma |u|^2 e^{-2\varphi} \gtrsim \int_{\mathbb{C}^2} (1 + |z|^a + |w|^b)^2 |u|^2 e^{-2\varphi},$$

for every $u \in C_c^\infty(\mathbb{C}^2)$.

Putting Corollary 10, Proposition 11, and Proposition 13 together, and recalling the discussion of Section 3, the reader can easily see that the proofs of Theorem 5 and Theorem 6 are completed.

Proof. We introduce the *uncertainty regions* $U_0 := \{0 \leq |z|, |w| \leq 2\}$,

$$U_r := \{|z| > 1, 0 \leq |w| \leq 2|z|^{-\sigma}\} \quad \text{and} \quad U_u := \{|w| > 1, 0 \leq |z| \leq 2|w|^{-\tau}\}.$$

If $u \in C_c^\infty(\mathbb{C}^2)$ and $\Omega \subseteq \mathbb{C}^2$, we put

$$F_\Omega(u) := \int_{\Omega} \left| \frac{\partial u}{\partial \bar{z}} \right|^2 e^{-2\varphi} + \int_{\Omega} \left| \frac{\partial u}{\partial \bar{w}} \right|^2 e^{-2\varphi} + 2 \int_{\Omega} \lambda_\Gamma |u|^2 e^{-2\varphi}.$$

We proceed by proving separately the estimate

$$(37) \quad F_\Omega(u) \gtrsim \int_{\Omega} (1 + |z|^a + |w|^b)^2 |u|^2 e^{-2\varphi},$$

when $\Omega = U_0$, U_r and U_u , the case $\Omega = E$ being trivially implied by the hypothesis (36).

Estimate for U_0 : By the trivial estimate $\sup_{(z, w) \in U_0} \varphi(z, w) \lesssim 1$, we have

$$F_{U_0}(u) \gtrsim \int_{U_0} \left(\left| \frac{\partial u}{\partial \bar{z}} \right|^2 + \left| \frac{\partial u}{\partial \bar{w}} \right|^2 \right) + \int_{U_0} \lambda_\Gamma |u|^2.$$

Since $U_0 = (2D) \times (2D)$, by Fubini's theorem and applying Lemma 12 twice, we get ($d\mathcal{L}$ denotes Lebesgue measure):

$$\begin{aligned}
& \int_{(2D) \times (2D)} \left(\left| \frac{\partial u}{\partial \bar{z}} \right|^2 + \left| \frac{\partial u}{\partial \bar{w}} \right|^2 \right) + \int_{(2D) \times (2D)} \lambda_\Gamma |u|^2 \\
&= \int_{2D} \int_{2D} \left(\left| \frac{\partial u}{\partial \bar{z}}(z, w) \right|^2 + \lambda_\Gamma(z, w) |u(z, w)|^2 \right) d\mathcal{L}(z) d\mathcal{L}(w) \\
&+ \int_{(2D) \times (2D)} \left| \frac{\partial u}{\partial \bar{w}} \right|^2 \\
&\gtrsim \int_{2D} \left\{ \left(\min_{1 < |z'| \leq 2} \lambda_\Gamma(z', w) \right) \int_{2D} |u(z, w)|^2 d\mathcal{L}(z) \right\} d\mathcal{L}(w) \\
&+ \int_{(2D) \times (2D)} \left| \frac{\partial u}{\partial \bar{w}} \right|^2 \\
&= \int_{2D} \int_{2D} \left(\left| \frac{\partial u}{\partial \bar{w}}(z, w) \right|^2 + \left(\min_{1 < |z'| \leq 2} \lambda_\Gamma(z', w) \right) |u(z, w)|^2 \right) d\mathcal{L}(w) d\mathcal{L}(z) \\
&\gtrsim \left(\min_{1 < |z'| \leq 2, 1 < |w'| \leq 2} \lambda_\Gamma(z', w') \right) \int_{2D} \int_{2D} |u(z, w)|^2 d\mathcal{L}(w) d\mathcal{L}(z).
\end{aligned}$$

Notice that $\{1 < |z'| \leq 2, 1 < |w'| \leq 2\} \subseteq E$, and hence (36) implies that the minimum above is ≈ 1 . Thus

$$F_{U_0}(u) \gtrsim \int_{U_0} |u|^2 \geq \int_{U_0} |u|^2 e^{-2\varphi} \gtrsim \int_{U_0} (1 + |z|^{2a} + |w|^{2b}) |u|^2 e^{-2\varphi},$$

where we used again the fact that $|z|, |w| \leq 2$ on U_0 .

Estimate for U_r : Notice that $\varphi = \varphi_{\Gamma_u} + \psi$, where ψ is a function of z alone.

By Fubini's theorem $F_{U_r}(u) \geq \int_{|z| \geq 1} I(z) e^{-2\psi(z)} d\mathcal{L}(z)$, where

$$I(z) := \int_{D(0, 2|z|^{-\sigma})} \left(\left| \frac{\partial u}{\partial \bar{w}}(z, w) \right|^2 + \lambda_\Gamma(z, w) |u(z, w)|^2 \right) e^{-2\varphi_{\Gamma_u}(z, w)} d\mathcal{L}(w).$$

By the definition of σ , if $(z, w) \in U_r$ and $(\alpha, \beta) \in \Gamma_u$, then

$$|z^\alpha w^\beta|^2 = \left(|z|^{\frac{\alpha}{\beta}} |w| \right)^{2\beta} \leq (|z|^\sigma |w|)^{2\beta} \lesssim 1,$$

where in the first inequality we used the fact that $|z| \geq 1$. Summing over $(\alpha, \beta) \in \Gamma_u$, we obtain

$$\sup_{(z, w) \in U_r} \varphi_{\Gamma_u}(z, w) \lesssim 1.$$

Using this bound and Lemma 12 we obtain, for every z of modulus greater than or equal to 1,

$$\begin{aligned}
I(z) &\gtrsim \int_{D(0, 2|z|^{-\sigma})} \left(\left| \frac{\partial u}{\partial \bar{w}}(z, w) \right|^2 + \lambda_\Gamma(z, w) |u(z, w)|^2 \right) d\mathcal{L}(w) \\
&\gtrsim \left(\min_{|z|^{-\sigma} < |w'| \leq 2|z|^{-\sigma}} \lambda_\Gamma(z, w') \right) \int_{D(0, 2|z|^{-\sigma})} |u(z, w)|^2 d\mathcal{L}(w).
\end{aligned}$$

Since the points (z, w) such that $|z| \geq 1$ and $|w| > |z|^{-\sigma}$ are contained in E , the hypothesis (36) gives

$$\min_{|z|^{-\sigma} < |w'| \leq 2|z|^{-\sigma}} \lambda_\Gamma(z, w') \approx \min_{|z|^{-\sigma} < |w'| \leq 2|z|^{-\sigma}} |z|^{2a} + |w'|^{2b} \approx |z|^{2a},$$

for every $|z| \geq 1$. We have

$$\begin{aligned} F_{U_r}(u) &\gtrsim \int_{|z| \geq 1} \left(|z|^{2a} \int_{D(0, 2|z|^{-\sigma})} |u(z, w)|^2 d\mathcal{L}(w) \right) e^{-2\psi(z)} d\mathcal{L}(z) \\ &\geq \int_{|z| \geq 1} \left(|z|^{2a} \int_{D(0, 2|z|^{-\sigma})} |u(z, w)|^2 e^{-2\varphi_{\Gamma_u}(z, w)} d\mathcal{L}(w) \right) e^{-2\psi(z)} d\mathcal{L}(z) \\ &\gtrsim \int_{U_r} |z|^{2a} |u(z, w)|^2 e^{-2\varphi(z, w)} d\mathcal{L}(z, w) \\ &\gtrsim \int_{U_r} (1 + |z|^{2a} + |w|^{2b}) |u(z, w)|^2 e^{-2\varphi(z, w)} d\mathcal{L}(z, w). \end{aligned}$$

The last step follows from the inequalities $|w| \leq 1$ and $|z| \geq 1$, which hold for any $(z, w) \in U_r$.

Estimate for U_u : This is done in complete analogy with the estimate for U_r , exchanging the role played by z and w , and replacing σ with τ .

The proof is complete. \square

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