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LYAPUNOV REGULARITY VIA SINGULAR VALUES

LUIS BARREIRA AND CLAUDIA VALLS

ABSTRACT. For a nonautonomous linear dynamics, we study the relation between Lyapunov regularity and the exponential growth rates of the singular values. In particular, for a tempered dynamics, we obtain lower and upper estimates for the Lyapunov exponents in terms of the growth rates. The proof is based on the somewhat unexpected existence of a structure of Oseledets type for any nonregular dynamics. Moreover, we show that any possible values of the Lyapunov exponent and of the growth rates are attained by some bounded sequence of matrices. As an application of our results, we give a simple proof of various characterizations of Lyapunov regularity as well as a new characterization. We consider both discrete and continuous time.

1. Introduction

1.1. **Lyapunov regularity and stability.** Our main aim is to describe how Lyapunov regularity relates to various other properties and quantities, such as the exponential growth rates of the singular values. The notion of regularity was introduced by Lyapunov and plays an important role in the Lyapunov stability theory. In particular, it allows studying when the type of stability of a linear dynamics persists under sufficiently small nonlinear perturbations. This is particularly effective in the context of ergodic theory—indeed, for a smooth dynamics preserving a finite measure and satisfying a certain integrability assumption, the linearization along almost all trajectories is Lyapunov regular, as a consequence of Oseledets' multiplicative ergodic theorem.

In order to illustrate the relation of Lyapunov regularity to the stability theory, consider a linear equation

$$v' = A(t)v \tag{1}$$

on $\mathbb{R}^q$, where the $q \times q$ matrices $A(t)$ vary continuously with $t \in \mathbb{R}$. Its Lyapunov exponent is the function $\lambda\colon \mathbb{R}^q \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\lambda(v_0) = \limsup_{t\to\infty} \frac{1}{t} \log\|v(t)\|, \tag{2}$$

where $v(t)$ denotes the solution of (1) with $v(0) = v_0$, with the convention that $\log 0 = -\infty$ (notice that all solutions of equation (1) are global). The *Lyapunov coefficient of regularity* of equation (1) is defined by

$$\sigma(A) = \min \sum_{i=1}^{q} \lambda(v_i) - \liminf_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} A(\tau)\, d\tau,$$

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where the minimum is taken over all bases $v_1, \dots, v_q$ of $\mathbb{R}^q$ (see [9]). One can easily verify that $\sigma(A) \ge 0$. The equation is said to be *Lyapunov regular* or simply *regular* if $\sigma(A) = 0$ or, equivalently, if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} A(\tau)\, d\tau = \sum_{i=1}^q \lambda(v_i)$$

for some basis $v_1, \dots, v_q$ of $\mathbb{R}^q$. There are various other regularity coefficients that can be used to define Lyapunov regularity, such as those introduced by Perron (see [11, 13]) and Grobman (see [8]). We refer the reader to [3, 8] for detailed accounts of the theory and to [6] for an infinite-dimensional version in Hilbert spaces.

If all the values of the Lyapunov exponent λ in (2) are negative, that is, if $\lambda(v_0) < 0$ for every $v_0 \in \mathbb{R}^q$, then equation (1) is asymptotically stable. This means that any solution $v(t)$ of equation (1) tends to zero when $t \to +\infty$ (see for example [7]). However, the equation need not be *uniformly* asymptotically stable (on the initial time), which would mean, in addition to being asymptotically stable, that for each $\varepsilon > 0$ there would exist $\delta = \delta(\varepsilon) > 0$ such that for all $t_0 > 0$, any solution $v(t)$ of equation (1) with $\|v(t_0)\| < \delta$ would satisfy

$$\|v(t)\| < \varepsilon \quad \text{for all } t > t_0.$$

This means that the type of stability of the linear equation may change under arbitrarily small nonlinear perturbations. For example, for

$$A(t) = \begin{pmatrix} -15 - 14(\sin\log t + \cos\log t) & 0 \\ 0 & -15 + 14(\sin\log t + \cos\log t) \end{pmatrix} \tag{3}$$

we have $\lambda(v_0) = -1$ for all v_0 and so equation (1) is asymptotically stable. However, as shown by Perron [12], for the perturbed equation

$$u' = A(t)u + (0, u_1^4),$$

where $u = (u_1, u_2)$, there exists a solution $u(t)$ with

$$\limsup_{t\to\infty} \frac{1}{t} \log\|u(t)\| > 0$$

(see [3]). We note that for $A(t)$ as in (3) the linear equation is not regular. On the other hand, Lyapunov [9] showed that for a regular equation the asymptotic stability persists under sufficiently small nonlinear perturbations, such as for the equation

$$u' = A(t)u + f(t, u),$$

with $f\colon \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ continuous such that

$$\|f(t,u) - f(t,v)\| \le K\|u - v\|(\|u\|^q + \|v\|^q)$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{R}^q$, for some constants $K, q > 0$ (we refer the reader to [3] for details and generalizations).

1.2. **Singular values and Lyapunov exponents.** Now we recall the exponential growth rates of the singular values. Write the solutions of equation (1) in the form $v(t) = X(t)v_0$, say with $X(0) = \mathrm{Id}$, and let

$$\rho_1(t) \le \cdots \le \rho_q(t)$$

be the eigenvalues of the symmetric positive-definite matrix $(X(t)^*X(t))^{1/2}$ (we note that all of them are positive real numbers). These are the *singular values* of the equation, and their exponential lower and upper growth rates are the numbers

$$a_i = \liminf_{t\to\infty} \frac{1}{t} \log \rho_i(t), \quad b_i = \limsup_{t\to\infty} \frac{1}{t} \log \rho_i(t)$$

for $i = 1, \dots, q$. The upper growth rates and the values of the Lyapunov exponent satisfy

$$b_i \le \lambda_i' \text{ for } i < q \quad \text{and} \quad b_q = \lambda_q', \tag{4}$$

where

$$\lambda_1' \le \lambda_2' \le \cdots \le \lambda_q'$$

are the values of the Lyapunov exponent λ counted with their multiplicities (see section 3.1). To the best of our knowledge, there are at present no results in the literature concerning upper bounds for the Lyapunov exponents in terms of the exponential growth rates of the singular values.

One can consider corresponding notions and make similar observations for discrete time (the same applies to the discussion in section 1.1). Namely, let $(A_m)_{m\in\mathbb{N}}$ be a sequence of $q \times q$ matrices. The eigenvalues

$$\rho_1(m) \le \cdots \le \rho_q(m)$$

of the symmetric positive-semidefinite matrix $(\mathcal{A}_m^*\mathcal{A}_m)^{1/2}$, where

$$\mathcal{A}_m = A_{m-1}A_{m-2}\cdots A_1,$$

are the *singular values* of the sequence (we note that all of them are nonnegative real numbers). Their exponential lower and upper growth rates are the numbers

$$a_i = \liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m), \quad b_i = \limsup_{m\to\infty} \frac{1}{m} \log \rho_i(m)$$

for $i = 1, \dots, q$. Again one can show that property (4) holds.

1.3. **Main results and applications.** In this work we describe various relations between Lyapunov regularity, which are expressed in terms of the values of the Lyapunov exponent, and the exponential growth rates of the singular values. In particular, we obtain:

(1) sequences of matrices for which any given values of the Lyapunov exponent and of the growth rates of the singular values are attained;
(2) upper bounds for the values of the Lyapunov exponent in terms of the exponential growth rates of the singular values;
(3) a structure of Oseledets type for any nonregular tempered dynamics that is analogous to that in the multiplicative ergodic theorem;
(4) a simple proof of various characterizations of Lyapunov regularity as well as a new characterization.

We shall formulate briefly our main results in the case of discrete time, leaving the technicalities to the main text. Full descriptions for discrete and continuous time are given in the remaining sections.

Our first main result shows that there exists a bounded sequence of matrices for each given value of the Lyapunov exponent and of the growth rates of the singular values satisfying property (4).

Theorem A. *Consider real numbers*

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_q \quad \text{and} \quad \beta_1 \le \beta_2 \le \cdots \le \beta_q$$

such that $\alpha_i \le \beta_i$ *for* $i = 1, \ldots, q-1$ *and* $\alpha_q = \beta_q$. *Then there exists a bounded sequence* $(A_m)_{m\in\mathbb{N}}$ *of diagonal* $q \times q$ *matrices such that* $b_i = \alpha_i$ *and* $\lambda'_i = \beta_i$ *for* $i = 1, \ldots, q$.

In other words, the relations in (4) are the best possible between the two families of numbers and can even be attained taking only diagonal matrices. The construction is explicit and is partly inspired by work of Barabanov and Fominykh [2] for the case of continuous time: consider the sets

$$S_n = \bigcup_{i=0}^{\infty} \big[(n + 2(q+1)i)!, (n + 1 + 2(q+1)i)!\big)$$

for $n = 0, \ldots, 2q+1$ and define $A_m = \operatorname{diag}(e^{b_1(m)}, \ldots, e^{b_q(m)})$, where

$$b_j(m) = \begin{cases} \beta_j & \text{if } m \in S_{2j-1}, \\ \alpha_j & \text{if } m \in S_{2q+1}, \\ \alpha_1 & \text{otherwise.} \end{cases}$$

In order to formulate our second main result, let $F_i(m)$ be the eigenspace of the matrix $(\mathcal{A}_m^* \mathcal{A}_m)^{1/2}$ associated to the eigenvalue $\rho_i(m)$. Moreover, we recall that a sequence of matrices $(A_m)_{m\in\mathbb{N}}$ is said to be *tempered* if

$$\limsup_{m\to\infty} \frac{1}{m} \log \|A_m\| \le 0.$$

The result shows that there exists a structure of Oseledets type even for a nonregular tempered dynamics. In other words, and somewhat surprisingly, part of the structure provided by the multiplicative ergodic theorem is in fact present for *all* trajectories and not only for almost all trajectories.

Theorem B. *Let* $(A_m)_{m\in\mathbb{N}}$ *be a tempered sequence of* $q \times q$ *matrices. If the intervals* $[a_i, b_i]$ *are pairwise disjoint, then for each* $i = 1, \ldots, q$ *the sequence* $\bigoplus_{j\le i} F_j(m)$ *converges to some subspace* H_i *when* $m \to \infty$. *Moreover, for each* $i = 1, \ldots, q$ *and* $v, w \in H_i$ *with* $w \notin H_{i-1}$ *(taking* $H_0 = \{0\}$*), we have*

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m v\| \le a_i + \sum_{j=1}^{q} (b_j - a_j) \tag{5}$$

and

$$\liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \ge a_i.$$

In view of (4), it is also known that

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \ge b_i,$$

with equality for $i = q$. To the best of our knowledge, the upper bounds for the values of the Lyapunov exponent in (5) are the first in the literature that are expressed in terms of the exponential growth rates of the singular values.

Theorem 5 considers the general case when some intervals $[a_i, b_i]$ are not pairwise disjoint. Again, there exists a structure of Oseledets type for an arbitrary tempered dynamics. When the dynamics is regular one recovers the usual Oseledets' filtration.

In particular, for a regular dynamics the spaces H_i in Theorem B are exactly those in the Oseledets' filtration for a single trajectory, that is, $H_i = E_i$ for $i = 1, \dots, q$, where

$$E_i = \{v \in \mathbb{R}^q : \lambda(v) \leq \lambda_i'\}.$$

The proofs are partly inspired by arguments of Raghunathan in [14] in his proof of Oseledets' multiplicative ergodic theorem (see also [4]).

Finally, we give a new characterization of Lyapunov regularity, expressed in terms of the eigenvectors of the matrix $(\mathcal{A}_m^* \mathcal{A}_m)^{1/2}$.

Theorem C. *For a tempered sequence $(A_m)_{m\in\mathbb{N}}$ of $q \times q$ matrices, the sequence $(A_m)_{m\in\mathbb{N}}$ is regular if and only if the limit*

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m v_j(m)\|$$

exists for $i = 1, \dots, q$ and for some orthonormal bases $v_1(m), \dots, v_q(m)$ of eigenvectors of $(\mathcal{A}_m^* \mathcal{A}_m)^{1/2}$ for the eigenvalues $\rho_1(m), \dots, \rho_q(m)$.

2. Lyapunov regularity

In this section we recall the notion of Lyapunov regularity for a sequence of matrices and its characterization in terms of the exponential growth rate of volumes.

2.1. **Basic notions.** Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of $q \times q$ matrices with real entries. For each $m \in \mathbb{N}$, let

$$\mathcal{A}_m = \begin{cases} A_{m-1}A_{m-2}\cdots A_1, & m > 1, \\ \mathrm{Id}, & m = 1. \end{cases}$$

The *Lyapunov exponent* $\lambda\colon \mathbb{R}^q \to [-\infty, +\infty]$ associated to the sequence of matrices $(A_m)_{m\in\mathbb{N}}$ is defined by

$$\lambda(v) = \limsup_{m\to\infty} \frac{1}{m} \log \|\mathcal{A}_m v\|, \tag{6}$$

with the convention that $\log 0 = -\infty$. We note that the numbers $\lambda(v)$ are independent of the particular norm considered on $\mathbb{R}^q$.

We always assume in the paper that λ does not take the value $+\infty$. This standing hypothesis is standard (see [3]) and allows one to compute various sums that otherwise would be undefined (it ensures that the term $\infty - \infty$ does not occur). For example, if $\sup_{m\in\mathbb{N}}\|A_m\| < +\infty$, then λ does not take the value $+\infty$.

By the abstract theory of Lyapunov exponents (see [3]), the function λ takes at most a number $r \leq q$ of distinct values on $\mathbb{R}^q \setminus \{0\}$, say

$$-\infty \leq \lambda_1 < \lambda_2 < \cdots < \lambda_r \tag{7}$$

(and $\lambda_r < +\infty$ since we are assuming that λ does not take the value $+\infty$). Moreover, for each $i = 1, \dots, r$ the set

$$E_i = \{v \in \mathbb{R}^q : \lambda(v) \leq \lambda_i\} \tag{8}$$

is a linear subspace of $\mathbb{R}^q$ and

$$\{0\} \subset E_1 \subset E_2 \subset \cdots \subset E_r = \mathbb{R}^q.$$

We say that the sequence $(A_m)_{m\in\mathbb{N}}$ is *(Lyapunov) regular* if

$$\lim_{m\to\infty}\frac{1}{m}\log|\det\mathcal{A}_m| = \sum_{i=1}^q \lambda(v_i) \tag{9}$$

for some basis $v_1,\dots,v_q$ of $\mathbb{R}^q$ (this includes the requirement that the limit on the left exists). Since λ does not take the value $+\infty$, the sum on the right is well-defined.

Now let C be the $q\times q$ invertible matrix whose columns are the vectors $v_1,\dots,v_q$. Then $|\det(\mathcal{A}_mC)|\le\prod_{i=1}^q\|\mathcal{A}_mv_i\|$ since the vectors $\mathcal{A}_mv_i$ are the columns of $\mathcal{A}_mC$. Therefore,

$$\limsup_{m\to\infty}\frac{1}{m}\log|\det\mathcal{A}_m| = \limsup_{m\to\infty}\frac{1}{m}|\log(\mathcal{A}_mC)|\le\sum_{i=1}^q\lambda(v_i),$$

and so

$$\limsup_{m\to\infty}\frac{1}{m}\log|\det\mathcal{A}_m|\le\min\sum_{i=1}^q\lambda(v_i),$$

with the minimum taken over all bases $v_1,\dots,v_q$ of $\mathbb{R}^q$. Hence condition (9) can be replaced by the requirement that

$$\liminf_{m\to\infty}\frac{1}{m}\log|\det\mathcal{A}_m|\ge\sum_{i=1}^q\lambda(v_i)$$

for some basis $v_1,\dots,v_q$ of $\mathbb{R}^q$ or, equivalently, that

$$\liminf_{m\to\infty}\frac{1}{m}\log|\det\mathcal{A}_m|\ge\min\sum_{i=1}^q\lambda(v_i)=\sum_{j=1}^r\lambda_j(\dim E_j-\dim E_{j-1}),$$

with the infimum taken over all bases $v_1,\dots,v_q$ of $\mathbb{R}^q$ and where $E_0=\{0\}$.

2.2. **Exterior powers and volumes.** For each integer $k\in[1,q]$, let $(\mathbb{R}^q)^{\wedge k}$ be the set of all alternating k-linear forms on $\mathbb{R}^q$. We define an inner product on $(\mathbb{R}^q)^{\wedge k}$ by requiring that

$$\langle v_1\wedge\cdots\wedge v_k, w_1\wedge\cdots\wedge w_k\rangle=\det J, \tag{10}$$

where J is the $k\times k$ matrix with entries $b_{ij}=\langle v_i,w_j\rangle$ for each i and j. Given a $q\times q$ matrix A, its *k-fold exterior power* $A^{\wedge k}$ is the unique linear transformation $A^{\wedge k}$ of $(\mathbb{R}^q)^{\wedge k}$ such that

$$A^{\wedge k}(v_1\wedge\cdots\wedge v_k)=Av_1\wedge\cdots\wedge Av_k$$

for all $v_1,\dots,v_k\in\mathbb{R}^q$. We recall that the norm

$$\|A^{\wedge k}\|=\sup_{\|v\|=1}\|A^{\wedge k}v\|$$

satisfies the identity

$$\|A^{\wedge k}\|=\prod_{j=1}^k\rho_{q-j+1}, \tag{11}$$

where $\rho_1\le\cdots\le\rho_q$ are the eigenvalues of the matrix $\big(A^*A\big)^{1/2}$ and A^* is the transpose of A. Since A^*A is symmetric and positive-semidefinite, each eigenvalue ρ_i is real and positive. In particular, equipping $\mathbb{R}^q$ with the 2-norm (that is obtained from (10) with $k=1$), we have $\|A\|=\rho_q$.

Now we consider the relation between exterior powers and volumes. Given vectors $v_1, \dots, v_k \in \mathbb{R}^q$, the k-volume of the parallelepiped

$$\left\{ \sum_{i=1}^{k} \alpha_i v_i : 0 \le \alpha_i \le 1 \text{ for } i = 1, \dots, k \right\}$$

determined by them is defined by

$$\operatorname{vol}(v_1, \dots, v_k) = \sqrt{\det(B^* B)},$$

where B is the $q \times k$ matrix whose columns are the vectors $v_1, \dots, v_k$. Comparing to (10) we obtain

$$\operatorname{vol}(v_1, \dots, v_k) = \|v_1 \wedge \cdots \wedge v_k\|. \tag{12}$$

The following result of Lyapunov [9] (see also [5, 8]) gives a characterization of the notion of regularity in terms of the exponential growth rate of volumes. More precisely, Lyapunov considered the case of continuous time, but it is simple to obtain a corresponding version for discrete time.

Proposition 1. *A tempered sequence $(A_m)_{m\in\mathbb{N}}$ is regular if and only if the limit*

$$\lim_{m\to\infty} \frac{1}{m} \log \operatorname{vol}(\mathcal{A}_m v_1, \dots, \mathcal{A}_m v_k)$$

exists for any basis $v_1, \dots, v_q$ of $\mathbb{R}^q$ and any integer $k \in [1, q]$.

Moreover, by (12), we have

$$\lim_{m\to\infty} \frac{1}{m} \log \operatorname{vol}(\mathcal{A}_m v_1, \dots, \mathcal{A}_m v_k) = \lim_{m\to\infty} \frac{1}{m} \log \|(\mathcal{A}_m)^{\wedge k}(v_1 \wedge \cdots \wedge v_k)\|$$

whenever any of the limits exist.

3. Lyapunov exponents and singular values

In this section we describe some relations between the values of the Lyapunov exponent of a sequence of matrices and the exponential growth rates of their singular values. Moreover, we show that any possible values of the Lyapunov exponent and of the growth rates are attained by some bounded sequence of matrices. In addition, when the sequence is tempered, we obtain lower and upper estimates for the (lower and upper) Lyapunov exponents in terms of the growth rates. To the best of our knowledge, no upper estimates have been obtained before in the literature.

3.1. **Growth rates of the singular values.** Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of $q \times q$ matrices. We notice that the matrix

$$T_m = (\mathcal{A}_m^* \mathcal{A}_m)^{1/2} \tag{13}$$

is symmetric and positive-semidefinite. Hence, its eigenvalues

$$\rho_1(m) \le \cdots \le \rho_q(m)$$

are real and nonnegative. These are the *singular values* of the matrix $\mathcal{A}_m$. For $i = 1, \dots, q$ we define

$$a_i = \liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m), \quad b_i = \limsup_{m\to\infty} \frac{1}{m} \log \rho_i(m). \tag{14}$$

We refer to these numbers, respectively, as the lower and upper exponential growth rates of the singular values.

Now consider the values

$$\lambda_1' \le \lambda_2' \le \cdots \le \lambda_q'$$

of the Lyapunov exponent λ in (6) counted with their multiplicities, that is, $\lambda_i' = \lambda_j$ (see (7)) for $i = \dim E_{j-1} + 1, \ldots, \dim E_j$ and $j = 1, \ldots, r$, with the convention that $E_0 = \{0\}$. We have

$$\lambda_i' = \min_{F \in \mathcal{L}_i} \max_{v \in F, \|v\|=1} \limsup_{m \to \infty} \frac{1}{m} \log\|\mathcal{A}_m v\| \tag{15}$$

and

$$b_i = \limsup_{m \to \infty} \min_{F \in \mathcal{L}_i} \max_{v \in F, \|v\|=1} \frac{1}{m} \log\|\mathcal{A}_m v\|, \tag{16}$$

where $\mathcal{L}_i$ is the set of all i-dimensional subspaces of $\mathbb{R}^q$ (see for example [1]). Then

$$b_i \le \lambda_i' \text{ for } i < q \quad \text{and} \quad b_q = \lambda_q'. \tag{17}$$

Indeed, it follows readily from (15) and (16) that $b_i \le \lambda_i'$ for $i = 1, \ldots, q$. Moreover,

$$\begin{aligned} b_q &= \limsup_{m \to \infty} \max_{v \in \mathbb{R}^q, \|v\|=1} \frac{1}{m} \log\|\mathcal{A}_m v\| \\ &= \limsup_{m \to \infty} \frac{1}{m} \log\|\mathcal{A}_m\| \ge \lambda_q' \end{aligned}$$

and so $b_q = \lambda_q'$, that is,

$$\limsup_{m \to \infty} \frac{1}{m} \log\|\mathcal{A}_m\| = \max_{v \ne 0} \limsup_{m \to \infty} \frac{1}{m} \log\|\mathcal{A}_m v\|.$$

This establishes property (17).

Using arguments inspired by work of Barabanov and Fominykh [2] for the case of continuous time, we show that all possible numbers

$$b_1 \le b_2 \le \cdots \le b_q \quad \text{and} \quad \lambda_1' \le \lambda_2' \le \cdots \le \lambda_q'$$

satisfying (17) are attained by some sequence of matrices.

Theorem 2. *Consider real numbers*

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_q \quad \textit{and} \quad \beta_1 \le \beta_2 \le \cdots \le \beta_q$$

such that $\alpha_i \le \beta_i$ for $i = 1, \ldots, q-1$ and $\alpha_q = \beta_q$. Then there exists a bounded sequence $(A_m)_{m \in \mathbb{N}}$ of $q \times q$ matrices such that $b_i = \alpha_i$ and $\lambda_i' = \beta_i$ for $i = 1, \ldots, q$.

Proof. Consider an increasing sequence $(s_k)_{k \ge 0} \subset \mathbb{R}_0^+$ such that

$$s_0 = 0, \quad \frac{s_k}{s_{k+1}} \to 0 \text{ when } k \to \infty. \tag{18}$$

We define sets

$$S_n = \bigcup_{i=0}^{\infty} (s_{n+pi}, s_{n+1+pi}] \cap \mathbb{N}$$

for $n = 0, \ldots, p-1$, where $p = 2q + 2$. Finally, for each $m \in \mathbb{N}$ let

$$b_j(m) = \begin{cases} \beta_j & \text{if } m \in S_{2j-1}, \\ \alpha_j & \text{if } m \in S_{p-1}, \\ \alpha_1 & \text{if } m \in \mathbb{N} \setminus (S_{2j-1} \cup S_{p-1}) \end{cases} \tag{19}$$

and consider the matrix

$$A_m = \operatorname{diag}(e^{b_1(m)}, \dots, e^{b_q(m)}). \tag{20}$$

We first show that the values of the Lyapunov exponent associated to the sequence of matrices A_m in (20) are $\beta_1, \dots, \beta_q$. Since the matrices are diagonal, the canonical basis $e_1, \dots, e_q$ is normal, and so the values of the Lyapunov exponent are

$$\lambda(e_j) = \limsup_{m\to\infty} \frac{1}{m} \log \prod_{l=1}^{m} e^{b_j(l)} = \limsup_{m\to\infty} \frac{1}{m} \sum_{l=1}^{m} b_j(l)$$

for $j = 1, \dots, q$. Since $\alpha_1 \le \alpha_j \le \beta_j$, it follows from (19) that $b_j(m) \le \beta_j$ for $m \ge 0$, and thus $\lambda(e_j) \le \beta_j$. On the other hand, for each $i \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{l=1}^{s_{2j+pi}} b_j(l) &= \sum_{l=1}^{s_{2j-1+pi}} b_j(l) + \sum_{l=s_{2j-1+pi}+1}^{s_{2j+pi}} b_j(l) \\ &\ge \alpha_1 s_{2j-1+pi} + \beta_j(s_{2j+pi} - s_{2j-1+pi}).\end{aligned}$$

Therefore, by (18),

$$\lambda(e_j) \ge \alpha_1 \lim_{i\to\infty} \frac{s_{2j-1+pi}}{s_{2j+pi}} + \beta_j \lim_{i\to\infty} \left(1 - \frac{s_{2j-1+pi}}{s_{2j+pi}}\right) = \beta_j$$

for $j = 1, \dots, q$. Hence, the values of the Lyapunov exponent are $\beta_1, \dots, \beta_q$.

Now we show that the exponential growth rates of the singular values associated to the sequence of matrices A_m in (20) are $\alpha_1, \dots, \alpha_q$. For each $j = 1, \dots, q$, by (19) we have $b_j(m) = \alpha_1$ for $m \in \bigcup_{l=0}^{q} S_{2l}$ and $b_j(m) \le \beta_j \le \beta_q$ for $m \ge 0$. Hence,

$$\alpha_1 s_{n+pi} \le \sum_{l=1}^{s_{n+pi}} b_j(l) \le \alpha_1(s_{n+pi} - s_{n-1+pi}) + \beta_q s_{n-1+pi}$$

for n odd and $j \in \{1, \dots, q\}$. By (18), this implies that

$$\sum_{l=1}^{s_{n+pi}} b_j(l) = \alpha_1 s_{n+pi} + B_i,$$

where $B_i = B_i(n, j)$ satisfies

$$\lim_{i\to\infty} \frac{B_i}{s_{n+(2q+2)i}} = 0.$$

Therefore, for n odd distinct from $p-1$: if $m \in (s_{n+pi}, s_{n+1+pi}] \cap \mathbb{N}$ for some $i \in \mathbb{N}$, then

$$\sum_{l=1}^{m} b_j(l) = \begin{cases} \beta_{(r+1)/2} m + (\alpha_1 - \beta_{(r+1)/2}) s_{n+pi} + C_m & \text{if } j = (r+1)/2, \\ \alpha_1 m + D_m & \text{if } j \ne (r+1)/2, \end{cases} \tag{21}$$

where $C_m = C_m(j)$, $D_m = D_m(j)$ and

$$\lim_{m\to\infty} \frac{C_m}{m} = 0, \quad \lim_{m\to\infty} \frac{D_m}{m} = 0.$$

Now we observe that since the matrices A_m are diagonal, the singular values $\rho_1(m), \dots, \rho_q(m)$ of T_m are simply

$$\rho_j(m) = \exp \sum_{l=1}^{m-1} b_j(l) \quad \text{for } j = 1, \dots, q.$$

Moreover, we write

$$\limsup_{m\in S} \frac{1}{m} \log \rho_i(m) = \inf_{m\in\mathbb{N}} \sup \left\{ \frac{1}{l} \log \rho_i(l) : l \in S \cap [m, +\infty) \right\}.$$

By (21) and the fact that $\alpha_1 \le \beta_j$ for $j = 1, \dots, q$, for n odd distinct from $p-1$ we obtain

$$\begin{aligned} &\limsup_{m\in S_n} \frac{1}{m} \log \rho_1(m) = \limsup_{m\in S_n} \frac{1}{m} \sum_{l=1}^{m-1} b_1(l) = \beta_{(r+1)/2}, \\ &\limsup_{m\in S_n} \frac{1}{m} \log \rho_j(m) = \alpha_1, \quad j \ge 2. \end{aligned} \tag{22}$$

Since $b_j(m) = \alpha_1$ for $m \in \bigcup_{l=0}^{q} S_{2l}$, it follows from (22) that

$$\begin{aligned} &\limsup_{m\in\mathbb{N}\setminus S_{p-1}} \frac{1}{m} \log \rho_1(m) = \beta_q, \\ &\limsup_{m\in\mathbb{N}\setminus S_{p-1}} \frac{1}{m} \log \rho_j(m) = \alpha_1, \quad j \ge 2. \end{aligned} \tag{23}$$

It remains to consider the case when $m \in S_{p-1}$. By (19) and proceeding as before, we find that if $m \in (s_{p-1+pi}, s_{p+pi}]$ for some $i \in \mathbb{N}$, then

$$\sum_{l=1}^{m} b_j(l) = \alpha_j m + (\alpha_1 - \alpha_j) s_{p-1+pi} + E_m$$

for $j = 1, \dots, q$, where $E_m = E_m(j)$ satisfies

$$\lim_{m\to\infty} \frac{E_m}{m} = 0.$$

Therefore,

$$\limsup_{m\in S_{p-1}} \frac{1}{m} \log \rho_j(m) = \alpha_j \tag{24}$$

for $j = 1, \dots, q$. The relations $\beta_q = \alpha_q$ and $\alpha_j \le \beta_j$ for $j = 1, \dots, q-1$ together with (23) and (24) readily imply that the exponential growth rates of the singular values are equal to $\alpha_1, \dots, \alpha_q$. □

3.2. **Bounds for tempered sequences.** In this section, for the class of tempered sequences of matrices, we describe general inequalities between the (lower and upper) Lyapunov exponents and the lower and upper exponential growth rates of the singular values, provided that the intervals $[a_i, b_i]$ (see (14)) are disjoint. In section 4 we obtain a corresponding result in the general situation when the intervals $[a_i, b_i]$ may intersect each other.

A sequence of matrices $(A_m)_{m\in\mathbb{N}}$ is said to be *tempered* if

$$\limsup_{m\to\infty} \frac{1}{m} \log \|A_m\| \le 0. \tag{25}$$

For example, if $\sup_{m\in\mathbb{N}} \|A_m\| < +\infty$, then the sequence is tempered. We continue to assume that the Lyapunov exponent does not take the value $+\infty$. We emphasize that none of these properties (temperedness of a sequence of matrices and finiteness of its Lyapunov exponent) imply the other.

Example 1. Consider the sequence of real numbers $A_m = m$. Clearly, it is tempered. However,

$$\frac{1}{m}\log(A_{m-1}A_{m-2}\cdots A_1) = \frac{1}{m}\log[(m-1)!] \to +\infty$$

when $m \to \infty$ and so $\lambda(v) = +\infty$ for $v \neq 0$.

Example 2. Consider the sequence of real numbers

$$A_m = \begin{cases} e^{1-2m}, & m \text{ even}, \\ e^{3+2m}, & m \text{ odd}, \end{cases}$$

that is, $A_m = e^{2-(-1)^m(2m+1)}$. One can verify that $\mathcal{A}_m = e^{(2+(-1)^m)m-1}$. Then

$$\limsup_{m\to\infty} \frac{1}{m}\log A_m = 2,$$

and so the sequence is not tempered, but

$$\limsup_{m\to\infty} \frac{1}{m}\log \mathcal{A}_m < +\infty.$$

On the other hand, there are plenty of unbounded tempered sequences for which λ does not take the value $+\infty$. This is illustrated by the next example.

Example 3. Consider the sequence of real numbers $A_m = e^{a_{m+1}-a_m}$ for some sequence $a_m \nearrow +\infty$ with $\limsup_{m\to\infty}(a_m/m) = 0$. Clearly, $(A_m)_{m\in\mathbb{N}}$ is tempered, and since $\mathcal{A}_m = e^{a_m - a_1}$, the Lyapunov exponent λ does not take the value $+\infty$.

The following result relates the lower and upper Lyapunov exponents and the numbers a_i and b_i in (14) when the intervals $[a_i, b_i]$ are disjoint. Let $F_i(m)$ be the eigenspace of the matrix T_m in (13) associated to the eigenvalue $\rho_i(m)$.

Theorem 3. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of $q \times q$ matrices. If $b_i < a_{i+1}$ for $i = 1, \ldots, q-1$, then for each $i = 1, \ldots, q$ we have:*

1. *$\bigoplus_{j\leq i} F_j(m) \to H_i$ when $m \to \infty$, for some subspace H_i;*
2. *for each $v, w \in H_i$ with $w \notin H_{i-1}$, we have*

$$\limsup_{m\to\infty} \frac{1}{m}\log\|\mathcal{A}_m v\| \leq a_i + \sum_{j=1}^{q}(b_j - a_j) \tag{26}$$

and

$$\liminf_{m\to\infty} \frac{1}{m}\log\|\mathcal{A}_m w\| \geq a_i. \tag{27}$$

Moreover, if $\sum_{j=1}^{q}(b_j - a_j) < a_{i+1} - a_i$ for $i = 1, \ldots, q-1$, then $r = q$ and $H_i = E_i$ for $i = 1, \ldots, q$.

We omit the proof of Theorem 3 since the statement is a particular case of Theorem 5 below that considers the general situation when the intervals $[a_i, b_i]$ may intersect each other. Under the assumptions of Theorem 3, we have

$$\limsup_{m\to\infty} \frac{1}{m}\log\|\mathcal{A}_m v\| - \liminf_{m\to\infty} \frac{1}{m}\log\|\mathcal{A}_m v\| \leq \sum_{j=1}^{q}(b_j - a_j)$$

for each $v \in H_i \setminus \{0\}$. When $a_j = b_j$ for $j = 1, \ldots, q$, that is, when the exponential growth rates of the singular values are limits, this implies that each value of the

Lyapunov exponent is a limit (in fact, the sequence of matrices is regular, as shown in Theorem 6 below).

As an application of Theorem 3 we obtain a stronger version of property (17) for tempered sequences when all the values of the Lyapunov exponent are distinct.

Theorem 4. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of $q\times q$ matrices with $b_i < a_{i+1}$ for $i = 1, \dots, q-1$. If $a_i = b_i$ for $i = 1, \dots, q$, then $b_i = \lambda'_i$ for $i = 1, \dots, q$.*

Proof. It follows from the last statement of Theorem 3 that

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m v\| = \lambda'_i$$

for $v \in H_i \setminus H_{i-1}$ and each i. The desired result follows now readily from (26) and (27). □

4. Main result

In this section we consider the general case when not all intervals $[a_i, b_i]$ (see (14)) are disjoint and we obtain corresponding lower and upper estimates for the Lyapunov exponents in terms of the exponential growth rates of the singular values.

We first introduce some notation. Let $i_1 < i_2 < \cdots < i_{\alpha-1}$ be the integers $i \in [1, q-1]$ such that $[a_i, b_i] \cap [a_{i+1}, b_{i+1}] = \emptyset$. For $j = 1, \dots, \alpha$ we define

$$I_j = \bigcup_{k=i_{j-1}+1}^{i_j} [a_k, b_k],$$

with the convention that $i_0 = 0$ and $i_\alpha = q$. Notice that each set I_j is a closed interval since

$$[a_k, b_k] \cap [a_{k+1}, b_{k+1}] \neq \emptyset \quad \text{for} \quad k = i_{j-1}+1, \dots, i_j - 1.$$

We write $I_j = [c_j, d_j]$ for $j = 1, \dots, \alpha$ (notice that by definition $d_j < c_{j+1}$ for $j = 1, \dots, \alpha$). Moreover, for $j = 1, \dots, \alpha$, we define

$$G_j(m) = \bigoplus_i F_i(m) \quad \text{and} \quad H_j(m) = \bigoplus_{i\le j} G_i(m),$$

where the first direct sum is taken over all i's such that

$$c_j \le \liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m) \le \limsup_{m\to\infty} \frac{1}{m} \log \rho_i(m) \le d_j. \tag{28}$$

Now we are ready to prove our main result.

Theorem 5. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of $q\times q$ matrices. Then for each $j = 1, \dots, \alpha$ we have:*

1. *$H_j(m) \to H_j$ when $m \to \infty$, for some subspace H_j;*
2. *for each $v, w \in H_j$ with $w \notin H_{j-1}$, we have*

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m v\| \le c_j + \sum_{k=1}^{\alpha} (d_k - c_k) \dim G_k(m) \tag{29}$$

and

$$\liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \ge c_j. \tag{30}$$

Proof. We divide the proof into steps.

Step 1. *Convergence of the sequences* $H_j(m)$. We observe that

$$\begin{aligned}\|\mathcal{A}_m v\|^2 &= v^*\mathcal{A}_m^*\mathcal{A}_m v = v^* T_m^2 v \\ &= v^* T_m^* T_m v = \|T_m v\|^2\end{aligned} \tag{31}$$

for any vector v. Moreover, by (25) and (28), given $\varepsilon > 0$, there exists an integer $m(\varepsilon)$ such that $\|A_m\| \le e^{\varepsilon m}$ and

$$e^{(c_k-\varepsilon)m} \le \|T_m|G_k(m)\| \le e^{(d_k+\varepsilon)m} \tag{32}$$

for $m \ge m(\varepsilon)$ and $k = 1, \dots, \alpha$, when $d_1 > -\infty$. The case when $d_1 = -\infty$ is analogous and requires only minor modifications. Namely, (32) must then be replaced by the property that given $\varepsilon > 0$ and $\gamma < 0$, there exist integers $m(\varepsilon)$ and $n(\gamma)$ such that

$$\|T_m|G_1(m)\| \le e^{-\gamma m}$$

for $m \ge n(\gamma)$ and

$$e^{(c_k-\varepsilon)m} \le \|T_m|G_k(m)\| \le e^{(d_k+\varepsilon)m}$$

for $m \ge m(\varepsilon)$ and $k = 2, \dots, \alpha$. In the sequel we shall always assume that $d_1 > -\infty$ since the arguments when $d_1 = -\infty$ are quite similar.

Now take $m \ge m(\varepsilon)$ and $v_m \in H_j(m)$ with $\|v_m\| = 1$. Then

$$\begin{aligned}\|\mathcal{A}_{m+1} v_m\| &= \|A_m \mathcal{A}_m v_m\| \le e^{\varepsilon m}\|\mathcal{A}_m v_m\| \\ &= e^{\varepsilon m}\|T_m v_m\| \le e^{(d_j+2\varepsilon)m}.\end{aligned} \tag{33}$$

We write the vector v_m in the form

$$v_m = a_{m+1} v_{m+1} + u,$$

where $a_{m+1} \ge 0$, $v_{m+1} \in H_j(m+1)$ with $\|v_{m+1}\| = 1$ and $u \in H_j(m+1)^\perp$. Since the eigenspaces $F_j(m+1)$ are pairwise orthogonal for each m (recall that the matrix T_{m+1} is symmetric), the orthogonal complement $H_j(m+1)^\perp$ is also a direct sum of eigenspaces. Namely,

$$H_j(m+1)^\perp = \bigoplus_i F_i(m+1) = \bigoplus_{k>j} G_k(m+1),$$

where the first direct sum is taken over all i's such that

$$\liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m) \ge c_{j+1}.$$

Therefore, the vectors $\mathcal{A}_{m+1} v_{m+1}$ and $\mathcal{A}_{m+1} u$ are orthogonal, since

$$\begin{aligned}\langle \mathcal{A}_{m+1} v_{m+1}, \mathcal{A}_{m+1} u\rangle &= \langle \mathcal{A}_{m+1}^* \mathcal{A}_{m+1} v_{m+1}, u\rangle = \langle T_{m+1}^2 v_{m+1}, u\rangle \\ &= \langle T_{m+1} v_{m+1}, T_{m+1} u\rangle = 0.\end{aligned}$$

This implies that

$$\begin{aligned}\|\mathcal{A}_{m+1} v_m\| &= \|a_{m+1}\mathcal{A}_{m+1} v_{m+1} + \mathcal{A}_{m+1} u\| \\ &\ge \|\mathcal{A}_{m+1} u\| = \|T_{m+1} u\|.\end{aligned}$$

Writing $u = u_{j+1} + \cdots + u_\alpha$, with $u_k \in G_k(m+1)$ for $k = j+1, \dots, \alpha$, we obtain

$$\|\mathcal{A}_{m+1} v_m\| \ge \|T_{m+1} u_k\| \ge e^{(c_k-\varepsilon)(m+1)}\|u_k\|. \tag{34}$$

By (33) and (34), we conclude that

$$\|u_k\| \le \eta e^{(d_j-c_k+3\varepsilon)m} \tag{35}$$

for $k = j+1, \ldots, \alpha$, where $\eta = e^{\varepsilon - c_1}$. Since $a_{m+1} v_{m+1} = v_m - u$, we obtain

$$1 - \|u\| \le a_{m+1} \le 1 + \|u\|.$$

It follows that $|1 - a_{m+1}| \le \|u\|$, and so

$$\begin{aligned}\|v_m - v_{m+1}\| &\le \|v_m - a_{m+1} v_{m+1}\| + \|(1 - a_{m+1}) v_{m+1}\| \\ &= \|u\| + |1 - a_{m+1}| \\ &\le 2\|u\| \le 2\sqrt{\alpha}\eta e^{(d_j - c_{j+1} + 3\varepsilon)m}.\end{aligned}$$

Proceeding inductively we obtain a sequence of vectors $v_{m+k} \in H_j(m+k)$ with $\|v_{m+k}\| = 1$ such that $v_{m+k-1} - a_{m+k} v_{m+k} \in H_j(m+k)^\perp$ for some $a_{m+k} \ge 0$ and

$$\|v_{m+k} - v_{m+k+1}\| \le 2\sqrt{\alpha}\eta e^{(d_j - c_{j+1} + 3\varepsilon)(m+k)}$$

for all $k \ge 0$. For $\ell > k \ge 0$, provided that ε is sufficiently small, we obtain

$$\begin{aligned}\|v_{m+k} - v_{m+\ell}\| &\le \sum_{i=k}^{\ell-1} \|v_{m+i} - v_{m+i+1}\| \\ &\le 2\sqrt{\alpha}\eta \sum_{i=m+k}^{\infty} e^{(d_j - c_{j+1} + 3\varepsilon)i} \\ &= Ce^{(d_j - c_{j+1} + 3\varepsilon)(m+k)},\end{aligned}$$

where $C = 2\sqrt{\alpha}\eta/(1 - e^{d_j - c_{j+1} + 3\varepsilon})$. In particular, $(v_p)_{p \ge m}$ is a Cauchy sequence, say with limit v, and

$$\|v_p - v\| \le Ce^{(d_j - c_{j+1} + 3\varepsilon)p} \quad \text{for } p \ge m.$$

Now let $v_m^1, \ldots, v_m^q$ be an orthonormal basis of $\mathbb{R}^q$ such that $v_m^1, \ldots, v_m^{n_j}$, where $n_j = \dim H_j(m)$, is a basis of $H_j(m)$ for $j = 1, \ldots, \alpha$. Proceeding as before, for each $i = 1, \ldots, q$ we obtain a Cauchy sequence $(v_p^i)_{p \ge m}$ such that

$$\|v_p^i - v_{p'}^i\| \le Ce^{(d_j - c_{j+1} + 3\varepsilon)p}$$

for $p' \ge p \ge m$ whenever $i \le n_j$. Therefore, if $v^i \in \mathbb{R}^q$ is the limit of the sequence $(v_p^i)_{p \ge m}$, then

$$\|v_p^i - v^i\| \le Ce^{(d_j - c_{j+1} + 3\varepsilon)p} \tag{36}$$

for $p \ge m$ whenever $i \le n_j$. Increasing m if necessary, it follows readily from (36) that $v_p^1, \ldots, v_p^q$, for each $p \ge m$, and $v^1, \ldots, v^q$ are bases of $\mathbb{R}^q$. This implies that the sequence of spaces $(H_j(m))_{m \in \mathbb{N}}$ converges to the space H_j spanned by the vectors $v^1, \ldots, v^{n_j}$, for $j = 1, \ldots, \alpha$.

Step 2. Estimates along subspaces. In order to obtain an upper bound for the Lyapunov exponents, we start by studying the behavior under certain projections. Let

$$P_{j,m} = P_{H_j(m)^\perp} \quad \text{and} \quad Q_{j,m} = \mathrm{Id} - P_{j,m} = P_{H_j(m)} \tag{37}$$

be the orthogonal projections associated to the splitting

$$\mathbb{R}^q = H_j(m)^\perp \oplus H_j(m).$$

Note that

$$P_{j,m+k} = P_{j,m+k}P_{j,m+k-1}\cdots P_{j,m+1} + \sum_{l=2}^{k} P_{j,m+k}P_{j,m+k-1}\cdots P_{j,m+l}Q_{j,m+l-1}. \tag{38}$$

For $v \in H_j(m)$ we obtain

$$\|P_{j,m+k}v\| \le \|P_{j,m+1}v\| + \sum_{l=2}^{k}\|P_{j,m+l}Q_{j,m+l-1}v\|,$$

since $\|P_{j,m}w\| \le \|w\|$ for all $m \in \mathbb{N}$ and $w \in \mathbb{R}^q$, in view of (37). Hence, by (35),

$$\begin{aligned}\|P_{j,m+k}v\| &\le \eta\sqrt{\alpha}\sum_{l=0}^{k-1} e^{(d_j-c_{j+1}+3\varepsilon)(m+l)}\|v\| \\ &\le Ke^{(d_j-c_{j+1}+3\varepsilon)m}\|v\|\end{aligned}$$

for some constant $K > 0$ independent of j, m and k (notice that in view of (37) we have $\|Q_{j,m+l-1}v\| \le \|v\|$). Therefore,

$$\|P_{j,m+k}|H_j(m)\| \le Ke^{(d_j-c_{j+1}+3\varepsilon)m}.$$

Similarly, one can use (35) and identity (38) with j replaced by $l \ge j$ to conclude that for $v \in H_j(m)$,

$$\begin{aligned}\|P_{l,m+k}v\| &\le \|P_{l,m+1}v\| + \sum_{l=2}^{k}\|P_{l,m+l}Q_{j,m+l-1}v\| \\ &\le \eta\sqrt{\alpha}\sum_{l=0}^{k-1} e^{(d_j-c_{l+1}+3\varepsilon)(m+l)}\|v\| \\ &\le Ke^{(d_j-c_{l+1}+3\varepsilon)m}\|v\|,\end{aligned}$$

taking, without loss of generality, the same constant K. Hence,

$$\|P_{l,m+k}|H_j(m)\| \le Ke^{(d_j-c_{l+1}+3\varepsilon)m}. \tag{39}$$

Now let $u_{m1}, \dots, u_{mq}$ be an orthonormal basis of $\mathbb{R}^q$ such that the first n_j elements form a basis of $H_j(m)$, for $j = 1, \dots, \alpha$. We write

$$u_{mi} = \sum_{l=1}^{q} c_{il}u_{m+k,l}$$

(the constants c_{il} depend on m and k, but for simplicity of the notation we shall not make this dependence explicit). Moreover, let

$$c'_1 \le c'_2 \le \cdots \le c'_q \quad \text{and} \quad d'_1 \le d'_2 \le \cdots \le d'_q$$

be the numbers c_i and d_i counted with multiplicities $n_j - n_{j-1}$; that is, we take $c'_i = c_j$ and $d'_i = d_j$ whenever $n_{j-1} \le i \le n_j$, with the convention that $n_0 = 0$. It follows readily from (39) that

$$|c_{il}| \le e^{(d'_i-c'_l+3\varepsilon)m} \tag{40}$$

whenever $n_{j-1} < i \le n_j$ and $n_t < l \le n_{t+1}$ for some $t \ge j$, that is, whenever $i \le l$.

In fact, inequality (40) holds for any i and l. First observe that since $u_{m1}, \dots, u_{mq}$ is an orthonormal basis, the matrix X_m having these vectors as columns is orthogonal. Moreover, since $X_m = CX_{m+k}$, the matrix C is also orthogonal and so all its entries have absolute value at most 1. On the other hand, if $i > l$, then $d'_i - c'_l \geq d'_i - c'_i \geq 0$. Hence,

$$|c_{il}| \leq 1 \leq e^{(d'_i - c'_l + 3\varepsilon)m} \quad \text{for } i > l.$$

Step 3. *Upper bound for the Lyapunov exponents.* Write

$$u_{m+k,l} = \sum_{t=1}^{q} d_{lt} u_{mt} \tag{41}$$

for some numbers d_{lt}. Notice that the matrix $D = (d_{lt})_{l,t}$ is the inverse of the matrix $C = (c_{il})_{i,l}$. We shall estimate the entries of D using the identity

$$D = \frac{1}{\det C} C',$$

where C' is the cofactor matrix of C.

Let S be the set of all permutations σ of $\{1, \dots, q\}$ with $\sigma(l) = i$. Each $\sigma \in S$ induces a permutation σ' of the set $\{1, \dots, q\} \setminus \{l\}$, and we have

$$d_{il} = \frac{1}{\det C} \sum_{\sigma \in S} \operatorname{sgn}(\sigma') \prod_{r \neq l} c_{r\sigma(r)}. \tag{42}$$

Note that for each $\sigma \in S$ there exists $p = p_\sigma \in \mathbb{N}$ such that $\sigma^p(i) = l$. Since all entries of C satisfy $|c_{il}| \leq 1$, we obtain

$$\begin{aligned}
\left| \prod_{r \neq l} c_{r\sigma(r)} \right| &\leq \prod_{r=0}^{p-1} |c_{\sigma^r(i)\sigma^{r+1}(i)}| \\
&\leq \prod_{r=0}^{p-1} e^{(d'_{\sigma^r(i)} - c'_{\sigma^{r+1}(i)} + 3\varepsilon)m} \\
&\leq e^{(d'_i + \sum_{j \notin \{i,l\}} (d'_j - c'_j) - c'_l + 3q\varepsilon)m} \\
&= e^{(c'_i + \sum_{j \neq l} (d'_j - c'_j) - c'_l + 3q\varepsilon)m},
\end{aligned}$$

and it follows from (42) that

$$|d_{il}| \leq (q-1)! e^{(c'_i + \sum_{j \neq l} (d'_j - c'_j) - c'_l + 3q\varepsilon)m}. \tag{43}$$

Now take $v \in H_j(m+k)$ with $\|v\| = 1$. We write it in the form $v = u_m + w_m$ with $u_m \in H_j(m)$ and $w_m \in H_j(m)^\perp$. It follows from (41) and (43) that

$$w_m = \sum_{t=n_j+1}^{q} \beta_{t,m,k} u_{mt}$$

with

$$|\beta_{t,m,k}| \leq (q-1)! e^{(c_j + \sum_{l \neq t} (d'_l - c'_l) - c'_t + 3q\varepsilon m)}.$$

Since the last upper bound is independent of k, replacing v by v^i_{m+k} (see (36)) and letting $k \to \infty$, we find that v^i (which is the limit of v^i_{m+k} when $k \to \infty$) can be written in the form

$$v^i = u^i_m + w^i_m \quad \text{with} \quad u^i_m \in H_j(m),\ w^i_m \in H_j(m)^\perp.$$

Notice that $\|u_m^i\| \le 1$ and

$$w_m^i = \sum_{t=n_j+1}^{q} \beta_{t,m,i} u_{mt}$$

with

$$|\beta_{t,m,i}| \le (q-1)! e^{(c_j + \sum_{l\ne t}(d_l' - c_l') - c_t' + 3q\varepsilon m)}.$$

Therefore,

$$\begin{aligned}
\|\mathcal{A}_m v^i\| &\le \|\mathcal{A}_m u_m^i\| + \|\mathcal{A}_m w_m^i\| \\
&\le e^{(d_j+\varepsilon)m}\|u_m^i\| + \sum_{t=n_j+1}^{q} |\beta_{t,m,i}| \cdot \|\mathcal{A}_m u_{mt}\| \\
&\le e^{(d_j+\varepsilon)m} + \sum_{t=n_j+1}^{q} (q-1)! e^{(c_j + \sum_{l\ne t}(d_l' - c_l') - c_t' + 3q\varepsilon m)} e^{(d_t'+\varepsilon)m} \\
&= e^{(d_j+\varepsilon)m} + \sum_{t=n_j+1}^{q} (q-1)! e^{(c_j + \sum_{l=1}^{q}(d_l' - c_l') + 3q\varepsilon m)}.
\end{aligned}$$

This implies that

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m v^i\| \le c_j + \sum_{l=1}^{q}(d_l' - c_l') + 3q\varepsilon,$$

and so

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m v^i\| \le c_j + \sum_{l=1}^{q}(d_l' - c_l') \tag{44}$$

whenever $i \le n_j$, due to the arbitrariness of ε.

Step 4. Lower bound for the Lyapunov exponents. Now take $w \in H_j \setminus H_{j-1}$ and write $w = w_m + u_m$ with $w_m \in H_{j-1}(m)^\perp$ and $u_m \in H_{j-1}(m)$. Since $H_{j-1}(m) \to H_{j-1}$ when $m \to \infty$, the vector w_m converges to the component $w^\perp \in H_{j-1}^\perp$ that is nonzero since $w \notin H_{j-1}$. In particular, there exists a constant $c > 0$ such that $\|w_m\| \ge c$ for all sufficiently large m. Therefore,

$$\begin{aligned}
\|\mathcal{A}_m w\| &\ge \|\mathcal{A}_m w_m\| - \|\mathcal{A}_m u_m\| \\
&\ge e^{(c_j-\varepsilon)m}\|w_m\| - e^{(d_{j-1}+\varepsilon)m}\|u_m\| \\
&\ge c e^{(c_j-\varepsilon)m} - e^{(d_{j-1}+\varepsilon)m}\|v\|
\end{aligned}$$

for any sufficiently large m. Taking ε sufficiently small such that $c_j - \varepsilon > d_{j-1} + \varepsilon$, we obtain

$$\liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \ge c_j - \varepsilon.$$

Finally, since ε is arbitrary, we conclude that

$$\liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \ge c_j. \tag{45}$$

This completes the proof of the theorem. □

5. Characterizations of regularity

In this section we use Theorem 5 to recover some characterizations of regularity, in particular in terms of the exponential growth rates of the singular values and in terms of the matrices $(\mathcal{A}_m^*\mathcal{A}_m)^{1/(2m)}$.

Theorem 6. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of $q\times q$ matrices. Then $(A_m)_{m\in\mathbb{N}}$ is regular if and only if the limits*

$$\lim_{m\to\infty}\frac{1}{m}\log\rho_i(m),\quad i=1,\ldots,q, \tag{46}$$

exist.

Proof. We first assume that the limits in (46) exist. Let $\rho'_1\le\cdots\le\rho'_\alpha$ be their distinct values and let $F_i(m)$ be the eigenspace of the matrix T_m in (13) associated to the eigenvalue $\rho_i(m)$. We note that $\rho'_i=c_i=d_i$, using the notation introduced right before Theorem 5. For $j=1,\ldots,\alpha$, we have

$$H_j(m)=\bigoplus_i F_i(m),$$

where the sum is taken over all i's such that

$$\lim_{m\to\infty}\frac{1}{m}\log\rho_i(m)\le\rho'_j.$$

It follows from (44) and (45) that

$$\lim_{m\to\infty}\frac{1}{m}\log\|\mathcal{A}_m v\|=\rho'_j$$

for $v\in H_j\setminus H_{j-1}$. In particular, $\alpha=r$, $H_j=E_j$ and $\rho'_j=\lambda_j$ for $j=1,\ldots,r$ (see (8) for the definition of E_j). Now let S_m be the matrix whose columns are the vectors $v_m^1,\ldots,v_m^q$ constructed in the proof of Theorem 5 (it is invertible because the vectors form a basis). Then $S_m^{-1}T_mS_m$ is the diagonal matrix with entries $\rho_1(m)\le\cdots\le\rho_q(m)$ on the diagonal. Since $v_m^i\to v^i$ when $m\to\infty$, for each i, we obtain

$$\begin{aligned}\lim_{m\to\infty}T_m^{1/m}&=\lim_{m\to\infty}\left[S_m\begin{pmatrix}\rho_1(m)&&0\\&\ddots&\\0&&\rho_q(m)\end{pmatrix}S_m^{-1}\right]^{1/m}\\&=\lim_{m\to\infty}S_m\begin{pmatrix}\rho_1(m)^{1/m}&&0\\&\ddots&\\0&&\rho_q(m)^{1/m}\end{pmatrix}S_m^{-1}\\&=S\begin{pmatrix}e^{\rho_1}&&0\\&\ddots&\\0&&e^{\rho_q}\end{pmatrix}S^{-1},\end{aligned}\tag{47}$$

where S is the matrix whose columns are the vectors $v^1,\ldots,v^q$ and

$$\rho_i=\lim_{m\to\infty}\frac{1}{m}\log\rho_i(m),\quad i=1,\ldots,q.$$

In particular, since $\det(T_m^{1/m}) = (\det T_m)^{1/m}$, it follows from (47) that

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| = \sum_{i=1}^{q} \rho_i = \sum_{j=1}^{r} \lambda_j (\dim E_j - \dim E_{j-1}),$$

and so the sequence $(A_m)_{m\in\mathbb{N}}$ is regular.

Now we assume that the sequence $(A_m)_{m\in\mathbb{N}}$ is regular. We first show that for $k = 1, \dots, q$ the limit

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m^{\wedge k}\| \tag{48}$$

exists. In view of (11) we have

$$\rho_q(m) = \|\mathcal{A}_m^{\wedge 1}\| \quad \text{and} \quad \rho_i(m) = \|\mathcal{A}_m^{\wedge (q-i+1)}\| / \|\mathcal{A}_m^{\wedge (q-i)}\|$$

for $i = 1, \dots, q-1$. Thus, it follows from (48) that the limit

$$\lim_{m\to\infty} \frac{1}{m} \log \rho_i(m)$$

exists for $i = 1, \dots, q$. Hence, in order to prove the theorem, it remains to show that the limit in (48) exists for $k = 1, \dots, q$.

Consider a vector $v \in (\mathbb{R}^q)^{\wedge k}$ with $\|v\| = 1$ and write it in the form

$$v = \sum_{i_1 < \cdots < i_k} c_{i_1\cdots i_k} v_{i_1\cdots i_k}$$

for some orthonormal basis $v_{i_1\cdots i_k}$ of $(\mathbb{R}^q)^{\wedge k}$. Then

$$\sum_{i_1 < \cdots < i_k} c_{i_1\cdots i_k}^2 = 1$$

and thus $|c_{i_1\cdots i_k}| \le 1$ for all coefficients. Then

$$\|\mathcal{A}_m^{\wedge k} v\| \le \sum_{i_1 < \cdots < i_k} |c_{i_1\cdots i_k}| \cdot \|\mathcal{A}_m^{\wedge k} v_{i_1\cdots i_k}\| \le \sum_{i_1 < \cdots < i_k} \|\mathcal{A}_m^{\wedge k} v_{i_1\cdots i_k}\|$$

and hence, by Proposition 1,

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m^{\wedge k}\| \le \max_{i_1 < \cdots < i_k} \lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m^{\wedge k} v_{i_1\cdots i_k}\|. \tag{49}$$

Now we consider a vector $v_{j_1\cdots j_k}$ for which

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m^{\wedge k} v_{j_1\cdots j_k}\|$$

attains the maximum in (49). For each $v \in (\mathbb{R}^q)^{\wedge k}$ with $\|v\| = 1$ we have $\|\mathcal{A}_m^{\wedge k}\| \ge \|\mathcal{A}_m^{\wedge k} v\|$, and so

$$\begin{aligned} \liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m^{\wedge k}\| &\ge \liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m^{\wedge k} v_{j_1\cdots j_k}\| \\ &= \max_{i_1 < \cdots < i_k} \lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m^{\wedge k} v_{i_1\cdots i_k}\|. \end{aligned} \tag{50}$$

Finally, it follows from (49) and (50) that

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m^{\wedge k}\|$$

exists. This concludes the proof of the theorem. □

As an application of Theorem 6 (and of the arguments in its proof), we provide two additional characterizations of the notion of regularity. We start by considering the sequence of matrices $(\mathcal{A}_m^*\mathcal{A}_m)^{1/(2m)}$ and its limiting behavior.

Theorem 7. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of $q\times q$ matrices. Then $(A_m)_{m\in\mathbb{N}}$ is regular if and only if the sequence of matrices $(\mathcal{A}_m^*\mathcal{A}_m)^{1/(2m)}$ converges when $m\to\infty$.*

Proof. Assume that the sequence $(A_m)_{m\in\mathbb{N}}$ is regular. By Theorem 6, the limits in (46) exist, and so it follows from the first part of the proof of that theorem that (47) holds. In particular, the sequence $T_m^{1/m}$ converges when $m\to\infty$, where $T_m=(\mathcal{A}_m^*\mathcal{A}_m)^{1/2}$.

Now we assume that the sequence $T_m^{1/m}$ converges when $m\to\infty$. The eigenvalues of $T_m^{1/m}$ are the nonnegative real numbers $\nu_i(m)=\rho_i(m)^{1/m}$, for $i=1,\dots,q$, where $\rho_i(m)$ are the eigenvalues of T_m. Since the eigenvalues of a matrix vary continuously with its entries, it follows readily from the convergence of the sequence $T_m^{1/m}$ that the limit

$$\lim_{m\to\infty}\nu_i(m)=\lim_{m\to\infty}\rho_i(m)^{1/m}$$

exists for $i=1,\dots,q$. This implies that the limit in (46) exists for $i=1,\dots,q$. Hence, it follows from Theorem 6 that the sequence $(A_m)_{m\in\mathbb{N}}$ is regular. □

The second characterization of regularity is expressed in terms of the eigenvectors of the matrices $T_m=(\mathcal{A}_m^*\mathcal{A}_m)^{1/2}$.

Theorem 8. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of $q\times q$ matrices. Then $(A_m)_{m\in\mathbb{N}}$ is regular if and only if the limit*

$$\lim_{m\to\infty}\frac{1}{m}\log\|\mathcal{A}_m v_j(m)\| \tag{51}$$

exists for $i=1,\dots,q$ and for some orthonormal bases $v_1(m),\dots,v_q(m)$ of eigenvectors of $(\mathcal{A}_m^*\mathcal{A}_m)^{1/2}$ for the eigenvalues $\rho_1(m),\dots,\rho_q(m)$.

Proof. By (31), we have

$$\|\mathcal{A}_m v_j(m)\|=\|T_m v_j(m)\|=\|\rho_j(m)v_j(m)\|=\rho_j(m).$$

Therefore, the limit in (51) exists for $i=1,\dots,q$ and some sequence of bases as in the theorem if and only if the limits in (46) exist. The desired result follows now readily from Theorem 6. □

We note that in Theorem 8 condition (51) can be replaced by the requirement that the limits

$$\lim_{m\to\infty}\frac{1}{m}\log\min_{v\in F_i(m)\setminus\{0\}}\|\mathcal{A}_m v\|=\lim_{m\to\infty}\frac{1}{m}\log\max_{v\in F_i(m)\setminus\{0\}}\|\mathcal{A}_m v\|$$

exist for $i=1,\dots,t$ (and are equal).

The following result combines Proposition 1 with Theorems 6, 7 and 8.

Theorem 9. *For a tempered sequence $(A_m)_{m\in\mathbb{N}}$ of $q\times q$ matrices, the following properties are equivalent:*

(1) *the sequence $(A_m)_{m\in\mathbb{N}}$ is regular;*

(2) *the limit*

$$\lim_{m\to\infty} \frac{1}{m} \log \operatorname{vol}(\mathcal{A}_m v_1, \dots, \mathcal{A}_m v_k)$$

exists for any basis $v_1, \dots, v_q$ *of* $\mathbb{R}^q$ *and any integer* $k \in [1, q]$*;*

(3) *the limits*

$$\lim_{m\to\infty} \frac{1}{m} \log \rho_i(m), \quad i = 1, \dots, q,$$

exist;

(4) *the limits*

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m^{\wedge i}\|, \quad i = 1, \dots, q,$$

exist;

(5) *the sequence of matrices* $(\mathcal{A}_m^* \mathcal{A}_m)^{1/(2m)}$ *converges when* $m \to +\infty$*;*

(6) *the limit*

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m v_j(m)\|$$

exists for $i = 1, \dots, q$ *and some orthonormal bases* $v_1(m), \dots, v_q(m)$ *of eigenvectors of* $(\mathcal{A}_m^* \mathcal{A}_m)^{1/2}$ *for the eigenvalues* $\rho_1(m), \dots, \rho_q(m)$*.*

The equivalence between properties (1) and (2) is due to Lyapunov [9] (see also [8]). Following arguments of Raghunathan [14], Ruelle showed in [15] that property (3) implies properties (1) and (5) (his formulation is slightly different, but this follows readily from what he proves). More recently, Barabanov [1] proved that property (1) implies property (3) (although he considers only the case of continuous time, it is simple to obtain a corresponding argument for discrete time). We note also that property (3) is a simple consequence of property (5) and that properties (3) and (4) are equivalent, in view of identity (11).

It was proved earlier by Oseledets in [10] that properties (1) and (3) hold for almost all trajectories of a cocycle over a measure-preserving transformation (whose generator satisfies a certain natural integrability assumption). Moreover, Raghunathan [14] proved earlier that property (5) holds under the same assumption.

6. The case of continuous time

In this section we obtain versions of the results in the former sections for evolution families, obtained for example from nonautonomous linear differential equations. The arguments are analogous to those for discrete time and so we partially omit them.

6.1. **Preliminaries.** Let $T(t, s)$, for $t \geq s \geq 0$, be an evolution family on the space $\mathbb{R}^q$, that is, a collection of $q \times q$ matrices $T(t, s)$, for $t \geq s \geq 0$, satisfying

$$T(t, t) = \mathrm{Id} \quad \text{and} \quad T(t, r)T(r, s) = T(t, s)$$

for $t, r, s \geq 0$ with $t \geq r \geq s$. The *Lyapunov exponent* $\lambda\colon \mathbb{R}^q \to [-\infty, +\infty]$ associated to the evolution family $T(t, s)$ is defined by

$$\lambda(v) = \limsup_{t\to\infty} \frac{1}{t} \log\|T(t, 0)v\|, \tag{52}$$

again with the convention that $\log 0 = -\infty$. We shall always assume that λ does not take the value $+\infty$.

As in section 2.1, by the abstract theory of Lyapunov exponents (see [3]), the function λ takes at most a number $r \le q$ of distinct values on $\mathbb{R}^q \setminus \{0\}$, say

$$-\infty \le \lambda_1 < \lambda_2 < \cdots < \lambda_r < +\infty.$$

Moreover, for each $i = 1, \ldots, r$ the set

$$E_i = \big\{ v \in \mathbb{R}^q : \lambda(v) \le \lambda_i \big\}$$

is a linear subspace of $\mathbb{R}^q$ and

$$\{0\} \subset E_1 \subset E_2 \subset \cdots \subset E_r = \mathbb{R}^q.$$

We say that the evolution family $T(t,s)$ is *(Lyapunov) regular* if

$$\lim_{t\to\infty} \frac{1}{t} \log|\det T(t,0)| = \sum_{i=1}^q \lambda(v_i) \tag{53}$$

for some basis $v_1, \ldots, v_q$ of $\mathbb{R}^q$. In a manner similar to that in section 2.1, condition (53) can be replaced by the requirement that

$$\liminf_{t\to\infty} \frac{1}{t} \log|\det T(t,0)| \ge \sum_{i=1}^q \lambda(v_i)$$

for some basis $v_1, \ldots, v_q$ of $\mathbb{R}^q$.

We say that an evolution family $T(t,s)$ is *tempered* if

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{s\in[0,1]} \|T(t+s,t)\| \le 0.$$

Before proceeding, we illustrate how temperedness can be achieved for an evolution family defined by a nonautonomous linear differential equation. Let M_q be the set of all $q \times q$ matrices. Given a piecewise-continuous function $A \colon \mathbb{R}_0^+ \to M_q$ with left-hand and right-hand limits at all points, consider the linear equation

$$v' = A(t)v. \tag{54}$$

The evolution family associated to equation (54) is the collection of $q \times q$ matrices $T(t,s)$ such that $v(t) = T(t,s)v(s)$ for any solution v of (54) and any $t \ge s \ge 0$.

Proposition 10. *If*

$$\lim_{t\to\infty} \frac{1}{t} \int_t^{t+1} \|A(s)\|\, ds = 0, \tag{55}$$

then the evolution family $T(t,s)$ associated to equation (54) *is tempered.*

Proof. It follows readily from the identity

$$v(t) = v(s) + \int_s^t A(\tau)v(\tau)\, d\tau$$

and Gronwall's lemma that

$$\|v(t)\| \le \|v(s)\| \exp\left(\int_s^t \|A(\tau)\|\, d\tau \right)$$

for $t \ge s$. Therefore,

$$\|T(t,s)\| \le \exp\left(\int_s^t \|A(\tau)\|\, d\tau \right)$$

for $t \geq s$, and so

$$\log \sup_{s\in[0,1]} \|T(t+s,t)\| \leq \sup_{s\in[0,1]} \int_t^{t+s} \|A(\tau)\|\,d\tau \leq \int_t^{t+1} \|A(\tau)\|\,d\tau.$$

This yields the desired result. □

Notice that condition (55) holds when $\sup_{t\geq 0}\|A(t)\| < +\infty$.

6.2. **Bounds for the Lyapunov exponents.** In this section we obtain versions of the results in section 3.2 for an evolution family. We first notice that the matrix

$$R(t) = (T(t,0)^*T(t,0))^{1/2} \tag{56}$$

is symmetric and positive-definite. We denote its (real and positive) eigenvalues by

$$\rho_1(t) \leq \cdots \leq \rho_q(t).$$

These are the *singular values* of the evolution family. Moreover, let

$$a_i = \liminf_{t\to\infty} \frac{1}{t}\log\rho_i(t), \quad b_i = \limsup_{t\to\infty}\frac{1}{t}\log\rho_i(t) \tag{57}$$

for $i = 1,\dots,q$ and let $F_i(t)$ be the eigenspace of the matrix $R(t)$ associated to the eigenvalue $\rho_i(t)$.

Theorem 11. *Let $T(t,s)$ be a tempered evolution family on $\mathbb{R}^q$. If $b_i < a_{i+1}$ for $i = 1,\dots,q-1$, then for each $i = 1,\dots,q$ we have:*

(1) $\bigoplus_{j\leq i} F_j(t) \to H_i$ *when $t\to\infty$, for some subspace H_i;*

(2) *for each $v,w \in H_i$ with $w \notin H_{i-1}$, we have*

$$\limsup_{t\to\infty}\frac{1}{t}\log\|T(t,0)v\| \leq a_i + \sum_{j=1}^{q}(b_j - a_j)$$

and

$$\liminf_{t\to\infty}\frac{1}{t}\log\|T(t,0)w\| \geq a_i.$$

Moreover, if $\sum_{j=1}^q (b_j - a_j) < a_{i+1} - a_i$ for $i = 1,\dots,q-1$, then $r = q$ and $H_i = E_i$ for $i = 1,\dots,q$.

Moreover, in a manner similar to that in section 3.2, we can establish a more precise relation between the values of the Lyapunov exponent and the numbers b_i in (57). Namely, let

$$\lambda_1' \leq \lambda_2' \leq \cdots \leq \lambda_q' \quad \text{and} \quad b_1 \leq b_2 \leq \cdots \leq b_q$$

be, respectively, the values of the Lyapunov exponent λ in (52), counted with their multiplicities, and the numbers b_i. Then

$$\lambda_i' = \min_{F\in\mathcal{L}_i}\max_{v\in F,\|v\|=1}\limsup_{t\to\infty}\frac{1}{t}\log\|T(t,0)v\|$$

and

$$b_i = \limsup_{t\to\infty}\min_{F\in\mathcal{L}_i}\max_{v\in F,\|v\|=1}\frac{1}{t}\log\|T(t,0)v\|,$$

where $\mathcal{L}_i$ is the set of all i-dimensional subspaces of $\mathbb{R}^q$. This implies that property (17) holds. It was shown by Barabanov and Fominykh [2] that for any real numbers α_i and β_i as in Theorem 2 there exists a piecewise-constant bounded function $t \mapsto A(t)$ such that the evolution family associated to equation (54) satisfies (17).

The following result is a version of Theorem 4 for evolution families.

Theorem 12. *Let $T(t,s)$ be a tempered evolution family with $b_i < a_{i+1}$ for $i = 1, \ldots, q-1$. If $a_i = b_i$ for $i = 1, \ldots, q$, then $b_i = \lambda_i'$ for $i = 1, \ldots, q$.*

6.3. **Main result for evolution families.** Theorem 11 is a particular case of a more general result that considers the general situation when the intervals $[a_i, b_i]$ may intersect. Let $i_1 < i_2 < \cdots < i_{\alpha-1}$ be the integers $i \in [1, q-1]$ such that $[a_i, b_i] \cap [a_{i+1}, b_{i+1}] = \emptyset$. For $j = 1, \ldots, \alpha$ we define

$$[c_j, d_j] = \bigcup_{k=i_{j-1}+1}^{i_j} [a_k, b_k], \tag{58}$$

with the convention that $i_0 = 0$ and $i_\alpha = q$. Moreover, for $j = 1, \ldots, \alpha$, we define

$$G_j(t) = \bigoplus_i F_i(t) \quad \text{and} \quad H_j(t) = \bigoplus_{i \le j} G_i(t),$$

where the first direct sum is taken over all i's such that

$$c_j \le \liminf_{t\to\infty} \frac{1}{t} \log \rho_i(t) \le \limsup_{t\to\infty} \frac{1}{t} \log \rho_i(t) \le d_j.$$

Theorem 13. *Let $T(t,s)$ be a tempered evolution family on $\mathbb{R}^q$. Then for each $j = 1, \ldots, \alpha$ we have:*

(1) *$H_j(t) \to H_j$ when $t \to \infty$, for some subspace H_j;*
(2) *for each $v, w \in H_j$ with $w \notin H_{j-1}$, we have*

$$\limsup_{t\to\infty} \frac{1}{t} \log\|T(t,0)v\| \le c_j + \sum_{k=1}^{\alpha} (d_k - c_k) \dim G_k(t) \tag{59}$$

and

$$\liminf_{t\to\infty} \frac{1}{t} \log\|T(t,0)w\| \ge c_j. \tag{60}$$

Proof. For each $m \in \mathbb{N}$, take a number $s_m \in [0,1)$ and write $p_m = m + s_m$. Since the evolution family $T(t,s)$ is tempered, given $\varepsilon > 0$, we have

$$\|T(p_m, m)\| \le e^{\varepsilon m}$$

for any sufficiently large m, and hence

$$\|T(p_m, 0)v\| \le \|T(p_m, m)\| \cdot \|T(m,0)v\| \le e^{\varepsilon m} \|T(m,0)v\|.$$

Similarly,

$$\begin{aligned}\|T(m+1, 0)v\| &\le \|T(m+1, p_m)\| \cdot \|T(p_m, 0)v\| \\ &\le e^{\varepsilon p_m} \|T(p_m,0)v\| \le e^{\varepsilon(m+1)} \|T(p_m, 0)v\|,\end{aligned}$$

and so

$$e^{-\varepsilon(m+1)} \|T(m+1,0)v\| \le \|T(p_m,0)v\| \le e^{\varepsilon m} \|T(m,0)v\|.$$

Since $p_m/m \to 1$ when $m \to \infty$ and ε is arbitrary, this implies that

$$\liminf_{m\to\infty} \frac{1}{p_m} \log\|T(p_m,0)v\| = \liminf_{m\to\infty} \frac{1}{m} \log\|T(m,0)v\| \tag{61}$$

and

$$\limsup_{m\to\infty}\frac{1}{p_m}\log\|T(p_m,0)v\| = \limsup_{m\to\infty}\frac{1}{m}\log\|T(m,0)v\|. \tag{62}$$

Moreover, since the right-hand sides of (61) and (62) are independent of the sequence s_m and since any number $t>0$ can be written in the form $t=m+s_m=p_m$, we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log\|T(t,0)v\| = \liminf_{m\to\infty}\frac{1}{p_m}\log\|T(p_m,0)v\| \tag{63}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log\|T(t,0)v\| = \limsup_{m\to\infty}\frac{1}{p_m}\log\|T(p_m,0)v\|. \tag{64}$$

Now we consider the sequence of matrices

$$A_m = T(p_m,p_{m-1}) \quad \text{for } m\in\mathbb{N}, \tag{65}$$

with the convention that $p_0=0$. Then

$$\mathcal{A}_{m+1} = A_mA_{m-1}\cdots A_1 = T(p_m,0). \tag{66}$$

Moreover,

$$\begin{aligned}\|A_m\| &= \|T(p_m,p_{m-1})\|\\ &\le \|T(p_m,m)\|\cdot\|T(m,p_{m-1})\|\\ &\le e^{\varepsilon m}e^{\varepsilon p_{m-1}} < e^{2\varepsilon m}\end{aligned}$$

for any sufficiently large m. Thus,

$$\limsup_{m\to\infty}\frac{1}{m}\log\|A_m\|\le\varepsilon,$$

and it follows from the arbitrariness of ε that the sequence $(A_m)_{m\in\mathbb{N}}$ is tempered. Hence, one can apply Theorem 5 to deduce that for each $j=1,\ldots,\alpha$ we have:

(1) $H_j(p_m)\to F_j$ when $m\to\infty$, for some subspace H_j;
(2) properties (29) and (30) hold for each $v,w\in H_j$ with $w\notin H_{j-1}$.

In view of (63), (64) and (66), conditions (29) and (30) are equivalent to (59) and (60). Hence, in order to establish the theorem it remains to prove that the space H_j in property (1) does not depend on the sequence $(s_m)_{m\in\mathbb{N}}$, since then $H_j(t)\to H_j$ when $t\to\infty$. For that we consider another sequence $s'_m\in[0,1)$ and the matrices

$$A'_m = T(p'_m,p'_{m-1}) \quad \text{for } m\in\mathbb{N},$$

where $p'_m=m+s'_m$, with the convention that $p'_0=0$. Applying Theorem 5 to this sequence we find that $H_j(p'_m)\to H'_j$ when $m\to\infty$, for some spaces H'_j. Now we consider the "mixed" sequence

$$s''_m = \begin{cases} s_m, & m \text{ odd},\\ s'_m, & m \text{ even}.\end{cases}$$

We apply Theorem 5 once more, now to the sequence of matrices

$$A''_m = T(p''_m,p''_{m-1}) \quad \text{for } m\in\mathbb{N},$$

where $p''_m=m+s''_m$, with the convention that $p''_0=0$. In particular, this tells us that the sequence $H_j(p''_m)$ converges when $m\to\infty$. But since

$$H_j(p''_{2m+1}) = H_j(p_{2m+1})\to H_j \quad \text{when} \quad m\to\infty$$

and

$$H_j(p''_{2m}) = H_j(p'_{2m}) \to H'_j \quad \text{when} \quad m \to \infty,$$

we obtain $H_j = H'_j$. In other words, the space H_j does not depend on the sequence $(s_m)_{m\in\mathbb{N}}$. This completes the proof of the theorem. □

6.4. **Characterizations of Lyapunov regularity.** Theorem 13 leads to various characterizations of the notion of regularity. We start with a characterization in terms of the exponential growth rates of the singular values.

Theorem 14. *Let $T(t,s)$ be a tempered evolution family on $\mathbb{R}^q$. Then $T(t,s)$ is regular if and only if the limits*

$$\lim_{t\to\infty} \frac{1}{t} \log \rho_i(t), \quad i = 1, \dots, q, \tag{67}$$

exist.

Proof. The proof is analogous to that of Theorem 6. We first assume that the limits in (67) exist. Let $\rho'_1 \le \dots \le \rho'_\alpha$ be their distinct values and let $F_i(t)$ be the eigenspace of the matrix $R(t)$ (see (56)) associated to the eigenvalue $\rho_i(t)$. Then $\rho'_i = c_i = d_i$, using the notation in (58). For $j = 1, \dots, \alpha$, we have

$$H_j(t) = \bigoplus_i F_i(t),$$

where the sum is taken over all i's such that

$$\lim_{t\to\infty} \frac{1}{t} \log \rho_i(t) \le \rho'_j.$$

It follows from (59) and (60) that

$$\lim_{t\to\infty} \frac{1}{t} \log\|T(t,0)v\| = \rho'_j \quad \text{for } v \in H_j \setminus H_{j-1}.$$

Therefore, $\alpha = r$, $H_j = E_j$ and $\rho'_j = \lambda_j$ for $j = 1, \dots, r$. On the other hand, by Theorem 13, we have $H_j(t) \to H_j$ when $t \to \infty$. Hence, there exist orthonormal bases $v_1(t), \dots, v^q(t)$ of $\mathbb{R}^q$ for $t \in \mathbb{R}$ such that:

(1) $v^1(t), \dots, v^{n_j}(t)$, where $n_j = \dim H_j(t)$, is a basis of $H_j(t)$ for each j;
(2) $v^i(t) \to v^i$ when $t \to \infty$, for $i = 1, \dots, q$, for some vector v^i.

Then $v^1, \dots, v^{n_j}$ is a basis of H_j for each j. Moreover, if $S(t)$ is the matrix whose columns are the vectors $v_1(t), \dots, v^q(t)$, then $S(t)^{-1}R(t)S(t)$ is the diagonal matrix with entries $\rho_1(t) \le \dots \le \rho_q(t)$ on the diagonal. Hence,

$$\begin{aligned}
\lim_{t\to\infty} R(t)^{1/t} &= \lim_{t\to\infty} \left[S(t) \begin{pmatrix} \rho_1(t) & & 0 \\ & \ddots & \\ 0 & & \rho_q(t) \end{pmatrix} S(t)^{-1} \right]^{1/t} \\
&= \lim_{t\to\infty} S(t) \begin{pmatrix} \rho_1(t)^{1/t} & & 0 \\ & \ddots & \\ 0 & & \rho_q(t)^{1/t} \end{pmatrix} S(t)^{-1} \\
&= S \begin{pmatrix} e^{\rho_1} & & 0 \\ & \ddots & \\ 0 & & e^{\rho_q} \end{pmatrix} S^{-1},
\end{aligned} \tag{68}$$

where S is the limit of $S(t)$ when $t \to \infty$ and

$$\rho_i = \lim_{t\to\infty} \frac{1}{t} \log \rho_i(t), \quad i = 1, \dots, q.$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t} \log|\det T(t,0)| = \sum_{i=1}^{q} \rho_i = \sum_{j=1}^{r} \lambda_j (\dim E_j - \dim E_{j-1}),$$

and the evolution family $T(t,s)$ is regular.

Now we assume that the evolution family $T(t,s)$ is regular. Proceeding as in the proof of Theorem 6 one can show that the limit

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m^{\wedge k}\|$$

exists for $k = 1, \dots, q$, again for the sequence of matrices A_m in (65). Also as in the proof of Theorem 6, this implies that the limits in (67) exist. □

The following result provides another characterization of regularity.

Theorem 15. *Let $T(t,s)$ be a tempered evolution family on $\mathbb{R}^q$. Then $T(t,s)$ is regular if and only if $R(t)^{1/t}$ converges when $t \to \infty$.*

Proof. Assume that the evolution family $T(t,s)$ is regular. By Theorem 14, the limits in (67) exist, and so it follows from the first part of the proof of that theorem that (68) holds. In particular, $R(t)^{1/t}$ converges when $t \to \infty$.

Now we assume that $R(t)^{1/t}$ converges when $t \to \infty$. The eigenvalues of $R(t)^{1/t}$ are the positive real numbers $\nu_i(t) = \rho_i(t)^{1/t}$, for $i = 1, \dots, q$. Since the eigenvalues of a matrix vary continuously with its entries, it follows readily from the convergence of $R(t)^{1/t}$ that the limit

$$\lim_{t\to\infty} \nu_i(t) = \lim_{t\to\infty} \rho_i(t)^{1/t}$$

exists for $i = 1, \dots, q$. Hence, the limit in (67) exists for $i = 1, \dots, q$, and it follows from Theorem 14 that the evolution family $T(t,s)$ is regular. □

References

[1] E. A. Barabanov, *Singular exponents and regularity criteria for linear differential systems*, Differ. Equ. **41** (2005), no. 2, 151–162, DOI 10.1007/s10625-005-0145-y. MR2202014

[2] E. A. Barabanov and E. I. Fominykh, *Description of the mutual arrangement of singular exponents of a linear differential systems and the exponents of its solutions*, Differ. Equ. **42** (2006), no. 12, 1657–1673, DOI 10.1134/S0012266106120019. MR2347114

[3] Luis Barreira and Yakov B. Pesin, *Lyapunov exponents and smooth ergodic theory*, University Lecture Series, vol. 23, American Mathematical Society, Providence, RI, 2002. MR1862379

[4] Luis Barreira and Yakov Pesin, *Nonuniform hyperbolicity: Dynamics of systems with nonzero Lyapunov exponents*, Encyclopedia of Mathematics and its Applications, vol. 115, Cambridge University Press, Cambridge, 2007. MR2348606

[5] Luis Barreira and Claudia Valls, *Stability theory and Lyapunov regularity*, J. Differential Equations **232** (2007), no. 2, 675–701, DOI 10.1016/j.jde.2006.09.021. MR2286395

[6] Luis Barreira and Claudia Valls, *Stability of nonautonomous differential equations*, Lecture Notes in Mathematics, vol. 1926, Springer, Berlin, 2008. MR2368551

[7] Luis Barreira and Claudia Valls, *Ordinary differential equations: Qualitative theory*, Graduate Studies in Mathematics, vol. 137, American Mathematical Society, Providence, RI, 2012. MR2931599

[8] B. F. Bylov, R. È. Vinograd, D. M. Grobman, and V. V. Nemyckiĭ, *Teoriya pokazatelei Lyapunova i ee prilozheniya k voprosam ustoichivosti* (Russian), Izdat. "Nauka", Moscow, 1966. MR0206415

[9] A. M. Lyapunov, *The general problem of the stability of motion*, reprint of Internat. J. Control **55** (1992), no. 3 [MR1154209 (93e:01035)]; Taylor & Francis, Ltd., London, 1992. MR1229075

[10] V. I. Oseledec, *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–221.

[11] Oskar Perron, *Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differenzengleichungen* (German), J. Reine Angew. Math. **161** (1929), 41–64, DOI 10.1515/crll.1929.161.41. MR1581191

[12] Oskar Perron, *Die Ordnungszahlen linearer Differentialgleichungssysteme* (German), Math. Z. **31** (1930), no. 1, 748–766, DOI 10.1007/BF01246445. MR1545146

[13] Oskar Perron, *Die Stabilitätsfrage bei Differentialgleichungen* (German), Math. Z. **32** (1930), no. 1, 703–728, DOI 10.1007/BF01194662. MR1545194

[14] M. S. Raghunathan, *A proof of Oseledec's multiplicative ergodic theorem*, Israel J. Math. **32** (1979), no. 4, 356–362, DOI 10.1007/BF02760464. MR571089

[15] David Ruelle, *Ergodic theory of differentiable dynamical systems*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 27–58. MR556581

Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal
E-mail address: barreira@math.tecnico.ulisboa.pt

Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal
E-mail address: cvalls@math.tecnico.ulisboa.pt