

ON RATIONAL FIXED POINTS OF FINITE GROUP ACTIONS ON THE AFFINE SPACE

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ABSTRACT. Consider a finite l -group acting on the affine space of dimension n over a field k , whose characteristic differs from l . We prove the existence of a fixed point, rational over k , in the following cases:

- The field k is p -special for some prime p different from its characteristic.
- The field k is perfect and fertile, and $n = 3$.

1. INTRODUCTION

The following question was popularised by Serre in [Ser09]:

Let k be a field, and l a prime number different from its characteristic. Let G be a finite l -group acting by k -automorphisms on the affine space \mathbb{A}^n . Does the action fix a k -rational point?

The structure of the automorphism group of the affine space \mathbb{A}^n over a field (the affine Cremona group) is quite mysterious when $n \geq 3$; see e.g. [Kra96]. The question above may be seen as an attempt to provide a small piece of information on this group. Of course the techniques developed to answer this question are likely to be useful to find fixed points for actions on other varieties, but the variety \mathbb{A}^n is a particularly interesting test case. The fact that it is not projective prevents the use of many intersection-theoretic techniques. At the same time, one can still hope to answer the question by exploiting the simplicity of \mathbb{A}^n from a cohomological point of view (its “acyclicity”).

Serre mentions that the answer to his question is unknown when $G = \mathbb{Z}/2$, $k = \mathbb{Q}$ and $n = 3$, but gives a positive answer in the two cases listed below.

- k is finite.
- k is algebraically closed.

He also proves the existence of a fixed point when k is algebraically closed and G is instead cyclic of order prime to the characteristic exponent of k .

Esnault and Nicaise extend in [EN11] the list of fields k for which the question has a positive answer as follows:

- k is separably closed.
- $k = \mathbb{R}$.
- k is a henselian discretely valued field of characteristic zero with algebraically closed residue field of characteristic unequal to l .

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- k is a henselian discretely valued field of characteristic zero with residue field finite of cardinality q , and such that $l - 1 \mid q$.

They also settle the question when

- k is arbitrary and $n \leq 2$.

In fact, they prove much more in this case: if G is a solvable group of order prime to char k , which acts on \mathbb{A}^n with $n \leq 2$, then $(\mathbb{A}^n)^G \in \{\mathbb{A}^0, \mathbb{A}^1, \mathbb{A}^2\}$.

In the present paper we further extend the list to the following cases:

- k is pseudo algebraically closed.
- k is q -special for a prime number $q \neq \text{char } k$.
- k is perfect and fertile, and $n = 3$.

The definitions of “ q -special” and “fertile” will be given below; a field is called *pseudo algebraically closed* if every geometrically integral variety has a rational point. In this case, the existence of fixed points follows at once from Smith’s theory (see Proposition 3.2.1), and will not be discussed further.

In fact, we answer the analog of Serre’s question where the existence of a rational point is replaced by that of a zero-cycle of degree one (at least over fields of characteristic zero). More precisely, we prove:

Theorem 1.1. *Let k be a field of characteristic exponent p . Let G be a finite group acting on \mathbb{A}^n . Assume that one of the following assumptions holds:*

- (i) G is cyclic of order prime to p .
- (ii) G is an l -group, with $l \neq p$.

Then the fixed locus $(\mathbb{A}^n)^G$ supports a zero-cycle whose degree is a power of p . If k is perfect and $n \leq 4$, then $(\mathbb{A}^n)^G$ supports a zero-cycle of degree one.

The proof is based on Hironaka’s resolution of singularities in characteristic zero (and recent results of Cossart-Piltant for threefolds in arbitrary characteristic), on Gabber’s theorem on alterations, and on results of K. Kato and T. Saito concerning wild ramification.

Let q be a prime number. A field k is *q -special* if the degree of every finite extension of k is a power of q . Over such a field, the presence of a zero-cycle of degree prime to q on a variety is equivalent to that of a rational point. Thus we obtain a positive answer to the original question when k is a q -special field with $q \neq \text{char } k$.

A related question concerns the existence of non-trivial separable forms of the affine space: if X is k -variety such that $X_L \simeq \mathbb{A}_L^n$ for some algebraic separable field extension L/k , do we have $X \simeq \mathbb{A}_k^n$? It is easy to see that the answer is positive when $n \in \{0, 1\}$. This is also the case when $n = 2$ by a result of Shafarevich (see [Sha66], [GD75, (5.8.1)], [Kam75]), but again, very little is known as soon as $n \geq 3$, even for $k = \mathbb{R}$ and $L = \mathbb{C}$ (see [Kra96, Remark 4]). The arguments of the proof of Theorem 1.1 allow us to show (in §3.5) that X at least supports a zero-cycle whose degree is a power of the characteristic exponent of k .

The second result of this paper concerns the case $n = 3$ in the question of Serre:

Theorem 1.2. *Let G be a finite l -group acting on \mathbb{A}^3 over a perfect field k , whose characteristic differs from l . Then the action fixes a $k((t))$ -rational point.*

The conclusion of Theorem 1.2 may be reformulated by saying that any compactification of $(\mathbb{A}^3)^G$ has a k -rational point. A field is called *fertile* if any dense

open subvariety of a smooth variety with a rational point has itself a rational point. This notion was introduced by Pop in [Pop96] (he calls such fields “large fields”); the terminology “fertile” is due to Moret-Bailly. Any finite extension of a fertile field is fertile, and every henselian field is fertile (more examples and references may be found e.g. in [Pop14]). Thus we obtain a positive answer to the question of Serre when k is perfect and fertile, and $n = 3$.

The notation and conventions used in the paper are given in §2. Theorem 1.1 is proved in §3, and Theorem 1.2 in §4 and §5.

2. NOTATION

In the whole paper, we work over a base field k , and fix a prime number l unequal to its characteristic.

2.1. Varieties. We denote by k_s a separable algebraic closure of k . A variety, or k -variety, is a reduced separated scheme of finite type over k . Closed subsets of a variety will be considered as closed subvarieties using the reduced structure. If L/k is a field extension and X a k -variety, we will write X_L for the L -variety obtained by extending scalars and taking the underlying reduced scheme. We will say that a variety X is geometrically irreducible, resp. connected, if X_{k_s} is irreducible, resp. connected (the empty set is neither connected nor irreducible). The residue field of a point x of X will be denoted by $\kappa(x)$. A variety is complete if it is proper over k . A variety is a compactification of X if it is complete and contains X as a dense open subscheme; a compactification always exists by a theorem of Nagata [Lüt93].

The index n_X of a variety X is the g.c.d. of the degrees of its closed points (it is zero when $X = \emptyset$).

The subset $\text{Sing}(X)$ of points $x \in X$ such that the local ring $\mathcal{O}_{X,x}$ is not regular is closed in X by a theorem of Zariski [Gro65, (6.12.5)]. The variety X is regular when $\text{Sing}(X) = \emptyset$.

When X is a smooth variety, we denote by T_X its tangent bundle.

We denote by \mathbb{A}^n the k -variety $\text{Spec } k[t_1, \dots, t_n]$.

2.2. Normal crossing divisors. An effective Cartier divisor $D \rightarrow X$ will be called an *snc divisor* if for every $x \in X$ there is a regular system of parameters f_1, \dots, f_n in $\mathcal{O}_{X,x}$ such that the ideal $\mathcal{O}(-D)_x$ of $\mathcal{O}_{X,x}$ is generated by the element $(f_1)^{r_1} \cdots (f_n)^{r_n}$ for some integers $r_i \geq 0$ (see [Liu02, Definition 9.1.6]). We will say that a closed subvariety of X is the *support of an snc divisor in X* if it underlies some snc divisor.

If Y is the support of an snc divisor in X , then X is regular. One also sees easily that each irreducible component of Y is regular, and that no point of codimension n in X belongs to more than n irreducible components of Y .

Lemma 2.2.1. *Let X be a k -variety, and L/k a separable algebraic field extension. If Y is the support of an snc divisor in X , then so is Y_L in X_L .*

Proof. It will suffice to prove that if $D \rightarrow X$ is an snc divisor, then so is $D_L \rightarrow X_L$. Let $y \in X_L$ and $x \in X$ be its image. Write $A = \mathcal{O}_{X,x}$ and $B = \mathcal{O}_{X_L,y}$. Let f_1, \dots, f_n be a regular system of parameters in A such that the ideal $\mathcal{O}_X(-D)_x$ of A is generated by the element $(f_1)^{r_1} \cdots (f_n)^{r_n}$. Then the ideal $\mathcal{O}_{X_L}(-D_L)_y$ of B is generated by the image of this element. It will therefore suffice to prove that (the images of) f_1, \dots, f_n form a regular system of parameters in B . Let

$C = A \otimes_k L$. The noetherian ring $\kappa(x) \otimes_k L = C/(f_1, \dots, f_n)$ is integral over $\kappa(x)$ (because L/k is algebraic) and reduced (because L/k is separable). Therefore the ring $C/(f_1, \dots, f_n)$ is reduced and artinian, and so is its localisation $B/(f_1, \dots, f_n)$. The latter is additionally local, hence must be a field, showing that f_1, \dots, f_n generate the maximal ideal of B . This concludes the proof, since $\dim B = \dim A = n$ by [Gro65, (6.1.3)] applied to the morphism $A \rightarrow B$. \square

2.3. Group actions. When G is a finite group, an action of G on a variety X will mean an action by k -automorphisms. The fixed locus X^G is a closed subvariety of X such that $X^G(L) = X(L)^G$ for any field extension L/k . In fact, there is a scheme-theoretic version of the fixed locus, defined as the intersection in X of the equalisers of id_X and the action of σ , where σ runs over G . The variety X^G is the underlying reduced scheme. When X is smooth and the order of G is prime to the characteristic of k , then the scheme-theoretic fixed locus is smooth [Edi92, Proposition 3.4], hence coincides with the variety X^G .

2.4. Cohomology groups. Let X be a variety. We will write $H^i(X_{k_s}, \mathbb{Q}_l)$ (resp. $H_c^i(X_{k_s}, \mathbb{Q}_l)$) for the étale cohomology groups (resp. with compact supports) with \mathbb{Q}_l -coefficients of the k_s -variety X_{k_s} .

2.5. Graphs. By a graph, we will mean a finite undirected graph. In other words, a graph Γ consists in a finite set of vertices $V(\Gamma)$, and for each unordered pair of vertices $\{v_1, v_2\}$ a finite set of edges $E(\{v_1, v_2\})$. If $e \in E(\{v_1, v_2\})$ we say that v_1 and v_2 are the extremities of the edge e , or that e is an edge between v_1 and v_2 . Declaring two vertices equivalent if there is an edge between them generates an equivalence relation on $V(\Gamma)$. The graph Γ will be called connected if there is exactly one equivalence class. A tree is a connected graph with $n - 1$ edges and n vertices, for some $n \geq 1$. A vertex v of a tree is called a leaf if there is at most one edge one of whose extremities is v .

3. ZERO-CYCLES OF DEGREE ONE

3.1. Euler characteristic and index.

Definition 3.1.1. The Euler characteristic (with compact supports) of a variety X is defined as:

$$\chi(X) = \sum_i (-1)^i \dim_{\mathbb{Q}_l} H_c^i(X_{k_s}, \mathbb{Q}_l).$$

If Z is a closed subvariety of X , then the long exact sequence of cohomology groups with compact supports [Mil80, III, Remark 1.30] yields the relation

$$\chi(X) = \chi(Z) + \chi(X - Z).$$

In the next proposition, we denote by c the total Chern class with values in the Chow group.

Proposition 3.1.2. *Let X be a smooth complete variety. Then $\chi(X) = \deg c(T_X)$. In particular X supports a zero-cycle of degree $\chi(X)$.*

Proof. This follows from the Lefschetz trace formula [Mil80, VI, Theorem 12.3] and the self-intersection formula [Ful98, Example 8.1.12]. \square

Lemma 3.1.3. *Let L/k be a purely inseparable field extension, and X a non-empty k -variety. Then n_X/n_{X_L} is a power of the characteristic exponent of k .*

Proof. Let p be the characteristic exponent of k . Let x be a closed point of X_L . Its image y in X is a closed point, because the extension L/k is algebraic. By multiplicativity of separable degrees [Lan02, V, Theorem 4.1], we have in $\mathbb{N} \cup \{\infty\}$

$$[\kappa(x) : L]_s \cdot [L : k]_s = [\kappa(x) : k]_s = [\kappa(x) : \kappa(y)]_s \cdot [\kappa(y) : k]_s.$$

By assumption we have $[L : k]_s = 1$, and since the extension $\kappa(x)/L$ is finite, all the above displayed separable degrees are finite. It follows that

$$[\kappa(y) : k]_s \mid [\kappa(x) : L]_s.$$

Since the extension $\kappa(y)/k$ is finite, there is an integer m such that $[\kappa(y) : k] = p^m \cdot [\kappa(y) : k]_s$. Thus we have a chain of divisibilities

$$n_X \mid [\kappa(y) : k] \mid p^m \cdot [\kappa(x) : L]_s \mid p^m \cdot [\kappa(x) : L].$$

Now let $x_1, \dots, x_n \in X_L$ be a finite family of closed points such that n_{X_L} is the g.c.d. of $[\kappa(x_1) : L], \dots, [\kappa(x_n) : L]$. As we have just seen, we may find for each i an integer m_i such that $n_X \mid p^{m_i} \cdot [\kappa(x_i) : L]$. Letting m be the maximum of the m_i 's, we obtain that $n_X \mid p^m \cdot n_{X_L}$.

On the other hand any zero-cycle on X gives rise by scalars extension to a zero-cycle of the same degree on the L -scheme $X \otimes_k L$, and therefore also on the underlying L -variety X_L . Thus $n_{X_L} \mid n_X$, and the statement follows. \square

Proposition 3.1.4. *Let X be a k -variety.*

- (i) *If $\text{char } k = 0$, or if k is perfect and $\dim X \leq 3$, then X supports a zero-cycle of degree $\chi(X)$.*
- (ii) *If $\text{char } k = p > 0$, then X supports a zero-cycle of degree $p^m \cdot \chi(X)$, for some integer m .*

Proof. (i): We proceed by induction on $\dim X$; if $X = \emptyset$, then $\chi(X) = 0$ and X supports the null zero-cycle. While proving the statement, one may replace X by any dense open subvariety. Indeed assume that such a subvariety U of X supports a zero-cycle of degree $\chi(U)$. Then so does X . Since by induction $X - U$ supports a zero-cycle of degree $\chi(X - U)$, so again does X . Therefore X supports a zero-cycle of degree $\chi(X) = \chi(U) + \chi(X - U)$.

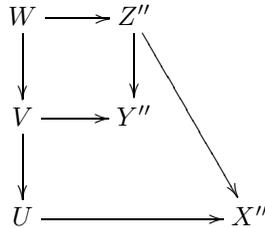
In particular we may replace X by a smooth dense open subvariety (which exists since k is perfect). Then we may find a smooth compactification X' of X ; this result is due to Hironaka [Hir64] when $\text{char } k = 0$, to Abhyankar (see e.g. Lipman [Lip78]) when $\dim X \leq 2$, and to Cossart and Piltant [CP14] when $\dim X \leq 3$. Then X' supports a zero-cycle of degree $\chi(X')$ by Proposition 3.1.2, and by induction $Z = X' - X$ (hence also X') supports a zero-cycle of degree $\chi(Z)$. It follows that X' supports a zero-cycle of degree $\chi(X') - \chi(Z) = \chi(X)$. The same is true for its dense open subvariety X , by a moving lemma [GLL13, Proposition 6.8] (see also [CT05, p. 599]).

(ii): We let q be a prime number different from p and denote by v_q the q -adic valuation on \mathbb{Z} . We need to prove that $v_q(n_X) \leq v_q(\chi(X))$. Replacing k by a perfect closure affects neither $\chi(X)$ [SGA72, VIII, Théorème 1.1] nor $v_q(n_X)$ (Lemma 3.1.3). Thus we may assume that k is perfect. We again proceed by induction on $\dim X$. Induction on the number of irreducible components of X shows that we may assume that X is irreducible (the argument is the same as the beginning of the proof of (i)). Let X' be a compactification of X . By the results of Gabber on alterations [ILO, Introduction, Theorem 3 (1), or X, Theorem 3.5 (iii)],

we may find a smooth complete variety Y' , a morphism $f: Y' \rightarrow X'$, a non-empty open subvariety U of X' such that $V = f^{-1}U$ is dense in Y' and $V \rightarrow U$ is étale and finite of degree d , with d prime to q . Shrinking U , we may assume that U is smooth and contained in X . It will suffice to prove that $v_q(n_U) \leq v_q(\chi(U))$ (see the argument at the beginning of the proof of (i)). Consider the variety $B = X' - U$ (with reduced structure). Let X'' be the blow-up of B in X' , and Y'' the blow-up of the (possibly non-reduced) closed subscheme $f^{-1}B$ of Y' . There is an induced morphism $Y'' \rightarrow X''$. Using once again the result of Gabber, we find a smooth complete variety Z'' , a morphism $g: Z'' \rightarrow Y''$ generically of degree prime to q and such that $W = g^{-1}V$ is the complement of an snc divisor in Z'' . We now use results of K. Kato and T. Saito. We have by [KS08, Lemma 3.4.5.1]

$$(1) \quad \deg D_{V/U}^{log} = d \cdot \chi(U) - \chi(V) \in \mathbb{Q},$$

where $D_{V/U}^{log} \in \text{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the wild different [KS08, Definition 3.4.1], and $\text{CH}_0(\overline{V} \setminus V)$ is the Chow group of zero-cycles on the boundary [KS08, Definition 3.1.1]. We apply [KS08, Theorem 3.2.3.1], where the diagram (3.4) of [KS08] is



and obtain that the image of $D_{V/U}^{log}$ under the natural morphism $\text{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \text{CH}_0(Y'' - V) \otimes_{\mathbb{Z}} \mathbb{Q}$ lies in the image of the morphism $\text{CH}_0(Y'' - V) \otimes_{\mathbb{Z}} \mathbb{Z}_{(q)} \rightarrow \text{CH}_0(Y'' - V) \otimes_{\mathbb{Z}} \mathbb{Q}$ (here $\mathbb{Z}_{(q)} \subset \mathbb{Q}$ denotes those fractions whose denominator is prime to q). Since there is a morphism $Y'' - V \rightarrow Y'$, we deduce that

$$(2) \quad v_q(n_{Y'}) \leq v_q(n_{Y''-V}) \leq v_q(\deg D_{V/U}^{log}).$$

On the other hand $v_q(n_{Y'}) \leq v_q(\chi(Y'))$ (by Proposition 3.1.2, since Y' is smooth), and by induction $v_q(n_{Y'-V}) \leq v_q(\chi(Y' - V))$. Using the relation $n_{Y'} \mid n_{Y'-V}$, we obtain

$$(3) \quad v_q(n_{Y'}) \leq v_q(\chi(Y') - \chi(Y' - V)) = v_q(\chi(V)).$$

Combining (1), (2), (3), we deduce that $v_q(n_{Y'}) \leq v_q(\chi(U))$. Finally $n_{Y'} = n_V$ (by the moving lemma [GLL13, Proposition 6.8], since V is dense in the smooth variety Y'), and $n_U \mid n_V$ (since there is a morphism $V \rightarrow U$). We conclude that $v_q(n_U) \leq v_q(\chi(U))$, as required. \square

Remark 3.1.5. Statement (i) could be improved (by removing the assumption $\dim X \leq 3$, for k perfect) if we could resolve singularities in positive characteristic. On the other hand, we may see that (ii) is optimal by considering the case when X is the spectrum of a finite purely inseparable field extension of k .

3.2. Actions of l -groups on \mathbb{A}^n .

Proposition 3.2.1. *Let l be a prime number unequal to $\text{char } k$. Let G be a finite l -group acting on \mathbb{A}^n over k , and $S = (\mathbb{A}^n)^G$. Then S is smooth, geometrically connected, and satisfies $\chi(S) = 1$. More precisely:*

$$H^i(S_{k_s}, \mathbb{Q}_l) = \begin{cases} 0 & \text{if } i \neq 0, \\ \mathbb{Q}_l & \text{if } i = 0, \end{cases} \quad \text{and} \quad H_c^i(S_{k_s}, \mathbb{Q}_l) = \begin{cases} 0 & \text{if } i \neq 2 \dim S, \\ \mathbb{Q}_l & \text{if } i = 2 \dim S. \end{cases}$$

Proof. The variety S is smooth (see §2.3), and by Poincaré duality it will suffice to prove the statements concerning $H^i(S_{k_s}, \mathbb{Q}_l)$. Replacing k by a perfect closure does not affect $H^i(S_{k_s}, \mathbb{Q}_l)$ by [SGA72, VIII, Théorème 1.1], so that we may assume that k is perfect.

Let us say that a variety X is A -acyclic if the étale cohomology groups with coefficients in A satisfy $H_{\text{ét}}^0(X_{k_s}, A) = A$ and $H_{\text{ét}}^i(X_{k_s}, A) = 0$ for $i \neq 0$. The variety \mathbb{A}^n is \mathbb{Z}/l -acyclic [Mil80, Corollary 4.20], and it follows from Smith’s theory [Ser09, Theorem 7.5 and its corollary] that S is also \mathbb{Z}/l -acyclic. Using the exact sequence

$$0 \rightarrow \mathbb{Z}/l^m \rightarrow \mathbb{Z}/l^{m+1} \rightarrow \mathbb{Z}/l \rightarrow 0$$

we see by induction on m that S is \mathbb{Z}/l^m -acyclic for every positive integer m . Taking the limit over m , it follows that S is \mathbb{Z}_l -acyclic, and tensoring with \mathbb{Q}_l over \mathbb{Z}_l , we conclude that S is \mathbb{Q}_l -acyclic. □

Theorem 3.2.2. *Let k be a field of characteristic exponent p , and l a prime number unequal to p . Let G be a finite l -group acting on \mathbb{A}^n over k . Then $(\mathbb{A}^n)^G$ supports a zero-cycle whose degree is a power of p . If k is perfect and $n \leq 4$, then $(\mathbb{A}^n)^G$ supports a zero-cycle of degree one.*

Proof. Since $\chi((\mathbb{A}^n)^G) = 1$ by Proposition 3.2.1, one sees that the statement follows from Proposition 3.1.4 (we may assume that G acts non-trivially, so that $\dim(\mathbb{A}^n)^G \leq 3$ when $n \leq 4$). □

3.3. Actions of cyclic groups on \mathbb{A}^n .

Theorem 3.3.1. *Let k be a field of characteristic exponent p . Let G be a finite cyclic group of order prime to p acting on \mathbb{A}^n over k . Then $(\mathbb{A}^n)^G$ supports a zero-cycle whose degree is a power of p . If k is perfect and $n \leq 4$, then $(\mathbb{A}^n)^G$ supports a zero-cycle of degree one.*

Proof. By Lemma 3.1.3 we may assume that k is perfect. Let g be a generator of G . By a result of Deligne-Lusztig [DL76, Theorem 3.2] we have

$$(4) \quad \sum_i (-1)^i \text{Tr} (g^* : H_c^i(\mathbb{A}_{k_s}^n, \mathbb{Q}_l)) = \chi((\mathbb{A}^n)^G).$$

Using [Mil80, Corollary 4.20] for $F = \mathbb{Z}/l^m$ and taking the limit over m and tensoring with \mathbb{Q}_l over \mathbb{Z}_l , and using Poincaré duality, we see that $H_c^i(\mathbb{A}_{k_s}^n, \mathbb{Q}_l) = 0$ if $i \neq 2n$ and $H_c^{2n}(\mathbb{A}_{k_s}^n, \mathbb{Q}_l) = \mathbb{Q}_l$. Moreover G acts trivially on the set of connected components of $\mathbb{A}_{k_s}^n$, and thus also on $H^0(\mathbb{A}_{k_s}^n, \mathbb{Q}_l) \simeq H_c^{2n}(\mathbb{A}_{k_s}^n, \mathbb{Q}_l)$. Therefore the value of the left hand side of (4) is 1, and we conclude using Proposition 3.1.4, as in the proof of Theorem 3.2.2. □

3.4. q -special fields. Let q be a prime number. The field k is called q -special if the degree of every finite extension of k is a power of q . The combination of Theorem 3.2.2 and Theorem 3.3.1 is Theorem 1.1 of the introduction, and has the following consequence.

Corollary 3.4.1 (of Theorem 1.1). *Let k be a field of characteristic exponent p . Assume that k is q -special, for some prime number q unequal to p . Then the action on \mathbb{A}^n of any finite l -group with $l \neq p$, or any finite cyclic group of order prime to p , fixes a k -rational point.*

This corollary applies to real-closed fields, which are 2-special and of characteristic zero. In particular, we obtain a purely algebraic proof for the case $k = \mathbb{R}$ (the classical proof uses algebraic topology; see e.g. [EN11, §5.4]).

3.5. Separable forms of \mathbb{A}^n . As mentioned in the introduction, it is currently unknown whether every \mathbb{R} -variety X such that $X_{\mathbb{C}} \simeq \mathbb{A}_{\mathbb{C}}^n$ is isomorphic to $\mathbb{A}_{\mathbb{R}}^n$ (for $n \geq 3$). A consequence of the next proposition is that X must have an \mathbb{R} -point.

Proposition 3.5.1. *Let k be a field of characteristic exponent p , and X a k -variety such that $X_{k_s} \simeq \mathbb{A}_{k_s}^n$. Then X supports a zero-cycle whose degree is a power of p . If k is perfect and $n \leq 3$, then X supports a zero-cycle of degree one. If k is q -special for some prime number q unequal to p , then $X(k) \neq \emptyset$.*

Proof. This follows from Proposition 3.1.4 and the fact that $\chi(\mathbb{A}^n) = 1$, already observed in the course of the proof of Theorem 3.3.1. \square

4. ACTIONS OF l -GROUPS ON \mathbb{A}^3

A field F is called *fertile* if any dense open subvariety of a smooth F -variety with an F -rational point has itself an F -rational point. When a (smooth) k -variety S admits a smooth compactification S' , the following conditions are equivalent:

- Every compactification of S has a k -rational point.
- The variety S has an F -rational point for any fertile field F containing k .
- The variety S has a $k((t))$ -rational point.

Indeed, any of these conditions is equivalent to the condition $S'(k) \neq \emptyset$ (this follows from Nishimura's Lemma, the valuative criterion of properness and the fact that $k((t))$ is fertile; see [Pop14, §1.A.2]).

Theorem 4.1. *Let k be a perfect field, and l a prime number different from its characteristic. Let G be a finite l -group acting on \mathbb{A}^3 over k . Then the variety $S = (\mathbb{A}^3)^G$ satisfies the three conditions above.*

Proof. By Proposition 3.2.1, the variety S is geometrically connected, smooth and satisfies $\chi(S) = 1$. If $\dim S = 0$, then S must be a single rational point. If $\dim S = 3$, then $S = \mathbb{A}^3$. In these two cases, the conclusion of the theorem holds.

We now assume that $\dim S \in \{1, 2\}$. Since k is perfect, the variety S admits a smooth compactification S' such that the closed subvariety $D = S' - S$ is the support of an snc divisor in X (§2.2), by [Lip78] and [Liu02, Theorem 9.2.26]. It will suffice to prove that $S'(k) \neq \emptyset$. The variety S' is geometrically connected because its dense open subvariety S is so. Moreover $H_c^1(S_{k_s}, \mathbb{Q}_l) = 0$ by Proposition 3.2.1.

The exact sequence of \mathbb{Q}_l -vector spaces

$$H^0(S'_{k_s}, \mathbb{Q}_l) \rightarrow H^0(D_{k_s}, \mathbb{Q}_l) \rightarrow H^1_c(S_{k_s}, \mathbb{Q}_l)$$

then shows that D is geometrically connected.

In case $\dim S = 1$, the variety D is non-empty because S is affine. Thus $\dim D = 0$, and D must be a single rational point (k is perfect), which concludes the proof in this case. But we can say more: the exact sequence of \mathbb{Q}_l -vector spaces

$$H^1_c(S_{k_s}, \mathbb{Q}_l) \rightarrow H^1(S'_{k_s}, \mathbb{Q}_l) \rightarrow H^1(D_{k_s}, \mathbb{Q}_l)$$

shows that $H^1(S'_{k_s}, \mathbb{Q}_l) = 0$. Thus the k -variety S' is a smooth complete curve of genus zero (see e.g. [SGA73, IX, §4]), that is, a conic. It has a rational point, hence is isomorphic to \mathbb{P}^1 , and $S = \mathbb{P}^1 - D \simeq \mathbb{A}^1$.

We now assume that $\dim S = 2$. By semi-purity [Mil80, VI, Lemma 9.1], the restriction morphisms $H^1_{\text{ét}}(S'_{k_s}, \mathbb{Z}/l^m) \rightarrow H^1_{\text{ét}}(S_{k_s}, \mathbb{Z}/l^m)$ are injective for all m , and by left-exactness of the inverse limit and flatness of \mathbb{Q}_l over \mathbb{Z}_l , it follows that $H^1(S'_{k_s}, \mathbb{Q}_l) \rightarrow H^1(S_{k_s}, \mathbb{Q}_l)$ is injective. Since by Proposition 3.2.1 we have $H^1(S_{k_s}, \mathbb{Q}_l) = 0$, we deduce that $H^1(S'_{k_s}, \mathbb{Q}_l) = 0$. Proposition 3.2.1 also yields $H^2_c(S_{k_s}, \mathbb{Q}_l) = 0$, and the exact sequence of \mathbb{Q}_l -vector spaces

$$H^1(S'_{k_s}, \mathbb{Q}_l) \rightarrow H^1(D_{k_s}, \mathbb{Q}_l) \rightarrow H^2_c(S_{k_s}, \mathbb{Q}_l)$$

shows that $H^1(D_{k_s}, \mathbb{Q}_l) = 0$. By Lemma 4.3, the variety D supports a zero-cycle of degree $\chi(S)$, and $\chi(S) = 1$ by Proposition 3.2.1. We prove in Proposition 5.3.1 below that $D(k) \neq \emptyset$, which implies that $S'(k) \neq \emptyset$. \square

Remark 4.2. Let k be an arbitrary field, and l a prime number different from its characteristic. Let G be a finite l -group acting on \mathbb{A}^n over k , with n arbitrary. Let $S = (\mathbb{A}^n)^G$. The proof of Theorem 4.1 shows the following:

- If $\dim S = n$, then $S = \mathbb{A}^n$.
- If $\dim S = 0$, then S is a single k -rational point.
- If $\dim S = 1$ and k is perfect, then $S \simeq \mathbb{A}^1$. In case $n = 2$, this was proved in [EN11, Theorem 5.12] without assuming that k is perfect.

However, our proof breaks down when $\dim S = 2$ and $n > 3$ because Lemma 4.3 below seems to be specific to subvarieties of codimension one in \mathbb{A}^n .

Lemma 4.3. *Assume that the field k is perfect. Let S be a smooth closed subvariety of pure codimension one in \mathbb{A}^n , with $2 \leq n \leq 4$. Let S' be a smooth compactification of S . Then $D = S' - S$ supports a zero-cycle of degree $\chi(S)$.*

Proof. The closed embedding $i: S \rightarrow \mathbb{A}^n$ is an effective Cartier divisor; let $L = \mathcal{O}(S)$ be the corresponding line bundle on \mathbb{A}^n . Then we have an exact sequence of vector bundles on S [Gro67, (17.13.2.1)]

$$0 \rightarrow T_S \rightarrow i^*T_{\mathbb{A}^n} \rightarrow i^*L \rightarrow 0.$$

Thus $[T_S] \in K_0(S)$ is in the image of the morphism $i^*: K_0(\mathbb{A}^n) \rightarrow K_0(S)$. By homotopy invariance [Qui73, §6, Corollary of Theorem 9], the natural morphism $\mathbb{Z} \rightarrow K_0(\mathbb{A}^n)$ is an isomorphism, and we deduce that $[T_S] = \text{rank } T_S = n - 1 \in K_0(S)$. This implies that $c_{n-1}(T_S) = 0 \in \text{CH}_0(S)$, because the Chern classes of a vector bundle depend only on its class in K_0 [Ful98, Example 3.2.7 (b)], and the $(n - 1)$ -st Chern class of a trivial bundle vanishes (since $n - 1 > 0$). Now $T_S = (T_{S'})|_S$, hence $c_{n-1}(T_{S'})|_S = 0 \in \text{CH}_0(S)$. By the localisation sequence for

Chow groups [Ful98, Proposition 1.8], it follows that the cycle class $c_{n-1}(T_{S'})$ lies in the image of the morphism $\text{CH}_0(D) \rightarrow \text{CH}_0(S')$. Thus D supports a zero-cycle of degree $\deg c_{n-1}(T_{S'})$, and $\deg c_{n-1}(T_{S'}) = \chi(S')$ by Proposition 3.1.2. Since the variety D also supports a zero-cycle of degree $\chi(D)$ by Proposition 3.1.4 (i) (because $n - 1 \leq 3$), it must support one of degree $\chi(S) = \chi(S') - \chi(D)$. \square

5. ONE-DIMENSIONAL SNC DIVISORS WITH NO H^1

5.1. The geometric number of components.

Definition 5.1.1. Let X be a variety. We define m_X , resp. μ_X , as the number of irreducible, resp. connected, components of the k_s -variety X_{k_s} .

When X is a variety, we have

$$(5) \quad \mu_X = \dim_{\mathbb{Q}_l} H^0(X_{k_s}, \mathbb{Q}_l).$$

Lemma 5.1.2. Let X be a variety of pure dimension n . Then

$$m_X = \dim_{\mathbb{Q}_l} H_c^{2n}(X_{k_s}, \mathbb{Q}_l).$$

Proof. Replacing k by a perfect closure affects neither m_X (this operation does not affect the topological space X_{k_s}) nor $H_c^{2n}(X_{k_s}, \mathbb{Q}_l)$ [SGA72, VIII, Théorème 1.1]. Thus we may assume that k is perfect. Let U be a smooth dense open subvariety of X , and Z its complement. We have an exact sequence of \mathbb{Q}_l -vector spaces

$$H_c^{2n-1}(Z_{k_s}, \mathbb{Q}_l) \rightarrow H_c^{2n}(U_{k_s}, \mathbb{Q}_l) \rightarrow H_c^{2n}(X_{k_s}, \mathbb{Q}_l) \rightarrow H_c^{2n}(Z_{k_s}, \mathbb{Q}_l),$$

where the two extreme groups vanish since $\dim Z < n$. Using Poincaré duality and (5), we deduce that

$$\dim_{\mathbb{Q}_l} H_c^{2n}(X_{k_s}, \mathbb{Q}_l) = \dim_{\mathbb{Q}_l} H_c^{2n}(U_{k_s}, \mathbb{Q}_l) = \dim_{\mathbb{Q}_l} H^0(U_{k_s}, \mathbb{Q}_l) = \mu_U.$$

The result follows, since $m_X = m_U$ (the set U_{k_s} is dense in the noetherian space X_{k_s}), and $m_U = \mu_U$ (the scheme U_{k_s} is locally irreducible, being smooth). \square

Lemma 5.1.3. Let X be a connected complete variety. Then $\mu_X \mid n_X$.

Proof. When $X = \text{Spec } F$ for a finite field extension F/k , the integer μ_X is the separable degree $[F : k]_s$, which divides the degree $[F : k] = n_X$.

In general, the k -algebra $F = H^0(X, \mathcal{O}_X)$ is reduced and finite, and moreover the k -variety $\text{Spec } F$ is connected. Thus F is a finite field extension of k . Since $\mu_{\text{Spec } F} = \mu_X$, and $n_{\text{Spec } F} \mid n_X$, we conclude using the special case treated above. \square

5.2. The dual graph.

Definition 5.2.1. Let X be a k -variety. The *dual graph* of X , denoted henceforth Γ_X , is the undirected graph defined as follows. Its vertices are the irreducible components of the k_s -variety X_{k_s} (its cardinality is thus m_X). The set of edges between two irreducible components is empty if they coincide, and equal to the set of irreducible components (over k_s) of their intersection otherwise.

The graph Γ_X is naturally endowed with an action of the absolute Galois group $\text{Gal}(k_s/k)$, and orbits in the set of vertices correspond bijectively to irreducible components of X by [Sta15, Tag 04KY].

Lemma 5.2.2. A variety X is geometrically connected if and only if its dual graph Γ_X is connected.

Proof. First note that $X = \emptyset$ if and only if $\Gamma_X = \emptyset$. Associating to a set of vertices V the union $\alpha(V)$ of the corresponding irreducible components yields a bijection between the subsets of $V(\Gamma_X)$ and the subsets of X_{k_s} which are unions of irreducible components of X . We have $\alpha(V_1 \cup V_2) = \alpha(V_1) \cup \alpha(V_2)$. Moreover $\alpha(V_1) \cap \alpha(V_2) = \emptyset$ if and only if $V_1 \cap V_2 = \emptyset$ and there are no edges between elements V_1 and V_2 . To conclude the proof, note that a closed and open subset of X_{k_s} is necessarily a union of irreducible components of X_{k_s} . \square

5.3. Zero-cycles of odd degree and rational points.

Proposition 5.3.1. *Assume that k is perfect. Let X be a complete, geometrically connected variety of dimension one such that $H^1(X_{k_s}, \mathbb{Q}_l) = 0$. Assume that X is the support of an snc divisor in some variety (§2.2). If X supports a zero-cycle of odd degree, then $X(k) \neq \emptyset$.*

Proof. Since $\dim X \leq 1$, the variety $I = \text{Sing}(X)$ is finite. It follows from [Gro65, (6.7.4)] that $I_{k_s} = \text{Sing}(X_{k_s})$. Let X_1, \dots, X_{m_X} be the irreducible components of X_{k_s} . Since X_{k_s} is the support of an snc divisor in some variety (Lemma 2.2.1), each point of I_{k_s} belongs to exactly two X_i 's. Conversely any point of $X_i \cap X_j$ with $i \neq j$ belongs to I_{k_s} . It follows that I_{k_s} is the disjoint union of the varieties underlying $X_i \cap X_j$ for $i < j$, and that $\chi(I)$ is the number of edges of Γ_X . By Lemma 5.3.2 below, it follows that

$$\chi(X) = \sum_i \chi(X_i) - \sum_{i < j} \chi(X_i \cap X_j) = \sum_i \chi(X_i) - \chi(I).$$

From the assumption on $H^1(X_{k_s}, \mathbb{Q}_l)$ and by Lemma 5.1.2, we deduce that

$$(6) \quad 1 + m_X = \sum_{i=1}^{m_X} \chi(X_i) - \chi(I).$$

Since for each i , the k_s -variety X_i is irreducible, we have

$$(7) \quad \chi(X_i) = 1 - \dim_{\mathbb{Q}_l} H^1(X_i, \mathbb{Q}_l) + 1 \leq 2.$$

From (6) and (7) we deduce that $\chi(I) \leq m_X - 1$ with equality if and only if $H^1(X_i, \mathbb{Q}_l) = 0$ for each i . But Γ_X is a connected graph with m_X vertices and $\chi(I)$ edges, hence $\chi(I) \geq m_X - 1$. Thus $\chi(I) = m_X - 1$ and Γ_X is a tree. In addition, for each i , the k_s -variety X_i is a smooth connected curve, whose genus is zero because $H^1(X_i, \mathbb{Q}_l) = 0$ by (7) (see e.g. [SGA73, IX, §4]). It follows that $X_i \simeq \mathbb{P}_{k_s}^1$ for each i . We conclude using Lemma 5.3.3 below. \square

Lemma 5.3.2. *Let X_1, \dots, X_n be closed subvarieties of X such that $X_i \cap X_j \cap X_l = \emptyset$ whenever i, j, l are pairwise distinct. If $X = X_1 \cup \dots \cup X_n$, then*

$$\chi(X) = \sum_i \chi(X_i) - \sum_{i < j} \chi(X_i \cap X_j).$$

Proof. We proceed by induction on n , the case $n = 0$ being clear. When $n > 0$, let $Y = X_1 \cup \dots \cup X_{n-1}$. Then $Y \cap X_n$ is the disjoint union of the varieties $X_i \cap X_n$

for $i < n$, and using the induction hypothesis,

$$\begin{aligned} \chi(X) &= \chi(X_n) + \chi(Y) - \chi(Y \cap X_n) \\ &= \chi(X_n) + \sum_{i < n} \chi(X_i) - \sum_{i < j < n} \chi(X_i \cap X_j) - \sum_{i < n} \chi(X_i \cap X_n) \\ &= \sum_i \chi(X_i) - \sum_{i < j} \chi(X_i \cap X_j). \end{aligned} \quad \square$$

Lemma 5.3.3. *Assume that k is perfect. Let X be a complete k -variety such that Γ_X is a tree and every irreducible component of the k_s -variety X_{k_s} is isomorphic to $\mathbb{P}^1_{k_s}$. If X supports a zero-cycle of odd degree, then $X(k) \neq \emptyset$.*

Proof. We first reduce to the case when X is irreducible by induction on the number of irreducible components of X . Let us assume that X is not irreducible, and let T be the irreducible component of X corresponding to the $\text{Gal}(k_s/k)$ -orbit of a leaf in Γ_X . Note that the closure Y of $X - T$ in X is non-empty. We may view the graphs Γ_T and Γ_Y as full subgraphs of Γ_X , compatibly with the $\text{Gal}(k_s/k)$ -actions, and the set of vertices of Γ_X is the disjoint union of the set of vertices of Γ_T and Γ_Y . Since each vertex of Γ_T is a leaf of the tree Γ_X , it follows that Γ_Y is a tree (removing a leaf from a tree yields a tree or the empty graph). The irreducible components of Y_{k_s} , being among those of X_{k_s} , are isomorphic to $\mathbb{P}^1_{k_s}$.

Let P be an irreducible component of T_{k_s} . We claim that in the graph Γ_X there is exactly one edge one of whose extremities is P , and that the other extremity is an irreducible component of Y_{k_s} . Since P is a leaf in the tree Γ_X , there is at most one such edge. If there are none, then P is the only vertex of Γ_X , hence $P = X_{k_s}$ and X is irreducible, a contradiction. Thus there is exactly one such edge; let Q be its other extremity. If Q is an irreducible component of T_{k_s} , it is a leaf of the tree Γ_X , and P, Q are the only two vertices of Γ_X . Then $T = X$ is again irreducible, a contradiction. Therefore Q is an irreducible component of Y_{k_s} , which proves the claim.

From the claim we deduce that the irreducible components of T_{k_s} are pairwise disjoint (so that $m_T = \mu_T$), and are in bijection with the points of the finite k_s -variety $(T \cap Y)_{k_s}$. Thus $\mu_{T \cap Y} = \mu_T$. If n_T is even, then n_Y must be odd. If n_T is odd, then so is μ_T by Lemma 5.1.3, and thus also $\mu_{T \cap Y}$. The latter is the sum of the separable degrees of the residue fields of the points of the finite variety $T \cap Y$, which are in particular closed points of Y . Since k is perfect, we deduce again that n_Y must be odd. Therefore in any case we may conclude the proof using the induction hypothesis for the variety Y .

Thus we may assume that X is irreducible. Since the group $\text{Gal}(k_s/k)$ acts transitively on the tree Γ_X , each of the vertices of Γ_X is a leaf. It follows that the graph Γ_X has no more than two vertices (and thus at most one edge). If X_{k_s} has two irreducible components, they meet in a single point (over k_s), which coincides with $\text{Sing}(X_{k_s}) = \text{Sing}(X)_{k_s}$ [Gro65, (6.7.4)]. Since k is perfect, it follows that $\text{Sing}(X)$ is a rational point of X . Otherwise, the variety X is geometrically irreducible. Thus $X_{k_s} \simeq \mathbb{P}^1_{k_s}$, hence X is a smooth conic over k . Since X supports a zero-cycle of odd degree, it must possess a rational point by Springer’s theorem. \square

Remark 5.3.4. It is not necessary to assume that X is the support of an snc divisor in Proposition 5.3.1. An earlier version of this paper contained a proof of this

more general statement, but as pointed out by Johannes Nicaise, Proposition 5.3.1 suffices for the proof of Theorem 4.1 and is substantially shorter to prove.

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