

COHOMOLOGY OF UNISERIAL p -ADIC SPACE GROUPS

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ABSTRACT. A decade ago, J. F. Carlson proved that there are finitely many cohomology rings of finite 2-groups of fixed coclass, and he conjectured that this result ought to be true for odd primes. In this paper, we prove the non-twisted case of Carlson’s conjecture for any prime and we show how to proceed in the twisted case.

1. INTRODUCTION

In [2] Carlson proved the astonishing result that there are finitely many isomorphism types of cohomology rings with coefficients in \mathbb{F}_2 of 2-groups of a fixed coclass. He also conjectured that an analogous result should hold for odd primes p . In this paper, we study this conjecture of Carlson, we solve it in the non-twisted case and we reduce the twisted case to a controllable situation. In order to present the results, we start recalling Leedham-Green’s coclass classification of p -groups. A p -group G of size p^n and nilpotency class m has coclass $c = n - m$. The main result in [12] states that there exist an integer $f(p, c)$ and a normal subgroup N of G with $|N| \leq f(p, c)$ such that G/N is *constructible*: we recall the definition of a constructible group in Subsection 3.1 below. These constructible groups arise from uniserial p -adic space groups and come in two flavors, either *twisted* or *non-twisted*, according to whether the defining twisting homomorphism is non-zero or zero respectively; see Remark 3.5. We say that G is *non-twisted* if for some normal subgroup N of bounded size as above, the quotient G/N is constructible non-twisted. Otherwise, we say that G is *twisted*. In the former case, G/N has a large abelian normal subgroup, and in the latter case, G/N has a large normal subgroup of nilpotency class 2.

Theorem 1.1. *Let p be any prime. Then there are finitely many isomorphism types of cohomology rings with coefficients in \mathbb{F}_p of non-twisted p -groups of a fixed coclass.*

Received by the editors May 10, 2016 and, in revised form, October 12, 2016.

2010 *Mathematics Subject Classification.* Primary 20J06, 55T10.

The first author was supported by MICINN grant RYC-2010-05663 and partially supported by MEC grant MTM2013-41768-P and Junta de Andalucía grant FQM-213.

The second author was supported by the Basque Government Ph.D. grant PRE.2015.2.0130 and partially supported by the Spanish Ministry of Economy and Competitivity grant MTM2014-53810-C2-2-P and by the Basque Government grants IT753-13 and IT974-16.

The third author acknowledges the support of grants MTM2011-28229-C02-01 and MTM2014-53810-C2-2-P from the Spanish Ministry of Economy and Competitivity, the Ramon y Cajal Programme of the Spanish Ministry of Science and Innovation, grant RYC-2011-08885, and of the Basque Government, grants IT753-13 and IT974-16.

For $p = 2$, there are no twists in Leedham-Green's classification and hence every 2-group is non-twisted. In particular, Theorem 1.1 includes Carlson's aforementioned result on 2-groups. We reach Theorem 1.1 in several steps starting from abelian p -groups. Next, we give a coarse description of the main milestones along this route.

The first step is to consider K and K' abelian p -groups of the same rank d . Then their cohomology rings with coefficients in \mathbb{F}_p are abstractly isomorphic regardless of the exponents (but for $p = 2$ and exponent 1). For small rank we can realize this isomorphism at the level of cochain complexes.

Proposition 1.2. *The ring isomorphism $H^*(K; \mathbb{F}_p) \cong H^*(K'; \mathbb{F}_p)$ can be realized by a zig-zag of quasi-isomorphisms in the category of cochain complexes when $d < p$.*

In the next step, we let an arbitrary p -group P act on these abelian p -groups, and hence we must be careful enough to make the previous quasi-isomorphisms P -invariant. When the action may be lifted to integral matrices (Definition 5.5) we still have control of the number of isomorphism types for semidirect products.

Proposition 1.3. *Let p be a prime and let $\{G_i = K_i \rtimes P\}_{i \in I}$ be a family of groups such that K_i is abelian of fixed rank $d < p$ for all i and that all actions of P have a common integral lifting. Then there are finitely many isomorphism types of rings in the collection of cohomology rings $\{H^*(G_i; \mathbb{F}_p)\}_{i \in I}$.*

We also show that all these rings are isomorphic as graded \mathbb{F}_p -modules. The family of maximal nilpotency class p -groups $\{C_{p^i} \times \cdots \times C_{p^i} \rtimes C_p\}_{i \geq 1}$ forms a family to which the proposition applies with $d = p - 1$. We surpass the bounded rank restriction by considering n -fold direct products.

Proposition 1.4. *Let p be a prime and let $\{G_i = L_i \rtimes Q\}_{i \in I}$ be a family of groups such that L_i is an n -fold direct product, $L_i = K_i \times \cdots \times K_i$, K_i is abelian of fixed rank $d < p$, $Q \leq P \wr S$ with $S \leq \Sigma_n$ and all actions of P on K_i have a common integral lifting. Then there are finitely many isomorphism types of rings in the collection of cohomology rings $\{H^*(G_i; \mathbb{F}_p)\}_{i \in I}$.*

Again, we also show here that all these rings are isomorphic as graded \mathbb{F}_p -modules. Next we move to uniserial p -adic space groups, as they give rise to constructible groups. So let R be a uniserial p -adic space group with translation group T and point group P , and let T_0 be the minimal P -lattice for which the extension $T_0 \rightarrow R_0 \rightarrow P$ splits (Subsection 3.1). We prove the next result either using Proposition 1.3 and Nakaoka's theorem on wreath products or more directly by employing Proposition 1.4. The connection with the aforementioned integral liftings is that, up to conjugation, P may be chosen to act by integral matrices on the lattice T . The relation to the symmetric group is that the standard uniserial p -adic space group has a point group that involves Σ_n .

Proposition 1.5. *There are finitely many isomorphism types of rings for the cohomology rings $H^*(T_0/U \rtimes P; \mathbb{F}_p)$ for the infinitely many P -invariant lattices $U < T$.*

From here, we prove Theorem 1.1 by using certain refinements of Carlson's counting arguments for spectral sequences (Section 4), Leedham-Green's classification and a detailed description of constructible groups. An immediate consequence of Proposition 1.5 is the following result, in which we have dropped the splitting condition.

Proposition 1.6. *There are finitely many possibilities for the ring structure of the graded \mathbb{F}_p -module $H^*(R/U; \mathbb{F}_p)$ for the infinitely many P -invariant lattices $U < T$.*

Regarding the twisted case of Carlson's conjecture, we reduce it to a similar problem about realizations of abstract isomorphism between cohomology rings of certain p -groups. In this case, we are concerned about an abelian p -group A and its twisted version A_λ (Subsection 3.2), and again we consider invariance under the action of a certain p -group P . Under mild assumptions that hold in the context of the Leedham-Green classification, the cohomology rings with coefficients in \mathbb{F}_p of A and A_λ are abstractly isomorphic (Subsection 6.3).

Conjecture 1.1. *The abstract isomorphism $H^*(A; \mathbb{F}_p) \cong H^*(A_\lambda; \mathbb{F}_p)$ can be realized in the category of cochain complexes via a zig-zag of P -invariant quasi-isomorphisms.*

We prove that Conjecture 1.1 implies the twisted case of Carlson's conjecture, and hence also Carlson's conjecture in full generality.

Theorem 1.7. *If Conjecture 1.1 holds, then for any prime p there are finitely many isomorphism types of cohomology rings with coefficients in \mathbb{F}_p of p -groups of a fixed coclass.*

Explicit isomorphisms among cohomology rings of certain families of p -groups are expected; see [2, Question 6.1]. In [5, Conjecture 3], Eick and Green refine this question via coclass families and prove that it holds asymptotically up to F -isomorphism. In our setting, we expect that there is just one isomorphism type for the collections in Propositions 1.3 and 1.4. Forgetting the ring structure, Ellis has proven in [6] by other methods that for a uniserial p -adic space group R with translation group T of dimension d , $H_n(R/T_i; \mathbb{F}_p)$ and $H_n(R/T_{i+d}; \mathbb{F}_p)$ are isomorphic \mathbb{F}_p -vector spaces for i big enough and any n . Here, $\{T_i\}_{i \geq 0}$ is the *uniserial filtration* of R ; see Subsection 3.1. We expect that the cohomology rings $H^*(R/T_i; \mathbb{F}_p)$ and $H^*(R/T_{i+d}; \mathbb{F}_p)$ are isomorphic, and Proposition 1.6 already points in that direction. Here it is an exhaustive account of the layout of this work.

Section 2: We fix notation for the homological algebra objects that we use (Subsection 2.1), including the standard resolution (Subsection 2.2) and resolutions for semidirect products (Subsection 2.3). We also prove a lemma (Subsection 2.4) that is fundamental to dealing with cohomology rings of semidirect products, and hence for Propositions 1.3 and 1.4.

Section 3: Here we summarize results about uniserial p -adic space groups and Leedham-Green's constructible groups (Subsection 3.1), twisted abelian p -groups (Subsections 3.2 and 3.4), powerful p -central groups (Subsection 3.3) and standard uniserial p -adic space groups (Subsection 3.5). We also introduce split constructible groups and give a very detailed description of them via twisted abelian p -groups (Lemma 3.11).

Section 4: We prove certain refinements of Carlson's counting arguments in [2].

Section 5: This section is devoted to cohomology of abelian p -groups of small rank (Subsection 5.1), their semidirect products (Subsection 5.2) and the unbounded rank case (Subsection 5.3). We prove Propositions 1.2, 1.3 and 1.4 as Corollary 5.4 and Propositions 5.8 and 5.10 respectively.

Section 6: Cohomology of uniserial p -adic groups is addressed (Subsections 6.1 and 6.2) and Propositions 1.5 and 1.6 are proven as Proposition 6.3 and

Corollary 6.4 respectively (see also Propositions 6.1 and 6.2). Moreover, cohomology of twisted abelian p -groups is studied (Subsection 6.3), a fully detailed version of Conjecture 1.1 is given as Conjecture 6.1 and some of its consequences are proven.

Section 7: In Theorem 7.1, we give a proof that Theorem 1.7 holds assuming Conjecture 1.1/6.1. The same proof shows that the non-twisted case Theorem 1.1 holds with no assumptions.

2. PRELIMINARIES: HOMOLOGICAL ALGEBRA

Throughout the paper we denote by k a field and by G a (possibly infinite) discrete group. We also denote the group algebra of G with coefficients in k by kG . We shall deal with (bounded below) single complexes and (first quadrant) double complexes in the category of kG -modules. We define a *quasi-isomorphism* as a map of cochain complexes which induce a k -isomorphism in cohomology in all degrees. Finally, for a prime p , we shall often use the cyclic group with p^i elements, which we denote by C_{p^i} . If p is even we further assume that $i \geq 2$. All the material presented in this section except Subsection 2.4 is standard and can be found in [1], [7], [21] and [17].

2.1. Notation and sign convention. If A_* is a chain complex of kG -modules with differential d_A and M is a kG -module, then $\text{Hom}_{kG}(A_*, M)$ becomes a cochain complex with $d(f)(a) = (-1)^{n+1}f(d_A(a))$, where $f \in \text{Hom}_{kG}(A_n, M)$ and $a \in A_{n+1}$. If A_* and B_* are chain or cochain complexes of kG -modules with differentials d_A and d_B , we denote by $C = A \otimes_{kG} B$ the double complex $C_{n,m} = A_n \otimes_{kG} B_m$ with differentials

$$d_h(a \otimes b) = d_A(a) \otimes b \text{ and } d_v(a \otimes b) = (-1)^n a \otimes d_B(b), \text{ for } a \otimes b \in A_n \otimes_{kG} B_m.$$

Similarly, let A_* be a chain complex of kG -modules and let B^* be a cochain complex of kG -modules with differentials d_A and d_B . Denote by $D = \text{Hom}_{kG}(A, B)$ the double complex $D^{n,m} = \text{Hom}_{kG}(A_n, B^m)$. The differentials are given in this case by

$$d^h(f)(a) = (-1)^{n+m+1}f(d_A(a)) \text{ and } d^v(f)(a) = d_B(f(a)), \text{ for } f \in \text{Hom}_{kG}(A_n, B^m).$$

As usual, the total complex $\text{Tot}(C)$ is a chain complex with differential $d_h + d_v$, and $\text{Tot}(D)$ is a cochain complex with differential $d^h + d^v$.

Assume now that $N \trianglelefteq G$, that A_* is a chain complex of $k(G/N)$ -modules and that B_* is a chain complex of kG -modules. Then for each n and m and each kG -module M we have an isomorphism of k -modules

$$\text{Hom}_{kG}(A_n \otimes_k B_m, M) \cong \text{Hom}_{k(G/N)}(A_n, \text{Hom}_{kN}(B_m, M)),$$

where G acts diagonally on $A_n \otimes_k B_m$ and G/N acts on $f \in \text{Hom}_{kN}(B_m, M)$ via $(\bar{g} \cdot f)(b) = g \cdot f(g^{-1} \cdot b)$ for $g \in G$ and $b \in B_m$. Consider the double complexes $C = A \otimes_k B$ and $D = \text{Hom}_{kG/N}(A, \text{Hom}_{kN}(B, M))$. Then, via the isomorphism above and with the sign conventions described, the two cochain complexes $\text{Hom}_{kG}(\text{Tot}(C), M)$ and $\text{Tot}(D)$ are identical.

Finally, by a *product* on a cochain complex B^* with differential d_B , we mean a degree preserving k -bilinear form

$$\cup: \text{Tot}(B^* \otimes B^*) \rightarrow B^*$$

such that \cup is associative with unit and satisfies the Leibnitz rule, i.e.:

$$d_B(b \cup b') = d_B(b) \cup b' + (-1)^n b \cup d_B(b'),$$

for $b \in B^n$ and $b' \in B^{n'}$. If B^* has a product we say that B^* is a *differential graded algebra*. For instance, if A^* and B^* are differential graded algebras, then $C = A \otimes_k B$ is also a differential graded algebra by defining

$$(1) \quad (a \otimes b) \cup (a' \otimes b') = (-1)^{n'm} (a \cup_A a') \otimes (b \cup_B b'), \text{ for } b \in B^m \text{ and } a' \in A^{n'}.$$

For the standard resolution of a group G , $B_*(G; k)$ (defined in Subsection 2.2) and a differential graded algebra of kG -modules K^* , we may define a product \cup on the total complex $\text{Tot}(D)$ for $D_{n,m} = \text{Hom}_{kG}(B_n(G; K), K^m)$. For

$$f_1 \otimes f_2 \in \text{Hom}_{kG}(B_{n_1}(G; k), K^{m_1}) \otimes \text{Hom}_{kG}(B_{n_2}(G; k), K^{m_2})$$

define $f_1 \cup f_2 \in \text{Hom}_{kG}(B_{n_1+n_2}(G; k), K^{m_1+m_2})$ as the function that evaluates to

$$(2) \quad (-1)^{n_1(n_2+m_2)} f_1(g_0, \dots, g_{n_1}) \cup_K f_2(g_{n_1}, \dots, g_{n_1+n_2})$$

on $(g_0, \dots, g_{n_1+n_2})$, where the sign implements the Koszul sign rule. This last construction is more general (see [17, p. 137]), but we need only this simple version here.

2.2. Standard resolution and standard cup product. We recall the standard resolution $B_*(G; k)$ of the trivial kG -module k : $B_n(G; k)$ is the free k -module kG^{n+1} with diagonal G -action

$$g \cdot (g_0, \dots, g_n) = (gg_0, \dots, gg_n).$$

The differential $\delta_n: B_n(G; k) \rightarrow B_{n-1}(G; k)$ is the alternate sum $\sum_{i=0}^n (-1)^i \partial_i$, where ∂_i is the i th-face map $B_n(G; k) \rightarrow B_{n-1}(G; k)$ with

$$\partial_i(g_0, \dots, g_n) = (g_0, \dots, \widehat{g}_i, \dots, g_n).$$

The augmentation $\epsilon: B_0(G; k) \rightarrow k$ sends $g_0 \mapsto 1$ for all $g_0 \in G$. Now let M be any kG -module. Then $C^*(G; M) = \text{Hom}_{kG}(B_*(G; k), M)$ is a cochain complex of k -modules whose differential $\delta^n: C^n(G, k) \rightarrow C^{n+1}(G, k)$ is given as in Subsection 2.1, i.e., by $\delta^n = (-1)^{n+1} \text{Hom}_{kG}(\delta_{n+1}, M)$. Its cohomology is $H^*(G; M)$.

The cup product at the cochain level for the standard resolution,

$$\cup: C^*(G; k) \otimes C^*(G; k) \rightarrow C^*(G; k),$$

takes $f_1 \otimes f_2 \in C^{n_1}(G; k) \otimes C^{n_2}(G; k)$ to $f_1 \cup f_2 \in C^{n_1+n_2}(G, k)$ defined by

$$(3) \quad (f_1 \cup f_2)(g_0, \dots, g_{n_1+n_2}) = (-1)^{n_1 n_2} f_1(g_0, \dots, g_{n_1}) f_2(g_{n_1}, \dots, g_{n_1+n_2}),$$

where the sign again implements the Koszul sign rule [1, p. 110]. It is well known that $(C^*(G; k), d, \cup)$ is a differential graded k -algebra (associative with unit). This product induces in $H^*(G; k)$ the usual cup product, which we denote by the same symbol $([f_1] \cup [f_2] = [f_1 \cup f_2])$.

2.3. Resolutions and cup products for semidirect products. Let $G = N \rtimes P$ be a semidirect product and denote by conjugation $p \cdot n = {}^p n = p n p^{-1}$ the action of P on N . Assume that N_* is a kN -resolution of k , that P_* is a kP -resolution of k and that P acts on N_* in such a way that:

- (1) The action of P commutes with the augmentation and the differentials of N_* .
- (2) For all $p \in P, n \in N$ and $z \in N_*, p \cdot (n \cdot z) = (p \cdot n) \cdot (p \cdot z)$.

Consider the double complex $C = C_{*,*} = P_* \otimes N_*$. Then there is an action of $G = N \rtimes P$ over C and over $\text{Tot}(C)$ described by

$$(n, p) \cdot (z_P \otimes z_N) = p \cdot z_P \otimes n \cdot (p \cdot z_N),$$

where $n \in N, p \in P, z_P \in P_*$ and $z_N \in N_*$. Moreover, following [7, p. 19], $\text{Tot}(C)$ is a kG -projective resolution of the trivial kG -module k and thus, for any kG -module M , we have $H^*(G; M) \cong H^*(\text{Hom}_{kG}(\text{Tot}(C), M))$ as graded k -modules.

As a particular case, consider $P_* = B_*(P; k)$ and $N_* = B_*(N; k)$. Then the action of P on $B_*(N; k)$ given by

$$p \cdot (n_0, \dots, n_m) = ({}^p n_0, \dots, {}^p n_m)$$

satisfies (1) and (2) above. So $H^*(G; M) \cong H^*(\text{Hom}_{kG}(\text{Tot}(C), M))$ as graded k -modules for $C = C_{*,*} = B_*(P; k) \otimes B_*(N; k)$. In fact, we can endow $\text{Hom}_{kG}(\text{Tot}(C), k)$ with a product as in equation (2):

$$\begin{aligned} \text{Hom}_{kP}(B_{n_1}(P; k), C^{m_1}(N; k)) \otimes \text{Hom}_{kP}(B_{n_2}(P; k), C^{m_2}(N; k)) \\ \rightarrow \text{Hom}_{kP}(B_{n_1+n_2}(P; k), C^{m_1+m_2}(N; k)) \end{aligned}$$

sends $f_1 \otimes f_2$ to the map that on $(p_0, \dots, p_{n_1+n_2})$ evaluates to the function that on $(n_0, \dots, n_{m_1+m_2})$ takes the following value:

$$(-1)^{n_1(n_2+m_2)+m_1 m_2} f_1(p_0, \dots, p_{n_1})(n_0, \dots, n_{m_1}) f_2(p_{n_1}, \dots, p_{n_1+n_2})(n_{m_1}, \dots, n_{m_1+m_2}),$$

where again the sign reflects the Koszul sign rule. That this product induces the usual cup product in $H^*(G; k)$ is a consequence of the criterion [16, XII.10.4].

2.4. A lemma for spectral sequences. In this subsection we prove the following easy and useful result:

Lemma 2.1. *Let G be a group, let K^* and K'^* be cochain complexes of kG -modules and let P_* be a chain complex of free kG -modules. If $\varphi: K^* \rightarrow K'^*$ is a quasi-isomorphism, then*

$$\text{Tot}(\text{Hom}_{kG}(P_*, K^*)) \xrightarrow{\text{Tot}(\varphi_*)} \text{Tot}(\text{Hom}_{kG}(P_*, K'^*))$$

is a quasi-isomorphism, and so

$$H^*(\text{Tot}(\text{Hom}_{kG}(P_*, K^*))) \cong H^*(\text{Tot}(\text{Hom}_{kG}(P_*, K'^*)))$$

as graded k -modules.

Proof. Consider the double complexes of k -modules C and C' given by $C_{n,m} = \text{Hom}_{kG}(P_n, K^m)$ and $C'_{n,m} = \text{Hom}_{kG}(P_n, K'^m)$. It is clear that the cochain map φ induces a morphism of double complexes $\varphi_*: C_{*,*} \rightarrow C'_{*,*}$ given by $f \mapsto \varphi \circ f$. Consider the total complexes $\text{Tot}(C)$ and $\text{Tot}(C')$. The filtrations by columns produce spectral sequences E and E' converging to the cohomology of $\text{Tot}(C)$ and

$\text{Tot}(C')$ respectively. The first page E_1 of E is obtained by taking cohomology with respect to the vertical differential:

$$E_1^{n,m} = H^m(\text{Hom}_{kG}(P_n, K^*)) = \text{Hom}_{kG}(P_n, H^m(K^*)),$$

where $\text{Hom}_{kG}(P_n, \cdot)$ commutes with the cohomology functor because P_n is a free kG -module and hence $\text{Hom}_{kG}(P_n, \cdot)$ is an exact functor. As the differential of K^* commutes with the G -action, $H^*(K)$ becomes a kG -module and $\text{Hom}_{kG}(P_*, H^*(K))$ is well-defined. The morphism of double complexes φ_* induces a morphism of spectral sequences $\Phi : E \rightarrow E'$. Between the first pages, the morphism

$$\Phi_1 : \text{Hom}_{kG}(P_*, H^*(K)) \rightarrow \text{Hom}_{kG}(P_*, H^*(K'))$$

is given by postcomposing with $H^*(\varphi)$, that is, $f \mapsto H^*(\varphi) \circ f$. This is an isomorphism as $H^*(\varphi)$ is an isomorphism by hypothesis. Therefore, all morphisms $\Phi_r : E_r \rightarrow E'_r$ are isomorphisms for $r \geq 1$ and

$$H^*(\text{Tot}(\varphi_*)) : H^*(\text{Tot}(C)) \cong H^*(\text{Tot}(C'))$$

is an isomorphism too. □

We shall need the following version of the previous lemma that involves products:

Lemma 2.2. *With the hypothesis of Lemma 2.1, suppose in addition that:*

- (1) $P_* = B_*(G; k)$ is the standard resolution of G .
- (2) K^* and K'^* are equipped with products \cup and \cup' .
- (3) $H^*(\varphi) : H^*(K^*) \rightarrow H^*(K'^*)$ preserves the induced products.

Then there are filtrations of the graded algebra $H^(\text{Tot}(\text{Hom}_{kG}(P_*, K^*)))$ and of the graded algebra $H^*(\text{Tot}(\text{Hom}_{kG}(P_*, K'^*)))$ such that the associated bigraded algebras are isomorphic.*

Proof. Consider the double complexes $C_{n,m} = \text{Hom}_{kG}(B_n(G; k), K^m)$ and $C'_{n,m} = \text{Hom}_{kG}(B_n(G; K), K'^m)$. By (2), there are products \cup_C and $\cup_{C'}$ on $\text{Tot}(C)$ and $\text{Tot}(C')$ that induce products in $H^*(\text{Tot}(C))$ and $H^*(\text{Tot}(C'))$ respectively. The products \cup_C and $\cup_{C'}$ are defined in such a way that the filtrations by columns preserve them. Hence, the associated spectral sequences are spectral sequences of algebras, and the bigraded algebra structures on E_∞ and E'_∞ arise from the induced filtrations of $H^*(\text{Tot}(C))$ and $H^*(\text{Tot}(C'))$ respectively. The morphism $\Phi_1 : E_1 \rightarrow E'_1$ is given by post-composition by $H^*(\varphi)$, and hence it preserves the bigraded algebra structures on E_1 and E'_1 because of hypothesis (3). From here, it is easy to see that $\Phi_r : E_r \rightarrow E'_r$ preserve the bigraded algebra structures for all $r \geq 1$, in particular, for $r = \infty$. This proves the claim. □

Remark 2.3. Note that we do not claim that the two graded k -algebras

$$H^*(\text{Tot}(\text{Hom}_{kG}(P_*, K^*))) \text{ and } H^*(\text{Tot}(\text{Hom}_{kG}(P_*, K'^*)))$$

are isomorphic.

3. PRELIMINARIES: UNISERIAL p -ADIC SPACE GROUPS

In this section, we state general facts about uniserial p -adic space groups (see [14], [18] or [4]), about powerful p -central groups with the Ω -extension property (see [20]), about twisted abelian p -groups, and about constructible groups and the Leedham-Green classification of finite p -groups of fixed coclass (see [12]).

3.1. Uniserial p -adic space groups and constructible groups. Let R be a uniserial p -adic space group with translation group T and point group P . Then R is a p -adic pro- p group that fits in the extension of groups

$$(4) \quad 1 \rightarrow T \rightarrow R \rightarrow P \rightarrow 1,$$

and T is the maximal normal abelian pro- p subgroup of R . The translation group is a \mathbb{Z}_p -lattice of rank $d_x = p^{x-1}(p-1)$, and P is a finite p -group which acts faithfully and uniserially on T . From the former condition, we have that $P \leq \text{GL}_{d_x}(\mathbb{Z}_p)$. From the latter condition, the group R has finite coclass c with $x \leq c$ and there exists a unique series of P -invariant lattices, the *uniserial filtration*. More precisely, for each integer $i \geq 0$, there exists a unique P -invariant sublattice T_i of T that satisfies $|T : T_i| = p^i$. Moreover, for $i = j + sd_x$ with $s \geq 0$, we have $T_i = p^s T_j$. For $\tilde{T} := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, we obtain a split extension of \tilde{T} by the point group P . By Lemma 10.4.3 in [14], there exists a minimal superlattice T_0 of T in \tilde{T} such that the subgroup R_0 generated by T_0 and P splits over T_0 :

$$(5) \quad 1 \rightarrow T_0 \rightarrow R_0 \rightarrow P \rightarrow 1.$$

Notice that the index of T in T_0 is finite, and therefore, the index of R in R_0 is also finite. The following result of Leedham-Green describes the structure of finite p -groups of fixed coclass.

Theorem 3.1 (Leedham-Green [12, Theorem 7.6, Theorem 7.7]). *For some function $f(p, c)$, almost every p -group G of coclass c has a normal subgroup N of order at most $f(p, c)$ such that G/N is constructible.*

A constructible group arises from the following data: a uniserial p -adic space group R with translation group T and point group P , two P -invariant sublattices of T , $U < V$, and $\gamma \in \text{Hom}_P(\Lambda^2(T_0/V), V/U)$. In the next paragraphs, we review the definition of the *constructible group* G_γ associated to R, U, V and γ by Leedham-Green [12, p. 60]. At the same time, we consider a related group $G_{\gamma,0}$ that we term *split constructible group*.

Remark 3.2. For $p = 2$, we assume, following Leedham-Green, that $\gamma = 0$; i.e., there is no twist in this case.

Consider the group X with underlying set $V/U \times T_0/V$ and operation

$$(a_1, b_1)(a_2, b_2) = (a_1 + a_2 + \frac{1}{2}\gamma(b_1, b_2), b_1 + b_2).$$

Then P acts on X coordinate-wise, and we may consider the group $X \rtimes P$ and the following extension:

$$(6) \quad 1 \rightarrow V/U \rightarrow X \rtimes P \rightarrow T_0/V \rtimes P = R_0/V \rightarrow 1.$$

Pulling back along the inclusion $R/V \leq R_0/V$, we get another extension Y :

$$(7) \quad 1 \rightarrow V/U \rightarrow Y \rightarrow R/V \rightarrow 1.$$

On the other hand, there are extensions of groups

$$(8) \quad 1 \rightarrow V/U \rightarrow R_0/U \rightarrow R_0/V \rightarrow 1$$

and

$$(9) \quad 1 \rightarrow V/U \rightarrow R/U \rightarrow R/V \rightarrow 1.$$

Definition 3.3. The *constructible group* associated to R, U, V and γ is the group G_γ obtained as the Baer sum of the extensions (7) and (9).

Definition 3.4. The *split constructible group* associated to R, U, V and γ is the group $G_{\gamma,0}$ obtained as the Baer sum of the extensions (6) and (8).

Remark 3.5. We say that G_γ and $G_{\gamma,0}$ are *twisted* or *non-twisted* according to the conditions $\gamma \neq 0$ or $\gamma = 0$ respectively. In the non-twisted case, the group X is the direct product $V/U \times T_0/V, G_\gamma = R/U, G_{\gamma,0} = R_0/U$.

3.2. Twisted abelian p -groups. In this subsection, we describe certain p -groups that are obtained by twisting the sum operation in an abelian p -group. They play a central role in the description of Leedham-Green constructible groups.

Definition 3.6. Let A and B be abelian p -groups. We denote by $\text{Hom}(\Lambda^2 A, B)$ the set of maps $\lambda : A \times A \rightarrow B$ which are biadditive and alternating. If, in addition, P is a p -group that acts on A and B , then we denote by $\text{Hom}_P(\Lambda^2 A, B)$ the subset of $\text{Hom}(\Lambda^2 A, B)$ consisting of the maps λ that satisfy

$$p \cdot \lambda(a, a') = \lambda(p \cdot a, p \cdot a') \text{ for all } a, a' \in A, \text{ all } p \in P.$$

The statements in the following definitions are easy to check.

Definition 3.7. Let $A = (A, +)$ be an abelian p -group for an odd prime p and let $\lambda \in \text{Hom}(\Lambda^2 A, A)$. The set A endowed with the operation

$$a +_\lambda a' = a + a' + \frac{1}{2}\lambda(a, a'), \text{ for } a, a' \in A,$$

defines a group structure that we denote by $A_\lambda = (A, +_\lambda)$.

Remark 3.8. Throughout the paper we assume that the map λ used to define A_λ (Definition 3.7) satisfies

$$\text{Im}(\lambda) \leq \text{Rad}(\lambda) = \{a \in A \mid \lambda(a, a') = 0 \text{ for all } a' \in A\}.$$

This ensures that A_λ has nilpotency class at most two; see Lemma 3.10(iii).

Definition 3.9. With the notation of Definition 3.7 and if we further assume that P is a p -group that acts on A and that $\lambda \in \text{Hom}_P(\Lambda^2 A, A)$, then P acts on A_λ and hence we may consider the semidirect product $A_\lambda \rtimes P$.

The group A_λ has the following properties:

Lemma 3.10. *With the notation of Definition 3.7, the following properties hold:*

- (i) For all $a \in A$ and all $n \in \mathbb{Z}$ we have $\overbrace{a +_\lambda a +_\lambda \cdots +_\lambda a}^{n \text{ times}} = \overbrace{a + a + \cdots + a}^{n \text{ times}}$.
- (ii) $\Omega_1(A) = \Omega_1(A_\lambda)$.
- (iii) A_λ is abelian or has nilpotency class two.
- (iv) If R is a subset of $\text{Rad}(\lambda)$, then R is a subgroup of A if and only if it is a subgroup of A_λ . Moreover, in that case, the identity map $(R, +) \rightarrow (R, +_\lambda)$ is an isomorphism.
- (v) If R is a subgroup satisfying $\text{Im}(\lambda) \subseteq R \subseteq \text{Rad}(\lambda)$, then R is central in both A and A_λ and the identity map $(A/R, +) \rightarrow (A_\lambda/R, +_\lambda)$ is an isomorphism.

Proof. Points (i) and (iii) follow from the fact that $\lambda(a, a) = 0$ for all $a \in A$ and from the condition $\text{Im}(\lambda) \subseteq \text{Rad}(\lambda)$ in Remark 3.8 respectively. Then (ii) is a consequence of (i) and the fact that $pa = pb = 0 \Rightarrow p(a +_\lambda b) = 0$. The fourth item (iv) is a consequence of the fact that $\lambda(a, A) = 0$ for all $a \in R$. Point (v) is also straightforward. □

In the next result, we give an alternative description of the group $G_{\gamma,0}$ (see Definition 3.4) in terms of a twisted abelian p -group. We refer the reader to the notation of Subsection 3.1. We denote by λ the element in $\text{Hom}_P(\Lambda^2(T_0/U), T_0/U)$ obtained from γ precomposing it with the projection $T_0/U \twoheadrightarrow T_0/V$ and post-composing it with the inclusion $V/U \hookrightarrow T_0/U$. Then we consider the p -group $(T_0/U)_\lambda = (T_0/U, +_\lambda)$ (Definition 3.7), and the semidirect product $(T_0/U, +_\lambda) \rtimes P$ (Definition 3.9).

Lemma 3.11. *The constructible group G_γ and the split constructible group $G_{\gamma,0}$ satisfy the following properties:*

- (1) G_γ is a subgroup of $G_{\gamma,0}$ of index $|T_0 : T|$.
- (2) There is an isomorphism $G_{\gamma,0} \cong (T_0/U, +_\lambda) \rtimes P$.

Proof. The first point is a consequence of the fact that the short exact sequences (7) and (9) are sub-short exact sequences of (6) and (8) respectively. The index is as indicated because $|R_0 : R| = |T_0 : T|$.

Before proving the second point of the statement, consider the following diagram, where the upper row is the extension (8):

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & V/U & \longrightarrow & R_0/U = T_0/U \rtimes P & \longrightarrow & R_0/V = T_0/V \rtimes P & \longrightarrow & 1 \\
 & & \parallel & & \uparrow & & \uparrow & & \\
 1 & \longrightarrow & V/U & \longrightarrow & T_0/U & \longrightarrow & T_0/V & \longrightarrow & 1.
 \end{array}$$

Choose a (set-theoretical) section $s: T_0/V \rightarrow T_0/U$ satisfying $s(1) = 1$. Then $s \times 1: R_0/V \rightarrow R_0/U$ is also a section. We define the corresponding cocycles $\theta \in C^2(T_0/V; V/U)$ and $\eta \in C^2(R_0/V; V/U)$ by

$$\begin{aligned}
 \theta(x_1, x_2) &= s(x_1) + s(x_2) - s(x_1 + x_2) \text{ and} \\
 \eta(x_1, p_1, x_2, p_2) &= (s \times 1)(x_1, p_1) + (s \times 1)(x_2, p_2) - (s \times 1)((x_1, p_1) + (x_2, p_2)) \\
 &= (s(x_1) + {}^{p_1}s(x_2) - s(x_1 + {}^{p_1}x_2), 1).
 \end{aligned}$$

A short calculation shows that they satisfy

$$(10) \quad \eta(x_1, p_1, x_2, p_2) = \theta(x_1, {}^{p_1}x_2) + {}^{p_1}s(x_2) - s({}^{p_1}x_2),$$

where $(x_i, p_i) \in R_0/V = T_0/V \rtimes P$. Now consider the group $X \rtimes P$ appearing in the extension (6). It has underlying set $V/U \times T_0/V \times P$ and operation given by

$$(z_1, x_1, p_1)(z_2, x_2, p_2) = (z_1 + {}^{p_1}z_2 + \frac{1}{2}\gamma(x_1, {}^{p_1}x_2), x_1 + {}^{p_1}x_2, p_1p_2),$$

where $(z_i, x_i, p_i) \in V/U \times T_0/V \times P$. Now, choose the section 0×1 for the extension (6) and define the corresponding cocycle by

$$(0 \times 1)(x_1, p_1) + (0 \times 1)(x_2, p_2) - (0 \times 1)((x_1, p_1) + (x_2, p_2)) = (\frac{1}{2}\gamma(x_1, {}^{p_1}x_2), 0, 1),$$

where $(x_i, p_i) \in T_0/V \rtimes P = R_0/V$. As the Baer sums correspond to adding extension cocycles, the group $G_{\gamma,0}$ has operation

$$(z_1, x_1, p_1)(z_2, x_2, p_2) = (z_1 + {}^{p_1}z_2 + \frac{1}{2}\gamma(x_1, {}^{p_1}x_2) + \eta(x_1, p_1, x_2, p_2), x_1 + {}^{p_1}x_2, p_1p_2).$$

Fix the bijection $T_0/U \xrightarrow{\cong} V/U \times T_0/V$ given by $y \mapsto (y - s(\pi(y)), \pi(y))$. Then the operation $+_\lambda$ in T_0/U (Definition 3.7) is readily checked to be

$$(z_1, x_1) +_\lambda (z_2, x_2) = (z_1 + z_2 + \frac{1}{2}\gamma(x_1, x_2) + \theta(x_1, x_2), x_1 + x_2),$$

where $(z_i, x_i) \in V/U \times T_0/V$. Moreover, the action of $p \in P$ on $(z, x) \in V/U \times T_0/V$ is given by

$${}^p(z, x) = ({}^p z + {}^p s(x) - s({}^p x), {}^p x).$$

It follows that the product in $(T_0/U, +_\lambda) \rtimes P$ of $(z_i, x_i, p_i) \in V/U \times T_0/V \times P$ for $i = 1, 2$ is exactly

$$(z_1 + {}^{p_1} z_2 + \frac{1}{2}\gamma(x_1, {}^{p_1} x_2) + \theta(x_1, {}^{p_1} x_2) + s({}^{p_1} x_2) - {}^{p_1} s(x_2), x_1 + {}^{p_1} x_2, p_1 p_2).$$

Now the lemma follows from equation (10). □

Remark 3.12. The above proof shows that, in fact, the extensions

$$V/U \rightarrow (T_0/U, +_\lambda) \rtimes P \rightarrow T_0/V \rtimes P \text{ and } V/U \rightarrow G_{\gamma,0} \rightarrow T_0/V \rtimes P$$

are isomorphic.

3.3. Powerful p -central groups and the Ω -extension property. In this subsection, we state the basic definitions and properties of powerful p -central groups as discussed in [20], as well as apply them to the context of constructible groups.

Definition 3.13. Let G be a finite group, let p be an odd prime and let L be a \mathbb{Z}_p -Lie algebra. Then we say that G or L is p -central if its elements of order p are contained in its centre:

$$\Omega_1(G) = \langle g \in G \mid g^p = 1 \rangle \subseteq Z(G) \text{ and } \Omega_1(L) = \langle a \in L \mid pa = 0 \rangle \subseteq Z(L).$$

Definition 3.14. Let p be an odd prime, let G be a finite p -group and let L be a \mathbb{Z}_p -Lie algebra. We say that G is *powerful* if $[G, G] \subseteq G^p$ and we say that L is *powerful* if $[L, L] \subseteq pL$.

Definition 3.15. Let p be an odd prime, let G be a p -central group and let L be a p -central \mathbb{Z}_p -Lie algebra. We say that G has the Ω -extension property (Ω EP for short) if there exists a p -central group H such that $G = H/\Omega_1(H)$. We say that L has the Ω EP if there exists a p -central Lie algebra A such that $L = A/\Omega_1(A)$.

The following result of Weigel [20, Theorem 2.1, Corollary 4.2] determines the cohomology ring of powerful, p -central p -groups with Ω EP. It uses the following subquotient of the cohomology ring:

Definition 3.16. For a finite group G , its *reduced mod p cohomology ring*, which we denote by $H^*(G; \mathbb{F}_p)_{\text{red}}$, is the quotient $H^*(G; \mathbb{F}_p)/\text{nil}(H^*(G; \mathbb{F}_p))$, where $\text{nil}(H^*(G; \mathbb{F}_p))$ is the ideal of all nilpotent elements in $H^*(G; \mathbb{F}_p)$.

Theorem 3.17. *Let p be an odd prime and let G be a powerful p -central p -group with the Ω EP and $d = \text{rk}_{\mathbb{F}_p}(\Omega_1(G))$. Then:*

- (1) $H^*(G; \mathbb{F}_p) \cong \Lambda[x_1, \dots, x_d] \otimes_{\mathbb{F}_p} [y_1, \dots, y_d]$ where $|x_i| = 1$ and $|y_i| = 2$.
- (2) The reduced restriction mapping $j_{\text{red}}: H^*(G; \mathbb{F}_p)_{\text{red}} \rightarrow H^*(\Omega_1(G); \mathbb{F}_p)_{\text{red}}$ is an isomorphism.

3.4. Properties of twisted abelian p -groups. We prove additional properties of the group A_λ introduced in Definition 3.7.

Lemma 3.18. *With the notation of Definition 3.7, the following properties hold:*

- (1) A_λ is powerful if and only if $\text{Im}(\lambda) \leq pA$.
- (2) A_λ is p -central if and only if $\Omega_1(A) \leq \text{Rad}(\lambda)$.

Proof. To prove (1), note that the commutator of $a, b \in A_\lambda$ is given by $[a, b] = \lambda(a, b)$. Then it follows from Lemma 3.10(i). For the last item (2), note that, by (ii) in Lemma 3.10, $\Omega_1(A) = \Omega_1(A_\lambda)$ and that $Z(A_\lambda) = \text{Rad}(\lambda)$ by the commutator description above. Then the claim holds. \square

Now, recall the notation of Subsection 3.1 and consider the twisted abelian p -group $(T_0/U, +_\lambda)$ already described in Lemma 3.11. Under mild assumptions this group has nice properties.

Lemma 3.19. *Assume that*

$$(11) \quad V \leq pT_0$$

and suppose that there is a P -invariant sublattice W of T such that $U \leq pW$ and $W \leq pV$. Define $\lambda' \in \text{Hom}_P(\Lambda^2(T_0/W), T_0/W)$ as γ precomposed with $T_0/W \rightarrow T_0/V$ and postcomposed with $V/U \rightarrow T_0/W$. Then

- (1) $(T_0/U, +_\lambda)$ is a powerful p -central group, and
- (2) $(T_0/W, +_{\lambda'})$ is a powerful p -central group with ΩEP .

Proof. We have $\text{Im}(\lambda) \leq V/U \leq \text{Rad}(\lambda)$ and $\text{Im}(\lambda') \leq V/W \leq \text{Rad}(\lambda')$. Moreover, $V/U \leq (pT_0)/U = p(T_0/U)$ and $V/W \leq (pT_0)/W = p(T_0/W)$. Finally, $\Omega_1(T_0/U) = (\frac{1}{p}U)/U \leq (\frac{1}{p^2}U)/U \leq V/U$ and $\Omega_1(T_0/W) = (\frac{1}{p}W)/W \leq V/W$. Then it follows from Lemma 3.18(1) and Lemma 3.18(2) that both $(T_0/U, +_\lambda)$ and $(T_0/W, +_{\lambda'})$ are powerful p -central groups. The same arguments show that $(T_0/W', +_{\lambda''})$ is powerful p -central for $W' = pW$ and λ'' defined analogously. Now, by Lemma 3.10(ii), $\Omega_1((T_0/W')_{\lambda''}) = \Omega_1(T_0/W')$, and this group is exactly W/W' . It is straightforward that

$$(T_0/W, +_{\lambda'}) \cong (T_0/W', +_{\lambda''})/(W/W')$$

and hence $(T_0/W, +_{\lambda'})$ has the ΩEP . \square

3.5. Standard uniserial p -adic space group. Recall that in Subsection 3.1 we considered a uniserial p -adic space group, $1 \rightarrow T \rightarrow R \rightarrow P \rightarrow 1$, and we set $\tilde{T} := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Since the action of P on T is faithful, P can be embedded in $\text{GL}(\tilde{T}) = \text{GL}_{d_x}(\mathbb{Q}_p)$, and hence P is contained in a maximal p -subgroup of $\text{GL}_{d_x}(\mathbb{Q}_p)$. For p odd, $\text{GL}_{d_x}(\mathbb{Q}_p)$ has a unique maximal p -subgroup, $W(x)$, up to conjugation, which is an iterated wreath product:

$$W(x) = C_p \wr \overbrace{C_p \wr \cdots \wr C_p}^{x-1}.$$

The action of $W(x)$ on the p -adic lattice $\mathbb{Z}_p^{d_x}$ is described as follows: the leftmost copy of C_p is generated by the companion matrix of the polynomial $y^{p-1} + \cdots + y + 1$, i.e., by the following matrix:

$$(12) \quad M = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \in \text{GL}_{p-1}(\mathbb{Z}).$$

The remaining $(x - 1)$ copies of C_p act by permutation matrices as $\overbrace{C_p \wr \cdots \wr C_p}^{x-1}$ is the Sylow p -subgroup of the symmetric group $\Sigma_{p^{x-1}}$.

Remark 3.20. For $p = 2$, we have $d_x = 2^{x-1}$, and there is another conjugacy class $\tilde{W}(x)$ in $\text{GL}_{2^{x-1}}(\mathbb{Q}_2)$ given as follows:

$$\tilde{W}(x) = Q_{16} \wr \overbrace{C_2 \wr \cdots \wr C_2}^{x-3},$$

where Q_{16} denotes the quaternion group of order 16. The action of $\tilde{W}(x)$ on $\mathbb{Z}_2^{d_x}$ is described in [15].

For odd p and after a suitable conjugation, we may assume that T_0 is a $W(x)$ -invariant lattice with $T_0 \leq \mathbb{Z}_p^{d_x}$. Hence, for p odd we have

$$(13) \quad R_0 = T_0 \rtimes P \leq \mathbb{Z}_p^{d_x} \rtimes W(x),$$

where both $T_0 \rtimes P$ and $\mathbb{Z}_p^{d_x} \rtimes W(x)$ are split uniserial p -adic space groups. We call $\mathbb{Z}_p^{d_x} \rtimes W(x)$ the *standard uniserial p -adic space group* of dimension d_x .

4. COUNTING THEOREMS

Throughout this section we use the sectional rank of a p -group G (see [11, §11]):

$$\begin{aligned} \text{rk}(G) &:= \max\{d(H) \mid H \leq G\} \\ &= \max\{\dim_{\mathbb{F}_p}(H/N) \mid N \trianglelefteq H \leq G, \text{ where } H/N \text{ is elementary abelian}\}, \end{aligned}$$

where $d(H)$ denotes the number of minimal generators of H . The following properties are standard and can be found in [11, §4 and §11]. We shall use them without further notice.

- (1) If G is a powerful p -group, then $\text{rk}(G) = d(G)$.
- (2) If N is a subgroup of G , then $\text{rk}(N) \leq \text{rk}(G)$.
- (3) If N is a normal subgroup of G , then $\text{rk}(G) \leq \text{rk}(G/N) + \text{rk}(N)$.

Next, we recall some results of J.F. Carlson from [2] and we prove some natural generalizations.

Theorem 4.1 ([2, Theorem 2.1]). *Let R be a finitely generated, graded-commutative \mathbb{F}_p -algebra and let S be the bigraded algebra induced by some filtration of R . Then the algebra structure of R is determined by the algebra structure of S within a finite number of possibilities.*

Theorem 4.2 ([2, Theorem 3.5]). *Let n be a positive integer. Suppose that S is a finitely generated k -algebra. Then, there are only finitely many k -algebras R with the property that $R \cong H^*(H, k)$ for H a subgroup of a p -group G with $H^*(G; k) \cong S$ and $|G : H| \leq p^n$.*

Now we need to generalize [2, Theorem 3.3]. We start with the following two lemmas.

Lemma 4.3. *Let L be a finite \mathbb{Z}_p -Lie algebra of nilpotency class two. Then there exists \tilde{L} , a finite powerful p -central nilpotent \mathbb{Z}_p -Lie algebra of nilpotency class 2, such that $pL \cong \tilde{L}/\Omega_1(\tilde{L})$.*

Proof. Consider the \mathbb{Z}_p -Lie algebra L as a d -generated abelian p -group. Then, L is isomorphic to a quotient of the free \mathbb{Z}_p -module M of rank d by a submodule I which is contained in pM . The Lie bracket in L is an antisymmetric bilinear form $[\cdot, \cdot]: L \times L \rightarrow L$, and therefore, it can be lifted to an antisymmetric bilinear form $\{\cdot, \cdot\}: M \times M \rightarrow M$. A direct computation shows that $\{\cdot, \cdot\}$ satisfies the Jacobi

identity in $\tilde{L} = pM/pI$. Furthermore, $(\tilde{L}, +, \{, \})$ is a powerful p -central \mathbb{Z}_p -Lie algebra. By the third isomorphism theorem

$$\frac{\tilde{L}}{\Omega_1(\tilde{L})} = \frac{pM/pI}{I/pI} \cong \frac{pM}{I} = pL,$$

and this concludes the proof. □

Lemma 4.4. *Let p be an odd prime and let G be a p -group with $\text{rk}(G) \leq r$, and let*

$$(14) \quad 1 \longrightarrow C_p \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1$$

be an extension of groups. Suppose that Q has a subgroup A of nilpotency class 2. Set $B = \pi^{-1}(A)^{p^2}$. Then B is a powerful p -central p -group of nilpotency class 2 with the Ω EP and $|G : B| \leq p^{4r}|Q : A|$.

Proof. Put $C = \pi^{-1}(A)$, $D = C^p$ and let N be the image of C_p in G ; then $[D, D, D] = [C^p, C^p, C^p] = [C, C, C]^{p^3} \subseteq N^{p^3} = 1$ (see [9, Theorem 2.4] and [8, Theorem 2.10]). In particular D is a p -group of nilpotency class 2. By the Lazard correspondence (for general theory see [11, §9 and §10] and for the explicit formulae see [3]), $D = \mathbf{exp}(L)$ where L is a \mathbb{Z}_p -Lie algebra of nilpotency class two. By Lemma 4.3 and again by the Lazard correspondence $B = D^p = \mathbf{exp}(pL)$ is a powerful p -central p -group of nilpotency class 2 with the Ω EP. Indeed, $pL \cong \tilde{L}/\Omega_1(\tilde{L})$ where \tilde{L} is a powerful p -central Lie algebra of nilpotency class 2, and by applying the functor \mathbf{exp} we obtain that $\mathbf{exp}(pL) = \mathbf{exp}(\tilde{L})/\mathbf{exp}(\Omega_1(\tilde{L}))$.

We also have $|G : B| = |G : C||C : D||D : B|$ and $|G : C| = |Q : A|$, where C/D and D/B have exponent p . Moreover, C has rank at most r and nilpotency class at most 3. We may write

$$|C : D| = |C : \Phi(C)||\Phi(C) : D| \text{ and } |D : B| = |D : \Phi(D)||\Phi(D) : B|.$$

Notice that by the Burnside Base Theorem in [11, Theorem 4.8], the Frattini factor group $C/\Phi(C)$ is an elementary abelian p -group and thus $|C : \Phi(C)| \leq p^r$. Also, it is not hard to check that $\Phi(C)/D$ is an elementary abelian group. Then, $|\Phi(C) : D| \leq p^r$. Hence, $|C : D| \leq p^{2r}$, and similarly, we obtain that $|D : B| \leq p^{2r}$. Then, the bound in the statement follows. □

Theorem 4.5. *Let p be an odd prime and suppose that*

$$(15) \quad 1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

is an extension of finite p -groups with $|H| \leq n$, $\text{rk}(G) \leq r$ and Q has a subgroup A of nilpotency class 2 with $|Q : A| \leq f$. Then the ring $H^(G; \mathbb{F}_p)$ is determined up to a finite number of possibilities (depending on p, n, r and f) by the ring $H^*(Q; \mathbb{F}_p)$.*

Proof. If $H \cong C_p$, then, by Lemma 4.4, there exists a powerful p -central subgroup B of G with Ω EP and whose index is bounded in terms of p, r and f . If H is not contained in B we consider $H \times B$ instead of B . In both situations, there exists an element $\eta \in H^2(B, \mathbb{F}_p)$ (resp. $\eta \in H^2(B \times H, \mathbb{F}_p)$) such that $\text{res}_H^B(\eta)$ is non-zero (resp. $\text{res}_H^{B \times H}(\eta)$) (see Theorem 3.17(2)). Then the spectral sequence arising from

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

stops at most at the page $2|G : B| + 1$ (cf. [2, proof of Lemma 3.2]). Now the theorem holds by [2, Proposition 3.1].

For general H , we proceed by induction on $|H|$. Suppose that the result holds for all the group extensions of the form

$$1 \rightarrow H' \rightarrow G \rightarrow Q,$$

where $|H'| < |H| \leq n$, $\text{rk}(G) \leq r$ and with $A \leq Q$ of nilpotency class 2 and $|Q : A| \leq f$. Choose a subgroup $H' \leq H$ with $H' \trianglelefteq G$ and $|H : H'| = p$. The quotients $G' = G/H'$ and $C_p \cong H/H'$ fit in a short exact sequence

$$(16) \quad 1 \rightarrow C_p \rightarrow G' \xrightarrow{\pi} Q \rightarrow 1,$$

and we also have the following extension of groups:

$$(17) \quad 1 \rightarrow H' \rightarrow G \rightarrow G' \rightarrow 1.$$

Applying Lemma 4.4 to the extension of groups (16), we know that G' has a p -subgroup B' of nilpotency class two. Moreover, as $\text{rk}(G') \leq \text{rk}(G)$, we have that $|G' : B'| \leq p^{4r}|Q : A|$. Also, by the previous case, we have that the cohomology algebra $H^*(G'; \mathbb{F}_p)$ is determined up to a finite number of possibilities (depending on p, n, r and f) by the algebra $H^*(Q; \mathbb{F}_p)$.

Now, we may apply the induction hypothesis to the extension (17) since $|H'| < |H|$. Then, the cohomology algebra $H^*(G; \mathbb{F}_p)$ is determined up to a finite number of possibilities (depending on p, n, r and f) by the algebra $H^*(G'; \mathbb{F}_p)$. In turn, the result holds. \square

5. COHOMOLOGY OF p -GROUPS THAT SPLIT OVER AN ABELIAN SUBGROUP

Assume that G is a p -group that splits over an abelian subgroup K of rank d , $K = C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$. In this section we show that if $d < p$ and the action $G/K \rightarrow \text{Aut}(K)$ has an integral lifting (defined below), then there are finitely many possibilities for the cohomology ring $H^*(G; \mathbb{F}_p)$ for all infinitely many choices of the exponents i_l . Here we are considering cohomology with trivial coefficients in the field of p elements, \mathbb{F}_p , and we assume that either p is odd or $p = 2$ and $i_l > 1$ for all l .

5.1. Cohomology of abelian p -groups of small rank. Consider the abelian groups $K = C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ and $K' = C_{p^{j_1}} \times \cdots \times C_{p^{j_d}}$ for some fixed rank $0 < d$. Then there is an abstract isomorphism of cohomology rings

$$(18) \quad H^*(K; \mathbb{F}_p) \cong H^*(K'; \mathbb{F}_p).$$

When $d < p$, we shall realize this isomorphism via quasi-isomorphisms in the category of cochain complexes. To that end, note that both cohomology rings above are isomorphic to the tensor product of a polynomial algebra with an exterior algebra: $\mathbb{F}_p[x_1, \dots, x_d] \otimes \Lambda(y_1, \dots, y_d)$, where $\text{deg}(x_l) = 2$ and $\text{deg}(y_l) = 1$.

The Prüfer p -group $C_{p^\infty} = \bigcup_{k \geq 1} C_{p^k}$ is an infinite discrete abelian group whose mod- p cohomology ring is polynomial with one generator in degree 2 (see [1, V.6.6] and recall that C_{p^∞} is divisible). Hence we also have $H^*(C_{p^\infty}^d; \mathbb{F}_p) = \mathbb{F}_p[x_1, \dots, x_d]$ with $\text{deg}(x_l) = 2$. Moreover, the group inclusion $K \hookrightarrow C_{p^\infty}^d$ induces a cochain map

$$(19) \quad C^*(C_{p^\infty}^d; \mathbb{F}_p) \xrightarrow{\varphi_e} C^*(K; \mathbb{F}_p)$$

that becomes an isomorphism on reduced cohomology (Definition 3.16), i.e., on cohomology modulo the ideal of the nilpotent elements. Note that the map (19) preserves the standard cup products in $C^*(C_{p^\infty}^d; \mathbb{F}_p)$ and $C^*(K; \mathbb{F}_p)$ (see Subsection 2.2).

By abusing notation, we denote by $\Lambda(y_1, \dots, y_d)$ both the exterior algebra (with product denoted by \cup) and the cochain complex obtained by equipping it with the 0 differential. Consider then the following cochain map introduced in [19, Proof of Proposition 2]:

$$(20) \quad \Lambda(y_1, \dots, y_d) \xrightarrow{\varphi_o} C^*(K; \mathbb{F}_p),$$

which on degree t , with $0 \leq t \leq d$, sends $y_{i_1} \cdots y_{i_t}$ to the element of $C^t(K; \mathbb{F}_p)$

$$\frac{1}{t!} \sum_{\sigma \in \Sigma_t} \text{sgn}(\sigma) Y_{i_{\sigma(1)}} \cup \cdots \cup Y_{i_{\sigma(t)}},$$

where Y_i are the representatives of cohomology classes generating $H^1(K; \mathbb{F}_p)$ defined by

$$(21) \quad Y_i(k_0, k_1) = \overline{(k_1 - k_0)_i},$$

where $\overline{k_l}$ denotes the image by the reduction $C_{p^{i_l}} \rightarrow C_p$ of the l th coordinate k_l of $k \in K$. Note that the condition $d < p$ is needed in the definition of this cochain map. When we pass to cohomology we obtain the identity on degree 1. Note that $H^*(\varphi_0)$ is a homomorphism of rings because of the equality

$$\left[\frac{1}{t!} \sum_{\sigma \in \Sigma_t} \text{sgn}(\sigma) Y_{i_{\sigma(1)}} \cup \cdots \cup Y_{i_{\sigma(t)}} \right] = [Y_{i_1} \cup \cdots \cup Y_{i_t}],$$

which in turn follows from the fact that $[Y_{i_{\sigma(1)}} \cup \cdots \cup Y_{i_{\sigma(t)}}] = \text{sgn}(\sigma)[Y_{i_1} \cup \cdots \cup Y_{i_t}]$ in the graded commutative algebra $H^*(K; \mathbb{F}_p)$.

Remark 5.1. Consider the cochain map defined by fixing an \mathbb{F}_p basis for $\Lambda(y_1, \dots, y_d)$ and extending linearly the map $y_{i_1} \cdots y_{i_t} \rightarrow Y_{i_1} \cup \cdots \cup Y_{i_t}$. It is simpler and good enough for the purposes of Lemma 5.3 below. Nevertheless, it is not invariant, a crucial condition that φ_o satisfies and that will be needed later in Lemma 5.7.

Definition 5.2. Let p be a prime and let $d < p$. Define $U(p, d)$ as the cochain complex $C^*(C_{p^\infty}^d; \mathbb{F}_p) \otimes \Lambda(y_1, \dots, y_d)$.

Lemma 5.3. *For every prime p and any abelian p -group K of rank $d < p$ there exists a quasi-isomorphism*

$$U(p, d) \xrightarrow{\varphi} C^*(K; \mathbb{F}_p)$$

that induces an isomorphism of rings $H^*(U(p, d)) \cong H^*(K; \mathbb{F}_p)$.

Proof. Define

$$U(p, d) = C^*(C_{p^\infty}^d; \mathbb{F}_p) \otimes \Lambda(y_1, \dots, y_d) \xrightarrow{\varphi_e \otimes \varphi_o} C^*(K; \mathbb{F}_p) \otimes C^*(K; \mathbb{F}_p) \xrightarrow{\cup} C^*(K; \mathbb{F}_p),$$

where the first arrow is the tensor product of the cochain maps (19) and (20) and the second map is the standard cup product (Subsection 2.2). Then the claim follows by the properties of the cochain maps (19) and (20). \square

Corollary 5.4. *For every prime p and any two abelian p -groups K and K' of rank $d < p$ there exists a zig-zag of quasi-isomorphisms*

$$C^*(K; \mathbb{F}_p) \leftarrow U(p, d) \rightarrow C^*(K'; \mathbb{F}_p)$$

that induces isomorphisms of rings $H^*(K; \mathbb{F}_p) \cong H^*(U(p, d)) \cong H^*(K'; \mathbb{F}_p)$.

Note that this zig-zag realizes the ring isomorphism (18).

5.2. Cohomology of p -groups that split over an abelian p -group of small rank. Suppose that P is a p -group acting on the abelian p -group $K = C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ via $P \xrightarrow{\alpha} \text{Aut}(K)$. Set R to be the subring of $\text{End}(\mathbb{Z}^d) = \mathcal{M}_d(\mathbb{Z})$ consisting of integral matrices $A = (a_{n,m})$ such that each entry $a_{n,m}$ is divisible by $\max(p^{i_n - i_m}, 1)$. Then there is a surjective ring homomorphism $w: R \rightarrow \text{End}(K)$ [10]. Moreover, if $A \in R$, then

$$(22) \quad w(A)(\pi(n_1, \dots, n_d)) = \pi\left(\sum_l a_{1,l}n_l, \dots, \sum_l a_{d,l}n_l\right) = \left(\sum_l a_{1,l}\overline{n_l}, \dots, \sum_l a_{d,l}\overline{n_l}\right),$$

where π is the surjection $\mathbb{Z}^d \rightarrow K$ that takes (n_1, \dots, n_r) to the element $\pi(n_1, \dots, n_d) = (\overline{n_1}, \dots, \overline{n_d})$ obtained by reducing modulo p^{i_l} the l th coordinate. For instance, if $i = i_1 = \cdots = i_d$, then w takes the matrix A to its C_{p^i} reduction in $\mathcal{M}_d(C_{p^i}) = \text{End}(K)$.

Definition 5.5. We say that the homomorphism $P \xrightarrow{\tilde{\alpha}} R$ is an *integral lifting* of the action $P \xrightarrow{\alpha} \text{Aut}(K)$ if the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\alpha}} & R \\ \alpha \downarrow & & \downarrow w \\ \text{Aut}(K) & \hookrightarrow & \text{End}(K), \end{array}$$

where the bottom horizontal arrow is the inclusion. Note that, as P is a group, the image of $\tilde{\alpha}$ lies in the group of units of R .

Remark 5.6. The group P acts on the chain complex $B_*(K; \mathbb{F}_p)$ via $p \cdot (g_0, \dots, g_n) = (p \cdot g_0, \dots, p \cdot g_n)$, where $p \in P$ and $(g_0, \dots, g_n) \in B_n(K; \mathbb{F}_p)$. There is also an action of P on $C^*(K; \mathbb{F}_p)$ given by $(p \cdot f)(g) = f(p^{-1} \cdot g)$, where $p \in P$, $g \in B_n(K; \mathbb{F}_p)$ and $f \in C^n(K; \mathbb{F}_p)$.

The next result is a P -invariant version of Lemma 5.3.

Lemma 5.7. *Let p be a prime, let K be an abelian p -group of rank $d < p$ and let P be a p -group that acts on K via $P \rightarrow \text{Aut}(K)$. If the action has an integral lifting, then P acts on $U(p, d)$ and the quasi-isomorphism*

$$U(p, d) \rightarrow C^*(K; \mathbb{F}_p)$$

is P -invariant.

Proof. Recall that $U(p, d) = C^*(C_{p^\infty}^d; \mathbb{F}_p) \otimes \Lambda(y_1, \dots, y_d)$ and consider the integral lifting $\tilde{\alpha}: P \rightarrow R \subseteq \mathcal{M}_d(\mathbb{Z})$. We shall show that P acts on $C^*(C_{p^\infty}^d; \mathbb{F}_p)$ and on $\Lambda(y_1, \dots, y_d)$ and hence diagonally on $U(p, d)$. Choose $p \in P$ and set $\tilde{\alpha}(p) = A = (a_{n,m})$. Then A acts on $C_{p^\infty}^d$ via

$$p \cdot (m_1, \dots, m_d) = \left(\sum_l a_{1,l}m_l, \dots, \sum_l a_{d,l}m_l\right),$$

on $B_n(C_{p^\infty}^d; \mathbb{F}_p)$ via $p \cdot (g_0, \dots, g_n) = (p \cdot g_0, \dots, p \cdot g_n)$ and on $f \in C^n(C_{p^\infty}^d; \mathbb{F}_p)$ via $(p \cdot f)(g) = f(p^{-1} \cdot g)$. With these definitions, the cochain map (19) $C^*(C_{p^\infty}^d; \mathbb{F}_p) \rightarrow C^*(K; \mathbb{F}_p)$ is P -invariant because, by (22), the inclusion $K \hookrightarrow C_{p^\infty}^d$ is P -invariant.

For the action of p on $c_1y_1 + \dots + c_dy_d \in \Lambda^1(y_1, \dots, y_d)$, where $c_l \in \mathbb{F}_p$, write $\tilde{\alpha}(p^{-1}) = B = (b_{n,m})$ and set

$$p \cdot (c_1y_1 + \dots + c_dy_d) = \left(\sum_l b_{l,1}c_l\right)y_1 + \dots + \left(\sum_l b_{l,d}c_l\right)y_d.$$

Observe that this action extends to $\Lambda(y_1, \dots, y_d)$. Moreover, on $C^*(K; \mathbb{F}_p)$ we also have that

$$(23) \quad p \cdot (c_1Y_1 + \dots + c_dY_d) = \left(\sum_l b_{l,1}c_l\right)Y_1 + \dots + \left(\sum_l b_{l,d}c_l\right)Y_d.$$

Then it turns out after routine computation that the cochain map (20) is P -invariant. □

Next we present the main result of this section.

Proposition 5.8. *Let p be a prime and let $\{G_i = K_i \rtimes P\}_{i \in I}$ be a family of groups such that K_i is abelian of fixed rank $d < p$ for all i and that all actions of P have a common integral lifting. Then the following hold:*

- (1) *The graded \mathbb{F}_p -modules $H^*(G_i; \mathbb{F}_p)$ and $H^*(G_{i'}; \mathbb{F}_p)$ are isomorphic for all i, i' .*
- (2) *There is a filtration of $H^*(G_i; \mathbb{F}_p)$ for each i , and the associated bigraded algebras are isomorphic for all i .*
- (3) *There are finitely many isomorphism types of rings in the collection of cohomology rings $\{H^*(G_i; \mathbb{F}_p)\}_{i \in I}$.*

Proof. Choose $i_0 \in I$ and set $G = G_{i_0}, K = K_{i_0}$. Then take any $i \in I$ and let $G' = G_i, K' = K_i$. By Subsection 2.3, we have $H^*(G; \mathbb{F}_p) \cong H^*(\text{Tot}(D_C))$, where D_C is the double complex

$$D_C^{*,*} = \text{Hom}_{\mathbb{F}_p P}(B_*(P; \mathbb{F}_p), C^*(K; \mathbb{F}_p)).$$

Similarly, we have $H^*(G'; \mathbb{F}_p) = H^*(\text{Tot}(D_{C'}))$ for $D_{C'}$ the analogous double complex. Now, by Lemma 5.7, there exists a zig-zag of P -invariant quasi-isomorphisms

$$C^*(K; \mathbb{F}_p) \xleftarrow{\varphi} U(p, d) \xrightarrow{\varphi'} C^*(K'; \mathbb{F}_p).$$

Then two applications of Lemma 2.1 give immediately that the maps

$$\text{Tot}(D_C) \xleftarrow{\text{Tot}(\varphi_*)} \text{Tot}(D_U) \xrightarrow{\text{Tot}(\varphi'_*)} \text{Tot}(D_{C'})$$

are both quasi-isomorphisms, where $D_U^{*,*} = \text{Hom}_{\mathbb{F}_p P}(B_*(P; \mathbb{F}_p), U^*(p, d))$. In particular, $H^*(G; \mathbb{F}_p) \cong H^*(G'; \mathbb{F}_p)$ as graded \mathbb{F}_p -vector spaces. Moreover, we can endow $\text{Tot}(D_C)$, $\text{Tot}(D_{C'})$ and $\text{Tot}(D_U)$ with a product by equation (2). Then, by Lemma 2.2, there are filtrations of $H^*(\text{Tot}(D_C))$, $H^*(\text{Tot}(D_U))$ and $H^*(\text{Tot}(D_{C'}))$ with isomorphic associated bigraded algebras. Indeed, by Subsection 2.3, the products in $H^*(\text{Tot}(D_C)) \cong H^*(G; \mathbb{F}_p)$ and $H^*(\text{Tot}(D_{C'})) \cong H^*(G'; \mathbb{F}_p)$ are the usual cup products for the cohomology rings of G and G' . Finally, the product in the corresponding bigraded algebras is induced by the products in $H^*(G; \mathbb{F}_p)$ and $H^*(G'; \mathbb{F}_p)$. Now, by Theorem 4.1, there are finitely many possibilities for the rings $H^*(G; \mathbb{F}_p)$ and $H^*(G'; \mathbb{F}_p)$. □

5.3. Cohomology of p -groups of unbounded rank. In this subsection, we generalize the last two subsections to a situation where the abelian subgroup has unbounded rank. So let $K = C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ with $d \leq p - 1$ and consider the n -fold direct product $L = K \times \cdots \times K$. Considering L as an iterated semidirect product with trivial action, we obtain from Subsection 2.3 that the cohomology of the cochain complex

$$C_{\times}^*(L; \mathbb{F}_p) = \text{Hom}_{\mathbb{F}_p L}(B_*(K; \mathbb{F}_p) \otimes \cdots \otimes B_*(K; \mathbb{F}_p), \mathbb{F}_p)$$

is exactly $H^*(L; \mathbb{F}_p)$. More explicitly, the action of L is given by

$$(k_1, \dots, k_n) \cdot (z_1 \otimes \cdots \otimes z_n) = k_1 \cdot z_1 \otimes \cdots \otimes k_n \cdot z_n,$$

where $k_i \in K$, $z_i \in B_*(K; \mathbb{F}_p)$ and the action of K on $B_*(K; \mathbb{F}_p)$ was defined in Subsection 2.2. Moreover, there is a product on $C_{\times}^*(L; \mathbb{F}_p)$ which induces the usual cup product in $H^*(L; \mathbb{F}_p)$. Here it is the generalization of Lemma 5.3.

Lemma 5.9. *There exist a cochain complex $U(p, d, n)$ and a cochain map*

$$U(p, d, n) \xrightarrow{\phi} C_{\times}^*(L; \mathbb{F}_p)$$

that induce an isomorphism of rings $H^*(U(p, d, n)) \cong H^*(L; \mathbb{F}_p)$.

Proof. Consider the analogous construction for $L_{\infty} = \bigotimes_{\times}^n C_{p^{\infty}}^d$, i.e.,

$$C_{\times}^*(L_{\infty}; \mathbb{F}_p) = \text{Hom}_{\mathbb{F}_p L_{\infty}}(B_*(C_{p^{\infty}}^d; \mathbb{F}_p) \otimes \cdots \otimes B_*(C_{p^{\infty}}^d; \mathbb{F}_p), \mathbb{F}_p).$$

Then there is a cochain map $\phi_e : C_{\times}^*(L_{\infty}; \mathbb{F}_p) \rightarrow C_{\times}^*(L; \mathbb{F}_p)$ induced by the n -fold tensor product of the group inclusion $K \hookrightarrow C_{p^{\infty}}^d$. This map becomes an isomorphism on reduced cohomology. Next, there are representatives of the generators of $H^1(L; \mathbb{F}_p)$ of the form $1 \otimes \cdots \otimes Y_i \otimes 1 \otimes \cdots \otimes 1$, where Y_i was defined in equation (21). Now consider the cochain map $\phi_o : \bigotimes_{\times}^n \Lambda(y_1, \dots, y_d) \rightarrow C_{\times}^*(L; \mathbb{F}_p)$ given by $\phi_o = \bigotimes_{\times}^n \varphi_o$ and where φ_o was defined in equation (20). Finally, the cochain map

$$U(p, d, n) = C_{\times}^*(L_{\infty}; \mathbb{F}_p) \otimes \bigotimes_{\times}^n \Lambda(y_1, \dots, y_d) \xrightarrow{\phi_e \otimes \phi_o} C_{\times}^*(L; \mathbb{F}_p) \otimes C_{\times}^*(L; \mathbb{F}_p) \xrightarrow{\cup} C_{\times}^*(L; \mathbb{F}_p)$$

induces an isomorphism of cohomology rings. □

Now we consider a semidirect product $L \rtimes Q$, where Q is a subgroup of $P \wr S$. Here $P \xrightarrow{\alpha} \text{Aut}(K)$ acts on K , S is a subgroup of the symmetric group Σ_n and the action of Q on L is given by

$$(p_1, \dots, p_n, \sigma) \cdot (k_1, \dots, k_n) = (p_1 \cdot k_{\sigma^{-1}(1)}, \dots, p_n \cdot k_{\sigma^{-1}(n)}).$$

Next we prove a generalization of Proposition 5.8.

Proposition 5.10. *Let p be a prime and let $\{G_i = L_i \rtimes Q\}_{i \in I}$ be a family of groups such that $L_i = K_i \times \cdots \times K_i$, K_i is abelian of fixed rank $d < p$, $Q \leq P \wr S$ and all actions of P have a common integral lifting. Then the following hold:*

- (1) *The graded \mathbb{F}_p -modules $H^*(G_i; \mathbb{F}_p)$ and $H^*(G_{i'}; \mathbb{F}_p)$ are isomorphic for all i, i' .*
- (2) *There is a filtration of $H^*(G_i; \mathbb{F}_p)$ for each i , and the associated bigraded algebras are isomorphic for all i .*

(3) *There are finitely many isomorphism types of rings in the collection of cohomology rings $\{H^*(G_i; \mathbb{F}_p)\}_{i \in I}$.*

Proof. The proof is identical to that of Proposition 5.8 once we prove that the cochain map $\phi: U(p, d, n) \rightarrow C^*_\times(L_i; \mathbb{F}_p)$ from Lemma 5.9 is Q -invariant for some action of Q on $U(p, d, n)$. First, Q acts on $B_*(K; \mathbb{F}_p) \otimes \cdots \otimes B_*(K; \mathbb{F}_p)$ by

$$(p_1, \dots, p_n, \sigma) \cdot (z_1 \otimes \cdots \otimes z_n) = (-1)^\epsilon (p_1 \cdot z_{\sigma^{-1}(1)} \otimes \cdots \otimes p_n \cdot z_{\sigma^{-1}(n)}),$$

where $z_i \in B_*(K; \mathbb{F}_p)$. Signs must be chosen appropriately in order for this action to commute with the differential, and it is enough to choose ϵ as for wreath products in [7, p. 49]. Moreover, this action satisfies points (1) and (2) in Subsection 2.3, and Q acts on $C^*_\times(L_i; \mathbb{F}_p)$ by $(q \cdot f)(z) = f(q^{-1} \cdot z)$. In the proof of Proposition 5.8, we showed how P acts, via its integral lifting, on $B_*(C^d_{p^\infty}; \mathbb{F}_p)$ and on $\Lambda(y_1, \dots, y_d)$. Then Q acts on $B_*(C^d_{p^\infty}; \mathbb{F}_p) \otimes \cdots \otimes B_*(C^d_{p^\infty}; \mathbb{F}_p)$ by

$$(p_1, \dots, p_n, \sigma) \cdot (z_1 \otimes \cdots \otimes z_n) = (-1)^\epsilon (p_1 \cdot z_{\sigma^{-1}(1)} \otimes \cdots \otimes p_n \cdot z_{\sigma^{-1}(n)}),$$

where $z_i \in B_*(C^d_{p^\infty}; \mathbb{F}_p)$ and the signs are chosen as above, and on $C^*_\times(L_\infty; \mathbb{F}_p)$ by $(q \cdot f)(z) = f(q^{-1} \cdot z)$. Moreover, Q acts on $\bigotimes_n \Lambda(y_1, \dots, y_d)$ by

$$(p_1, \dots, p_n, \sigma) \cdot (z_1 \otimes \cdots \otimes z_n) = (-1)^\epsilon (p_1 \cdot z_{\sigma^{-1}(1)} \otimes \cdots \otimes p_n \cdot z_{\sigma^{-1}(n)}),$$

where $z_i \in \Lambda(y_1, \dots, y_d)$ and the signs are chosen as above. Finally, Q acts diagonally on $U(p, d, n)$, and it is straightforward that ϕ is Q -invariant. \square

6. COHOMOLOGY OF UNISERIAL SPACE GROUPS AND TWISTED ABELIAN p -GROUPS

In this section, we study cohomology rings of quotients of uniserial space groups (see Subsections 3.1 and 3.5) and cohomology rings of twisted abelian p -groups (see Subsection 3.2). Let $\mathbb{Z}_p^{d_x} \rtimes W(x)$ denote the standard uniserial p -adic space group of dimensions $d_x = (p - 1)p^{x-1}$. For p odd, $W(x)$ is the unique maximal p -subgroup of $\text{GL}_{d_x}(\mathbb{Z}_p)$, up to conjugation. Recall that for the $p = 2$ case, there is another maximal 2-subgroup (up to conjugacy) $\tilde{W}(x)$ in $\text{GL}_{d_x}(\mathbb{Q}_2)$ described in Remark 3.20. Carlson shows that the cohomology of the quotients of $\mathbb{Z}_2^{d_x} \rtimes \tilde{W}(x)$ is determined (up to a finite number of possibilities) by the cohomology of the quotients of $\mathbb{Z}_2^{d_x} \rtimes W(x)$ (see [2, pp. 259-260]).

More precisely, in the proof of [2, Lemma 4.6], it is shown that the cohomology of the quotients of $\mathbb{Z}_2^4 \rtimes W(3)$ determines the cohomology of the quotients of $\mathbb{Z}_2^4 \rtimes C_8$, where $C_8 \leq Q_{16} = \tilde{W}(3)$ is the maximal subgroup of order 8 in Q_{16} and it is conjugate to an element of order 8 in $W(3)$. Then, by the proof of [2, Proposition 4.5], the cohomology of the quotients of $\mathbb{Z}_2^4 \rtimes C_8$ determines the cohomology of the quotients of $\mathbb{Z}_2^4 \rtimes \tilde{W}(3)$. Finally, [2, Proof of Proposition 4.7] proves that the cohomology of the quotients of $\mathbb{Z}_2^4 \rtimes \tilde{W}(3)$ determines the cohomology of the quotients of $\mathbb{Z}_2^{d_x} \rtimes \tilde{W}(x)$ for $x \geq 4$. So, throughout the following subsections, we shall only state the results for point groups $P \leq W(x)$, although the same results hold for point groups $P \leq \tilde{W}(x)$.

6.1. Cohomology of standard uniserial p -adic space group. Consider the standard uniserial space group of dimension $d_x = p^{x-1}(p - 1)$, $\mathbb{Z}_p^{d_x} \rtimes W(x)$, defined in Subsection 3.5.

Proposition 6.1. *For fixed x , there are finitely many possibilities for the ring structure of the graded \mathbb{F}_p -module $H^*(\mathbb{Z}_p^{d_x}/U \rtimes W(x); \mathbb{F}_p)$ for the infinitely many $W(x)$ -invariant lattices $U < \mathbb{Z}_p^{d_x}$.*

Proof. Step 1: We first prove that there are finitely many ring structures for the infinite collection of graded \mathbb{F}_p -modules $\{H^*(\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p)\}_{s \geq 1}$. Recall that, from Subsection 3.5, the group $\mathbb{Z}_p^{d_x} \rtimes W(x)$ is the wreath product

$$(24) \quad (\mathbb{Z}_p^{p-1} \rtimes C_p) \wr S \text{ with } S = \overbrace{C_p \wr \cdots \wr C_p}^{x-1},$$

where the leftmost group C_p is generated by the matrix M of equation (12) and S is the Sylow p -subgroup of the symmetric group $\Sigma_{p^{x-1}}$. Now, let $s \geq 1$ and $s' \geq 1$. Then, $\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x)$ and $\mathbb{Z}_p^{d_x}/p^{s'}\mathbb{Z}_p^{d_x} \rtimes W(x)$ may be written as the wreath products

$$(25) \quad (\mathbb{Z}_p^{p-1}/p^s\mathbb{Z}_p^{p-1} \rtimes C_p) \wr S \text{ and } (\mathbb{Z}_p^{p-1}/p^{s'}\mathbb{Z}_p^{p-1} \rtimes C_p) \wr S$$

respectively. Set $G = \mathbb{Z}_p^{p-1}/p^s\mathbb{Z}_p^{p-1} \rtimes C_p$ and $G' = \mathbb{Z}_p^{p-1}/p^{s'}\mathbb{Z}_p^{p-1} \rtimes C_p$. Then $G = K \rtimes C_p$ and $G' = K' \rtimes C_p$ where $K = C_{p^s} \times \cdots \times C_{p^s}$ and $K' = C_{p^{s'}} \times \cdots \times C_{p^{s'}}$ are abelian groups of rank $p - 1$. Moreover, the action of the generator of C_p on K and K' is given by the matrix M . Hence, by Proposition 5.8, $H^*(G; \mathbb{F}_p)$ and $H^*(G'; \mathbb{F}_p)$ are isomorphic \mathbb{F}_p -modules, and there are finitely many possibilities for their ring structures when s and s' run over all integers greater than or equal to 1. Finally, by Nakaoka's theorem [7, Theorem 5.3.1], there is an isomorphism of rings

$$H^*(\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p) \cong H^*(S; \bigotimes_{p^{x-1}} H^*(G; \mathbb{F}_p)),$$

where S acts by permutations on the p^{x-1} (tensor) copies of $H^*(G; \mathbb{F}_p)$. We conclude that there are finitely many possibilities for the ring structure of the graded \mathbb{F}_p -module $H^*(\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p)$ for the infinitely many choices of s .

Step 2: Now we consider the cohomology ring $H^*(\mathbb{Z}_p^{d_x}/U \rtimes W(x); \mathbb{F}_p)$, where U is any $W(x)$ -invariant sublattice of $\mathbb{Z}_p^{d_x}$. By uniseriality, there exists $s \geq 0$ such that $p^{s+1}\mathbb{Z}_p^{d_x} \leq U \leq p^s\mathbb{Z}_p^{d_x}$. In fact, leaving out a finite number of invariant sublattices we may assume that $s \geq 1$. Consider the short exact sequence of groups:

$$0 \rightarrow p^s\mathbb{Z}_p^{d_x}/U \rightarrow \mathbb{Z}_p^{d_x}/U \rtimes W(x) \rightarrow \mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x) \rightarrow 1.$$

We have that $|p^s\mathbb{Z}_p^{d_x}/U| < p^{d_x}$, that $\text{rk}(\mathbb{Z}_p^{d_x}/U \rtimes W(x)) \leq d_x + \text{rk}(W(x))$ and that, by Step 1 above, there are finitely many ring structures for the quotient group $\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x)$. Then by Theorem 4.5 there are finitely many ring structures for $H^*(\mathbb{Z}_p^{d_x}/U \rtimes W(x); \mathbb{F}_p)$. \square

6.2. Cohomology of uniserial p -adic space groups. Let R be a uniserial p -adic space group of dimension $d_x = p^{x-1}(p - 1)$ and cclass at most c ($x \leq c$) and denote by T its translation group and by P its point group. We define T_0 as the minimal P -lattice for which the extension $T_0 \rightarrow R_0 \rightarrow P$ splits (see Subsection 3.1).

Proposition 6.2. *Let p be an odd prime. For each $s \geq 1$, there is a filtration of the ring $H^*(T_0/p^sT_0 \rtimes P; \mathbb{F}_p)$ such that the associated bigraded algebras are isomorphic for all s .*

Proof. By equation (13), $T_0 \rtimes P$ is a subgroup of the standard uniserial p -adic space group of dimension d_x , $\mathbb{Z}_p^{d_x} \rtimes W(x)$. The quotient $T_0/p^s T_0$ is the p^{x-1} -fold direct product $K \times \cdots \times K$, where K is the $(p-1)$ -fold direct product $K = C_{p^s} \times \cdots \times C_{p^s}$. As $P \leq W(x)$, the action of P on T_0 is given by integral matrices, and there is a common integral lifting that does not depend on s . Then the result follows from Proposition 5.10(5.10). \square

Proposition 6.3. *There are finitely many possibilities for the ring structure of the graded \mathbb{F}_p -module $H^*(T_0/U \rtimes P; \mathbb{F}_p)$ for the infinitely many P -invariant lattices $U < T$.*

Proof. By equation (13), $T_0 \rtimes P$ is a subgroup of the standard uniserial p -adic space group of dimension d_x , $\mathbb{Z}_p^{d_x} \rtimes W(x)$. We give two proofs.

Proof 1: By uniseriality, the P -sublattice U is also a $W(x)$ -sublattice. Then we have $T_0/U \rtimes P \leq \mathbb{Z}_p^{d_x}/U \rtimes W(x)$. By Proposition 6.1, there are finitely many cohomology rings $H^*(\mathbb{Z}_p^{d_x}/U \rtimes W(x))$ when U runs over the infinitely many P -sublattices. The index $|\mathbb{Z}_p^{d_x}/U \rtimes W(x) : T_0/U \rtimes P|$ equals $|\mathbb{Z}_p^{d_x} : T_0| |W(x) : P|$ and does not depend on U . So by Theorem 4.2, there are finitely many ring structures for $H^*(T_0/U \rtimes P; \mathbb{F}_p)$.

Proof 2: By uniseriality, there exists $s \geq 0$ such that $p^{s+1}T_0 \leq U \leq p^s T_0$, and we may assume that $s \geq 1$. Consider the short exact sequence of groups:

$$0 \rightarrow p^s T_0/U \rightarrow T_0/U \rtimes P \rightarrow T_0/p^s T_0 \rtimes P \rightarrow 1.$$

As $|p^s T_0/U| < p d_x$ and $\text{rk}(T_0/U \rtimes P) \leq d_x + \text{rk}(P)$, Theorem 4.5 shows that it is enough to prove that there are finitely many ring structures for the quotient group $T_0/p^s T_0 \rtimes P$. This follows from Proposition 6.2 and Theorem 4.1. \square

Corollary 6.4. *There are finitely many possibilities for the ring structure of the graded \mathbb{F}_p -module $H^*(R/U; \mathbb{F}_p)$ for the infinitely many P -invariant lattices $U < T$.*

Proof. We have $R \leq R_0 = T_0 \rtimes P$, $R/U \leq T_0/U \rtimes P$ and $|R_0 : R| = |T_0 : T|$. The result follows from Proposition 6.3 and Theorem 4.2. \square

6.3. Cohomology of twisted abelian p -groups. Let A be an abelian p -group for an odd prime p , let P be a p -group that acts on A , and let $\lambda \in \text{Hom}_P(\Lambda^2 A, A)$. Consider then the group A_λ constructed in Definition 3.7. Under the assumption that A_λ is powerful p -central and has the Ω EP, Theorem 3.17(1) shows at once that A and A_λ have isomorphic cohomology rings. We show below how to compare the cohomology rings of $A \rtimes P$ and $A_\lambda \rtimes P$ (see Definition 3.9). Before that, we give a precise statement of the conjecture in the Introduction.

Conjecture 6.1 (Detailed statement of Conjecture 1.1). *Let A be an abelian p -group for an odd prime p , let P be a p -group, let $P \rightarrow \text{Aut}(A)$ be an action with an integral lifting, and let $\lambda \in \text{Hom}_P(\Lambda^2 A, A)$. Assume that A_λ is powerful p -central and has the Ω EP. Then there is a zig-zag of quasi-isomorphisms in the category of cochain complexes,*

$$C^*(A; \mathbb{F}_p) \leftarrow U_1 \rightarrow U_2 \leftarrow \cdots \leftarrow U_r \rightarrow C^*(A_\lambda; \mathbb{F}_p),$$

where each cochain complex U_i has a product and a P -action, and each morphism is P -invariant and induces a ring isomorphism in cohomology.

Here, $C^*(A; \mathbb{F}_p)$ and $C^*(A_\lambda; \mathbb{F}_p)$ are the standard cochain complexes for A and A_λ and they already have products (see Subsection 2.2). In fact, $C^*(A; \mathbb{F}_p)$ could be replaced by $\text{Hom}_{kA}(A_*, M)$ for A_* any $\mathbb{F}_p A$ -projective resolution (and similarly for A_λ). As in Subsection 5.2, we do not assume that the morphisms in the zig-zag commute with the products.

Proposition 6.5. *Assume that Conjecture 6.1 holds. Then there are filtrations of the graded algebras $H^*(A_\lambda \rtimes P; \mathbb{F}_p)$ and $H^*(A \rtimes P; \mathbb{F}_p)$ such that the associated bigraded algebras are isomorphic.*

Proof. The argument is similar to that in the proof of Proposition 5.8. Applying $\text{Tot}(\text{Hom}_{\mathbb{F}_p P}(B_*(P; \mathbb{F}_p), -))$ to the zig-zag in Conjecture 6.1 we get a zig-zag of morphisms of cochain complexes:

$$\text{Tot}(D_C) \leftarrow \text{Tot}(D_{U_1}) \rightarrow \cdots \leftarrow \text{Tot}(D_{U_r}) \rightarrow \text{Tot}(D_{C'})$$

Here $D_C = \text{Hom}_{\mathbb{F}_p P}(B_*(P; \mathbb{F}_p), C^*(A; \mathbb{F}_p))$, $D_{C'}$ is defined analogously and $D_{U_i} = \text{Hom}_{\mathbb{F}_p P}(B_*(P; \mathbb{F}_p), U_i)$. Then Lemma 2.1 gives that $H^*(\text{Tot}(D_C))$, $H^*(\text{Tot}(D_{C'}))$ and $H^*(\text{Tot}(D_{U_i}))$ are all isomorphic graded \mathbb{F}_p -vector spaces. Using now the products on $C^*(A; \mathbb{F}_p)$, $C^*(A_\lambda; \mathbb{F}_p)$ and each U_i , the proposition follows from Lemma 2.2. □

7. CARLSON'S CONJECTURE

Now we are ready to prove the main result of this work. The non-twisted case ($\gamma = 0$; see Remark 3.5) follows verbatim from the same proof below and without the assumption that Conjecture 6.1 holds.

Theorem 7.1. *Let p be an odd prime and let c be an integer. If Conjecture 6.1 holds, then there are finitely many ring isomorphism types for the cohomology rings $H^*(G; \mathbb{F}_p)$ when G runs over the set of all finite p -groups of coclass c .*

Proof. Let G be a finite p -group of coclass c . By Theorem 3.1, there exists an integer $f(p, c)$ and a normal subgroup N of G with $|N| \leq f(p, c)$ such that G/N is constructible. A constructible group G_γ arises from a quadruple R, U, V and γ (see Definition 3.3). Here, R is a uniserial p -adic space group of dimension $d_x = p^{x-1}(p - 1)$ and coclass at most c ($x \leq c$), and there are finitely many such groups. We denote by T the translation group of R and by P its point group. We also define T_0 as the minimal P -lattice for which the extension $T_0 \rightarrow R_0 \rightarrow P$ splits (see Subsection 3.1). So we have $G/N \cong G_\gamma$ and, by Lemma 3.11(1), G_γ is a subgroup of $G_{\gamma,0}$ (see Definition 3.4). Then, by Lemma 3.11(2), we have $G_{\gamma,0} \cong (T_0/U, +_\lambda) \rtimes P$.

We shall show that

$$(26) \quad \text{there are finitely many ring structures for } H^*(G_{\gamma,0}; \mathbb{F}_p)$$

considering three different cases. Before doing so, notice that

$$(27) \quad \text{rk}((T_0/U, +_\lambda) \rtimes P) \text{ is bounded}$$

when G_γ runs over the collection of constructible groups above. This is a consequence of the inequality

$$\begin{aligned} \text{rk}((T_0/U, +_\lambda) \rtimes P) &\leq \text{rk}(T_0/U, +_\lambda) + \text{rk}(P) \\ &\leq \text{rk}(T_0/V, +_\lambda) + \text{rk}(V/U, +_\lambda) + \text{rk}(P) \\ &\leq \text{rk}(T_0/V, +) + \text{rk}(V/U, +) + \text{rk}(P) \\ &\leq 2d_x + \text{rk}(P) \end{aligned}$$

and the fact that there are finitely many point groups P (see also Lemma 3.10(ii) for the properties of $+_\lambda$).

Case 1 ($\frac{1}{p^3}U \leq V \leq pT_0$). This implies that, in particular, hypothesis (11) holds. Let $i \geq 2$ be the only integer satisfying $p^{i+2}T_0 < U \leq p^{i+1}T_0$. We have the short exact sequence

$$(28) \quad (p^i T_0/U, +_\lambda) \longrightarrow (T_0/U, +_\lambda) \rtimes P \longrightarrow (T_0/p^i T_0, +_{\lambda'}) \rtimes P,$$

where, by Lemma 3.19(2) with $W = p^i T_0$, $(T_0/p^i T_0, +_{\lambda'})$ is a powerful p -central group with ΩEP . By Proposition 6.5, the cohomology rings $H^*((T_0/p^i T_0, +_{\lambda'}) \rtimes P; \mathbb{F}_p)$ and $H^*(T_0/p^i T_0 \rtimes P; \mathbb{F}_p)$ have filtrations with isomorphic associated bigraded algebras. By Proposition 6.2, this bigraded algebra does not depend on i . Hence, there are finitely many cohomology rings for $H^*((T_0/p^i T_0, +_{\lambda'}) \rtimes P; \mathbb{F}_p)$ for the infinitely many i 's and γ 's. In the short exact sequence (28), we have:

- (Case 1.a) the kernel has size at most p^{2d_x} ,
- (Case 1.b) $\text{rk}((T_0/U, +_\lambda) \rtimes P)$ is bounded by equation (27), and
- (Case 1.c) $(T_0/p^i T_0, +_{\lambda'})$ has nilpotency class 2 by Lemma 3.10(iii).

By Theorem 4.5, there are finitely many choices for the ring $H^*((T_0/U, +_\lambda) \rtimes P; \mathbb{F}_p)$.

Case 2 ($pT_0 \leq V$). If $U \geq pT_0$, then we are leaving out a finite number of cases. Otherwise $U \leq pT_0$ and we proceed as follows: Note first that

$$p \cdot \gamma(T_0/V, T_0/V) = \gamma(pT_0/V, T_0/V) \leq \gamma(V/V, T_0/V) = 0.$$

Then, $\text{Im}(\lambda) \subseteq (\frac{1}{p}U)/U$. Let $i \geq 1$ denote the largest integer such that $\frac{1}{p}U \leq p^i T_0$. In the short exact sequence

$$(p^i T_0/U, +_\lambda) \longrightarrow (T_0/U, +_\lambda) \rtimes P \longrightarrow T_0/p^i T_0 \rtimes P,$$

we have $(T_0/p^i T_0, +) = (T_0/p^i T_0, +_\lambda)$ as $\text{Im}(\lambda) \subseteq (\frac{1}{p}U)/U \subseteq p^i T_0/U$, and:

- (Case 2.a) the kernel has size at most p^{2d_x} ,
- (Case 2.b) $\text{rk}((T_0/U, +_\lambda) \rtimes P)$ is bounded by equation (27), and
- (Case 2.c) $T_0/p^i T_0$ is an abelian subgroup of finite index of $T_0/p^i T_0 \rtimes P$.

By Proposition 6.2, there are finitely many ring structures for $H^*(T_0/p^i T_0 \rtimes P; \mathbb{F}_p)$, and there are finitely many choices for the ring $H^*((T_0/U, +_\lambda) \rtimes P; \mathbb{F}_p)$ by Theorem 4.5.

Case 3 ($p^3V \leq U$). As $U \leq V$ we have $|V : U| \leq |V : p^3V| \leq p^{3d_x}$. We also have a short exact sequence

$$(29) \quad V/U \longrightarrow (T_0/U, +_\lambda) \rtimes P \longrightarrow T_0/V \rtimes P,$$

where the description of the rightmost group is a consequence of Lemma 3.10(v). By Proposition 6.3, there are finitely many ring structures for $H^*(T_0/V \rtimes P; \mathbb{F}_p)$.

Moreover, in the short exact sequence (29), we have:

(Case 3.a) the kernel has size bounded by p^{3d_x} ,

(Case 3.b) $\text{rk}((T_0/U, +_\lambda) \rtimes P)$ is bounded by equation (27), and

(Case 3.c) T_0/V has nilpotency class 1 (abelian).

By Theorem 4.5, there are finitely many choices for the ring $H^*((T_0/U, +_\lambda) \rtimes P; \mathbb{F}_p)$.

So (26) is proven. Now, by Lemma 3.11(1), the index of G_γ in $G_{\gamma,0}$ is exactly $|T_0 : T|$, and this is again bounded when R runs over the finitely many uniserial p -adic space groups of coclass $\leq c$. So, by Theorem 4.2, there are finitely many ring structures for $H^*(G_\gamma; \mathbb{F}_p)$. Thus, G fits in an extension of groups,

$$1 \rightarrow N \rightarrow G \rightarrow G_\gamma \rightarrow 1,$$

where:

- (a) the kernel has size $|N| \leq f(p, c)$,
- (b) $\text{rk}(G) \leq \text{rk}(N) + \text{rk}(G_\gamma) \leq |N| + \text{rk}(G_{\gamma,0}) \leq f(p, c) + \text{rk}((T_0/U, +_\lambda) \rtimes P)$ is bounded by equation (27), where $G_\gamma \leq G_{\gamma,0} \cong (T_0/U, +_\lambda) \rtimes P$, and
- (c) $G_\gamma \cap (T_0/U, +_\lambda)$ is a nilpotency class 2 (normal) subgroup of G_γ by Lemma 3.10(iii), and it has index bounded by $|P|$.

Then, by Theorem 4.5, there are finitely many ring structures for $H^*(G; \mathbb{F}_p)$. We are done. \square

ACKNOWLEDGEMENTS

The authors would like to thank Alex González, Lenny Taelman and Antonio Viruel for some discussions and advice related to this work. The authors are also very grateful to the referee for performing a meticulous reading and for helpful comments.

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