

BOUNDARY HARNACK INEQUALITY FOR THE LINEARIZED MONGE-AMPÈRE EQUATIONS AND APPLICATIONS

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ABSTRACT. In this paper, we obtain boundary Harnack estimates and comparison theorem for nonnegative solutions to the linearized Monge-Ampère equations under natural assumptions on the domain, Monge-Ampère measures and boundary data. Our results are boundary versions of Caffarelli and Gutiérrez’s interior Harnack inequality for the linearized Monge-Ampère equations. As an application, we obtain sharp upper bound and global L^p -integrability for Green’s function of the linearized Monge-Ampère operator.

1. INTRODUCTION

This paper is concerned with boundary Harnack estimates for solutions to the linearized Monge-Ampère equations

$$L_\phi u := - \sum_{i,j=1}^n \Phi^{ij} u_{ij} = 0$$

where $\Phi = (\Phi^{ij}) \equiv (\det D^2\phi)(D^2\phi)^{-1}$ denotes the cofactor matrix of the Hessian matrix $D^2\phi$ of a strictly convex function ϕ . The operator L_ϕ appears in many contexts, including the affine maximal surface equation in affine geometry [37–39], Abreu’s equation in the problem of finding Kähler metrics of constant scalar curvature in complex geometry [1, 13, 14], and the semigeostrophic equations in fluid mechanics [3, 11, 29].

The regularity theory for the linearized Monge-Ampère equations was initiated in the fundamental paper [9] by Caffarelli and Gutiérrez. They developed an interior Harnack inequality theory for nonnegative solutions of the homogeneous equations $L_\phi u = 0$ in terms of the pinching of the Hessian determinant

$$\lambda \leq \det D^2\phi \leq \Lambda.$$

Accordingly, they obtained Hölder continuity for solutions. Their approach is based on that of Krylov and Safonov [20, 21] on Hölder estimates for linear, uniformly elliptic equations in general form with measurable coefficients, with sections replacing Euclidean balls. The notion of sections (or cross sections) of solutions to the Monge-Ampère equation was first introduced and studied by Caffarelli [4–7] and plays an important role in his fundamental interior $W^{2,p}$ estimates [5]. Sections are defined as sublevel sets of convex solutions after subtracting their supporting hyperplanes.

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They have the same role as Euclidean balls have in the classical theory. A Euclidean ball of radius r is a section with height $r^2/2$ of the quadratic function $|x|^2/2$ whose Hessian determinant is 1.

This theory of Caffarelli and Gutiérrez is an affine invariant version of the classical Harnack inequality for uniformly elliptic equations with measurable coefficients. In fact, since the linearized Monge-Ampère operator L_ϕ can be written in both divergence form and nondivergence form, Caffarelli-Gutiérrez's Hölder regularity theorem is the affine invariant analogue of De Giorgi-Nash-Moser's theorem and also Krylov-Safonov's theorem on Hölder continuity of solutions to uniformly elliptic equations in divergence and nondivergence form, respectively. Caffarelli and Gutiérrez's theory has already played a crucial role in Trudinger and Wang's resolution [37] of Chern's conjecture in affine geometry concerning affine maximal hypersurfaces in \mathbb{R}^3 and in Donaldson's interior estimates for Abreu's equation in complex geometry [13], paving the way for his resolution of the constant scalar curvature problem for toric surfaces [14]. It was also used by Caffarelli and Silvestre in their pioneering paper on nonlocal equations [10] to prove several regularity results for problems involving the fractional powers of the Laplacian or other integro-differential equations.

The theory of Caffarelli and Gutiérrez is a basic building block in developing higher regularity results ($C^{1,\alpha}$, $W^{2,p}$ estimates) for the linearized Monge-Ampère equations, both in the interior and at the boundary; see for example the works of Gutiérrez and Nguyen [17, 18], Savin and the author [27], Nguyen and the author [25]. It is worth noting that these boundary regularity estimates for the linearized Monge-Ampère equations found interesting applications to nonlinear, fourth order, geometric PDEs. These include: establishing global second derivative estimates for the second boundary value problem of the prescribed affine maximal surface and Abreu's equations under optimal integrability conditions on the affine mean curvature [22] and proving global regularity for minimizers with prescribed determinant of certain functionals motivated by the Mabuchi functional in complex geometry in two dimensions [28].

Recently, Maldonado [31, 32] extended the Caffarelli-Gutiérrez theory and established the interior Harnack inequality for nonnegative solutions of the linearized Monge-Ampère equations, with and without lower-order terms, under minimal geometric conditions on the Monge-Ampère measure $\det D^2\phi$ of ϕ ; namely, it is doubling with respect to the center of mass on the sections of ϕ .

In this paper, we establish the corresponding boundary Harnack inequality for nonnegative solutions of the linearized Monge-Ampère equations under natural assumptions on the domain, Monge-Ampère measures and boundary data. We give two applications of this boundary Harnack inequality. In the first application, we prove a comparison theorem for two positive solutions to the linearized Monge-Ampère equations; here we assume furthermore that one of the two solutions vanishes on the boundary. In the second application, we obtain a sharp upper bound for Green's function of the linearized Monge-Ampère operator and its uniform L^p -integrability for the same integrability range as that of Green's function of the Laplace operator.

The rest of the paper is organized as follows. In section 2, we state our main results (Theorems 2.1, 2.3 and 2.5). The main tools used in the proofs of the main results will be recalled and established in the next two sections. In section 3, we recall basic geometric properties of solutions to the Monge-Ampère

equations and their rescalings using the boundary Localization Theorem, the Localization Theorem at the boundary for solutions to the Monge-Ampère equations, and Caffarelli-Gutiérrez’s interior Harnack inequality for solutions to the linearized Monge-Ampère equations. We establish some fine geometric properties of boundary sections of solutions to the Monge-Ampère equation in section 4. These include Theorem 2.8 and Lemma 4.3 concerning a chain property of sections of ϕ . The proofs of Theorems 2.1 and 2.3 will be given in section 5. Theorem 2.5 will be proved in the final section, section 6.

2. STATEMENT OF THE MAIN RESULTS

We assume the following global information on the convex domain Ω and the convex function ϕ .

$$(2.1) \quad \Omega \subset B_{1/\rho}(0), \text{ and for each } y \in \partial\Omega \text{ there is a ball } B_\rho(z) \subset \Omega \\ \text{that is tangent to } \partial\Omega \text{ at } y.$$

Let $\phi : \bar{\Omega} \rightarrow \mathbb{R}$, $\phi \in C^{0,1}(\bar{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying

$$(2.2) \quad \det D^2\phi = g, \quad 0 < \lambda \leq g \leq \Lambda \quad \text{in } \Omega.$$

Assume further that on $\partial\Omega$, ϕ separates quadratically from its tangent planes, namely

$$(2.3) \quad \rho |x - x_0|^2 \leq \phi(x) - \phi(x_0) - \nabla\phi(x_0) \cdot (x - x_0) \leq \rho^{-1} |x - x_0|^2, \quad \forall x, x_0 \in \partial\Omega.$$

We note that, given (2.2), the quadratic separation from tangent planes on the boundary (2.3) holds if $\phi|_{\partial\Omega}, \partial\Omega \in C^3$, and Ω is uniformly convex (see [34, Proposition 3.2]).

The section of ϕ centered at $x \in \bar{\Omega}$ with height h is defined by

$$S_\phi(x, h) := \{y \in \bar{\Omega} : \phi(y) < \phi(x) + \nabla\phi(x) \cdot (y - x) + h\}.$$

For $x \in \Omega$, we denote by $\bar{h}(x)$ the maximal height of all sections of ϕ centered at x and contained in Ω , that is,

$$\bar{h}(x) := \sup\{h \geq 0 \mid S_\phi(x, h) \subset \Omega\}.$$

In this case, $S_\phi(x, \bar{h}(x))$ is called the maximal interior section of ϕ with center $x \in \Omega$.

Denote by $\Phi = (\Phi^{ij}) \equiv (\det D^2\phi)(D^2\phi)^{-1}$ the cofactor matrix of the Hessian matrix $D^2\phi = (\phi_{ij})$. Φ is divergence-free. Then, the linearized operator of the Monge-Ampère equation (2.2) is given by

$$L_\phi v := - \sum_{i,j=1}^n \Phi^{ij} v_{ij} \equiv - \sum_{i,j=1}^n (\Phi^{ij} v)_{ij}.$$

We denote by $c, \bar{c}, c_1, c_2, C, \bar{C}, C_1, C_2, \theta_0, \theta_*, M, M_0, M_1, \dots$, positive constants depending only on $\rho, \lambda, \Lambda, n$, and their values may change from line to line whenever there is no possibility of confusion. We refer to such constants as *universal constants*.

We can assume that all functions ϕ, u in this paper are smooth, and thus solutions can be interpreted in the classical sense. However, our estimates do not depend on the assumed smoothness but only on the given structural constants.

Our first main result is concerned with boundary Harnack inequality for non-negative solutions to the linearized Monge-Ampère equations.

Theorem 2.1 (Boundary Harnack inequality or Carleson estimate). *Assume that Ω and ϕ satisfy (2.1)–(2.3). Let $x_0 \in \partial\Omega$ and let $0 < t \leq c$ be universally small. Suppose that $u \geq 0$ is a continuous solution of $L_\phi u = 0$ in $\Omega \cap S_\phi(x_0, t)$ with $u = 0$ on $\partial\Omega \cap S_\phi(x_0, t)$. Let $P_0 \in \partial S_\phi(x_0, t/4)$ be any point satisfying $\text{dist}(P_0, \partial\Omega) \geq c_1 t^{1/2}$ for some universal constant c_1 . Then there is a universal constant $M_0 > 0$ depending only on $\rho, \lambda, \Lambda, n$ such that*

$$\max_{x \in S_\phi(x_0, t/2)} u(x) \leq M_0 u(P_0).$$

Remark 2.2. The point P_0 in the statement of Theorem 2.1 always exists by Lemma 4.4. Also by this lemma, $\text{dist}(P_0, \partial\Omega)$ is comparable, with universal constants, to the largest distance from a point on $\partial S_\phi(x_0, t/2)$ to the boundary $\partial\Omega$. By abuse of notation, we call P_0 the *apogee* of $S_\phi(x_0, t/2)$.

Theorem 2.1 is a special case of the following comparison theorem. A particular case of this theorem asserts that any two positive solutions of $L_\phi u = 0$ in Ω which vanish on a portion of the boundary must vanish at the same rate.

Theorem 2.3 (Comparison Theorem). *Assume that Ω and ϕ satisfy (2.1)–(2.3). Let $x_0 \in \partial\Omega$ and let $0 < t \leq c$ be universally small. Suppose that $u, v \geq 0$ are continuous solutions of $L_\phi u = L_\phi v = 0$ in $\Omega \cap S_\phi(x_0, t)$ with $u = 0$ on $\partial\Omega \cap S_\phi(x_0, t)$ and $v > 0$ in $\Omega \cap S_\phi(x_0, t)$. Let $P \in \partial S_\phi(x_0, t/4)$ be any point satisfying $\text{dist}(P, \partial\Omega) \geq c_1 t^{1/2}$ for some universal constant c_1 . Then there is a universal constant $M_0 > 0$ depending only on $\rho, \lambda, \Lambda, n$ such that*

$$\sup_{x \in S_\phi(x_0, t/2) \cap \Omega} \frac{u(x)}{v(x)} \leq M_0 \frac{u(P)}{v(P)}.$$

The proofs of Theorems 2.1 and 2.3 will be given in section 5.

Theorem 2.3 is an affine invariant analogue of the comparison theorems in [2, 8]. In [8], Caffarelli, Fabes, Mortola and Salsa proved a comparison theorem for positive solutions of linear, uniform elliptic equations in divergence form. In [2], Bauman proved a comparison theorem for positive solutions of linear, uniform elliptic equations in nondivergence form with continuous coefficients.

Remark 2.4. We require t to be universally small in Theorems 2.1 and 2.3 for convenience. A simple covering argument with the help of the boundary Localization Theorem 3.7 and the interior Harnack inequality in Theorem 3.3 shows that the above theorems hold for any t satisfying $S_\phi(x_0, t) \not\supset \bar{\Omega}$.

As an application of Theorem 2.1, we obtain a sharp upper bound for Green’s function of the linearized Monge-Ampère operator L_ϕ . Let $g_V(x, y)$ be Green’s function of L_ϕ in V with pole $y \in V \cap \Omega$ where $V \subset \bar{\Omega}$; that is $g_V(\cdot, y)$ is a positive solution of

$$L_\phi g_V(\cdot, y) = \delta_y \text{ in } V \cap \Omega \text{ and } g_V(\cdot, y) = 0 \text{ on } \partial V.$$

Theorem 2.5 (Bound on Green’s function). *Assume that Ω and ϕ satisfy (2.1)–(2.3). Let $x_0 \in \Omega$ and $0 < t_0 < c(n, \lambda, \Lambda, \rho)$. If $x \in \partial S_\phi(x_0, t_0)$, then*

$$g_\Omega(x, x_0) \leq \begin{cases} C(n, \lambda, \Lambda, \rho) t_0^{-\frac{n-2}{2}} & \text{if } n \geq 3, \\ C(n, \lambda, \Lambda, \rho) |\log t_0| & \text{if } n = 2. \end{cases}$$

Theorem 2.5 is a natural global counterpart of [23, Theorem 1.1], where we established a sharp upper bound for Green’s function of L_ϕ in the interior domain V of Ω (see also [36, Lemma 3.3] and [30, Theorem 3] for related interior results). From Theorem 2.5, we obtain the uniform L^p bound ($p < \frac{n}{n-2}$) of Green’s function.

Corollary 2.6 (L^p -integrability of Green’s function). *Assume that Ω and ϕ satisfy (2.1)–(2.3). Let $p \in (1, \frac{n}{n-2})$ if $n \geq 3$ and $p \in (1, \infty)$ if $n = 2$. Then for all $x_0 \in \Omega$, we have*

$$\int_{\Omega} g_{\Omega}^p(x, x_0) dx \leq C(n, \lambda, \Lambda, \rho, p).$$

More generally, for all $x_0 \in \Omega$ and $t \leq c_1(n, \lambda, \Lambda, \rho, p)$ we have

$$\sup_{x \in S_{\phi}(x_0, t) \cap \Omega} \int_{S_{\phi}(x_0, t)} g_{S_{\phi}(x_0, t)}^p(y, x) dy \leq C(n, \lambda, \Lambda, \rho, p) |S_{\phi}(x_0, t)|^{1 - \frac{n-2}{n} p}.$$

The proofs of Theorem 2.5 and Corollary 2.6 will be given in section 6.

Remark 2.7. Note that, under the assumptions (2.1)–(2.3), $D^2\phi$ is not bounded. In fact, the best regularity we can have from (2.1)–(2.3) is only $D^2\phi \in L^{1+\varepsilon}(\Omega)$ where ε is a small constant depending on n, λ and Λ . This follows from De Philippis-Figalli-Savin’s interior $W^{2,1+\varepsilon}$ estimates [12] for the Monge-Ampère equation combined with Savin’s techniques [35] in obtaining global regularity.

Therefore, Corollary 2.6 is interesting. It establishes the same global integrability of Green’s function for the linearized Monge-Ampère operator as Green’s function of the Laplace operator which corresponds to $\phi(x) = |x|^2/2$.

Moreover, Corollary 2.6 also says that as a degenerate and singular nondivergence form operator, L_ϕ has Green’s function with global L^p -integrability higher than that of a typical uniformly elliptic operator in nondivergence form as established by Fabes and Stroock [15, Corollary 2.4].

Our boundary Harnack estimates in Theorems 2.1 and 2.3 depend only on the bounds on the Hessian determinant $\det D^2\phi$, the quadratic separation of ϕ from its tangent planes on the boundary $\partial\Omega$ and the geometry of Ω . Under these assumptions, the linearized Monge-Ampère operator L_ϕ is in general not uniformly elliptic; i.e., the eigenvalues of $\Phi = (\Phi^{ij})$ are not necessarily bounded away from 0 and ∞ . Moreover, L_ϕ can be possibly singular near the boundary (see, for example, [27, 40]). The degeneracy and singularity of L_ϕ are the main difficulties in establishing our boundary Harnack inequalities. We handle the degeneracy of L_ϕ by working as in [9] with sections of solutions to the Monge-Ampère equations. These sections have the same role as Euclidean balls have in the classical theory. To overcome the singularity of L_ϕ near the boundary, we use a localization theorem at the boundary for solutions to the Monge-Ampère equations which was obtained by Savin in [33, 34].

Our main tools in the proofs of the main results are the boundary localization theorem for the Monge-Ampère equations [33, 34], the interior Harnack estimates for solutions to the linearized Monge-Ampère equations which were established in [9] and fine geometric properties of solutions to the Monge-Ampère equation which were obtained in [24] and further elaborated on in this paper. Among these, we would like to point out the following geometric property of sections of ϕ which is a crucial ingredient in the proof of Theorem 2.1. It says quantitatively that sections of solutions to the Monge-Ampère equations share many properties as Euclidean

balls. It is a global version of [16, Theorem 3.3.10] and could be of independent interest.

Theorem 2.8. *Assume that Ω and ϕ satisfy (2.1)–(2.3). Fix a universal constant M such that any section of ϕ with height M contains $\overline{\Omega}$. Then, there exist universal constants $c_0 > 0$ and $p_1 \geq 1$ such that*

(i) *if $0 < r < s \leq 3, s - r \leq 1, 0 < t \leq M$ and $x_1 \in S_\phi(x_0, rt)$ where $x_0 \in \overline{\Omega}$, then*

$$S_\phi(x_1, c_0(s - r)^{p_1}t) \subset S_\phi(x_0, st);$$

(ii) *if $0 < r < s < 1, 0 < t \leq M$ and $x_1 \in S_\phi(x_0, t) \setminus S_\phi(x_0, st)$ where $x_0 \in \overline{\Omega}$, then*

$$S_\phi(x_1, c_0(s - r)^{p_1}t) \cap S_\phi(x_0, rt) = \emptyset.$$

For the existence of the universal constant M in the statement of Theorem 2.8, see Lemma 3.6. The proof of Theorem 2.8 will be given in section 4.

Remark 2.9. The results of this paper, in particular, Theorem 2.1 and Corollary 2.6, have recently been used by Nguyen and the author [26] to establish global $W^{1,p}$ estimates for all $p < \frac{nq}{n-q}$ for solutions to the nonhomogeneous linearized Monge-Ampère equations $L_\phi u = f$ when f belongs to L^q where $n/2 < q \leq n$.

3. GEOMETRY OF THE MONGE-AMPÈRE EQUATIONS

In this section, we recall the main tools used in the proofs of our main results. These include properties of sections of solutions to the Monge-Ampère equation and their rescalings using the boundary localization theorem, the localization theorem at the boundary for solutions to the Monge-Ampère equation, and Caffarelli-Gutiérrez’s interior Harnack inequality for solutions to the linearized Monge-Ampère equation.

3.1. Geometry of sections of solutions to the Monge-Ampère equations.

In this section, we assume that

$$\lambda \leq \det D^2\phi \leq \Lambda \text{ in } \Omega.$$

Throughout, we use the following volume growth for compactly supported sections:

Lemma 3.1 ([16, Corollary 3.2.4]). *If $S_\phi(x, t) \Subset \Omega$, then*

$$c_1(n, \lambda, \Lambda)t^{\frac{n}{2}} \leq |S_\phi(x, t)| \leq C_1(n, \lambda, \Lambda)t^{\frac{n}{2}}.$$

We will use the following consequence of $C^{1,\alpha}$ estimates:

Lemma 3.2 ([16, Theorem 3.3.8]). *If $B(0, 1) \subset S_\phi(x, r) \subset B(0, n)$, then for any $t \leq r/2$, we have*

$$S_\phi(x, t) \subset B(x, C_1t^\mu)$$

for universal constants $\mu(n, \lambda, \Lambda) \in (0, 1)$ and C_1 depending only on n, λ , and Λ .

The Caffarelli-Gutiérrez Harnack inequality for the linearized Monge-Ampère equations states:

Theorem 3.3 ([9, Theorem 5]). *For each compactly supported section $S_\phi(x, t) \Subset \Omega$ and any nonnegative solution v of $L_\phi v = 0$ in $S_\phi(x, t)$, we have*

$$\sup_{S_\phi(x, 2\tau t)} v \leq C_1(n, \lambda, \Lambda) \inf_{S_\phi(x, 2\tau t)} v$$

for a universal $\tau \in (0, 1/2)$ depending only on n, λ and Λ .

We also need the following Vitali type covering lemma.

Lemma 3.4 ([35, Lemma 2.3]). *Let D be a compact set in Ω and assume that for each $x \in D$ we associate a corresponding section $S_\phi(x, h) \Subset \Omega$. Then we can find a finite number of these sections $S_\phi(x_i, h_i), i = 1, \dots, m$, such that*

$$D \subset \bigcup_{i=1}^m S_\phi(x_i, h_i), \text{ with } S_\phi(x_i, \delta h_i) \text{ disjoint,}$$

where $\delta > 0$ is a small constant that depends only on λ, Λ and n .

We recall the following global results on sections.

Theorem 3.5 (Engulfing property of sections, [24, Theorem 2.1]). *Assume that Ω and ϕ satisfy (2.1)–(2.3). Then, there exists a universal constant $\theta_* > 1$ such that if $y \in S_\phi(x, t)$ with $x \in \bar{\Omega}$ and $t > 0$, then $S_\phi(x, t) \subset S_\phi(y, \theta_* t)$.*

Lemma 3.6 ([27, Proposition 2.6], [24, Lemma 3.4 and Corollary 2.4]). *Assume that Ω and ϕ satisfy (2.1)–(2.3). Then, there are universal constants $\alpha \in (0, 1)$, C_α, c_0, C_1 and M such that*

- (i) $[\nabla\phi]_{C^\alpha(\bar{\Omega})} \leq C_\alpha;$
- (ii) $S_\phi(x, M) \supset \bar{\Omega}$ for all $x \in \bar{\Omega};$
- (iii) for any section $S_\phi(x, t)$ with $x \in \bar{\Omega}$ and $t \leq c_0$, we have

$$C_1^{-1}t^{n/2} \leq |S_\phi(x, t)| \leq C_1t^{n/2}.$$

3.2. The localization theorem and properties of the rescaled functions. In this section, we assume that Ω and ϕ satisfy (2.1)–(2.3). We now focus on sections centered at a point on the boundary $\partial\Omega$ and describe their geometry. Assume this boundary point to be 0, and by (2.1), we can also assume that

$$(3.1) \quad B_\rho(\rho e_n) \subset \Omega \subset \{x_n \geq 0\} \cap B_{\frac{1}{\rho}},$$

where $\rho > 0$ is the constant given by condition (2.1). After subtracting a linear function, we can assume further that

$$(3.2) \quad \phi(0) = 0, \quad \nabla\phi(0) = 0.$$

By (2.3), the boundary data ϕ has quadratic growth near the hyperplane $\{x_n = 0\}$. Hence, as $h \rightarrow 0$, $S_\phi(0, h)$ is equivalent to a half-ellipsoid centered at 0. This follows from the localization theorem proved by Savin in [33, 34]. Precisely, this theorem reads as follows.

Theorem 3.7 (Localization Theorem [33, 34]). *Assume that Ω satisfies (3.1) and ϕ satisfies (2.2), (3.2), and*

$$\rho|x|^2 \leq \phi(x) \leq \rho^{-1}|x|^2 \quad \text{on } \partial\Omega \cap \{x_n \leq \rho\}.$$

Then, for each $h \leq k$ there exists an ellipsoid E_h of volume $\omega_n h^{n/2}$ such that

$$kE_h \cap \bar{\Omega} \subset S_\phi(0, h) \subset k^{-1}E_h \cap \bar{\Omega}.$$

Moreover, the ellipsoid E_h is obtained from the ball of radius $h^{1/2}$ by a linear transformation A_h^{-1} (sliding along the $x_n = 0$ plane)

$$A_h E_h = h^{1/2} B_1, \quad \det A_h = 1,$$

$$A_h(x) = x - \tau_h x_n, \quad \tau_h = (\tau_1, \tau_2, \dots, \tau_{n-1}, 0),$$

with $|\tau_h| \leq k^{-1} |\log h|$. The constant k above depends only on $\rho, \lambda, \Lambda, n$.

Throughout, we denote by $B_r = B_r(0)$ the Euclidean ball centered at 0 with radius r and $B_r^+ = B_r \cap \{x \in \mathbb{R}^n : x_n \geq 0\}$.

Let ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.7 at the origin. We know that for all $h \leq k$, $S_\phi(0, h)$ satisfies

$$kE_h \cap \bar{\Omega} \subset S_\phi(0, h) \subset k^{-1}E_h,$$

with A_h being a linear transformation and

$$\begin{aligned} \det A_h &= 1, & E_h &= A_h^{-1}B_{h^{1/2}}, & A_h x &= x - \tau_h x_n, \\ \tau_h \cdot e_n &= 0, & \|A_h^{-1}\|, \|A_h\| &\leq k^{-1}|\log h|. \end{aligned}$$

This gives

$$(3.3) \quad \bar{\Omega} \cap B_{\frac{c_2 h^{1/2}}{|\log h|}}^+ \subset S_\phi(0, h) \subset B_{C_2 h^{1/2} |\log h|}^+.$$

We denote the rescaled functions by

$$\phi_h(x) := \frac{\phi(h^{1/2}A_h^{-1}x)}{h}.$$

The function ϕ_h is continuous and is defined in $\bar{\Omega}_h$ with $\Omega_h := h^{-1/2}A_h\Omega$ and solves the Monge-Ampère equation

$$\det D^2\phi_h = g_h(x), \quad \lambda \leq g_h(x) := g(h^{1/2}A_h^{-1}x) \leq \Lambda.$$

The section at height 1 for ϕ_h centered at the origin satisfies $S_{\phi_h}(0, 1) = h^{-1/2}A_h S_\phi(0, h)$, and by Theorem 3.7, we obtain

$$B_k \cap \bar{\Omega}_h \subset S_{\phi_h}(0, 1) \subset B_{k^{-1}}^+.$$

Some properties of the rescaled function ϕ_h were established in [27, Lemma 4.2 and Lemma 5.4]. For later use, we record them here.

Lemma 3.8. *If $h \leq c$, then*

- (i) ϕ_h and $S_{\phi_h}(0, 1)$ satisfy the hypotheses of the Localization Theorem 3.7 at all $x_0 \in \partial\Omega_h \cap B(0, c)$, for a small constant $\tilde{\rho}$ depending only on $\rho, n, \lambda, \Lambda$;
- (ii) if $r \leq c$ small, we have

$$|\nabla\phi_h| \leq Cr|\log r|^2 \quad \text{in } \bar{\Omega}_h \cap B_r;$$

- (iii) $\partial\Omega_h \cap B_{2/k}$ is a graph in the e_n direction whose $C^{1,1}$ norm is bounded by $Ch^{1/2}$;
- (iv) if δ is universally small, then for any $x \in S_{\phi_h}(0, \delta) \cap \Omega_h$, the maximal interior section $S_{\phi_h}(x, h(x))$ of ϕ_h centered at x is tangent to $\partial\Omega_h$ at $z \in \partial\Omega_h \cap B_{c/2}$.

4. GEOMETRIC PROPERTIES OF BOUNDARY SECTIONS

In this section, we prove Theorem 2.8 and establish a chain property for sections of solutions to the Monge-Ampère equations in Lemma 4.3.

For the proof of Theorem 2.8, we need two additional results: Proposition 4.1 and Lemma 4.2. Proposition 4.1 is concerned with the shape of maximal interior sections. Lemma 4.2 estimates the size of a section of ϕ in terms of its height.

Proposition 4.1. *Let ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.7 at the origin. Assume that for some $y \in \Omega$ the maximal interior section $S_\phi(y, \bar{h}(y)) \subset \Omega$ is tangent to $\partial\Omega$ at 0, that is, $\partial S_\phi(y, \bar{h}(y)) \cap \partial\Omega = \{0\}$. Then, if $h := \bar{h}(y) \leq c$ with c universal,*

(i) *there exists a small positive constant $k_0 < k$ depending only on ρ, λ, Λ and n such that*

$$\nabla\phi(y) = ae_n \quad \text{for some } a \in [k_0h^{1/2}, k_0^{-1}h^{1/2}],$$

$$k_0E_h \subset S_\phi(y, h) - y \subset k_0^{-1}E_h, \quad k_0h^{1/2} \leq \text{dist}(y, \partial\Omega) \leq k_0^{-1}h^{1/2},$$

with E_h and k the ellipsoid and constant defined in the Localization Theorem 3.7;

(ii) *if $h/2 < t \leq c$, then $S_\phi(y, 2t) \subset S_\phi(0, (2k_0^{-4} + 4)t)$;*

(iii) *if $t \leq h/2$, then $S(y, t) \subset B(y, C_4t^\mu)$ for universal constants $\bar{\mu}(n, \lambda, \Lambda) \in (0, 1/8)$ and C_4 .*

Lemma 4.2. *Assume that Ω and ϕ satisfy (2.1)–(2.3). Let M be as in Lemma 3.6 and let $\bar{\mu}$ be as in Proposition 4.1. Then, there is a universal constant $\bar{C} > 0$ such that for all $t \leq M$ and $y \in \bar{\Omega}$,*

$$S_\phi(y, t) \subset B(y, \bar{C}t^{\bar{\mu}}).$$

Proof of Proposition 4.1. (i) is a consequence of Theorem 3.7 and was proved in [35] (see also [27, Proposition 3.2]). (ii) comes from the proof of [24, Proposition 2.3] (see equation (4.11) there).

We now prove (iii). By (i), $S_\phi(y, h)$ is equivalent to an ellipsoid E_h , i.e.,

$$k_0E_h \subset S_\phi(y, h) - y \subset k_0^{-1}E_h,$$

where

$$(4.1) \quad E_h := h^{1/2}A_h^{-1}B_1, \quad \text{with } \|A_h\|, \|A_h^{-1}\| \leq C|\log h|; \det A_h = 1.$$

The rescaling $\tilde{\phi}$ of ϕ ,

$$\tilde{\phi}(\tilde{x}) := \frac{1}{h} \left[\phi(y + h^{1/2}A_h^{-1}\tilde{x}) - \phi(y) - \nabla\phi(y) \cdot (h^{1/2}A_h^{-1}\tilde{x}) \right],$$

satisfies

$$\det D^2\tilde{\phi}(\tilde{x}) = \tilde{g}(\tilde{x}) := g(y + h^{1/2}A_h^{-1}\tilde{x}) \in [\lambda, \Lambda]$$

and

$$(4.2) \quad B_{k_0} \subset S_{\tilde{\phi}}(0, 1) \subset B_{k_0^{-1}}, \quad S_{\tilde{\phi}}(0, 1) = h^{-1/2}A_h(S_\phi(y, h) - y),$$

where we recall that $S_{\tilde{\phi}}(0, 1)$ represents the section of $\tilde{\phi}$ at the origin with height 1.

With (4.2), we apply Lemma 3.2 to obtain $\mu(n, \lambda, \Lambda) \in (0, 1)$ and $C_4(n, \lambda, \Lambda, \rho)$ such that

$$h^{-1/2}A_h(S_\phi(y, t) - y) = S_{\tilde{\phi}}(0, \frac{t}{h}) \subset B(0, C_4(\frac{t}{h})^\mu).$$

Using (4.1), $t/h \leq \frac{1}{2}$ and $h \leq c$, we can take $\bar{\mu} = \min\{\mu, \frac{1}{10}\}$ and obtain

$$S_\phi(y, t) - y \subset h^{1/2}A_h^{-1}(B(0, C_4(\frac{t}{h})^\mu)) \subset h^{1/4}B(0, C_4(\frac{t}{h})^{\bar{\mu}}) \subset B(0, C_4t^{\bar{\mu}}).$$

Hence (iii) is proved. □

Proof of Lemma 4.2. In this proof, we drop the dependence on ϕ of sections when no confusion arises. We first prove the lemma for the case where y is a boundary point which can be assumed to be $0 \in \partial\Omega$. By rotating coordinates and subtracting a linear function from ϕ , we can assume that ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.7 at the origin. Thus, if $t \leq k(\leq 1)$, then from (3.3), we have

$$S(0, t) \subset B(0, C_2 t^{1/2} |\log t|) \subset B(0, C_3 t^{1/4})$$

for some C_3 universal, depending only on ρ, n, λ and Λ . By increasing C_3 if necessary, we find that

$$(4.3) \quad S(0, t) \subset B(0, C_3 t^{\bar{\mu}}) \text{ for all } t \leq M.$$

Next, we prove the lemma for $y \in \Omega$ away from the boundary, that is, $r := \text{dist}(y, \partial\Omega) \geq c$, for c universal. In this case, we can use Lemma 3.2 and the strict convexity of ϕ which follows from the boundary $C^{1,\alpha}$ regularity of ϕ for all $\alpha \in (0, 1)$ as observed in [27, Lemma 4.1] and Caffarelli's Localization Theorem [4]. We then find a $\mu \in (0, 1)$ depending only on n, λ, Λ and C_0 depending on $\rho, n, \lambda, \Lambda$ such that $S_\phi(y, t) \subset B(y, C_0 t^\mu)$ for all $t \leq M$. By increasing C_0 if necessary, we find that

$$(4.4) \quad S_\phi(y, t) \subset B(y, C_0 t^{\bar{\mu}}) \text{ for all } t \leq M.$$

Finally, we prove the lemma for $y \in \Omega$ near the boundary $\partial\Omega$. Let $y \in \Omega$ with $r := \text{dist}(y, \partial\Omega) \leq c$, for c universal. Consider the section $S(y, t)$ with $t \leq M$. Then, there exists either $z \in \partial\Omega$ such that $z \in S(y, 2t)$ or $S(y, 2t) \subset \Omega$. In the first case, by Theorem 3.5, we have $S(y, 2t) \subset S(z, 2\theta_* t)$. Thus, as in (4.3), we find

$$S(y, t) \subset S(z, 2\theta_* t) \subset B(z, C_3 t^{\bar{\mu}}).$$

It follows that $|y - z| \leq C_3 t^{\bar{\mu}}$ and therefore

$$(4.5) \quad S(y, t) \subset B(y, 2C_3 t^{\bar{\mu}}) \text{ for all } t \leq M.$$

In the remaining case $S(y, 2t) \subset \Omega$, we obtain from Proposition 4.1(iii),

$$(4.6) \quad S(y, t) \subset B(y, C_4 t^{\bar{\mu}}).$$

The lemma now follows from (4.3)-(4.6) by choosing $\bar{C} = C_0 + 2C_3 + C_4$. □

Proof of Theorem 2.8. We prove that the conclusions of the theorem hold for $p_1 = \bar{\mu}^{-1}$ where $0 < \bar{\mu}(n, \lambda, \Lambda) < 1/8$ is the universal constant in Lemma 4.2.

(i) Let $0 < r < s \leq 3, s - r \leq 1$ and $0 < t \leq M$ where M is as in Lemma 3.6. Let c, δ be as in Lemma 3.8 and let θ_* be as in Theorem 3.5. We first consider the case $t \leq \frac{c\delta}{4\theta_*}$. If $S_\phi(x_0, 4t) \subset \Omega$, then the theorem follows from the interior result established in [16, Theorem 3.3.10]. Suppose now $S_\phi(x_0, 4t) \cap \partial\Omega \neq \emptyset$. Without loss of generality, we can assume that $0 \in S_\phi(x_0, 4t) \cap \partial\Omega$ and that ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.7 at 0. By Theorem 3.5, we have

$$(4.7) \quad S_\phi(x_0, 4t) \subset S_\phi(0, 4\theta_* t).$$

We use the Localization Theorem 3.7 at the origin with height $h = 4\theta_* \delta^{-1} t \leq c$ and consider the rescaled function

$$\phi_h(x) = \frac{\phi(h^{1/2} A_h^{-1} x)}{h} \text{ where } x \in \Omega_h = h^{-1/2} A_h \Omega.$$

Denote $x_{i,h} = h^{-1/2}A_h x_i$ for $i = 0, 1$. Let $\bar{t} = \frac{\delta}{4\theta_*}$. Then, since $x_1 \in S_\phi(x_0, rt)$, we have $x_{1,h} \in S_{\phi_h}(x_{0,h}, r\bar{t})$ and

$$\begin{aligned} h^{-1/2}A_h S_\phi(x_1, c_0(s-r)^{p_1}t) &= S_{\phi_h}(x_{1,h}, c_0(s-r)^{p_1}\bar{t}), \\ h^{-1/2}A_h S_\phi(x_0, st) &= S_{\phi_h}(x_{0,h}, s\bar{t}). \end{aligned}$$

We will show that for some universally small $c_0 > 0$ and $p_1 \geq 1$,

$$(4.8) \quad S_{\phi_h}(x_{1,h}, c_0(s-r)^{p_1}\bar{t}) \subset S_{\phi_h}(x_{0,h}, s\bar{t}).$$

Suppose that $y \in S_{\phi_h}(x_{1,h}, c_0(s-r)^{p_1}\bar{t})$ and $x_{1,h} \in S_{\phi_h}(x_{0,h}, r\bar{t})$. Then

$$\begin{aligned} \phi_h(y) &< \phi_h(x_{1,h}) + \nabla\phi_h(x_{1,h})(y - x_{1,h}) + c_0(s-r)^{p_1}\bar{t} \\ &< \phi_h(x_{0,h}) + \nabla\phi_h(x_{0,h})(x_{1,h} - x_{0,h}) + r\bar{t} + \nabla\phi_h(x_{1,h})(y - x_{1,h}) \\ &\quad + c_0(s-r)^{p_1}\bar{t} \\ &= \phi_h(x_{0,h}) + \nabla\phi_h(x_{0,h})(y - x_{0,h}) + [\nabla\phi_h(x_{1,h}) - \nabla\phi_h(x_{0,h})](y - x_{1,h}) \\ (4.9) \quad &\quad + c_0(s-r)^{p_1}\bar{t} + r\bar{t}. \end{aligned}$$

We note from (4.7) that $x_{1,h} \in S_{\phi_h}(x_{0,h}, r\bar{t}) \subset S_{\phi_h}(0, \delta)$. By Lemma 3.8(ii),

$$(4.10) \quad |\nabla\phi_h(x_{0,h})| \leq C \text{ and } |\nabla\phi_h(x_{1,h})| \leq C.$$

On the other hand, from $y \in S_{\phi_h}(x_{1,h}, c_0(s-r)^{p_1}\bar{t})$, we can estimate for some universal constant C

$$(4.11) \quad |y - x_{1,h}| \leq C(c_0(s-r)^{p_1}\bar{t})^\mu$$

where we recall that $\bar{\mu}(n, \lambda, \Lambda) \in (0, 1/8)$ is also the constant in Proposition 4.1.

Indeed, if $x_{1,h} \in \partial\Omega_h$, then from Lemma 3.8(i), ϕ_h and $S_{\phi_h}(0, 1)$ satisfy the hypotheses of the Localization Theorem 3.7 at $x_{1,h}$, and hence (4.11) follows from the estimate (3.3).

Consider now the case $x_{1,h} \in \Omega_h$. By Lemma 3.8(iv), $S_{\phi_h}(x_{1,h}, \bar{h}(x_{1,h}))$, the maximal interior section of ϕ_h centered at $x_{1,h}$, is tangent to $\partial\Omega_h$ at $z \in \partial\Omega_h \cap B_{c/2}$. Thus, by Lemma 3.8(i), the Localization Theorem 3.7 is applied at z , and we can apply Proposition 4.1 to the tangent point z .

If $c_0(s-r)^{p_1}\bar{t} > \frac{1}{2}\bar{h}(x_{1,h})$, then from Proposition 4.1(ii), we have

$$S_{\phi_h}(x_{1,h}, 2c_0(s-r)^{p_1}\bar{t}) \subset S_{\phi_h}(z, \bar{c}c_0(s-r)^{p_1}\bar{t}), \quad \bar{c} = 2k_0^{-4} + 4.$$

It follows from (3.3) that

$$(4.12) \quad |y - x_{1,h}| \leq \text{diam}(S_{\phi_h}(z, \bar{c}c_0(s-r)^{p_1}\bar{t})) \leq C(\bar{c}c_0(s-r)^{p_1}\bar{t})^{1/3}.$$

Now, it remains to consider the case $c_0(s-r)^{p_1}\bar{t} \leq \frac{1}{2}\bar{h}(x_{1,h})$. Then, by Proposition 4.1(iii),

$$(4.13) \quad |y - x_{1,h}| \leq C_4(c_0(s-r)^{p_1}\bar{t})^\mu.$$

Combining (4.12) and (4.13), we obtain (4.11) for the case $x_{1,h} \in \Omega_h$, as asserted.

It follows from (4.9)-(4.11) and $0 < s-r \leq 1$ that

$\phi_h(y) - \phi_h(x_{0,h}) - \nabla\phi_h(x_{0,h})(y - x_{0,h}) < C(c_0(s-r)^{p_1}\bar{t})^\mu + c_0(s-r)^{p_1}\bar{t} + r\bar{t} < s\bar{t}$
if we choose c_0 universally small and $p_1 = \bar{\mu}^{-1}$. Therefore $S_{\phi_h}(x_{1,h}, c_0(s-r)^{p_1}\bar{t}) \subset S_{\phi_h}(x_{0,h}, s\bar{t})$, proving (4.8) as claimed.

Finally, we consider the case $M \geq t \geq \frac{c\delta}{4\theta_*}$. Suppose that $z \in S_\phi(x_1, c_0(s-r)^{p_1}t)$ and $x_1 \in S_\phi(x_0, rt)$. Then, as in (4.9), we have

$$\phi(z) < \phi(x_0) + \nabla\phi(x_0)(z - x_0) + [\nabla\phi(x_1) - \nabla\phi(x_0)](z - x_1) + c_0(s-r)^{p_1}t + rt.$$

We show that if c_0 is universally small and $p_1 = \bar{\mu}^{-1}$ as above, then

$$(4.14) \quad \phi(z) - \phi(x_0) - \nabla\phi(x_0)(z - x_0) < [\nabla\phi(x_1) - \nabla\phi(x_0)](z - x_1) + c_0(s-r)^{p_1}t + rt < st,$$

which implies that $S_\phi(x_1, c_0(s-r)^{p_1}t) \subset S_\phi(x_0, st)$ as asserted in the theorem. Indeed, by Lemma 4.2, we have

$$|z - x_1| \leq \bar{C}(c_0(s-r)^{p_1}t)^{\bar{\mu}} \text{ and } |x_1 - x_0| \leq \bar{C}(rt)^{\bar{\mu}}.$$

It follows from the global $C^{1,\alpha}$ estimate in Lemma 3.6 that

$$|[\nabla\phi(x_1) - \nabla\phi(x_0)](z - x_1)| \leq C_\alpha|x_1 - x_0|^\alpha|z - x_1| \leq C(rt)^{\alpha\bar{\mu}}(c_0(s-r)^{p_1}t)^{\bar{\mu}}.$$

If $\frac{c\delta}{4\theta_*} \leq t \leq M$, then (4.14) easily follows from

$$C(rt)^{\alpha\bar{\mu}}(c_0(s-r)^{p_1}t)^{\bar{\mu}} + c_0(s-r)^{p_1}t + rt < st$$

if we choose c_0 universally small and $p_1 = \bar{\mu}^{-1}$.

(ii) The proof, based on convexity, is quite similar to that of (i). We include here for example the proof for the case $M \geq t \geq \frac{c\delta}{4\theta_*}$. From convexity and $x_1 \in S_\phi(x_0, t) \setminus S_\phi(x_0, st)$, we have for all $z \in \bar{\Omega}$,

$$\begin{aligned} \phi(z) &\geq \phi(x_1) + \nabla\phi(x_1)(z - x_1) \\ &\geq \phi(x_0) + \nabla\phi(x_0)(x_1 - x_0) + st + \nabla\phi(x_1)(z - x_1) \\ &= \phi(x_0) + \nabla\phi(x_0)(z - x_0) + [\nabla\phi(x_1) - \nabla\phi(x_0)](z - x_1) + st. \end{aligned}$$

We will show that for all $z \in S(x_1, c_0(s-r)^{p_1}t)$, we have

$$(4.15) \quad \phi(z) - \phi(x_0) - \nabla\phi(x_0)(z - x_0) > [\nabla\phi(x_1) - \nabla\phi(x_0)](z - x_1) + st > rt,$$

which implies that $S_\phi(x_1, c_0(s-r)^{p_1}t) \cap S_\phi(x_0, rt) = \emptyset$.

Suppose now that $z \in S(x_1, c_0(s-r)^{p_1}t)$. By Lemma 4.2, we have

$$|z - x_1| \leq \bar{C}(c_0(s-r)^{p_1}t)^{\bar{\mu}} \text{ and } |x_1 - x_0| \leq \bar{C}t^{\bar{\mu}}.$$

It follows from the global $C^{1,\alpha}$ estimate in Lemma 3.6 that

$$|[\nabla\phi(x_1) - \nabla\phi(x_0)](z - x_1)| \leq C_\alpha|x_1 - x_0|^\alpha|z - x_1| \leq Ct^{\alpha\bar{\mu}}(c_0(s-r)^{p_1}t)^{\bar{\mu}}$$

and thus

$$\phi(z) - \phi(x_0) - \nabla\phi(x_0)(z - x_0) \geq st - Ct^{\alpha\bar{\mu}}(c_0(s-r)^{p_1}t)^{\bar{\mu}} \geq rt,$$

proving (4.15), provided that

$$Ct^{\alpha\bar{\mu}}(c_0(s-r)^{p_1}t)^{\bar{\mu}} < (s-r)t.$$

Since $M \geq t \geq \frac{c\delta}{4\theta_*}$ and $0 < s-r \leq 1$, this can be achieved by choosing $p_1 = \bar{\mu}^{-1}$ and c_0 small. □

We now establish a chain property for sections of solutions to the Monge-Ampère equations.

Lemma 4.3. *Assume ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.7 at z . Let $\tau = \tau(n, \lambda, \Lambda)$ be as in Theorem 3.3. Assume that $x \in \Omega$ with $\text{dist}(x, \partial\Omega) \leq c_0$ is universally small and satisfies $\partial\Omega \cap \partial S_\phi(x, \bar{h}(x)) = \{z\}$. Then*

(i) *we can find a sequence $x_0 = x, x_1, \dots, x_m$ in $S_\phi(z, \bar{K} \text{dist}^2(x, \partial\Omega))$ with $m \leq K$ such that*

$$x_i \in S_\phi(x_{i+1}, \tau \bar{h}(x_{i+1})) \text{ for all } i = 0, 1, \dots, m - 1$$

and

$$\text{dist}(x_m, \partial\Omega) \geq 2k_0^{-2} \text{dist}(x, \partial\Omega)$$

for some universal constants $K, \bar{K} > 0$, and k_0 is the small constant in Proposition 4.1.

(ii) *Conversely, for any universal constants K_1 and K_2 and any $y \in S_\phi(z, K_1 \text{dist}^2(x, \partial\Omega))$ with $\text{dist}(y, \partial\Omega) \geq K_2 \text{dist}(x, \partial\Omega)$, we can find a sequence $x_0 = x, x_1, \dots, x_m = y$ in the section $S_\phi(z, K_1 \text{dist}^2(x, \partial\Omega))$ with m universally bounded and*

$$x_i \in S_\phi(x_{i+1}, \tau \bar{h}(x_{i+1})) \text{ for all } i = 0, 1, \dots, m - 1.$$

Proof of Lemma 4.3. We give here the proof of (i) for k_0 as in Proposition 4.1, while that of (ii) follows similarly. Assume $z = 0$ for simplicity. By Proposition 4.1,

$$k_0 \bar{h}^{1/2}(x) \leq \text{dist}(x, \partial\Omega) \leq k_0^{-1} \bar{h}^{1/2}(x).$$

Let c be as in Proposition 4.1. We first prove the lemma for $c_0 c \leq \text{dist}(x, \partial\Omega) \leq c_0$. By (2.1) and the first inclusion in (3.3), we can find $y \in S_\phi(0, c)$ such that

$$\text{dist}(y, \partial\Omega) \geq c_3(\rho, n, \lambda, \Lambda).$$

Thus if c_0 is universally small, then for $K_1 := 6k_0^{-2}$, we have

$$(4.16) \quad \text{dist}(y, \partial\Omega) \geq K_1 \text{dist}(x, \partial\Omega).$$

Let

$$D = \{y \in \overline{S_\phi(0, c)} : \text{dist}(y, \partial\Omega) \geq \text{dist}(x, \partial\Omega)\} \subset S_\phi(0, \frac{\bar{K}}{4} \text{dist}^2(x, \partial\Omega))$$

where $\bar{K} = 4(cc_0^2)^{-1}$ is a universal constant. Since $D \subset \bigcup_{y \in D} S_\phi(y, \tau \bar{h}(y))$, by Lemma 3.4, we can find a covering

$$(4.17) \quad D \subset \bigcup_{i=1}^K S_\phi(y_i, \tau \bar{h}(y_i))$$

such that $\{S_\phi(y_i, \delta \tau \bar{h}(y_i))\}$ is a disjoint family of sections. Hence, with y as in (4.16), we can find a sequence $x_0 = x, x_1, \dots, x_m$ in $S_\phi(0, \frac{\bar{K}}{4} \text{dist}^2(x, \partial\Omega))$ with $m \leq K$ such that $x_i \in S_\phi(x_{i+1}, \tau \bar{h}(x_{i+1}))$ for all $i = 0, 1, \dots, m - 1$. The conclusion of the lemma follows once we prove that K is universally bounded. However, this follows from the volume estimate. Indeed, we first note that $|D| \leq C(n, \rho, \lambda, \Lambda)$. On the other hand, from Proposition 4.1(i), we have for each $i = 1, \dots, K$,

$$\bar{h}(y_i) \geq k_0^2 \text{dist}^2(y_i, \partial\Omega) \geq k_0^2 \text{dist}^2(x, \partial\Omega) \geq (k_0 c c_0)^2.$$

Hence, by (4.17) and the lower bound on volume of sections in Lemma 3.1, we deduce that K is universally bounded.

We now prove the lemma for $dist(x, \partial\Omega) \leq cc_0$. Let $h = dist^2(x, \partial\Omega)/(c_0^2c) \leq c$. We use the Localization Theorem 3.7 at the origin and consider the rescaled function

$$\phi_h(x) = \frac{\phi(h^{1/2}A_h^{-1}x)}{h} \text{ where } x \in \Omega_h = h^{-1/2}A_h\Omega.$$

By Lemma 3.8, ϕ_h and $S_{\phi_h}(0, 1)$ satisfy the hypotheses of the Localization Theorem 3.7 at all points $x_0 \in \partial\Omega_h \cap B(0, c)$. By [27, Proposition 4.3], we have for all $y_h = h^{-1/2}A_hy \in \Omega_h \cap B_{k^{-1}}^+$,

$$(4.18) \quad 1 - Ch^{1/2}|\log h|^2 \leq \frac{h^{-1/2}dist(y, \partial\Omega)}{dist(y_h, \partial\Omega_h)} \leq 1 + Ch^{1/2}|\log h|^2.$$

If $h \leq c$ universally small, we have

$$(4.19) \quad Ch^{1/2}|\log h|^2 \leq 1/2.$$

This in particular implies that for $x_h = h^{-1/2}A_hx$,

$$dist(x_h, \partial\Omega_h) \in [\frac{1}{2}h^{-1/2}dist(x, \partial\Omega), \frac{3}{2}h^{-1/2}dist(x, \partial\Omega)] = [\frac{1}{2}c_0c^{1/2}, \frac{3}{2}c_0c^{1/2}].$$

As in the case $c_0c \leq dist(x, \partial\Omega) \leq c_0$, we can find a sequence

$$\{x_{0,h} = x_h, x_{1,h}, \dots, x_{m,h}\} \subset S_{\phi_h}(0, \frac{\bar{K}}{4}dist^2(x_h, \partial\Omega_h))$$

with $m \leq K$ universal such that

$$x_{i,h} \in S_{\phi_h}(x_{i+1,h}, \tau\bar{h}(x_{i+1,h})) \text{ for all } i = 0, 1, \dots, m - 1$$

and

$$dist(x_{m,h}, \partial\Omega_h) \geq K_1dist(x_h, \partial\Omega_h).$$

It follows that for $x_i = h^{1/2}A_h^{-1}x_{i,h}$, we have $x_i \in S_{\phi}(x_{i+1}, \tau\bar{h}(x_{i+1}))$ for all $i = 0, 1, \dots, m - 1$. Recalling (4.18) and (4.19), we find

$$dist(x_m, \partial\Omega) \geq \frac{K_1}{3}dist(x, \partial\Omega) = 2k_0^{-2}dist(x, \partial\Omega).$$

Furthermore, for all $i = 0, 1, \dots, m$, we have from (4.18) and (4.19)

$$x_i \in S_{\phi}(0, h(\bar{K}/4)dist^2(x_h, \partial\Omega_h)) \subset S_{\phi}(0, \bar{K}dist^2(x, \partial\Omega)).$$

The proof of the lemma is complete. □

We end this section with the following lemma, which justifies the definition of the *apogee* of sections in Remark 2.2 and its use in this paper.

Lemma 4.4. *Assume that Ω and ϕ satisfy (2.1)–(2.3). Suppose that $x \in \partial\Omega$ and $t \leq c_1$ is universally small. Then, for some universal constant $C > 0$,*

- (i) $dist(y, \partial\Omega) \leq Ct^{1/2}$ for all $y \in S_{\phi}(x, t)$;
- (ii) *there exists $y \in \partial S_{\phi}(x, t) \cap \Omega$ such that $dist(y, \partial\Omega) \geq C^{-1}t^{1/2}$.*

Proof of Lemma 4.4. Without loss of generality, we assume that $x = 0$ and ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.7 at the origin. Let k be as in Theorem 3.7. Let c be as in Lemma 3.8 and let θ be universally large such that

$$k^{-1}(\frac{1}{\theta})^{1/2} \log \frac{1}{\theta} \leq c.$$

Suppose $t \leq c_1 := \frac{c}{\theta}$. Let $h = \theta t \leq c$. We use the Localization Theorem 3.7 at the origin and consider the rescaled function

$$\phi_h(x) = \frac{\phi(h^{1/2}A_h^{-1}x)}{h} \text{ where } x \in \Omega_h = h^{-1/2}A_h\Omega.$$

By Lemma 3.8, ϕ_h and $S_{\phi_h}(0, 1)$ satisfy the hypotheses of the Localization Theorem 3.7 at all points $x_0 \in \partial\Omega_h \cap B(0, c)$. Note that

$$h^{-1/2}A_hS_{\phi}(0, t) = S_{\phi_h}(0, \frac{1}{\theta}).$$

Therefore, from (3.3), we have

$$(4.20) \quad \text{dist}(Z, \partial\Omega_h) \leq C \text{ for all } Z \in S_{\phi}(0, \frac{1}{\theta}).$$

Because $h \leq c$, we have $Ch^{1/2}|\log h|^2 \leq 1/2$. Thus, for all $y \in S_{\phi}(0, t)$, we have $y_h = h^{-1/2}A_hy \in \Omega_h \cap B_{k-1}^+$ and by (4.20) and (4.18),

$$\text{dist}(y, \partial\Omega) \leq (1 + Ch^{1/2}|\log h|^2)h^{1/2}\text{dist}(y_h, \partial\Omega_h) \leq Ch^{1/2} \leq Ct^{1/2}.$$

Hence (i) is proved.

For the proof of (ii), we use (3.3) and the fact that the curvature of $\partial\Omega_h \cap B_c$ is bounded by $Ch^{1/2}$ from Lemma 3.8(iii) to find $Y \in \partial S_{\phi_h}(0, \frac{1}{\theta}) \cap \Omega_h$ such that $\text{dist}(Y, \partial\Omega_h) \geq c'$ with c' universally small. Then (ii) follows easily because for $y = h^{1/2}A_h^{-1}Y \in \partial S_{\phi}(0, t) \cap \Omega$, we have from (4.18)

$$\text{dist}(y, \partial\Omega) \geq (1 - Ch^{1/2}|\log h|^2)h^{1/2}\text{dist}(Y, \partial\Omega_h) \geq C^{-1}t^{1/2}.$$

□

5. PROOF OF THE BOUNDARY HARNACK INEQUALITY

This section is devoted to the proofs of Theorems 2.1 and 2.3.

Throughout this section, we assume that the convex domain Ω and the convex function ϕ satisfy (2.1)–(2.3). We will use the following universal constants from the previous sections:

$$\theta_* > 1 \text{ in Theorem 3.5, } c_0 > 0 \text{ and } p_1 = \bar{\mu}^{-1} > 1 \text{ in Theorem 2.8,} \\ \bar{\mu} \text{ in Proposition 4.1 and Lemma 4.2.}$$

5.1. Boundary properties of solutions to the linearized Monge-Ampère equations. In this section, we state several ingredients used in the proof of Theorem 2.1. They are concerned with boundary properties of solutions to the linearized Monge-Ampère equations.

The first ingredient in the proof of Theorem 2.1 states that the maximum value of a positive solution to the linearized Monge-Ampère equations on a boundary section of ϕ increases by a factor of 2 when we pass to a concentric boundary section of universally larger height.

Lemma 5.1. *Assume that $x_0 \in \partial\Omega$. Suppose that $u \geq 0$ is a continuous solution of $L_{\phi}u = 0$ in $\Omega \cap S_{\phi}(x_0, c)$ and that $u = 0$ on $\partial\Omega \cap S_{\phi}(x_0, c)$. Then, there is a universal constant $H > 1$ such that for all $0 < h \leq c/2$, we have*

$$\max_{x \in S_{\phi}(x_0, h)} u(x) \geq 2 \max_{x \in S_{\phi}(x_0, h/H)} u(x).$$

In the next ingredient in the proof of Theorem 2.1, we show that for any interior point x of the domain Ω we can find another interior point y whose distance to the boundary is larger than that of x and the values of any nonnegative solution to the linearized Monge-Ampère equation $L_\phi u = 0$ are universally comparable at x and y .

Lemma 5.2. *Let ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.7 at the origin. There exist universal constants $\hat{K} > 1$ and $M_1 > 1$ with the following property. Suppose that $u \geq 0$ is a continuous solution of $L_\phi u = 0$ in $\Omega \cap S_\phi(0, c)$ and that $u = 0$ on $\partial\Omega \cap S_\phi(0, c)$. Assume that for some $x \in \Omega$ we have $\partial\Omega \cap \partial S_\phi(x, \bar{h}(x)) = \{0\}$ and $\text{dist}^2(x, \partial\Omega)/c \leq r$ where r is universally small. Then, we can find $y \in S_\phi(0, \hat{K}\bar{h}(x))$ such that $M_1^{-1}u(x) \leq u(y) \leq M_1u(x)$, and $\bar{h}(y) \geq 2\bar{h}(x)$.*

The final ingredient in the proof of Theorem 2.1 quantifies how close a point x is to the boundary when a positive solution u to the linearized Monge-Ampère equation $L_\phi u = 0$ is large at x given a bound of u at another point in the interior.

Lemma 5.3. *Let ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.7 at the origin. Let \hat{K} and M_1 be as in Lemma 5.2. Suppose that $u \geq 0$ is a continuous solution of $L_\phi u = 0$ in $\Omega \cap S_\phi(0, c)$ and that $u = 0$ on $\partial\Omega \cap S_\phi(0, c)$ where $0 \in \partial\Omega$. Let $t/c \leq c_2$ be universally small. Let P be the apogee of $S_\phi(0, c_3t)$ for some universal c_3 small with $\theta_*^2(1 + \hat{K})\bar{h}(P) < c_0t(1 - 2^{-1/p_1})^{p_1}$ (this P exists by Lemma 4.4). Let $D_k = S_\phi(0, (2 + 2^{-k/p_1})t)$. If $x \in D_k$ and*

$$u(x) \geq \hat{C}M_1^k u(P),$$

for some universally large constant \hat{C} , then

$$\bar{h}(x) \leq 2^{-k}\bar{h}(P).$$

Remark 5.4. By Proposition 4.1, $\bar{h}(x)$ and $\text{dist}^2(x, \partial\Omega)$ are comparable. However, $\bar{h}(x)$ is very well-behaved under affine transformations. That is why we use \bar{h} to measure the distance to the boundary in Lemmas 5.2, 5.3 and the proof of Theorem 2.1.

To prove the above ingredients and to implement the proof of Theorem 2.1, we use geometric properties of boundary sections established in section 4. Moreover, we will use some geometric properties of the quasi-distance δ_ϕ generated by ϕ . For $x \in \bar{\Omega}$, let us introduce the function $\delta_\phi(x, \cdot) : \bar{\Omega} \rightarrow [0, \infty)$ defined by

$$(5.1) \quad \delta_\phi(x, y) = \phi(y) - \phi(x) - \nabla\phi(x)(y - x).$$

Then, we have the following quasi-metric inequality for δ_ϕ .

Lemma 5.5. *Assume that Ω and ϕ satisfy (2.1)–(2.3). Then, for all $x, y, z \in \bar{\Omega}$, we have*

$$\delta_\phi(x, y) \leq \theta_*^2(\delta_\phi(x, z) + \delta_\phi(z, y)).$$

Proof of Lemma 5.5. Let $r = \delta_\phi(x, z)$ and $s = \delta_\phi(z, y)$. Then for any $\varepsilon > 0$, we have $y \in S_\phi(z, s + \varepsilon) \subset S_\phi(z, \theta_*(r + s + \varepsilon))$ and $z \in S_\phi(x, r + \varepsilon)$. Thus, by Theorem 3.5, we have

$$S_\phi(x, r + \varepsilon) \subset S_\phi(z, \theta_*(r + \varepsilon)) \subset S_\phi(z, \theta_*(r + s + \varepsilon)).$$

Again, by Theorem 3.5, we have

$$S_\phi(z, \theta_*(r + s + \varepsilon)) \subset S_\phi(x, \theta_*^2(r + s + \varepsilon)).$$

It follows that $y \in S_\phi(x, \theta_*^2(r + s + \varepsilon))$. Hence $\delta_\phi(x, y) \leq \theta_*^2(r + s + \varepsilon)$ and the conclusion of the lemma follows by letting $\varepsilon \rightarrow 0$. □

With the help of Theorem 2.8, we can now follow Maldonado’s proof of [32, Lemma 4] to show that $\delta_\phi(x, \cdot)$ satisfies a uniform Hölder property as stated in the following lemma. Recall that the constant p_1 in Theorem 2.8 satisfies $p_1 = \bar{\mu}^{-1}$ where $\bar{\mu}$ is the constant in Lemma 4.2.

Lemma 5.6. *Assume that Ω and ϕ satisfy (2.1)–(2.3). Let $\bar{\mu}$ be as in Proposition 4.1 and Lemma 4.2. Then, there exists a universal constant $C_2 \geq 1$ such that for all $x, y, z \in \bar{\Omega}$, we have*

$$|\delta_\phi(x, z) - \delta_\phi(x, y)| \leq C_2[\delta_\phi(y, z)]^{\bar{\mu}} (\delta_\phi(x, z) + \delta_\phi(x, y))^{1-\bar{\mu}}.$$

We are now ready to prove the first main result of the paper.

Proof of Theorem 2.1. We can assume that $x_0 = 0 \in \partial\Omega$ and ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.7 at 0. Let \hat{K} and c_2 be as in Lemma 5.2; \hat{C} , c_3 and M_1 as in Lemma 5.3. Let $\bar{t} = c_2t$. Then the *apogee* P of $S_\phi(0, c_3\bar{t})$ satisfies $\theta_*^2(1 + \hat{K})\bar{h}(P) < c_0\bar{t}(1 - 2^{-1/p_1})^{p_1}$. Without loss of generality, we assume that $u > 0$ in Ω . We prove that

$$(5.2) \quad u(x) < \hat{C}M_1^k u(P) \text{ for all } x \in S_\phi(0, c_3\bar{t}/2)$$

for some universally large constant k to be determined later. Then by using Lemma 4.3 and the interior Harnack inequality for the linearized Monge-Ampère equation in Theorem 3.3, we easily obtain the stated estimate of the theorem.

We will use a contradiction argument, following the main lines of the proof of the Carleson estimate in [8]. Suppose that (5.2) is false. Then we can find $x_1 \in S_\phi(0, c_3\bar{t}/2) \subset S_\phi(0, (2 + 2^{-k/p_1})\bar{t})$ satisfying $u(x_1) \geq \hat{C}M_1^k u(P)$. Invoking Lemma 5.3, we find

$$(5.3) \quad \bar{h}(x_1) \leq 2^{-k}\bar{h}(P).$$

We fix a universally large integer s so that $2^s \geq M_1$. Let $z = \partial\Omega \cap \partial S_\phi(x_1, \bar{h}(x_1))$. Then, by Theorem 3.5,

$$S_\phi(x_1, \bar{h}(x_1)) \subset S_\phi(z, \theta_*\bar{h}(x_1)) \subset S_\phi(z, H^s\theta_*\bar{h}(x_1)),$$

where H is as in Lemma 5.1. Reducing c_3 if necessary, we have for k universally large

$$(5.4) \quad S_\phi(z, 2H^s\theta_*\bar{h}(x_1)) \subset S_\phi(0, \bar{t}/2).$$

Indeed, if $y \in S_\phi(z, 2H^s\theta_*\bar{h}(x_1))$, then $\delta_\phi(z, y) \leq 2H^s\theta_*\bar{h}(x_1)$. From $\delta_\phi(x_1, z) \leq \bar{h}(x_1)$, $\bar{h}(P) \leq \bar{t}$, and Lemma 5.5, we have in view of (5.3),

$$\delta_\phi(x_1, y) \leq \theta_*^2(\delta_\phi(x_1, z) + \delta_\phi(z, y)) \leq \theta_*^2(1 + 2H^s\theta_*)\bar{h}(x_1) \leq \theta_*^2(1 + 2H^s\theta_*)2^{-k}\bar{t}.$$

Thus, from $\delta_\phi(0, x_1) \leq c_3\bar{t}/2$ and Lemma 5.5, we obtain

$$\delta_\phi(0, y) \leq \theta_*^2(\delta_\phi(0, x_1) + \delta_\phi(x_1, y)) \leq \theta_*^2[c_3\bar{t}/2 + \theta_*^2(1 + 2H^s\theta_*)2^{-k}\bar{t}] < \bar{t}/2$$

if c_3 is universally small and k universally large.

With (5.4), applying Lemma 5.1 to $S_\phi(z, H^m\theta_*\bar{h}(x_1))$ successively for $m = 0, \dots, s - 1$, we can find $x_2 \in \overline{S_\phi(z, H^s\theta_*\bar{h}(x_1))}$ such that

$$(5.5) \quad u(x_2) \geq 2^s u(x_1) \geq \hat{C}M_1^{k+1} u(P).$$

From $\delta_\phi(x_1, z) \leq \bar{h}(x_1)$, $\delta_\phi(z, x_2) \leq H^s \theta_* \bar{h}(x_1)$ and Lemma 5.5, we have

$$\delta_\phi(x_1, x_2) \leq \theta_*^2 (1 + \theta_* H^s) \bar{h}(x_1).$$

Let

$$d_0 = \delta_\phi(0, x_1), d_1 = \delta_\phi(0, x_2).$$

Now, using $\bar{h}(P) \leq \bar{t}$, (5.3) and the Hölder property of $\delta_\phi(0, \cdot)$ in Lemma 5.6, we find

$$\begin{aligned} d_1 &\leq d_0 + C_2 (\delta_\phi(x_1, x_2))^\mu (d_0 + d_1)^{1-\mu} \\ &\leq d_0 + C_2 (\theta_*^2 (1 + \theta_* H^s) 2^{-k} \bar{h}(P))^\mu (d_0 + d_1)^{1-\mu} \\ (5.6) \quad &\leq d_0 + \theta_0^\mu (d_0 + d_1)^{1-\mu} \end{aligned}$$

where $\{\theta_j\}_{j=0}^\infty$ is the sequence defined by

$$(5.7) \quad \theta_j = C_2^{1/\mu} \theta_*^2 (1 + \theta_* H^s) 2^{-(k+j)} \bar{t}.$$

Clearly, if k is universally large, we have $\theta_{j+1} < \theta_j < c_3 \bar{t}/2$ for all j and furthermore

$$(5.8) \quad \frac{2}{1 - (\frac{\theta_0}{\bar{t}/2})^\mu} \sum_{j=0}^\infty \left(\frac{\theta_j}{\bar{t}/2}\right)^\mu < \log \frac{3}{2}.$$

Given this k , we know from (5.6) together with $d_0 < c_3 \bar{t}/2$ and Lemma 5.7 that $\delta_\phi(0, x_2) < 3c_3 \bar{t}/4$. Hence $x_2 \in S_\phi(0, 3c_3 \bar{t}/4)$. Recalling (5.5), we conclude from Lemma 5.3 that

$$\bar{h}(x_2) \leq 2^{-(k+1)} \bar{h}(P).$$

With k satisfying (5.8), we repeat the above process to obtain a sequence $\{x_j\} \subset S_\phi(0, 3c_3 \bar{t}/4)$ such that

$$(5.9) \quad u(x_j) \geq \hat{C} M_1^{k+j-1} u(P)$$

with

$$(5.10) \quad \bar{h}(x_j) \leq 2^{-(k+j-1)} \bar{h}(P) \text{ and } \delta_\phi(x_j, x_{j+1}) \leq \theta_*^2 (1 + \theta_* H^s) \bar{h}(x_j)$$

provided that $x_j \in S_\phi(0, 3c_3 \bar{t}/4)$. But this follows from the choice of k . Indeed, recalling θ_j in (5.7) and the Hölder property of $\delta_\phi(0, \cdot)$ in Lemma 5.6, we obtain for $d_j = \delta_\phi(0, x_{j+1})$ the estimate

$$d_{j+1} - d_j \leq C_2 (\delta_\phi(x_{j+1}, x_{j+2}))^\mu (d_j + d_{j+1})^{1-\mu} \leq \theta_j^\mu (d_j + d_{j+1})^{1-\mu}.$$

Hence, from $d_0 \leq c_3 \bar{t}/2$ and (5.8), we find from Lemma 5.7 that

$$d_{j+1} \leq \frac{c_3 \bar{t}}{2} \exp \left[\sum_{j=0}^\infty \frac{2}{1 - (\frac{\theta_0}{\bar{t}/2})^\mu} \left(\frac{\theta_j}{\bar{t}/2}\right)^\mu \right] < \frac{3c_3}{4} \bar{t}.$$

We now let $j \rightarrow \infty$ in (5.9) and (5.10) to obtain $x_\infty \in \partial\Omega \cap S_\phi(0, c_3 \bar{t})$ with $u(x_\infty) = \infty$. This is a contradiction to $u(x_\infty) = 0$. Hence (5.2) holds and the proof of Theorem 2.1 is complete. \square

In the proof of Theorem 2.1, we used the following lemma, taken from [19, Lemma 1].

Lemma 5.7 ([19, Lemma 1]). *Let $R > 0$, $\alpha \in (0, 1)$, $\{d_n\}_{n=0}^\infty \subset (0, \infty)$, $\{\theta_n\}_{n=0}^\infty \subset (0, R)$ such that $d_0 \leq R$,*

$$d_{n+1} \leq d_n + \theta_n^\alpha (d_n + d_{n+1})^{1-\alpha}, \text{ and } \theta_{n+1} < \theta_n < R \text{ for all } n.$$

Then

$$d_{n+1} \leq R \exp \left[\frac{2}{1 - (\theta_0/R)^\alpha} \sum_{j=0}^n \left(\frac{\theta_j}{R} \right)^\alpha \right].$$

Proof of Theorem 2.3. We just sketch the proof. By multiplying u and v by suitable constants, we assume that $u(P) = v(P) = 1$. Without loss of generality, we assume that $x_0 = 0$, ϕ and Ω satisfy the hypotheses of Localization Theorem 3.7 at the origin. By Theorem 2.1,

$$(5.11) \quad u(x) \leq M_0 \text{ for all } x \in S_\phi(0, 3t/4).$$

By [27, Proposition 6.1], we have

$$(5.12) \quad u(x) \leq CM_0 t^{-1/2} \text{dist}(x, \partial\Omega) \text{ for all } x \in S_\phi(0, t/2).$$

Using Lemma 4.3 and the interior Harnack inequality in Theorem 3.3, we can find a universal constant c_1 such that

$$v(x) \geq c_1 \text{ for all } x \in S_\phi(0, 3t/4) \text{ with } \text{dist}(x, \partial\Omega) \geq c_1 t^{1/2}.$$

It follows from [27, Proposition 5.7] that

$$(5.13) \quad v(x) \geq c_2 t^{-1/2} \text{dist}(x, \partial\Omega) \text{ for all } x \in S_\phi(0, t/2).$$

The theorem follows from (5.12) and (5.13).

For the sake of completeness, we indicate how to obtain (5.12) and (5.13). Let $h = t$. We use Localization Theorem 3.7 at the origin and consider the rescaled functions

$$\phi_h(x) = \frac{\phi(h^{1/2}A_h^{-1}x)}{h}, u_h(x) = u(h^{1/2}A_h^{-1}x) \text{ where } x \in \Omega_h = h^{-1/2}A_h\Omega.$$

Then u_h is a positive solution of

$$L_{\phi_h} u_h = 0 \text{ in } \Omega_h \cap S_{\phi_h}(0, 1) \text{ and } u_h = 0 \text{ on } \partial\Omega_h \cap S_{\phi_h}(0, 1).$$

By [27, Proposition 6.1], we know that

$$u_h(x) \leq C \left(\max_{S_{\phi_h}(0, 3/4)} u_h \right) \text{dist}(x, \partial\Omega_h) \text{ for all } x \in S_{\phi_h}(0, 1/2).$$

Using (5.11) and (4.18), we obtain (5.12). The proof of (5.13), using rescaling, is similar. □

5.2. Proofs of boundary properties of solutions to the linearized Monge-Ampère.

Proof of Lemma 5.1. Without loss of generality, we assume that $x_0 = 0$ and ϕ and Ω satisfy the hypotheses of Localization Theorem 3.7 at 0. We use Localization Theorem 3.7 at the origin and consider the rescaled functions

$$\phi_h(x) = \frac{\phi(h^{1/2}A_h^{-1}x)}{h}, u_h(x) = u(h^{1/2}A_h^{-1}x) \text{ where } x \in \Omega_h = h^{-1/2}A_h\Omega.$$

Then u_h is a positive solution of

$$L_{\phi_h} u_h = 0 \text{ in } \Omega_h \cap S_{\phi_h}(0, 2) \text{ and } u_h = 0 \text{ on } \partial\Omega_h \cap S_{\phi_h}(0, 2).$$

We need to show that for H universally large

$$\max_{x \in S_{\phi_h}(0,1)} u_h(x) \geq 2 \max_{x \in S_{\phi_h}(0,1/H)} u_h(x).$$

Indeed, by Lemma 3.8, ϕ_h and $S_{\phi_h}(0,1)$ satisfy the hypotheses of Localization Theorem 3.7 at all points $x_0 \in \partial\Omega_h \cap B(0,c)$. Hence, if $1/H \leq c$, then from (3.3), we have

$$\max_{x \in S_{\phi_h}(0,1/H)} \text{dist}(x, \partial\Omega_h) \leq \max_{x \in S_{\phi_h}(0,1/H)} |x| \leq C_2 \left(\frac{1}{H}\right)^{1/2} \log\left(\frac{1}{H}\right).$$

On the other hand, by [27, Proposition 6.1], we know that

$$\max_{x \in S_{\phi_h}(0,1/H)} u_h(x) \leq C \max_{x \in S_{\phi_h}(0,1)} u_h(x) \max_{x \in S_{\phi_h}(0,1/H)} \text{dist}(x, \partial\Omega_h)$$

for some $C = C(n, \lambda, \Lambda, \rho)$. Therefore, for H universally large,

$$\max_{x \in S_{\phi_h}(0,1/H)} u_h(x) \leq CC_2 \left(\frac{1}{H}\right)^{1/2} \log\left(\frac{1}{H}\right) \max_{x \in S_{\phi_h}(0,1)} u_h(x) \leq \frac{1}{2} \max_{x \in S_{\phi_h}(0,1)} u_h(x).$$

□

Proof of Lemma 5.2. Let $\{x_0 = x, x_1, \dots, x_m\} \subset S_\phi(0, \bar{K} \text{dist}^2(x, \partial\Omega))$ and \bar{K} be as in Lemma 4.3(i). We deduce from Proposition 4.1 that $k_0^2 \bar{h}(x) \leq \text{dist}^2(x, \partial\Omega) \leq k_0^{-2} \bar{h}(x)$. Hence, for $\hat{K} := \bar{K} k_0^{-2}$,

$$x_m \in S_\phi(0, \hat{K} \bar{h}(x)), \text{ and } \bar{h}(x_m) \geq k_0^2 \text{dist}^2(x_m, \partial\Omega) \geq 4k_0^{-2} \text{dist}^2(x, \partial\Omega) \geq 4\bar{h}(x).$$

If $\text{dist}^2(x, \partial\Omega)/c \leq r$ where r is universally small, then

$$S_\phi(0, \hat{K} \bar{h}(x)) \subset S_\phi(0, c/2) \text{ and } S_\phi(x_i, \bar{h}(x_i)) \subset S_\phi(0, c/2) \text{ for all } i.$$

Since $u \geq 0$ satisfies $L_\phi u = 0$ in $S_\phi(x_{i+1}, \frac{1}{2} \bar{h}(x_{i+1})) \Subset S_\phi(0, c/2) \cap \Omega$ and $x_i \in S_\phi(x_{i+1}, \tau \bar{h}(x_{i+1}))$, we obtain from the Harnack inequality in Theorem 3.3

$$(5.14) \quad C_1^{-1} u(x_i) \leq u(x_{i+1}) \leq C_1(n, \lambda, \Lambda) u(x_i) \text{ for each } i = 0, 1, \dots, m-1.$$

We now take $y = x_m$. Then $\bar{h}(y) \geq 4\bar{h}(x)$. From the chain condition $x_i \in S_\phi(x_{i+1}, \tau \bar{h}(x_{i+1}))$ for all $i = 0, 1, \dots, m-1$ with m universally bounded and (5.14), we obtain for some universal M_1 :

$$M_1^{-1} u(x) \leq u(y) \leq M_1 u(x).$$

□

Proof of Lemma 5.3. By using Lemma 4.3 and the interior Harnack inequality for the linearized Monge-Ampère equation in Theorem 3.3, we can find a universally large constant \hat{C} such that for all $x \in S_\phi(0, 3t)$ with $\bar{h}(x) \geq 2^{-1} \bar{h}(P)$, we have $u(x) \leq (\hat{C}/2)u(P)$. With this choice of \hat{C} , we can now prove the statement of the lemma by induction. Clearly, the statement is true for $k = 1$. Suppose it is true for $k \geq 1$. We prove it for $k + 1$. Let $x \in D_{k+1}$ be such that

$$u(x) \geq \hat{C} M_1^{k+1} u(P).$$

Let $z = \partial\Omega \cap \partial S_\phi(x, \bar{h}(x))$. If $t/c \leq c_2$ is small, then we are in the setting of Lemma 5.2 with z replacing 0. Indeed, we first note from $x \in D_{k+1} \subset S_\phi(0, 3t)$ and Lemma 4.4 that

$$(5.15) \quad \text{dist}^2(x, \partial\Omega) \leq Ct.$$

By Proposition 4.1,

$$\delta_\phi(x, z) = \bar{h}(x) \leq k_0^{-2} \text{dist}^2(x, \partial\Omega) \leq Ck_0^{-2}t.$$

Using Lemma 5.5 and $\delta_\phi(0, x) \leq 3t$, we find

$$\delta_\phi(0, z) \leq \theta_*^2[\delta_\phi(0, x) + \delta_\phi(x, z)] \leq C't$$

for some universal C' . Now, let $M = C/r$ where C is as in (5.15) and r is as in Lemma 5.2. If $y \in S_\phi(z, Mt)$, then from Lemma 5.5, we have

$$\delta_\phi(0, y) \leq \theta_*^2[\delta_\phi(0, z) + \delta_\phi(z, y)] \leq \theta_*^2[C't + Mt] \leq c/2$$

provided that $t/c \leq c_2$ is universally small. It follows that if c_2 is small, then $S_\phi(z, Mt) \subset S_\phi(0, c/2)$. This combined with (5.15) and the choice of M shows that the hypotheses of Lemma 5.2 are satisfied.

By Lemma 5.2, we can find $y \in S_\phi(z, \hat{K}\bar{h}(x))$ such that

$$u(y) \geq M_1^{-1}u(x) \geq \hat{C}M_1^k u(P) \text{ and } \bar{h}(y) \geq 2\bar{h}(x).$$

The lemma follows if we can show $y \in D_k$. Indeed, by the induction hypothesis for k , we have $\bar{h}(y) \leq 2^{-k}\bar{h}(P)$ and hence

$$\bar{h}(x) \leq \frac{1}{2}\bar{h}(y) \leq 2^{-(k+1)}\bar{h}(P).$$

It remains to show that $y \in D_k$. From $\delta_\phi(x, z) = \bar{h}(x)$, $\delta_\phi(z, y) \leq \hat{K}\bar{h}(x)$ and Lemma 5.5, we have

$$\delta_\phi(x, y) \leq \theta_*^2(1 + \hat{K})\bar{h}(x).$$

Let us denote for simplicity $r_k = 2 + 2^{-k/p_1}$. Then $D_k = S_\phi(0, r_k t)$. Recalling $x \in D_{k+1} = S_\phi(0, r_{k+1} t)$, we have from Theorem 2.8(i),

$$S_\phi(x, c_0(r_k - r_{k+1})^{p_1} t) \subset S_\phi(0, r_k t) = D_k.$$

By our choice of P , we have $\theta_*^2(1 + \hat{K})\bar{h}(P) < c_0 t(1 - 2^{-1/p_1})^{p_1}$ and hence

$$\begin{aligned} c_0(r_k - r_{k+1})^{p_1} t &= c_0 t 2^{-k}(1 - 2^{-1/p_1})^{p_1} > \theta_*^2(1 + \hat{K})2^{-k}\bar{h}(P) \\ &\geq \theta_*^2(1 + \hat{K})\bar{h}(x) \geq \delta_\phi(x, y). \end{aligned}$$

Therefore, $y \in S_\phi(x, c_0(r_k - r_{k+1})^{p_1} t) \subset S_\phi(0, r_k t) = D_k$. □

6. BOUNDING GREEN'S FUNCTION FOR THE LINEARIZED MONGE-AMPÈRE OPERATOR

In this section, we prove Theorem 2.5 and Corollary 2.6 using the Harnack inequalities in Theorems 2.1, 3.3 and properties of Green's function for the linearized Monge-Ampère operator L_ϕ as stated in the next subsection.

Throughout, we assume Ω and ϕ satisfy (2.1)–(2.3). Recall that $g_V(x, y)$ denotes Green's function of L_ϕ in V with pole $y \in V \cap \Omega$ where $V \subset \bar{\Omega}$; that is, $g_V(\cdot, y)$ is a positive solution of

$$L_\phi g_V(\cdot, y) = \delta_y \text{ in } V \cap \Omega \text{ and } g_V(\cdot, y) = 0 \text{ on } \partial V.$$

6.1. Properties of Green’s function. First, from the Aleksandrov-Bakelman-Pucci maximum principle, we observe the following uniform $L^{\frac{n}{n-1}}$ and L^1 bounds for Green’s function.

Lemma 6.1. *If $V \subset \Omega$ is a convex domain, then for all $x_0 \in V$, we have the uniform $L^{\frac{n}{n-1}}$ bound*

$$\left(\int_V [g_V(x, x_0)]^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq C(n, \lambda) |V|^{1/n}.$$

As a consequence, we have the uniform L^1 bound

$$\int_V g_V(x, x_0) dx \leq C(n, \lambda) |V|^{2/n}.$$

Given an L^1 bound for positive solution to the linearized Monge-Ampère equation, we can use the interior Harnack inequality in Theorem 3.3 to obtain a pointwise upper bound in compactly supported subsets via the following general estimate.

Lemma 6.2. *Suppose $S_\phi(x_0, t) \Subset \Omega$. If $\sigma \geq 0$ satisfies $L_\phi \sigma = 0$ in $S_\phi(x_0, t) \setminus \{x_0\}$ and*

$$\int_{S_\phi(x_0, t)} \sigma(x) dx \leq A,$$

then

$$\sigma(x) \leq C(n, \lambda, \Lambda) A t^{-\frac{n}{2}} \quad \forall x \in \partial S_\phi(x_0, t/2).$$

Using the Harnack inequalities for the linearized Monge-Ampère equations in Theorems 2.1 and 3.3, we can give in the next lemma a sharp upper bound for Green’s function $g_{S_\phi(x_0, 2t)}(z, x_0)$ in the concentric section with half-height. It is a global version of [23, Lemma 3.1].

Lemma 6.3. *If $x_0 \in \Omega$ and $t \leq c(n, \lambda, \Lambda, \rho)$, then*

$$(6.1) \quad \max_{z \in \partial S_\phi(x_0, t)} g_{S_\phi(x_0, 2t)}(z, x_0) \leq C(n, \lambda, \Lambda, \rho) t^{-\frac{n-2}{2}}.$$

The following lemma estimates the growth of Green’s function near the pole. It is a variant at the boundary of [23, Lemma 3.2].

Lemma 6.4. *If $x_0 \in \Omega$, then*

$$(6.2) \quad \max_{x \in \partial S_\phi(x_0, t)} g_\Omega(x, x_0) \leq C(n, \lambda, \Lambda, \rho) t^{-\frac{n-2}{2}} + \max_{z \in \partial S_\phi(x_0, 2t)} g_\Omega(z, x_0).$$

With the above properties of Green’s function, we are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. Let m be a nonnegative integer such that $c/2 < 2^m t_0 \leq c$. Iterating Lemma 6.4, we find

$$(6.3) \quad \max_{x \in \partial S_\phi(x_0, t_0)} g_\Omega(x, x_0) \leq \max_{x \in \partial S_\phi(x_0, 2^m t_0)} g_\Omega(x, x_0) + C t_0^{-\frac{n-2}{2}} \sum_{l=1}^m (2^l)^{-\frac{n-2}{2}}.$$

The desired upper bound for $\max_{x \in \partial S_\phi(x_0, t_0)} g_\Omega(x, x_0)$ follows from

Claim. $\sigma := g_\Omega(\cdot, x_0)$ is universally bounded by $C(n, \lambda, \Lambda, \rho)$ in $\Omega \setminus S_\phi(x_0, c/2)$.

Indeed, if $n \geq 3$, then

$$\sum_{l=1}^m (2^l)^{-\frac{n-2}{2}} \leq C(n),$$

and hence, the Claim and (6.3) give

$$\max_{x \in \partial S_\phi(x_0, t_0)} g_\Omega(x, x_0) \leq C(n, \lambda, \Lambda, \rho) + CC(n)t_0^{-\frac{n-2}{2}} \leq C(n, \lambda, \Lambda, \rho)t_0^{-\frac{n-2}{2}}.$$

If $n = 2$, then since $m \leq \log_2(c/t_0)$, the Claim and (6.3) give

$$\max_{x \in \partial S_\phi(x_0, t_0)} g_\Omega(x, x_0) \leq C(n, \lambda, \Lambda, \rho) + C \log_2(c/t_0) \leq C(n, \lambda, \Lambda, \rho)|\log t_0|.$$

It remains to prove the Claim. If $S_\phi(x_0, c/4) \subset \Omega$, then by Hölder’s inequality and Lemma 6.1,

$$\int_{S_\phi(x_0, c/4)} \sigma \leq \|\sigma\|_{L^{\frac{n}{n-1}}(\Omega)} |S_\phi(x_0, c/4)|^{\frac{1}{n}} \leq C(n, \lambda) |\Omega|^{\frac{1}{n}} |S_\phi(x_0, c/4)|^{\frac{1}{n}} \leq C.$$

We used the volume estimate in the last inequality. By Lemma 6.2,

$$\max_{x \in \partial S_\phi(x_0, c/8)} \sigma(x) \leq C.$$

The Claim follows from the maximum principle.

Suppose now $S_\phi(x_0, c/4) \not\subset \Omega$. We consider the ring $A = S_\phi(x_0, c) \setminus S_\phi(x_0, c/4)$ and focus on the values of σ at points $z \in \partial S_\phi(x_0, c/2) \cap \Omega$. From the geometric properties of sections in Theorem 2.8, we can find a section

$$(6.4) \quad S_\phi(z, c_2) \Subset A \cap \Omega$$

for $z \in \Omega$ and some c_2 universally small. For the reader’s convenience, we indicate how to construct this section. Since $S_\phi(x_0, c/4) \not\subset \Omega$, we can find $y \in S_\phi(x_0, c/2) \cap \partial\Omega$. Then Theorem 2.8 gives $S_\phi(y, 2c_3) \subset S_\phi(x_0, 2c/3) \setminus S_\phi(x_0, c/3)$ for some universal constant $c_3 > 0$. Let z be the *apogee* of $S_\phi(y, 2c_3)$ (see Lemma 4.4), that is, $z \in \partial S_\phi(y, c_3) \cap \Omega$ with $dist(z, \partial\Omega) \geq c_4$ for some universal $c_4 > 0$. We now use Proposition 4.1 to obtain a section $S_\phi(z, c_2) \Subset A \cap \Omega$ as asserted in (6.4).

The uniform L^1 bound for σ from Lemma 6.1 gives

$$(6.5) \quad \int_{S_\phi(z, c_2)} \sigma(x) dx \leq \int_\Omega \sigma(x) dx \leq C(n, \lambda) |\Omega|^{2/n}.$$

Note that $L_\phi \sigma = 0$ in $S_\phi(z, c_2)$. By the interior Harnack inequality in Theorem 3.3 and the lower volume bound in Lemma 3.1, we obtain from (6.5)

$$(6.6) \quad \sigma(z) \leq C(n, \lambda, \Lambda, \rho).$$

On the other hand, using the chain property of sections in Lemma 4.3 and the boundary Harnack inequality in Theorem 2.1, we find that

$$\sigma(y) \leq C(n, \lambda, \Lambda, \rho) \sigma(z) \text{ for all } y \in \partial S_\phi(x_0, c/2) \cap \Omega.$$

Combining this with (6.6), we find that

$$\sigma(y) \leq C(n, \lambda, \Lambda, \rho) \text{ for all } y \in \partial S_\phi(x_0, c/2) \cap \Omega.$$

The Claim now also follows from the maximum principle. □

We now give the proof of Corollary 2.6.

Proof of Corollary 2.6. We give the proof for $n \geq 3$; the case $n = 2$ is similar. Let $x_0 \in \Omega$. Let C, c be the constants in Theorem 2.5. Denote $C_* = C/c^{\frac{n-2}{2}}$. For $s \geq C_*$, we have $(\frac{C}{s})^{\frac{2}{n-2}} \leq c$. Hence, from Theorem 2.5,

$$\{x \in \Omega : g_\Omega(x, x_0) > s\} \subset S_\phi(x_0, (C/s)^{\frac{2}{n-2}}).$$

Then, by the volume estimate in Lemma 3.6(iii), we have

$$|\{x \in \Omega : g_\Omega(x, x_0) > s\}| \leq |S_\phi(x_0, (C/s)^{\frac{2}{n-2}})| \leq C_1|(C/s)^{\frac{2}{n-2}}|^{\frac{n}{2}} = C_3s^{-\frac{n}{n-2}}.$$

It follows from the layer cake representation that

$$\begin{aligned} \int_\Omega g_\Omega^p(x, x_0) dx &= \int_0^\infty ps^{p-1}|\{x \in \Omega : g_\Omega(x, x_0) > s\}| ds \\ &= \int_0^{C_*} ps^{p-1}|\{x \in \Omega : g_\Omega(x, x_0) > s\}| ds \\ &\quad + \int_{C_*}^\infty ps^{p-1}|\{x \in \Omega : g_\Omega(x, x_0) > s\}| ds \\ &\leq C_*^p|\Omega| + \int_{C_*}^\infty ps^{p-1}C_3s^{-\frac{n}{n-2}} ds = C_*^p|\Omega| + \frac{pC_3}{\frac{n}{n-2} - p}C_*^{p-\frac{n}{n-2}} \\ &\leq C(n, \lambda, \Lambda, \rho, p). \end{aligned}$$

We now prove the last inequality stated in the corollary. Fix $x \in S_\phi(x_0, t) \cap \Omega$ where $t \leq c_1 := \frac{c}{2\theta_*}$. Then, by the engulfing property of sections in Theorem 3.5, we have $S_\phi(x_0, t) \subset S_\phi(x, \theta_*t)$. Therefore, if $y \in S_\phi(x_0, t)$, then $y \in S_\phi(x, \theta_*t) \subset S_\phi(x, 2\theta_*t)$. Note that

$$(6.7) \quad \int_{S_\phi(x_0, t)} g_{S_\phi(x_0, t)}^p(y, x) dy \leq \int_{S_\phi(x, 2\theta_*t)} g_{S_\phi(x, 2\theta_*t)}^p(y, x) dy,$$

and it suffices to bound from above the term on the right hand side of (6.7).

As in the above upper bound for $\int_\Omega g_\Omega^p(x, x_0) dx$, if we replace Ω by $S_\phi(x, 2\theta_*t)$ where we recall $2\theta_*t \leq c$ and C_* by $C_0 = C/t^{\frac{n-2}{2}}$, then for $s \geq C_0$,

$$\begin{aligned} \{y \in S_\phi(x, 2\theta_*t) : g_{S_\phi(x, 2\theta_*t)}(y, x) > s\} &\subset \{y \in \Omega : g_\Omega(y, x) > s\} \\ &\subset S_\phi(x, (C/s)^{\frac{2}{n-2}}). \end{aligned}$$

Consequently, we also obtain

$$(6.8) \quad \begin{aligned} \int_{S_\phi(x, 2\theta_*t)} g_{S_\phi(x, 2\theta_*t)}^p(y, x) dy &\leq C_0^p|S_\phi(x, 2\theta_*t)| + \frac{pC_3}{\frac{n}{n-2} - p}C_0^{p-\frac{n}{n-2}} \\ &\leq C(n, \lambda, \Lambda, \rho, p)t^{\frac{n}{2} - \frac{n-2}{2}p} \end{aligned}$$

by the upper bound on the volume of sections in Lemma 3.6. Since $t \leq c_1$, we can use the lower bound on the volume of sections in Lemma 3.6 to deduce from (6.7), (6.8) and $p < \frac{n}{n-2}$ that

$$\begin{aligned} \sup_{x \in S_\phi(x_0, t) \cap \Omega} \int_{S_\phi(x_0, t)} g_{S_\phi(x_0, t)}^p(y, x) dy &\leq C(n, \lambda, \Lambda, \rho, p)t^{\frac{n}{2} - \frac{n-2}{2}p} \\ &\leq C(n, \lambda, \Lambda, \rho, p)|S_\phi(x_0, t)|^{1 - \frac{n-2}{n}p}. \end{aligned}$$

□

6.2. **Proofs of the properties of Green’s function.**

Proof of Lemma 6.1. Let $x_0 \in V$ be given and $\sigma = g_V(\cdot, x_0)$. By the ABP estimate, for any $\varphi \in L^n(V)$, the solution ψ to

$$-\Phi^{ij}\psi_{ij} = \varphi \text{ in } V, \psi = 0 \text{ on } \partial V$$

satisfies

$$\left| \int_V \sigma(x)\varphi(x)dx \right| = |\psi(x_0)| \leq C(n)|V|^{1/n} \left\| \frac{\varphi}{\det \Phi} \right\|_{L^n(V)} \leq C(n, \lambda)|V|^{1/n} \|\varphi\|_{L^n(V)}.$$

Here we used the identity $\det \Phi = (\det D^2\phi)^{n-1}$ and $\det D^2\phi \geq \lambda$. By duality, we obtain

$$\left(\int_V [g_V(x, x_0)]^{\frac{n-1}{n}} dx \right)^{\frac{n}{n-1}} = \|\sigma\|_{L^{\frac{n}{n-1}}(V)} \leq C(n, \lambda)|V|^{1/n}.$$

The uniform L^1 bound for $g_V(\cdot, x_0)$ on V follows from the Hölder inequality and the uniform $L^{\frac{n}{n-1}}$ bound. □

Proof of Lemma 6.2. Let c_1 and C_1 be as in Lemma 3.1. Denote

$$D = (S_\phi(x_0, t) \setminus S_\phi(x_0, r_2t)) \cup S_\phi(x_0, r_1t)$$

where $0 < r_1 < 1/2 < r_2 < 1$. Then, by [16, Lemma 6.5.1] and Lemma 3.1, we can estimate

$$|D| \leq n(1-r_2)|S_\phi(x_0, t)| + |S_\phi(x_0, r_1t)| \leq C_1n(1-r_2)t^{n/2} + C_1(r_1t)^{n/2} \leq (c_1/2)^n t^{n/2}$$

if $r_1, 1 - r_2$ are universally small. Then by Lemma 3.1,

$$(6.9) \quad \frac{c_1}{2} t^{n/2} \leq |S_\phi(x_0, t) \setminus D|.$$

Given $0 < r_1 < r_2 < 1$ as above, we have

$$(6.10) \quad \sup_{S_\phi(x_0, t) \setminus D} \sigma \leq C(n, \lambda, \Lambda) \inf_{S_\phi(x_0, t) \setminus D} \sigma.$$

Combining (6.9) and (6.10) with the L^1 bound on σ , we find that

$$\sigma(x) \leq C(n, \lambda, \Lambda) A t^{-\frac{n}{2}} \quad \forall x \in S_\phi(x_0, t) \setminus D.$$

Since $r_2 > 1/2 > r_1$, we obtain the desired upper bound for $\sigma(x) = g_V(x, x_0)$ when $x \in \partial S_\phi(x_0, t/2)$.

For completeness, we include the details of (6.10). By [16, Theorem 3.3.10], we can find a universal $\alpha \in (0, 1)$ such that for each $x \in S_\phi(x_0, t) \setminus D$, the section $S_\phi(x, \alpha t)$ satisfies

$$x_0 \notin S_\phi(x, \alpha t) \text{ and } S_\phi(x, \alpha t) \subset S_\phi(x_0, t).$$

Using Lemma 3.4, we can find a collection of sections $S_\phi(x_i, \tau\alpha t)$ with $x_i \in S_\phi(x_0, t) \setminus D$ such that

$$S_\phi(x_0, t) \setminus D \subset \bigcup_{i \in I} S_\phi(x_i, \tau\alpha t)$$

and $S_\phi(x_i, \delta\tau\alpha t)$ are disjoint for some universal $\delta \in (0, 1)$. By using the volume estimates in Lemma 3.1, we find that $|I|$ is universally bounded. Now, we apply the Harnack inequality in Theorem 3.3 to σ in each $S_\phi(x_i, \alpha t)$ to obtain (6.10). □

Proof of Lemma 6.3. Let $\sigma = g_{S_\phi(x_0, 2t)}(\cdot, x_0)$. We consider two cases for a universally large K .

Case 1. $S_\phi(x_0, t/K) \subset \Omega$. By Hölder’s inequality and Lemma 6.1, we have

$$\begin{aligned} \int_{S_\phi(x_0, t/K)} \sigma &\leq \|\sigma\|_{L^{\frac{n}{n-1}}(S_\phi(x_0, 2t))} |S_\phi(x_0, t/K)|^{\frac{1}{n}} \\ &\leq C(n, \lambda) |S_\phi(x_0, 2t)|^{\frac{1}{n}} |S_\phi(x_0, t/K)|^{\frac{1}{n}} \leq Ct. \end{aligned}$$

We used the volume estimate in Lemma 3.1 in the last inequality. By Lemma 6.2, we have

$$\max_{x \in \partial S_\phi(x_0, \frac{t}{2K})} \sigma(x) \leq Ct^{-\frac{n-2}{2}}.$$

Now, the bound (6.1) follows from the maximum principle.

Case 2. $S_\phi(x_0, t/K) \not\subset \Omega$. Let $\bar{\theta} = 2\theta_*^2, K = \bar{\theta}^6$. Then $c \geq t \geq K\bar{h}(x_0)$. Suppose that $\partial S_\phi(x_0, \bar{h}(x_0)) \cap \partial\Omega = 0$. By the engulfing property in Theorem 3.5 and $\theta > \theta_*$, we have

$$\begin{aligned} S(x_0, \frac{t}{K}) &\subset S(0, \frac{\bar{\theta}t}{K}) \subset S(x_0, \frac{\bar{\theta}^2t}{K}) \subset S(0, \frac{\bar{\theta}^3t}{K}) \\ &\subset S(x_0, \frac{\bar{\theta}^4t}{K}) \subset S(0, \frac{\bar{\theta}^5t}{K}) \subset S(x_0, \frac{\bar{\theta}^6t}{K}) \subset S(x_0, t). \end{aligned}$$

By Lemma 4.4, there is $y \in \partial S_\phi(0, \frac{\bar{\theta}^3t}{K}) \cap \Omega$ such that $dist(y, \partial\Omega) \geq ct^{1/2}$. Hence, from Proposition 4.1, we can find a universal constant $c_2 \leq \frac{\bar{\theta}}{\theta_*K}$ such that $S_\phi(y, c_2t) \subset \Omega$. Then

$$(6.11) \quad S(0, \frac{\bar{\theta}t}{K}) \cap S_\phi(y, c_2t) = \emptyset; S_\phi(y, c_2t) \subset S(0, \frac{\bar{\theta}^5t}{K}) \cap \Omega.$$

Indeed, suppose there is $z \in S(0, \frac{\bar{\theta}t}{K}) \cap S_\phi(y, c_2t)$. Then, recalling the quasi-distance δ_ϕ in (5.1), we have

$$\delta_\phi(0, y) = \frac{\bar{\theta}^3t}{K}, \delta_\phi(0, z) \leq \frac{\bar{\theta}t}{K}, \text{ and } \delta_\phi(y, z) \leq c_2t.$$

On the other hand, we observe from the engulfing property in Theorem 3.5 that

$$\delta_\phi(z, y) \leq \theta_*\delta_\phi(y, z) \leq \theta_*c_2t.$$

By Lemma 5.5, we find that

$$\delta_\phi(0, y) \leq \theta_*^2(\delta_\phi(0, z) + \delta_\phi(z, y)) \leq \theta_*^2[\frac{\bar{\theta}t}{K} + \theta_*c_2t] \leq \frac{\bar{\theta}}{2}[\frac{\bar{\theta}t}{K} + \frac{\bar{\theta}t}{K}] = \frac{\bar{\theta}^2t}{K} < \frac{\bar{\theta}^3t}{K}.$$

This contradicts $\delta_\phi(0, y) = \frac{\bar{\theta}^3t}{K}$.

Suppose now that there is $z \in \partial S(0, \frac{\bar{\theta}^5t}{K}) \cap S_\phi(y, c_2t)$. Then, from $S_\phi(y, c_2t) \subset \Omega$, we have $\delta_\phi(0, z) = \frac{\bar{\theta}^5t}{K}$ and $\delta_\phi(y, z) \leq c_2t \leq \frac{\bar{\theta}t}{K}$. But from Lemma 5.5, we have a contradiction because

$$\delta_\phi(0, z) \leq \theta_*^2(\delta_\phi(0, y) + \delta_\phi(y, z)) \leq \theta_*^2[\frac{\bar{\theta}^3t}{K} + \frac{\bar{\theta}t}{K}] \leq \frac{\bar{\theta}}{2}[\frac{\bar{\theta}^3t}{K} + \frac{\bar{\theta}^3t}{K}] < \frac{\bar{\theta}^5t}{K} = \delta_\phi(0, z).$$

From (6.11), the $L^{\frac{n}{n-1}}$ bound on Green’s function in Lemma 6.1, and Lemma 3.1, we obtain

$$\begin{aligned} \int_{S_\phi(y, c_2t)} \sigma(x)dx &\leq \int_{S_\phi(x_0, 2t)} \sigma(x)dx \leq \|\sigma\|_{L^{\frac{n}{n-1}}(S_\phi(x_0, 2t))} |S_\phi(x_0, 2t)|^{\frac{1}{n}} \\ &\leq C(n, \lambda) |S_\phi(x_0, 2t)|^{\frac{1}{n}} |S_\phi(x_0, 2t)|^{\frac{1}{n}} \leq Ct. \end{aligned}$$

By Lemma 6.2, we have $\max_{x \in \partial S_\phi(y, c_2 t/2)} \sigma(x) \leq Ct^{-\frac{n-2}{2}}$. Thus, by the maximum principle, we have at the *apogee* y of $S(0, \frac{2\bar{\theta}^3 t}{K})$ the estimate

$$\sigma(y) \leq Ct^{-\frac{n-2}{2}}.$$

With this estimate, we use Lemma 4.3 and the boundary Harnack inequality in Theorem 2.1 together with the interior Harnack inequality in Theorem 3.3 to conclude that σ is bounded by $Ct^{-\frac{n-2}{2}}$ on $\partial S(0, \frac{\bar{\theta}^3 t}{K}) \cap \Omega$. Hence, the bound (6.1) follows from the maximum principle. \square

Proof of Lemma 6.4. To prove (6.2), we consider

$$w(x) = g_\Omega(x, x_0) - g_{S_\phi(x_0, 2t)}(x, x_0).$$

It satisfies $L_\phi w = 0$ in $S_\phi(x_0, 2t) \cap \Omega$. In $\overline{S_\phi(x_0, 2t)}$, w attains its maximum value on the boundary $\partial S_\phi(x_0, 2t)$. Thus, for $x \in \partial S_\phi(x_0, t)$, we have

$$g_\Omega(x, x_0) - g_{S_\phi(x_0, 2t)}(x, x_0) \leq \max_{z \in \partial S_\phi(x_0, 2t)} w(z) = \max_{z \in \partial S_\phi(x_0, 2t)} g_\Omega(z, x_0)$$

since $g_{S_\phi(x_0, 2t)}(z, x_0) = 0$ for z on $\partial S_\phi(x_0, 2t)$. This together with Lemma 6.3 gives

$$\begin{aligned} \max_{x \in \partial S_\phi(x_0, t)} g_\Omega(x, x_0) &\leq \max_{z \in \partial S_\phi(x_0, t)} g_{S_\phi(x_0, 2t)}(z, x_0) + \max_{z \in \partial S_\phi(x_0, 2t)} g_\Omega(z, x_0) \\ &\leq Ct^{-\frac{n-2}{2}} + \max_{z \in \partial S_\phi(x_0, 2t)} g_\Omega(z, x_0). \end{aligned}$$

Therefore, (6.2) is proved. \square

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