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TRANSITION MATRIX THEORY

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Dedicated to the memory of James Francis Reineck

ABSTRACT. In this article we present a unification of the theory of algebraic, singular, topological and directional transition matrices by introducing the (generalized) transition matrix which encompasses each of the previous four. Some transition matrix existence results are presented as well as verification that each of the previous transition matrices are cases of the (generalized) transition matrix. Furthermore we address how applications of the previous transition matrices to the Conley index theory carry over to the (generalized) transition matrix.

1. Introduction

The Conley index theory has been a valuable topological technique for detecting global bifurcations in dynamical systems [2], [1], [3], [4], [8], [7] and [11]. This index is a standard tool in the analysis of invariant sets in dynamical systems, and its significance owes partly to the fact that it is invariant under local perturbation of a flow (the continuation property). Typically, one does not investigate a single invariant set in a dynamical system but rather works with decompositions of a larger invariant set into invariant subsets and connecting orbits between them. The Morse decomposition is the standard such decomposition in the Conley index theory. Within the index theory there are matrices of maps defined between the Conley indices of invariant sets in a Morse decomposition, and these matrices (the connection matrices) provide information about connections that exist between sets in the decomposition. The connection matrices also have local invariance properties under continuation. Nevertheless, under global continuation sets of connection matrices can undergo change which usually means that the dynamical system has undergone a global bifurcation.

An initial approach to identify bifurcations via the interplay between local invariance and global change in connection matrices was due to Reineck in [14](1988). By introducing an artificial slow flow on the parameter space in a continuous family of dynamical systems, he obtained a map between the Conley indices of Morse decomposition invariant sets at the initial parameter value and those at the final parameter value in a continuation. This map is known as a singular transition matrix, and it has the feature that a nonzero entry can identify a change of the connecting orbit structure of the Morse decomposition under the continuation.

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Following Reineck's work on singular transition matrices, McCord and Mischaikow in [11](1992) used the continuation property of the Conley index, without introducing an artificial slow flow, to define matrices of maps between the Conley indices of Morse decomposition invariant sets at the initial and final parameter values in a continuation. Their maps, known as topological transition matrices, are naturally-defined maps on Conley indices that arise from the topological structure of invariant sets in the flow on the larger "phase-cross-parameter" space. In order to define these maps on the indices of the Morse decomposition sets, it was necessary to assume that connection matrices were trivial at the end parameter values. Nonetheless, as with the singular transition matrices, they were able to demonstrate that nontrivial topological transition matrix entries identify potential bifurcations that exist in the overall continuation. Furthermore, in [12] they established an equivalence between the singular transition matrices and the topological transition matrices in instances where both are defined. Later Franzosa, de Rezende and Vieira [5](2016) defined a new (general) topological transition matrix that extends the previous one, not requiring the assumption that the connection matrices are trivial at the end parameter values. In their case, the general topological transition matrix is defined to cover naturally-defined Conley index maps rather than being defined directly by them.

In [6](1995), Franzosa and Mischaikow introduced the concept of an algebraic transition matrix. Given that connection matrices for a Morse decomposition are not unique, they raised the question of whether the nonuniqueness could be understood via similarity transformations between connection matrices. Such transformations are algebraically defined, and - besides being associated with nonuniqueness of connection matrices for a particular Morse decomposition - can be exploited to track changes in connection matrices under flow continuation. They developed an existence result for algebraic transition matrices under the assumption of a "stackable" underlying partial order, and they demonstrated how algebraic transition matrices could also be used to identify global bifurcations in a dynamical system under continuation.

In [10] and [9], the authors developed the directional transition matrix, a transformation that is similar in nature to both the singular transition matrix and the topological transition matrix. As in the singular transition matrix case, a slow flow is added to the parameter space, but it is more general than the specific flow used in defining the singular transition matrix. The advantage to the more general approach is that it allows us to detect broader families of bifurcation orbits under continuation than those that are detected by the singular and topological transition matrices. The directional transition matrix is a transformation between indices of Morse decomposition sets at each end of the continuation, but not simply from those at one end of the continuation to those at the other (as in the other types of transition matrices). Instead, it maps the indices of those sets on either end that have an outward slow-flow direction to the indices of those sets with an inward slow-flow direction. As with the classical topological transition matrix, it is assumed that on each end of the continuation there are no connecting orbits between the Morse sets, so that natural flow-defined maps can be used to define the directional transition matrix. And, as in each of the above cases, the authors demonstrate how nontrivial directional transition entries identify bifurcations that occur under continuation.

While these four types of transition matrices are each defined differently and in different settings, they have in common that each is a Conley index based algebraic transformation that tracks changes in index information under continuation and thereby identifies global bifurcations that could occur during the continuation. It is natural to expect that the theories could be unified in an overarching transition matrix theory, and that is the main purpose of this paper. The basic idea for this general transition matrix is that it covers natural flow-defined index isomorphisms that arise under a continuation. We discuss this aspect of the transition matrix further at the start of the next section, and then we provide our definition of the transition matrix. First, though, we present briefly the necessary background material from Conley index theory, Morse decompositions, homology index braids, connection matrices, etc. (see [2], [4], [8], [7], [13] and [15]).

Throughout the paper $\mathbf{P}$ represents a finite set with a partial order $<$. An *interval* in $\mathbf{P}$ is a set $\mathbf{I} \subseteq \mathbf{P}$ which is such that if $p, q \in \mathbf{I}$ and $p < r < q$, then $r \in \mathbf{I}$. The set of intervals in $<$ is denoted by $\mathbf{I}(<)$.

An *adjacent n-tuple of intervals* in $<$ is an ordered collection $(\mathbf{I}_1, \ldots, \mathbf{I}_n)$ of mutually disjoint nonempty intervals in $<$ satisfying:

- $\bigcup_{i=1}^n \mathbf{I}_i \in \mathbf{I}(<)$;
- $\pi \in \mathbf{I}_j, \pi' \in \mathbf{I}_k$, $j < k$ imply $\pi' \not< \pi$.

The collection of adjacent n-tuples of intervals in $<$ is denoted $\mathbf{I}_n(<)$. An adjacent 2-tuple of intervals is also called an *adjacent pair* of intervals. If $<'$ is either an extension of $<$ or the restriction of $<$ to an interval in $<$, then $\mathbf{I}_n(<') \subseteq \mathbf{I}_n(<)$. If $(\mathbf{I}, \mathbf{J})$ is an adjacent pair (2-tuple) of intervals, then $\mathbf{I} \cup \mathbf{J}$ is denoted $\mathbf{IJ}$. If $(\mathbf{I}_1, \ldots, \mathbf{I}_n) \in \mathbf{I}_n(<)$ and $\bigcup_{i=1}^n \mathbf{I}_i = \mathbf{I}$, then $(\mathbf{I}_1, \ldots, \mathbf{I}_n)$ is called a decomposition of $\mathbf{I}$.

Definition 1.1. A graded module braid over $<$ is a collection $\mathcal{G} = \mathcal{G}(<)$ of graded modules and maps between the graded modules satisfying:

(1) for each $\mathbf{I} \in \mathbf{I}(<)$, there is a graded module $G(\mathbf{I})$,
(2) for each $(\mathbf{I}, \mathbf{J}) \in \mathbf{I}_2(<)$, there are maps: $i(\mathbf{I}, \mathbf{IJ}) : G(\mathbf{I}) \to G(\mathbf{IJ})$ of degree 0; $p(\mathbf{IJ}, \mathbf{J}) : G(\mathbf{IJ}) \to G(\mathbf{J})$ of degree 0; $\partial(\mathbf{J}, \mathbf{I}) : G(\mathbf{J}) \to G(\mathbf{I})$ of degree 1, that satisfy:
   - $\cdots \to G(\mathbf{I}) \overset{i}{\to} G(\mathbf{IJ}) \overset{p}{\to} G(\mathbf{J}) \overset{\partial}{\to} G(\mathbf{I}) \to \cdots$ is exact,
   - if $\mathbf{I}$ and $\mathbf{J}$ are noncomparable, then $p(\mathbf{JI}, \mathbf{I}) i(\mathbf{I}, \mathbf{IJ}) = id_{|G(\mathbf{I})}$,
   - if $(\mathbf{I}, \mathbf{J}, \mathbf{K}) \in \mathbf{I}_3(<)$, then the braid diagram shown in Figure 1 commutes.

Assume that $\mathcal{G}$ and $\mathcal{G}'$ are graded module braids over $<$.

Definition 1.2. An r-degree map $\theta : \mathcal{G} \to \mathcal{G}'$ is a collection $\{\theta(\mathbf{I})\}_{\mathbf{I} \in I(<)}$ of r-degree module homomorphisms $\theta(\mathbf{I}) : G(\mathbf{I}) \to G'(\mathbf{I})$ such that the following diagram commutes for each $(\mathbf{I}, \mathbf{J}) \in \mathbf{I}_2(<)$:

$$\begin{tikzcd}
\cdots \arrow[r] & G_k(\mathbf{I}) \arrow[r, "i"] \arrow[d, "\theta(\mathbf{I})"] & G_k(\mathbf{IJ}) \arrow[r, "p"] \arrow[d, "\theta(\mathbf{IJ})"] & G_k(\mathbf{J}) \arrow[r, "\partial_\lambda(\mathbf{J},\mathbf{I})"] \arrow[d, "\theta(\mathbf{J})"] & G_{k-1}(\mathbf{I}) \arrow[r] \arrow[d, "\theta(\mathbf{I})"] & \cdots \\
\cdots \arrow[r] & G'_{k-r}(\mathbf{I}) \arrow[r, "i"] & G'_{k-r}(\mathbf{IJ}) \arrow[r, "p"] & G'_{k-r}(\mathbf{J}) \arrow[r, "\partial_\mu(\mathbf{J},\mathbf{I})"] & G'_{k-r-1}(\mathbf{I}) \arrow[r] & \cdots
\end{tikzcd}$$

If, futhermore, $\theta(\mathbf{I})$ is an isomorphism for each $\mathbf{I} \in \mathbf{I}(<)$, then θ is called an r-degree isomorphism and $\mathcal{G}$ and $\mathcal{G}'$ are said to be r-isomorphic.

$$\begin{tikzcd}
 & {} \arrow[dl, dashed] \arrow[dr, dashed] & \\
G(I) \arrow[dr, "i"] \arrow[dd, bend right, "i"'] & & G(K) \arrow[dl, "\partial"'] \arrow[dd, bend left, "\partial"] \\
 & G(IJ) \arrow[dl, "i"'] \arrow[dr, "p"] & \\
G(IJK) \arrow[dr, "p"] \arrow[dd, bend right, "p"'] & & G(J) \arrow[dl, "i"'] \arrow[dd, bend left, "\partial"] \\
 & G(JK) \arrow[dl, "p"'] \arrow[dr, "\partial"] & \\
G(K) \arrow[dr, "\partial"] \arrow[dd, bend right, "\partial"'] & & G(I) \arrow[dl, "i"'] \arrow[dd, bend left, "i"] \\
 & G(IJ) \arrow[dl, "p"'] \arrow[dr, "i"] & \\
G(J) \arrow[dr, dashed] \arrow[d, dashed] & & G(IJK) \arrow[dl, dashed] \arrow[d, dashed] \\
{} & {} & {}
\end{tikzcd}$$

FIGURE 1

$$\begin{tikzcd}
C(I) \arrow[dr, "i"] \arrow[dd, bend right, "i"'] & & \\
 & C(IJ) \arrow[dl, "i"'] \arrow[dr, "p"] & \\
C(IJK) \arrow[dr, "p"] \arrow[dd, bend right, "p"'] & & C(J) \arrow[dl, "i"'] \\
 & C(JK) \arrow[dl, "p"'] & \\
C(K) & &
\end{tikzcd}$$

FIGURE 2

Definition 1.3. A chain complex braid over $<$ is a collection $C = C(<)$ of chain complexes and chain maps satisfying:

(1) for each $\mathbf{I} \in \mathbf{I}(<)$, there is a chain complex $C(\mathbf{I})$,
(2) for each $(\mathbf{I}, \mathbf{J}) \in \mathbf{I}_2(<)$, there are 0 degree maps $i(\mathbf{I}, \mathbf{IJ}) : C(\mathbf{I}) \to C(\mathbf{IJ})$ and $p(\mathbf{IJ}, \mathbf{J}) : C(\mathbf{IJ}) \to C(\mathbf{J})$ which satisfy:
   - $C(\mathbf{I}) \xrightarrow{i} C(\mathbf{IJ}) \xrightarrow{p} C(\mathbf{J})$ is weakly exact,
   - if $\mathbf{I}$ and $\mathbf{J}$ are noncomparable, then $p(\mathbf{JI}, \mathbf{I})i(\mathbf{I}, \mathbf{IJ}) = id_{|C(\mathbf{I})}$,
   - if $(\mathbf{I}, \mathbf{J}, \mathbf{K}) \in \mathbf{I}_3(<)$, then the braid diagram shown in Figure 2 commutes.

Now assume that C and C' are chain complex braids over $<$.

Definition 1.4. An r-degree chain map $T : C \to C'$ is a collection of r-degree maps $T(\mathbf{I}) : C(\mathbf{I}) \to C'(\mathbf{I}), \mathbf{I} \in \mathbf{I}(<)$, such that for each $(\mathbf{I}, \mathbf{J}) \in \mathbf{I}_2(<)$ the following diagram commutes:

$$\begin{tikzcd}
C_k(\mathbf{I}) \arrow[r, "i"] \arrow[d, "T(\mathbf{I})"] & C_k(\mathbf{IJ}) \arrow[r, "p"] \arrow[d, "T(\mathbf{IJ})"] & C_k(\mathbf{J}) \arrow[d, "T(\mathbf{J})"] \\
C'_{k-r}(\mathbf{I}) \arrow[r, "i"] & C'_{k-r}(\mathbf{IJ}) \arrow[r, "p"] & C'_{k-r}(\mathbf{J})
\end{tikzcd}$$

Assume throughout the paper that if the degree of a map is not specified, then it is 0-degree. The following proposition guarantees sufficient conditions for a map $T : C \to C'$ to be a chain map.

Proposition 1.1. *Let $C = \{C(p)\}_{p\in\mathbf{P}}$ and $C' = \{C'(p)\}_{p\in\mathbf{P}}$ be collections of graded modules, and $\Delta : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C(p)$ and $\Delta' : \bigoplus_{\mathbf{P}} C'(p) \to \bigoplus_{\mathbf{P}} C'(p)$ be $<$-upper triangular boundary maps. If $T : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C'(p)$ is $<$-upper triangular and $T\Delta = \Delta' T$, then $\{T(\mathbf{I})\}_{\mathbf{I}\in\mathbf{I}(<)}$ is a chain map from $\mathcal{C}\Delta$ to $\mathcal{C}\Delta'$.*

Let φ be a continuous flow on a locally compact Hausdorff space and let S be a compact invariant set under φ. A *Morse decomposition* of S is a collection of mutually disjoint compact invariant subsets of S,

$$\mathcal{M}(S) = \{M(\pi) \mid \pi \in \mathbf{P}\}$$

indexed by a finite set $\mathbf{P}$, where each set $M(p)$ is called a *Morse set.* A partial order $<$ on $\mathbf{P}$ is called an *admissible ordering* if for $x \in S\setminus \bigcup_{\pi\in\mathbf{P}} M(\pi)$ there exists $p < q$ such that $\alpha(x) \subseteq M(q)$ and $\omega(x) \subseteq M(p)$. The flow defines an admissible ordering of M, called the *flow ordering* of M, denoted $<_F$, and such that $M(\pi) <_F M(\pi')$ if and only if there exists a sequence of distinct elements of $P : \pi = \pi_0, \dots, \pi_n = \pi'$, where $C(M(\pi_j), M(\pi_{j-1}))$, the set of connecting orbits from $M(\pi_j)$ to $M(\pi_{j-1})$, is nonempty for each $j = 1, \dots, n$. Note that every admissible ordering of M is an extension of $<_F$.

In the Conley theory one begins with the Conley index for isolated invariant sets, i.e., $S \subseteq X$ is an *isolated invariant set* if there exists a compact set $N \subseteq X$ such that $S \subseteq intN$ and

$$S = Inv(N, \varphi) = \{x \in N|\ \mathcal{O}(x) \subseteq N\}.$$

The *homological Conley index of* S, $CH_*(S)$ is the homology of the pointed space (N/L), where (N, L) is an index pair for S.

Given a Morse decomposition $\mathcal{M}(S)$, setting

$$M(\mathbf{I}) = \bigcup_{\pi\in\mathbf{I}} M(\pi) \cup \bigcup_{\pi,\pi'\in\mathbf{I}} C(M(\pi'), M(\pi)),$$

the Conley index of $M(\mathbf{I})$, $CH_*(M(\mathbf{I}))$, in short $H_*(\mathbf{I})$, is well defined, since $M(\mathbf{I})$ is an isolated invariant set for all $\mathbf{I} \in \mathbf{I}(<)$.

The existence of an admissible ordering on $\mathcal{M}(S)$ implies that any recurrent dynamics in S must be contained within the Morse sets, thus the dynamics off the Morse sets must be gradient-like. For this reason, Conley index theory refers to the dynamics within a Morse set as local dynamics and off the Morse sets as global dynamics. We briefly introduce the connection matrix theory, which addresses this latter aspect.

Definition 1.5. Given $\mathcal{G}$, a graded module braid over $<$, and $C = \{C(p)\}_{p\in\mathbf{P}}$, a collection of graded modules, let $\Delta : \bigoplus_{p\in\mathbf{P}} C(p) \to \bigoplus_{p\in\mathbf{P}} C(p)$ be a $<$-upper triangular boundary map. Then:

(1) If $\mathcal{H}$, the graded module braid generated by Δ, is isomorphic to $\mathcal{G}$, then Δ is called a $C-$***connection matrix*** of $\mathcal{G}$.
(2) If, furthermore, $C(p)$ is isomorphic to $G(p)$ for each $p \in \mathbf{P}$, then Δ is called a ***connection matrix*** of $\mathcal{G}$.

To simplify notation, for $\mathbf{I} \in \mathbf{I}(<)$ we denote $\bigoplus_{\pi\in\mathbf{I}} C(\pi)$ by $C(\mathbf{I})$, and the corresponding homology module in $\mathcal{H}\Delta$ by $H(\mathbf{I})$. In particular, the homology index braid of an admissible ordering of a Morse decomposition $\mathcal{G} = \{H_*(\mathbf{I})\}_{\mathbf{I}\in\mathbf{I}(<)}$ is an example of a graded module braid. In this setting a $<$-upper triangular boundary map

$$\Delta : \bigoplus_{\pi\in\mathbf{P}} CH_*(M(\pi)) \to \bigoplus_{\pi\in\mathbf{P}} CH_{*-1}(M(\pi))$$

satisfying Definition 1.5 for $\mathcal{C}\Delta = \{CH_*(M(\pi))\}_{\pi\in\mathbf{P}}$ is called the *connection matrix for a Morse decomposition.* Moreover, let $\mathcal{CM}(<)$ denote the set of all connection matrices for a given ($<$-ordered) Morse decomposition $\mathcal{M}(S)$.

One of the key features in Conley theory is its invariance under continuation. Since the connection matrices for Morse decompositions are algebraically derived from the homology Conley index braid, this seems to indicate that connecting orbits that persist over open sets in parameter space are identified by connection matrices. We now define Conley index continuation.

Let Γ be a Hausdorff topological space, Λ a compact, locally contractible, connected metric space and X a locally compact metric space. Assume that $X\times\Lambda \subseteq \Gamma$ is a local flow. $X \times \Lambda$ is called a parameterized flow if for each $\lambda \in \Lambda$, $X \times \lambda$ is a local flow.

Lemma 1.1 (Salamon)**.** *For any compact set $N \subseteq X$ the set $\Lambda(N) = \{\lambda \in \Lambda \mid N\times\lambda$ is an isolating neighborhood in $X \times \lambda\}$ is open in Λ.*

Definition 1.6. The space of isolated invariant sets is

$$\mathscr{S} = \{S \times \lambda \mid \lambda \in \Lambda \text{ and } S \times \lambda \text{ is an isolated invariant compact set in } X \times \lambda\}.$$

For all compact sets $N \subseteq X \times \Lambda$ define the map $\varrho_N : \Lambda(N) \to \mathscr{S}$ by $\varrho_N(\lambda) = Inv(N \times \lambda)$. Then consider the topology on the space $\mathscr{S}$ generated by the sets $\{\varrho_N(U) \mid N \subseteq X \times \Lambda \text{ compact}, U \subseteq \Lambda(N) \text{ open}\}$.

A map $\gamma : \Lambda \to \mathscr{S}$ is called a *section* of the space of isolated invariant sets if $\Pi_\Lambda \circ \gamma = id|_\Lambda$.

We are interested in the situation where the homology index braids of admissible orderings of Morse decompositions at parameters λ and μ are isomorphic. That is, it is not enough that a Morse decomposition continues over Λ, it must also continue with a partial order, more specifically:

Definition 1.7. Let $\mathcal{M}(S) = \{M(\pi) \mid \pi \in (\mathbf{P}, <)\}$ be an ordered Morse decomposition of the isolated invariant set $S \subseteq X \times \Lambda$. Let $M_\lambda = \{M_\lambda(\pi)\}_{\pi\in(\mathbf{P},<_\lambda)}$, $M_\mu = \{M_\mu(\pi)\}_{\pi\in(\mathbf{P},<_\mu)}$, S_λ and S_μ be the sets obtained by intersection of $\mathcal{M}(S)$ and S by the fibers $X \times \lambda$ and $X \times \mu$, respectively, where $<_\nu$ is the restriction of the order $<$ to the parameter $\nu \in \Lambda$.

- We say that $\mathcal{M}(S)$ with its order $<$ continues over Λ if there exist sections σ and $\varsigma_\pi : \Lambda \to \mathscr{S}$ such that $\{\varsigma_\pi(\nu) | \, \pi \in (\mathbf{P}, <_\nu)\}$ is a Morse decomposition for $\sigma(\nu)$, $\forall\nu \in \Lambda$.

- If, furthermore, there exists a path $\omega : [0,1] \to \Lambda$ from λ to μ; $\sigma(\lambda) = S_\lambda$; $\sigma(\mu) = S_\mu$; $\varsigma_\pi(\lambda) = M_\lambda(\pi)$; $\varsigma_\pi(\mu) = M_\mu(\pi)$; and if $\mathcal{M}(S)$ continues at least over $\omega([0,1])$, then we say that the admissible orderings $<_\lambda$ and $<_\mu$ are related by continuation or continue from one to the other. See Figure 3.

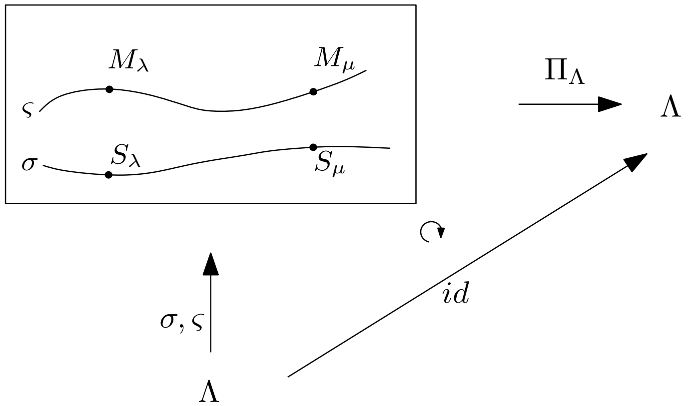

Figure 3. Sections from Definition 1.7

The following lemma is helpful to understand the previous Definition 1.7.

Lemma 1.2 (McCord, Mischaikow, Salamon)**.**

- *Let* $\gamma : \Lambda \to \mathscr{S}$ *be a section. Then* γ *is continuous if and only if*
$$S = \bigcup_{\lambda \in \Lambda} \gamma(\lambda)$$
is an isolated invariant set in $X \times \Lambda$*.*
- *Let*
$$S = \bigcup_{\lambda \in \Lambda} \sigma(\lambda), \quad M(\pi) = \bigcup_{\lambda \in \Lambda} \varsigma_\pi(\lambda) \text{ for any } \pi \in \mathbf{P}.$$
Then, S *is an isolated invariant set in* $X \times \Lambda$ *under* ϕ *and* $\mathcal{M}(S) = \{M(\pi) \mid \pi \in (\mathbf{P}, <)\}$ *is its Morse decomposition if, and only if,* $\mathcal{M}(S)$ *with its order continues.*

Suppose that S_0 and S_1 are invariant sets related by continuation in X_{λ_0} and X_{λ_1}. Hence, there exists a map $\omega : [0,1] \to \Lambda$ such that $\omega(0) = \lambda_0$ and $\omega(1) = \lambda_1$ and an isolated invariant set S over $\omega(I)$ such that $S_{\lambda_i} = S_i$. The inclusion $f_i : X_{\lambda_i} \to X \times \omega(I)$ induces an isomorphism $CH_*(S_i) \xrightarrow{f_{i*}} CH_*(S)$, where $CH_*(S_i)$ and $CH_*(S)$ indicates the Conley homology indices of S_i in X_{λ_i} and of S in $X \times \omega(I)$, respectively. Thus, there is an isomorphism, called the ***Conley flow-defined isomorphism***

$$F_\omega : CH_*(S_0) \xrightarrow{f_{1*}^{-1} \circ f_{0*}} CH_*(S_1),$$

that depends on the endpoint-preserving homotopy class ω. If $\pi_1(\Lambda) = 0$, then F_ω is independent of the path ω and one writes F_{λ_1,λ_2} instead of F_ω. The Conley flow-defined isomorphism is well-behaved with respect to composition of paths: $F_{\lambda,\lambda} = id$, $F_{\mu,\nu} \circ F_{\lambda,\mu} = F_{\lambda,\nu}$ and $F_{\lambda,\mu} = F^{-1}_{\mu,\lambda}$. For more details see [12] and [15]. To simplify notation, we denote $CH_*(M_\nu(\mathbf{I})) = H_{*,\nu}(\mathbf{I})$ or just $CH(M_\nu(\mathbf{I})) = H_\nu(\mathbf{I})$, where $\mathbf{I} \in \mathbf{I}(<_\nu)$ and $\nu \in \{\lambda, \mu\}$.

2. Transition matrix

Given a Morse decomposition of an isolated invariant set S in a flow, an associated homology index braid contains a significant amount of algebraic information about the overall structure of the Morse sets and connecting orbits in S. Generally speaking, the homology index braid is a relatively unmanageable structure to examine and analyze in order to draw conclusions about the structure of S. Overcoming this difficulty is part of the motivation for the connection matrix. The connection matrix is a matrix of maps between the indices of the Morse sets that produces a graded module braid that is isomorphic to the homology index braid. In this way we can think of the connection matrix as "covering" the homology index braid. Consequently, it is reasonable to expect that the connection matrix itself contains information from which the invariant set structure can be understood.

The transition matrices discussed in the introduction can be viewed similarly, and this perspective motivates our general transition matrix defined below. Given an invariant set and ordered Morse decomposition that continues over a parameter interval, there is a family of continuation-defined isomorphisms between indices of the invariant subsets that continue. If bifurcations occur during the continuation, then this will be reflected in a change in the flow-defined mappings between the homology index braids on either end of the continuation. The idea behind the general transition matrix is that it is a matrix of mappings between indices of Morse sets from either end of the continuation that "covers" the continuation-defined isomorphisms. By covering the continuation-defined isomorphisms, the transition matrix then assists in detecting change that occurs in the homology index braids, and thus it is reasonable to expect that the transition matrix contains information revealing change in the invariant set structure under continuation.

In this section we introduce our general definition of the transition matrix, and we address some straightforward properties of it. It is important to have the viewpoint that the transition matrix theory that we present here is an algebraic theory. In fact, the same could be said for the various previous versions of transition matrices. The primary motivation for introducing and studying transition matrices has been in applications to the Conley index theory, but there are important questions that need to be addressed regarding algebraic aspects of transition matrices. In section 3 we show how previous transition matrix existence results can be adapted to provide specific existence results for the transition matrix introduced here. In sections 4-7 we examine applications of the transition matrix in the Conley index theory, demonstrating how previous results using specific versions of the transition matrix carry over to our general setting.

Definition 2.1. Given chain complex braids $\mathcal{C}$ and $\mathcal{C}'$ and graded module braids $\mathcal{G}$ and $\mathcal{G}'$, an r-degree chain map $\mathcal{T} : \mathcal{C} \to \mathcal{C}'$ is said to **cover** an r-degree isomorphism $\theta : \mathcal{G} \to \mathcal{G}'$ (relative to Φ and Φ') if for all $\mathbf{I} \in \mathbf{I}(<)$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{HC}(\mathbf{I}) \arrow[r, "\mathcal{T}_*(\mathbf{I})"] \arrow[d, "\Phi(\mathbf{I})"'] & \mathcal{HC}'(\mathbf{I}) \arrow[d, "\Phi'(\mathbf{I})"] \\
\mathcal{G}(\mathbf{I}) \arrow[r, "\theta(\mathbf{I})"] & \mathcal{G}'(\mathbf{I})
\end{tikzcd}$$

where $\mathcal{T}_*(\mathbf{I})$ is the homology map induced from the chain map $\mathcal{T}(\mathbf{I})$, and $\Phi : \mathcal{HC} \to \mathcal{G}$ and $\Phi' : \mathcal{HC}' \to \mathcal{G}'$ are graded module braid isomorphisms.

Definition 2.2. If, in Definition 2.1, $\mathcal{C}$ and $\mathcal{C}'$ arise from connection matrices $\Delta : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C(p)$ and $\Delta' : \bigoplus_{\mathbf{P}} C'(p) \to \bigoplus_{\mathbf{P}} C'(p)$, respectively, and $\mathcal{T}$ arises from a matrix $T : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C'(p)$, then T is called an ***r*-degree (*generalized*) transition matrix** for Δ and Δ'.

Beyond this point in the paper, "transition matrix" refers to the transition matrix as defined here unless we specifically refer to a particular previous type such as "algebraic transition matrix", "topological transition matrix", etc. Also, if no degree is specified, assume that a transition matrix is a 0-degree map.

Theorem 2.1. *Let $\Delta : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C(p)$ and $\Delta' : \bigoplus_{\mathbf{P}} C'(p) \to \bigoplus_{\mathbf{P}} C'(p)$ be connection matrices of $\mathcal{G}$ and $\mathcal{G}'$, respectively. Let $T : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C'(p)$ be a transition matrix for Δ and Δ'. Assume that $\Phi : \mathcal{HC} \to \mathcal{G}$ and $\Phi' : \mathcal{HC}' \to \mathcal{G}'$ are graded module braid isomorphisms, and that T covers an isomorphism θ (relative to Φ and Φ'). Then the transition matrix T satisfies the following properties:*

(i) *$T \circ \Delta = \Delta' \circ T$;*
(ii) *$T(\{p\})$ is an isomorphism and T is upper triangular with respect to $<$;*
(iii) *T is an isomorphism;*
(iv) *T^{-1} covers θ^{-1} (relative to Φ' and Φ). Moreover, suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ cover $\theta_1 : \mathcal{G} \to \mathcal{G}'$ (relative to Φ and Φ') and $\theta_2 : \mathcal{G}' \to \mathcal{G}''$ (relative to Φ' and Φ''), and arise from T_1 and T_2, respectively. Then $\mathcal{T}_2 \circ \mathcal{T}_1$ covers $\theta_2 \circ \theta_1$ (relative to Φ and Φ'') and arises from $T_2 \circ T_1$.*

Proof. (i) Since $\Delta(\mathbf{I})$ and $\Delta'(\mathbf{I})$ are boundary maps and $T(\mathbf{I})$ is a chain map, we have that $T(\mathbf{I}) \circ \Delta(\mathbf{I}) = \Delta'(\mathbf{I}) \circ T(\mathbf{I})$ for all $\mathbf{I}$.

(ii) Change F to θ in the proof of Theorem 2.8 in [5].

(iii) By item (ii) we have that T is an isomorphism on the diagonal and upper triangular, thus T is an isomorphism.

(iv) It follows directly by the composition of the maps. □

The following theorem states that the properties (i) and (ii) of Theorem 2.1 are sufficient conditions for a chain map T to be a transition matrix.

Theorem 2.2. *Let $\Delta : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C(p)$ and $\Delta' : \bigoplus_{\mathbf{P}} C'(p) \to \bigoplus_{\mathbf{P}} C'(p)$ be connection matrices of $\mathcal{G}$ and $\mathcal{G}'$, respectively. If $T : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C'(p)$ is a map such that:*

- *$T \circ \Delta = \Delta' \circ T$,*
- *$T(\{p\})$ is an isomorphism for all $p \in \mathbf{P}$,*
- *T is upper triangular with respect to $<$,*

then T is a transition matrix. □

The proof of Theorem 2.2 is straightforward from Proposition 1.1. Furthermore, for $T(\{p\})$ being an isomorphism for all $p \in \mathbf{P}$ guarantees that T covers an isomorphism. More specifically, with Φ, Φ', and $\mathcal{T}$ as above, T covers $\theta = \Phi' \circ \mathcal{T}_* \circ \Phi^{-1}$.

As a consequence of Theorem 2.2, establishing the existence of transition matrices only requires proving the existence of T satisfying the requirements in the theorem. Such is the case with the existence of algebraic transition matrices discussed in the next section. Specifying a particular θ for T to cover is important in the application of transition matrices to parameterized flows. In that setting (for example, with singular and topological transition matrices) the transition matrix covers natural flow-defined Conley index isomorphisms and thereby potentially contains information about properties of the parameterized flow.

3. Algebraic existence results

As indicated in the previous section, the transition matrix theory is an algebraic theory, and there are important questions to be addressed regarding the theory solely from the algebraic viewpoint. The most basic is the existence question: Given connection matrices Δ and Δ', does there exist a transition matrix for them? The answer is unknown for this most general situation, however, the previous existence results for the specific types of transition matrices provide us with existence results under appropriate conditions. In this section, we present a few such results.

To begin, we consider the situation that is the algebraic basis behind the existence of topological transition matrices in [11]. In this case we have trivial connection matrices $\Delta : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C(p)$ and $\Delta' : \bigoplus_{\mathbf{P}} C'(p) \to \bigoplus_{\mathbf{P}} C'(p)$. So, for each $\mathbf{I} \in \mathcal{I}(<)$ we have $\mathcal{HC}(\mathbf{I}) = \bigoplus_{\mathbf{I}} C(p)$ and $\mathcal{HC}'(\mathbf{I}) = \bigoplus_{\mathbf{I}} C'(p)$. Suppose that the associated graded module braid isomorphisms are $\Phi : \mathcal{HC} \to \mathcal{G}$ and $\Phi' : \mathcal{HC}' \to \mathcal{G}'$. Then for each $\mathbf{I} \in \mathcal{I}(<)$, the corresponding isomorphisms are mappings $\Phi(\mathbf{I}) : \bigoplus_{\mathbf{I}} C(p) \to \mathcal{G}(\mathbf{I})$ and $\Phi'(\mathbf{I}) : \bigoplus_{\mathbf{I}} C'(p) \to \mathcal{G}'(\mathbf{I})$. We have the following transition matrix existence result for this case. The proof is straightforward.

Proposition 3.1. *Let Δ and Δ' be connection matrices of $\mathcal{G}$ and $\mathcal{G}'$, respectively. Assume that Δ and Δ' are trivial and that $\Phi : \mathcal{HC} \to \mathcal{G}$ and $\Phi' : \mathcal{HC}' \to \mathcal{G}'$ are graded module braid isomorphisms. If $\theta : \mathcal{G} \to \mathcal{G}'$ is an isomorphism between graded module braids, then $T = \Phi^{-1}(\mathbf{P}) \circ \theta(\mathbf{P}) \circ \Phi(\mathbf{P}) : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C'(p)$ is a transition matrix for Δ and Δ', and T covers θ relative to Φ and Φ'.* ☐

Next we address an existence result that is based on the algebraic transition matrix existence result in [6]. In this case the result holds where the underlying partial order is of a particular type.

A partial order $<$ on $\mathbf{P}$ is said to be *N-free* if $\mathbf{P}$ contains no N's, where an N in $<$ is a four-element subset $\{a, b, c, d\}$ of $\mathbf{P}$ such that the only relations in $<$ between a, b, c, and d are: $a < b, c < b$, and $c < d$.

Theorem 3.1. *Let $\Delta : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C(p)$ and $\Delta' : \bigoplus_{\mathbf{P}} C'(p) \to \bigoplus_{\mathbf{P}} C'(p)$ be connection matrices of $\mathcal{G}$ and $\mathcal{G}'$. If the order $<$ is N-free, then there is a transition matrix $T : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C'(p)$, for Δ and Δ'.*

Proof. It follows from Theorem 3.5 in [6] and Theorem 2.2. ☐

This theorem is the basis for the existence of algebraic transition matrices in [6].

Remark 3.1. Observe that Theorem 3.1 does not specify which isomorphism T covers. Nevertheless, the constructive proof of Theorem 3.5 [6] can be applied to any interval $\mathbf{I}$ such that $<_{\mathbf{I}}$ is N-free in order to obtain a chain map $T(\mathbf{I})$ that makes the following diagram commute for a given isomorphism $\theta(\mathbf{I})$:

$$\begin{array}{ccc} \mathcal{HC}(\mathbf{I}) & \xrightarrow{\mathcal{T}_*(\mathbf{I})} & \mathcal{HC}'(\mathbf{I}) \\ {\scriptstyle \Phi(\mathbf{I})}\downarrow & & \downarrow{\scriptstyle \Phi'(\mathbf{I})} \\ \mathcal{G}(\mathbf{I}) & \xrightarrow{\theta(\mathbf{I})} & \mathcal{G}'(\mathbf{I}). \end{array}$$

Even though such diagrams do not necessarily commute for subintervals of $\mathbf{I}$, it is possible to get information for $p, q \in \mathbf{P}$ by considering a subinterval $\mathbf{J}$ such that $<_{\mathbf{J}}$ is N-free and $p, q \in \mathbf{J}$.

Now we explore a special chain complex braid over a particular order $<_k$ that leads us to another existence result by using an approach arising from working with Morse-Smale gradient-like flows.

Consider a chain complex braid $C(<_k) = \{C(p)\}_{p\in(\mathbf{P},<_k)}$ over $<_k$ such that for all $p \in \mathbf{P}$ there is at most one $k(p)$ such that $C_{k(p)}(p) \neq 0$, and $<_k$ is a partial order that has the following property:

$$p <_k p' \Leftrightarrow k(p) < k(p'). \tag{3.1}$$

Such a chain complex braid $C(<_k)$ is called a *unidimensional chain complex braid.*

The next lemma is an obvious property that a unidimensional chain complex braid possesses, but it plays the main role in the proof of the existence and uniqueness theorem of a transition matrix for this setting.

Lemma 3.1. *If $C(<_k) = \{C(p)\}_{p\in(\mathbf{P},<_k)}$ is a unidimensional chain complex braid that arises from a connection matrix $\Delta : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C(p)$ of $\mathcal{G}(<_k)$, then $\Delta(p,p') = 0$ for all $p, p' \in \mathbf{P}$ such that $|k(p) - k(p')| \neq 1$ or $p' < p$.*

Proof. Since Δ is a boundary map and $C(p) = C'(p) = 0$ except in dimension $k(p)$ and $k(p')$, we obtain that $\Delta(p,p') = 0$ when $|k(p) - k(p')| \neq 1$. If $p' < p$, it follows by upper triangularity of Δ that $\Delta(p,p') = 0$. □

Theorem 3.2. *Let $C(<_k) = \{C(p)\}_{p\in(\mathbf{P},<_k)}$ and $C'(<_k) = \{C'(p)\}_{p\in(\mathbf{P},<_k)}$ be unidimensional chain complex braids that arise from connection matrices*

$$\Delta : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C(p) \text{ of } \mathcal{G}(<_k) \text{ and } \Delta' : \bigoplus_{\mathbf{P}} C'(p) \to \bigoplus_{\mathbf{P}} C'(p) \text{ of } \mathcal{G}'(<_k),$$

respectively. Assume that $C_\ell(p) \simeq C'_\ell(p)$ for all dimensions ℓ and for all $p \in \mathbf{P}$. Then given a graded module braid isomorphism $\theta : \mathcal{G} \to \mathcal{G}'$ there exists a unique transition matrix $T : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C'(p)$ that covers θ.

Proof. Without loss of generality, assume that there is no $p \in \mathbf{P}$ such that $C_\ell(p) = 0$ for all $\ell \in \mathbb{N} \cup \{0\}$. Let $\mathbf{I}_i$ be a subset of $\mathbf{P}$ such that $C_i(p) \neq 0$ for all $p \in \mathbf{I}_i$ and let $n \in \mathbb{N}$ be such that $\mathbf{P} = \mathbf{I}_0 \cup \cdots \cup \mathbf{I}_n$.

By Lemma 3.1, Δ and Δ' can be regarded as in block form where only the $\Delta(\mathbf{I}_i, \mathbf{I}_{i-1})$ and $\Delta'(\mathbf{I}_i, \mathbf{I}_{i-1})$ blocks, for all $i \in \{1, \dots, n\}$, could be nontrivial.

Restricting to $\mathbf{I}_i$, by Proposition 3.1 there exists a unique transition matrix $T(\mathbf{I}_i)$ between $\Delta(\mathbf{I}_i)$ and $\Delta'(\mathbf{I}_i)$ that covers $\theta(\mathbf{I}_i)$. If we define T to be block diagonal, equal to $T(\mathbf{I}_i)$ on the corresponding diagonal blocks, and trivial otherwise, then it is straightforward to show that T defines a chain map.

By Theorem 2.2, T is a transition matrix and it is unique since $T(\mathbf{I}_i)$ are unique for all $i = 0, \dots, n$. Furthermore, from the fact that $T(\mathbf{I}_i)$ covers $\theta(\mathbf{I}_i)$ for all $i = 0, \dots, n$, one can use the same approach as the proof of Theorem 3.1 in [5] to prove that T covers θ. □

Note that T obtained in the previous theorem is a matrix in block form as in Theorem 3.1 in [5]. Furthermore, it is possible to consider a generic partial order $<$ instead of $<_k$ in Theorem 3.2; nevertheless the proof splits in various cases where $<$ fails to be $<_k$. However, in such a case T may be neither unique nor in block form.

Observe that Theorem 3.2 can be applied in a more general context than Theorem 3.1 in [5]. For instance, the flow on parameters λ and μ do not need to be Morse-Smale without periodic orbits; we only need that the Morse decomposition has to meet the hypothesis that there is at most one $k(p)$ such that $CH_{k(p)}(M(p)) \neq 0$, for all $p \in \mathbf{P}$. Such a property is not satisfied in the presence of some simple Morse sets; for example repeller periodic orbits or attractor periodic orbits. However, this property is satisfied for critical point Morse sets, including situations where there are saddle-saddle connections between them.

4. Algebraic transition matrix

The algebraic transition matrix theory is developed in [6]. There the authors define the algebraic transition matrix in the setting of a parameterized family of flows after having earlier developed an algebraic theory that they employ. Their algebraic transition matrix is an example of a similarity transformation defined between connection matrices.

Let $\Delta : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C(p)$ be a connection matrix for a graded module braid $\mathcal{G}$ over a partial order $<$. If $T : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C'(p)$ is a $<$-upper triangular matrix such that $T(p)$ is an isomorphism for all $p \in \mathbf{P}$, then it follows that $\Delta' := T\Delta T^{-1}$ is also a connection matrix for $\mathcal{G}$. In this way (as referred to above) T can be thought of as a similarity transformation between connection matrices Δ and Δ'.

Here, for the purpose of unifying the various transition matrix theories–and being most general–we define the above-mentioned similarity transformation between connection matrices as follows:

Definition 4.1. Let $\Delta : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C(p)$ and $\Delta' : \bigoplus_{\mathbf{P}} C'(p) \to \bigoplus_{\mathbf{P}} C'(p)$ be connection matrices. If $T : \bigoplus_{\mathbf{P}} C(p) \to \bigoplus_{\mathbf{P}} C'(p)$ is $<$-upper triangular such that $T(p)$ is an isomorphism for all $p \in \mathbf{P}$ and such that $\Delta' T = T\Delta$, then T is called an **algebraic transition matrix** from Δ to Δ'.

The algebraic transition matrix as defined in [6] then is an example of an algebraic transition matrix in Definition 4.1 above. With the current definition of algebraic transition matrix, we then show that it is a transition matrix.

Note that, given Δ, Δ' and T as above, T is a chain map between $C\Delta$ and $C\Delta'$ (see Proposition 1.1). Furthermore T induces homology isomorphisms $T_*(\mathbf{I}) : H\Delta(\mathbf{I}) \to H\Delta'(\mathbf{I})$ for all $\mathbf{I} \in \mathbf{I}(<)$. Now, let Δ be a connection matrix and let $\Phi : H\Delta \to \mathcal{G}$ be a graded module braid isomorphism associated with Δ. Then we can define $\Phi'(\mathbf{I}) = \Phi(\mathbf{I}) \circ T_*^{-1}(\mathbf{I})$ for each $\mathbf{I} \in \mathbf{I}(<)$. The isomorphisms $\Phi'(\mathbf{I})$ define a graded module braid isomorphism Φ' between $H\Delta'$ and $\mathcal{G}$. Also, clearly, for each $I \in \mathbf{I}(<)$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}\Delta(\mathbf{I}) \arrow[r, "T_*(\mathbf{I})"] \arrow[d, "\Phi(\mathbf{I})"'] & \mathcal{H}\Delta'(\mathbf{I}) \arrow[d, "\Phi'(\mathbf{I})"] \\
\mathcal{G}(\mathbf{I}) \arrow[r, "id(\mathbf{I})"] & \mathcal{G}(\mathbf{I})
\end{tikzcd}$$

where $id(\mathbf{I})$ is the identity map on $\mathcal{G}(\mathbf{I})$. Thus T covers $id : \mathcal{G} \to \mathcal{G}$ (relative to Φ and Φ'), and it follows that T is a transition matrix.

It follows from above, that given a connection matrix Δ for a graded module braid $\mathcal{G}$, then other connection matrices for $\mathcal{G}$ can be obtained via transition matrices. The converse question is significant; that is, given connection matrices, Δ and Δ', is there a transition matrix for them? This question was addressed with some initial positive results in [6] and in the previous section 3.

Define ATM($<$) to be the set of all algebraic transition matrices that are similarity transformations between connection matrices $\Delta : \bigoplus_{(\mathbf{P},<)} C(p) \to \bigoplus C(p)$ and $\Delta' : \bigoplus_{(\mathbf{P},<)} C'(p) \to \bigoplus C'(p)$. Also define $<_m$ to be the *minimal order* such that $\mathcal{M}(S)$ continues over Λ.

From the existence theorems in section 3, one can improve Theorem 4.3 in [6].

Theorem 4.1. *Let $M_\lambda = \{M_\lambda(\pi)\}_{\pi \in (\mathbf{P},<_\lambda)}$ and $M_\mu = \{M_\mu(\pi)\}_{\pi \in (\mathbf{P},<_\mu)}$ be Morse decompositions; let Δ_λ and Δ_μ be their respective connection matrices. Moreover, assume that M_λ and M_μ are related by continuation with an admissible ordering $<$. If $\mathrm{ATM}(<_m) \neq \emptyset$ and $T_{\lambda,\mu}(p,q) \neq 0$ for all $T_{\lambda,\mu} \in \mathrm{ATM}(<)$, then there exists a finite sequence $0, s_1, s_2, \ldots, s_n, 1$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q,\ p_n = p$ and the set of connecting orbits $C\left(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i)\right)$ is nonempty.*

Proof. There is a $T_{\lambda,\mu} \in \mathrm{ATM}(<)$ such that $T_{\lambda,\mu} \in \mathrm{ATM}(<_m)$ since $<$ is an extension of the minimal order $<_m$ that continues. Thus $T_{\lambda,\mu}(p,q) \neq 0$ for the order $<_m$, i.e., $p <_m q$. Therefore, by definition of $<_m$, we have that $p <_m q$ implies that there exists a finite sequence $0, s_1, s_2, \ldots, s_n, 1$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q,\ p_n = p$ and the set of connecting orbits $C\left(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i)\right)$ is nonempty. □

Under the hypothesis of Theorem 4.1 we have the following two corollaries.

Corollary 4.1. *If the order is $<_k$ and $T_{\lambda,\mu}(p,q) \neq 0$, then there exists a finite sequence $0, s_1, s_2, \ldots, s_n, 1$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q,\ p_n = p$ and the set of connecting orbits $C\left(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i)\right)$ is nonempty.*

Proof. If the order is $<_k$ (see Lemma 3.1), then by Theorem 3.2 we have the existence of a unique algebraic transition matrix. Hence, by uniqueness in Theorem 3.2, we have $T_{\lambda,\mu}$ is the same for $<_k$ and $<_m$, since $<_k$ is an extension of the minimal order $<_m$ that continues. Thus, by Theorem 4.1, we have the result. □

Corollary 4.2. *If the order $<_m$ is N-free and $T_{\lambda,\mu}(p,q) \neq 0$ for all $T_{\lambda,\mu} \in \mathrm{ATM}(<)$, then there exists a finite sequence $0, s_1, s_2, \ldots, s_n, 1$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q,\ p_n = p$ and the set of connecting orbits $C\left(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i)\right)$ is nonempty.*

Proof. Since the order $<_m$ is N-free, by Theorem 3.1, we have that there exists an algebraic transition matrix $T_{\lambda,\mu}$ for the order $<_m$, hence $\mathrm{ATM}(<_m) \neq 0$. Therefore, the proof follows by Theorem 4.1. □

Notice that, under the hypotheses of Theorem 4.3 in [6], by choosing an adjacent pair $(\mathbf{I}_{j-1}, \mathbf{I}_j)$ such that $p \in \mathbf{I}_{j-1}$ and $q \in \mathbf{I}_j$, Theorem 4.3 in [6] is a particular case of Corollary 4.2.

The connecting-orbit existence result in Theorem 4.1 has corresponding results in theorems in the following sections. Those results are stronger, providing information about the sequence of the connecting orbits along the continuation. That they are stronger is partially a consequence of the fact that in those cases (in contrast to Theorem 4.1) the transition matrices are assumed to cover the flow-defined Conley index isomorphisms that naturally arise from the continuation of the flow across the parameter values.

5. Singular transition matrix

In most cases singular transition matrices can only be computed via the dynamics of the slow system, since these transition matrices are essentially just submatrices of a connection matrix; see [14]. Thus, in practice, one may find difficulty in obtaining those matrices if the objective is to understand the dynamics of the parametrized family. However, by showing that those matrices are transition matrices that cover an isomorphism, we actually can use the singular transition theory to assist in the development of other transition matrices, as we will show in the following sections.

Let

$$\dot{x} = f(x, \lambda), \tag{5.1}$$

be a parametrized family of ordinary differential equations defined in $\mathbb{R}^n$, where the parameter space $\Lambda = \mathbb{R}$. Assume that the Morse decomposition $\mathcal{M}(S_\lambda) = \{M_\lambda(p) \mid p \in \mathbf{P}\}$ continues over $\mathbb{R}$ and that connection matrices Δ_{-1} and Δ_1 for the Morse decompositions $\mathcal{M}(S_{-1})$ and $\mathcal{M}(S_1)$, respectively, are known. Moreover, let $N \subseteq \mathbb{R}^n$ be an isolating neighborhood for S_λ, $\lambda \in \mathbb{R}$.

One introduces slow dynamics in the parameter space of (5.1) with the purpose of comprehending the bifurcations that occur for $-1 < \lambda < 1$. Hence (5.1) can be written as:

$$\begin{aligned} \dot{x} &= f(x, \lambda), \\ \dot{\lambda} &= \epsilon(\lambda^2 - 1), \end{aligned} \tag{5.2}$$

where $\epsilon > 0$. Define

$$M(p^+) := M_1(p), \quad M(p^-) := M_{-1}(p)$$

and $M(p^\pm) := M(p^+) \cup M(p^-) \subseteq \mathbb{R}^n \times \{\pm 1\}$. For $\epsilon > 0$ sufficiently small, $N \times [-2, 2]$ is an isolating neighborhood for the flow ϕ_ϵ generated by (5.2). Let $K_\epsilon := Inv(N \times [-2,2], \phi_\epsilon)$. Now observe that since $\dot{\lambda} < 0$ for $\lambda \in (-1, 1)$, then for each $\epsilon > 0$,

$$\mathcal{M}(K_\epsilon) = \{M(p^\pm) \mid p \in \mathbf{P}\}$$

is a Morse decomposition, and there is an admissible ordering given by

$$\begin{gathered} q^- < p^+, \\ q^- < p^- \;\Leftrightarrow\; q <_{-1} p, \\ q^+ < p^+ \;\Leftrightarrow\; q <_1 p, \end{gathered}$$

where $<_{-1}$ and $<_1$ are admissible orderings for $\mathcal{M}(S_{-1})$ and $\mathcal{M}(S_1)$, respectively. Denoting a connection matrix for $\mathcal{M}(K_\epsilon)$ by Δ_ϵ, since the dynamics on the subspaces $\mathbb{R}^n \times \{\pm 1\}$ are given exactly by the flows generated by $\dot{x} = f(x, \pm 1)$, we have that

$$\Delta_\epsilon : \bigoplus_{p \in \mathbf{P}} H_*\left(M(p^-)\right) \bigoplus_{p \in \mathbf{P}} H_*\left(M(p^+)\right) \to \bigoplus_{p \in \mathbf{P}} H_{*-1}\left(M(p^-)\right) \bigoplus_{p \in \mathbf{P}} H_{*-1}\left(M(p^+)\right)$$

takes the form

$$\Delta_\epsilon = \begin{pmatrix} \Delta_- & T_\epsilon \\ 0 & \Delta_+ \end{pmatrix} \tag{5.3}$$

where Δ_- is the connection matrix for $\mathcal{M}(S_-)$, S_- is an isolated invariant set for the drift flow at parameter -1 and Δ_+ is the conjugation of a connection matrix of $\mathcal{M}(S_1)$ (this conjugation is necessary because on $\lambda = 1$ we have an increase by one in the dimension of the unstable manifold, which introduces a suspension of the Conley index). A contribution of Reineck in [14] was to formalize the expression (5.3). Note that T_ϵ is a 1-degree chain map. When $\epsilon \to 0$ the limit T_ϵ is well defined and the resulting matrix is referred to as the ***R-singular transition matrix***.

Theorem 5.1 (Reineck)**.** *An R-singular transition matrix T from $\lambda = 1$ to $\lambda = -1$ satisfies the following properties:*

(i) $\Delta_- T + T\Delta_+ = 0$.

(ii) *T is an isomorphism.*

(iii) *T is an upper triangular matrix with respect to $<$.*

(iv) *If $T(p,q) : H(M(p)) \to H(M(q))$ is nonzero, then there exists a finite sequence $1 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq -1$ and corresponding $p_i \in \mathbf{P}$ such that $p_i >_{\lambda_i} p_{i+1}$, $p_0 = q$, and $p_{k+1} = p$, where $>_{\lambda_i}$ is the flow defined order under ϕ_{λ_i}.*

Now we define the singular transition matrix presented in [12]. As one can note, in [14] and [12], different suspension isomorphisms were used to define the singular transition matrix. This difference is important to note because, as we point out in Remark 5.1 below, those matrices are transition matrices that cover different isomorphisms.

Following the same idea as in Reineck's development, McCord and Mischaikow in [12] created a new parameter space that incorporates the drift flows and the one-parameter families in Λ. More specifically, let $\mathcal{D} = \mathcal{P}(\Lambda) \times \mathcal{G}$ be a parameter space for flows on $X \times [-1, 2]$, where $\mathcal{P}(\Lambda) = \{\alpha : [0, 1] \to \mathbb{R}\}$ is the set of paths in Λ and

$$\begin{aligned}\mathcal{G} = \{g : [-1, 2] \to \mathbb{R} \mid g \in C^0, \quad & g((-1, 0) \cup (1, 2)) \geq 0 \geq g(0, 1), \\ & g(i) = 0 \text{ for } i = -1, 0, 1, 2\}.\end{aligned}$$

The choice of $[-1, 2]$ instead of $[0, 1]$ is convenient, since the drift flow $\dot{s} = g(s)$, with $\{0, 1\}$ a hyperbolic attractor-reppeler pair for $[0, 1]$, is dense in $\mathcal{G}$. Note that the domain of any path in $\mathcal{P}$ is $[0, 1]$. We define

$$\tau(s) = \begin{cases} -s & \text{for} \quad -1 \leq s \leq 0, \\ s & \text{for} \quad 0 \leq s \leq 1, \\ 2 - s & \text{for} \quad 1 \leq s \leq 2. \end{cases}$$

For any $(\alpha, g) \in \mathcal{D}$ consider a flow on $X \times [-1, 2]$ over (α, g) given by

$$\begin{aligned} \dot{x} &= f(x, \alpha\tau(s)), \\ \dot{s} &= g(s). \end{aligned}$$

Observe that if α is a constant path λ, then we obtain a product flow

$$\begin{aligned} \dot{x} &= f(x, \lambda), \\ \dot{s} &= g(s). \end{aligned}$$

When $||g|| \to 0$, we have the original parameterized family of flows

$$\dot{x} = f(x, \alpha\tau(s))$$

restricted to the image of α.

We are interested in studying the behavior on $\mathcal{D}([0, 1])$, where the flow on $X \times [0, 1]$ over (α, g) is the restriction of the flow on $X \times \Lambda$ to the one-parameter family of flows picked out by α. Let $C \subseteq [0, 1]$ be a connected isolated set of zeros for $g \in \mathcal{G}$ restricted to $(-1, 2)$. Thus if S is an isolated invariant set that continues over $\alpha[0,1]$ for some α, then $S_\alpha = \{(x,s) \mid s \in C, x \in S_{\alpha(s)}\}$ is an isolated invariant set for the flow over (α, g). See [12].

Let $S_\alpha(C)$ denote the restriction of S_α to $C \subseteq [0,1]$. The following proposition gives us a way to compute Conley indices for isolated invariant sets over $\mathcal{D}$.

Proposition 5.1 (McCord-Mischaikow)**.** *If C is a connected, isolated set of zeros of g, then $h(C)$, the homotopy Conley index of C in $[-1,2]$, is either Σ^1, Σ^0 or $\bar{0}$. Let $\mathcal{C}_g$ denote the component of $\mathcal{G}$ that contains g. If S is an isolated invariant set that continues over Λ, then, over $\mathcal{P} \times \mathcal{C}_g$, the Conley index of $S_\alpha(C)$ in $X \times [-1,2]$ is $h(S) \wedge h(C)$.*

By the previous proposition, we have that the homology Conley index of $S_\alpha(C)$ is the tensor product

$$CH(S_\alpha(C)) = CH(S) \otimes CH(C).$$

In this setting, we present the *index suspension isomorphism* $\Sigma(S)$ defined in [12] by the compositions

$$CH_k(S_{\alpha(c)}) \xrightarrow{\otimes\sigma_n} CH_k(S_{\alpha(c)}) \otimes CH_n(C) \xrightarrow{\times} CH_{n+k}(S_{\alpha(c)} \times C) = CH_{n+k}(S_{\bar{\alpha}(c)}(C))$$

and

$$CH_{n+k}(S_{\bar{\alpha}(c)}(C)) \xrightarrow{F_{\alpha,\bar{\alpha}(c)}} CH_{k+n}(S_\alpha(C)),$$

where $c \in C$, α is a path, $\bar{\alpha}(c)$ is the constant path $\alpha(c), \sigma_n$ is the generator of $CH_n(C)$ and $F_{\alpha,\bar{\alpha}(c)}$ is the continuation isomorphism along a path in $\mathcal{D}^+$ from $(\bar{\alpha}(c), g)$ to (α, g). More concisely, we have

$$\Sigma(S) : CH_k(S_{\alpha(c)}) \xrightarrow{F_{\alpha,\bar{\alpha}(c)} \circ \times \circ \otimes \sigma_n} CH_{k+n}(S_\alpha(C)).$$

Given that $M(\mathbf{I})$ is an isolated invariant set that also continues, we can define

$$\Sigma(\mathbf{I}) = \Sigma(M(\mathbf{I})) : CH_k(M(\mathbf{I})_{\alpha(c)}) \xrightarrow{F_{\alpha,\bar{\alpha}(c)} \circ \times \circ \otimes \sigma_n} CH_{k+n}(M(\mathbf{I})_\alpha(C)).$$

Define $\mathcal{G}^+ \subseteq \mathcal{G}$ to be the set of functions with $g^{-1}(0) = \{-1, 0, 1, 2\}$, and let $\mathcal{D}^+ = \mathcal{P}(\Lambda) \times \mathcal{G}^+$. From the dynamics in $[0,1]$, note that $S_{(\alpha,g)}$ has $S_{\alpha(0)}$ as an attractor and $S_{\alpha(1)}$ as a repeller. These characteristics will be the same for all $g \in \mathcal{G}^+$, even though the structure of the connecting orbit set may vary with g.

By Corollary 5.3 in [12], there is an open set $\mathcal{V}^+ \subseteq \mathcal{D}^+$, whose closure contains $\mathcal{D}([0,1])$, such that $\mathcal{M}_0 \cup \mathcal{M}_1$ continues across $\mathcal{V}^+$. Computing the connection matrix for the flow in $X \times \alpha[0,1]$ and $g \in \mathcal{V}^+_\alpha$, we have

$$\begin{aligned} \Delta_g : \bigoplus_{p \in \mathbf{P}} CH\left(M(p)_{\alpha(0)}\right) \quad \bigoplus_{p \in \mathbf{P}} CH\left(M(p)_{\alpha(1)}\right) \longrightarrow \\ \longrightarrow \bigoplus_{p \in \mathbf{P}} CH\left(M(p)_{\alpha(0)}\right) \bigoplus_{p \in \mathbf{P}} CH\left(M(p)_{\alpha(1)}\right) \end{aligned}$$

takes the form

$$\Delta_g = \begin{pmatrix} \Delta_{\alpha(0)} & T_g \\ 0 & \Delta^{\Sigma}_{\alpha(1)} \end{pmatrix} \tag{5.4}$$

where $\Delta_{\alpha(0)}$ is a connection matrix for $\mathcal{M}(S)_{\alpha(0)}$ and $\Delta^{\Sigma}_{\alpha(1)}$ is the conjugation by Σ of a connection matrix of $\mathcal{M}(S)_{\alpha(1)}$; see Lemma 5.1. Note that T_g is a 1-degree chain map.

When $||g|| \to 0$ the limit T_g gives us an **MM-singular transition matrix** T_s. Observe that Δ_g may differ from Δ_ϵ since we used different suspension isomorphisms to define these singular transition matrices.

The following theorem points out an important difference between the structure of Δ_g and the structure of Δ_ϵ.

Proposition 5.2 (McCord-Mischaikow)**.** *The connection homomorphism for the attractor-repeller decomposition* $\big(\mathcal{M}_{\alpha(0)}(\mathbf{I}), \mathcal{M}_{\alpha(1)}(\mathbf{I})\big)$ *of* $\mathcal{M}_{(\alpha,g)}(\mathbf{I})$ *is an isomorphism that is computed by continuation of* $\mathcal{M}_{(\alpha,g)}(\mathbf{I})$ *across* Λ*. That is, there is a commutative diagram:*

$$\begin{tikzcd}
H_{k,\alpha(1)}(\mathbf{I}) \arrow[r, "\Sigma(\mathbf{I})"] \arrow[d, "id"] & H_{k+1,\alpha(1)}(\mathbf{I}) \arrow[r, "\delta"] & H_{k,\alpha(0)}(\mathbf{I}) \arrow[d, "id"] \\
H_{k,\alpha(1)}(\mathbf{I}) \arrow[rr, "F_{\alpha(1),\alpha(0)}(\mathbf{I})"] & & H_{k,\alpha(0)}(\mathbf{I})
\end{tikzcd}$$

As a consequence we have that the flow-defined attractor-repeller map δ is an isomorphism.

The following theorem addresses both the R-singular transition matrix and the MM-singular transition matrix.

Theorem 5.2. *A singular transition matrix is a* 1*-degree transition matrix.*

Remark 5.1*.* As we can see in the following proof, Theorem 5.2 works for both the R-singular transition matrix and the MM-singular transition matrix. More specifically, they are 1-degree transition matrices that cover the 1-degree isomorphisms δ_Ψ and $F \circ \Sigma^{-1}$, respectively.

Proof. In order to simplify notation, we define $\lambda = \alpha(0)$ and $\mu = \alpha(1)$. Let T^1_s and T_s be the R-singular and MM-singular transition matrices (defined by Ψ and Σ), respectively. Choose an interval $\mathbf{I} \in \mathcal{I}(<)$, where $<$ is the flow-defined order for the product flow. Hence $(\mathbf{I}_\mu, \mathbf{I}_\lambda)$ is an attractor-repeller pair and, by Δ_g being a connection matrix, we have that the following diagram commutes:

$$\begin{tikzcd}
H_{k+1}\Delta^{\Sigma}_{\lambda}(\mathbf{I}) \arrow[r, "{[T_s]}"] \arrow[d, "\Phi^{\Sigma}_{\lambda}(\mathbf{I})"] & H_k\Delta_\mu(\mathbf{I}) \arrow[d, "\Phi_\mu(\mathbf{I})"] \\
H_{k+1,\lambda}(\mathbf{I}) \arrow[r, "\delta"] & H_{k,\mu}(\mathbf{I})
\end{tikzcd}$$

Therefore T_s is a 1-degree transition matrix that covers $\delta = F \circ \Sigma^{-1}$ since T_s is a 1-degree chain map and δ is defined by a 1-degree graded module braid isomorphism. If one changes Σ for Ψ we obtain that T^1_s is a 1-degree transition matrix covering δ_Ψ. Note that in this case we may not have $\delta_\Psi = F \circ \Psi^{-1}$ since δ_Ψ is a flow-defined map of the attractor-repeller pair for which the Conley index of the repeller was suspended by Ψ. ∎

Lemma 5.1. *$\Sigma : \mathcal{H}_{*,\lambda} \to \mathcal{H}_{*+1,\lambda}$ is a braid isomorphism between Conley index braids and Δ_λ^Σ is Σ conjugated to a connection matrix Δ_λ at parameter λ. In other words the following diagram commutes:*

$$\begin{tikzcd} H_k\Delta_\lambda(\mathbf{I}) \arrow[r, "{[\oplus_{\pi\in\mathbf{I}}\Sigma(\pi)]}"] \arrow[d, "\Phi_\lambda(\mathbf{I})"] & H_{k+1}\Delta_\lambda^\Sigma(\mathbf{I}) \arrow[d, "\Phi_\lambda^\Sigma(\mathbf{I})"] \\ H_{k,\lambda}(\mathbf{I}) \arrow[r, "\Sigma(\mathbf{I})"] & H_{k+1,\lambda}(\mathbf{I}) \end{tikzcd}$$

Proof. It is straightforward to prove that Σ is a graded module braid isomorphism, since it is defined via the braid isomorphisms $\times \circ \otimes\sigma$ and F.

Furthermore, by defining $\Delta_\lambda(\pi', \pi) := \Sigma^{-1}(\pi) \circ \Delta^\Sigma(\pi', \pi) \circ \Sigma(\pi')$, we obtain that Δ_λ is a connection matrix at parameter λ. Therefore the diagram in Lemma 5.1 commutes. □

The MM-singular transition matrix T_s defined via Σ has another feature as follows by the next theorem.

Theorem 5.3. *The composition of the MM-singular transition matrix T_s with the induced map of $\bigoplus_{\pi\in\mathbf{P}} \Sigma(\pi)$ is a transition matrix*

$$T = T_s \circ \left(\bigoplus_{\pi\in\mathbf{P}} \Sigma(\pi) \right)$$

that covers the Conley flow-defined isomorphism F.

Proof. The proof follows from the same idea of Theorem 7.1 in [12] and by using Lemma 5.1 and Theorem 5.2 instead of Lemma 7.2 and Lemma 7.3 in [12]. □

It is worthwhile to note that Lemma 5.1 and Theorem 5.2 are fundamental in the proof of Theorem 5.3 given that the connections matrices in Theorem 7.1, Lemma 7.2 and Lemma 7.3 in [12] are all trivial.

Note that Σ depends on the flow-defined isomorphism F; therefore Δ_λ^Σ and T_s depend on F too. The following corollary provides a circumstance when Σ does not depend on F.

Corollary 5.1. *The suspension isomorphism Σ does not depend on F when the flow at parameter λ is structurally stable. Moreover,*

$$T = T_s \circ \times \circ \otimes\sigma,$$

where T is a transition matrix that covers the Conley flow-defined isomorphism F, and T_s is an MM-singular transition matrix.

Proof. Let α be a path from λ to μ. If $\lambda = \mu$, let α be a constant path, and hence we have that $\Sigma = \times \circ \otimes\sigma$. Therefore Σ does not depend on F and by Theorem 5.3 the result follows. Suppose that α is a nonconstant path. Since the flow in parameter λ is structurally stable there is an $\epsilon > 0$ sufficiently small such that $\alpha[0, \epsilon]$ does not have bifurcations. Thus $F_{\overline{\alpha}(0)[0,\epsilon]} = id$, where $\overline{\alpha}(0)$ is a constant path evaluated in $\alpha(0)$.

By Λ being simply connected, we have that there exists a homotopy $A : \alpha \simeq \xi$, where $\xi = \overline{\alpha}(0)[0, \epsilon] * \gamma * \alpha[\epsilon, 1]$ and γ is a path between $\alpha(0)$ and $\alpha(\epsilon)$. Hence $F_{\alpha[0,\epsilon]} = id$. Thus Σ does not depend on F, since we can suspend the Conley index by Σ locally. Therefore, by Theorem 5.3, the result follows. □

Theorem 6.3 in the next section shows us the importance when Σ does not depend on F, i.e., when one can apply Corollary 5.1.

6. Topological transition matrix

In this section, we develop more proprieties of generalized topological transition matrices from [5] given that such matrices are transition matrices that cover flow-defined Conley index isomorphisms. Therefore we can use the relationship between singular and topological transition matrix theory in order to obtain refined properties. For instance, Theorem 6.2 is an improvement of Theorem 2.8 in [5] by not requiring existence assumptions. Moreover, Theorem 6.3 describes appropriately the importance of the unified theory of transition matrix.

Let $M_\lambda = \{M_\lambda(\pi)\}_{\pi\in\mathbf{P}}$ and $M_\mu = \{M_\mu(\pi)\}_{\pi\in\mathbf{P}}$ be Morse decompositions, related by continuation, for the isolated invariant sets $S_\lambda \subseteq X_\lambda$ and $S_\mu \subseteq X_\mu$, respectively.

When there are no connections in the λ and μ parameters, then $\Delta_\lambda = 0 = \Delta_\mu$. By Conley's theory we have that there is an isomorphism $\Phi_\lambda : C_*\Delta_\lambda(\mathbf{P}) \to H_{*,\lambda}(\mathbf{P})$ for $\lambda \in \Lambda'$, where $C_*\Delta_\lambda(\mathbf{P}) = \bigoplus_{\pi\in\mathbf{P}} CH(M_\lambda(\pi))$ is the chain complex with connection matrix Δ_λ.

Therefore, we can carry out the continuation along the path ω in two ways: first by continuing S_λ along the path ω using the isomorphism $F_{\lambda,\mu}$; second by continuing $\bigcup_{p\in\mathbf{P}} M_\lambda(p)$ along the path ω by using the isomorphism $E_{\lambda,\mu} = \bigoplus_{p\in\mathbf{P}} F_{\lambda,\mu}(M(p))$. More precisely, we have the following diagram:

$$\begin{tikzcd} C\Delta_\lambda(\mathbf{P}) \arrow[r, "E_{\lambda,\mu}"] \arrow[d, "\Phi_\lambda"] & C\Delta_\mu(\mathbf{P}) \arrow[d, "\Phi_\mu"] \\ H_\lambda(\mathbf{P}) \arrow[r, "F_{\lambda,\mu}"] & H_\mu(\mathbf{P}) \end{tikzcd}$$

In general the diagram above is not commutative. In order to obtain a commutative diagram we replace $E_{\lambda,\mu}$ with the map $T_{\lambda,\mu} = \Phi_\mu^{-1} \circ F_{\lambda,\mu} \circ \Phi_\lambda$. Then we have that the following diagram commutes:

$$\begin{tikzcd} C\Delta_\lambda(\mathbf{P}) \arrow[r, "T_{\lambda,\mu}(\mathbf{P})"] \arrow[d, "\Phi_\lambda"] & C\Delta_\mu(\mathbf{P}) \arrow[d, "\Phi_\mu"] \\ H_\lambda(\mathbf{P}) \arrow[r, "F_{\lambda,\mu}(\mathbf{P})"] & H_\mu(\mathbf{P}) \end{tikzcd}$$

In this particular case, such a matrix $T_{\lambda,\mu}$ is called a **(*classical*) topological transition matrix**.

In contrast to the classical case, we do not insist that there are no connections at the initial and final parameters of a continuation when defining a (generalized) topological transition matrix. As a consequence, we can apply this theory to a much broader family of dynamical systems.

Definition 6.1. If T is a transition matrix that covers the Conley flow-defined isomorphism F, then we refer to T as a **(generalized) topological transition matrix**.

In order to simplify one may omit the term "generalized" from Definition 6.1; nevertheless be aware that this definition is an extension of the topological transition matrix defined in [11] where there are no connections at the initial and final parameters of a continuation.

Denote GTTM($<$) as the set of all generalized topological transition matrices with the partial order $<$.

Although the existence of generalized topological transition matrices is asserted for some cases (see section 2 or [5]), it is possible to obtain an improved result by using singular transition matrix theory. More specifically, we apply Theorem 5.3 and Remark 6.1 to obtain stronger existence results in the remainder of this section.

The following remark, that follows from the inductive construction process in the connection matrix existence proof in [8], is a useful extension result. It asserts that a connection matrix for an "attracting" interval can be extended to a connection matrix over the whole underlying set.

Remark 6.1. Let $\mathcal{G}$ be a graded module braid over $<$, $C = \{C(p)\}_{p\in\mathbf{P}}$ be a collection of graded modules such that $C(p)$ is a free chain complex and $C(p) \simeq G(p)$, and let $\Delta : \bigoplus_{p\in\mathbf{P}} C(p) \to \bigoplus_{p\in\mathbf{P}} C(p)$ be a $<$-upper triangular boundary map. Assume that $\mathbf{I}$ and $\mathbf{J}$ are an adjacent pair of intervals such that $\mathbf{P} = \mathbf{IJ}$. If $\Delta(\mathbf{I})$ is a connection matrix, then there is a connection matrix $\Delta(\mathbf{J})$ of $\mathcal{G}(\mathbf{J})$ and a matrix $\Delta(\mathbf{J},\mathbf{I})$ such that

$$\Delta(\mathbf{P}) = \begin{pmatrix} \Delta(\mathbf{I}) & \Delta(\mathbf{J},\mathbf{I}) \\ 0 & \Delta(\mathbf{J}) \end{pmatrix}$$

is a connection matrix of $\mathcal{G}$.

Remark 6.2. Applying Remark 6.1 to equations (5.3) and (5.4), one can freely choose a connection matrix (in their respective set of connection matrices) to be Δ_λ, Δ_- and $\Delta_{\alpha(0)}$ respectively. However, Δ_μ, Δ_+ and $\Delta^{\Sigma}_{\alpha(1)}$ are not free to be chosen. In fact they come from the inductive process of the connection matrix existence proof in [8]. In other words, existence results coming from the singular transition matrix theory give us liberty to choose a connection matrix at the parameter that was not suspended, yet we cannot choose freely a connection matrix at the parameter that was suspended.

The previous remark gives us a clear idea of how to apply Theorem 5.3 and Remark 6.1 to obtain an existence result for topological transition matrices given a connection matrix for the parameter at the beginning of a continuation.

Theorem 6.1. *Let $M_\lambda = \{M_\lambda(\pi)\}_{\pi\in(\mathbf{P},<_\lambda)}$ and $M_\mu = \{M_\mu(\pi)\}_{\pi\in(\mathbf{P},<_\mu)}$ be Morse decompositions. Assume that M_λ and M_μ are related by continuation with an admissible ordering $<$. Given $\Delta_\lambda \in \mathcal{CM}(<_\lambda)$, there exists a generalized topological transition matrix T related to Δ_λ and a $\Delta_\mu \in \mathcal{CM}(<_\mu)$.*

Proof. From Theorem 5.3 we have the existence of

$$T_{\nu,\mu} = T_s \circ \left(\bigoplus_{\pi\in\mathbf{P}} \Sigma(\pi) \right)$$

related to some $\bar{\Delta}_\lambda \in \mathcal{CM}(<_\lambda)$ and $\bar{\Delta}_\mu \in \mathcal{CM}(<_\mu)$. Note that the existence of the singular transition matrix T_s is in fact related to existence of the connection matrix, as in equation (5.4),

$$\Delta = \begin{pmatrix} \Delta_{\alpha(0)} & T_s \\ 0 & \Delta^{\Sigma}_{\alpha(1)} \end{pmatrix}.$$

Hence, we can use Remark 6.1 with $\Delta(\mathbf{I}) = \Delta_\lambda$ to obtain

$$\Delta = \begin{pmatrix} \Delta_\lambda & T'_s \\ 0 & \Delta'_\mu \end{pmatrix}, \tag{6.1}$$

with given $\Delta_\lambda \in \mathcal{CM}(<_\lambda)$, thus this new topological transition matrix

$$T = T'_s \circ \left(\bigoplus_{\pi \in \mathbf{P}} \Sigma(\pi) \right)$$

is related to Δ_λ and a $\Delta'_\mu \in \mathcal{CM}(<_\mu)$. $\square$

The following theorem (whose proof employs Theorem 6.1) strengthens Theorem 2.8 in [5] by removing the existence hypotheses "GTTM($<_m$) $\neq \emptyset$ for all $\omega[s,t]$" from item (v) in the statement of Theorem 2.8 in [5].

Theorem 6.2. *Let $M_\lambda = \{M_\lambda(\pi)\}_{\pi \in (\mathbf{P}, <_\lambda)}$ and $M_\mu = \{M_\mu(\pi)\}_{\pi \in (\mathbf{P}, <_\mu)}$ be Morse decompositions, Δ_λ and Δ_μ their respective connection matrices. Moreover, assume that M_λ and M_μ are related by continuation with an admissible ordering $<$. Then the generalized topological transition matrix T satisfies the following properties:*

(i) *$T \circ \Delta_\lambda = \Delta_\mu \circ T$.*
(ii) *$T_{\lambda,\mu}(\{p\}) = id$ and T is upper triangular with respect to $<$.*
(iii) *T is an isomorphism.*
(iv) *$T_{\lambda,\lambda} = id$, $T_{\lambda,\nu}(\mathbf{I}) = T_{\mu,\nu} \circ T_{\lambda,\mu}(\mathbf{I})$ and $T_{\mu,\lambda}(\mathbf{I}) = T^{-1}_{\lambda,\mu}(\mathbf{I})$ are generalized topological transition matrices, for all intervals $\mathbf{I} \in \mathcal{I}$ and $p \in \mathbf{P}$, in particular $T = T(\mathbf{P})$.*
(v) *Let $\omega : [0,1] \to \Lambda$ be a path that continues M_λ to M_μ. Assume that $T_{\lambda,\mu}(p,q) \neq 0$ for all generalized topological transition matrices. Then there exists a finite sequence $0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq 1$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q$, $p_n = p$ and the set of connecting orbits $C\left(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i)\right)$ is nonempty.*

Proof. From Theorem 6.1 we have the existence of $T_{\lambda,\mu}$ related to Δ_λ and a $\bar{\Delta}_\mu \in \mathcal{CM}(<_\mu)$. However, $\bar{\Delta}_\mu$ may be different from Δ_μ since Theorem 6.1 does not specify to which connection matrix (at parameter μ) T is related. Therefore, one needs to check in the proof of Theorem 2.8 in [5] that all connection matrices at parameters $\omega(0)$, $\omega(s_{\xi_1})$, $\ldots$, $\omega(1)$ agree.

Without loss of generality, assume that we are in the case where the connection matrices at parameters $\omega(0) = \lambda$, $\omega(s_{\xi_1}) = \nu$, $\omega(1) = \mu$ must agree. Indeed, by Theorem 6.1 one obtains a $T_{\lambda,\nu}$ related to Δ_λ and a $\Delta_\nu \in \mathcal{CM}(<_\nu)$, and fixing Δ_ν, one also obtains a $T_{\nu,\mu}$ related to Δ_ν and Δ'_μ. Since $T_{\lambda,\nu}$ and $T_{\nu,\mu}$ are related to the very same matrix Δ_ν at parameter ν, the composition of the maps $T'_{\lambda,\mu} = T_{\lambda,\nu} \circ T'_{\nu,\mu}$ is a well-defined topological transition matrix related to Δ_λ and Δ'_μ.

Note that, $T'_{\lambda,\mu}$ and Δ'_μ may be different from $T_{\lambda,\mu}$ and Δ_μ, respectively. But it is not a problem because by hypothesis $T'_{\lambda,\mu}(p,q) \neq 0$. $\square$

Using both the topological and singular transition matrices, one is able to obtain richer dynamical information in Theorem 6.2 without assuming that $T_{\lambda,\mu}(p,q) \neq 0$

for all generalized topological transition matrices. In other words, it is enough to assume $T_{\lambda,\mu}(p,q) \neq 0$ for at least one matrix in GTTM(<).

Theorem 6.3. *Let $M_\lambda = \{M_\lambda(\pi)\}_{\pi\in(\mathbf{P},<_\lambda)}$ and $M_\mu = \{M_\mu(\pi)\}_{\pi\in(\mathbf{P},<_\mu)}$ be Morse decompositions related by continuation with an admissible ordering $<$. Moreover, assume that the flow at parameter λ is structurally stable and a generalized topological transition matrix T has a nonzero entry $T_{\lambda,\mu}(p,q) \neq 0$. Then there exists a finite sequence $0 < s_1 \leq s_2 \leq \ldots \leq s_n \leq 1$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q$, $p_n = p$, and the set of connecting orbits $C\left(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i)\right)$ is nonempty, where $\omega : [0,1] \to \Lambda$ is a path that continues M_λ to M_μ.*

Proof. By Corollary 5.1 a singular transition matrix is related to a generalized topological transition matrix by

$$T_{\lambda,\mu}(p,q) = T_s \circ \times \circ \otimes\sigma(p,q).$$

Therefore $T_{\lambda,\mu}(p,q) \neq 0$ implies that $T_s \neq 0$. Since T_s comes from a connection matrix Δ_g for a flow-defined order it follows that a nonzero entry in T_s implies the result. $\square$

7. Directional transition matrix

In this section, we prove that the directional transition matrix is a generalized transition matrix that covers an isomorphism defined via the flow-defined Conley index isomorphism F.

Consider the fast-slow systems of the form

$$\begin{array}{rcl} \dot{x} & = & f(x,y), \\ \dot{y} & = & \epsilon g(x,y), \end{array} \tag{7.1}$$

where $x \in \mathbb{R}^2$ and $y \in \mathbb{R}$. Observe that this fast-slow system is more general than the one used to define the singular transition matrix. In contrast to equation (5.2), the slow variable depends also on the fast variable x. Thus for $\epsilon > 0$ Morse sets may have different directions of slow flow. Assume that when $\epsilon = 0$, for all $y \in [0,1]$ the parameterized system has an isolated invariant set S_y that continues over $[0,1]$, and each such S_y has a Morse decomposition

$$\mathcal{M}_y = \{M_y(\pi) \mid \pi \in \mathbf{P}\}$$

that also continues over $[0,1]$. Furthermore, suppose that $g(M_y(\pi), y) \neq 0$ for all $y \in (0,1)$ and $\pi \in \mathbf{P}$. Note that, depending on g, the slow dynamics introduced when $\epsilon > 0$ may not go in the same direction for each Morse set. For this setting, we define an isolating neighborhood as follows.

Definition 7.1. A set $\mathcal{B}$ is a box if:

(1) There exists an isolating neighborhood $\mathcal{B} \subseteq \mathbb{R}^n \times [0,1]$ for the parameterized flow $\psi^{\mathcal{B}}$ defined by

$$\begin{array}{rcl} \psi^{\mathcal{B}} : \mathbb{R} \times \mathbb{R}^n \times [0,1] & \to & \mathbb{R}^n \times [0,1] \\ (t,x,y) & \mapsto & (\psi_y(t,x), y), \end{array}$$

where ψ_y is the flow of $\dot{x} = f(x,y)$ with fixed y.

(2) Let $S(\mathcal{B}) = Inv(\mathcal{B}, \psi^{\mathcal{B}})$. There is a Morse decomposition

$$\mathcal{M}(S(\mathcal{B})) = \{M(p,\mathcal{B}) \mid p = 1, \ldots, P_{\mathcal{B}}\},$$

with the usual ordering on the integers as an admissible ordering. Let $\mathcal{B}_y = \mathcal{B} \cap (\mathbb{R}^n \times \{y\})$ and $S_y(\mathcal{B}) = Inv(\mathcal{B}_y, \psi_y)$, and let $\{M_y(p, \mathcal{B} \mid p = 1, \ldots, P_{\mathcal{B}}\}$ be the corresponding Morse decomposition of $S_y(\mathcal{B})$. Then

$$S_0(\mathcal{B}) = \bigcup_{p=1}^{P_{\mathcal{B}}} M_0(p, \mathcal{B}) \quad \text{and} \quad S_1(\mathcal{B}) = \bigcup_{p=1}^{P_{\mathcal{B}}} M_1(p, \mathcal{B}).$$

(3) There are isolating neighborhoods $V(p, \mathcal{B})$ for $M(p, \mathcal{B})$ such that

$$V(p, \mathcal{B}) \subseteq \mathcal{B} \quad \text{and} \quad V(p, \mathcal{B}) \cap V(q, \mathcal{B}) = \emptyset$$

for $p \neq q$ with $p, q = 1, \ldots, P_{\mathcal{B}}$, and for every $y \in [0, 1]$

$$V_y(p, \mathcal{B}) \subseteq Int(\mathcal{B}_y).$$

Furthermore, there are $\delta(p, \mathcal{B}) \in \{-1, 1\}, p = 1, \ldots, P_{\mathcal{B}})$ such that

$$\delta(p, \mathcal{B}) g(x, y) > 0 \quad \text{for all } (x, y) \in V(p, \mathcal{B}).$$

From the last property, one can decompose the finite index set of the Morse decomposition as

$$\mathbf{P} = \mathbf{P}_+ \cup \mathbf{P}_-$$

where

$$\mathbf{P}_\pm = \{p \in \mathbf{P} \mid \pm \delta(p) > 0\},$$

and correspondingly, one can define $M_{in}(p, \mathcal{B})$ and $M_{out}(p, \mathcal{B})$ as follows:

$$M_{in}(p, \mathcal{B}) = \begin{cases} M_0(p, \mathcal{B}) & \text{if } p \in \mathbf{P}_+, \\ M_1(p, \mathcal{B}) & \text{if } p \in \mathbf{P}_-; \end{cases}$$

$$M_{out}(p, \mathcal{B}) = \begin{cases} M_1(p, \mathcal{B}) & \text{if } p \in \mathbf{P}_+, \\ M_0(p, \mathcal{B}) & \text{if } p \in \mathbf{P}_-. \end{cases}$$

Notice, by property (2) in Definition 7.1 there are no connecting orbits among the Morse sets at $y = 0$ and at $y = 1$ and, by construction, the sets $S_0(\mathcal{B})$ and $S_1(\mathcal{B})$ are related by continuation. Hence in this setting the topological transition matrix T is well defined and unique.

A box with bidirectional slow dynamics can naturally occur, for instance in the FitzHugh-Nagumo equation (see [9] and [10]). In this situation, neither the singular nor the topological transition matrix applies since they are both defined for essentially unidirectional continuations.

Proposition 7.1. *Let V, V' and W, W' be mutually isomorphic finitely generated free Abelian groups, and let*

$$A : V \otimes W \to V' \otimes W'$$

be an isomorphism. Suppose A is an upper triangular with the following block decomposition:

$$A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

where $X : V \to V'$ and $Z : W \to W'$ are isomorphisms. Then the following maps are all upper triangular isomorphisms:

$$A_1 = \begin{pmatrix} X & YZ^{-1} \\ 0 & Z^{-1} \end{pmatrix} : V \otimes W' \to V' \otimes W,$$
$$A_2 = \begin{pmatrix} X^{-1} & -X^{-1}Y \\ 0 & Z \end{pmatrix} : V' \otimes W \to V \otimes W',$$
$$A_3 = \begin{pmatrix} X^{-1} & -X^{-1}YZ^{-1} \\ 0 & Z^{-1} \end{pmatrix} : V' \otimes W' \to V \otimes W.$$

In order to have a map from M_{out} to M_{in}, one can repeatedly apply Proposition 7.1 to the topological transition matrix until one obtains an isomorphism

$$D : \bigoplus_{p \in \mathbf{P}} CH_*(M_{out}(p)) \to \bigoplus_{p \in \mathbf{P}} CH_*(M_{in}(p)).$$

The matrix representation of this isomorphism is called a *directional transition matrix*, and it has the following connecting-orbit existence property.

Theorem 7.1 ([10]). *Let D be the directional transition matrix for a box in the fast-slow system* (7.1). *If its (p,q)-entry $D(p,q)$ is nonzero, then there exists a finite sequence $\{y_i\}_{i=1}^{k+1}$ in $[0,1]$ and a sequence $\{p_i\}$ in $\mathbf{P}$ satisfying*

$$\partial(p_{i+1})(y_{i+1} - y_i) > 0 \text{ for all } i = 1, \dots, k-1$$

and

$$p = p_1 > p_2 > \dots > p_k > p_{k+1} = q$$

such that the corresponding parameterized flow at $y = y_i$ has a connecting orbit from $M_{y_i}(p_i)$ to $M_{y_i}(p_{i+1})$.

Simple examples show us that D depends on the choices that arise when applying Proposition 7.1 repeatedly and therefore led us to the question: which way should we obtain D from the topological transition matrix T? Theorem 7.2 answers this question by demonstrating that different D (obtained from the same T) cover different isomorphisms that arise as a result of the choice made by applying Proposition 7.1 repeatedly.

In this sense, the definition of the directional transition matrix D is deeper than just a rearrangement of Morse sets from a topological transition matrix. At first glance, one might think that is an artificial definition; however D is actually a transition matrix and thus covers a graded module braid isomorphism.

The next proposition seems to be a redundant way to define D, nevertheless this new way is helpful in the continuation context as one can see in Theorem 7.2.

Proposition 7.2. *A directional transition matrix*

$$D : \bigoplus_{p \in \mathbf{P}} CH_*(M_{out}(p)) \to \bigoplus_{p \in \mathbf{P}} CH_*(M_{in}(p))$$

can be represented by an isomorphism

$$\bar{D} : \bigoplus_{p \in \mathbf{P}} CH_*(M_1(p)) \to \bigoplus_{p \in \mathbf{P}} CH_*(M_0(p))$$

after a change of base. Moreover D and $\bar{D}$ are represented by the same matrix.

Proof. Recall

$$M_{in}(p,\mathcal{B}) = \begin{cases} M_0(p,\mathcal{B}) & \text{if } p \in \mathbf{P}_+, \\ M_1(p,\mathcal{B}) & \text{if } p \in \mathbf{P}_-, \end{cases}$$

and

$$M_{out}(p,\mathcal{B}) = \begin{cases} M_1(p,\mathcal{B}) & \text{if } p \in \mathbf{P}_+, \\ M_0(p,\mathcal{B}) & \text{if } p \in \mathbf{P}_-, \end{cases}$$

and define

$$R_{1,out} = \begin{cases} id(p) & \text{if } M_{out}(p) = M_1(p), \\ F_{01}(p) & \text{if } M_{out}(p) = M_0(p), \end{cases}$$

and

$$R_{in,0} = \begin{cases} id(p) & \text{if } M_{in}(p) = M_0(p), \\ F_{10}(p) & \text{if } M_{in}(p) = M_1(p). \end{cases}$$

Note that D is defined via a topological transition matrix, which is an isomorphism from a base $\mathcal{B}_0$ to the base $\bigoplus_{p\in\mathbf{P}} F_{01}(p)(\mathcal{B}_0)$. Thus $\bar{D} = R_{in,0} \circ D \circ R_{1,out} : \bigoplus_{p\in\mathbf{P}} CH_*(M_1(p)) \to \bigoplus_{p\in\mathbf{P}} CH_*(M_0(p))$ is just a change of base since $R_{1,out}^{-1} = R_{in,0}$. Observe that id covers $\bigoplus_{p\in\mathbf{P}} F_{01}(p)$. In other words, $\bigoplus_{p\in\mathbf{P}} F_{01}(p)$ does not give information about connecting orbits between Morse sets. Therefore $\bar{D}$ and D are represented by the same matrix. □

The same idea works in Proposition 7.2 when one needs to change the map $\bar{D} : \bigoplus_{p\in\mathbf{P}} CH_*(M_1(p)) \to \bigoplus_{p\in\mathbf{P}} CH_*(M_0(p))$ for $\bar{D}' : \bigoplus_{p\in\mathbf{P}} CH_*(M_0(p)) \to \bigoplus_{p\in\mathbf{P}} CH_*(M_1(p))$ by choosing another path orientation in the parameter space $[0,1]$. This can happen in Theorem 7.2 when n is even.

Applying Proposition 7.2 to Proposition 7.1 in the fast-slow systems setting, the next lemma identifies the isomorphisms that the matrices in Proposition 7.1 cover.

Lemma 7.1. *Let*

$$T = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

be the topological transition matrix for the fast-slow systems in (7.1) *when* $\epsilon = 0$. *For* $A = T$ *in Proposition* 7.1 *the matrices* A_1, A_2 *and* A_3 *are transition matrices that cover*

$$\left(\bigoplus_{p\in\mathbf{I}} F_{10}(p) \oplus F_{10}(\mathbf{J})\right) \circ F_{01}(\mathbf{P}) \circ \left(\bigoplus_{p\in\mathbf{I}} F_{10}(p) \oplus F_{10}(\mathbf{J})\right),$$

$$\left(F_{10}(\mathbf{I}) \circ \bigoplus_{p\in\mathbf{J}} F_{10}(p)\right) \circ F_{01}(\mathbf{P}) \circ \left(F_{10}(\mathbf{I}) \circ \bigoplus_{p\in\mathbf{J}} F_{10}(p)\right),$$

and F_{10} *respectively, where* $\mathbf{I}$ *and* $\mathbf{J}$ *are intervals such that* $\mathbf{P} = \mathbf{IJ}$,

$$X : \bigoplus_{p\in\mathbf{I}} CH_*(M_0(p)) \to \bigoplus_{p\in\mathbf{I}} CH_*(M_1(p))$$

and $Z : \bigoplus_{p\in\mathbf{J}} CH_*(M_0(p)) \to \bigoplus_{p\in\mathbf{J}} CH_*(M_1(p))$.

Proof. Apply the same change of base in Proposition 7.2 to the matrices A_1, A_2 and A_3, and also note that the connection matrices at parameters 0 and 1 are equal to zero. Thus A_1, A_2 and A_3 are chain maps.

Since $A_3 = T^{-1}$, it follows that A_3 is a transition matrix that covers F_{10}. For the others, first, let $\alpha : [0,1] \to [0,1]$ be a path such that $\alpha_{|_{[0,1/3]}}$ and $\alpha_{|_{[2/3,1]}}$ are subpaths from the parameter 1 to 0 and $\alpha_{|_{[1/3,2/3]}}$ is a subpath from 0 to 1. Observe that

$$A_1 = \begin{pmatrix} X & YZ^{-1} \\ 0 & Z^{-1} \end{pmatrix} = \begin{pmatrix} id & 0 \\ 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \begin{pmatrix} id & 0 \\ 0 & Z^{-1} \end{pmatrix},$$

$$\begin{aligned} A_2 &= \begin{pmatrix} X^{-1} & -X^{-1}Y \\ 0 & Z \end{pmatrix} \\ &= \begin{pmatrix} X^{-1} & -X^{-1}Y \\ 0 & Z \end{pmatrix} \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \begin{pmatrix} X^{-1} & -X^{-1}Y \\ 0 & Z \end{pmatrix}. \end{aligned}$$

Thus, for the path α, A_1 and A_2 cover

$$\left(\bigoplus_{p \in \mathbf{I}} F_{10}(p) \oplus F_{10}(\mathbf{J}) \right) \circ F_{01}(\mathbf{P}) \circ \left(\bigoplus_{p \in \mathbf{I}} F_{10}(p) \oplus F_{10}(\mathbf{J}) \right),$$

$$\left(F_{10}(\mathbf{I}) \circ \bigoplus_{p \in \mathbf{J}} F_{10}(p) \right) \circ F_{01}(\mathbf{P}) \circ \left(F_{10}(\mathbf{I}) \circ \bigoplus_{p \in \mathbf{J}} F_{10}(p) \right),$$

respectively, since X, X^{-1}, Z, Z^{-1}, $id(\mathbf{I})$ and $id(\mathbf{J})$ are transition matrices that cover $F_{01}(\mathbf{I})$, $F_{10}(\mathbf{I})$, $F_{01}(\mathbf{J})$, $F_{10}(\mathbf{J})$, $\bigoplus_{p \in \mathbf{I}} F_{10}(p)$ and $\bigoplus_{p \in \mathbf{J}} F_{10}(p)$, respectively. $\square$

The next theorem describes how the directional transition matrix fits in the transition matrix theory. Furthermore it shows us the relation between the choices made by applying Proposition 7.1 on T to obtain D and which isomorphism D must cover.

Theorem 7.2. *The directional transition matrix is a generalized transition matrix that covers*

$$G_n \circ \cdots \circ G_1 \circ F_{01} \circ G_1 \circ \cdots \circ G_n,$$

where G_i is the isomorphism defined in Lemma 7.1 *after applying it i times, and n is the number of times needed to apply Proposition* 7.1 *on the topological transition matrix T in order to obtain the directional transition matrix D.*

Proof. Let T be the topological transition matrix for a box $\mathcal{B}$ for a fast-slow system (7.1). Suppose that one needs to apply Proposition 7.1 n times on T in order to obtain the directional transition matrix D. Instead of using Proposition 7.1, one can use Lemma 7.1, but be aware that what was done by using Proposition 7.1 must be done in the same way for Lemma 7.1. On the i-th time that one applies Lemma 7.1, the new matrix will cover

$$G_i \circ \cdots \circ G_1 \circ F_{01} \circ G_1 \circ \cdots \circ G_i,$$

and thus the process ends for $i = n$. And to recover D, one just needs to apply Proposition 7.2. $\square$

8. Summary

We have presented an algebraic definition that unifies the theory of algebraic, singular, topological and directional transition matrices by introducing the generalized transition matrix theory, an algebraic theory of which the previous types of transition matrices are examples. We introduced existence results for the transition matrix in some specific settings: when the connection matrices are trivial (Proposition 3.1), when the underlying partial order is N-free (Theorem 3.1), and when the associated chain complex braids are unidimensional (Theorem 3.2). Further investigation, in a general algebraic setting, is needed for a satisfactory completion of the existence results, either into a theorem addressing existence of transition matrices for an arbitrary pair of connection matrices or into a theorem indicating exactly when transition matrices do and do not exist.

In the Conley index theory setting, we presented transition matrix existence results in Theorems 5.3 and 6.1 that were subsequently applied to parameterized families of flows. Broadening the algebraic existence theory for transition matrices should naturally lead to further applications within the Conley index theory.

Furthermore, we have provided results (Theorem 4.1, Corollaries 4.1 and 4.2, Theorems 6.2 and 6.3) indicating how transition matrices enable one to identify bifurcations that occur in a parameterized family of flows. Such results are a primary step in employing algebraic information from the transition matrices to understand structure in a family of flows. A further investigation is the next step to extend the body of results that provide an understanding of the structure in flows based on the algebra in the associated transition matrices.

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