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EIGENVALUES AND EIGENFUNCTIONS OF DOUBLE LAYER POTENTIALS

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ABSTRACT. Eigenvalues and eigenfunctions of two- and three-dimensional double layer potentials are considered. Let Ω be a C^2 bounded region in $\mathbf{R}^n$ $(n = 2, 3)$. The double layer potential $K : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is defined by

$$(K\psi)(x) \equiv \int_{\partial\Omega} \psi(y) \cdot \nu_y E(x,y)\, ds_y,$$

where

$$E(x,y) = \begin{cases} \frac{1}{\pi} \log \frac{1}{|x-y|}, & \text{if } n = 2, \\ \frac{1}{2\pi} \frac{1}{|x-y|}, & \text{if } n = 3, \end{cases}$$

ds_y is the line or surface element and ν_y is the outer normal derivative on $\partial\Omega$. It is known that K is a compact operator on $L^2(\partial\Omega)$ and consists of at most a countable number of eigenvalues, with 0 as the only possible limit point. This paper aims to establish some relationships among the eigenvalues, the eigenfunctions, and the geometry of $\partial\Omega$.

1. Introduction and results

Let Ω be a C^2 bounded region in $\mathbf{R}^n$ $(n = 2, 3)$. Consider the double layer potential $K : L^2(\partial\Omega) \to L^2(\partial\Omega)$:

$$(K\psi)(x) \equiv \int_{\partial\Omega} \psi(y) \cdot \nu_y E(x,y)\, ds_y,$$

where

$$E(x,y) = \begin{cases} \frac{1}{\pi} \log \frac{1}{|x-y|}, & \text{if } n = 2, \\ \frac{1}{2\pi} \frac{1}{|x-y|}, & \text{if } n = 3, \end{cases}$$

ds_y is the line or surface element and ν_y is the outer normal derivative on $\partial\Omega$. We know that K is a compact operator on $L^2(\partial\Omega)$ and consists of at most a countable number of eigenvalues, with 0 as the only possible limit point. It is also known that the eigenvalues of the double layer potential integral operator lie in the interval $[-1, 1)$ and that the eigenvalue -1 corresponds to constant eigenfunctions (see [Pl] and see also [Ta] for some recent progress).

We set the ordered eigenvalues and eigenfunctions counting multiplicities as

$$\sigma_p(K) = \{\ \lambda_j \mid |\lambda_0| > |\lambda_1| \geq |\lambda_2| \geq \ \cdots \}$$

where

$$Ke_{\lambda_j}(x) = \lambda_j e_{\lambda_j}(x).$$

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Recall also that every compact operator K on Hilbert space takes the canonical form

$$K\psi = \sum_{j=1}^{\infty} \alpha_j \langle \psi, v_j \rangle u_j$$

for some orthonormal basis $\{u_j\}$ and $\{v_j\}$, where α_j are singular values of K (i.e., the eigenvalues of $(K^*K)^{1/2}$), and $\langle \cdot, \cdot \rangle$ is the $L^2(\partial\Omega)$ inner product. The singular values are non-negative, and we denote the ordered singular values by

$$\sigma_{sing}(K) = \{ \alpha_j \mid \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \ \cdots \}.$$

With this in mind, we have the following two concerns:

(i) What can we say about the geometry of $\partial\Omega$ given the eigen- or singular values?

(ii) What can we say about the eigenvalues, singular values, and eigenfunctions given the geometry?

In this paper, we examine some selected aspects of these questions: Isoperimetric eigen- and singular value problems, decay rates of eigen- and singular values, and nodal sets of eigenfunctions. These questions concern the spectral geometry of elliptic operators. As will be mentioned at the end of this section, there have been many studies in this direction. In other words, we aim to develop the spectral geometry of double layer potentials.

For this purpose, in §2, we begin studying two questions for two-dimensional double layer potentials:

(Q1) What types of eigen- and singular values give rise to the isoperimetric property of $\partial\Omega$?

(Q2) What types of sequences can occur as eigen- and singular values?

Our answer to (Q1) is the following.

Theorem 2.7. *Let $n = 2$ and let Ω be a simply connected region with a C^2 boundary. Then*

$$\sigma_{sing}(K)\backslash\{0\} = \{1\} \text{ is necessary and sufficient for } \partial\Omega = S^1.$$

It follows that $(K\psi)(x) = 0$ for all $\psi(x) \in L_0^2(\partial\Omega)$, and then, $\partial\Omega = S^1$ (see Corollary 2.8). There are proofs of this theorem (see e.g. [Li]). In §2.1, a short proof is presented that uses the Hilbert-Schmidt norm of K. For $\partial\Omega = S^1$, K has an eigenvalue -1 of multiplicity 1 and an eigenvalue 0 of countably infinite multiplicities (see [Ah1] and §2.1). Hence, the condition $\sigma_{sing}(K)\backslash\{0\} = \{1\}$ can be replaced by $\sigma_{sing}(K) = \{1, 0\}$. Moreover, the theory of quasiconformal mapping states that $\sigma_p(K) = \{-1, 0\}$ is also necessary and sufficient for $\partial\Omega = S^1$. We mention this result at the end of §2.1.

To answer (Q2), we consider the Schatten norm of K and estimate the decay rate[1] of eigen- and singular values by the regularity of $\partial\Omega$:

Theorem 2.12. *Let $n = 2$ and let Ω be a C^k ($k \geqq 2$) region. Then, for any $\alpha > -2k + 3$,*

$$\alpha_j = o(j^{\alpha/2}) \text{ and } \lambda_j = o(j^{\alpha/2}) \quad \text{as } j \to \infty,$$

where o indicates the small order.

[1]When considering the asymptotics of eigenvalues and eigenfunctions, we hereafter assume that the number of eigenvalues counting multiplicities is infinite.

It follows that if Ω is a C^∞ region, then $\lambda_j = o(j^{-\infty})$. For an ellipse $\partial\Omega$, for instance, direct calculations yield $\lambda_j = O(e^{-cj})$ (see [Ah1], [KPS, §8.3] and Example 2.2). Note that ellipses are analytic curves, and hence, the eigenvalues are presumed to have stronger decay properties than in the case of smooth curves. These considerations shed some new light on eigenvalue asymptotics.

Few studies have focused on the eigenfunctions. In §3, we consider the question for two-dimensional double layer potentials.

(Q3) What can we say about the nodal sets of eigenfunctions?
Here we establish the holomorphic extension of e_{λ_j} for analytic curves and provide the growth of zeros of analytic eigenfunctions.

Theorem 3.10. *Let $n = 2$ and let Ω be a real analytic domain and $\{e_{\lambda_j}(x)\} \subset C^\omega(\partial\Omega)$ be real analytic eigenfunctions. There exists $C > 0$, depending only on Ω, such that the zeros $N(e_{\lambda_j}(x))$ satisfy*

$$\sharp N(e_{\lambda_j}(x)) \leqq C|\log|\lambda_j||.$$

From Theorem 3.10, one can expect that the positive eigenfunctions correspond to eigenvalue -1; hence, the positive eigenfunction is constant. This fact holds true even for a much more general case (see Theorem 3.1).

Quite aside from $n = 2$, the analogy of the above theorems is difficult to handle for the case of $n = 3$. Thus, we discuss only some remarks and conjectures in §4. The behavior of $\sigma_p(K)$, for instance, becomes drastic.

Remark 4.1. Let $n = 3$ and let Ω be a smooth region. For $\alpha > -\frac{1}{2}$,

$$\lambda_j = o(j^\alpha) \quad \text{as } j \to \infty.$$

For $n = 3$, no satisfactory solution to isoperimetric eigenvalue problems has yet been found. Instead, in the context of studying the eigenvalue problems, we propose reasonable conjectures.

Conjecture 1. *Let $n = 3$ and $\underline{\lambda} \equiv \min \sigma_p(K)\backslash\{-1\}$. We have*

$$\sup_{\partial\Omega} \underline{\lambda} = -\frac{1}{3},$$

where the supremum is taken over all C^∞ simply connected closed surfaces. The supremum is achieved if and only if $\partial\Omega = S^2$.

Conjecture 2. *Let $n = 3$. For $p > 1$,*

$$\inf_{\partial\Omega} \mathrm{tr}\{(K^*K)^p\} = \left(1 - \frac{1}{2^{2p-1}}\right)\zeta(2p-1),$$

where the infimum is taken over all C^∞ simply connected closed surfaces and $\zeta(s)$ is the Riemann zeta function. The infimum is achieved if and only if $\partial\Omega = S^2$.

We confirm the validity of these conjectures. When C^∞ closed surfaces are replaced by ellipsoids, these conjectures will be proved (see Theorem 4.3).

We end the introduction by comparing the above results with the spectral geometry of the Laplacian on manifolds. In case of the Laplacian, the isoperimetric properties of manifolds are characterized by the first or second eigenvalue or eigenvalue asymptotics. (See e.g. [Be] and the references therein.) Theorem 2.7 and Conjectures 1 and 2 correspond to these results. Theorem 2.12 can be viewed as eigenvalue asymptotics, called Weyl's law. For the Laplacian, Weyl's law includes information about the dimension and volume of manifolds (see e.g. [CH] and [ANPS] and the references therein). Theorems regarding the zeros of Laplace eigenfunctions include Courant's nodal line theorem and Donnelly-Fefferman's results ([CH], [DF] and [Ze]). Roughly speaking, they estimate the Haussdorf dimension and measure of nodal sets by the eigenvalues. Indeed, we prove Theorem 3.10 by using the modified Donnelly-Fefferman value distribution theory.

2. Eigenvalues of two-dimensional double layer potentials

In §2, we restrict ourselves to two-dimensional double layer potentials. Such a situation allows us to treat the Hilbert-Schmidt and Schatten norms of K. Using these norms, we obtain isoperimetric properties of singular values and decay estimates in §2.1 and §2.2, respectively.

2.1. **The trace of K^*K and its application to isoperimetric problems.**

We consider the boundary integral equation

$$(K\psi)(x) \equiv \frac{1}{\pi}\int_{\partial\Omega} \psi(y)\cdot \nu_y \log\frac{1}{|x-y|}\, ds_y, \tag{1}$$

where Ω is a C^2 bounded region in $\mathbf{R}^2$ and ν_y is the outer normal derivative on $\partial\Omega$. $\partial\Omega \in C^2 \subset C^{1,\alpha}$ is a Lyapunov curve, and K and K^* are compact operators on $L^2(\partial\Omega)$. Moreover, the spectra in $L^2(\partial\Omega)$ and in $C^0(\partial\Omega)$ are identical ([Mik, Theorem 7.3.2]). A standard result in two-dimensional potential theory (see [Tr, pp. 78-80] and see also Lemma 2.13) states that for closed C^2 curves $\partial\Omega$,

$$\lim_{\substack{x\to y\\ x\in\partial\Omega}} \nu_y \log\frac{1}{|x-y|} = -\frac{1}{2}\kappa(y),$$

where $\kappa(y)$ is the curvature of $\partial\Omega$. Consequently, unlike the singular nature of the double layer potentials in $\mathbf{R}^3$, the double layer kernel in $\mathbf{R}^2$ is continuous for all points x and y on $\partial\Omega$, including when $x = y$. It is also known that the eigenvalues of the integral operator K, defined in equation (1), lie in the interval $[-1, 1)$ and are symmetric with respect to the origin ([BM], [Sh]). The only exception is the eigenvalue -1 corresponding to constant eigenfunctions. Summarizing these results, we have the "formal" trivial trace formula for K:[2]

$$\operatorname{tr}(K) \equiv \sum_{\substack{\lambda_i:\text{eigenvalue of } K\\ |\lambda_0|>|\lambda_1|=|\lambda_2|\ge|\lambda_3|=|\lambda_4|\ge\cdots}} \lambda_i = \int_{\partial\Omega} -\frac{1}{2\pi}\kappa(y)ds_y = -1.$$

Here, K is not self-adjoint or even normal, but K^*K is a self-adjoint trace class operator. Thus, the trace of K^*K,

$$\operatorname{tr}(K^*K),$$

is also considered. Consequently, we obtain some asymptotic properties of the eigenvalue of K. In §2, we begin by rapidly going over some basic examples of $\operatorname{tr}(K^*K)$.

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[2] K is not always the usual trace class operator. The above "formal" trace formula is defined by only a conditional summation. Note that if $\partial\Omega$ is a C^3 curve, then K is the usual trace class operator (see the proof of Theorem 2.12 and Remark 2.16).

Example 2.1 (The circle (see [Ah1]))**.** Let $\partial\Omega$ be a circle of radius R. We find

$$\sigma_p(K) = \{-1, 0\},$$
$$\operatorname{tr}(K^*K) = 1,$$

where $\sigma_p(K)$ is the set of eigenvalues of K.

In the case of an ellipse, $\operatorname{tr}(K^*K) > 1$.

Example 2.2 (The ellipse (see [Ah1], [KPS, §8.3]))**.** For $R > 0$ and $c > 0$, we define an ellipse by $\partial\Omega = \{(x, y)|\ x = \frac{1}{2}c\cosh R\cos\theta,\ y = \frac{1}{2}c\sinh R\sin\theta\}$. Then

$$\sigma_p(K) = \{-1, \pm e^{-2mR} \mid m \in \mathbf{N}\},$$
$$\operatorname{tr}(K^*K) > 1.$$

Seeing this, we want to characterize the region for which $\operatorname{tr}(K^*K) = 1$.

Lemma 2.3. *K^*K is a trace class operator on $L^2(\partial\Omega)$, i.e., K is a Hilbert-Schmidt class operator and*

$$\operatorname{tr}(K^*K) = \frac{1}{\pi^2}\int_{\partial\Omega_y}\int_{\partial\Omega_x} |\nu_y \log|x-y||^2\, ds_x\, ds_y.$$

Proof.

$$(KK^*\psi)(x) = \frac{1}{\pi^2}\int_{\partial\Omega_z}\int_{\partial\Omega_y} (\nu_z \log|x-z|)\,(\nu_z \log|y-z|)\,\psi(y)\, ds_y\, ds_z.$$

The kernel $K(x, y)$ of KK^* is continuous symmetric non-negative definite on $\partial\Omega_x \times \partial\Omega_y$. By Mercer's theorem ([CH, p. 138], [FM, Theorem 1.1] and [Kö]), there is an orthonormal set $\{u_i\}_i$ of $L^2(\partial\Omega)$ that consists of eigenfunctions of KK^* such that the corresponding eigenvalues $\{\mu_i\}_i$ are non-negative. The eigenfunctions corresponding to non-zero eigenvalues are continuous on $\partial\Omega$, and $K(x, y)$ has the representation

$$K(x, y) = \sum_{i=1}^{\infty} \mu_i u_i(x) u_i(y),$$

where the convergence is absolute and compactly uniform. This leads to

$$\operatorname{tr}(K^*K) = \operatorname{tr}(KK^*) = \frac{1}{\pi^2}\int_{\partial\Omega_y}\int_{\partial\Omega_x} |\nu_y \log|x-y||^2\, ds_x\, ds_y.$$

□

Recall that every compact operator K on Hilbert space takes the canonical form

$$K\psi = \sum_{j=1}^{\infty} \alpha_j \langle \psi, v_j \rangle u_j$$

for some orthonormal basis $\{u_j\}$ and $\{v_j\}$, where α_i are singular values of K (i.e., the eigenvalues of $(K^*K)^{1/2}$), and $\langle\cdot,\cdot\rangle$ is the L^2 inner product. In addition, the usual operator norm is $\|K\| = \sup_j(\alpha_j)$. $\operatorname{tr}(K^*K) = \sum_{j=1}^{\infty} |\alpha_j|^2$, and then, Weyl's inequality ([Si], [Te])

$$\sum_{j=1}^{\infty} |\alpha_j|^2 \geqq \sum_{\lambda_j \in \sigma_p(K)} |\lambda_j|^2,$$

yields the following lemma.

Lemma 2.4.

$$\operatorname{tr}(K^*K) \geqq \|K\|^2, \quad \operatorname{tr}(K^*K) \geqq \sum_{\lambda_j \in \sigma_p(K)} |\lambda_j|^2 \geqq 1.$$

Remark 2.5. From Lemma 2.4, the ordered eigenvalues satisfy

$$\sum_{\lambda_j \in \sigma_p(K)} |\lambda_j|^2 < \infty.$$

Thus, for all $\epsilon > 0$, there exists N such that

$$(n-N)|\lambda_n|^2 \leqq \sum_{N+1}^{n} |\lambda_j|^2 < \epsilon,$$

and hence,

$$n|\lambda_n|^2 < 2\epsilon \quad \text{for all } n > 2N.$$

Accordingly, $\lambda_j = o(j^{-1/2})$. This is not the best possible estimate (see Examples 2.1 and 2.2 and §2.2).

Now, we apply the trace for the analysis of singular values. The minimizer of $\operatorname{tr}(K^*K)$ is attained by $\partial\Omega = S^1$.

Theorem 2.6. *Let Ω be a simply connected domain with C^2 boundary. Then*

$$\operatorname{tr}(K^*K) = 1 \textit{ is necessary and sufficient for } \partial\Omega = S^1.$$

Proof. We note that $\int_{\partial\Omega_x} \nu_y \log|x-y|\, ds_y = \pi$. Letting $C \equiv \pi \cdot (\text{length of } \partial\Omega)^{-1}$,

$$\begin{aligned}
\operatorname{tr}(K^*K) &= \frac{1}{\pi^2}\int_{\partial\Omega_y}\int_{\partial\Omega_x} |\nu_y \log|x-y||^2\, ds_x\, ds_y \\
&= \frac{1}{\pi^2}\int_{\partial\Omega_y}\int_{\partial\Omega_x} |\nu_y \log|x-y| - C|^2\, ds_x\, ds_y \\
&\quad + \frac{2C}{\pi^2}\int_{\partial\Omega_y}\int_{\partial\Omega_x} \nu_y \log|x-y|\, ds_x\, ds_y \\
&\quad - \frac{C^2}{\pi^2}\int_{\partial\Omega_y}\int_{\partial\Omega_x} ds_x\, ds_y \\
&= \frac{1}{\pi^2}\int_{\partial\Omega_y}\int_{\partial\Omega_x} |\nu_y \log|x-y| - C|^2\, ds_x\, ds_y + 1.
\end{aligned}$$

It follows that $\operatorname{tr}(K^*K) = 1 \;\Rightarrow\; \nu_y \log|x-y| = C$ for all $(x,y) \in \partial\Omega_x \times \partial\Omega_y$.

By the continuity of $\nu_y \log|x-y|$,

$$\frac{1}{2}\kappa(x) = C \quad (\text{constant})$$

as desired. □

Suppose $\operatorname{tr}(K^*K) = 1$. Then, only one singular value takes 1; otherwise, $\alpha_j = 0$. Thus, we obtain the following theorem.

Theorem 2.7. *Let Ω be a simply connected region with C^2 boundary. Then*

$$\sigma_{sing}(K)\backslash\{0\} = \{1\} \textit{ is necessary and sufficient for } \partial\Omega = S^1,$$

where $\sigma_{sing}(K)$ denotes the set of singular values.

From the canonical form of K, the ball symmetry property of the double layer potential is also obtained.

Corollary 2.8 (See [Li, Theorem 1.3]). *Let Ω be a simply connected domain with a C^2 boundary, and $L_0^2(\partial\Omega) = \{\psi \in L^2(\partial\Omega) \mid \int_{\partial\Omega} \psi\, ds = 0\}$. If $(K\psi)(x) = 0$ for all $\psi(x) \in L_0^2(\partial\Omega)$, then $\partial\Omega = S^1$.*

M. Lim proved that if K is self-adjoint, then $\partial\Omega$ is a circle. Lim's work is essentially based on the "moving hyperplane" method of Alexandroff and Serrin, but we are not aware of any studies in this direction (see [Re], [Se]). It is also known [S] that the disc is the only planar domain for which K has a finite rank.

Remark 2.9. For higher dimensions, K^*K is not always a trace class operator (see §4).

In the following, we introduce well-known classical results on $\sigma_p(K)$ ([Scho1]). Even when $\sigma_{sing}(K)$ is replaced by $\sigma_p(K)$, Theorem 2.7 holds. The points $\lambda \in \sigma_p(K)\backslash\{-1\}$ are known as the Fredholm eigenvalues of $\partial\Omega$. The largest eigenvalue $\overline{\lambda}$ is often of interest. By the symmetry of eigenvalues, we have $\overline{\lambda} = -\underline{\lambda} \equiv -\inf \sigma_p(K)\backslash\{-1\}$.

Let $\partial\Omega$ be on the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then, $\partial\Omega$ divides $\widehat{\mathbf{C}}$ into complementary simply connected domains Ω and $\bar{\Omega}^c$. Let $\mathcal{H}$ be the family of all functions u continuous in $\widehat{\mathbf{C}}$ and harmonic in $\Omega\cup\bar{\Omega}^c$, with $0 < \mathcal{D}_\Omega(u)+\mathcal{D}_{\bar{\Omega}^c}(u) < \infty$. Here, $\mathcal{D}_A(u)$ is the Dirichlet integral on A:

$$\mathcal{D}_A(u) = \iint_A u_x^2 + u_y^2\, dxdy.$$

Ahlfors [Ahl] showed the relationships among the Fredholm eigenvalues, the Dirichlet integral, and quasiconformal mappings. In particular, the value $\overline{\lambda}$ can be represented in terms of the Dirichlet integral:

$$\overline{\lambda} = \sup_{u\in\mathcal{H}} \frac{|\mathcal{D}_\Omega(u) - \mathcal{D}_{\bar{\Omega}^c}(u)|}{\mathcal{D}_\Omega(u) + \mathcal{D}_{\bar{\Omega}^c}(u)}.$$

Since conformal mappings preserve harmonic functions and Dirichlet integrals, $\overline{\lambda}$ is invariant under linear fractional transformations. Let $f : \Omega\cup\bar{\Omega}^c \to \Omega\cup\bar{\Omega}^c$ of $\widehat{\mathbf{C}}$ be an orientation-preserving homeomorphism whose distributional partial derivatives are in L^2_{loc}. If f preserves the curve $\partial\Omega$, then the reflection coefficient of f is defined by

$$q_{\partial\Omega} = \inf \|\partial_{\bar{z}} f/\partial_z f\|_\infty,$$

where the infimum is taken over all quasireflections across $\partial\Omega$, provided these exist, and is attained by some quasireflection f_0. The number M satisfying

$$q_{\partial\Omega} = \frac{M+1}{M-1}$$

is called the quasiconformal constant. The M-quasiconformal mapping is an orientation-preserving diffeomorphism whose derivative maps infinitesimal circles to infinitesimal ellipses with eccentricity at most M. A basic ingredient for estimating $\overline{\lambda}$ is Ahlfors inequality [Ahl]:

$$\overline{\lambda} \geq \frac{1}{q_{\partial\Omega}}.$$

If $\overline{\lambda} = 0$, then $q_{\partial\Omega} = \infty$ and $M = 1$. Thus, f_0 is 1-conformal, and hence, conformal. The conformal mapping of $\Omega \cup \bar{\Omega}^c$ onto $\Omega \cup \bar{\Omega}^c$ can be extended to a 1-conformal mapping of $\widehat{\mathbf{C}}$ onto $\widehat{\mathbf{C}}$. The only such mappings are linear fractional transformations, and hence, since $\partial\Omega$ is mapped onto $\partial\Omega$, it must itself be S^1. Thus, $q_{\partial\Omega} = \infty, \overline{\lambda} = 0$ and $\sigma_p(K)\backslash\{-1\} = \{0\}$ only for the circle.

Remark 2.10. Many authors have conducted research in this direction. We mention only some fascinating results.

1. If $\partial\Omega$ is convex, then $\overline{\lambda} \geqq \{1 - (|\partial\Omega|/2\pi R)\}^{-1}$, where R is the supremum of radii of all circles intersecting $\partial\Omega$ in at least 3 points. (If $\partial\Omega$ is smooth, then R is the maximum radius of curvature.) This is due to C. Neumann ([Scho2], [Wa]).

2. Recently, Krushkal proved the celebrated inequality (see [Kr1, p. 358] and the references in [Kr2]):

$$\frac{3}{2\sqrt{2}} \frac{1}{q_{\partial\Omega}} \geq \overline{\lambda} \geq \frac{1}{q_{\partial\Omega}}.$$

3. For higher dimensions, Fredholm eigenvalues are also characterized by Dirichlet integrals ([S], [KPS]).

Remark 2.11. Taking the limit $R \to \infty$ in Example 2.2, $\sup_{\partial\Omega} \overline{\lambda} = 1$, where the supremum is taken over all C^∞ domain Ω.

2.2. **Asymptotic properties of $\sigma_p(K)$.** In the preceding subsection, we considered the Hilbert-Schmidt norm of K. More generally, K is in Schatten classes of $r > \frac{2}{2k-3}$ for a C^k ($k \geqq 2$) closed curve $\partial\Omega$. (For details on the notion of the Schatten classes, see [Mc].)

Let λ_n be the eigenvalues of K, satisfying

$$|\lambda_0| > |\lambda_1| = |\lambda_2| \geq |\lambda_3| = |\lambda_4| \geq \cdots .$$

In the case of an ellipse,

$$\lambda_j = O(e^{-cj})$$

where $\lambda_j = O(e^{-cj})$ indicates that there exists a constant $C > 0$ such that $\lambda_j \leq Ce^{-cj}$ for large $j \in \mathbf{N}$. For a general C^k closed curve $\partial\Omega$, we obtain the following theorem.

Theorem 2.12. *Let $n = 2$ and Ω be a C^k ($k \geqq 2$) bounded region. For any $\alpha > -2k + 3$,*

$$\alpha_j = o(j^{\alpha/2}) \text{ and } \lambda_j = o(j^{\alpha/2}) \quad \text{as } j \to \infty.$$

Thus, the boundary regularity is essential to the decay rate of eigenvalues. To prove Theorem 2.12, we first prepare a fundamental lemma. For the reader's convenience, we also prove the following.

Lemma 2.13. *If $k \geqq 2$, then $E \in C^{k-2}(\partial\Omega \times \partial\Omega)$. In particular,*

$$\lim_{\substack{x \to y \\ x \in \partial\Omega}} \nu_y \log \frac{1}{|x - y|} = -\frac{1}{2}\kappa(y),$$

where $\kappa(y)$ is the curvature of $\partial\Omega$.

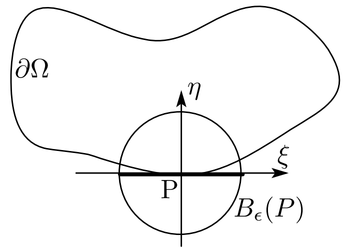

FIGURE 1

Proof. For every point P on $\partial\Omega$, there exists a small neighborhood $B_\epsilon(P)$ such that the part of $B_\epsilon(P) \cap \partial\Omega$ for some orientation of the axes of the coordinate system (ξ, η) admits the representation (see Figure 1)

$$\partial\Omega \cap B_\epsilon(P) = \{(\xi, \eta) \mid \eta = F(\xi),\ |\xi| < \epsilon'\}$$

where $F(\xi) \in C^k$.

For $x = (\xi_1, \eta_1)$ and $y = (\xi_2, \eta_2)$, ν_y and $\log|x - y|$ are given by

$$\nu_y = \Big(\frac{F'(\xi_2)}{\{1 + (F'(\xi_2))^2\}^{1/2}} \frac{\partial}{\partial \xi_2},\ \frac{-1}{\{1 + (F'(\xi_2))^2\}^{1/2}} \frac{\partial}{\partial \eta_2}\Big),$$

$$\log|x - y| = \frac{1}{2} \log\{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2\}.$$

Now,

$$\begin{aligned}\nu_y \log|x - y| &= \frac{(\xi_2 - \xi_1)F'(\xi_2) - (\eta_2 - \eta_1)}{\{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2\}\{1 + (F'(\xi_2))^2\}^{1/2}} \\ &= \frac{(\xi_2 - \xi_1)F'(\xi_2) - (F(\xi_2) - F(\xi_1))}{\{(\xi_1 - \xi_2)^2 + (F(\xi_1) - F(\xi_2))^2\}\{1 + (F'(\xi_2))^2\}^{1/2}}.\end{aligned}$$

Since

$$F(\xi_1) - F(\xi_2) - (\xi_1 - \xi_2)F'(\xi_2) = (\xi_1 - \xi_2)^2 \int_0^1 tF''(\xi_2 + (\xi_1 - \xi_2)t)\, dt$$

and

$$F(\xi_1) - F(\xi_2) = (\xi_1 - \xi_2) \int_0^1 F'(\xi_2 + (\xi_1 - \xi_2)t)\, dt,$$

we obtain

$$\begin{aligned}&\nu_y \log|x - y| \\ &\quad = \frac{(\xi_1 - \xi_2)^2 \int_0^1 tF''(\xi_2 + (\xi_1 - \xi_2)t)\, dt}{[(\xi_1 - \xi_2)^2 + (\xi_1 - \xi_2)^2\{\int_0^1 F'(\xi_2 + (\xi_1 - \xi_2)t)\, dt\}^2]\{1 + (F'(\xi_2))^2\}^{1/2}} \\ &\quad = \frac{\int_0^1 tF''(\xi_2 + (\xi_1 - \xi_2)t)\, dt}{[1 + \{\int_0^1 F'(\xi_2 + (\xi_1 - \xi_2)t)\, dt\}^2]\{1 + (F'(\xi_2))^2\}^{1/2}}.\end{aligned}$$

The positive denominator is of class C^{k-1}, and the numerator is of class C^{k-2}, including when $x = y$. Moreover,

$$\lim_{\substack{x\to y\\ x\in\partial\Omega}} \nu_y \log\frac{1}{|x-y|} = -\frac{\frac{1}{2}F''(\xi_2)}{\{1+(F'(\xi_2))^2\}^{3/2}} = -\frac{1}{2}\kappa(y).$$

□

For $p < 2$, the Schatten class $S_p(L^2)$ cannot be characterized as in the case $p = 2$ by a property analogous to the square integrability of integral kernels. To obtain criteria for operators belonging to Schatten classes for $p < 2$, we use the following result of J. Delgado and M. Ruzhansky.

Theorem 2.14 ([DR], Theorem 3.6). *Let M be a closed smooth manifold of dimension n and let $\mu_1,\ \mu_2 \geqq 0$. Let $K \in L^2(M\times M)$ be such that $E(x,y) \in H^{\mu_1,\mu_2}_{x,y}(M\times M)$. Then the integral operator K on $L^2(M)$, defined by*

$$(Kf)(x) = \int_M E(x,\ y)f(y)\,dy,$$

is in the Schatten classes $S_r(L^2(M))$ for $r > \frac{2n}{n+2(\mu_1+\mu_2)}$.

Proof of Theorem 2.12. Taking a C^∞ atlas on $M = \partial\Omega$, as in Lemma 2.13, we see

$$E(x,y) \in C^{k-2}_{x,y}(M\times M).$$

Let $n = \dim\partial\Omega = 1$ and $\mu_1 + \mu_2 = k-2$. From Theorem 2.14,

$$K \in S_r(L^2(M)) \quad \text{for all } r > \frac{2}{2k-3}.$$

Using Weyl's inequality again ([Si], [Te]),

$$\{\sum_{j=1}^{\infty}|\alpha_j|^r\}^{1/r} \geqq \{\sum_{\lambda_j\in\sigma_p(K)}|\lambda_j|^r\}^{1/r}.$$

The left-hand side is the Schatten norm of K, which is finite. □

Corollary 2.15. *Let $n = 2$ and Ω be a C^∞ region. Then*

$$\alpha_j = o(j^{-\infty}) \text{ and } \lambda_j = o(j^{-\infty}) \quad \text{as } j\to\infty.$$

Remark 2.16. If Ω is a C^6 region, then $E(x,y) \in C^{2,2}_{x,y}(M\times M)$. From [DR, Corollary 4.4], K is a trace class operator, and its trace is

$$\sum_{\lambda_i\in\sigma_p(K)}\lambda_i \equiv \mathrm{tr}(K) = \int_{\partial\Omega} -\frac{1}{2\pi}\kappa(y)ds_y = -1.$$

-1 is an eigenvalue of K; hence, the sum of Fredholm eigenvalues is 0.

The $L^p \to L^q$ estimate of eigenfunctions is one of the principal interests in spectral geometry. From Lemma 2.13, we obtain a fundamental estimate of eigenfunctions:

Remark 2.17. Let $n = 2$ and Ω be a C^2 region. There exists a constant C, depending only on Ω, such that

$$\|e_{\lambda_j}\|_{L^\infty(\partial\Omega)} \leqq C{\lambda_j}^{-1}\|e_{\lambda_j}\|_{L^1(\partial\Omega)}.$$

Presumably, this is the best $L^1 \to L^\infty$ estimate of eigenfunctions.

3. Nodal sets of eigenfunctions

Few studies have focused on eigenfunctions. In this section, we introduce some fundamental estimates for nodal sets of eigenfunctions of two-dimensional double layer potentials.

3.1. **Basic properties of nodal sets.** The nodal set $N(e_\lambda(x))$ of eigenfunction $e_\lambda(x)$ is defined by

$$N(e_\lambda(x)) \equiv \{\ x \in \partial\Omega \mid e_\lambda(x) = 0\ \}.$$

We note that the nodal set of non-constant eigenfunctions is not empty.

Theorem 3.1. *Let Ω be a bounded C^2 region in $\mathbf{R}^n$ and $0 < \phi(x) \in C(\partial\Omega)$ be an eigenfunction of K. Then $\phi(x) = \text{const}$.*

This theorem holds even for $n \geqq 3$. To prove Theorem 3.1, we closely follow [KPS] and introduce the properties of symmetrizable operators. The proposition below is aimed at and will be directly applicable to double layer potentials K. We know that K is in some Schatten classes (see §2 and §4 for the case of $n = 3$). Moreover, the eigenvalues of symmetrizable Schatten class operators can be determined using min-max methods ([KPS, §3 and Proposition 3]).

Proposition 3.2 (Min-max principle for double layer potentials)**.** *Let $\lambda_1^+ \geqq \lambda_2^+ \geqq \cdots \geqq 0 \geqq \cdots \geqq \lambda_1^- > \lambda_0^- = -1$ be the eigenvalues of K repeated according to their mulitplicity, and let $\phi_k^+,\ \phi_k^-$ be the corresponding eigenfunctions. Then*

$$\lambda_k^+ = \max_{f \perp \{\phi_1^+, \cdots, \phi_{k-1}^+\}} \frac{\langle SKf,\ f\rangle}{\langle Sf,\ f\rangle}$$

and similarly

$$\lambda_k^- = \min_{f \perp \{\phi_0^-, \cdots, \phi_{k-1}^-\}} \frac{\langle SKf,\ f\rangle}{\langle Sf,\ f\rangle}.$$

Here we can use the single layer potential S defined by

$$(S\psi)(x) \equiv \int_{\partial\Omega} E(x,y)\psi(y)\, dS_y$$

where $f \perp g$ means $\langle f, Sg\rangle = 0$, especially if $\lambda \neq -1$, $e_\lambda(x) \in \{\phi(x) \in L^2(\partial\Omega) \mid \langle \phi, S1\rangle = 0\}$.

Proof of Theorem 3.1. From the min-max principle for double layer potentials, non-constant eigenfunctions $\{e_\lambda(x)\}$ satisfy $e_\lambda(x) \in \{\phi(x) \in L^2(\partial\Omega) \mid \langle \phi, S1\rangle = 0\}$. Because $f(x) = S1(x) > 0$ for $n \geqq 3$ and

$$\int_{\partial\Omega} f(x)\phi(x)\, dS_x = 0,$$

there exists a subset $N^- \subset \partial\Omega$ such that $\phi(x) < 0$ on N^-.

For $n = 2$, eigenfunctions and eigenvalues are equivalent under the self-similar transformations. Indeed, letting $x_\epsilon = \epsilon x,\ y_\epsilon = \epsilon y$, $\Omega_\epsilon = \{x_\epsilon \mid x \in \Omega\}$ and $\psi(x_\epsilon) \equiv \psi(x)$, we have

$$(K_\epsilon\psi)(x_\epsilon) \equiv \int_{\partial\Omega_\epsilon} \psi(y_\epsilon) \cdot \nu_{y_\epsilon} E(x_\epsilon, y_\epsilon)\, ds_{y_\epsilon} = \int_{\partial\Omega} \psi(y) \cdot \nu_y E(x,y)\, ds_y = (K\psi)(x),$$

since $S1(x) > 0$ for the shrinking region Ω_ϵ. Again, using the min-max principle, there exists a subset $N^- \subset \partial\Omega$ such that $\phi(x) < 0$ on N^-. □

For a convex region, we can provide another short proof of Theorem 3.1 without Proposition 3.2.

Remark 3.3. Let Ω be a convex region in $\mathbf{R}^n$ and $\phi(x) > 0$ an eigenfunction of K. Then $\phi(x) = \text{const}$.

Proof. From a convex separation theorem,

$$\nu_y E(x,y) = C\frac{x-y}{|x-y|^{n-1}} \cdot \mathbf{n}_y \le 0 \quad (\forall x,\ y \in \partial\Omega).$$

Because $(K1)(x) \equiv \int_{\partial\Omega} \nu_y E(x,y)\, ds_y = -1$ and as a result of the first mean value theorem for integration, for all $x \in \partial\Omega$, there exists $x' \in \partial\Omega$ satisfying

$$(K\phi)(x) = -\phi(x').$$

For non-constant eigenfunction $\phi(x) > 0$, we know $(K\phi)(x) = \lambda\phi(x)$ with $|\lambda| < 1$. Thus,

$$\inf_{x\in\partial\Omega} |(K\phi)(x)| = \inf_{x\in\partial\Omega} |\lambda\phi(x)| < \inf_{x'\in\partial\Omega} |\phi(x')| = \inf_{x\in\partial\Omega} |(K\phi)(x)|.$$

This is a contradiction. □

In the following example, the nodal set of the second eigenfunction of double layer potential divides $\partial\Omega$ into many pieces.

Example 3.4. Let $\partial\Omega = S^1$. For an arbitrary non-empty closed set $A \subsetneqq \partial\Omega$, there exists an eigenfunction $e_0(x) \neq 0$ such that

$$A \subset N(e_0(x)).$$

Proof. From Corollary 2.8, we just choose the non-constant function $e_0(x) \in C^\infty(\partial\Omega) \cap L^2_0(\partial\Omega)$ to satisfy $e_0(x) = 0$ on A. □

We recall Courant's nodal line theorem (CNLT), which states that if the eigenvalues λ_n of a Laplacian are ordered increasingly, then each eigenfunction $u_n(x)$ corresponding to λ_n divides the region by its nodal set into at most n subdomains. In contrast to the CNLT, we find that the nodal set of double layer eigenfunction e_n is characterized not by n but by λ_n.

3.2. **Two-dimensional analytic boundary.** In this subsection, we only consider the analytic domains $\Omega \subset \mathbf{R}^2$ and real analytic eigenfunctions $\{e_\lambda(x)\} \subset C^\omega(\partial\Omega)$. This assumption is reasonable since the continuous eigenfunction $e_\lambda(x)$ is also analytic for $\lambda \neq 0$ (see Remark 3.8).

We prove that the boundary zeros $N(e_\lambda(x))$ satisfy

$$\sharp N(e_\lambda(x)) < C|\log|\lambda||,$$

where $Ke_\lambda(x) = \lambda e_\lambda(x)$.

3.2.1. *Holomorphic extensions of eigenfunctions.* The following notation and results borrow heavily from Garabedian (see [Ga]), Millar (see [Mi1], [Mi2], [Mi3]) and Toth-Zelditch (see [TZ]). We denote points $\mathbf{R}^2$ and also in $\mathbb{C}^2$ by (x,y). We further write $z = x + iy$, $z^* = x - iy$. Note that z, z^* are independent holomorphic coordinates on $\mathbb{C}^2$ and are characteristic coordinates for the Laplacian $\frac{1}{4}\triangle$, and that the Laplacian extends analytically to $\frac{\partial^2}{\partial z\partial z^*}$. When dealing with the kernel functions of two variables, we use (ξ,η) in the same way as (x,y) for the second variable.

When the boundary is real analytic, the complexification $\partial\Omega \subset \mathbb{C}$ is the image of analytic continuation of a real analytic parametrization. For simplicity and without loss of generality, we assume that the length of $\partial\Omega = 2\pi$. We denote a real parametrization by arc-length by $Q : \mathbf{S}^1 \to \partial\Omega \subset \mathbb{C}$ and write the parametrization as a periodic function

$$q(t) = Q(e^{it}) \ : \ [0, 2\pi] \to \partial\Omega$$

on $[0, 2\pi]$. We then set the complex conjugate of $q(s) = q_1(s) + iq_2(s)$, $\bar{q}(s) = q_1(s) - iq_2(s)$ for $s \in [0, 2\pi]$.

We complexify $\partial\Omega$ by extending the parametrization holomorphically to $Q^{\mathbb{C}}$ on the annulus

$$A(\epsilon) \equiv \{\tau \in \mathbb{C} \ : \ e^{-\epsilon} < |\tau| < e^{\epsilon}\}$$

for a small enough $\epsilon > 0$. Note that the complex conjugate parametrization $\bar{Q}$ extends holomorphically to $A(\epsilon)$ as $Q^{*\mathbb{C}}$. The $q(t)$ parametrization continues analytically to a periodic function $q^{\mathbb{C}}(t)$ on $[0, 2\pi] + i[-\epsilon, \epsilon]$. The complexification $\partial\Omega_{\mathbb{C}}(\epsilon)$ of $\partial\Omega$ is denoted by

$$\partial\Omega_{\mathbb{C}}(\epsilon) \equiv Q^{\mathbb{C}}(A(\epsilon)) \ \subset \mathbb{C}.$$

Next, we write $r^2((x, y); (\xi, \eta)) = (\xi - x)^2 + (\eta - y)^2$. For $s \in \mathbf{R}$ and $t \in \mathbb{C}$, we have $q(s) = \xi(s) + i\eta(s)$, $q^{\mathbb{C}}(t) = x(t) + iy(t)$, $q^{\mathbb{C}*}(t) = x(t) - iy(t)$, and we write $r^2(q(s); q^{\mathbb{C}}(t))$. Thus,

$$r^2((x, y); (\xi, \eta)) = (q(s) - q^{\mathbb{C}}(t))(\bar{q}(s) - q^{\mathbb{C}*}(t)) \ \in \mathbb{C}.$$

To clarify the notation, we consider two examples.

Example 3.5 (The circle)**.** Let $\partial\Omega = S^1$. Then $q(s) = e^{is}$, $t = \theta + i\xi$, $q^{\mathbb{C}}(t) = e^{i(\theta+i\xi)}$, $q^{\mathbb{C}*}(t) = e^{-i(\theta+i\xi)}$, $\bar{q}^{\mathbb{C}*}(t) = e^{i(\theta-i\xi)}$, and

$$r^2(s, t) = (e^{i(\theta+i\xi)} - e^{is})(e^{-i(\theta+i\xi)} - e^{-is}) = 4\sin^2 \frac{\theta - s + i\xi}{2}.$$

Thus, $\log r^2 = \log(4\sin^2 \frac{\theta-s+i\xi}{2})$.

Example 3.6 (The ellipse)**.** Let $\partial\Omega = \{(x, y) \mid \frac{x^2}{9} + y^2 = 1\}$. Then $q(s) = 2e^{is} + e^{-is}$, $t = \theta + i\xi$, $q^{\mathbb{C}}(t) = 2e^{i(\theta+i\xi)} + e^{-i(\theta+i\xi)}$, $q^{\mathbb{C}*}(t) = 2e^{-i(\theta+i\xi)} + e^{i(\theta+i\xi)}$, $\bar{q}^{\mathbb{C}*}(t) = 2e^{i(\theta-i\xi)} + e^{-i(\theta-i\xi)}$, and

$$r^2(s, t) = (2e^{i(\theta+i\xi)} + e^{-i(\theta+i\xi)} - e^{is})(2e^{-i(\theta+i\xi)} + e^{i(\theta+i\xi)} - e^{-is}).$$

We denote by $\frac{\partial}{\partial n}$ the not-necessarily unit normal derivative in the direction $iq'(s)$. Thus, in terms of the notation $\frac{\partial}{\partial \nu}$ above, $\frac{\partial}{\partial n} = |q'(s)|\frac{\partial}{\partial \nu}$. When we are using an arc-length parametrization, $\frac{\partial}{\partial n} = \frac{\partial}{\partial \nu}$. One has

$$\begin{aligned}
\frac{d}{ds}\log r &= \frac{1}{2}\Big[\frac{q'(s)}{q(s) - q^{\mathbb{C}}(t)} + \frac{\bar{q}'(s)}{\bar{q}(s) - q^{\mathbb{C}*}(t)}\Big],\\
\frac{\partial}{\partial n}\log r &= \frac{-i}{2}\Big[\frac{q'(s)}{q(s) - q(t)} - \frac{\bar{q}'(s)}{\bar{q}(s) - q^{*}(t)}\Big].
\end{aligned}$$

3.2.2. *Analytic continuation of eigenfunctions through layer potential representation.* Since $r^2(s,t)=0$ when $s=t$, the logarithmic factor in K now gives rise to a multi-valued integrand. Nevertheless, any derivative of $\log r^2$ is unambiguously defined, and the analytic continuation of the complex representation was provided by Millar (see [Mi1, p. 508 (7.2)]).

Proposition 3.7. *The integral*

$$Ke_\lambda(q(s)) = \frac{1}{\pi}\int_0^{2\pi} e_\lambda(q(s))\frac{\partial}{\partial\nu}\log r(s,t)\,ds = \frac{1}{\pi}\int_0^{2\pi} e_\lambda(q(s))\frac{1}{r}\frac{\partial r}{\partial \nu}(s,t)\,ds$$

is real analytic on the parameter interval S^1 parametrizing $\partial\Omega$ and holomorphically extended to an annulus $A(\epsilon)$ by the formula

$$Ke_\lambda(q^{\mathbb{C}}(t)) = \frac{1}{2\pi i}\int_0^{2\pi} e_\lambda(q(s))\left(\frac{q'(s)}{q(s)-q^{\mathbb{C}}(t)} - \frac{\bar{q}'(s)}{\bar{q}(s)-q^{\mathbb{C}*}(t)}\right)\,ds.$$

Proof. We first remark that $\frac{\partial}{\partial\nu} = |q'(s)|^{-1}\frac{\partial}{\partial n}$; hence, the integral representation is invariant under reparametrization. Any derivative of $\log r^2$ is unambiguously defined, and we already have

$$\frac{1}{r}\frac{\partial r}{\partial n} = \frac{\partial \log r}{\partial n} = \frac{1}{2i}\Big[\frac{q'(s)}{q(s)-q^{\mathbb{C}}(t)} - \frac{\bar{q}'(s)}{\bar{q}(s)-q^{\mathbb{C}*}(t)}\Big].$$

In the real domain, $q^{\mathbb{C}*}(t) = \bar{q}^{\mathbb{C}}(t)$, so

$$\frac{1}{r}\frac{\partial r}{\partial n} = \mathrm{Im}\frac{q'(s)}{q(s)-q(t)}.$$

Here Im z is the imaginary part of z. We recall that in terms of real parametrization, $\frac{1}{r}\frac{\partial r}{\partial\nu}$ is real and continuous (see Lemma 2.13).

In complex notation, the same statement follows from the fact that

$$\lim_{t\to s}\frac{q(s)-q^{\mathbb{C}}(t)}{s-t} = q'(s) \quad\Rightarrow\quad \frac{q'(s)}{q(s)-q^{\mathbb{C}}(t)} = \frac{1}{s-t} + O(1),\ (s\to t),$$

where $\frac{1}{s-t}$ is real when $s,t\in\mathbf{R}$. Hence, $\mathrm{Im}\frac{q'(s)}{q(s)-q^{\mathbb{C}}(t)}$ is continuous for $s,t\in[0,2\pi]$, and since $q(s),\ q(t)$ are real analytic, the map

$$s \rightarrow \Big[\frac{q'(s)}{q(s)-q^{\mathbb{C}}(t)} - \frac{\bar{q}'(s)}{\bar{q}(s)-q^{\mathbb{C}*}(t)}\Big]$$

is continuous from $s\in[0,2\pi]$ to the space of holomorphic functions of t. Thus, the integral admits a holomorphic extension. □

Remark 3.8. We notice that the continuous eigenfunction satisfies

$$e_\lambda(q(s)) = \frac{1}{\lambda}Ke_\lambda(q(s)) \quad \text{for } s\in\mathbf{S}^1.$$

From Proposition 3.7, if $\lambda\neq 0$, the continuous eigenfunction is also analytic.

3.2.3. *Growth of zeros and growth of* $e_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t))$. The main purpose of this subsection is to provide an upper bound for the number of complex zeros of $e_\lambda^{\mathbb{C}}$ in $\partial\Omega_{\mathbb{C}}(\epsilon)$ in terms of the growth of $|e_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t))|$. For the eigenvalue λ and for a region $D \subset \partial\Omega_{\mathbb{C}}(\epsilon)$, we denote the number of complex zeros by

$$n(\lambda, D) = \sharp\{q^{\mathbb{C}}(t) \in D \ : \ e_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t)) = 0\}.$$

For the reader's convenience, we recall that the classical distribution theory of holomorphic functions is concerned with the relationship between the growth of the number of zeros of a holomorphic function f and the growth of $\max_{|z|=r} \log|f(z)|$ on discs of increasing radius. The following estimate, suggested by Lemma 6.1 of Donnelly-Fefferman (see [DF]), provides an upper bound on the number of zeros in terms of the growth of the family.

Proposition 3.9. *Normalize* e_λ *such that* $\|e_\lambda\|_{L^2(\partial\Omega)} = 2\pi$. *There exists a constant* $C(\epsilon) > 0$ *such that for any* $\epsilon > 0$,

$$n(\lambda, \partial\Omega_{\mathbb{C}}(\epsilon/2)) \leqq C(\epsilon) \max_{q^{\mathbb{C}}(t)\in\partial\Omega_{\mathbb{C}}(\epsilon)} \Big|\log|e_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t))|\Big|.$$

Proof. Let G_ϵ be the Dirichlet Green's function of $\frac{2}{\pi}\frac{\partial^2}{\partial z \partial z^*}$ in the "annulus" $\partial\Omega_{\mathbb{C}}(\epsilon)$. In addition, let $\{a_k\}_{k=1}^{n(\lambda,\partial\Omega_{\mathbb{C}}(\epsilon/2))}$ be the zeros of $e_\lambda^{\mathbb{C}}$ in the subannulus $\partial\Omega_{\mathbb{C}}(\epsilon/2)$. Let $f_\lambda = \frac{e_\lambda^{\mathbb{C}}}{\|e_\lambda^{\mathbb{C}}\|_{\partial\Omega_{\mathbb{C}}(\epsilon)}}$, where $\|u\|_{\partial\Omega_{\mathbb{C}}(\epsilon)} = \max_{\zeta\in\partial\Omega_{\mathbb{C}}(\epsilon)}|u(\zeta)|$. Then, $\log|f_\lambda(q^{\mathbb{C}}(t))|$ can be separated into two terms:

$$\begin{aligned}\log|f_\lambda(q^{\mathbb{C}}(t))| &= \int_{\partial\Omega_{\mathbb{C}}(\epsilon/2)} G_\epsilon(q^{\mathbb{C}}(t), w)\frac{i}{\pi}\partial\bar{\partial}\log|e_\lambda^{\mathbb{C}}(w)| + F_\lambda(q^{\mathbb{C}}(t)) \\ &= \sum_{a_k\in\partial\Omega_{\mathbb{C}}(\epsilon/2)\ :\ e_\lambda^{\mathbb{C}}(a_k)=0} G_\epsilon(q^{\mathbb{C}}(t), a_k) + F_\lambda(q^{\mathbb{C}}(t)),\end{aligned}$$

because $\frac{i}{\pi}\partial\bar{\partial}\log|e_\lambda^{\mathbb{C}}(w)| = \sum_{a_k\in\partial\Omega_{\mathbb{C}}(\epsilon/2)\ :\ e_\lambda^{\mathbb{C}}(a_k)=0}\delta_{a_k}$, which is called the Poincaré-Lelong formula of holomorphic functions ([De, p. 139 (3.6)]). Moreover, the function F_λ is subharmonic on $\partial\Omega_{\mathbb{C}}(\epsilon)$ in the sense of distribution:

$$\begin{aligned}\frac{i}{\pi}\partial\bar{\partial}F_\lambda &= \frac{i}{\pi}\partial\bar{\partial}\log|f_\lambda(q^{\mathbb{C}}(t))| - \sum_{a_k\in\partial\Omega_{\mathbb{C}}(\epsilon/2)\ :\ e_\lambda^{\mathbb{C}}(a_k)=0}\frac{i}{\pi}\partial\bar{\partial}G_\epsilon(q^{\mathbb{C}}(t), a_k) \\ &= \sum_{a_k\in\partial\Omega_{\mathbb{C}}(\epsilon)\setminus\partial\Omega_{\mathbb{C}}(\epsilon/2)\ :\ e_\lambda^{\mathbb{C}}(a_k)=0}\delta_{a_k} > 0.\end{aligned}$$

Thus, by the maximum principle for subharmonic functions,

$$\max_{\partial\Omega_{\mathbb{C}}(\epsilon)} F_\lambda(q^{\mathbb{C}}(t)) \leqq \max_{\partial(\partial\Omega_{\mathbb{C}}(\epsilon))} F_\lambda(q^{\mathbb{C}}(t)) = \max_{\partial(\partial\Omega_{\mathbb{C}}(\epsilon))}\log|f_\lambda(q^{\mathbb{C}}(t))| = 0.$$

It follows that

$$\log|f_\lambda(q^{\mathbb{C}}(t))| \leqq \sum_{a_k\in\partial\Omega_{\mathbb{C}}(\epsilon/2)\ :\ e_\lambda^{\mathbb{C}}(a_k)=0} G_\epsilon(q^{\mathbb{C}}(t), a_k),$$

and hence, that

$$\max_{q^{\mathbb{C}}(t)\in\partial\Omega_{\mathbb{C}}(\epsilon)}\log|f_\lambda(q^{\mathbb{C}}(t))| \leqq \left(\max_{z,w\in\partial\Omega_{\mathbb{C}}(\epsilon/2)} G_\epsilon(z,w)\right) n(\lambda, \partial\Omega_{\mathbb{C}}(\epsilon/2)).$$

Now, $G_\epsilon(z,w) \leqq \max_{w\in\partial(\partial\Omega_{\mathbb{C}}(\epsilon))} G_\epsilon(z,w) = 0$, and thus, $G_\epsilon(z,w) < 0$ for $z, w \in \partial\Omega_{\mathbb{C}}(\epsilon/2)$. Thus, there exists a constant $\nu(\epsilon) < 0$ such that $\max_{z,w\in\partial\Omega_{\mathbb{C}}(\epsilon/2)} G_\epsilon(z,w) \leqq \nu(\epsilon)$. Hence,

$$\max_{q^{\mathbb{C}}(t)\in\partial\Omega_{\mathbb{C}}(\epsilon/2)} \log|f_\lambda(q^{\mathbb{C}}(t))| \leqq \nu(\epsilon) n(\lambda, \partial\Omega_{\mathbb{C}}(\epsilon/2)).$$

Since both sides are negative,

$$\begin{aligned} n(\lambda, \partial\Omega_{\mathbb{C}}(\epsilon/2)) &\leqq \frac{1}{|\nu(\epsilon)|}\Big| \max_{q^{\mathbb{C}}(t)\in\partial\Omega_{\mathbb{C}}(\epsilon/2)} \log|f_\lambda(q^{\mathbb{C}}(t))|\Big| \\ &\leqq \frac{1}{|\nu(\epsilon)|}\Big(\max_{q^{\mathbb{C}}(t)\in\partial\Omega_{\mathbb{C}}(\epsilon)} \log|e^{\mathbb{C}}_\lambda(q^{\mathbb{C}}(t))| - \max_{q^{\mathbb{C}}(t)\in\partial\Omega_{\mathbb{C}}(\epsilon/2)} \log|e^{\mathbb{C}}_\lambda(q^{\mathbb{C}}(t))|\Big) \\ &\leqq \frac{1}{|\nu(\epsilon)|} \max_{q^{\mathbb{C}}(t)\in\partial\Omega_{\mathbb{C}}(\epsilon)} \log|e^{\mathbb{C}}_\lambda(q^{\mathbb{C}}(t))|, \end{aligned}$$

where, in the last inequality, we use the fact that $\max_{q^{\mathbb{C}}(t)\in\partial\Omega_{\mathbb{C}}(\epsilon/2)} \log|e^{\mathbb{C}}_\lambda(q^{\mathbb{C}}(t))| \geqq 0$, which holds since $|e^{\mathbb{C}}_\lambda| \geqq 1$ at some point in $\partial\Omega_{\mathbb{C}}(\epsilon/2)$. Indeed, by our normalization, $\|e_\lambda\|_{L^2(\partial\Omega)} = 2\pi$, and hence, there must already exist a point on $\partial\Omega$ with $|e_\lambda| > 1$. Writing $C(\epsilon) = \frac{1}{|\nu(\epsilon)|}$, we have the desired result. □

We obtain the principal theorem.

Theorem 3.10. *Let $\Omega \subset \mathbf{R}^2$ be a real analytic domain and $|\lambda| \neq 0$. For real analytic eigenfunctions $e_\lambda(x)$,*

$$\sharp N(e_\lambda(x)) < C|\log|\lambda||.$$

Proof. For real $t \in S^1 = [0, 2\pi]/\sim$,

$$e_\lambda(q(t)) = \frac{1}{\lambda} K e_\lambda = \frac{1}{\lambda}\int_0^{2\pi} e_\lambda(q(s)) \frac{1}{r}\frac{\partial r}{\partial\nu}(s,t)\, ds.$$

The holomorphic extension of $e_\lambda(q(s)) \in C^\omega(S^1)$ to $C^\omega(A(\epsilon))$ is unique, and hence, from Proposition 3.7,

$$e^{\mathbb{C}}_\lambda(q^{\mathbb{C}}(t)) = \frac{1}{2\pi i\lambda}\int_0^{2\pi} e_\lambda(q(s))\left(\frac{q'(s)}{q(s)-q^{\mathbb{C}}(t)} - \frac{\bar{q}'(s)}{\bar{q}(s)-q^{\mathbb{C}*}(t)}\right) ds,$$

remarking that the function $(\cdots)$ is continuous and bounded on $A(\epsilon)$ from the proof of Proposition 3.7. Hence, using the Cauchy-Schwarz inequality, there exists $C_{A(\epsilon)} > 0$ such that

$$\begin{aligned} |e^{\mathbb{C}}_\lambda(q^{\mathbb{C}}(t))| &\leqq \left|\frac{1}{2\pi i\lambda}\right| \cdot \|e_\lambda(q(s))\|_{L^2(\partial\Omega)} \int_0^{2\pi}\left|\frac{q'(s)}{q(s)-q^{\mathbb{C}}(t)} - \frac{\bar{q}'(s)}{\bar{q}(s)-q^{\mathbb{C}*}(t)}\right|^2 ds \\ &\leqq \left|\frac{1}{\lambda}\right| \cdot C_{A(\epsilon)} \cdot \|e_\lambda(q(s))\|_{L^2(\partial\Omega)}. \end{aligned}$$

Letting $\|e_\lambda(q(s))\|_{L^2(\partial\Omega)} = 2\pi$ and by Proposition 3.9,

$$\begin{aligned} n(\lambda, \partial\Omega_{\mathbb{C}}(\epsilon/2)) &\leqq C(\epsilon) \max_{q^{\mathbb{C}}(t)\in\partial\Omega_{\mathbb{C}}(\epsilon)} \Big|\log|e^{\mathbb{C}}_\lambda(q^{\mathbb{C}}(t))|\Big| \\ &\leqq C(\epsilon)\Big|\log\Big[\Big|\frac{1}{\lambda}\Big| \cdot C_{A(\epsilon)} \cdot \|e_\lambda(q(s))\|_{L^2(\partial\Omega)}\Big]\Big| \\ &\leqq \tilde{C}(\epsilon)|\log|\lambda|| \end{aligned}$$

as desired. □

4. Double layer potentials in $\mathbf{R}^3$

Plemeij [Pl] derived a fundamental result on the double layer potential in $\mathbf{R}^3$, stating that the eigenvalues of K satisfy the following inequality:

$$-1 \leq \lambda_j < 1.$$

For the case of a sphere, however, it is known that the eigenvalue of K is negative, and by a straightforward calculation, it can be shown that the eigenvalues are

$$\lambda_j = -\frac{1}{2j+1} \quad (j = 0, 1, 2, \cdots),$$

with multiplicity $2j+1$. Hence, $\sigma_p(K)$ for $n = 3$ is very different from that for $n = 2$ (see Examples 2.1 and 2.2 and Theorem 2.7). Furthermore, Ahner and Arenstorf [AA] have shown that when $\partial\Omega$ is a prolate spheroid, the corresponding eigenvalues are also negative. Consequently, for this geometry, the spectrum of K also lies in the closed interval $[-1, 0]$.

Apart from these calculations, for the case of a special oblate spheroid, Ahner [Ah2, p. 333] finds the positive eigenvalue $\lambda = 0.0598615 \cdots < 1$. This is an example of positive eigenvalues. Unfortunately, determining the supremum of eigenvalues becomes a formidable task for a general region.

Nevertheless, for $\overline{\lambda} = \sup\{\lambda_j \mid \lambda_j \in \sigma_p(K)\}$, we know the supremum of the boundary variation ([ADR, Lemma 3.2, Theorem 3.4])

$$\sup_{\partial\Omega} \overline{\lambda} = 1,$$

where the supremum is taken over all C^∞ domain Ω. Letting

$$\underline{\lambda} = \inf\{\lambda_j \mid \lambda_j \in \sigma_p(K) \backslash \{-1\}\},$$

we also know ([ADR, Lemma 3.2], [KPS, Theorem 5]) that

$$\inf_{\partial\Omega} \underline{\lambda} = -1.$$

Here we introduce a result regarding $\underline{\lambda}$. Steinbach and Wendland proved that

$$(1 - \sqrt{1 - c_0})\|w\|_{S^{-1}} \leqq \|(I \pm K)w\|_{S^{-1}} \leqq (1 + \sqrt{1 - c_0})\|w\|_{S^{-1}},$$

where $c_0 = \inf_{w \in H_*^{1/2}} \frac{\langle Kw, w\rangle}{\langle S^{-1}w, w\rangle}$ and $\|w\|_{S^{-1}} = \sqrt{\langle S^{-1}w, w\rangle}_{L^2(\partial\Omega)}$ for $w \in H^{1/2}(\partial\Omega)$. They showed that $c_0 \leqq 1$. (These constants are slightly different from those in the original papers. For more information, see [SW, Theorem 3.2].) In particular, for the negative eigenvalue $\underline{\lambda}$,

$$\underline{\lambda} \geqq -\sqrt{1 - c_0}.$$

Thus, the shape-dependent constant c_0 controls the eigenvalue $\underline{\lambda}$. Pechstein recently found the lower bound of c_0 by using the isoperimetric constant $\gamma(\Omega)$ and Sobolev extension constants (see [Pe, Corollary 6.14], [KRW]). In the case of $\partial\Omega = S^2$, $c_0 = \frac{8}{9}$ and $\underline{\lambda} = -\sqrt{1 - c_0} = -\frac{1}{3}$.

4.1. **Asymptotic properties of $\sigma(K)$ for $n = 3$.** For the case of $n = 3$, D. Khavinson, M. Putinar and H. S. Shapiro briefly mentioned the following result: K is in the Schatten class $\mathcal{S}_p(L^2(\partial\Omega))$, $p > 2$ (see [KPS, p. 150]). We explain it in more detail for smooth $\partial\Omega$. Following [Ke, p. 303], the nature of the diagonal singularity of the kernel $\nu_y E(x,y)$ shows that

$$E_2(x,y) = \int_{\Omega_z} \nu_z E(z,x) \cdot \nu_z E(z,y) dS_z = A(x,y) + B(x,y)\log(|x-y|),$$

where $A(x,y), B(x,y) \in C^\infty(\partial\Omega \times \partial\Omega)$. Since $E_2(x,y) \in H^{\mu_1,\mu_2}_{x,y}$ with $\mu_1 + \mu_2 < 1$, applying Theorem 2.14 to E_2 yields

$$K^*K \ \in \mathcal{S}_r(L^2(\partial\Omega)) \quad \text{for } r > \frac{4}{2+2} = 1.$$

This means that K is in the Schatten-von Neumann class $\mathcal{S}_p(L^2(\partial\Omega))$, $p > 2$.

Note that the regularity of $A(x,y)$ and $B(x,y)$ is essential to this result. Immediately, a decay rate of $\sigma_p(K)$ is obtained.

Remark 4.1. Let $n = 3$ and let Ω be a smooth region. For $\alpha > -\frac{1}{2}$,

$$\lambda_j = o(j^\alpha) \quad \text{as } j \to \infty.$$

In the case of a sphere, this is the best possible estimate.

4.2. **Isoperimetric properties of K.** We want to characterize the isoperimetric properties by $\sigma_p(K)$. For the case of $n = 3$, however, the explicit formula is yet to be obtained. In this subsection, some expected properties and conjectures are introduced. Seeing the case of $n = 2$, $-\underline{\lambda}$ and the Schatten norm are expected to be minimized by $\partial\Omega = S^2$. Hence, we expect the following conjectures:

Conjecture 1. *Let $n = 3$ and $\underline{\lambda} \equiv \min \sigma_p(K)\backslash\{-1\}$. We have*

$$\sup_{\partial\Omega} \underline{\lambda} = -\frac{1}{3},$$

where the supremum is taken over all C^∞ simply connected closed surfaces. The supremum is achieved if and only if $\partial\Omega = S^2$.

For the case of $\partial\Omega = S^2$, $\underline{\lambda} = -\frac{1}{3}$ is obtained by direct calculations.

Conjecture 2. *Let $n = 3$. For $p > 1$,*

$$\inf_{\partial\Omega} \operatorname{tr}\{(K^*K)^p\} = \left(1 - \frac{1}{2^{2p-1}}\right)\zeta(2p-1),$$

where the infimum is taken over all C^∞ simply connected closed surfaces and $\zeta(x)$ is the Riemann zeta function. The infimum is achieved if and only if $\partial\Omega = S^2$.

To confirm the validity of the conjectures, henceforth, we consider the case of ellipsoids. For the case of ellipsoids $\{(x,y,z) \in \mathbf{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$, Ritter [Ri1], [Ri2] showed that $\sigma_p(K)$ is completely solved by ellipsoidal harmonics (Lamé polynomials); there are exactly $2l+1$ linearly independent Lamé polynomials of order $l \geq 0$ (see [H]). In addition, Martensen [Ma, Theorem 1] proved the following proposition.

Proposition 4.2. *For any $2l+1$ linearly independent Lamé polynomials of order $l \geq 0$, considered as eigenfunctions of K, the sum of the corresponding eigenvalues is equal to -1.*

We denote these eigenvalues by $\lambda_{k,l}$ ($k = 1, 2, \cdots, 2l+1$), and hence,

$$\sum_{k=1}^{2l+1} \lambda_{k,l} = -1.$$

Furthermore, deformation of the sphere into a triaxial ellipsoid yields to bifurcation $-\frac{1}{2l+1}$ into $2l+1$ different eigenvalues of order l, say $\lambda_{k,l}$, $k = 1, 2, \cdots, 2l+1$, each with multiplicity one (see [Ri2]).

Consequently, proofs of the conjectures for ellipsoids are provided.

Theorem 4.3. *Let $n = 3$ and $p > 1$. For the case of ellipsoids $\partial\Omega$,*

$$\sup_{\partial\Omega} \underline{\lambda} = -\frac{1}{3} \quad \textit{and} \quad \inf_{\partial\Omega} \mathrm{tr}\{(K^*K)^p\} = \left(1 - \frac{1}{2^{2p-1}}\right)\zeta(2p-1).$$

The supremum and infimum are achieved if and only if $\partial\Omega = S^2$.

Proof. For $l = 1$,

$$\lambda_{1,1} + \lambda_{2,1} + \lambda_{3,1} = -1.$$

Thus, $\underline{\lambda} \leqq \min(\lambda_{1,1}, \lambda_{2,1}, \lambda_{3,1}) \leqq -\frac{1}{3}$. Equality holds if and only if $\lambda_{1,1} = \lambda_{2,1} = \lambda_{3,1} = -\frac{1}{3}$. Thus, we have the first equation.

To prove the second equation, we note that from Hölder's inequality

$$1 = |(1, 1, \cdots, 1) \cdot (\lambda_{1,l}, \lambda_{2,l}, \cdots, \lambda_{2l+1,l})| \leqq (2l+1)^{(2p-1)/2p} (\sum_{k=0}^{2l+1} |\lambda_{k,l}|^{2p})^{1/2p}.$$

This leads to $s_l = \sum_{k=0}^{2l+1} |\lambda_{k,l}|^{2p} \geqq \left(\frac{1}{2l+1}\right)^{2p-1}$. Remarking that $(K^*K)^p$ is in a trace class, and using Weyl's inequality,

$$\begin{aligned}
\mathrm{tr}\{(K^*K)^p\} &\geqq \sum_{l=0}^{\infty} \sum_{k=0}^{2l+1} |\lambda_{k,l}|^{2p} \\
&= \sum_{l=0}^{\infty} s_l \\
&\geqq \sum_{l=0}^{\infty} \left(\frac{1}{2l+1}\right)^{2p-1} = \left(1 - \frac{1}{2^{2p-1}}\right)\zeta(2p-1)
\end{aligned}$$

as desired. □

For the general smooth surfaces, we mention equivalent statements of the conjectures. We infer the Schatten norm of single layer potentials:

$$\mathrm{tr}\{(K^*K)^p\} \leqq \mathrm{tr}\{(\frac{1}{4}S^*S)^p\} + \mathrm{tr}[\{(K - \frac{1}{2}S)^*(K - \frac{1}{2}S)\}^p].$$

The following theorem is also known ([KPS, Theorem 8]; see also [EKS], [Re], [Ra1], and [Ra2]).

Theorem 4.4. *For a ball in $\mathbf{R}^3$, the kernel of K is symmetric with $K = \frac{1}{2}S$, and balls are the only domains with this property.*

Thus, if one proves the single layer version of the above conjectures, we obtain the proof for the double layer potentials simultaneously.

5. Conclusion

Some fundamental properties of the eigenvalue and eigenfunctions of double layer potentials were discussed. Characteristic properties of a ball are provided by the Hilbert-Schmidt and Schatten norms of double layer potentials. The fundamental estimates of decay rates of eigenvalues are also provided by the regularity of the boundary.

With respect to eigenfunctions, the growth rates of nodal sets are characterized by the eigenvalues. Even less is known in the case of $n = 3$. We intend to discuss this in the future.

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