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MAXIMAL OPERATORS ASSOCIATED WITH SOME SINGULAR SUBMANIFOLDS

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Abstract. Let U be a bounded open subset of $\mathbb{R}^d$ and let Ω be a Lebesgue measurable subset of U. Let $\gamma = (\gamma_1, \cdots, \gamma_n) : \mathrm{U} \setminus \Omega \to \mathbb{R}^n$ be a Lebesgue measurable function, and let μ be a Borel measure on $\mathbb{R}^{d+n}$ defined by

$$\langle \mu, f \rangle = \int_{\mathbb{R}^d} f(y, \gamma(y))\psi(y)\, \chi_{\mathrm{U}\setminus\Omega}(y)\, dy,$$

where ψ is a smooth function supported in U. In this paper we give some conditions under which the Fourier decay estimates $|\widehat{\mu}(\xi)| \le C(1+|\xi|)^{-\epsilon}$ hold for some $\epsilon > 0$. As a corollary we obtain the L^p-boundedness properties of the maximal operators M_S associated with a certain class of possibly non-smooth n-dimensional submanifolds of $\mathbb{R}^{d+n}$, i.e.,

$$\mathrm{M}_S f(x) = \sup_{r>0} r^{-d} \int_{|y|<r} \big|f\big(x - (y, \gamma(y))\big)\big|\, \chi_{\mathbb{R}^d\setminus\Omega_{\mathrm{sym}}}\, dy,$$

where Ω_{sym} is a radially symmetric Lebesgue measurable subset of $\mathbb{R}^d$, $\gamma(y) = (\gamma_1(y), \cdots, \gamma_n(y))$, $\gamma_i(ty) = t^{a_i}\gamma_i(y)$ for each $t > 0$ where $a_i \in \mathbb{R}$, and the function $\gamma_i : \mathbb{R}^d \setminus \Omega_{\mathrm{sym}} \to \mathbb{R}$ satisfies some singularity conditions over a certain subset of $\mathbb{R}^d$. Also we investigate the endpoint (*parabolic* $H^1, L^{1,\infty}$) mapping properties of the maximal operators M_H associated with a certain class of possibly non-smooth hypersurfaces, i.e.,

$$\mathrm{M}_H f(x) = \sup_{r>0} \left| \int_{\mathbb{R}^d} f\big(x - (y, \gamma(y))\big) r^{-d} \psi(r^{-1}y)\, dy \right|,$$

where the function $\gamma : \mathbb{R}^d \to \mathbb{R}$ satisfies some singularity conditions over a certain subset of $\mathbb{R}^d$ and $\gamma(ty) = t^m\gamma(y)$ for each $t > 0$ where $m > 0$.

1. Introduction and statements of results

In this paper we consider the L^p boundedness for the maximal operators associated with some possibly non-smooth n-dimensional submanifolds of $\mathbb{R}^{d+n}$. In particular, we obtain near L^1 boundedness for the maximal operators associated with some possibly non-smooth hypersurfaces by constructing appropriate exceptional sets. Our maximal operators are related to lacunary maximal operators which were mainly studied by Ricci and Stein [9] (see also [3]).

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Theorem (Ricci and Stein [9]). *Let ν be a positive measure supported on the unit cube, such that*

$$|\widehat{\nu}(\xi)| \le C(1+|\xi|)^{-\epsilon} \quad \text{for some } \epsilon > 0.$$

Then the multi-parameter lacunary maximal operator

$$Mf(x) = \sup_{(i_1,\cdots,i_n)\in\mathbb{Z}^n} \Big| \int_{\mathbb{R}^n} f\big(x-(2^{i_1}y_1,\cdots,2^{i_n}y_n)\big)\, d\nu(y)\Big|, \quad y=(y_1,\cdots,y_n),$$

is bounded on $L^p(\mathbb{R}^n)$ for $1<p\le\infty$.

When $p=1$, even for the one-parameter case, a complete picture has not been drawn for the lacunary maximal operator M. Most of the results for the case $p=1$ have been focused on one-parameter lacunary maximal operators with suitable conditions for hypersurfaces. The first meaningful breakthrough was made by M. Christ [2] in 1988.

• **Model case** (t,t^2). Let ν be a measure on $\mathbb{R}^2$ given by

$$\langle \nu, f\rangle = \int f(t,t^2)\,\psi(t)\,dt$$

where ψ is a smooth function supported in the open interval $(1,2)$. Then the maximal operator

$$Tf(x) = \sup_{i\in\mathbb{Z}} \Big| \int_{\mathbb{R}^2} f\big(x-(2^i y_1, 2^{2i}y_2)\big)\, d\nu(y)\Big|$$

maps the parabolic Hardy space $H^1(\mathbb{R}^2)$ into weak-$L^1(\mathbb{R}^2)$. In fact, M. Christ [2] obtained celebrated results for full maximal operators. Also it was stated that the one-parameter lacunary maximal operator

$$Mf(x) = \sup_{i\in\mathbb{Z}} \Big| \int_{\mathbb{R}^d} f(x-2^i y)\, d\nu(y)\Big| \tag{1.1}$$

maps the standard Hardy space $H^1(\mathbb{R}^d)$ into weak-$L^1(\mathbb{R}^d)$ when ν is the Lebesgue measure σ_{d-1} on the unit sphere $\mathbb{S}^{d-1}$. It is not known whether the maximal operator T or M maps L^1 into weak-L^1 or not.

• **Fourier decay and geometric conditions.** In 1996, D. M. Oberlin [8] obtained $(H^1, \text{weak-}L^1)$ results for lacunary maximal operators M in (1.1) when ν is a compactly supported finite Borel measure on $\mathbb{R}^d$ with the Fourier decay condition

$$|\widehat{\nu}(\xi)| \le C(1+|\xi|)^{-d/2}.$$

In 2001, Y. Heo [4] obtained $(H^1, \text{weak-}L^1)$ results for lacunary maximal operators M in (1.1) when ν is a compactly supported finite Borel measure on $\mathbb{R}^d$ with the conditions:

$$\sup_{0<\epsilon<1} \epsilon^n \mathcal{N}(\text{supp}(\nu);\epsilon) < \infty \quad \text{and} \quad |\widehat{\nu}(\xi)| \le C(1+|\xi|)^{-n/2} \quad \text{for some} \quad 0<n\le d,$$

where $\mathcal{N}(A;\epsilon)$ denotes the smallest number of ϵ-balls to cover A.

In 2011, Seeger and Wright [12] obtained $(H^1, \text{weak-}L^1)$ results for lacunary maximal operators M in (1.1) when ν is a compactly supported finite Borel measure on $\mathbb{R}^d$ that satisfies the conditions:

(1) $\sup_{0<\epsilon<1} \epsilon^{d-s}\,\mathcal{N}(\text{supp}(\nu);\ \epsilon) < \infty$;
(2) $\sup_{k>0} 2^{k(d-s)(1-\frac{1}{p})}\, \|P_k\mathcal{A}\|_{L^p\to L^p} < \infty$ for some $p>1$ and $0<s\le d$,

where the operator P_k is defined by $\widehat{P_k f}(\xi) = \psi(2^{-k}|\xi|)\,\widehat{f}(\xi)$ with $\psi \in C_c^\infty(\mathbb{R})$ and $\mathcal{A}$ denotes the convolution operator $f \to f * \nu$.

• **Curvature conditions.** Let $\mathfrak{S}$ be a compact smooth hypersurface of $\mathbb{R}^d$ and let ν be a compactly supported smooth density on $\mathfrak{S}$, that is, $\nu = \chi d\sigma$ where $\chi \in C_c^\infty(\mathbb{R}^d)$ and $d\sigma$ is the surface measure on $\mathfrak{S}$. In 2004, A. Seeger, T. Tao and J. Wright [11] obtained weak type $L \log\log L$ results for one-parameter maximal operators (1.1) associated with ν when the Gaussian curvature K of $\mathfrak{S}$ does not vanish to infinite order.

Notation. Given two quantities A and B we write $A \lesssim B$ if $A \le CB$ for some positive constant C, depending on the dimension and possibly other parameters apparent from the context. We write $A \sim B$ if $A \lesssim B$ and $A \gtrsim B$. For each rectangle $Q \subset \mathbb{R}^d$, Q^* denotes the double of the rectangle Q. The Lebesgue measure on $\mathbb{R}^d$ of a subset E will be denoted by $\mathrm{meas}(E)$ or $|E|$. For each finite set F, $\mathrm{card}(F)$ denotes the number of elements in the set F.

In the above results, regularity of submanifolds up to appropriate order is necessary. The purpose of this paper is to develop arguments to handle the case that the submanifold has possible singularities. Let U be a bounded open subset of $\mathbb{R}^d$ and Ω be a Lebesgue measurable subset of U. Let $\gamma = (\gamma_1, \cdots, \gamma_n) : \mathrm{U} \setminus \Omega \to \mathbb{R}^n$ be a Lebesgue measurable function, and let μ be a Borel measure on $\mathbb{R}^{d+n}$ defined by

$$\langle \mu, f \rangle = \int_{\mathbb{R}^d} f(y, \gamma(y))\psi(y)\chi_{\mathrm{U}\setminus\Omega}(y)\, dy, \tag{1.2}$$

where ψ is a smooth function supported in U and χ_A is the characteristic function of A. In this paper we describe sufficient conditions under which the inequality $|\widehat{\mu}(\xi)| \le C(1+|\xi|)^{-\epsilon}$ holds for some $\epsilon > 0$. Let $\theta \in \mathbb{S}^{n-1}$. For each $1 \le \ell \le d$ and $1 \le i_1 < \cdots < i_\ell \le d$, let

$$\mathbf{D}_{(i_1,\cdots,i_\ell)}(\theta, y) = \begin{pmatrix} \partial^2_{y_{i_1}y_{i_1}}[\theta\cdot\gamma(y)] & \partial^2_{y_{i_1}y_{i_2}}[\theta\cdot\gamma(y)] & \cdots & \partial^2_{y_{i_1}y_{i_\ell}}[\theta\cdot\gamma(y)] \\ \partial^2_{y_{i_2}y_{i_1}}[\theta\cdot\gamma(y)] & \partial^2_{y_{i_2}y_{i_2}}[\theta\cdot\gamma(y)] & \cdots & \partial^2_{y_{i_2}y_{i_\ell}}[\theta\cdot\gamma(y)] \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2_{y_{i_\ell}y_{i_1}}[\theta\cdot\gamma(y)] & \partial^2_{y_{i_\ell}y_{i_2}}[\theta\cdot\gamma(y)] & \cdots & \partial^2_{y_{i_\ell}y_{i_\ell}}[\theta\cdot\gamma(y)] \end{pmatrix}.$$

We assume that the Lebesgue measurable function γ satisfies the following singularity conditions.

1.1. **Singularity conditions for γ.** Let U be a bounded open subset of $\mathbb{R}^d$ and Ω be a Lebesgue measurable subset of U. Let Σ be a closed subset of U that contains Ω.

(i) **Geometric conditions for the domain.** We assume $\gamma \in C^4_{\mathrm{loc}}(\mathrm{U} \setminus \Sigma)$. If Σ is not empty, then there exists $d_0 > 0$ such that

$$\mathrm{meas}\,(\Sigma_\delta \setminus \Omega) \le C\delta^{d_0} \quad \text{for all } 0 < \delta \ll 1, \tag{1.3}$$

where $\Sigma_\delta := \{y : \mathrm{dist}(\Sigma, y) < \delta\}$.

(ii) **Derivative conditions.** If $\Sigma \ne \emptyset$, then there exists $N > 0$ such that

$$|\partial_y^\alpha \gamma(y)| \le C_\alpha\, \mathrm{dist}(y, \Sigma)^{-N(|\alpha|+1)} \tag{1.4}$$

for all $y \in \mathrm{U} \setminus \Sigma$ and $|\alpha| \le 4$.

(iii) **Directional sub-Hessian conditions.** There exists $\eta > 0$ such that for every $\theta \in \mathbb{S}^{n-1}$, there exist $1 \le \ell = \ell(\theta) \le d$ and $1 \le i_1(\theta) < i_2(\theta) < \cdots < i_\ell(\theta) \le d$ satisfying

$$\int_{\mathrm{U}\setminus\Sigma} \left|\det\big(\mathbf{D}_{(i_1,\cdots,i_\ell)}(\theta,y)\big)\right|^{-\eta} dy < C \tag{1.5}$$

with C independent of θ.

Definition 1.1. We denote by $\mathcal{SH}_{\mathbb{R}^d\to\mathbb{R}^n}(\mathrm{U},\Omega,\Sigma)$ the collection of Lebesgue measurable functions $\gamma : \mathrm{U}\setminus\Omega \to \mathbb{R}^n$ which satisfy the singularity conditions (i), (ii), and (iii) for some d_0, N, and η.

We now state the main theorems in the following subsection.

1.2. Statements of main theorems.

Theorem 1. *Let $\gamma \in \mathcal{SH}_{\mathbb{R}^d\to\mathbb{R}^n}(\mathrm{U},\Omega,\Sigma)$ and let μ be the Borel measure as in* (1.2). *Then there exists $\epsilon > 0$ such that*

$$|\widehat{\mu}(\xi)| \le C_\epsilon(1+|\xi|)^{-\epsilon},$$

where $\widehat{\mu}$ denotes the Fourier transform of μ.

Next we assume that Ω_{sym} is a radially symmetric Lebesgue measurable subset of $\mathbb{R}^d$, that is,

$$y \in \Omega_{\mathrm{sym}} \;\Rightarrow\; ry \in \Omega_{\mathrm{sym}} \quad \text{for all } r > 0.$$

For each $1 \le i \le n$, let $\gamma_i : \mathbb{R}^d\setminus\Omega_{\mathrm{sym}} \to \mathbb{R}$ be Lebesgue measurable functions. Then we consider the maximal operators M_S given by

$$\mathrm{M}_S f(x) = \sup_{r>0} r^{-d} \int_{|y|<r} \big|f\big(x-(y,\gamma(y))\big)\big|\, \chi_{\mathbb{R}^d\setminus\Omega_{\mathrm{sym}}}(y)\, dy, \tag{1.6}$$

where $\gamma(y) = (\gamma_1(y),\cdots,\gamma_n(y))$. If f is bounded and Borel measurable, then $\mathrm{M}_S f$ is a well-defined Borel measurable function (see Appendix 6.1). The maximal operators $\mathrm{M}_S f$ are pointwisely dominated by some lacunary maximal operators associated with the Borel measure μ on $\mathbb{R}^{d+n}$ defined by

$$\langle \mu, f\rangle = \int_{\mathbb{R}^d} f(y,\gamma(y))\psi(y)\chi_{\mathrm{U}\setminus\Omega}(y)\, dy,$$

where ψ is a positive smooth function supported on the annulus $\mathrm{U} = \{y \in \mathbb{R}^d : 1/2 < |y| < 4\}$ and $\Omega = \mathrm{U}\cap\Omega_{\mathrm{sym}}$.

Corollary 1. *For each $1 \le i \le n$, we assume that $\gamma_i(ty) = t^{a_i}\gamma_i(y)$ for all $t > 0$ where $a_i \in \mathbb{R}$. Suppose that $\gamma \in \mathcal{SH}_{\mathbb{R}^d\to\mathbb{R}^n}(\mathrm{U},\Omega,\Sigma)$ with $\mathrm{U} = \{y \in \mathbb{R}^d : 1/2 < |y| < 4\}$, $\Omega = \mathrm{U}\cap\Omega_{\mathrm{sym}}$, and*

$$\sup_{y\in\mathrm{U}\setminus\Omega} |\gamma(y)| \le C.$$

Then for $1 < p < \infty$ we have the a priori inequality

$$\|\mathrm{M}_S f\|_{L^p(\mathbb{R}^{d+n})} \le C_p \|f\|_{L^p(\mathbb{R}^{d+n})},$$

whenever f is bounded and Borel measurable.

Proof of Corollary 1. By using that $\Omega_{\rm sym}$ is radially symmetric, we have

$$\mathrm{M}_S f(x) \le \frac{1}{1-2^{-d}} \sup_{k\in\mathbb{Z}} 2^{-kd} \int \big|f\big(x-(y,\gamma(y))\big)\big|\, \psi(2^{-k}y)\, \chi_{\mathbb{R}^d\setminus\Omega_{\rm sym}}(y)\, dy \tag{1.7}$$

where ψ is a positive smooth function supported on $\mathrm{U} := \{y \in \mathbb{R}^d : 1/2 < |y| < 4\}$ and $\psi \equiv 1$ on $1 \le |y| < 2$. Let μ be the finite Borel measure on $\mathbb{R}^{d+n}$ that is defined by

$$\langle \mu, f\rangle = \int_{\mathbb{R}^d} f\big(y,\gamma(y)\big)\, \psi(y)\, \chi_{\mathrm{U}\setminus\Omega}(y)\, dy.$$

Since $\sup_{y\in\mathrm{U}\setminus\Omega} |\gamma(y)| < \infty$, the measure μ is compactly supported. For each $1 \le i \le n$, if we assume that $\gamma_i(ty) = t^{a_i}\gamma_i(y)$ for all $t > 0$ where $a_i \in \mathbb{R}$, then by (1.7) together with the fact that $\Omega_{\rm sym}$ is radially symmetric, we have

$$\mathrm{M}_S f(x) \le \frac{1}{1-2^{-d}} \sup_{k\in\mathbb{Z}} \int \big|f\big(x-(2^k y, 2^{a_1 k} z_1, \cdots, 2^{a_n k} z_n)\big)\big|\, d\mu(y,z),$$

where $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$ and $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$. By Theorem 1, we have

$$|\widehat{\mu}(\xi)| \le C(1+|\xi|)^{-\epsilon} \quad \text{for some } \epsilon > 0.$$

And by applying a theorem of Ricci and Stein [9], M_S is an L^p-bounded operator for $p > 1$. □

Next let $\gamma : \mathbb{R}^d \to \mathbb{R}$ be a Lebesgue measurable function that satisfies some singularity conditions over a certain subset of $\mathbb{R}^d$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then we consider the maximal operators given by

$$\mathrm{M}_H f(x) = \sup_{r>0} \Big| \int_{\mathbb{R}^d} f\big(x-(y,\gamma(y))\big)\, r^{-d}\psi(r^{-1}y)\, dy\Big|.$$

If f is bounded and Borel measurable, then $\mathrm{M}_H f$ is a Borel measurable function (see Appendix 6.1).

Notation. Let $m > 0$ be a real number. As shown by A.-P. Calderón [1], let $H^1_m(\mathbb{R}^{d+1})$ be the parabolic $H^1(\mathbb{R}^{d+1})$, that is, $H^1_m(\mathbb{R}^{d+1})$ denotes the one-parameter Hardy space with respect to the scaling

$$x = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \to (tx', t^m x_{d+1}), \quad t > 0.$$

For the sake of self-containedness we state the definition in [1]:

For each non-zero $x = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R}$, let $\rho(x)$ denote the unique $t > 0$ such that

$$\big(|t^{-1}x'|^2 + |t^{-m}x_{d+1}|^2\big)^{1/2} = 1.$$

Let $\rho(0) = 0$; then the function $\rho : \mathbb{R}^{d+1} \to \mathbb{R}$ is a norm which satisfies $\rho(tx', t^m x_{d+1}) = t\rho(x', x_{d+1})$ for $t > 0$. For $r > 0$, put $B_r(x) = \{y : \rho(x-y) < r\}$. For $\varphi \in \mathcal{S}(\mathbb{R}^{d+1})$ and $t > 0$, we define $\varphi_t(x) = t^{-(d+m)}\varphi(t^{-1}x', t^{-m}x_{d+1})$. When $\int \varphi \ne 0$ and f is a distribution, we define a maximal function

$$M_\alpha f(x) = \sup_{\rho(y)<\alpha t} |(f * \varphi_t)(x+y)|, \quad \alpha > 0.$$

We say that $f \in H^1_m(\mathbb{R}^{d+1})$ if $M_\alpha f \in L^1$ and $\|f\|_{H^1_m} := \|M_\alpha f\|_{L^1}$. It is known that for any other choice of φ and α we obtain the same space H^1_m and an equivalent

norm $\|\cdot\|_{H^1_m}$. An H^1_m atom is a function a which is supported on a ball $B_r(x_0)$ and satisfies

$$|a(x)| \le |B_r(x_0)|^{-1}, \qquad \int a(x)dx = 0.$$

Theorem 2. *Let $m > 0$ be a real number, and assume that $\gamma(ty) = t^m\gamma(y)$ for each $t > 0$. Suppose that $\gamma \in \mathcal{SH}_{\mathbb{R}^d\to\mathbb{R}}(\mathrm{U}, \Omega, \Sigma)$ with $\mathrm{U} = \{y \in \mathbb{R}^d : 1/2 < |y| < 4\}$, $\Omega = \emptyset$, and satisfies the following conditions:*

(iii′) ***Hessian condition.*** *There exists $\eta > 0$ such that*

$$\int_{\mathrm{U}\setminus\Sigma} |\det(\gamma''(y))|^{-\eta}\, dy < \infty. \tag{1.8}$$

(iv) ***Geometric conditions for the range.***

$$\mathrm{meas}\Big(B(0,\delta) + \big\{(y,\gamma(y)) : |y| \le 1\big\}\Big) \le C\delta \quad \textit{for } 0 < \delta \ll 1. \tag{1.9}$$

Then

$$\|\mathrm{M}_H f\|_{L^{1,\infty}(\mathbb{R}^{d+1})} \le C\|f\|_{H^1_m(\mathbb{R}^{d+1})},$$

whenever f is bounded and Borel measurable.

For the proof of Theorem 2, we develop the ideas in [2] to define a certain collection of dyadic rectangles $\mathcal{R}_{\sigma,\tau}$ for each $\sigma, \tau \in \mathbb{Z}$ with $\sigma \le \tau$. For each $q \in \mathcal{R}_{\sigma,\tau}$ we construct the "*tendril*" $T(q)$ that reflects the geometry of the graph $(y, \gamma(y))$. We then establish appropriate exceptional sets by using stopping time arguments based on the "*tendrils*". For the case $m > 1$, the constructions of the tendrils and exceptional sets are similar to those in [2]. But for the case $0 < m < 1$, the geometry of the graph $(y, \gamma(y))$ is different from that of the case $m > 1$, so it is necessary to modify the "*tendril*" constructions of the previous case $m > 1$. Let μ be a measure defined by

$$\langle f, \mu\rangle = \int_{\mathbb{R}^d} f\big(y, \gamma(y)\big)\, \psi(y)\, dy,$$

where ψ is a smooth function supported in $|y| \sim 1$. As in [2], outside the exceptional sets we use L^2 estimates based on the decay estimates of $\mu * \widetilde{\mu}$ and $\nabla(\mu * \widetilde{\mu})$ where $\widetilde{\mu}$ denotes the reflection of μ about the origin. If $\{(y, \gamma(y)) : |y| \sim 1\}$ is a smooth hypersurface with non-vanishing Gaussian curvature, then $|\mu * \widetilde{\mu}(x)| \le C|x|^{-1}$ and $|\nabla(\mu * \widetilde{\mu})(x)| \le C|x|^{-2}$. Thus the operators of these types can be handled by similar arguments as in [2]. But our hypersurfaces may not be smooth, and may have singularities on a subset Σ. To conquer these difficulties, for each $\beta > 0$ we construct a partition of unity

$$\mu = \sum_{\upsilon\in\zeta} \mu_\upsilon$$

with a finite index set ζ, where $\mathrm{supp}(\mu_\upsilon) \subset \big\{(y, \gamma(y)) : y \in B(x_\upsilon, \delta\beta^{N'})\big\}$ with fixed $N' \ge N(3d + 1) + 3$ and $\delta > 0$. We then split the finite index set ζ into disjoint subsets ζ_1, ζ_2, and ζ_3 having the following properties:

$$\begin{cases} \zeta_1 := \big\{\upsilon\in\zeta : \mathrm{dist}(\Sigma, x_\upsilon) \le \beta\big\}; \\ \zeta_2 := \big\{\upsilon\in\zeta : \mathrm{dist}(\Sigma, x_\upsilon) > \beta \text{ and } |\det(\gamma''(y))| \le \beta \text{ for all } y \in B(x_\upsilon, \delta\beta^{N'})\big\}; \\ \zeta_3 := \big\{\upsilon\in\zeta : \mathrm{dist}(\Sigma, x_\upsilon) > \beta \text{ and } |\det(\gamma''(y))| > \beta \text{ for some } y \in B(x_\upsilon, \delta\beta^{N'})\big\}. \end{cases}$$

For ζ_1 we use the geometric conditions for the domain (1.3) to obtain $\sum_{\upsilon\in\zeta_1} \|\mu_\upsilon\|_{\mathcal{M}} \le C\beta^{d_0}$ where $\mathcal{M}$ denotes the space of finite Borel measures. For ζ_2 we apply the

Hessian condition (1.8) to obtain $\sum_{\upsilon\in\zeta_2}\|\mu_\upsilon\|_{\mathcal{M}}\le C\beta^\eta$. For ζ_3 we estimate $\mu_\upsilon*\widetilde{\mu_\upsilon}$ by using the derivative conditions (1.4). For the estimate $\mu_\upsilon*\widetilde{\mu_\upsilon}\,(\upsilon\in\zeta_3)$, see Lemma 3.5 which is one of our key estimates.

Structure of the paper. In Section 2 we will prove Theorem 1. In Section 3, we will develop two Calderón-Zygmund type arguments whose proofs are based on those of M. Christ [2], and prove Lemma 3.5 that is crucial for the proofs. The proof of Theorem 2 will be considered in two cases, $m\ge 1$ and $0<m<1$. The case $m\ge 1$ will be considered in Section 3, and the case $0<m<1$ in Section 4. In Section 5 we give several examples related to each theorem.

2. Proof of Theorem 1

Since the proof of the case $\Sigma=\emptyset$ is easier than that of the case $\Sigma\ne\emptyset$, we only consider the case $\Sigma\ne\emptyset$. Let $\xi=(\xi',\xi'')\in\mathbb{R}^d\times\mathbb{R}^n$ be fixed with $|\xi|\gg 1$. Since ψ is supported in U, $\widehat{\mu}$ can be written as

$$\widehat{\mu}(\xi)=\int_{\mathbb{R}^d}\mathrm{e}^{-2\pi\mathrm{i}(\xi'\cdot y+\xi''\cdot\gamma(y))}\psi(y)\,\chi_{\mathbb{R}^d\setminus\Omega}(y)\,dy.$$

Let $N>0$ be as in (1.4) and fix $N'>4N+2$. For a given $0<\delta\ll 1$, it is clear we have a partition of unity $\big\{\big(\psi_\upsilon,B(y_\upsilon,\delta^{N'})\big)\big\}_{\upsilon\in\zeta}$ with a finite index set ζ satisfying

(1) $\mathrm{card}(\zeta)\lesssim\delta^{-N'd}$,
(2) ψ_υ is a smooth function supported in $B(y_\upsilon,\delta^{N'})$,
(3) $\|\partial^\alpha\psi_\upsilon\|_\infty\le C_\alpha\,(\delta^{N'})^{-|\alpha|}$ for each multi-index α,
(4) $\sum_{\upsilon\in\zeta}\psi_\upsilon=\psi$, and
(5) the balls $\{B(y_\upsilon,\delta^{N'}):\upsilon\in\zeta\}$ have finite overlapping property, that is,

$$\sum_{\upsilon\in\zeta}\chi_{B(y_\upsilon,\delta^{N'})}(y)\le C,$$

where the constant C depends only on the dimension d.

We then rewrite $\widehat{\mu}$ as $\widehat{\mu}=\sum_{\upsilon\in\zeta}\widehat{\mu_\upsilon}$ where

$$\widehat{\mu_\upsilon}(\xi):=\int_{\mathbb{R}^d}\mathrm{e}^{-2\pi\mathrm{i}(\xi'\cdot y+\xi''\cdot\gamma(y))}\,\psi_\upsilon(y)\,\chi_{\mathbb{R}^d\setminus\Omega}(y)\,dy.$$

We split ζ into two disjoint subsets ζ_1 and ζ_2 defined by

$$\begin{cases}\zeta_1:=\big\{\upsilon\in\zeta:\ \mathrm{dist}\big(B(y_\upsilon,\delta^{N'}),\Sigma\big)\le\delta\big\},\\ \zeta_2:=\big\{\upsilon\in\zeta:\ \mathrm{dist}\big(B(y_\upsilon,\delta^{N'}),\Sigma\big)>\delta\big\}.\end{cases}$$

If $\upsilon\in\zeta_1$, then $B(y_\upsilon,\delta^{N'})\subset\Sigma_{2\delta}$, so the geometric conditions for the domain (1.3) lead us to the following estimates:

$$\Big|\sum_{\upsilon\in\zeta_1}\widehat{\mu_\upsilon}(\xi)\Big|\le\sum_{\upsilon\in\zeta_1}\|\mu_\upsilon\|_{\mathcal{M}}\lesssim\mathrm{meas}(\Sigma_{2\delta}\setminus\Omega)\lesssim\delta^{d_0}. \tag{2.1}$$

Since $\mathrm{supp}(\psi_\upsilon)\cap\Sigma=\emptyset$ for $\upsilon\in\zeta_2$, we have

$$\widehat{\mu_\upsilon}(\xi)=\int_{\mathbb{R}^d}\mathrm{e}^{-2\pi\mathrm{i}(\xi'\cdot y+\xi''\cdot\gamma(y))}\psi_\upsilon(y)\,dy,\quad \xi=(\xi',\xi'')\in\mathbb{R}^d\times\mathbb{R}^n.$$

If $\xi''=0$, then $|\widehat{\mu}_\upsilon(\xi)|\le C_M(\delta^{-N'})^M(1+|\xi|)^{-M}$ for any M, and so

$$\Big|\sum_{\upsilon\in\zeta_2}\widehat{\mu_\upsilon}(\xi)\Big|\le C_M\,\delta^{-N'(d+M)}(1+|\xi|)^{-M}. \tag{2.2}$$

Therefore if $\xi''=0$, then by (2.1) and (2.2) we have

$$|\widehat{\mu}(\xi)|\le C_M\left[\delta^{d_0}+\delta^{-N'(d+M)}(1+|\xi|)^{-M}\right],$$

and by choosing $\delta=|\xi|^{-\epsilon}$ for some appropriate $0<\epsilon<\frac{M}{N'(d+M)}$, we obtain the desired result.

We therefore assume $\xi''\neq 0$ in the remaining part of this section. Set $\theta=\xi''/|\xi''|$; then $\xi''\cdot\gamma(y)=|\xi''|\,\theta\cdot\gamma(y)$. Without loss of generality we may assume $(i_1(\theta),\cdots,i_\ell(\theta))=(1,2,\cdots,\ell)$ in the directional sub-Hessian conditions (1.5). We split ζ_2 into two disjoint subsets $\zeta_2'(\theta)$ and $\zeta_2''(\theta)$ based on the sizes of the directional sub-Hessian, namely,

$$\begin{cases}\zeta_2'(\theta):=\big\{\upsilon\in\zeta_2:|\det\big(\mathbf{D}_{(1,\cdots,\ell)}(\theta,y)\big)|\le\delta \text{ for all } y\in B(y_\upsilon,\delta^{N'})\big\},\\ \zeta_2''(\theta):=\big\{\upsilon\in\zeta_2:|\det\big(\mathbf{D}_{(1,\cdots,\ell)}(\theta,y)\big)|>\delta \text{ for some } y\in B(y_\upsilon,\delta^{N'})\big\}.\end{cases}$$

By the directional sub-Hessian conditions (1.5), for a given $\delta>0$ we have

$$\big|\big\{y\in \mathrm{U}\backslash\Sigma:|\det\big(\mathbf{D}_{(1,\cdots,\ell)}(\theta,y)\big)|\le\delta\big\}\big|\le\int_{\mathrm{U}\backslash\Sigma}\delta^\eta\,|\det\big(\mathbf{D}_{(1,\cdots,\ell)}(\theta,y)\big)|^{-\eta}\,dy\lesssim\delta^\eta. \tag{2.3}$$

By combining (2.3) and the finite overlapping property of the balls $\{B(y_\upsilon,\delta^{N'}):\upsilon\in\zeta\}$, we obtain

$$\begin{aligned}\Big|\sum_{\upsilon\in\zeta_2'(\theta)}\widehat{\mu_\upsilon}(\xi)\Big|&\le\sum_{\upsilon\in\zeta_2'(\theta)}\|\mu_\upsilon\|_{\mathcal{M}}\le C\sum_{\upsilon\in\zeta_2'(\theta)}|B(y_\upsilon,\delta^{N'})\cap\mathrm{U}|\\&\le C\Big\|\sum_{\upsilon\in\zeta_2'(\theta)}\chi_{B(y_\upsilon,\delta^{N'})}(y)\Big\|_{L^1(\mathrm{U})}\le C\Big|\bigcup_{\upsilon\in\zeta_2'(\theta)}B(y_\upsilon,\delta^{N'})\cap\mathrm{U}\Big|\\&\le C\Big|\big\{y\in\mathrm{U}\setminus\Sigma:|\det\big(\mathbf{D}_{(1,\cdots,\ell)}(\theta,y)\big)|\le\delta\big\}\Big|\le C\,\delta^\eta.\end{aligned} \tag{2.4}$$

For $\upsilon\in\zeta_2''(\theta)$, we have

$$\widehat{\mu_\upsilon}(\xi)=\int_{\mathbb{R}^d}\mathrm{e}^{-2\pi\mathrm{i}\left(\xi'\cdot y+|\xi''|\,[\theta\cdot\gamma(y)]\right)}\,\psi_\upsilon(y)\,dy. \tag{2.5}$$

We separately consider (2.5) in two cases, $|\xi'|\gtrsim\delta^{-2N}|\xi''|$ and $|\xi'|\lesssim\delta^{-2N}|\xi''|$.

Case 1. $|\xi'|\gtrsim\delta^{-2N}|\xi''|$.

Since $\upsilon\in\zeta_2$, $\mathrm{dist}(B(y_\upsilon,\delta^{N'}),\Sigma)>\delta$. Thus by the derivative conditions (1.4), for all $y\in B(y_\upsilon,\delta^{N'})$ we have $|\nabla_y[\theta\cdot\gamma(y)]|\lesssim\mathrm{dist}(y,\Sigma)^{-2N}\lesssim\delta^{-2N}$ and

$$|\nabla_y(\xi'\cdot y+\xi''\cdot\gamma(y))|=\big|\,\xi'+|\xi''|\,\nabla_y[\theta\cdot\gamma(y)]\,\big|\gtrsim|\xi'|\gtrsim|\xi|. \tag{2.6}$$

If we integrate (2.5) by parts with the condition (2.6), then we have

$$|\widehat{\mu_\upsilon}(\xi)|\lesssim\delta^{(d-1)N'}|\xi|^{-1},$$

and so

$$\sum_{\upsilon\in\zeta_2''(\theta)}|\widehat{\mu_\upsilon}(\xi)|\lesssim\delta^{(d-1)N'}|\xi|^{-1}\mathrm{card}(\zeta_2''(\theta))\lesssim|\xi|^{-1}\delta^{-N'}. \tag{2.7}$$

Case 2. $|\xi'| \lesssim \delta^{-2N}|\xi''|$.

Since $\upsilon \in \zeta_2''(\theta)$, there exists $z_\upsilon \in B(y_\upsilon, \delta^{N'})$ such that $|\det \mathbf{D}_{(1,\cdots,\ell)}(\theta, z_\upsilon)| > \delta$. In view of the symmetry of $\mathbf{D}_{(1,\cdots,\ell)}(\theta, z_\upsilon)$, it can be written as

$$\mathbf{D}_{(1,\cdots,\ell)}(\theta, z_\upsilon) = P^{-1}DP$$

for some orthogonal matrix P and $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_\ell)$. Since $|\det \mathbf{D}_{(1,\cdots,\ell)}(\theta, z_\upsilon)| = |\lambda_1 \cdots \lambda_\ell| \geq \delta$, there exists i such that $|\lambda_i| \geq \delta^{1/\ell}$. Without loss of generality, we may set $|\lambda_1| \geq \delta^{1/\ell}$. Let $\mathrm{v}_1 \in \mathbb{S}^{\ell-1}$ be a unit vector such that $(\mathrm{v}_1)^T$ is an eigenvector of $\mathbf{D}_{(1,\cdots,\ell)}(\theta, z_\upsilon)$ corresponding to λ_1, and let $\mathrm{v}_1' = (\mathrm{v}_1, 0, \cdots, 0) \in \mathbb{S}^{d-1}$. We form an orthonormal basis $\{\mathrm{v}_1', \cdots, \mathrm{v}_d'\}$ of $\mathbb{R}^d$, and apply the change of variables $y = z_\upsilon + \mathrm{v}_1' t_1 + \cdots + \mathrm{v}_d' t_d := m_\upsilon(t)$ to obtain

$$\widehat{\mu_\upsilon}(\xi) = \int \mathrm{e}^{-2\pi \mathrm{i}\left(\xi' \cdot m_\upsilon(t) + |\xi''|[\theta \cdot \gamma(m_\upsilon(t))]\right)} \psi_\upsilon(m_\upsilon(t))\, dt.$$

We note that

$$\begin{aligned} \frac{\partial^2}{\partial t_1^2}\Big(\xi' \cdot m_\upsilon(t) + |\xi''|\,[\theta \cdot \gamma(m_\upsilon(t))]\Big) &= |\xi''|\, \mathrm{v}_1'\, \mathbf{D}_{(1,\cdots,d)}(\theta, m_\upsilon(t))(\mathrm{v}_1')^T \\ &= |\xi''|\, \mathrm{v}_1\, \mathbf{D}_{(1,\cdots,\ell)}(\theta, m_\upsilon(t))(\mathrm{v}_1)^T, \end{aligned} \tag{2.8}$$

and

$$|\, \mathrm{v}_1'\, \mathbf{D}_{(1,\cdots,d)}(\theta, z_\upsilon)(\mathrm{v}_1')^T\, | = |\, \mathrm{v}_1\, \mathbf{D}_{(1,\cdots,\ell)}(\theta, z_\upsilon)(\mathrm{v}_1)^T\, | = |\lambda_1| \geq \delta^{1/\ell}. \tag{2.9}$$

Since $\mathrm{dist}\big(B(y_\upsilon, \delta^{N'}), \Sigma\big) > \delta$, we can apply the mean value theorem together with the derivative conditions (1.4) with $|\alpha| = 3$ to obtain

$$\big\|\mathbf{D}_{(1,\cdots,d)}(\theta, m_\upsilon(t)) - \mathbf{D}_{(1,\cdots,d)}(\theta, z_\upsilon)\big\| \lesssim \delta^{-4N}\delta^{N'} \lesssim \delta^2 \quad \text{if } N' > 4N+2.$$

We combine this with (2.9) to obtain that if $N' > 4N+2$, then

$$\begin{aligned} &\big|\, \mathrm{v}_1'\, \mathbf{D}_{(1,\cdots,d)}(\theta, m_\upsilon(t))(\mathrm{v}_1')^T\, \big| \\ &\geq \big|\, \mathrm{v}_1'\, \mathbf{D}_{(1,\cdots,d)}(\theta, z_\upsilon)(\mathrm{v}_1')^T\, \big| - \Big|\, \mathrm{v}_1'\, \big[\mathbf{D}_{(1,\cdots,d)}(\theta, m_\upsilon(t)) - \mathbf{D}_{(1,\cdots,d)}(\theta, z_\upsilon)\big](\mathrm{v}_1')^T\, \Big| \\ &\gtrsim \delta^{1/\ell} \geq \delta. \end{aligned}$$

Therefore by (2.8) together with the condition $|\xi| \lesssim \delta^{-2N}|\xi''|$ we have

$$\left|\frac{\partial^2}{\partial t_1^2}\Big(\xi' \cdot m_\upsilon(t) + |\xi''|\,[\theta \cdot \gamma(m_\upsilon(t))]\Big)\right| \gtrsim |\xi''|\,\delta \gtrsim |\xi|\,\delta^{2N+1}.$$

If we apply the van der Corput lemma with respect to the t_1 variable, and then integrate in $t_2, \cdots, t_d$ variables, we have

$$|\widehat{\mu_\upsilon}(\xi)| \lesssim (|\xi|\,\delta^{2N+1})^{-1/2}(\delta^{N'})^{d-1} \lesssim |\xi|^{-1/2}\delta^{-N-1/2}(\delta^{N'})^{d-1}.$$

Thus if $N' > 4N+2$, then

$$\sum_{\upsilon \in \zeta_2''(\theta)} |\widehat{\mu_\upsilon}(\xi)| \lesssim |\xi|^{-1/2}\delta^{-N-1/2}(\delta^{N'})^{d-1}\mathrm{card}(\zeta_2''(\theta)) \lesssim |\xi|^{-1/2}\delta^{-N-1/2-N'}. \tag{2.10}$$

By (2.1), (2.4), (2.7) and (2.10) if $N' > 4N+2$, then

$$|\widehat{\mu}(\xi)| \lesssim \delta^{d_0} + \delta^{\eta} + |\xi|^{-1}\delta^{-N'} + |\xi|^{-1/2}\delta^{-N-1/2-N'},$$

and by choosing $\delta = |\xi|^{-\epsilon}$ for an appropriate $\epsilon > 0$, we obtain the desired result. $\square$

3. Proof of Theorem 2: The case $m \geq 1$

In this section we only consider the case $m > 1$; because the case $m = 1$ can be treated by a small modification of Lemmas 3.2 and 3.3. First we dominate M_H by the summation of two lacunary maximal operators. To make this precise, we write

$$r^{-d}\psi(r^{-1}y) - 2^{-jd}\psi(2^{-j}y) = \int_{2^j}^{r} \frac{d}{dt}\left(t^{-d}\psi(t^{-1}y)\right) dt = \int_{2^j}^{r} t^{-d}\Psi(t^{-1}y)\,\frac{dt}{t},$$

where j is the integer satisfying $r \in [2^j, 2^{j+1})$ and $\Psi(y) = -d\,\psi(y) - y \cdot \nabla\psi(y)$. Therefore

$$\mathrm{M}_H f(x) \leq \mathrm{M}'_H f(x) + \mathrm{M}''_H f(x),$$

where

$$\mathrm{M}'_H f(x) = \sup_{j\in\mathbb{Z}} \Big| \int_{\mathbb{R}^d} f\big(x - (y, \gamma(y))\big)\, 2^{-jd}\psi(2^{-j}y)\, dy \Big|,$$

$$\mathrm{M}''_H f(x) = \sup_{j\in\mathbb{Z}} \left[\int_{2^j}^{2^{j+1}} \Big| \int_{\mathbb{R}^d} f\big(x - (y, \gamma(y))\big)\, t^{-d}\Psi(t^{-1}y)\, dy \Big|^2 \frac{dt}{t} \right]^{1/2}.$$

We will only prove that M'_H maps the parabolic H^1 to weak-L^1. Similar arguments can be applied for M''_H. As was pointed out in [2], in proving M'_H maps the parabolic H^1 to weak-L^1, we may make an additional assumption that

$$\operatorname{supp}(\psi) \subset \{y : 1 < |y| < 2\}. \tag{3.1}$$

Tendrils and stopping time construction. Let $m > 1$ be a real number. For each $\tau \in \mathbb{Z}$, we define

$$\mathbb{F}(\tau) := \max\left\{\tau' \in \mathbb{Z} : \lfloor (m-1)\tau \rfloor = \lfloor (m-1)\tau' \rfloor\right\},$$

where $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x. Let

$$\mathbb{F} = \{\mathbb{F}(\tau) : \tau \in \mathbb{Z}\}.$$

In view of the definition of $\mathbb{F}(\tau)$, it is easy to see that there exists $0 \leq \theta_\tau < 1$ such that $(m-1)\tau + \theta_\tau = (m-1)\mathbb{F}(\tau)$ for all $\tau \in \mathbb{Z}$, which implies $|\mathbb{F}(\tau) - \tau| < 1/(m-1)$. For $\tau \in \mathbb{F}$, we define τ_+ as the smallest integer in $\mathbb{F}$ that is greater than τ, and τ_- as the greatest integer in $\mathbb{F}$ that is smaller than τ. We then see that

$$|\tau - \tau_+|,\ |\tau - \tau_-| \leq 1/(m-1).$$

For $\sigma, \tau \in \mathbb{Z}$ with $\tau \geq \sigma$, we let $\mathcal{D}_\sigma(\mathbb{R}^n)$ denote the collection of all dyadic cubes in $\mathbb{R}^n$ with sidelengths 2^σ and let $\mathcal{R}_{\sigma,\tau}$ denote the set of all rectangles in $\mathbb{R}^{d+1}$ of the form

$$I \times J \in \mathcal{D}_\sigma(\mathbb{R}^d) \times \mathcal{D}_{\sigma + \lfloor (m-1)\tau \rfloor}(\mathbb{R}).$$

Let $\mathcal{R}$ be the union of all the classes $\mathcal{R}_{\sigma,\tau}$. Note that

$$\mathcal{R} = \bigcup_{\sigma \leq \tau} \mathcal{R}_{\sigma,\tau} = \biguplus_{\sigma \leq \tau, \tau \in \mathbb{F}} \mathcal{R}_{\sigma,\tau},$$

where $\biguplus$ is the disjoint union. For $q \in \mathcal{R}$, there are unique $\sigma \in \mathbb{Z}$ and $\tau \in \mathbb{F}$ with $\sigma \leq \tau$ such that $q \in \mathcal{R}_{\sigma,\tau}$, by which we set

$$\sigma(q) := \sigma, \quad \tau(q) := \tau \quad \text{if } q \in \mathcal{R}_{\sigma,\tau} \text{ with } \tau \in \mathbb{F}.$$

Let μ be the measure given by

$$\langle f, \mu \rangle = \int_{\mathbb{R}^d} f(y, \gamma(y))\psi(y) dy,$$

where ψ is as in (3.1). For each $j \in \mathbb{Z}$, we define a measure μ_j by

$$\langle f, \mu_j \rangle = 2^{-jd} \int_{\mathbb{R}^d} f(y, \gamma(y)) \psi(2^{-j} y) dy. \tag{3.2}$$

For each $q \in \mathcal{R}_{\sigma,\tau}$ with $\tau \in \mathbb{F}$, we define the tendril $T(q)$ to be the set

$$T(q) = 2q + \left\{ (y, \gamma(y)) : |y| \le 2^{\tau(q)_+} \right\}, \tag{3.3}$$

where $2q$ denotes the rectangle with the same center and the double of the sidelength of q. It is worth noting that $T(q)$ is defined so that the functions $\chi_q * \mu_j$ with $j \le \tau(q)_+$ are supported on it.

Lemma 3.1. *If $q \in \mathcal{R}_{\sigma,\tau}$ with $\tau \in \mathbb{F}$ and the tendril $T(q)$ is as in* (3.3)*, then*

$$|T(q)| \lesssim 2^{\sigma + (d+m-1)\tau}.$$

Proof. This can be easily proved via the geometric conditions for the range (1.9) together with $2^{-\tau} \circ T(q)$ where $2^{-\tau} \circ (y, y_{d+1}) = (2^{-\tau} y, 2^{-m\tau} y_{d+1})$ with $(y, y_{d+1}) \in \mathbb{R}^d \times \mathbb{R}$. We omit the details. □

The following lemma is the standard Calderón-Zygmund decomposition.

Lemma 3.2 (Standard Calderón-Zygmund decomposition)**.** *By "parabolic dyadic cubes" we shall mean a collection $\mathcal{R}_\sigma := \mathcal{R}_{\sigma,\mathbb{F}(\sigma)}$ $(\sigma \in \mathbb{Z})$ of rectangles in $\mathbb{R}^{d+1}$. For any $\alpha > 0$ and any finite collection $\mathbf{Q}$ of parabolic dyadic cubes Q in $\mathbb{R}^{d+1}$ with associated positive scalars λ_Q, there exists a collection of pairwise disjoint parabolic dyadic cubes $\{S\}_{S \in \mathbf{S}}$ such that*

(1) $\sum_{Q : Q \subset S} \lambda_Q \le C\alpha |S|$ *for all* $S \in \mathbf{S}$,

(2) $\sum_{S : S \in \mathbf{S}} |S| \le \alpha^{-1} \sum \lambda_Q$,

(3) $\left\| \sum_{Q : \text{not contained in any } S} \lambda_Q |Q|^{-1} \chi_Q \right\|_\infty \le \alpha$.

Let $\mathbf{C}$ be the collection of parabolic dyadic cubes $Q \in \mathbf{Q}$ which is contained in some $S \in \mathbf{S}$. For $Q \in \mathbf{C}$ we define $S(Q)$ as the unique $S \in \mathbf{S}$ containing Q. The following lemma is a variant of the standard Calderón-Zygmund decomposition which can be proved by using stopping time arguments based on the tendrils $T(q)$, $q \in \mathcal{R}_{\sigma,\tau}$. For the sake of the reader's convenience we attach its proof in Appendix 6.3. See also [2, 4–6, 10, 12].

Lemma 3.3 (A variant of the Calderón-Zygmund decomposition)**.** *Given $\alpha > 0$ there exists a function $\Gamma : \mathbf{C} \to \mathbb{Z}$ and a measurable set E such that*

(1) $|E| \le C \left(\alpha^{-1} \sum_{Q \in \mathbf{Q}} \lambda_Q + \sum_{S \in \mathbf{S}} |S| \right)$,

(2) $\chi_Q * \mu_j$ *is supported in E for all $j < \Gamma(Q)$ and all $Q \in \mathbf{C}$,*

(3) $\Gamma(Q) \ge \sigma(S(Q))$,

(4) *for any $\sigma, \tau \in \mathbb{Z}$ with $\tau \ge \sigma$, and any $q \in \mathcal{R}_{\sigma,\tau}$,*

$$\sum_{Q \subset q : \Gamma(Q) \le \tau} \lambda_Q \le C\alpha 2^{\sigma + (d+m-1)\tau}.$$

Proof of Theorem 2 *for the case $m > 1$.* As is pointed out in [2, 10], limiting arguments in Appendix 6.2 enable us to assume that $f \in H^1_m(\mathbb{R}^{d+1})$ has the form of a finite sum $f = \sum \lambda_Q a_Q$ where $\lambda_Q > 0$ and a_Q are H^1_m-atoms. Let $Q = I \times J$ where $I \subset \mathbb{R}^d$ is a cube with sidelength $r > 0$, and $J \subset \mathbb{R}$ is an interval with

length r^m. Note that if $2^\sigma \le r < 2^{\sigma+1}$ for some $\sigma \in \mathbb{Z}$, then there is a unique $I' \in \mathcal{D}_{\sigma+2}(\mathbb{R}^d)$ containing I, and since $r^m < 2^{(\sigma+2)+\lfloor(m-1)(\sigma+2)\rfloor}$, there is a unique $J' \in \mathcal{D}_{(\sigma+2)+\lfloor(m-1)(\sigma+2)\rfloor}(\mathbb{R})$ containing J. We thus have

$$Q = I \times J \subset I' \times J' \in \mathcal{D}_{\sigma+2}(\mathbb{R}^d) \times \mathcal{D}_{(\sigma+2)+\lfloor(m-1)(\sigma+2)\rfloor}(\mathbb{R}) = \mathcal{R}_{\sigma+2,\mathbb{F}(\sigma+2)}.$$

Since $|I \times J| \sim |I' \times J'|$, we may assume that a_Q is supported in a parabolic dyadic cube $Q \in \bigcup_{\sigma\in\mathbb{Z}} \mathcal{R}_{\sigma,\mathbb{F}(\sigma)}$, and satisfies the following conditions:

$$\|a_Q\|_{L^\infty} \le |Q|^{-1}; \quad \int_{\mathbb{R}^{d+1}} a_Q(x)dx = 0.$$

For $\alpha > 0$, we now show that

$$\big|\{x : \mathrm{M}'_H f(x) > 2\alpha\}\big| \lesssim \alpha^{-1} \sum \lambda_Q. \tag{3.4}$$

We first apply Lemma 3.2 to the collection of parabolic dyadic cubes Q and associated scalars λ_Q appearing in the definition of f. Let $\mathbf{S}$ be as in Lemma 3.2 and define

$$b = \sum_{S\in\mathbf{S}} \sum_{Q\subset S} \lambda_Q a_Q, \quad g = f - b.$$

Then $\|g\|_{L^\infty} \le \alpha$ from (3) of Lemma 3.2 and so we have $\|\mathrm{M}'_H g\|_{L^\infty} \le \alpha$. Thus (3.4) follows from the inequality

$$\big|\{x : \mathrm{M}'_H b(x) > \alpha\}\big| \lesssim \alpha^{-1} \sum \lambda_Q. \tag{3.5}$$

Let $\mathbf{S}$ be as above and $\mathbf{C}$ be the collection of Q's appearing in the definition of b. Then apply Lemma 3.2 to $\mathbf{S}$ and $\mathbf{C}$, and let Γ and E be as in Lemma 3.3. By Lemmas 3.2 and 3.3 we have $|E| \lesssim \alpha^{-1} \sum \lambda_Q$. Thus (3.5) will follow from

$$\big|\big\{x \in \mathbb{R}^{d+1} \setminus E : \mathrm{M}'_H b(x) > \alpha\big\}\big| \lesssim \alpha^{-1} \sum \lambda_Q. \tag{3.6}$$

Let μ_j be as in (3.2); then we have

$$\mathrm{M}'_H b(x) = \sup_{j\in\mathbb{Z}} |b * \mu_j(x)|.$$

By (2) of Lemma 3.3, for each $Q \in \mathbf{C}$, $a_Q * \mu_j$ is supported in E unless $j \ge \Gamma(Q)$. Thus for each $x \in \mathbb{R}^{d+1} \setminus E$, we have

$$\begin{aligned} \mathrm{M}'_H b(x) &= \sup_j \Big|\Big(\sum_{\Gamma(Q)\le j} \lambda_Q a_Q\Big) * \mu_j(x)\Big| \\ &= \sup_j \Big|\sum_{s=0}^{\infty} \Big(\sum_{\Gamma(Q)=j-s} \lambda_Q a_Q\Big) * \mu_j(x)\Big|. \end{aligned} \tag{3.7}$$

For further discussions, if $\Sigma = \emptyset$, then we will use the following Lemma 3.4 which is taken from [11], and if $\Sigma \ne \emptyset$, then we will use the following Lemma 3.5.

Lemma 3.4 (Lemma 2.5 in [11]). *Let γ be a $C^4([-1,1]^d)$ function so that*

$$|\partial^\alpha \gamma(x)| \le C_\alpha \quad \textit{for all } |\alpha| \le 4.$$

For each k, let $D_k := \max_{|\alpha|\le k}(C_\alpha)$. Suppose that $|\det\gamma''(y_0')| \ge \beta$ and $B_{\delta\beta} \subset [-1,1]^d$ is a ball of radius $\delta\beta$ containing y_0', here $0 < \delta \le [10(d)^4(D_3)^d]^{-1}$. Let ψ

be a $C^\infty(\mathbb{R}^d)$ function supported on $B_{\delta\beta}$ so that $\|\partial^\alpha\psi\|_\infty \le c_\alpha(\delta\beta)^{-|\alpha|}$. Define the measure ν by

$$\langle \nu, f\rangle = \int_{\mathbb{R}^d} f(y', \gamma(y'))\psi(y')dy'$$

and define the reflection $\langle \widetilde{\nu}, f\rangle = \langle \nu, f(-\cdot)\rangle$. Then $\nu * \widetilde{\nu}(x)$ is a function which is not zero only if

$$|x_{d+1}| \le D_1|x'| \quad \text{where } x = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R},$$

and for each multi-index α with $|\alpha| \le 1$, there are constants $C_{\alpha,\delta}$ so that

$$\left|\partial_x^\alpha[\nu * \widetilde{\nu}](x)\right| \le C_{\alpha,\delta}\beta^{d-2-2|\alpha|}|x|^{-1-|\alpha|}.$$

Lemma 3.5. *Let Σ be a non-empty closed subset in $[-1,1]^d$, and let γ be a $C^4_{\mathrm{loc}}([-1,1]^d \setminus \Sigma)$ function so that*

$$|\partial^\alpha\gamma(x)| \le C_\alpha \operatorname{dist}(x,\Sigma)^{-N(|\alpha|+1)} \quad \text{for all } |\alpha| \le 4. \tag{3.8}$$

For each k, let $D_k := \max_{|\alpha|\le k}(C_\alpha)$. Let $N' \ge N(3d+1)+3$, $0 < \beta < 1$, and $0 < \delta \le [10d^{d+2}(D_3)^d]^{-1}$. Suppose that $B_{\delta\beta^{N'}} \subset [-1,1]^d \setminus \Sigma$ is a ball of radius $\delta\beta^{N'}$ containing y_0' so that

$$|\det\gamma''(y_0')| \ge \beta, \quad \operatorname{dist}(B_{\delta\beta^{N'}}, \Sigma) \ge \beta. \tag{3.9}$$

Let ψ be a $C^\infty(\mathbb{R}^d)$ function supported on $B_{\delta\beta^{N'}}$ so that $\|\partial^\alpha\psi\|_\infty \le c_\alpha(\delta\beta^{N'})^{-|\alpha|}$. Define the measure ν by

$$\langle \nu, f\rangle = \int_{\mathbb{R}^d} f(y', \gamma(y'))\psi(y')dy'.$$

Then $\nu * \widetilde{\nu}(x)$ is a function which is not zero only if

$$|x_{d+1}| \le D_1\beta^{-2N}|x'| \quad \text{where } x = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R},$$

and for each multi-index α with $|\alpha| \le 1$, there are constants $C_{\alpha,\delta}$ so that

$$\left|\partial_x^\alpha[\nu * \widetilde{\nu}](x)\right| \le C_{\alpha,\delta}\beta^{-(1+4|\alpha|)N'}|x|^{-1-|\alpha|}.$$

Proof of Lemma 3.5. We assume $d \ge 2$. The case $d = 1$ follows with a small modification. Since $\nu * \widetilde{\nu}$ does not change by a translation of the measure ν, we may assume $y_0' = 0$,

$$\begin{aligned}\langle \nu * \widetilde{\nu}, f\rangle &= \iint f(x-y)\, d\nu(x)d\nu(y)\\ &= \sum_k \iint \psi(x'+y')\psi(y')\rho_k(x')f\big(x', \gamma(x'+y') - \gamma(y')\big)\, dx'dy'\\ &:= \sum_k V_k(f),\end{aligned}$$

where the ρ_k form a partition of unity on the unit sphere $\mathbb{S}^{d-1}$ which is extended to a homogeneous function of degree 0. Also we assume that the restriction of ρ_k to the unit sphere is supported on a set of diameter $\le \delta\beta^{N'}$ and so the summation is over $O((\delta\beta^{N'})^{1-d})$ terms. The ρ_k can be chosen so that it satisfies the natural estimates

$$|\partial_{x'}^\alpha\rho_k(x')| \le C_\alpha(\delta\beta^{N'})^{-|\alpha|}|x'|^{-|\alpha|}.$$

Now we pick a unit vector $\theta_k \in \mathrm{supp}(\rho_k)$ and set

$$b_{k,1} = \frac{\gamma''(0)(\theta_k)^T}{|\gamma''(0)(\theta_k)^T|}, \tag{3.10}$$

with which we form an orthonormal basis $\{b_{k,1}^T, b_{k,2}^T, \cdots, b_{k,d}^T\}$ of $\mathbb{R}^d$. We then make the change of variables

$$y' = b_{k,1}^T t_1 + \cdots + b_{k,d}^T t_d = m_k(t'),$$

to rewrite $V_k(f)$ as

$$V_k(f) = \iint \psi(x' + m_k(t'))\psi(m_k(t'))\rho_k(x') f(x', \Gamma_k(t', x'))\, dt' dx'$$

where

$$\Gamma_k(t', x') = \gamma(x' + m_k(t')) - \gamma(m_k(t')) = \Big\langle x', \int_0^1 \nabla\gamma(m_k(t') + sx') ds \Big\rangle. \tag{3.11}$$

We now observe that

$$\begin{aligned} \frac{\partial}{\partial t_1}\Gamma_k(t', x') &= \int_0^1 x'\, \gamma''(m_k(t') + sx') b_{k,1}\, ds \\ &= |x'| \left(\theta_k\, \gamma''(0) b_{k,1}\right) + e_1(t', x') + e_2(t', x'), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \mathrm{e}_1(t', x') &= \int_0^1 |x'|\, \theta_k\, [\gamma''(m_k(t') + sx') - \gamma''(0)] b_{k,1}\, ds, \\ \mathrm{e}_2(t', x') &= \int_0^1 (\, x' - |x'|\theta_k)\, \gamma''(m_k(t') + sx') b_{k,1}\, ds. \end{aligned} \tag{3.13}$$

Let $\mathrm{A} = [a_{ij}]_{1\le i,j\le d} = \gamma''(0)$ and $\mathrm{A}^{-1} = [c_{ij}]_{1\le i,j\le d}$; then by Cramer's rule we have

$$c_{ij} = \frac{(-1)^{i+j}}{\mathrm{detA}} \mathrm{detA}_{ji}$$

where A_{ji} is obtained from A by removing the j-th row and the i-th column. We combine (3.8) and (3.9) to obtain

$$\begin{aligned} 1 = |\theta_k^T| = |\mathrm{A}^{-1}\mathrm{A}\, \theta_k^T| &\le d \max\{|c_{ij}|\} |\, \mathrm{A}\theta_k^T| \\ &\le d^d\, (\mathrm{detA})^{-1}\, (\max\{|a_{ij}|\})^{d-1}\, |\mathrm{A}\theta_k^T| \\ &\le d^d\, \beta^{-1} (D_3 \beta^{-3N})^{d-1} |\mathrm{A}\theta_k^T|. \end{aligned}$$

Hence we have $|\gamma''(0)\theta_k^T| = |\mathrm{A}\theta_k^T| \ge d^{-d}(D_3)^{-d+1}\beta^{3N(d-1)+1}$ and by (3.10) we obtain

$$|x'| \left(\theta_k\, \gamma''(0) b_{k,1}\right) = |x'|\, |\gamma''(0)\theta_k^T| \ge |x'|\, d^{-d}(D_3)^{-d+1}\beta^{3N(d-1)+1}. \tag{3.14}$$

From (3.13), by the mean value theorem we have

$$\begin{aligned} |\mathrm{e}_1(t', x')| &\le \int_0^1 |x'|\, |m_k(t') + sx'|\, d^2 \sup_{|\alpha|=3} (|\partial^\alpha \gamma|)\, ds \\ &\le |x'| (2\delta\beta^{N'}) d^2 (D_3 \beta^{-4N}). \end{aligned} \tag{3.15}$$

Since $x', \theta_k \in \mathrm{supp}(\rho_k)$, we have $|x' - |x'|\theta_k| \le |x'|\delta\beta^{N'}$, and so from (3.13)

$$|\mathrm{e}_2(t', x')| \le |x'|\delta\beta^{N'} d^2 D_3 (\beta^{-4N}). \tag{3.16}$$

Hence by (3.12), (3.14), (3.15) and (3.16), if $0 < \delta \le [10d^{d+2}(D_3)^d]^{-1}$ and $N' \ge N(3d+1)+1$, then we have

$$\frac{\partial}{\partial t_1}\Gamma_k(t',x') \ge \frac{7}{10}(d^{d+2})^{-1}\,\beta^{3N(d-1)+1}(D_3)^{-d+1}\,|x'|.$$

Therefore we may perform the change of variables $t_1 \longmapsto x_{d+1} = \Gamma_k(t',x')$ with inverse

$$t_1(x_{d+1};x',t'')$$

where $t'' = (t_2, t_3, \cdots, t_d)$, and obtain

$$\langle \nu * \widetilde{\nu}, f\rangle = \sum_k \iiint f(x', x_{d+1}) H_k(x', x_{d+1}, t'')\, dt'' dx' dx_{d+1}$$

where

$$H_k(x', x_{d+1}, t'') = \frac{\psi(x' + m_k(t'))\psi(y')\rho_k(x')}{\partial_{t_1}\Gamma_k(t_1(x_{d+1};x',t''),t'',x')}.$$

From (3.11) we have $|x_{d+1}| \le D_1\beta^{-2N}|x'|$ and $|t''| \le |y'| \le \delta\beta^{N'}$. Hence

$$|H_k(x', x_{d+1}, t'')| \le \frac{C}{|\partial_{t_1}\Gamma_k|} \le \frac{C}{\beta^{3N(d-1)+1}|x'|} \le \frac{C}{\beta^{N(3d-1)+1}|x|}.$$

Integrating in t'' yields a factor $O(\beta^{N'(d-1)})$, and since there are $((\delta\beta^{N'})^{1-d})$ terms of ρ_k's so that $\sum_k \rho_k = O(1)$ we obtain the claimed estimate for $\alpha = 0$. The estimates for the derivatives will follow from the facts

$$\frac{\partial t_1}{\partial x_{d+1}} = \frac{1}{\partial_{t_1}\Gamma_k}, \quad \frac{\partial t_1}{\partial x_j} = -\frac{\partial_{x_j}\Gamma_k}{\partial_{t_1}\Gamma_k} \quad \text{for} \quad j = 1, 2, \cdots, d,$$

which will be obtained from the identity $x_{d+1} = \Gamma_k(t',x')$ and differentiation of both sides by x_j for $j = 1, 2, \cdots, d+1$. □

Now we return to the proof of (3.6). We will only consider the case $\Sigma \ne \emptyset$. The proofs for the case $\Sigma = \emptyset$ follow similarly by using Lemma 3.4. Given $\delta > 0$, we use a partition of unity with a finite index set ζ to write

$$\mu = \sum_{\upsilon\in\zeta} \mu_\upsilon$$

where $\mathrm{supp}(\mu_\upsilon) \subset \big\{(y, \gamma(y)) : y \in B(x_\upsilon, \delta\beta^{N'})\big\}$ with $N' \ge N(3d+1)+3$. Then by Lemma 3.5, we can choose δ (not depending on β) such that if there exists $y_\upsilon \in B(x_\upsilon, \delta\beta^{N'})$ with $|\det\gamma''(y_\upsilon)| \ge \beta$ and $\mathrm{dist}(x_\upsilon, \Sigma) \ge \beta$, then for $|\alpha| \le 1$ we have

$$\big|\partial_x^\alpha[\mu_\upsilon * \widetilde{\mu_\upsilon}](x)\big| \le C_{\alpha,\delta}\beta^{-(1+4|\alpha|)N'}|x|^{-1-|\alpha|}, \tag{3.17}$$

here the constant $C_{\alpha,\delta}$ does not depend on υ. Now for each $s = 0, 1, 2, \cdots$, we use a partition of unity with a finite index set ζ^s to write

$$\mu = \sum_{\upsilon\in\zeta^s} \mu_\upsilon^s$$

where $\mathrm{supp}(\mu_\upsilon^s) \subset \big\{(y, \gamma(y)) : y \in B_\upsilon^s\big\}$, B_υ^s is a ball of diameter $\delta 2^{-\epsilon N' s}$(here δ is as in (3.17) and $0 < \epsilon < 1$ will be determined later) and $\{B_\upsilon^s\}_{\upsilon\in\zeta^s}$ have bounded overlap independent of s. Note that

$$\mathrm{card}(\zeta^s) \lesssim 2^{\epsilon N' s d}. \tag{3.18}$$

We split ζ^s into disjoint subsets ζ_1^s, ζ_2^s and ζ_3^s having the following properties:

$$\begin{cases} \zeta_1^s := \big\{\upsilon \in \zeta^s : \mathrm{dist}(\Sigma, B_\upsilon^s) \le 2^{-\epsilon s}\big\}; \\ \zeta_2^s := \big\{\upsilon \in \zeta^s : \mathrm{dist}(\Sigma, B_\upsilon^s) > 2^{-\epsilon s} \text{ and } |\det(\gamma''(y))| \le 2^{-\epsilon s} \text{ for all } y \in B_\upsilon^s\big\}; \\ \zeta_3^s := \big\{\upsilon \in \zeta^s : \mathrm{dist}(\Sigma, B_\upsilon^s) > 2^{-\epsilon s} \text{ and } |\det(\gamma''(y))| > 2^{-\epsilon s} \text{ for some } y \in B_\upsilon^s\big\}. \end{cases}$$

By the geometric conditions for the domain (1.3) we have

$$\sum_{\upsilon \in \zeta_1^s} \|\mu_\upsilon^s\|_{\mathcal{M}} \le C\big|\big\{y \in \mathbb{R}^d : \mathrm{dist}(\Sigma, y) \le 2^{-\epsilon s}\big\}\big| \le C 2^{-\epsilon s d_0}, \tag{3.19}$$

and by the Hessian condition (1.8) we obtain

$$\begin{aligned} \sum_{\upsilon \in \zeta_2^s} \|\mu_\upsilon^s\|_{\mathcal{M}} &\lesssim \Big|\big\{y \in \mathrm{U} \setminus \Sigma \,:\, |\det(\gamma''(y))| \le 2^{-\epsilon s}\,\big\}\Big| \\ &\le \int_{\mathrm{U}\setminus\Sigma} (2^{-\epsilon s})^\eta \, |\det(\gamma''(y))|^{-\eta}\, dy \\ &\lesssim 2^{-\epsilon s \eta}. \end{aligned} \tag{3.20}$$

Let

$$\langle f, \mu_{\upsilon,j}^s\rangle = \int f(2^j y, 2^{mj}\gamma(y))\, \psi_\upsilon^s(y)\, dy$$

where ψ_υ^s is a smooth function supported in B_υ^s and satisfies

$$|\partial^\alpha \psi_\upsilon^s(y)| \le C_\alpha (\delta 2^{-\epsilon N' s})^{-|\alpha|}$$

for each α. Then by (3.7) we have

$$\mathrm{M}'_H b(x) \le \mathrm{M}'_{H1} b(x) + \mathrm{M}'_{H2} b(x) + \mathrm{M}'_{H3} b(x)$$

where

$$\mathrm{M}'_{Hi} b(x) = \sup_j \Big| \sum_{s=0}^\infty \Big(\sum_{\Gamma(Q)=j-s} \lambda_Q a_Q \Big) * \sum_{\upsilon \in \zeta_i^s} \mu_{\upsilon,j}^s(x) \Big|.$$

For $\mathrm{M}'_{H1} b$, we use (3.19) and Chebyshev's inequality to obtain

$$\begin{aligned} \big|\{x : \mathrm{M}'_{H1} b(x) > \alpha\}\big| &\le \alpha^{-1} \|\mathrm{M}'_{H1} b\|_{L^1} \\ &\le \alpha^{-1} \sum_j \sum_{s=0}^\infty \Big\| \sum_{\Gamma(Q)=j-s} \lambda_Q a_Q \Big\|_{L^1} \Big\| \sum_{\upsilon \in \zeta_1^s} \mu_{\upsilon,j}^s \Big\|_{\mathcal{M}} \\ &\lesssim \alpha^{-1} \sum_j \sum_{s=0}^\infty \sum_{\Gamma(Q)=j-s} \lambda_Q 2^{-\epsilon s d_0} \\ &\lesssim \alpha^{-1} \sum \lambda_Q. \end{aligned}$$

Similarly by using (3.20) we have

$$\big|\{x : \mathrm{M}'_{H2} b(x) > \alpha\}\big| \lesssim \alpha^{-1} \sum \lambda_Q.$$

For $\mathrm{M}'_{H3} b$, if we show

$$\|\mathrm{M}'_{H3} b\|_{L^2}^2 \lesssim \alpha \sum \lambda_Q, \tag{3.21}$$

then by Chebyshev's inequality we have

$$\big|\{x : \mathrm{M}'_{H3} b(x) > \alpha\}\big| \le \alpha^{-2} \|\mathrm{M}'_{H3} b\|_{L^2}^2 \lesssim \alpha^{-1} \sum \lambda_Q.$$

Thus (3.6) will follow from (3.21). From now on we prove (3.21). For M'_{H3}, we have

$$|\mathrm{M}'_{H3}b(x)|^2 \le \sum_j \Big|\sum_{s=0}^{\infty}\sum_{\upsilon\in\zeta_3^s}\Big(\sum_{\Gamma(Q)=j-s}\lambda_Q a_Q\Big) * \mu^s_{\upsilon,j}(x)\Big|^2,$$

and by Minkowski's inequality we obtain

$$|\mathrm{M}'_{H3}b(x)| \le \sum_{s=0}^{\infty}\sum_{\upsilon\in\zeta_3^s}\Big(\sum_j\Big|\Big(\sum_{\Gamma(Q)=j-s}\lambda_Q a_Q\Big) * \mu^s_{\upsilon,j}(x)\Big|^2\Big)^{\frac12}.$$

Hence from (3.18) we have

$$\begin{aligned}\|\mathrm{M}'_{H3}b\|_{L^2} &\le \sum_{s=0}^{\infty}\sum_{\upsilon\in\zeta_3^s}\Big\|\Big(\sum_j\Big|\Big(\sum_{\Gamma(Q)=j-s}\lambda_Q a_Q\Big) * \mu^s_{\upsilon,j}(x)\Big|^2\Big)^{\frac12}\Big\|_{L^2}\\ &\lesssim \sum_{s=0}^{\infty}2^{\epsilon dN's}\max_{\upsilon\in\zeta_3^s}\Big\|\Big(\sum_j\Big|\Big(\sum_{\Gamma(Q)=j-s}\lambda_Q a_Q\Big) * \mu^s_{\upsilon,j}(x)\Big|^2\Big)^{\frac12}\Big\|_{L^2}.\end{aligned} \tag{3.22}$$

We fix some $\upsilon\in\zeta_3^s$ in the subsequent discussions. Let $L^s_\upsilon = \mu^s_\upsilon * \widetilde{\mu^s_\upsilon}$. By the derivative conditions (1.4) we have

$$\begin{cases}\operatorname{supp}(\mu^s_\upsilon)\subset\big\{(y,\gamma(y)) : y\in B(x^s_\upsilon, 2^{-\epsilon N's})\big\},\\ |\gamma(y)-\gamma(z)| \le |\gamma(y)-\gamma(x^s_\upsilon)| + |\gamma(z)-\gamma(x^s_\upsilon)|\\ \quad\lesssim (2^{2\epsilon Ns})(2^{-\epsilon N's}), \quad \text{if } y,z\in B(x^s_\upsilon, 2^{-\epsilon N's}).\end{cases}$$

Thus we see that

$$\operatorname{supp}(L^s_\upsilon)\subset\big\{x : |x| < 2^{\epsilon(2N-N')s}\big\}. \tag{3.23}$$

Note that for each $\sigma\le\tau$ and $Q\in\mathcal{R}_{\sigma,\mathbb{F}(\sigma)}$, there exists a unique $q\in\mathcal{R}_{\sigma,\tau}$ so that $Q\subset q$. Thus for each $q\in\mathcal{R}_{\sigma,j-s}$ with $\sigma\le j-s$, if we define

$$A_q(x) = \sum\lambda_Q a_Q(x) \quad\text{and}\quad \lambda_q = \sum\lambda_Q \tag{3.24}$$

where the sums are taken over all $Q\subset q$ with $\sigma(Q)=\sigma(q)$ and $\Gamma(Q)=j-s$, then

$$\sum_{\Gamma(Q)=j-s}\lambda_Q a_Q = \sum_{\sigma\le j-s}\ \sum_{q\in\mathcal{R}_{\sigma,j-s}}\ \sum_{\substack{Q\subset q,\ \sigma(Q)=\sigma\\ \Gamma(Q)=j-s}}\lambda_Q a_Q = \sum_{\sigma\le j-s}\ \sum_{q\in\mathcal{R}_{\sigma,j-s}} A_q.$$

For each $r>0$ and $x=(x',x_{d+1})\in\mathbb{R}^{d+1}$, let

$$r\circ x = (rx', r^m x_{d+1}).$$

Then by change of variable $x\to 2^j\circ x$, we have

$$\Big\|\sum_{\Gamma(Q)=j-s}\lambda_Q a_Q * \mu^s_{\upsilon,j}\Big\|_2^2 = 2^{j(d+m)}\Big\|\sum_\sigma\sum_{q\in\mathcal{R}_{\sigma,j-s}} A^j_q * \mu^s_\upsilon\Big\|_2^2,$$

where $A^j_q(x) := A_q(2^j\circ x)$. Note that

$$\Big\|\sum_\sigma\sum_{q\in\mathcal{R}_{\sigma,j-s}} A^j_q * \mu^s_\upsilon\Big\|_2^2 \le 2\sum_{\substack{\sigma,\sigma'\\ \sigma\le\sigma'}}\ \sum_{\substack{q\in\mathcal{R}_{\sigma,j-s}\\ q'\in\mathcal{R}_{\sigma',j-s}}}\Big|\langle A^j_q, A^j_{q'} * L^s_\upsilon\rangle\Big| = 2(\mathbf{I}+\mathbf{II}),$$

where

$$\mathbf{I} = \sum_{q'} \sum_{\operatorname{dist}(2^{-j}\circ q, 2^{-j}\circ q') \le 2^{\sigma(q')-j}} \Big|\langle A_q^j, A_{q'}^j * L_\upsilon^s\rangle\Big|,$$
$$\mathbf{II} = \sum_{q'} \sum_{\operatorname{dist}(2^{-j}\circ q, 2^{-j}\circ q') > 2^{\sigma(q')-j}} \Big|\langle A_q^j, A_{q'}^j * L_\upsilon^s\rangle\Big|.$$

Note that $\operatorname{supp}(A_q^j) \subset 2^{-j}\circ q$. Since $\sigma(q)$, $\sigma(q') \le j-s$, we see that A_q^j and $A_{q'}^j$ are supported in balls of radius $\lesssim 2^{-s}$. And so by (3.23) if $\epsilon(N'-2N) < 1$, then

$$\big|\langle A_q^j, A_{q'}^j * L_\upsilon^s\rangle\big| = 0 \quad \text{when} \quad \operatorname{dist}(2^{-j}\circ q, 2^{-j}\circ q') > 2^{\epsilon(2N-N')s}. \tag{3.25}$$

For each rectangle $R \subset \mathbb{R}^{d+1}$, let R^* denote the double of the rectangle R, and λ_q be as in (3.24).

Lemma 3.6. *For each $q \in \mathcal{R}_{\sigma(q),j-s}$, $q' \in \mathcal{R}_{\sigma(q'),j-s}$ with $\sigma(q) \le \sigma(q')$, we have the following:*

(1) *If $d=1$, then $\big\|A_{q'}^j * \mu_\upsilon^s\big\|_{L^\infty} \lesssim 2^{-(1+m)j}\, 2^{-\sigma(q')+j}\, \lambda_{q'}$.*
(2) *If $d \ge 2$, then $\big\|A_{q'}^j * L_\upsilon^s\big\|_{L^\infty} \lesssim 2^{\epsilon N' s} 2^{-(d+m)j}\, 2^{(-\sigma(q')+j)}\, \lambda_{q'}$.*
(3) *If $(2^{-j}\circ q) \cap (2^{-j}\circ q')^* = \emptyset$, then*

$$\big|\langle A_q^j, A_{q'}^j * L_\upsilon^s\rangle\big| \lesssim 2^{5\epsilon N' s} 2^{-2(d+m)j} \frac{2^{\sigma(q')-j}}{\operatorname{dist}(2^{-j}\circ q, 2^{-j}\circ q')^2}\, \lambda_q \lambda_{q'}.$$

Proof of Lemma 3.6. For (1), we note that

$$\gamma(t) = \begin{cases} |t|^m \gamma(1), & \text{if } t > 0, \\ -|t|^m \gamma(1), & \text{if } t < 0. \end{cases} \tag{3.26}$$

Let $a_Q^j(x) := a_Q(2^j \circ x)$ where $Q \subset q'$ with $\sigma(Q) = \sigma(q')$; then from (3.26) and the support condition (3.1) it is easy to see that

$$\big|a_Q^j * \mu_\upsilon^s(x)\big| \le |Q|^{-1} \int_{|t|\sim 1} \chi_{2^{-j}\circ Q}(x - (t, \gamma(t)))dt \lesssim 2^{-(1+m)j}\, 2^{-\sigma(q')+j}.$$

For (2), let (z', z_{d+1}) be the center of $2^{-j}\circ Q$; then by applying (3.17) with $\alpha = 0$ we have

$$\begin{aligned} \big|a_Q^j * L_\upsilon^s(x)\big| &\le 2^{\epsilon N' s} |Q|^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \chi_{2^{-j}\circ Q}(x - (y', y_{d+1}))|y'|^{-1} dy' dy_{d+1} \\ &\lesssim 2^{\epsilon N' s} |Q|^{-1} \int \int_{\substack{|x'-y'-z'| \lesssim 2^{\sigma(q')-j} \\ |x_{d+1}-y_{d+1}-z_{d+1}| \lesssim 2^{m(\sigma(q')-j)}}} |y'|^{-1} dy' dy_{d+1} \\ &\lesssim 2^{\epsilon N' s} |Q|^{-1} 2^{(\sigma(q')-j)(d+m-1)}. \end{aligned}$$

For (3), let x_0 be the center of $2^{-j}\circ q$; then by applying (3.17) with $|\alpha|=1$, we have

$$\begin{aligned}\big|\langle A_q^j, A_{q'}^j * L_\upsilon^s\rangle\big| &= \Big|\iint A_q^j(x)A_{q'}^j(y)\big(L_\upsilon^s(x-y)-L_\upsilon^s(x_0-y)\big)dxdy\Big|\\ &\le \|A_q^j\|_{L^1}\sup_{x\in 2^{-j}\circ q}\int |A_{q'}^j(y)||x-x_0||\nabla L_\upsilon^s(\bar{x}_0-y)|dy\\ &\lesssim \|A_q^j\|_{L^1}\|A_{q'}^j\|_{L^1}\frac{2^{5\epsilon N's}\,2^{\sigma(q')-j}}{\mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q')^2}\end{aligned}$$

where $\bar{x}_0$ lies in the line connecting x_0 and x. □

Estimation of I. By (1) and (2) of Lemma 3.6 we have

$$\begin{aligned}\mathbf{I} &\lesssim \sum_{q'}\sum_{\mathrm{dist}(2^{-j}\circ q,2^{-j}\circ q')\le 2^{\sigma(q')-j}} 2^{\epsilon N's}2^{-2(d+m)j}2^{(-\sigma(q')+j)}\lambda_q\lambda_{q'}\\ &\lesssim 2^{\epsilon N's}2^{-2(d+m)j}\sum_{q'}2^{(-\sigma(q')+j)}\lambda_{q'}\Big(\sum_{\mathrm{dist}(2^{-j}\circ q,2^{-j}\circ q')\le 2^{\sigma(q')-j}}\lambda_q\Big).\end{aligned}$$

For each fixed $q'\in\mathcal{R}_{\sigma(q'),j-s}$ there are at most $O(2^{(m-1)s})$ number of $q''\in\mathcal{R}_{\sigma(q'),j-s}$ such that

$$\mathrm{dist}(2^{-j}\circ q'', 2^{-j}\circ q')\le 2^{\sigma(q')-j+1}.$$

Therefore by (4) of Lemma 3.3 we have

$$\sum_{\substack{q\,:\,\sigma(q)\le\sigma(q')\\ \mathrm{dist}(2^{-j}\circ q,2^{-j}\circ q')\le 2^{\sigma(q')-j}}}\lambda_q \lesssim 2^{(m-1)s}\Big(\alpha 2^{\sigma(q')+(d+m-1)(j-s)}\Big)$$

and so

$$\mathbf{I}\lesssim 2^{(\epsilon N'-d)s}2^{-(d+m)j}\alpha\sum_{\Gamma(Q)=j-s}\lambda_Q. \tag{3.27}$$

Estimation of II. By using (3) of Lemma 3.6 and (3.25), we have

$$\begin{aligned}\mathbf{II} &\lesssim 2^{\epsilon(5N')s}2^{-2(d+m)j}\sum_{q'}\sum_{2^{\sigma(q')-j}<\mathrm{dist}(2^{-j}\circ q,2^{-j}\circ q')}\lambda_q\lambda_{q'}\frac{2^{\sigma(q')-j}}{\mathrm{dist}(2^{-j}\circ q,2^{-j}\circ q')^2}\\ &\lesssim 2^{\epsilon(5N')s}2^{-2(d+m)j}\sum_{q'}\Big(2^{\sigma(q')-j}\lambda_{q'}\Big)\\ &\quad\times\Big(\sum_{2^{\sigma(q')-j}<\mathrm{dist}(2^{-j}\circ q,2^{-j}\circ q')}\frac{\lambda_q}{\mathrm{dist}(2^{-j}\circ q,2^{-j}\circ q')^2}\Big)\\ &\lesssim 2^{\epsilon(5N')s}2^{-2(d+m)j}\sum_{q'}2^{\sigma(q')-j}\lambda_{q'}\big(\mathbf{II}_1+\mathbf{II}_2\big)\end{aligned}$$

where

$$\mathbf{II}_1=\sum_{\ell\ge 0}\sum_{\substack{q:\ \mathrm{dist}(2^{-j}\circ q,2^{-j}\circ q')\sim 2^{\ell+\sigma(q')-j}\\ \ell+\sigma(q')-j\le -s+(d+1)}}\frac{\lambda_q}{\mathrm{dist}(2^{-j}\circ q,2^{-j}\circ q')^2}$$

and

$$\mathbf{II}_2 = \sum_{\ell\geq 0} \sum_{\substack{q:\ \mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q')\sim 2^{\ell+\sigma(q')-j} \\ -s+(d+1)\leq \ell+\sigma(q')-j\leq (2N-N')\epsilon s}} \frac{\lambda_q}{\mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q')^2},$$

where $\mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q') \sim 2^{\ell+\sigma(q')-j}$ denotes

$$2^{\ell+\sigma(q')-j} \leq \mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q') < 2^{\ell+1+\sigma(q')-j}.$$

For $\mathbf{II}_1$, for each positive integer ℓ, we consider the contribution of λ_q over all q disjoint from q'^* with $\sigma(q)\leq\sigma(q')$ and $\mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q') \sim 2^{\ell+\sigma(q')-j}$. There are at most $O(2^{(m-1)s})$ number of $q''\in\mathcal{R}_{\ell+\sigma(q'),j-s}$ containing all such q's. To see this, note that $2^{-j}\circ q'$ is contained in a cube of sidelength $2^{\sigma(q')-j}$. If $q'''\in\mathcal{R}_{\ell+\sigma(q'),j}$, then $2^{-j}\circ q'''$ is a rectangle of sidelengths $\sim 2^{\ell+\sigma(q')-j}$. Thus there are fixed finite number of rectangles $q'''\in\mathcal{R}_{\ell+\sigma(q'),j}$ so that $\mathrm{dist}(2^{-j}\circ q', 2^{-j}\circ q''') \sim 2^{\ell+\sigma(q')-j}$. Since each $q'''\in\mathcal{R}_{\ell+\sigma(q'),j}$ can be written as a union of $O(2^{(m-1)s})$ number of $q''\in\mathcal{R}_{\ell+\sigma(q'),j-s}$, we have the result. Therefore when $\ell+\sigma(q')\leq j-s+(d+1)$, we can use (4) of Lemma 3.3 to obtain

$$\begin{aligned}\mathbf{II}_1 &= \sum_{\ell\geq 0} \sum_{\substack{q:\ \mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q')\sim 2^{\ell+\sigma(q')-j} \\ \ell+\sigma(q')-j\leq -s+(d+1)}} \frac{\lambda_q}{\mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q')^2} \\ &\lesssim \sum_{\ell\geq 0} \left(2^{(m-1)s}\right)\left(\alpha 2^{(\ell+\sigma(q'))+(j-s)(d+m-1)}\right) 2^{-2(\ell+\sigma(q')-j)} \\ &\lesssim \alpha 2^{-\sigma(q')+j(d+m+1)-ds}.\end{aligned} \tag{3.28}$$

For $\mathbf{II}_2$, we recall the definition

$$\lambda_q = \sum_{\substack{Q\subset q \\ \sigma(Q)=\sigma, \Gamma(Q)=j-s}} \lambda_Q$$

where $q\in\mathcal{R}_{\sigma,j-s}$. For each $\ell\geq 0$ with $-s+(d+1)\leq \ell+\sigma(q')-j\leq (2N-N')\epsilon s$, we consider all $Q\subset q$ that appear in the definition of λ_q where $q\in\mathcal{R}_{\sigma,j-s}$ with $\mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q') \sim 2^{\ell+\sigma(q')-j}$. For each $q\in\mathcal{R}_{\sigma,j-s}$ with $\sigma\leq j-s$, $2^{-j}\circ q$ is contained in a cube of sidelength 2^{-s} and so

$$\mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q') \leq \mathrm{dist}(2^{-j}\circ Q, 2^{-j}\circ q') \leq \sqrt{d+1}\, 2^{-s} + \mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q'). \tag{3.29}$$

Since each $Q\in\mathbf{C}$ is contained in some $S(Q)\in\mathbf{S}$ and $\sigma(S(Q))<\Gamma(Q)=j-s$, $2^{-j}\circ S(Q)$ is contained in a cube of sidelength 2^{-s} and so

$$\begin{aligned}\mathrm{dist}(2^{-j}\circ S(Q), 2^{-j}\circ q') &\leq \mathrm{dist}(2^{-j}\circ Q, 2^{-j}\circ q') \\ &\leq \sqrt{d+1}\, 2^{-s} + \mathrm{dist}(2^{-j}\circ S(Q), 2^{-j}\circ q').\end{aligned} \tag{3.30}$$

By (3.29) and (3.30) we have

$$\mathrm{dist}(2^{-j}\circ q, 2^{-j}\circ q') \sim \mathrm{dist}(2^{-j}\circ Q, 2^{-j}\circ q') \sim \mathrm{dist}(2^{-j}\circ S(Q), 2^{-j}\circ q'). \tag{3.31}$$

By (1) of Lemma 3.2, for each $S \in \mathbf{S}$ we have $\sum_{Q\subset S} \lambda_Q \le C\alpha|S|$ and so by (3.31)

$$\begin{aligned}
\mathbf{II}_2 &\lesssim \sum_{\ell\ge 0} \sum_{\substack{Q:\ \mathrm{dist}(2^{-j}\circ Q, 2^{-j}\circ q')\sim 2^{\ell+\sigma(q')-j} \\ -s+(d+1)\le \ell+\sigma(q')-j\le (2N-N')\epsilon s}} \frac{\lambda_Q}{\mathrm{dist}(2^{-j}\circ Q, 2^{-j}\circ q')^2} \\
&\lesssim \sum_{\substack{S\in\mathbf{S} \\ 2^{-s}\lesssim \mathrm{dist}(2^{-j}\circ S, 2^{-j}\circ q')\lesssim 2^{(2N-N')\epsilon s}}} \sum_{Q\subset S} \frac{\lambda_Q}{\mathrm{dist}(2^{-j}\circ S, 2^{-j}\circ q')^2} \\
&\lesssim \sum_{\substack{S\in\mathbf{S} \\ 2^{-s}\lesssim \mathrm{dist}(2^{-j}\circ S, 2^{-j}\circ q')\lesssim 2^{(2N-N')\epsilon s}}} \frac{\alpha|S|}{\mathrm{dist}(2^{-j}\circ S, 2^{-j}\circ q')^2}.
\end{aligned}$$

Since $2^{-j}\circ S$ and $2^{-j}\circ q'$ are contained in cubes of sidelengths 2^{-s} and if $\mathrm{dist}(2^{-j}\circ S, 2^{-j}\circ q') \gtrsim 2^{-s}$, then

(3.32)
$$|2^{-j}\circ(x-y)| \sim \mathrm{dist}(2^{-j}\circ S, 2^{-j}\circ q') \sim \mathrm{dist}(2^{-j}\circ x, 2^{-j}\circ q') \quad \text{for all } x\in S \text{ and } y\in q'.$$

Fix $y\in q'$. Since parabolic dyadic cubes $S\in\mathbf{S}$ are pairwise disjoint, by (3.32) we have

$$\begin{aligned}
\mathbf{II}_2 &\lesssim \alpha \int_{2^{-s}\lesssim |2^{-j}\circ(x-y)|\lesssim 2^{(2N-N')\epsilon s}} |2^{-j}\circ(x-y)|^{-2}\,dx \\
&\lesssim \alpha 2^{j(d+m)} \int_{\left(x\in\mathbb{R}^{d+1}\,:\,2^{-s}\lesssim |x|\lesssim 2^{(2N-N')\epsilon s}\right)} |x|^{-2}\,dx \\
&\lesssim \alpha 2^{j(d+m)} 2^{(2N-N')(d-1)\epsilon s}\log(s).
\end{aligned} \tag{3.33}$$

Hence, by (3.28) and (3.33) with the condition $\sigma(q')\le j-s$, we have

$$\begin{aligned}
\mathbf{II} &\lesssim 2^{\epsilon(5N')s} 2^{-2(d+m)j} \sum_{q'} 2^{\sigma(q')-j}\ \lambda_{q'}\big(\mathbf{II}_1+\mathbf{II}_2\big) \\
&\lesssim \alpha 2^{\epsilon(5N')s} 2^{-(d+m)j}\Big(2^{-ds} + 2^{[(2N-N')(d-1)\epsilon-1]s}\log(s)\Big) \sum_{\Gamma(q')=j-s} \lambda_{q'}.
\end{aligned} \tag{3.34}$$

Thus if ϵ is sufficiently small, then from (3.22), (3.27), and (3.34) we have

$$\begin{aligned}
\|\mathrm{M}'_{H3} b\|_{L^2} &\lesssim \sum_{s=0}^{\infty} 2^{\epsilon dN's}\Big(\sum_j 2^{(d+m)j}(\mathbf{I}+\mathbf{II})\Big)^{\frac12} \\
&\lesssim \big(\alpha\sum\lambda_Q\big)^{\frac12}
\end{aligned}$$

and (3.21) follows. This completes the proof. □

4. Proof of Theorem 2: The case $0<m<1$

In the case $0<m<1$, the geometry of the graph $(y,\gamma(y))$ is different from that of the case $m\ge 1$, and so we modify the previous tendril constructions. Even though the context is changed, we abuse the notation to give a convenient environment to compare arguments of each case. First, we dominate M_H by the summation of two lacunary maximal operators. For each $r\in[2^{j/m}, 2^{(j+1)/m})$ we write

$$r^{-d}\psi(r^{-1}y) - 2^{-jd/m}\psi(2^{-j/m}y) = \int_{2^{j/m}}^{r} \frac{d}{dt}\left(t^{-d}\psi(t^{-1}y)\right)dt = \int_{2^{j/m}}^{r} t^{-d}\Psi(t^{-1}y)\,\frac{dt}{t},$$

where $\Psi(y) = -d\psi(y) - y \cdot \nabla\psi(y)$. Therefore

$$\mathrm{M}_H f(x) \le \mathrm{M}'_H f(x) + \mathrm{M}''_H f(x),$$

where

$$\mathrm{M}'_H f(x) = \sup_{j\in\mathbb{Z}} \Big| \int_{\mathbb{R}^d} f(x-(y,\gamma(y)))2^{-jd/m}\psi(2^{-j/m}y)dy\Big|,$$

$$\mathrm{M}''_H f(x) = \sup_{j\in\mathbb{Z}} \left[\int_{2^{j/m}}^{2^{(j+1)/m}} \Big|\int_{\mathbb{R}^d} f(x-(y,\gamma(y)))t^{-d}\Psi(t^{-1}y)dy\Big| \frac{dt}{t}\right]^{1/2}.$$

As in the case $m \ge 1$, we will only prove that M'_H maps the parabolic H^1 to weak-L^1 under the assumption $\mathrm{supp}(\psi) \subset \{y : 1 < |y| < 2\}$.

Let $0 < m < 1$ be a real number. For each $\tau \in \mathbb{Z}$, we define

$$\mathbb{F}(\tau) := \max\Big\{\tau' \in \mathbb{Z} : \Big\lfloor\Big(\frac{1}{m}-1\Big)\tau\Big\rfloor = \Big\lfloor\Big(\frac{1}{m}-1\Big)\tau'\Big\rfloor\Big\}$$

and we set $\mathbb{F} = \{\mathbb{F}(\tau) : \tau \in \mathbb{Z}\}$. Then for each $\tau \in \mathbb{Z}$, there exists $0 \le \theta_\tau < 1$ such that $(\frac{1}{m}-1)\tau + \theta_\tau = (\frac{1}{m}-1)\mathbb{F}(\tau)$, and so $|\mathbb{F}(\tau) - \tau| < m/(m-1)$. For $\tau \in \mathbb{F}$ and $\tau_\pm$ as in the previous section, it is easy to see that

$$|\tau - \tau_+|,\ |\tau - \tau_-| \le m/(m-1).$$

For each $\sigma, \tau \in \mathbb{Z}$ with $\tau \ge \sigma$, let $\mathcal{R}_{\sigma,\tau}$ denote the set of all rectangles in $\mathbb{R}^{d+1}$ with sides parallel to the axes, of the form

$$I \times \Big[i2^\sigma, (i+1)2^\sigma\Big]$$

where I's are dyadic cubes in $\mathbb{R}^d$ with sidelengths $2^{\sigma+\lfloor(\frac{1}{m}-1)\tau\rfloor}$ and $i \in \mathbb{Z}$. Let $\mathcal{R}$ denote the union of all the classes $\mathcal{R}_{\sigma,\tau}$. Note that

$$\mathcal{R} = \bigcup_{\sigma\le\tau} \mathcal{R}_{\sigma,\tau} = \biguplus_{\sigma\le\tau,\tau\in\mathbb{F}} \mathcal{R}_{\sigma,\tau}.$$

For each $q \in \mathcal{R}$, there are unique $\sigma \in \mathbb{Z}$, $\tau \in \mathbb{F}$ with $\sigma \le \tau$ such that $q \in \mathcal{R}_{\sigma,\tau}$. Let

$$\sigma(q) := \sigma, \quad \tau(q) := \tau \quad \text{if } q \in \mathcal{R}_{\sigma,\tau} \text{ with } \tau \in \mathbb{F}.$$

Let μ be the measure given by

$$\langle f, \mu\rangle = \int_{\mathbb{R}^d} f(y,\gamma(y))\psi(y)dy.$$

Also for each $j \in \mathbb{Z}$ we define a measure μ_j by

$$\langle f, \mu_j\rangle = 2^{-jd/m}\int_{\mathbb{R}^d} f(y,\gamma(y))\psi(2^{-j/m}y)dy.$$

For each $q \in \mathcal{R}_{\sigma,\tau}$ with $\tau \in \mathbb{F}$, we define the tendril $T(q)$ to be the set

$$T(q) = 2q + \Big\{\,(y,\gamma(y))\ :\ |y| \le 2^{\frac{\tau(q)_+}{m}}\,\Big\}. \tag{4.1}$$

Thus $T(q)$ is defined so that the functions $\chi_q * \mu_j$, $j \le \tau(q)_+$ are supported on it.

Lemma 4.1. *For each $q \in \mathcal{R}_{\sigma,\tau}$ with $\tau \in \mathbb{F}$, let the tendril $T(q)$ be as in* (4.1)*; then*

$$|T(q)| \lesssim 2^{\sigma+(d\tau)/m}.$$

Lemma 4.2. *By "parabolic dyadic cubes" we shall mean a collection $\mathcal{R}_\sigma := \mathcal{R}_{\sigma,\sigma}$ ($\sigma \in \mathbb{Z}$) of rectangles in $\mathbb{R}^{d+1}$. For any $\alpha > 0$ and any finite collection $\mathbf{Q}$ of parabolic dyadic cubes Q in $\mathbb{R}^{d+1}$ with associated positive scalars λ_Q, there exists a collection of pairwise disjoint parabolic dyadic cubes $\{S\}_{S\in\mathbf{S}}$ such that*

(1) $\sum_{Q:Q\subset S} \lambda_Q \le C\alpha|S|$ *for all* $S \in \mathbf{S}$,

(2) $\sum_{S:S\in\mathbf{S}} |S| \le \alpha^{-1} \sum \lambda_Q$,

(3) $\Big\|\sum_{Q:\text{not contained in any } S} \lambda_Q |Q|^{-1}\chi_Q\Big\|_\infty \le \alpha$.

Let $\mathbf{C}$ be the collection of parabolic dyadic cubes $Q \in \mathbf{Q}$ which is contained in some $S \in \mathbf{S}$. And for each $Q \in \mathbf{C}$ define $S(Q)$ as the unique $S \in \mathbf{S}$ containing Q.

Lemma 4.3. *Given $\alpha > 0$ there exists a function $\Gamma : \mathbf{C} \to \mathbb{Z}$ and a measurable set E such that*

(1) $|E| \le C\left(\alpha^{-1}\sum_{Q\in\mathbf{Q}} \lambda_Q + \sum_{S\in\mathcal{S}} |S|\right)$,

(2) $\chi_Q * \mu_j$ *is supported in E for all $j < \Gamma(Q)$ and all $Q \in \mathbf{C}$,*

(3) $\Gamma(Q) \ge \sigma(S(Q))$,

(4) *for any $\sigma, \tau \in \mathbb{Z}$ with $\tau \ge \sigma$, and any $q \in \mathcal{R}_{\sigma,\tau}$,*

$$\sum_{Q\subset q\,:\,\Gamma(Q)\le\tau} \lambda_Q \le C\alpha 2^{\sigma+d\tau/m}.$$

As in Section 3, it suffices to consider the maximal operator

$$\mathrm{M}'_H b(x) = \sup_j \Big|\sum_{s=0}^{\infty}\Big(\sum_{\Gamma(Q)=j-s} \lambda_Q a_Q\Big) * \mu_j(x)\Big|.$$

Given $\delta > 0$, we use a partition of unity with a finite index set ζ to write

$$\mu = \sum_{\upsilon\in\zeta} \mu_\upsilon$$

where $\mathrm{supp}(\mu_\upsilon) \subset \big\{(y, \gamma(y)) : y \in B(x_\upsilon, \delta\beta^{N'})\big\}$ with $N' \ge N(3d+1)+3$. Then by Lemma 3.5, we can choose δ (not depending on β) such that if there exists $y_\upsilon \in B(x_\upsilon, \delta\beta^{N'})$ with $|\gamma''(y_\upsilon)| \ge \beta$ and $\mathrm{dist}(B_{\delta\beta^{N'}}, \Sigma) \ge \beta$, then $\mu_\upsilon * \widetilde{\mu_\upsilon}(x)$ is a function which is not zero only if

$$|x_{d+1}| \le D_1\beta^{-2N}|x'| \quad \text{where } x = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R}, \tag{4.2}$$

and for $|\alpha| \le 1$

$$\big|\partial_x^\alpha[\mu_\upsilon * \widetilde{\mu_\upsilon}](x)\big| \le C_{\alpha,\delta}\beta^{-(1+4|\alpha|)N'}|x|^{-1-|\alpha|}. \tag{4.3}$$

Here the constant $C_{\alpha,\delta}$ does not depend on υ. Now for each $s = 0, 1, 2, \cdots$, we use a partition of unity with a finite index set ζ^s to write

$$\mu = \sum_{\upsilon\in\zeta^s} \mu_\upsilon^s$$

where $\mathrm{supp}(\mu_\upsilon^s) \subset \big\{(y, \gamma(y)) : y \in B_\upsilon^s\big\}$, B_υ^s is a ball of diameter $\delta 2^{-\epsilon N's}$(here δ is as in (4.3) and $0 < \epsilon < 1$ will be determined later) and $\{B_\upsilon^s\}_{\upsilon\in\zeta^s}$ have bounded overlap independent of s. Note that

$$\mathrm{card}(\zeta^s) \lesssim 2^{\epsilon N' sd}. \tag{4.4}$$

We split ζ^s into disjoint subsets ζ_1^s, ζ_2^s and ζ_3^s having the following properties:

$$\begin{cases} \zeta_1^s := \big\{\upsilon \in \zeta^s : \operatorname{dist}(\Sigma, B_\upsilon^s) \le 2^{-\epsilon s}\big\}; \\ \zeta_2^s := \big\{\upsilon \in \zeta^s : \operatorname{dist}(\Sigma, B_\upsilon^s) > 2^{-\epsilon s} \text{ and } |\det(\gamma''(y))| \le 2^{-\epsilon s} \text{ for all } y \in B_\upsilon^s\big\}; \\ \zeta_3^s := \big\{\upsilon \in \zeta^s : \operatorname{dist}(\Sigma, B_\upsilon^s) > 2^{-\epsilon s} \text{ and } |\det(\gamma''(y))| > 2^{-\epsilon s} \text{ for some } y \in B_\upsilon^s\big\}. \end{cases}$$

By the geometric conditions for the domain (1.3)

$$\sum_{\upsilon \in \zeta_1^s} \|\mu_\upsilon^s\|_{\mathcal{M}} \le C\big|\big\{y \in \mathbb{R}^d : \operatorname{dist}(\Sigma, y) \le 2^{-\epsilon s}\big\}\big| \le C2^{-\epsilon s d_0}. \tag{4.5}$$

Also by the Hessian condition (1.8), we have

$$\sum_{\upsilon \in \zeta_2^s} \|\mu_\upsilon^s\|_{\mathcal{M}} \le C2^{-\epsilon s \eta}. \tag{4.6}$$

Let

$$\langle f, \mu_{\upsilon,j}^s\rangle = \int f(2^{j/m}y, 2^j\gamma(y))\psi_\upsilon^s(y)\,dy,$$

where ψ_υ^s is a smooth function supported in B_υ^s and satisfies $|\partial^\alpha \psi_\upsilon^s(y)| \le C_\alpha(\delta 2^{-\epsilon N' s})^{-|\alpha|}$ for each α. Then we have

$$\mathrm{M}'_H b(x) \le \mathrm{M}'_{H1} b(x) + \mathrm{M}'_{H2} b(x) + \mathrm{M}'_{H3} b(x)$$

where

$$\mathrm{M}'_{Hi} b(x) = \sup_j \Big|\sum_{s=0}^\infty \Big(\sum_{\Gamma(Q)=j-s} \lambda_Q a_Q\Big) * \sum_{\upsilon \in \zeta_i^s} \mu_{\upsilon,j}^s(x)\Big|.$$

For part $\mathrm{M}'_{H1} b$, by (4.5) and Chebyshev's inequality we have

$$\big|\{x : \mathrm{M}'_{H1} b(x) > \alpha\}\big| \lesssim \alpha^{-1} \sum \lambda_Q.$$

Similarly by (4.6), we have

$$\big|\{x : \mathrm{M}'_{H2} b(x) > \alpha\}\big| \lesssim \alpha^{-1} \sum \lambda_Q.$$

For part $\mathrm{M}'_{H3} b$, by Minkowski's inequality we have

$$\mathrm{M}'_{H3} b(x) \le \sum_{s=0}^\infty \sum_{\upsilon \in \zeta_3^s} \Big(\sum_j \Big|\Big(\sum_{\Gamma(Q)=j-s} \lambda_Q a_Q\Big) * \mu_{\upsilon,j}^s(x)\Big|^2\Big)^{\frac12}.$$

And so from (4.4), we have

$$\begin{aligned} \|\mathrm{M}'_{H_3} b\|_{L^2} &\le \sum_{s=0}^\infty \sum_{\upsilon \in \zeta_3^s} \Big\|\Big(\sum_j \Big|\Big(\sum_{\Gamma(Q)=j-s} \lambda_Q a_Q\Big) * \mu_{\upsilon,j}^s(x)\Big|^2\Big)^{\frac12}\Big\|_{L^2} \\ &\lesssim \sum_{s=0}^\infty 2^{\epsilon d N' s} \max_{\upsilon \in \zeta_3^s} \Big\|\Big(\sum_j \Big|\Big(\sum_{\Gamma(Q)=j-s} \lambda_Q a_Q\Big) * \mu_{\upsilon,j}^s(x)\Big|^2\Big)^{\frac12}\Big\|_{L^2}. \end{aligned}$$

We fix some $\upsilon \in \zeta_3^s$ in the subsequent discussions. Let $L_\upsilon^s = \mu_\upsilon^s * \widetilde{\mu_\upsilon^s}$. By the derivative conditions (1.4), we have

$$\begin{cases} \operatorname{supp}(\mu_\upsilon^s) \subset \big\{(y, \gamma(y)) : y \in B(x_\upsilon^s, 2^{-\epsilon N' s})\big\}, \\ |\gamma(y) - \gamma(z)| \le |\gamma(y) - \gamma(x_\upsilon^s)| + |\gamma(z) - \gamma(x_\upsilon^s)| \\ \quad \lesssim (2^{2\epsilon N s})(2^{-\epsilon N' s}), \quad \text{if } y, z \in B(x_\upsilon^s, 2^{-\epsilon N' s}), \end{cases}$$

and so

$$\operatorname{supp}(L_\upsilon^s) \subset \big\{x : |x| < 2^{\epsilon(2N-N')s}\big\}. \tag{4.7}$$

Let q denote an element of any $\mathcal{R}_{\sigma,j-s}$ with $\sigma \le j-s$. Define $A_q(x) = \sum \lambda_Q a_Q(x)$ and $\lambda_q = \sum \lambda_Q$, the sums being taken over all $Q \subset q$ with $\sigma(Q) = \sigma(q)$ and $\Gamma(Q) = j-s$. Then by the change of variable $x \to 2^{\frac{j}{m}} \circ x$, we have

$$\Big\| \sum_{\Gamma(Q)=j-s} \lambda_Q a_Q * \mu_{\upsilon,j}^s \Big\|_2^2 = 2^{j(d+m)/m} \Big\| \sum_\sigma \sum_{q\in\mathcal{R}_{\sigma,j-s}} A_q^j * \mu_\upsilon^s \Big\|_2^2,$$

where $A_q^j(x) := A_q(2^{\frac{j}{m}} \circ x)$. Note that

$$\Big\| \sum_\sigma \sum_{q\in\mathcal{R}_{\sigma,j-s}} A_q^j * \mu_\upsilon^s \Big\|_2^2 \le 2 \sum_{\substack{\sigma,\sigma' \\ \sigma\le\sigma'}} \sum_{\substack{q\in\mathcal{R}_{\sigma,j-s} \\ q'\in\mathcal{R}_{\sigma',j-s}}} \Big|\langle A_q^j, A_{q'}^j * L_\upsilon^s\rangle\Big| = 2(\mathbf{I}+\mathbf{II}),$$

where

$$\mathbf{I} = \sum_{q'} \sum_{\operatorname{dist}(2^{-\frac{j}{m}}\circ q, 2^{-\frac{j}{m}}\circ q') \le 2^{\sigma(q')-j}} \Big|\langle A_q^j, A_{q'}^j * L_\upsilon^s\rangle\Big|,$$

$$\mathbf{II} = \sum_{q'} \sum_{\operatorname{dist}(2^{-\frac{j}{m}}\circ q, 2^{-\frac{j}{m}}\circ q') > 2^{\sigma(q')-j}} \Big|\langle A_q^j, A_{q'}^j * L_\upsilon^s\rangle\Big|.$$

Since $\sigma(q), \sigma(q') \le j-s$, $2^{-j}\circ q$ and $2^{-j}\circ q'$ are supported in balls of radius $\lesssim 2^{-s}$. Thus by (4.7) if $\epsilon(N'-2N) < 1$, then

$$\big|\langle A_q^j, A_{q'}^j * L_\upsilon^s\rangle\big| = 0 \quad \text{when} \quad \operatorname{dist}(2^{-\frac{j}{m}}\circ q, 2^{-\frac{j}{m}}\circ q') > 2^{\epsilon(2N-N')s}.$$

Lemma 4.4. *For each $q \in \mathcal{R}_{\sigma,j-s}$ and $q' \in \mathcal{R}_{\sigma',j-s}$ with $\sigma(q) \le \sigma(q')$, we have the following:*

(1) *If $d = 1$, then* $\big\|A_{q'}^j * \mu_\upsilon^s\big\|_{L^\infty} \lesssim 2^{-(1+\frac{1}{m})j}\, 2^{(-\sigma(q')+j)}\, \lambda_{q'}$.

(2) *If $d \ge 2$, then* $\big\|A_{q'}^j * L_\upsilon^s\big\|_{L^\infty} \lesssim 2^{3\epsilon N's}\, 2^{-(d+m)j/m} 2^{-\sigma(q')+j}\, \lambda_{q'}$.

(3) *If* $(2^{-\frac{j}{m}}\circ q) \cap (2^{-\frac{j}{m}}\circ q')^* = \emptyset$, *then*

$$\big|\langle A_q^j, A_{q'}^j * L_\upsilon^s\rangle\big| \lesssim 2^{5\epsilon N's} 2^{-2(d+m)j/m} \frac{2^{\sigma(q')-j}}{\operatorname{dist}(2^{-\frac{j}{m}}\circ q, 2^{-\frac{j}{m}}\circ q')^2} \lambda_q \lambda_{q'}.$$

Proof of Lemma 4.4. The proofs of (1) and (3) are similar to those of Lemma 3.6. For (2), by (4.2) and (4.3) with $\alpha = 0$,

$$\begin{cases} L_\upsilon^s(y) \ne 0 \quad \text{only if} \quad |y_{d+1}| \le D_1\, 2^{2\epsilon N s}|y'| \quad \text{where } y = (y', y_{d+1}) \in \mathbb{R}^d \times \mathbb{R}, \\ |L_\upsilon^s(y)| \le C_\delta 2^{\epsilon N's}|y|^{-1}. \end{cases}$$

Let $a_Q^j(x) := a_Q(2^{\frac{j}{m}} \circ x)$ where $Q \subset q'$ with $\sigma(Q) = \sigma(q')$; then $a_Q^j(x) \neq 0$ only if $x \in 2^{-\frac{j}{m}} \circ Q$. Let (z', z_{d+1}) be the center of $2^{-\frac{j}{m}} \circ Q$; then $a_Q^j(x', x_{d+1}) \neq 0$ only if $|x' - z'| \lesssim 2^{(\sigma(q')-j)/m}$. Hence

$$\begin{aligned}
&\left|a_Q^j * L_v^s(x)\right| \\
&\quad \lesssim 2^{\epsilon N' s}|Q|^{-1} \iint_{\substack{|y_{d+1}| \le D_1 2^{2\epsilon N s}|y'| \\ |x'-y'-z'| \lesssim 2^{(\sigma(q')-j)/m}}} \chi_{2^{-\frac{j}{m}} \circ Q}(x - (y', y_{d+1}))|y'|^{-1}\, dy_{d+1}\, dy' \\
&\quad \lesssim 2^{3\epsilon N' s}|Q|^{-1} 2^{(\sigma(q')-j)d/m}.
\end{aligned}$$

This completes the proof of the lemma. □

We resume the proof of the theorem. By the same arguments as in the case $m \geq 1$, we have

$$\begin{aligned}
\mathbf{I} &\lesssim 2^{(3\epsilon N' - d)s} 2^{-(d+m)j/m} \alpha \sum_{\Gamma(Q)=j-s} \lambda_Q, \\
\mathbf{II} &\lesssim \alpha 2^{\epsilon(5N')s} 2^{-(d+m)j/m} \Big(2^{-ds} + 2^{[(2N-N')(d-1)\epsilon - 1]s} \log(s)\Big) \sum_{\Gamma(Q)=j-s} \lambda_Q.
\end{aligned}$$

Therefore if ϵ is sufficiently small, then

$$\begin{aligned}
\|\mathrm{M}'_{H3} b\|_{L^2} &\lesssim \sum_{s=0}^{\infty} 2^{\epsilon d N' s} \Big(\sum_j 2^{(d+m)j/m} (\mathbf{I} + \mathbf{II}) \Big)^{\frac{1}{2}} \\
&\lesssim \big(\alpha \sum \lambda_Q\big)^{\frac{1}{2}}
\end{aligned}$$

and this completes the proof. □

4.1. **Remarks.**

(1) It would be interesting to determine whether the maximal operator

$$\mathrm{M}_H f(x) = \sup_{r>0} \Big| \int_{\mathbb{R}^2} f\big(x - (y, y_1^2 + y_2^3)\big)\, r^{-5} \psi(r^{-3} y_1, r^{-2} y_2)\, dy \Big|,$$

maps the parabolic H^1 into weak-L^1 or not, where the parabolic H^1 denotes the one-parameter Hardy space with respect to the scaling $x = (x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to (t^3 x_1, t^2 x_2, t^6 x_3)$.

(2) As was suggested by M. Christ [2], it is not known whether the maximal operator

$$\mathrm{M}_S f(x) = \sup_{r>0} \Big| \int_{\mathbb{R}} f\big(x - (t, t^2, t^3)\big)\, r^{-1} \psi(r^{-1} t)\, dt \Big|$$

maps the parabolic H^1 into weak-L^1 or not, where the parabolic H^1 denotes the one-parameter Hardy space with respect to the scaling $x = (x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to (t x_1, t^2 x_2, t^3 x_3)$.

5. Examples

In this section we give several examples related to each theorem.

5.1. Examples for Theorem 1.

(1) Let $\gamma(y_1, y_2) = (y_1^2 + y_1 y_2 + y_2^2)/(y_1 - y_2)$; then $\gamma \in \mathcal{SH}_{\mathbb{R}^2 \to \mathbb{R}}(\mathrm{U}, \Omega, \Sigma)$ where $\mathrm{U} = B(0,1)$, $\Omega = \{y \in \mathrm{U} : y_1 = y_2\}$ and $\Sigma = \Omega$. Note that
   (a) $\mathrm{meas}(\Sigma_\delta \setminus \Omega) \le C\delta$ for all $0 < \delta \ll 1$,
   (b) $|\partial_y^\alpha \gamma(y)| \le C_\alpha |y_1 - y_2|^{-(|\alpha|+1)} \le C_\alpha \mathrm{dist}(y, \Sigma)^{-(|\alpha|+1)}$ for all $y \in \mathrm{U} \setminus \Sigma$ and each multi-index α,
   (c) $\det\big(\mathbf{D}_{(1,2)}(\theta, y)\big) = \theta \det(\gamma''(y_1, y_2)) = \theta\left(38 y_1 y_2/(y_1 - y_2)^4\right)$ for each $\theta \in \mathbb{S}^0 = \{\pm 1\}$.

   Let μ be the measure on $\mathbb{R}^{2+1}$ defined by
$$\langle \mu, f\rangle = \int_{\mathbb{R}^2} f(y, \gamma(y))\psi(y)\chi_{\mathrm{U}\setminus\Omega}\, dy,$$
   where ψ is a smooth function supported in U. Then $|\widehat{\mu}(\xi)| \le C_\epsilon(1 + |\xi|)^{-\epsilon}$ for some $\epsilon > 0$.

(2) Let $\gamma(y_1, y_2) = (y_1^4 + y_2^4)/(y_1 - y_2)(y_1 + y_2)$. Then $\gamma \in \mathcal{SH}_{\mathbb{R}^2 \to \mathbb{R}}(\mathrm{U}, \Omega, \Sigma)$ where $\mathrm{U} = B(0,1)$, $\Omega = \{y \in \mathrm{U} : y_1 = \pm y_2\}$ and $\Sigma = \Omega$. Note that
   (a) $\mathrm{meas}(\Sigma_\delta \setminus \Omega) \le C\delta$ for all $0 < \delta \ll 1$,
   (b) $|\partial_y^\alpha \gamma(y)| \le C_\alpha |(y_1 - y_2)(y_1 + y_2)|^{-(|\alpha|+1)} \le C_\alpha \mathrm{dist}(y, \Sigma)^{-2(|\alpha|+1)}$ for all $y \in \mathrm{U} \setminus \Sigma$ and each multi-index α,
   (c) $\det(\gamma''(y_1, y_2)) = 4[y_1^8 + 8y_1^6 y_2^2 - 6y_1^4 y_2^4 + 8y_1^2 y_2^6 + y_2^8]/(y_1 - y_2)^4 (y_1 + y_2)^4$.

   Let μ be the measure on $\mathbb{R}^{2+1}$ given by
$$\langle \mu, f\rangle = \int_{\mathbb{R}^2} f(y, \gamma(y))\psi(y)\chi_{\mathrm{U}\setminus\Omega}\, dy,$$
   where ψ is a smooth function supported in U. Then $|\widehat{\mu}(\xi)| \le C_\epsilon(1 + |\xi|)^{-\epsilon}$ for some $\epsilon > 0$.

(3) Let $\gamma : \mathbb{R}^d \to \mathbb{R}$ be given by $\gamma(y) = |y|^m$ for some $m > 0$ $(m \ne 1)$. Then $\gamma \in \mathcal{SH}_{\mathbb{R}^d \to \mathbb{R}}(\mathrm{U}, \Omega, \Sigma)$ where $\mathrm{U} = B(0,2)$, $\Omega = B(0,2) \setminus B(0,1)$ and $\Sigma = \Omega \cup \{0\}$. Note that
   (a) $\mathrm{meas}(\Sigma_\delta \setminus \Omega) \le C\delta$ for all $0 < \delta \ll 1$,
   (b) $|\partial_y^\alpha \gamma(y)| \le C_\alpha |y|^{m-|\alpha|} \le C_\alpha \mathrm{dist}(y, \Sigma)^{-|\alpha|}$ for all $y \in \mathrm{U} \setminus \Sigma = B(0,1) \setminus \{0\}$ and each multi-index α,
   (c) $\det(\gamma''(y)) = (m-1)m^d |y|^{(m-2)d}$.

   Let μ be the measure on $\mathbb{R}^{d+1}$ defined by
$$\langle \mu, f\rangle = \int_{|y|<1} f(y, \gamma(y))\, dy = \int_{\mathbb{R}^d} f(y, \gamma(y))\psi(y)\chi_{\mathrm{U}\setminus\Omega}(y)\, dy,$$
   where ψ is a smooth function supported in U and $\psi = 1$ on $B(0,1)$. Then $|\widehat{\mu}(\xi)| \le C_\epsilon(1 + |\xi|)^{-\epsilon}$ for some $\epsilon > 0$.

(4) Let $\gamma : B(0,1) \subset \mathbb{R}^2 \to \mathbb{R}$ be given by $\gamma(y) = (1 - |y|)^{-m}$ for some $m > 0$. Then $\gamma \in \mathcal{SH}_{\mathbb{R}^2 \to \mathbb{R}}(\mathrm{U}, \Omega, \Sigma)$ where $\mathrm{U} = B(0,2)$, $\Omega = B(0,2) \setminus B(0,1)$ and $\Sigma = \Omega \cup \{0\}$. Note that
   (a) $\mathrm{meas}(\Sigma_\delta \setminus \Omega) \le C\delta$ for all $0 < \delta \ll 1$,
   (b) $|\partial_y^\alpha \gamma(y)| \le C_\alpha (1 - |y|)^{-m-|\alpha|} |y|^{-|\alpha|} \le C_\alpha \mathrm{dist}(y, \Sigma)^{-m-2|\alpha|}$ for all $y \in \mathrm{U} \setminus \Sigma = B(0,1) \setminus \{0\}$ and each multi-index α,
   (c) $\det(\gamma''(y)) = m^2(m+1)(1 - |y|)^{-2m-3}|y|^{-1}$.

Let μ be the measure on $\mathbb{R}^{d+1}$ defined by

$$\langle \mu, f\rangle = \int_{|y|<1} f(y,\gamma(y))\,dy = \int_{\mathbb{R}^2} f(y,\gamma(y))\psi(y)\chi_{\mathrm{U}\setminus\Omega}(y)\,dy,$$

where ψ is a smooth function supported in U and $\psi = 1$ on $B(0,1)$. Then $|\widehat{\mu}(\xi)| \le C_\epsilon(1+|\xi|)^{-\epsilon}$ for some $\epsilon > 0$.

(5) Let V be an open set in $\mathbb{R}^d$ such that $V \subset B(0,1)$, and let $\gamma : V \to \mathbb{R}^n$ be a smooth function. Assume the following:

(a) Let ∂V denote the boundary of V and satisfy

$$\mathrm{meas}((\partial V)_\delta) \le C\delta^{d_0} \quad \text{for some } d_0 > 0,$$

where $(\partial V)_\delta := \{y \in \mathbb{R}^d : \mathrm{dist}(\partial V, y) \le \delta\}$.

(b) For each $\theta \in \mathbb{S}^{n-1}$, there exist $1 \le \ell = \ell(\theta) \le d$ and $1 \le i_1(\theta) < i_2(\theta) < \cdots < i_\ell(\theta) \le d$ such that

$$\int_V \left|\det\big(\mathbf{D}_{(i_1,\cdots,i_\ell)}(\theta,y)\big)\right|^{-\eta} dy < C_\eta$$

for some $\eta > 0$ that is independent of θ and ℓ.

Let $\mathrm{U} = B(0,2)$, $\Omega = \mathrm{U}\setminus V$ and $\Sigma = \Omega$. Then by noting $\mathrm{U}\setminus\Sigma = V$ and $\Sigma_\delta \setminus \Omega \subset (\partial V)_\delta \cup (\partial \mathrm{U})_\delta$, we have $\gamma \in \mathcal{SH}_{\mathbb{R}^d\to\mathbb{R}^n}(\mathrm{U},\Omega,\Sigma)$. Let μ be the measure defined by

$$\langle \mu, f\rangle = \int_V f(y,\gamma(y))\,dy = \int_{\mathbb{R}^d} f(y,\gamma(y))\psi(y)\chi_{\mathrm{U}\setminus\Omega}(y)\,dy,$$

where ψ is a smooth function supported in U and $\psi = 1$ on $B(0,1)$. Then $|\widehat{\mu}(\xi)| \le C_\epsilon(1+|\xi|)^{-\epsilon}$ for some $\epsilon > 0$.

5.2. **Examples for Corollary 1.**

(1) If $\gamma(y_1,y_2) = (y_1y_2, y_1^2 \pm y_2^2)$, then $\gamma \in \mathcal{SH}_{\mathbb{R}^2\to\mathbb{R}^2}(\mathrm{U},\Omega,\Sigma)$ where $\mathrm{U} = \{y \in \mathbb{R}^2 : 1/2 < |y| < 4\}$, $\Omega = \emptyset$, and $\Sigma = \emptyset$. For each $\theta = (\theta_1,\theta_2) \in \mathbb{S}^1$,

$$\det\big(\mathbf{D}_{(1)}(\theta,y)\big) = 2\theta_2, \quad \det\big(\mathbf{D}_{(1,2)}(\theta,y)\big) = -\theta_1^2 \pm 4\theta_2^2.$$

Hence if $|\theta_2| \ge 1/4$, then $|\det\big(\mathbf{D}_{(1)}(\theta,y)\big)| \ge 1/2$, and if $|\theta_2| \le 1/4$, then $|\theta_1| \ge \sqrt{15}/4$ and so $|\det\big(\mathbf{D}_{(1,2)}(\theta,y)\big)| \ge 15/16$. Thus we see that M_S maps $L^p \cap \mathcal{S}$ to L^p for $1 < p \le \infty$.

(2) If $\gamma(t) = (|t|^a, |t|^b)$ with $a,b \ne 0,1$ and $a \ne b$, then $\gamma \in \mathcal{SH}_{\mathbb{R}\to\mathbb{R}^2}(\mathrm{U},\Omega,\Sigma)$ where $\mathrm{U} = \{t \in \mathbb{R} : 1/2 < |t| < 4\}$, $\Omega_{\mathrm{sym}} = \{0\}$, $\Omega = \mathrm{U} \cap \Omega_{\mathrm{sym}} = \emptyset$, and $\Sigma = \emptyset$. For each $\theta = (\theta_1,\theta_2) \in \mathbb{S}^1$, we have

$$|\det\big(\mathbf{D}_{(1)}(\theta,t)\big)| = |t|^{b-2}\left|\theta_1 a(a-1)|t|^{a-b} + \theta_2 b(b-1)\right|,$$

and so $\int_{\mathrm{U}} |\det\big(\mathbf{D}_{(1)}(\theta,t)\big)|^{-\eta}\,dt \le C_\eta$ if $0 < \eta < 1$. Hence M_S maps $L^p \cap \mathcal{S}$ to L^p for $1 < p \le \infty$.

5.3. **Examples for Theorem 2.**

(1) If $\gamma(y_1,\cdots,y_d) = |y|^m$, then $\det(\gamma''(y)) = (m-1)m^d|y|^{(m-2)d}$. Hence if $m > 0$ ($m \ne 1$), then M_H maps $H^1_m(\mathbb{R}^{d+1})$ to weak-$L^1(\mathbb{R}^{d+1})$ by taking $\Sigma = \emptyset$. The case $d = 1$ and $\gamma(y) = y^2$ was considered by M. Christ [2], and it is proved that M_H maps $H^1_2(\mathbb{R}^2)$ to weak-$L^1(\mathbb{R}^2)$. Moreover it was stated that if $d = 2$ and $\gamma(y) = y_1^2 + y_2^2$, then M_H maps $H^1_2(\mathbb{R}^3)$ to weak-$L^1(\mathbb{R}^3)$. Also if $d = 1$ and $\gamma(y) = |y|^m$ with $m > 1$, then A. Seeger and T. Tao [10]

proved that M_H maps $H^1_{\mathrm{prod}}(\mathbb{R}^2)$ to $L^{1,2}(\mathbb{R}^2)$. See [10] for the definition of the space H^1_{prod}.

(2) If $\gamma(y_1, y_2) = |y_1|^{1/2} \pm |y_2|^{1/2}$, then $\det(\gamma''(y_1, y_2)) = \pm\frac{1}{16}(|y_1||y_2|)^{-\frac{3}{2}}$. Hence M_H maps $H^1_{1/2}(\mathbb{R}^3)$ to weak-$L^1(\mathbb{R}^3)$ by taking $\Sigma = \{(y_1, y_2) \in \mathrm{U} : y_1 = 0, \text{or } y_2 = 0\}$.

(3) If $\gamma(y_1, y_2) = |y_1|^a|y_2|^b$, then $\det(\gamma''(x, y)) = ab(a+b-1)|y_1|^{2(a-1)}|y_2|^{2(b-1)}$. Hence if $a, b > 0$ and $a + b \neq 1$, then M_H maps $H^1_{a+b}(\mathbb{R}^3)$ to weak-$L^1(\mathbb{R}^3)$ by taking $\Sigma = \{(y_1, y_2) \in \mathrm{U} : y_1 = 0, \text{or } y_2 = 0\}$.

6. Appendix

6.1. In this subsection we prove that the maximal operator $\mathrm{M}_S f$ in (1.6) is a Borel measurable function if f is a bounded Borel measurable function. Let

$$\mathcal{B}_m = \text{the collection of Borel sets in } \mathbb{R}^m,$$
$$\mathcal{L}_m = \text{the collection of Lebesgue measurable sets in } \mathbb{R}^m.$$

If $f : \mathbb{R}^{d+n} \to \mathbb{R}$ is a $\mathcal{B}_{d+n}$-measurable function, then the function $f(y, \gamma(y))\chi_{\Omega^c_{\mathrm{sym}}}(y)$ is an $\mathcal{L}_d$-measurable function. To see this, let

$$\gamma^{\mathrm{ex}}(y) = \begin{cases} \gamma(y) & \text{if } y \in \Omega^c_{\mathrm{sym}}, \\ 0 & \text{if } y \in \Omega_{\mathrm{sym}}; \end{cases}$$

then γ^{ex} is a Lebesgue measurable function. Let $g(y) = (y, \gamma^{\mathrm{ex}}(y))$ and

$$\mathcal{B} := \{E \in \mathcal{B}_{d+n} \mid g^{-1}(E) \in \mathcal{L}_d\};$$

then $\mathcal{B}$ forms a σ algebra. Let $I \times J \subset \mathbb{R}^d \times \mathbb{R}^n$ be a rectangle and $J = J_1 \times \cdots \times J_n$; then $g^{-1}(I \times J) = I \cap (\gamma^{\mathrm{ex}})^{-1}(J) = I \cap_{i=1}^n (\gamma_i^{\mathrm{ex}})^{-1}(J_i)$ which is $\mathcal{L}_d$-measurable and so $I \times J \in \mathcal{B}$. Since any open set $U \subset \mathbb{R}^{d+n}$ can be written as a countable union of rectangles we have $U \in \mathcal{B}$. Therefore $\mathcal{B}_{d+n} \subset \mathcal{B}$. Since f is $\mathcal{B}_{d+n}$-measurable, for each $t \in \mathbb{R}$,

$$\{y : f(y, \gamma^{\mathrm{ex}}(y)) > t\} = g^{-1}\left(f^{-1}((t, \infty))\right) \in \mathcal{L}_d.$$

Therefore for each $B \in \mathcal{B}_{d+n}$, the function

$$\chi_B(y, \gamma^{\mathrm{ex}}(y))\chi_{\Omega^c_{\mathrm{sym}}}(y) = \chi_B(y, \gamma(y))\chi_{\Omega^c_{\mathrm{sym}}}(y)$$

is $\mathcal{L}_d$-measurable. For each $r > 0$, let μ_r be the measure defined by

$$\mu_r(B) = r^{-d} \int_{|y|<r} \chi_B(y, \gamma(y))\chi_{\Omega^c_{\mathrm{sym}}}(y)\, dy, \quad B \in \mathcal{B}_{d+n}.$$

Then μ_r is a finite Borel measure. Thus if f is a bounded Borel measurable function in $\mathbb{R}^{d+n}$, then $f * d\mu_r(x)$ is well defined for every $x \in \mathbb{R}^{d+n}$ and continuous in $r > 0$. Also if f is a bounded Borel measurable function in $\mathbb{R}^{d+n}$, then the function $F(x, y) = f(x - y)$ is $\mathcal{B}_{d+n} \times \mathcal{B}_{d+n} = \mathcal{B}_{2d+2n}$-measurable, and for any compact subset $C \subset \mathbb{R}^{d+n}$

$$F(x, y)\chi_C(x) \in L^1\big(\mathbb{R}^{d+n} \times \mathbb{R}^{d+n}, \mathcal{B}_{d+n} \times \mathcal{B}_{d+n}, dx \times d\mu_r(y)\big).$$

Hence by Fubini's theorem, we have

$$f * d\mu_r(x)\chi_C(x) = \int F(x, z)\chi_C(x) d\mu_r(z) \in L^1(\mathbb{R}^{d+n}, \mathcal{B}_{d+n}, dx),$$

and so $f * d\mu_r(x)$ is a Borel measurable function. Since $f * d\mu_r(x)$ is a continuous function of $r > 0$, we have

$$\mathrm{M}_S f(x) = \sup_{r\in\mathbb{Q}^+} r^{-d} \int_{|y|<r} |f(x-(y,\gamma(y)))|\, \chi_{\Omega^c_{\mathrm{sym}}}(y)dy,$$

where $\mathbb{Q}^+$ denotes the collection of positive rational numbers. Hence $\mathrm{M}_S f$ is Borel measurable. Similarly we can prove that $\mathrm{M}_H f$ is Borel measurable if f is a bounded Borel measurable function.

6.2. **Limiting arguments.** We prove the limiting arguments which were assumed in Section 3. By [7], for $f \in H^1_m$, there exist a sequence a_i of H^1_m-atoms and a sequence $\lambda_i > 0$ such that

$$\sum_{i=1}^{\infty} \lambda_i \le C\|f\|_{H^1_m}, \quad \text{and} \quad f = \sum_{i=1}^{\infty} \lambda_i a_i \quad \text{in } H^1_m,$$

where the constant C does not depend on f. Furthermore if f is bounded and Borel measurable, we may assume the atoms a_i are bounded and Borel measurable. Hence we can find a sequence $\{f_k\}$ of bounded and Borel measurable functions such that:

(1) each $f_k \in H^1_m$ has finite sum of atoms;
(2) $\|f_1\|_{H^1_m} \le C\|f\|_{H^1_m}$ and $\|f_k\|_{H^1_m} \le C2^{-2k}\|f\|_{H^1_m}$ for $k \ge 2$;
(3) $\sum f_k \to f$ in H^1_m.

Let $g_m = \sum_{k=1}^m f_k$; then for each j

$$g_m * d\mu_j \to f * d\mu_j \quad \text{in } L^1.$$

Hence $g_{m_k} * d\mu_j \to f * d\mu_j$ almost everywhere for some subsequence $\{m_k\}$ and

$$|f * d\mu_j| = \big|\lim_{m_k\to\infty} g_{m_k} * d\mu_j\big| \le \limsup_{m_k\to\infty} |g_{m_k} * d\mu_j| \le \sum |f_m * d\mu_j| \quad \text{a.e.}$$

Thus we have

$$\mathrm{M}'_H f(x) = \sup_j |f * d\mu_j(x)| \le \sum \sup_j |f_m * d\mu_j(x)| \quad \text{a.e.},$$

and by (2)

$$\begin{aligned} |\{\mathrm{M}'_H f > \alpha\}| &\le \sum_m \big|\{\sup_j |f_m * d\mu_j| > 2^{-m}\alpha\}\big| \\ &\le \sum_m C2^m\alpha^{-1}\, \|f_m\|_{H^1_m} \le 2C\alpha^{-1}\|f\|_{H^1_m}. \end{aligned}$$

6.3. **Proof of Lemma 3.3.** For (4), it suffices to show that

$$\sum_{Q\subset q:\, \Gamma(Q)\le\tau} \lambda_Q \le \alpha 2^{\sigma+(d+m-1)\tau_+}$$

for any $\sigma \in \mathbb{Z}$, $\tau \in \mathbb{F}$ with $\tau \ge \sigma$, and any $q \in \mathcal{R}_{\sigma,\tau}$. Let $\sigma_0 = \min\{\sigma(Q) : Q \in \mathbf{C}\}$. Select an integer $\tau_0 \in \mathbb{F}$ such that

$$\tau_0 > \max\{\sigma(Q) : Q \in \mathbf{C}\}, \quad \sum_{Q\in\mathbf{C}} \lambda_Q < \alpha 2^{\sigma_0+(d+m-1)\tau_0}.$$

For each fixed $\tau \in \mathbb{F}$ with $\tau \le \tau_0$, we will define a sequence of functions $\Lambda_{\sigma,\tau} : \mathcal{R}_{\sigma,\tau} \to \mathbb{R}$ by a descending induction on $\sigma \in \mathbb{Z}$ with $\sigma \le \tau$. We proceed with the same construction by a descending induction on $\tau \in \mathbb{F}$. At each step, we will define

subsets $\mathbf{C}_1$, $\mathbf{C}_2$ of $\mathbf{C}$ which will increase as we proceed. Let $\mathbf{C}_1, \mathbf{C}_2 \subset \mathbf{C}$ and $\tau \in \mathbb{F}$ be fixed for the moment, and define STC as follow:

Definition 6.1. STC (Stopping time construction). Define $\Lambda_{\sigma,\tau} : \mathcal{R}_{\sigma,\tau} \to \mathbb{R}$ with $\sigma \le \tau$. For each $q \in \mathcal{R}_{\sigma,\tau}$ define

$$\Lambda_{\sigma,\tau}(q) = \sum_{Q \subset q:\ Q \notin \mathbf{C}_1 \cup \mathbf{C}_2} \lambda_Q.$$

Begin with $\sigma = \tau$. If $\Lambda_{\sigma,\tau}(q) > \alpha 2^{\sigma+(d+m-1)\tau}$, then we say that "$q$ is selected at step (σ,τ)". Put into $\mathbf{C}_1$ every Q such that $Q \subset q$ and define $\Gamma(Q) = \tau_+$. Repeat until $\sigma < \min\{\sigma(Q) : Q \in \mathbf{C}\}$. Actually this part of the process will be terminated once σ is smaller than σ_0. We then put into $\mathbf{C}_2$ every $Q \in \mathbf{C} \setminus \mathbf{C}_1$ such that $\sigma(Q) \ge \tau$ and for such Q define $\Gamma(Q) = 1 + \sigma(S(Q))$. Actually every $Q \in \mathbf{C} \setminus \mathbf{C}_1 \cup \mathbf{C}_2$ satisfies $\sigma(Q) \le \tau_-$.

Perform STC with $\mathbf{C}_1 = \emptyset = \mathbf{C}_2$ and $\tau = \tau_0$. Next replace τ by τ_- and repeat STC. Repeat until $\tau = \sigma_0 - 1$. After this process we obtain $\mathbf{C} = \mathbf{C}_1 \cup \mathbf{C}_2$. Clearly all selected q are disjoint, and Γ is well defined. Note that there is the usual stopping-time condition

$$\Lambda_{\sigma,\tau}(q) \le \alpha 2^{\sigma+(d+m-1)\tau_+} \tag{6.1}$$

which holds for all $q \in \mathcal{R}_{\sigma,\tau}$ when $\sigma \le \tau \le \tau_0$. This is because, if $\tau = \tau_0$, then the condition is clear from the choice of τ_0 and σ_0. And when $\sigma \le \tau \le \tau_0$, suppose this fails. Then $\Lambda_{\sigma,\tau_+}(q) \ge \Lambda_{\sigma,\tau}(q) > \alpha 2^{\sigma+(d+m-1)(\tau_+)}$. This means q is selected at step (σ,τ_+), hence $\Lambda_{\sigma,\tau}(q) = 0$ and we have the contradiction.

Next we show (4), which says, for each $q \in \mathcal{R}_{\sigma,\tau}$ with $\sigma \le \tau$, $\tau \in \mathbb{F}$,

$$\sum_{Q \subset q:\ \Gamma(Q) \le \tau} \lambda_Q \le \alpha 2^{\sigma+(d+m-1)\tau_+}.$$

When $\tau \ge \tau_0$, then the condition is clear from the initial condition of τ_0. When $\tau \le \tau_0$, then we note the fact that, for each $q \in \mathcal{R}_{\sigma,\tau}$ with $\sigma \le \tau < \tau_0$

$$\Lambda_{\sigma,\tau}(q) = \sum_{Q \subset q:\ Q \notin \mathbf{C}_1 \cup \mathbf{C}_2} \lambda_Q \ge \sum_{Q \subset q:\ \Gamma(Q) \le \tau} \lambda_Q. \tag{6.2}$$

Combining (6.1) and (6.2), we have (4) when $\sigma \le \tau < \tau_0$. (6.2) will follow from the definition

$$\Lambda_{\sigma,\tau}(q) = \sum_{Q \subset q:\ Q \notin \mathbf{C}_1 \cup \mathbf{C}_2} \lambda_Q,$$

and the fact that if $Q \in \mathbf{C}_1 \cup \mathbf{C}_2$ at the beginning of step (σ,τ), then $\Gamma(Q) > \tau$. This is because, if $Q \in \mathbf{C}_1$, then $\Gamma(Q) \ge \tau_+ > \tau$, and if $Q \in \mathbf{C}_2$, then $\Gamma(Q) = 1 + \sigma(S(Q)) \ge 1 + \tau_+ > \tau$. Hence we have (6.2). Next, we construct an exceptional set E. If q is selected at step (σ,τ), then we define $\tau(q) = \tau$ and

$$T(q) = \bigcup_{j \le \tau(q)_+} \{q + \operatorname{supp}(\mu_j)\}, \quad E = E_1 \cup E_2, \quad E_1 = \bigcup_{S \in \mathbf{S}} S^*, \quad E_2 = \bigcup_{q:\ \text{selected}} T(q).$$

Thus we have

$$|E_1| \lesssim \sum_{S \in \mathbf{S}} |S|, \quad \text{and} \quad T(q) = \bigcup_{j \le \tau(q)_+} \{q + \operatorname{supp}(\mu_j)\}.$$

By Lemma 3.1 we have $|T(q)| \lesssim 2^{\sigma(q)+(d+m-1)\tau(q)}$ and so

$$|E_2| \le \sum_{q:\text{ selected}} |T(q)| \lesssim \sum_{q:\text{ selected}} 2^{\sigma(q)+(d+m-1)\tau(q)} \lesssim \frac{1}{\alpha} \sum_{q:\text{ selected}} \Lambda_{\sigma,\tau}(q) \lesssim \frac{1}{\alpha} \sum \lambda_Q.$$

So we obtain (1). For (2), we observe that if $Q \in \mathbf{C}_1$, then Q belongs to some selected q and $\Gamma(Q) = \tau(q)_+$, hence we have

$$\bigcup_{j<\Gamma(Q)} \{Q + \operatorname{supp}(\mu_j)\} \subset T(q) \subset E_2,$$

and if $Q \in \mathbf{C}_2$, then Q belongs to some $S(Q) \in \mathbf{S}$ and $\Gamma(Q) = 1 + \sigma(S(Q))$, hence

$$\bigcup_{j<\Gamma(Q)} \{Q + \operatorname{supp}(\mu_j)\} \subset S^*(Q) \subset E_2.$$

For (3), we replace Γ by Γ' and define

$$\Gamma(Q) = \max\{\Gamma'(Q), 1 + \sigma(S(Q))\}.$$

Then (1) and (2) are satisfied. We must check (2) and (4). For (2), if $\Gamma(Q) = \Gamma'(Q)$, then there is no problem. If $\Gamma(Q) = 1 + \sigma(S(Q)) > j$, then the argument is the same as above. (4) follows from

$$\sum_{Q\subset q;\ \Gamma(Q)\le\tau} \lambda_Q \le \sum_{Q\subset q;\ \Gamma'(Q)\le\tau} \lambda_Q$$

and we have Lemma 3.3.

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