

STRONGLY SELF-ABSORBING C^* -DYNAMICAL SYSTEMS

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ABSTRACT. We introduce and study strongly self-absorbing actions of locally compact groups on C^* -algebras. This is an equivariant generalization of a strongly self-absorbing C^* -algebra to the setting of C^* -dynamical systems. The main result is the following equivariant McDuff-type absorption theorem: A cocycle action $(\alpha, u) : G \curvearrowright A$ on a separable C^* -algebra is cocycle conjugate to its tensorial stabilization with a strongly self-absorbing action $\gamma : G \curvearrowright \mathcal{D}$, if and only if there exists an equivariant and unital $*$ -homomorphism from \mathcal{D} into the central sequence algebra of A . We also discuss some non-trivial examples of strongly self-absorbing actions.

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0. INTRODUCTION

Group actions on C^* -algebras and von Neumann algebras are one of the most fundamental subjects in the theory of operator algebras. Because of this, the problem of classifying group actions has a long history within the theory of operator algebras. A typical example is Connes' classification of injective factors, which involves the classification of cyclic group actions [4, 5]. This has sparked a lot of work towards classifying group actions on von Neumann algebras, of which the classification of actions of countable, amenable groups on injective factors (done by many hands) can be regarded as a highlight; see [4, 5, 21, 22, 24, 44, 48]. Although this has historically been developed as a case-by-case study by the factor type, a unified approach for McDuff factors has been found in [35] building on an intertwining argument [8] of Evans-Kishimoto originally invented for the C^* -setting.

Classification of group actions on C^* -algebras is, however, still a far less developed subject. This is not least because of substantial K -theoretical difficulties, which are already apparent in the seemingly harmless case of finite groups: these

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can act on a K -theoretically trivial algebra such as \mathcal{O}_2 and produce a variety of possible K -groups in the crossed product; see [1, 15, 16]. However, despite such difficulties, there is more and more important work going on to make progress on the C^* -algebraic side. Finite group actions with the Rokhlin property were successfully classified by Izumi [15, 16] on classifiable classes of C^* -algebras. Phillips is currently developing a theory for the classification of pointwise outer, finite group actions on Kirchberg algebras via equivariant KK -theory. In his pioneering work [28, 30–32], Kishimoto has outlined a method to use the Rokhlin property in order to classify single automorphisms (i.e., \mathbb{Z} -actions) on certain inductive limit C^* -algebras such as AT algebras. This has been developed further by work of Matui [37] for AH algebras, Nakamura [43] for Kirchberg algebras, and more recently Lin [34] for TAI algebras. These techniques and results are being pushed to \mathbb{Z}^d -actions and even beyond; see [18, 19, 23, 36–38, 42]. The reader is recommended to also consult Izumi’s survey article [17] for an informative summary of these developments.

In the non-equivariant classification theory of C^* -algebras, also known as the Elliott program, a certain class of C^* -algebras can be conceptually easier to handle under the assumption that they absorb certain C^* -algebras tensorially. In the general form of the Elliott program, this relates to Jiang-Su stability of C^* -algebras and the Toms-Winter conjecture [7, 51, 54]. A famous result by Kirchberg-Phillips [27] asserts that all Kirchberg algebras absorb the Cuntz algebra \mathcal{O}_∞ and are absorbed by the Cuntz algebra \mathcal{O}_2 tensorially, which Phillips’ approach [46] to the classification of these algebras made use of. (See also [25] for Kirchberg’s more direct approach to this classification.) In this way, one can regard the C^* -algebras \mathcal{O}_2 and \mathcal{O}_∞ as cornerstones of the classification of Kirchberg algebras. Even in order to classify stably finite C^* -algebras, the method of *localizing* the classification at a strongly self-absorbing C^* -algebra, as outlined by Winter in [53], has become the state-of-the-art. Strongly self-absorbing C^* -algebras have historically been looked at by example, but for the first time conceptually fleshed out by Toms-Winter in [50]. A very useful tool from [50] in studying \mathcal{D} -absorbing C^* -algebras for a strongly self-absorbing C^* -algebra \mathcal{D} is a McDuff-type result, a variant of which was also proved by Kirchberg in [26]. We note that the K_1 -injectivity criterion in both [26] and [50] has turned out to be redundant due to the main result of [52].

Theorem. *Let \mathcal{D} be a strongly self-absorbing C^* -algebra. A separable C^* -algebra A is \mathcal{D} -stable if and only if its (corrected) central sequence algebra*

$$F_\infty(A) = (A_\infty \cap A') / \text{Ann}(A, A_\infty)$$

contains a unital copy of \mathcal{D} .

We note that although Toms-Winter’s criterion [50, 2.3] is not directly stated in these terms, it is equivalent to the above criterion. It is also important to note that a precursor to this result has been known before due to Rørdam [47, 7.2.2], and his result recovers the above theorem in the unital case. Characterizing \mathcal{D} -stability in this manner has a number of interesting consequences, for instance permanence properties for the class of \mathcal{D} -stable C^* -algebras, which were studied both in [50] and [26]. The reason for calling this a *McDuff-type* absorption result comes from the following famous result of McDuff [41], which characterizes when a II_1 -factor absorbs the hyperfinite II_1 -factor tensorially.

Theorem. *Let M be a II_1 -factor with separable predual. Let \mathcal{R} be the hyperfinite II_1 -factor. Let ω be a free ultrafilter on \mathbb{N} . Then the following are equivalent:*

- (i) $M \cong M \overline{\otimes} \mathcal{R}$.
- (ii) $M^\omega \cap M'$ contains a unital copy of \mathcal{R} .
- (iii) $M^\omega \cap M'$ contains a unital copy of M_k , for some $k \in \mathbb{N}$.

In this paper, we introduce strongly self-absorbing actions of locally compact groups on C*-algebras, extending the notion of a strongly self-absorbing C*-algebra to the setting of C*-dynamical systems. Our main result is the following equivariant McDuff-type absorption result for strongly self-absorbing actions.

Theorem (cf. Theorem 3.7). *Let A be a separable C*-algebra, and let \mathcal{D} be a separable, unital C*-algebra. Let $\alpha : G \curvearrowright A$ be an action and $\gamma : G \curvearrowright \mathcal{D}$ a strongly self-absorbing action. Then (A, α) is cocycle conjugate to $(A \otimes \mathcal{D}, \alpha \otimes \gamma)$ if and only if there exists a unital, equivariant *-homomorphism from \mathcal{D} to $F_\infty(A)$. (Here, $F_\infty(A)$ is endowed with the G -action that is induced by componentwise application of α to representing sequences.)*

The main result, Theorem 3.7, is in fact more general than stated here, because we also consider cocycle actions $(\alpha, u) : G \curvearrowright A$ for possibly non-trivial 2-cocycles u . In this context, cocycle conjugacy turns out to be equivalent to strong cocycle conjugacy.

One should note that special kinds of equivariant tensorial absorption theorems have already existed in the theory of von Neumann algebras for a long time. For instance, in Ocneanu's classification paper [44], an important step consisted in showing that any pointwise outer action of an amenable group on the hyperfinite II_1 -factor \mathcal{R} absorbs the trivial G -action on \mathcal{R} tensorially.

Moreover, one should also note that the above equivariant McDuff-type absorption result in the C*-setting is at least folklore for cocycle actions of discrete groups. For finite group actions, Izumi has alluded to special cases of such an absorption theorem in [15]. As an important tool, it has been used to prove equivariant absorption theorems by Izumi-Matui in [19], Goldstein-Izumi in [12], and Matui-Sato in [39, 40].

Let us now summarize how this paper is organized. In section 1, we remind the reader of some standard definitions from the literature, introduce terminology and recall some sequence algebra techniques.

In section 2, we prove a preliminary version of the main result, which only treats the unital case and is easier to prove. In order to do this, we start the section by proving an equivariant version of a well-known one-sided intertwining argument, which will also be crucial in the later sections. In the second section, we must also deal with an important technical obstacle: If one wishes to apply a typical sequence algebra argument (commonly referred to as the *reindexation trick*) in the equivariant context, a non-discrete group may cause problems. This stems from the fact that for a non-discrete group G and a continuous action $\alpha : G \curvearrowright A$ on a C*-algebra, the induced (algebraic) action $\alpha_\infty : G \curvearrowright A$ on the sequence algebra will typically fail to be point-norm continuous. Moreover, given an element $x \in A_\infty$ satisfying some desirable dynamical property involving the action α_∞ , it is at first unclear what kind of *approximate* version of this property holds along a representing sequence $(x_n)_n \in \ell^\infty(\mathbb{N}, A)$ of x . After all, naively lifting relations to the representing sequence will only allow keeping track of some property on finitely

many group elements of G at a time, and this is not good enough. This may be even more troublesome for a representing sequence of some element $x \in F_\infty(A)$ in the central sequence algebra of A . However, we solve this problem by borrowing an idea of Guentner-Higson-Trout [13] to apply a Baire-category argument on representing sequences, which (simply put) enables us to keep track of a property on compact subsets of the group at a time, instead of only finite sets.

In section 3, we introduce and study strongly self-absorbing actions for a second-countable, locally compact group. We first establish some equivariant generalizations of some partial results from [50], and then prove the main result (see above), which is an equivariant McDuff-type absorption theorem for strongly self-absorbing actions.

In section 4, we study a property for group actions, which is a slight weakening of being strongly self-absorbing; we will call such actions semi-strongly self-absorbing. As it turns out, actions that are genuinely semi-strongly self-absorbing can only exist in cases where the acting group is non-compact. However, there are some advantages to consider this weaker notion: First, the property of being semi-strongly self-absorbing is more common and easier to verify than the property of being strongly self-absorbing. Second, we will see that the above equivariant McDuff-type theorem can be extended to the case where γ is only assumed to be semi-strongly self-absorbing. In particular, one may argue that the class of semi-strongly self-absorbing actions deserves just as much attention as the class of strongly self-absorbing actions, at least from the point of view of tensorial absorption.

In section 5, we will discuss non-trivial examples of (semi-)strongly self-absorbing actions on C^* -algebras, and present some open problems regarding certain classes of model actions.

We are confident that the main results and techniques of this paper will become important tools in some kind of equivariant Elliott program for nice group actions on classifiable C^* -algebras. Given the astounding importance that strongly self-absorbing C^* -algebras have in the Elliott program, we expect the same to become true for (semi-)strongly self-absorbing actions in the classification theory of group actions.

1. PRELIMINARIES

Notation 1.1. Unless specified otherwise, we will stick to the following notational conventions in this paper:

- A and B denote C^* -algebras.
- $\mathcal{M}(A)$ denotes the multiplier algebra of A and \tilde{A} denotes the smallest unitalization of A .
- Let $m_\lambda \in \mathcal{M}(A)$ be a net and $m \in \mathcal{M}(A)$ an element. If m_λ converges to m in the strict topology, i.e., $m_\lambda \cdot a \rightarrow m \cdot a$ and $a \cdot m_\lambda \rightarrow a \cdot m$ in norm for every $a \in A$, then we write $m_\lambda \xrightarrow{\text{str}} m$.
- G denotes a second-countable, locally compact group.
- The symbol α is used either for a continuous action $\alpha : G \curvearrowright A$ or more generally for a point-norm continuous map $\alpha : G \rightarrow \text{Aut}(A)$, e.g., if α belongs to a cocycle action, as defined below. By slight abuse of notation, we will also write $\alpha : G \rightarrow \text{Aut}(\mathcal{M}(A))$ for the unique strictly continuous extension.

- If $\alpha : G \curvearrowright A$ is an action, then A^α denotes the fixed-point algebra of A .
- If (X, d) is some metric space with elements $a, b \in X$, then we write $a =_\varepsilon b$ as a shortcut for $d(a, b) \leq \varepsilon$.
- ω denotes some free filter on \mathbb{N} . Given a bounded sequence of real numbers $a_n \in \mathbb{R}$, one defines $\limsup_{n \rightarrow \omega} a_n = \inf_{J \in \omega} \sup_{n \in J} a_n$. The \liminf is defines analogously, and if it agrees with the \limsup , then the limit along ω exists and is denoted $\lim_{n \rightarrow \omega} a_n$.

Definition 1.2 (cf. [3, 2.1] or [45, 2.1]). A cocycle action $(\alpha, u) : G \curvearrowright A$ is a point-norm continuous map $\alpha : G \rightarrow \text{Aut}(A)$, $g \mapsto \alpha_g$ together with a strictly continuous map $u : G \times G \rightarrow \mathcal{U}(\mathcal{M}(A))$ satisfying the properties

- $\alpha_1 = \text{id}_A$,
- $\alpha_r \circ \alpha_s = \text{Ad}(u(r, s)) \circ \alpha_{rs}$,
- $u(1, t) = u(t, 1) = \mathbf{1}$,
- $\alpha_r(u(s, t)) \cdot u(r, st) = u(r, s) \cdot u(rs, t)$

for all $r, s, t \in G$.

Remark. If $u = \mathbf{1}$ is trivial, then this just recovers the definition of an ordinary action $\alpha : G \curvearrowright A$.

Definition 1.3 (cf. [45, 3.2] for (i)+(ii)). Let $\alpha : G \curvearrowright A$ be an action. Consider a strictly continuous map $w : G \rightarrow \mathcal{U}(\mathcal{M}(A))$.

- w is called an α -1-cocycle, if one has $w_g \alpha_g(w_h) = w_{gh}$ for all $g, h \in G$. In this case, the map $\alpha^w : G \rightarrow \text{Aut}(A)$ given by $\alpha_g^w = \text{Ad}(w_g) \circ \alpha_g$ is again an action, and is called a cocycle perturbation of α . Two G -actions on A are called exterior equivalent if one of them is a cocycle perturbation of the other.
- Assume that w is an α -1-cocycle. It is called a coboundary, if there exists a unitary $v \in \mathcal{U}(\mathcal{M}(A))$ with $w_g = v \alpha_g(v^*)$ for all $g \in G$.
- Assume that w is an α -1-cocycle. It is called an approximate coboundary, if there exists a sequence of unitaries $x_n \in \mathcal{U}(\mathcal{M}(A))$ such that $x_n \alpha_g(x_n^*) \xrightarrow{\text{str}} w_g$ for all $g \in G$ and uniformly on compact sets. Two G -actions on A are called strongly exterior equivalent, if one of them is a cocycle perturbation of the other via an approximate coboundary.

Definition 1.4 (cf. [45, 3.1] for (i)). Let $(\alpha, u), (\beta, w) : G \curvearrowright A$ be two cocycle actions.

- The pairs (α, u) and (β, w) are called exterior equivalent, if there is a strictly continuous map $v : G \rightarrow \mathcal{U}(\mathcal{M}(A))$ satisfying $\beta_g = \text{Ad}(v_g) \circ \alpha_g$ and $w(s, t) = v_s \alpha_s(v_t) u(s, t) v_{st}^*$ for all $g, s, t \in G$.
- The pairs (α, u) and (β, w) are called strongly exterior equivalent, if there is a map $v : G \rightarrow \mathcal{U}(\mathcal{M}(A))$ as above such that there is a sequence of unitaries $x_n \in \mathcal{U}(\mathcal{M}(A))$ with $x_n \alpha_g(x_n^*) \xrightarrow{\text{str}} v_g$ for all $g \in G$ and uniformly on compact sets.

Definition 1.5. Let $(\alpha, u) : G \curvearrowright A$ and $(\beta, w) : G \curvearrowright B$ be two cocycle actions. A non-degenerate $*$ -homomorphism $\varphi : A \rightarrow B$ is called equivariant, and is written $\varphi : (A, \alpha, u) \rightarrow (B, \beta, w)$, if one has $\beta_g \circ \varphi = \varphi \circ \alpha_g$ and $w(s, t) = \varphi(u(s, t))$ for all $g, s, t \in G$.

If $u(\cdot, \cdot) = w(\cdot, \cdot) = \mathbf{1}$, i.e., one has actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$, then we omit the cocycles in the notation and write $\varphi : (A, \alpha) \rightarrow (B, \beta)$.

Let us now recall the notions of conjugacy, cocycle conjugacy and strong cocycle conjugacy. The latter notion is somewhat less prominent in the literature, and was first explicitly introduced by Izumi-Matui in [19, 2.1] as a relation between discrete group actions. In [40, 2.1(3)], Matui-Sato extended this notion to be a relation between cocycle actions of discrete groups. It is important to note that implicitly, the notion of strong cocycle conjugacy has played a role in earlier work of Kishimoto; see [8, 28, 30–32].

We are going to consider a suitable generalization of strong cocycle conjugacy to cocycle actions of locally compact groups. We note, however, that the definition given below only extends the definition from [19, 40] in the case of unital C^* -algebras, and is weaker in the non-unital case.

Definition 1.6. Two cocycle actions $(\alpha, u) : G \curvearrowright A$ and $(\beta, w) : G \curvearrowright B$ are called:

- (i) conjugate, if there is an equivariant isomorphism $\varphi : (A, \alpha, u) \rightarrow (B, \beta, w)$. In this case, we write $(A, \alpha, u) \cong (B, \beta, w)$.
- (ii) cocycle conjugate, if there is an isomorphism $\varphi : A \rightarrow B$ such that $(\varphi \circ \alpha \circ \varphi^{-1}, \varphi \circ u)$ is exterior equivalent to (β, w) . In this case, we write $(A, \alpha, u) \simeq_{cc} (B, \beta, w)$.
- (iii) strongly cocycle conjugate, if there is an isomorphism $\varphi : A \rightarrow B$ such that $(\varphi \circ \alpha \circ \varphi^{-1}, \varphi \circ u)$ is strongly exterior equivalent to (β, w) . In this case, we write $(A, \alpha, u) \simeq_{scc} (B, \beta, w)$.

If α and β are genuine actions, we omit the cocycles from the notation and write $(A, \alpha) \cong (B, \beta)$, $(A, \alpha) \simeq_{cc} (B, \beta)$ or $(A, \alpha) \simeq_{scc} (B, \beta)$.

As we mentioned before, the above given definition of strong cocycle conjugacy does not extend the definition from [19, 40] in the case of non-unital C^* -algebras. Our definition is weaker, and the difference is that the approximate coboundary condition is expressed in the strict topology rather than the norm topology.

As was outlined in the introduction, we aim to characterize equivariant absorption of certain actions via a property of the central sequence algebra. In particular, we shall spend most of our effort on studying sequence algebras, (relative) central sequence algebras and actions induced on them. Let us begin by recalling the definition of a *corrected* relative central sequence algebra, i.e., a quotient of a relative central sequence algebra by an Annihilator ideal. This idea originates from Kirchberg's important work [26] on central sequences of C^* -algebras.

Definition 1.7 (cf. [26, 1.1]). Let A be a C^* -algebra and ω a free filter on \mathbb{N} . Consider the (ω) -sequence algebra of A as the quotient

$$A_\omega = \ell^\infty(\mathbb{N}, A) / \left\{ (x_n)_n \mid \lim_{n \rightarrow \omega} \|x_n\| = 0 \right\}.$$

There is a standard embedding of A into A_ω by sending an element to its constant sequence. For some C^* -subalgebra $B \subset A_\omega$, the relative central sequence algebra is defined as

$$A_\omega \cap B' = \{x \in A_\omega \mid xb = bx \text{ for all } b \in B\}.$$

Observe that the two-sided annihilator

$$\text{Ann}(B, A_\omega) = \{x \in A_\omega \mid xb = bx = 0 \text{ for all } b \in B\}$$

is an ideal in $A_\omega \cap B'$, and one can thus define

$$F(B, A_\omega) = A_\omega \cap B' / \text{Ann}(B, A_\omega).$$

Of particular importance is the central sequence algebra of A , i.e., $A_\omega \cap A'$, or $F_\omega(A) = F(A, A_\omega)$, or the relative central sequence algebra $\mathcal{M}(A)_\omega \cap A'$.

Remark 1.8. If $\varphi \in \text{Aut}(A)$ is an automorphism, then componentwise application on representative sequences yields a well-defined automorphism $\varphi_\omega \in \text{Aut}(A_\omega)$ or $\varphi_\omega \in \text{Aut}(\mathcal{M}(A)_\omega)$. If $B \subset A_\omega$ is φ_ω -invariant, then φ_ω restricts to a well-defined automorphism on $A_\omega \cap B'$ and $\text{Ann}(B, A_\omega)$. This further induces an automorphism $\tilde{\varphi}_\omega$ on $F(B, A_\omega)$.

In the special case that $\varphi = \text{Ad}(u)$ is an inner automorphism for some $u \in \mathcal{M}(A)$ and we have $uB + Bu \subset B$, the induced automorphism $\tilde{\varphi}_\omega$ on $F(B, A_\omega)$ is the identity map.

Proof. We have $u \cdot b, b \cdot u \in B$ for all $b \in B$ by assumption. Given $x \in A_\omega \cap B'$ and $b \in B$, we thus have

$$(uxu^*)b = ux(u^*b) = u(u^*b)x = bx = xb.$$

But this implies that $x - uxu^* \in \text{Ann}(B, A_\omega)$, which shows that $\text{Ad}(u)$ indeed acts trivially on $F(B, A_\omega)$. \square

Notation 1.9. Let $(\alpha, u) : G \curvearrowright A$ be a cocycle action and ω a free filter on \mathbb{N} . Then in view of Remark 1.8, we may consider the induced map $\alpha_\omega : G \rightarrow \text{Aut}(A_\omega)$, $g \mapsto \alpha_{\omega, g} := (\alpha_g)_\omega$. In general, this map is not point-norm continuous. So let

$$A_{\omega, \alpha} = \{x \in A_\omega \mid [g \mapsto \alpha_{\omega, g}] \text{ is continuous}\}$$

be the C*-subalgebra of elements in A_ω on which this map acts continuously.

Remark 1.10. Let G be a second-countable, locally compact group. Let $(\alpha, u) : G \curvearrowright A$ be a cocycle action. Assume that $B \subset A_\omega$ is a C*-subalgebra that is invariant under multiplication with $\{u(s, t)\}_{s, t \in G}$ and under applying $\{\alpha_{\omega, s}\}_{s \in G}$. Then it follows from Remark 1.8 that the induced family of automorphisms $\{\tilde{\alpha}_{\omega, g}\}_{g \in G}$ defines a genuine G -action on $F(B, A_\omega)$. However, this action will in general fail to be point-norm continuous. We can thus consider the continuous part

$$F_\alpha(B, A_\omega) = \{x \in F(B, A_\omega) \mid [G \ni g \mapsto \tilde{\alpha}_{\omega, g}(x)] \text{ is continuous}\}.$$

Now let B be another C*-algebra and $(\beta, v) : G \curvearrowright B$ another cocycle action. Assume that (A, α, u) and (B, β, v) are cocycle conjugate via an isomorphism $\psi : A \rightarrow B$. Then for every $g \in G$, the automorphisms β_g and $\varphi \circ \alpha_g \circ \varphi^{-1}$ agree up to an inner automorphism, so their induced maps on $F_\omega(B)$ are the same. In particular, componentwise application of φ on representing sequences yields an equivariant isomorphism

$$(F_\omega(A), \tilde{\alpha}_\omega) \cong (F_\omega(B), \tilde{\beta}_\omega).$$

Remark 1.11. If $B \subset A_\omega$ is a C*-subalgebra, then obviously

$$\text{Ann}(B, A_\omega) = \text{Ann}(B, \mathcal{M}(A)_\omega) \cap A_\omega \quad \text{and} \quad A_\omega \cap B' = (\mathcal{M}(A)_\omega \cap B') \cap A_\omega.$$

So we have a natural embedding $F(B, A_\omega) \rightarrow F(B, \mathcal{M}(A)_\omega)$. If B is σ -unital, then this embedding is an isomorphism.

The same is true if we replace $\mathcal{M}(A)$ by \tilde{A} .

Proof. Let $\tilde{x} = x + \text{Ann}(B, \mathcal{M}(A)_\omega) \in F(B, \mathcal{M}(A)_\omega)$ for some $x \in \mathcal{M}(A)_\omega \cap B'$ be represented by a bounded sequence $x_n \in \mathcal{M}(A)$. Since B is σ -unital, let $h \in B$ be a strictly positive element. Let h be represented by a bounded sequence $(h_n)_n$

in A . Using that A is strictly dense in $\mathcal{M}(A)$, we choose elements $y_n \in A$ with $\|y_n\| \leq \|x_n\|$ satisfying

$$x_n h_n =_{1/n} y_n h_n \text{ and } h_n x_n =_{1/n} h_n y_n \text{ for all } n \in \mathbb{N}.$$

For the resulting bounded sequence and its induced element $y = [(y_n)] \in A_\omega$, we thus have $yh = xh$ and $hy = hx$. But since h is strictly positive in B , this implies $yb = xb$ and $by = bx$ for all $b \in B$. In particular,

$$y \in A_\omega \cap B' \text{ and } bx = by \text{ for all } b \in B.$$

But this means that $\tilde{x} = y + \text{Ann}(B, \mathcal{M}(A)_\omega) \in F(B, \mathcal{M}(A)_\omega)$ is in the image of $F(B, A_\omega)$ under the natural embedding. This shows our claim. \square

Remark 1.12 (cf. [26, 1.9(1)+(4)]). For a C^* -algebra A , we denote $D_{\omega, A} = \overline{A \cdot A_\omega \cdot A}$, the hereditary C^* -subalgebra of A_ω generated by A . Let $B \subset A$ be a non-degenerate C^* -subalgebra. Then one checks easily that for any $y \in A_\omega \cap B'$, we have $yD_{\omega, A} + D_{\omega, A}y \subset D_{\omega, A}$. By the universal property of the multiplier algebra, we have a natural $*$ -homomorphism

$$\pi : A_\omega \cap B' \rightarrow \mathcal{M}(D_{\omega, A}), \quad y \mapsto [x \mapsto yx].$$

The kernel coincides with $\text{Ann}(B, A_\omega)$ by definition, and thus we get a natural embedding

$$F(B, A_\omega) = (A_\omega \cap B') / \text{Ann}(B, A_\omega) \rightarrow \mathcal{M}(D_{\omega, A}).$$

In particular, we can view $F(B, A_\omega)$ as a subset of $\mathcal{M}(D_{\omega, A})$ in a natural way. Given $d \in D_{\omega, A}$ and $x \in F(B, A_\omega)$, we can multiply $d \cdot x = d \cdot y$ for some element y with $\pi(y) = x$. Analogously, the product $x \cdot d$ is well-defined, and both give well-defined values in $D_{\omega, A} \subset A_\omega$.

Last, let us observe that the technique of Elliott intertwining (see [47, Chapter 2, Section 3]) can be extended to the setting of equivariant $*$ -homomorphisms between C^* -dynamical systems.

Definition 1.13 (cf. [47, 2.3.1]). Let A_n and B_n be sequences of separable C^* -algebras. Let G be a second-countable, locally compact group and for each $n \in \mathbb{N}$, let two actions $\alpha^{(n)} : G \curvearrowright A_n$ and $\beta^{(n)} : G \curvearrowright B_n$ be given. Consider two equivariant, inductive systems $\{(A_n, \alpha^{(n)}), \varphi_n\}$ and $\{(B_n, \beta^{(n)}), \psi_n\}$. Consider also two sequences of $*$ -homomorphisms $\kappa_n : B_n \rightarrow A_n$ and $\theta_n : A_n \rightarrow B_{n+1}$ fitting into the (not necessarily commutative) diagram:

$$(e1.1) \quad \begin{array}{ccccccc} \dots & \longrightarrow & A_n & \xrightarrow{\varphi_n} & A_{n+1} & \longrightarrow & \dots \\ & & \nearrow \kappa_n & & \nearrow \kappa_{n+1} & & \\ & & & \searrow \theta_n & & \searrow & \\ \dots & \longrightarrow & B_n & \xrightarrow{\psi_n} & B_{n+1} & \longrightarrow & \dots \end{array}$$

Let us call diagram (e1.1) an approximate G -equivariant intertwining, if the following holds: There exist finite sets $F_n^A \subset A_n$, $F_n^B \subset B_n$, increasing compact sets $K_n \subset G$ and $\delta_n > 0$ satisfying:

- (i) $\varphi_n(a) =_{\delta_n} \kappa_{n+1} \circ \theta_n(a)$ for all $a \in F_n^A$;
- (ii) $\psi_n(b) =_{\delta_n} \theta_n \circ \kappa_n(b)$ for all $b \in F_n^B$;
- (iii) $\kappa_n \circ \beta_g^{(n)}(b) =_{\delta_n} \alpha_g^{(n)} \circ \kappa_n(b)$ for all $b \in F_n^B$ and $g \in K_n$;
- (iv) $\theta_n \circ \alpha_g^{(n)}(a) =_{\delta_n} \beta_g^{(n+1)} \circ \theta_n(a)$ for all $a \in F_n^A$ and $g \in K_n$;
- (v) $\bigcup_{n \in \mathbb{N}} K_n = G$;

- (vi) $\kappa_n(F_n^B) \subset F_n^A$, $\psi_n(F_n^B), \theta_n(F_n^A) \subset F_{n+1}^B$ and $\varphi_n(F_n^A) \subset F_{n+1}^A$;
- (vii) $\bigcup_{m \geq n} \varphi_{n,m}^{-1}(F_m^A) \subset A_n$ and $\bigcup_{m \geq n} \psi_{n,m}^{-1}(F_m^B) \subset B_n$ are dense for all n ;
- (viii) $\sum_{n \in \mathbb{N}} \delta_n < \infty$.

Proposition 1.14 (cf. [47, 2.3.2]). *Let the diagram (e1.1) describe an approximate G -equivariant intertwining. Consider the G -equivariant limits $(A, \alpha) = \varinjlim \{(A_n, \alpha^{(n)}), \varphi_n\}$ and $(B, \beta) = \varinjlim \{(B_n, \beta^{(n)}), \psi_n\}$. Then the formulas*

$$\theta(\varphi_{n,\infty}(a)) = \lim_{k \rightarrow \infty} (\psi_{k+1,\infty} \circ \theta_k \circ \varphi_{n,k})(a), \quad a \in A_n,$$

and

$$\kappa(\psi_{n,\infty}(b)) = \lim_{k \rightarrow \infty} (\varphi_{k,\infty} \circ \kappa_k \circ \psi_{n,k})(b), \quad b \in B_n,$$

define mutually inverse, equivariant isomorphisms $\theta : (A, \alpha) \rightarrow (B, \beta)$ and $\kappa : (B, \beta) \rightarrow (A, \alpha)$.

Proof. The fact that these formulas define mutually inverse isomorphisms between A and B is proved in [47, 2.3.2]. Omitting the details, it is straightforward to check that conditions (iii) to (viii) above imply that the maps θ and κ are in fact G -equivariant. \square

Definition 1.15. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions of a second-countable, locally compact group on separable C*-algebras. Let $\varphi_1, \varphi_2 : (A, \alpha) \rightarrow (B, \beta)$ be two equivariant *-homomorphisms. The maps φ_1, φ_2 are called approximately G -unitarily equivalent, if there is a sequence $v_n \in \mathcal{U}(\mathcal{M}(B))$ with $\varphi_2 = \lim_{n \rightarrow \infty} \text{Ad}(v_n) \circ \varphi_1$ and $\beta_g(v_n) - v_n \xrightarrow{\text{str}} 0$ uniformly on compact subsets of G . We denote $\varphi_1 \approx_{u,G} \varphi_2$.

With the tool Proposition 1.14 at hand, we can obtain a G -equivariant analogue of [47, 2.3.3]. Since the proof is virtually identical to the original proof in [47], let us just record the result:

Corollary 1.16 (cf. [47, 2.3.3]). *Let A_n, B_n be sequences of separable C*-algebras. Let G a second-countable, locally compact group and for each $n \in \mathbb{N}$, let two actions $\alpha^{(n)} : G \curvearrowright A_n$ and $\beta^{(n)} : G \curvearrowright B_n$ be given. Consider two equivariant, inductive systems $\{(A_n, \alpha^{(n)}), \varphi_n\}$ and $\{(B_n, \beta^{(n)}), \psi_n\}$ with limits (A, α) and (B, β) , respectively. Consider two sequences of equivariant *-homomorphisms $\kappa_n : (B_n, \beta^{(n)}) \rightarrow (A_n, \alpha^{(n)})$ and $\theta_n : (A_n, \alpha^{(n)}) \rightarrow (B_{n+1}, \beta^{(n+1)})$ fitting into the (not necessarily commutative) diagram:*

$$\begin{array}{ccccccc} \dots & \longrightarrow & (A_n, \alpha^{(n)}) & \xrightarrow{\varphi_n} & (A_{n+1}, \alpha^{(n+1)}) & \longrightarrow & \dots \\ & & \nearrow \kappa_n & & \nearrow \kappa_{n+1} & & \\ \dots & \longrightarrow & (B_n, \beta^{(n)}) & \xrightarrow{\psi_n} & (B_{n+1}, \beta^{(n+1)}) & \longrightarrow & \dots \\ & & \searrow \theta_n & & \searrow \theta_{n+1} & & \end{array}$$

Assume that $\theta_n \circ \kappa_n \approx_{u,G} \psi_n$ and $\kappa_{n+1} \circ \theta_n \approx_{u,G} \varphi_n$ for all $n \in \mathbb{N}$. Then there exist mutually inverse, equivariant isomorphisms $\theta : (A, \alpha) \rightarrow (B, \beta)$ and $\kappa : (B, \beta) \rightarrow (A, \alpha)$ satisfying

$$\theta \circ \varphi_{n,\infty} \approx_{u,G} \psi_{n+1,\infty} \circ \theta_n \quad \text{and} \quad \kappa \circ \psi_{n,\infty} \approx_{u,G} \varphi_{n,\infty} \circ \kappa_n$$

for all $n \in \mathbb{N}$.

2. A ONE-SIDED INTERTWINING ARGUMENT

In the last section, we treated equivariant analogues of two-sided approximate intertwining between inductive limits. However, certain one-sided versions are more useful for McDuff-type absorption results. The following is an equivariant analogue of a well-known lemma in Rørdam's book [47].

A variant of the next lemma was shown by Matui, and we are grateful to him for kindly allowing its inclusion in this paper.

Lemma 2.1 (cf. [47, 2.3.5]). *Let G be a second-countable, locally compact group. Let $(\alpha, u) : G \curvearrowright A$ and $(\beta, w) : G \curvearrowright B$ be two cocycle actions on separable C^* -algebras. Let $\varphi : (A, \alpha, u) \rightarrow (B, \beta, w)$ be an injective, non-degenerate and equivariant $*$ -homomorphism. Assume the following:*

For every $\varepsilon > 0$, compact subset $K \subset G$ and finite subsets $F_A \subset A, F_B \subset B$, there exists a unitary $z \in \mathcal{U}(\mathcal{M}(B))$ satisfying:

- (2.1a) $\| [z, \varphi(a)] \| \leq \varepsilon$ for every $a \in F_A$.
- (2.1b) $\text{dist}(z^*bz, \varphi(A)) \leq \varepsilon$ for every $b \in F_B$.
- (2.1c) $b\beta_g(z) =_\varepsilon bz$ for every $g \in K$ and $b \in F_B$.

Then φ is approximately multiplier unitarily equivalent to an isomorphism $\psi : A \rightarrow B$ inducing strong cocycle conjugacy between (A, α, u) and (B, β, w) .

If moreover z may always be chosen to be a unitary in \tilde{B} , then φ is approximately unitarily equivalent to an isomorphism inducing strong cocycle conjugacy between (A, α, u) and (B, β, w) .

Proof. Let $\{a_n\}_{n \in \mathbb{N}} \subset A$ and $\{b_n\}_{n \in \mathbb{N}} \subset B$ be dense sequences. Since G is σ -compact, write $G = \bigcup_{n \in \mathbb{N}} K_n$ for an increasing union of compact subsets $1_G \in K_n$. We are going to add unitaries to φ step by step:

In the first step, choose some $a_{1,1} \in A$ and $z_1 \in \mathcal{U}(\mathcal{M}(B))$ such that

- $z_1^* b_1 z_1 =_{1/2} \varphi(a_{1,1})$,
- $\| [z_1, \varphi(a_1)] \| \leq 1/2$,
- $b_1 \beta_g(z_1) =_{1/2} b_1 z_1$ for all $g \in K_1$.

In the second step, choose $a_{2,1}, a_{2,2} \in A$ and $z_2 \in \mathcal{U}(\mathcal{M}(B))$ such that

- $z_2^* (z_1^* b_j z_1) z_2 =_{1/4} \varphi(a_{2,j})$ for $j = 1, 2$,
- $\| [z_2, \varphi(a_j)] \| \leq 1/4$ for $j = 1, 2$,
- $\| [z_2, \varphi(a_{1,1})] \| \leq 1/4$,
- $(b_j z_1) \beta_g(z_2) =_{1/4} (b_j z_1) z_2$ for all $g \in K_2$ and $j = 1, 2$.

Now assume that for some $n \in \mathbb{N}$, we have found $z_1, \dots, z_n \in \mathcal{U}(\mathcal{M}(B))$ and $\{a_{m,j}\}_{m,j \leq n} \subset A$ satisfying

- (c1) $z_n^* (z_{n-1}^* \cdots z_1^* b_j z_1 \cdots z_{n-1}) z_n =_{2^{-n}} \varphi(a_{n,j})$ for $j \leq n$,
- (c2) $\| [z_n, \varphi(a_j)] \| \leq 2^{-n}$ for $j \leq n$,
- (c3) $\| [z_n, \varphi(a_{m,j})] \| \leq 2^{-n}$ for $m < n$ and $j < m$,
- (c4) $(b_j z_1 \cdots z_{n-1}) \beta_g(z_n) =_{2^{-n}} (b_j z_1 \cdots z_{n-1}) z_n$ for all $g \in K_n$ and $j \leq n$.

Then we can again apply our assumptions to find $z_{n+1} \in \mathcal{U}(\mathcal{M}(B))$ and $\{a_{n+1,j}\}_{j \leq n+1}$ with

- $z_{n+1}^* (z_n^* \cdots z_1^* b_j z_1 \cdots z_n) z_{n+1} =_{2^{-(n+1)}} \varphi(a_{n+1,j})$ for $j \leq n+1$,
- $\| [z_{n+1}, \varphi(a_j)] \| \leq 2^{-(n+1)}$ for $j \leq n+1$,
- $\| [z_{n+1}, \varphi(a_{m,j})] \| \leq 2^{-(n+1)}$ for $m < n+1$ and $j < m+1$,

- $(b_j z_1 \cdots z_n) \beta_g(z_{n+1}) =_{2^{-(n+1)}} (b_j z_1 \cdots z_n) z_{n+1}$ for all $g \in K_{n+1}$ and $j \leq n+1$.

Carry on inductively. Then we define $\psi_n : A \rightarrow B$ by the formula $\psi_n = \text{Ad}(z_1 \cdots z_n) \circ \varphi$. Now let us observe a number of facts:

By condition (c2), the sequence $(\psi_n(a_j))_{n \in \mathbb{N}}$ is Cauchy for all $j \in \mathbb{N}$. Since the a_j are dense, this implies that the ψ_n converge to some $*$ -homomorphism $\psi : A \rightarrow B$ in the point-norm topology. Then ψ is clearly approximately multiplier unitarily equivalent to φ .

By condition (c3), we have for every $n \in \mathbb{N}$ and $j \leq n$ the estimate

$$\|\psi(a_{n,j}) - \psi_n(a_{n,j})\| \leq \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n}.$$

Combined with condition (c1), we have for every $n \in \mathbb{N}$ and $j \leq n$ that

$$\|b_j - \psi(a_{n,j})\| \leq 2^{-n} + \|b_j - \psi_n(a_{n,j})\| \leq 2^{-n} + 2^{-n} = 2^{1-n}.$$

Since the $\{b_j\}_j \subset B$ are dense, it follows that $\psi(A)$ is dense, hence ψ is surjective. Since it is also injective, it is an isomorphism.

By condition (c4), we have that the sequences $b_j \cdot z_1 \cdots z_n \beta_g(z_n^* \cdots z_1^*)$ are Cauchy sequences for every $j \in \mathbb{N}$ and $g \in G$, with uniformity on compact sets. Since the b_j form a dense subset of B by choice, it follows that every sequence of functions of the form $[g \mapsto b \cdot z_1 \cdots z_n \beta_g(z_n^* \cdots z_1^*)]$ (for $b \in B$) converges uniformly on compact sets. Since β is point-norm continuous, it follows that the functions

$$g \mapsto \beta_g \left(\beta_{g^{-1}}(b)^* \cdot z_1 \cdots z_n \beta_{g^{-1}}(z_n^* \cdots z_1^*) \right)^* = z_1 \cdots z_n \beta_g(z_n^* \cdots z_1^*) \cdot b$$

must also converge uniformly on compact sets of G , for every $b \in B$.

Denote $x_n = z_1 \cdots z_n$. It follows that the strict limit $v_g = \lim_{n \rightarrow \infty} x_n \beta_g(x_n^*)$ exists for every $g \in G$, and that this convergence is uniform on compact subsets of G . In particular, the assignment $g \mapsto v_g \in \mathcal{U}(\mathcal{M}(B))$ is strictly continuous.

We calculate for each $g \in G$ that

$$\begin{aligned} \psi \circ \alpha_g &= \lim_{n \rightarrow \infty} \text{Ad}(x_n) \circ \varphi \circ \alpha_g \\ &= \lim_{n \rightarrow \infty} \text{Ad}(x_n) \circ \beta_g \circ \varphi \\ &= \lim_{n \rightarrow \infty} \text{Ad}(x_n) \circ \beta_g \circ \text{Ad}(x_n^*) \circ \text{Ad}(x_n) \circ \varphi \\ &= \lim_{n \rightarrow \infty} \text{Ad}(x_n \beta_g(x_n^*)) \circ \beta_g \circ \text{Ad}(x_n) \circ \varphi \\ &= \text{Ad}(v_g) \circ \beta_g \circ \psi. \end{aligned}$$

Keeping in mind that $\varphi, \psi_n : A \rightarrow B$ are non-degenerate, we may consider the unique, strictly continuous extensions $\varphi, \psi_n : \mathcal{M}(A) \rightarrow \mathcal{M}(B)$. Observe that we have convergence $\psi_n(m) \xrightarrow{\text{str}} \psi(m)$ for every $m \in \mathcal{M}(A)$: Given some $b \in B$ and $\delta > 0$, if we choose n large enough such that $\psi_n(m \cdot \psi^{-1}(b)) =_{\delta} \psi(m \cdot \psi^{-1}(b)) = \psi(m)b$ and $b =_{\delta} \psi_n(\psi^{-1}(b))$, then we get

$$\psi_n(m)b =_{\delta} \psi_n(m) \cdot \psi_n(\psi^{-1}(b)) = \psi_n(m \cdot \psi^{-1}(b)) =_{\delta} \psi(m) \cdot b.$$

In particular, we see that

$$x_n w(s, t) x_n^* = x_n \varphi(u(s, t)) x_n^* = \psi_n(u(s, t)) \xrightarrow{\text{str}} \psi(u(s, t))$$

for every $s, t \in G$.

Using this, let us now observe for every $s, t \in G$ that

$$\begin{aligned} v_s \beta_s(v_t) w(s, t) v_{st}^* &= \operatorname{str}\text{-}\lim_{n \rightarrow \infty} x_n \beta_s(x_n^*) \beta_s(x_n \beta_t(x_n^*)) w(s, t) (x_n \beta_{st}(x_n^*))^* \\ &= \operatorname{str}\text{-}\lim_{n \rightarrow \infty} x_n (\beta_s \circ \beta_t)(x_n^*) \underbrace{w(s, t) \beta_{st}(x_n)}_{=(\beta_s \circ \beta_t)(x_n) w(s, t)} x_n^* \\ &= \operatorname{str}\text{-}\lim_{n \rightarrow \infty} x_n w(s, t) x_n^* = \psi(u(s, t)). \end{aligned}$$

This finishes the proof of the first statement. However, reviewing the proof, it is clear that if z_n may always be chosen to be in \bar{B} , then φ is approximately unitarily equivalent to ψ . So the second statement follows as well. \square

For the next few results, recall the conventions introduced in Remark 1.8:

Lemma 2.2. *Let G be a second-countable, locally compact group. Let $(\alpha, u) : G \curvearrowright A$ and $(\beta, w) : G \curvearrowright B$ be two cocycle actions. Let $\varphi : (A, \alpha, u) \rightarrow (B, \beta, w)$ be a non-degenerate and equivariant $*$ -homomorphism.*

Let $x \in \mathcal{M}(B)_\infty \cap \varphi(A)'$ with the property that the map $[g \mapsto \varphi(a) \cdot \beta_{\infty, g}(x)]$ is continuous for all $a \in A$. Let $(x_n)_n \in \ell^\infty(\mathbb{N}, \mathcal{M}(B))$ be a bounded sequence representing x . Then for every $g_0 \in G$, $b \in B$ and $\delta > 0$, there exists $\ell \in \mathbb{N}$ and an open neighbourhood U of g_0 such that

$$\sup_{k \geq \ell} \sup_{g \in U} \|b(\beta_g(x_k) - \beta_{g_0}(x_k))\| \leq \delta.$$

Proof (see Note added in proof, p. 128). Since $\varphi(A) \subset B$ is assumed to be a non-degenerate subalgebra, it suffices to show the assertion for all $b \in \varphi(A)$.

We are going to apply a Baire category argument in the spirit of [13, 1.8]. However, we first make a few basic observations concerning cocycle action formulas in general. For $g \in G$, the cocycle identity of u implies

$$\operatorname{Ad}(w(g^{-1}, g)) = \operatorname{Ad}(w(g^{-1}, g)) \circ \beta_1 = \beta_{g^{-1}} \circ \beta_g.$$

In particular, we have $\beta_g^{-1} = \operatorname{Ad}(w(g^{-1}, g)^*) \circ \beta_{g^{-1}}$ for every $g \in G$. If now $g_0 \in G$ is a possibly different group element, then this implies

$$\beta_g^{-1} \circ \beta_{g_0} = \operatorname{Ad}(w(g^{-1}, g)^*) \circ \beta_{g^{-1}} \circ \beta_{g_0} = \operatorname{Ad}(w(g^{-1}, g)^* w(g^{-1}, g_0)) \circ \beta_{g^{-1} g_0}.$$

Now let us prove the assertion. Let $g_0 \in G$, $a \in A$ and $\delta > 0$ be given. Without loss of generality, assume that all of x , the elements x_k and a are contractions. Recall that $\beta : G \rightarrow \operatorname{Aut}(B)$ is point-norm continuous. Hence let us find a compact neighbourhood $V_1 = V_1^{-1} \ni 1_G$ of the unit such that

$$(e2.1) \quad \beta_g(\varphi(\alpha_{g_0}^{-1}(a))) =_\delta \varphi(\alpha_{g_0}^{-1}(a))$$

for all $g \in V_1$.

Recall that $u : G \times G \rightarrow \mathcal{M}(A)$ is strictly continuous. So as V_1 is compact, the set

$$K_A = \left\{ (\alpha_{g_0} \circ \alpha_{g_1} \circ \alpha_{g_2})^{-1} \left(a \cdot u(g_0, g_1 g_2)^* \alpha_{g_0} (u(g_1, g_2)^*) \right) \mid g_1, g_2 \in V_1 \right\}$$

is compact in A . By equivariance of φ , this means that the set

$$K_B = \left\{ (\beta_{g_0} \circ \beta_{g_1} \circ \beta_{g_2})^{-1} \left(\varphi(a) \cdot w(g_0, g_1 g_2)^* \beta_{g_0} (w(g_1, g_2)^*) \right) \mid g_1, g_2 \in V_1 \right\}$$

is equal to $\varphi(K_A)$ and thus compact in $\varphi(A)$. Since $x \in \mathcal{M}(B)_\infty \cap \varphi(A)'$, we can find some $\ell_0 \in \mathbb{N}$ such that

$$\|[x_k, d]\| \leq \delta \quad \text{for all } k \geq \ell_0 \text{ and } d \in K_B.$$

So if $d \in K_B$ is an element of the above form, one may apply $(\beta_{g_0} \circ \beta_{g_1} \circ \beta_{g_2})$ to this commutator and obtain that

$$(e2.2) \quad \left\| \left[\varphi(a) \cdot w(g_0, g_1 g_2)^* \beta_{g_0}(w(g_1, g_2)^*), (\beta_{g_0} \circ \beta_{g_1} \circ \beta_{g_2})(x_k) \right] \right\| \leq \delta$$

for all $g_1, g_2 \in V_1$ and $k \geq \ell_0$.

By the continuity assumption on x , we can find a neighbourhood $V = V^{-1}$ of $1_G \in G$ such that

$$(e2.3) \quad \|\varphi(\alpha_{g_0}^{-1}(a))(\beta_{\omega, g}(x) - x)\| = \limsup_{n \rightarrow \infty} \|\varphi(\alpha_{g_0}^{-1}(a))(\beta_g(x_n) - x_n)\| \leq \delta$$

for all $g \in V$. In other words, we have

$$\inf_{\ell \in \mathbb{N}} \sup_{n \geq \ell} \|\varphi(\alpha_{g_0}^{-1}(a))(\beta_g(x_n) - x_n)\| \leq \delta$$

for all $g \in V$. For every $\ell \in \mathbb{N}$, we define

$$(e2.4) \quad W_\ell = \{g \in G \mid \|\varphi(\alpha_{g_0}^{-1}(a))(\beta_g(x_k) - x_k)\| \leq 2\delta \text{ for all } k \geq \ell\}.$$

As the extension of β to a map from G to $\text{Aut}(\mathcal{M}(B))$ is strictly continuous, these sets are closed subsets of G . By (e2.3), it follows that the countable union $\bigcup_{\ell \in \mathbb{N}} W_\ell \cap W_\ell^{-1}$ contains the neighbourhood $V \cap V_1$ of the unit. By local compactness of G , it follows from the Baire category theorem that there exists $\ell \geq \ell_0$ such that $W_\ell \cap W_\ell^{-1}$ contains an open subset of V_1 . Then $U_0 = (V_1 \cap W_\ell \cap W_\ell^{-1})^{-1} \cdot (V_1 \cap W_\ell \cap W_\ell^{-1})$ is a neighbourhood of the unit. We then have for every $g_1, g_2 \in V_1 \cap W_\ell \cap W_\ell^{-1}$ and all $k \geq \ell$

$$\begin{aligned} & \varphi(a) \cdot \beta_{g_0 g_1^{-1} g_2}(x_k) \\ &= \varphi(a) \left(\text{Ad}(w(g_0, g_1^{-1} g_2)^*) \circ \beta_{g_0} \circ \beta_{g_1^{-1} g_2}(x_k) \right) \\ &= \varphi(a) \left(\text{Ad}(w(g_0, g_1^{-1} g_2)^* \cdot \beta_{g_0}(w(g_1^{-1}, g_2)^*)) \circ \beta_{g_0} \circ \beta_{g_1^{-1}} \circ \beta_{g_2}(x_k) \right) \\ &\stackrel{(e2.2)}{=} \beta_{g_0} \circ \beta_{g_1^{-1}} \circ \beta_{g_2}(x_k) \cdot \varphi(a) \\ &\stackrel{(e2.2)}{=} \varphi(a) \cdot (\beta_{g_0} \circ \beta_{g_1^{-1}} \circ \beta_{g_2})(x_k) \\ &= \beta_{g_0} \left(\varphi(\alpha_{g_0}^{-1}(a)) \cdot (\beta_{g_1^{-1}} \circ \beta_{g_2})(x_k) \right) \\ &\stackrel{(e2.1)}{=} \beta_{g_0} \circ \beta_{g_1^{-1}} \left(\varphi(\alpha_{g_0}^{-1}(a)) \cdot \beta_{g_2}(x_k) \right) \\ &\stackrel{(e2.4)}{=} \beta_{g_0} \circ \beta_{g_1^{-1}} \left(\varphi(\alpha_{g_0}^{-1}(a)) \cdot x_k \right) \\ &\stackrel{(e2.1)}{=} \beta_{g_0} \left(\varphi(\alpha_{g_0}^{-1}(a)) \cdot \beta_{g_1^{-1}}(x_k) \right) \\ &\stackrel{(e2.4)}{=} \beta_{g_0} \left(\varphi(\alpha_{g_0}^{-1}(a)) \cdot x_k \right) \\ &= \varphi(a) \cdot \beta_{g_0}(x_k). \end{aligned}$$

In particular, if we set $U = g_0 \cdot U_0$, then this shows

$$\sup_{k \geq \ell} \sup_{g \in U} \|\varphi(a) \cdot (\beta_{g_0}(x_k) - \beta_g(x_k))\| \leq 8\delta.$$

As $g_0 \in G$ and $\delta > 0$ are arbitrary, this finishes the proof. \square

Lemma 2.3. *Let G be a second-countable, locally compact group. Let $(\alpha, u) : G \curvearrowright A$ and $(\beta, w) : G \curvearrowright B$ be two cocycle actions. Let $\varphi : (A, \alpha, u) \rightarrow (B, \beta, w)$ be a non-degenerate and equivariant $*$ -homomorphism. Let $z \in \mathcal{M}(B)_\infty \cap \varphi(A)'$ be an element such that $[g \mapsto \varphi(a)\beta_{\infty, g}(z)]$ is continuous for every $a \in A$. Let $\varepsilon > 0$, $K \subset G$ a compact set and assume that $\|\varphi(a)(z - \beta_{\infty, g}(z))\| \leq \varepsilon$ for all $g \in K$ and $a \in A$ with $\|a\| \leq 1$. If $(z_n)_n$ is a bounded sequence representing z , then*

$$\limsup_{n \rightarrow \infty} \max_{g \in K} \|b(z_n - \beta_g(z_n))\| \leq \varepsilon$$

for all $b \in B$ with $\|b\| \leq 1$.

Proof. Given any $b \in B$, $\|b\| \leq 1$, $\delta > 0$ and $g_0 \in K$, we can apply Lemma 2.2 and find $\ell_0 \in \mathbb{N}$ and a neighbourhood $U \ni g_0$ such that

$$\sup_{k \geq \ell_0} \sup_{g \in U} \|b(\beta_{g_0}(z_k) - \beta_g(z_k))\| \leq \delta.$$

By compactness of K , we can find $N \in \mathbb{N}$, elements $\{g_j\}_{j=1}^N \subset K$ and a covering $K \subset \bigcup_{j=1}^N U_j$ with $g_j \in U_j$ and some $\ell \in \mathbb{N}$ such that

$$\sup_{k \geq \ell} \sup_{g \in U_j} \|b(\beta_{g_j}(z_k) - \beta_g(z_k))\| \leq \delta$$

for all $j = 1, \dots, N$. Since $\varphi(A)$ is non-degenerate in B , it follows from our assumptions on z that $\|b(\beta_{\infty, g_j}(z) - z)\| \leq \varepsilon$ for all $j = 1, \dots, N$. By enlarging ℓ , if necessary, we can thus assume that

$$\sup_{k \geq \ell} \|b(z_k - \beta_{g_j}(z_k))\| \leq \varepsilon$$

for all $j = 1, \dots, N$. So if $g \in K$ is arbitrary, then there is some j with $g \in U_j$, and so

$$\|b(\beta_g(z_k) - z_k)\| \leq \|b(\beta_{g_j}(z_k) - \beta_g(z_k))\| + \|b(z_k - \beta_{g_j}(z_k))\| \leq \delta + \varepsilon$$

for all $k \geq \ell$. Since δ and b were chosen arbitrary, this finishes the proof. \square

Remark 2.4. By inserting $b = \mathbf{1}_B$ in Lemma 2.3, we obtain the following unital variant:

Let B be unital and let $z \in B_\infty \cap A'$ be an element such that $[g \mapsto \beta_{\infty, g}(z)]$ is continuous. Let $\varepsilon > 0$, $K \subset G$ a compact set and assume that $\|z - \beta_{\infty, g}(z)\| \leq \varepsilon$ for all $g \in K$. If $(z_n)_n$ is a bounded sequence representing z , then

$$\limsup_{n \rightarrow \infty} \max_{g \in K} \|z_n - \beta_g(z_n)\| \leq \varepsilon.$$

Definition 2.5 (cf. [19, Section 6] and [12, 2.3]). Let $\gamma : G \curvearrowright D$ be an action of a second-countable, locally compact group on a separable, unital C^* -algebra. We say that the action γ (or the system (D, γ)) has approximately G -inner half-flip, if the two equivariant $*$ -homomorphisms $\text{id}_D \otimes \mathbf{1}, \mathbf{1} \otimes \text{id}_D : (D, \gamma) \rightarrow (D \otimes D, \gamma \otimes \gamma)$ are approximately G -unitarily equivalent.

We are now going to prove an equivariant McDuff-type absorption result in the unital case, modelling the well-known McDuff-type results [47, 7.2.2], [50, 2.3] or [26, 4.11] from the non-equivariant setting. We note that the result given below is folklore for discrete acting groups, and was already used by Izumi-Matui in [19, 6.3], Goldstein-Izumi in [12, 2.3] and Matui-Sato in [39, 4.8] and [40, proof of 4.9].

Theorem 2.6. *Let A and D be separable, unital C*-algebras and G a second-countable, locally compact group. Assume that $(\alpha, u) : G \curvearrowright A$ is a cocycle action. Let $\gamma : G \curvearrowright D$ be an action with approximately G -inner half-flip. If there exists an equivariant and unital *-homomorphism*

$$(D, \gamma) \rightarrow (A_{\infty, \alpha} \cap A', \alpha_{\infty}),$$

then (A, α, u) is strongly cocycle conjugate to $(D \otimes A, \gamma \otimes \alpha, \mathbf{1}_D \otimes u)$.

Proof. Consider the canonical unital *-homomorphisms

$$(A_{\infty} \cap A') \otimes_{\max} A \rightarrow A_{\infty}, \quad x \otimes a \mapsto x \cdot a.$$

Note that this is $(\alpha_{\infty} \otimes \alpha)$ -to- α_{∞} equivariant. This induces another unital and equivariant *-homomorphism (disregarding the cocycles for now)

$$(D \otimes (A_{\infty, \alpha} \cap A') \otimes_{\max} A, \gamma \otimes \alpha_{\infty} \otimes \alpha) \rightarrow ((D \otimes A)_{\infty, \gamma \otimes \alpha}, (\gamma \otimes \alpha)_{\infty})$$

via $d \otimes x \otimes a \mapsto d \otimes (x \cdot a) \in D \otimes (A_{\infty, \alpha}) \subset (D \otimes A)_{\infty, \gamma \otimes \alpha}$. By our assumption, it follows that there is a unital and equivariant *-homomorphism

$$\Phi : (D \otimes D \otimes A, \gamma \otimes \gamma \otimes \alpha) \rightarrow ((D \otimes A)_{\infty, \gamma \otimes \alpha}, (\gamma \otimes \alpha)_{\infty})$$

with $\Phi(d \otimes \mathbf{1}_D \otimes a) = d \otimes a$ for all $a \in A$ and $d \in D$, and with $\Phi(\mathbf{1}_D \otimes D \otimes A) \subset \mathbf{1}_D \otimes A_{\infty}$. Since (D, γ) has approximately G -inner half-flip, choose a sequence $v_n \in D \otimes D$ with $\max_{g \in K} \|v_n - (\gamma \otimes \gamma)_g(v_n)\| \rightarrow 0$ for every compact $K \subset G$, and $v_n^*(d \otimes \mathbf{1}_D)v_n \rightarrow \mathbf{1}_D \otimes d$ for all $d \in D$. The sequence of unitaries $u_n = \Phi(v_n \otimes \mathbf{1}_A) \in (D \otimes A)_{\infty, \gamma \otimes \alpha}$ then satisfies:

- $[u_n, \mathbf{1}_D \otimes a] = \Phi([v_n \otimes \mathbf{1}_A, \mathbf{1}_{D \otimes D} \otimes a]) = 0$ for all $a \in A$;
- $u_n^*(d \otimes a)u_n = \Phi((v_n \otimes \mathbf{1}_A)(d \otimes \mathbf{1}_D \otimes a)(v_n^* \otimes \mathbf{1}_A)) \rightarrow \Phi(\mathbf{1}_D \otimes d \otimes a) \in \mathbf{1}_D \otimes A_{\infty}$ for all $a \in A$ and $d \in D$;
- $\|u_n - (\gamma \otimes \alpha)_{\infty, g}(u_n)\| = \|\Phi((v_n - (\gamma \otimes \gamma)_g(v_n)) \otimes \mathbf{1}_A)\| \rightarrow 0$ for all $g \in G$, and uniformly on compact sets.

Lifting each u_n to a sequence of unitaries $(u_k^{(n)})_k \in \ell^{\infty}(\mathbb{N}, D \otimes A)$ and applying a diagonal sequence argument using Remark 2.4, we can find a sequence of unitaries $z_n = u_{k_n}^{(n)} \in D \otimes A$ satisfying:

- $[z_n, \mathbf{1}_D \otimes a] \rightarrow 0$ for all $a \in A$;
- $\text{dist}(z_n^*(d \otimes a)z_n, \mathbf{1} \otimes A) \rightarrow 0$ for all $a \in A$ and $d \in D$;
- $\max_{g \in K} \|z_n - (\gamma \otimes \alpha)_g(z_n)\| \rightarrow 0$ for all compact $K \subset G$.

In particular, we have met the conditions of (2.1a), (2.1b) and (2.1c) for the equivariant embedding $\mathbf{1}_D \otimes \text{id}_A : (A, \alpha, u) \rightarrow (D \otimes A, \gamma \otimes \alpha, \mathbf{1}_D \otimes u)$. This finishes the proof. \square

Remark. In the non-equivariant situation, a suitable replacement for $A_{\infty} \cap A'$ in Theorem 2.6 can be $\mathcal{M}(A)_{\infty} \cap A'$ to cover the non-unital case; see [47, 7.2.2]. In the equivariant case, the above result, Theorem 2.6, can be shown to generalize with using this C*-algebra, but only if α is assumed to be a genuine action. If $(\alpha, u) : G \curvearrowright A$ is a cocycle action with non-trivial cocycles, then $\mathcal{M}(A)_{\infty} \cap A'$ might not be the right object to consider, since α does not necessarily induce a genuine action on this C*-algebra. The more current (non-equivariant) McDuff type theorems for strongly self-absorbing C*-algebras [50, 2.3] and [26, 4.1] shift away from considering $\mathcal{M}(A)_{\infty} \cap A'$ in the non-unital case, in order to instead consider (either implicitly or explicitly) its quotient C*-algebra $F_{\infty}(A)$. As seen in

Remark 1.8, a cocycle action on A gives rise to a genuine action on $F_\infty(A)$ in a natural way, thus indicating that $F_\infty(A)$ is the more natural object to look at.

Both in [50] and [26], the cost of working with $F_\infty(A)$ in the non-unital case is that one has to use a strengthened version of an approximate inner half-flip, which essentially gives rise to all the K_1 -injectivity assumptions in both sources (however, we note that K_1 -injectivity later turned out to be redundant by virtue of [52]). In the next section, we are going to generalize this approach to the equivariant situation, and prove an optimal McDuff-type theorem by considering equivariant embeddings into $F_\infty(A)$.

3. STRONGLY SELF-ABSORBING C^* -DYNAMICAL SYSTEMS

In this section, we will introduce strongly self-absorbing actions and generalize some partial results from the seminal work of Toms and Winter [50] on strongly self-absorbing C^* -algebras.

Definition 3.1. Let \mathcal{D} be a separable, unital C^* -algebra and G a second-countable, locally compact group. Let $\gamma : G \curvearrowright \mathcal{D}$ be a point-norm continuous action. We say that (\mathcal{D}, γ) is a strongly self-absorbing C^* -dynamical system, or that γ is a strongly self-absorbing action, if the equivariant first-factor embedding

$$\text{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} : (\mathcal{D}, \gamma) \rightarrow (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$$

is approximately G -unitarily equivalent to an isomorphism.

Example 3.2. Let G be a second-countable, locally compact group and \mathcal{D} a strongly self-absorbing C^* -algebra. Then the trivial G -action on \mathcal{D} is a strongly self-absorbing action.

Remark. Trivially, any two equivariant $*$ -homomorphisms, which are approximately G -unitarily equivalent, are also approximately unitarily equivalent as ordinary $*$ -homomorphisms. So if (\mathcal{D}, γ) is a strongly self-absorbing C^* -dynamical system, then \mathcal{D} must be a strongly self-absorbing C^* -algebra; see [50].

We now make some important observations concerning strongly self-absorbing actions on the one hand, and actions with approximately G -inner half-flip on the other hand. This mimics the situation in the non-equivariant case [50]:

Proposition 3.3 (cf. [50, 1.9(ii)]). *Let \mathcal{D} be a separable, unital C^* -algebra. Let $\gamma : G \curvearrowright \mathcal{D}$ be an action with approximately G -inner half-flip. Then the infinite tensor power $(\bigotimes_{\mathbb{N}} \mathcal{D}, \bigotimes_{\mathbb{N}} \gamma)$ is strongly self-absorbing.*

Proof. For each $k \in \mathbb{N}$, let

$$\iota_k = \text{id}_{\mathcal{D}^{\otimes 2^k}} \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2^k}} : \mathcal{D}^{\otimes 2^k} \rightarrow \mathcal{D}^{\otimes 2^k} \otimes \mathcal{D}^{\otimes 2^k} = \mathcal{D}^{\otimes 2^{k+1}}$$

be the first-factor embedding. Consider the following equivariant inductive system: $\{(D^{\otimes 2^k}, \gamma^{\otimes 2^k}), \iota_k\}$. The limit of this system is, by definition, the system $(\bigotimes_{\mathbb{N}} \mathcal{D}, \bigotimes_{\mathbb{N}} \gamma)$. Denote by $\iota_{k,\infty} : (D^{\otimes 2^k}, \gamma^{\otimes 2^k}) \rightarrow (\bigotimes_{\mathbb{N}} \mathcal{D}, \bigotimes_{\mathbb{N}} \gamma)$ the embedding into the limit for every $k \in \mathbb{N}$. The tensor product of this system with itself is then given by the limit

$$\lim_{\rightarrow} \left\{ (D^{\otimes 2^k} \otimes D^{\otimes 2^k}, \gamma^{\otimes 2^k} \otimes \gamma^{\otimes 2^k}), \iota_k \otimes \iota_k \right\}.$$

Consider the two equivariant $*$ -homomorphisms

$$\iota_k \otimes \iota_k, \iota_{k+1} : ((D^{\otimes 2^k})^{\otimes 2}, (\gamma^{\otimes 2^k})^{\otimes 2}) \rightarrow ((D^{\otimes 2^k})^{\otimes 4}, (\gamma^{\otimes 2^k})^{\otimes 4}).$$

Then we see $\iota_{k+1} = \sigma_{2,3} \circ (\iota_k \otimes \iota_k)$, where $\sigma_{2,3}$ is the automorphism on $(D^{\otimes 2^k})^{\otimes 4}$ that flips the second and the third tensor factors. Since (D, γ) has approximately G -inner half-flip, so has $(D, \gamma)^{\otimes 2^k}$. It follows that

$$\iota_{k+1} = \text{id} \otimes \text{id} \otimes \mathbf{1} \otimes \mathbf{1} \approx_{u,G} \text{id} \otimes \mathbf{1} \otimes \text{id} \otimes \mathbf{1} = \iota_k \otimes \iota_k.$$

From Corollary 1.16, we can thus deduce that there exists an isomorphism

$$\varphi : \left(\bigotimes_{\mathbb{N}} D, \bigotimes_{\mathbb{N}} \gamma \right) \rightarrow \left(\bigotimes_{\mathbb{N}} D, \bigotimes_{\mathbb{N}} \gamma \right)^{\otimes 2},$$

with $\varphi \circ \iota_{k+1, \infty} \approx_{u,G} \iota_{k, \infty} \otimes \iota_{k, \infty}$ for all $k \in \mathbb{N}$. By repeating the same argument as before, we see that $\iota_{k, \infty} \otimes \iota_{k, \infty} \approx_{u,G} \iota_{k+1, \infty} \otimes \mathbf{1}_D^{\otimes \infty}$ for all $k \in \mathbb{N}$. But this implies that φ is approximately G -unitarily equivalent to the first-factor embedding, which verifies that the system $(\bigotimes_{\mathbb{N}} D, \bigotimes_{\mathbb{N}} \gamma)$ is strongly self-absorbing. \square

Proposition 3.4 (cf. [50, 1.10(iv), 1.5, 1.12]). *Let (\mathcal{D}, γ) be a strongly self-absorbing C^* -dynamical system. Then:*

- (i) (\mathcal{D}, γ) is conjugate to its infinite tensor power $(\bigotimes_{\mathbb{N}} \mathcal{D}, \bigotimes_{\mathbb{N}} \gamma)$.
- (ii) (\mathcal{D}, γ) has approximately G -inner half-flip.
- (iii) Let (A, α) be a C^* -dynamical system on a unital C^* -algebra such that (A, α) is cocycle conjugate to $(A \otimes \mathcal{D}, \alpha \otimes \gamma)$. Then any two equivariant, unital $*$ -homomorphisms from (\mathcal{D}, γ) to (A, α) are approximately G -unitarily equivalent.

Proof. Let us show (i). For each $k \in \mathbb{N}$, let $\iota_k = \text{id}_{\mathcal{D}^{\otimes k}} \otimes \mathbf{1}_{\mathcal{D}} : \mathcal{D}^{\otimes k} \rightarrow \mathcal{D}^{\otimes k+1}$ be the first-factor embedding. Let $\varphi : (\mathcal{D}, \gamma) \rightarrow (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$ be an isomorphism with $\varphi \approx_{u,G} \iota_1$. Consider the two (equivariant) inductive systems $\{(\mathcal{D}^{\otimes k}, \gamma^{\otimes k}), \iota_k\}$ and $\{(\mathcal{D}^{\otimes k}, \gamma^{\otimes k}), \text{id}_{\mathcal{D}^{\otimes k-1}} \otimes \varphi\}$. The limit of the first inductive system is precisely $(\bigotimes_{\mathbb{N}} \mathcal{D}, \bigotimes_{\mathbb{N}} \gamma)$. The limit of the second inductive system is conjugate to its first building block (\mathcal{D}, γ) because each connecting map is an isomorphism. Now because of

$$\iota_k = \text{id}_{\mathcal{D}^{\otimes k-1}} \otimes \iota_1 \approx_{u,G} \text{id}_{\mathcal{D}^{\otimes k-1}} \otimes \varphi$$

for each k , it follows from Corollary 1.16 that (\mathcal{D}, γ) and $(\bigotimes_{\mathbb{N}} \mathcal{D}, \bigotimes_{\mathbb{N}} \gamma)$ are conjugate.

In order to show (ii), one carries out Toms and Winter's original proof from the non-equivariant case [50, 1.5] verbatim, only replacing approximate unitary equivalence by approximate G -unitary equivalence.

Let us show (iii). Let $\varphi_1, \varphi_2 : (\mathcal{D}, \gamma) \rightarrow (A, \alpha)$ be two equivariant, unital $*$ -homomorphisms. Since (A, α) is cocycle conjugate to $(A \otimes \mathcal{D}, \alpha \otimes \gamma)$, it follows from (i) that we can find a unital $*$ -homomorphism $\psi : (\mathcal{D}, \gamma) \rightarrow (A_{\infty, \alpha} \cap A', \alpha_{\infty})$. This induces two equivariant $*$ -homomorphisms

$$\kappa_1, \kappa_2 : (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma) \rightarrow (A_{\infty, \alpha}, \alpha_{\infty}) \text{ via } \kappa_j(a \otimes b) = \varphi_j(a) \cdot \psi(b), \quad j = 1, 2.$$

By (ii), we know that (\mathcal{D}, γ) has approximately G -inner half-flip. Let $K \subset G$ be compact, $F \subset \mathcal{D}$ and $\varepsilon > 0$. There exists a unitary $v \in \mathcal{U}(\mathcal{D} \otimes \mathcal{D})$ with

$$v(a \otimes \mathbf{1})v^* =_{\varepsilon} \mathbf{1} \otimes a \text{ and } v =_{\varepsilon} (\gamma \otimes \gamma)_g(v)$$

for all $a \in F$ and $g \in K$. The unitary $w = \kappa_2(v^*) \cdot \kappa_1(v) \in \mathcal{U}(A_{\infty, \alpha})$ is then a continuous element under α_{∞} , and satisfies

$$\alpha_{\infty, g}(w) =_{2\varepsilon} w$$

and

$$\begin{aligned} w\varphi_1(a)w^* &= \kappa_2(v^*)\kappa_1(v)\kappa_1(a \otimes \mathbf{1})\kappa_1(v^*)\kappa_2(v) \\ &= \kappa_2(v^*)\kappa_1(v(a \otimes \mathbf{1})v^*)\kappa_2(v) \\ &=_{\varepsilon} \kappa_2(v^*)\kappa_1(\mathbf{1} \otimes a)\kappa_2(v) \\ &= \kappa_2(v^*)\kappa_2(\mathbf{1} \otimes a)\kappa_2(v) \\ &= \kappa_2(v^*(\mathbf{1} \otimes a)v) \\ &=_{\varepsilon} \kappa_2(a \otimes \mathbf{1}) = \varphi_2(a) \end{aligned}$$

for all $g \in K$ and $a \in F$. Representing w by a sequence of unitaries in A , we can apply Remark 2.4 (with A in place of B and \mathbb{C} in place of A) and find a unitary $z \in \mathcal{U}(A)$ satisfying

$$\alpha_g(z) =_{3\varepsilon} z \text{ and } z\varphi_1(a)z^* =_{3\varepsilon} \varphi_2(a)$$

for all $g \in K$ and $a \in F$. As K, F, ε are arbitrary, we have $\varphi_1 \approx_{u, G} \varphi_2$. \square

Proposition 3.5 (cf. [50, 1.13]). *Let (\mathcal{D}, γ) be a strongly self-absorbing C^* -dynamical system. Then there exist sequences of unitaries $u_n, v_n \in \mathcal{U}(\mathcal{D} \otimes \mathcal{D})$ satisfying*

$$\max_{g \in K} \left(\|u_n - (\gamma \otimes \gamma)_g(u_n)\| + \|v_n - (\gamma \otimes \gamma)_g(v_n)\| \right) \xrightarrow{n \rightarrow \infty} 0$$

for every compact set $K \subset G$ and

$$\text{Ad}(u_n v_n u_n^* v_n^*)(d \otimes \mathbf{1}_{\mathcal{D}}) \xrightarrow{n \rightarrow \infty} \mathbf{1}_{\mathcal{D}} \otimes d$$

for all $d \in \mathcal{D}$.

Proof. Let $K \subset G$ be a compact subset and $\varepsilon > 0$. Applying Proposition 3.4(i), we may replace (\mathcal{D}, γ) by its infinite tensor power. So let $k \in \mathbb{N}$ be a number and $F \subset \mathcal{D}^{\otimes k}$ a finite set consisting of contractions. By Proposition 3.4(ii), we know that (\mathcal{D}, γ) , and hence all of its tensorial powers, have approximately G -inner half-flip. So choose a unitary $\tilde{u} \in \mathcal{U}(\mathcal{D}^{\otimes k} \otimes \mathcal{D}^{\otimes k})$ with

$$\max_{g \in K} \|\tilde{u} - (\gamma^{\otimes k} \otimes \gamma^{\otimes k})_g(\tilde{u})\| \leq \varepsilon \text{ and } \tilde{u}(a \otimes \mathbf{1}_{\mathcal{D}^{\otimes k}})\tilde{u}^* =_{\varepsilon} \mathbf{1}_{\mathcal{D}^{\otimes k}} \otimes a$$

for all $a \in F$. Applying approximately G -inner half-flip again, find a unitary $\tilde{v} \in \mathcal{U}(\mathcal{D}^{\otimes 2k} \otimes \mathcal{D}^{\otimes 2k})$ with

$$\max_{g \in K} \|\tilde{v} - (\gamma^{\otimes 2k} \otimes \gamma^{\otimes 2k})_g(\tilde{v})\| \leq \varepsilon \text{ and } \tilde{v}(\tilde{u}^* \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}})\tilde{v}^* =_{\varepsilon} \mathbf{1}_{\mathcal{D}^{\otimes 2k}} \otimes \tilde{u}^*.$$

This yields

$$\begin{aligned} &(\tilde{u} \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}})\tilde{v}(\tilde{u} \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}})^*\tilde{v}^*(a \otimes \mathbf{1}_{\mathcal{D}^{\otimes 3k}})[(\tilde{u} \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}})\tilde{v}(\tilde{u} \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}})^*\tilde{v}^*]^* \\ &=_{2\varepsilon} (\tilde{u} \otimes \tilde{u}^*)(a \otimes \mathbf{1}_{\mathcal{D}^{\otimes 3k}})(\tilde{u} \otimes \tilde{u}^*) \\ &= (\tilde{u}(a \otimes \mathbf{1}_{\mathcal{D}^{\otimes k}})\tilde{u}) \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}} \\ &=_{\varepsilon} (\mathbf{1}_{\mathcal{D}^{\otimes k}} \otimes a) \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}} \end{aligned}$$

for all $a \in F$. If now $\sigma_{2,3} \in \text{Aut}((\mathcal{D}^{\otimes k})^{\otimes 4})$ is the (equivariant) automorphism that flips the second a third tensorial factor, then the unitaries $u = \sigma_{2,3}(\tilde{u} \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}})$ and

$v = \sigma_{2,3}(\tilde{v})$ satisfy

$$\begin{aligned} & \text{Ad}(uvu^*v^*)\left((a \otimes \mathbf{1}_{\mathcal{D}^{\otimes k}}) \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}}\right) \\ &= \sigma_{2,3}\left((\tilde{u} \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}})\tilde{v}(\tilde{u} \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}})^*\tilde{v}^*(a \otimes \mathbf{1}_{\mathcal{D}^{\otimes 3k}})[(\tilde{u} \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}})\tilde{v}(\tilde{u} \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}})^*\tilde{v}^*]^*\right) \\ &=_{3\varepsilon} \sigma_{2,3}\left((\mathbf{1}_{\mathcal{D}^{\otimes k}} \otimes a) \otimes \mathbf{1}_{\mathcal{D}^{\otimes 2k}}\right) \\ &= \mathbf{1}_{\mathcal{D}^{\otimes 2k}} \otimes (a \otimes \mathbf{1}_{\mathcal{D}^{\otimes k}}) \end{aligned}$$

for all $a \in F$. Since k, F, K, ε are arbitrary, this shows our claim. □

In particular, this means that in strongly self-absorbing C*-dynamical systems, the approximate G -inner half-flip may be chosen to be implemented by unitaries representing the zero-class in K_1 , similarly as in [50]. By K_1 -injectivity of strongly self-absorbing C*-algebras, which follows from [20, 52], we get the following consequence.

Corollary 3.6 (cf. [20] and [52, 3.1, 3.3]). *Let (\mathcal{D}, γ) be a strongly self-absorbing C*-dynamical system. Then there exists a sequence of unitaries $v_n \in \mathcal{U}_0(\mathcal{D} \otimes \mathcal{D})$ satisfying*

$$\max_{g \in K} \|v_n - (\gamma \otimes \gamma)_g(v_n)\| \xrightarrow{n \rightarrow \infty} 0$$

for every compact set $K \subset G$ and

$$v_n(d \otimes \mathbf{1}_{\mathcal{D}})v_n^* \xrightarrow{n \rightarrow \infty} \mathbf{1}_{\mathcal{D}} \otimes d$$

for all $d \in \mathcal{D}$.

The following constitutes the main result of this paper.

Theorem 3.7. *Let G be a second-countable, locally compact group. Let A be a separable C*-algebra and $(\alpha, u) : G \curvearrowright A$ a cocycle action. Let \mathcal{D} be a separable, unital C*-algebra and let $\gamma : G \curvearrowright \mathcal{D}$ be an action such that (\mathcal{D}, γ) is strongly self-absorbing. The following are equivalent:*

- (i) (A, α, u) is strongly cocycle conjugate to $(A \otimes \mathcal{D}, \alpha \otimes \gamma, u \otimes \mathbf{1}_{\mathcal{D}})$.
- (ii) (A, α, u) is cocycle conjugate to $(A \otimes \mathcal{D}, \alpha \otimes \gamma, u \otimes \mathbf{1}_{\mathcal{D}})$.
- (iii) There exists a unital, equivariant *-homomorphism from (\mathcal{D}, γ) to $(F_{\infty, \alpha}(A), \tilde{\alpha}_{\infty})$.
- (iv) The first-factor embedding $\text{id}_A \otimes \mathbf{1}_{\mathcal{D}} : (A, \alpha, u) \rightarrow (A \otimes \mathcal{D}, \alpha \otimes \gamma, u \otimes \mathbf{1}_{\mathcal{D}})$ is approximately unitarily equivalent to an isomorphism inducing strong cocycle conjugacy.

Proof. The implications from (iv) to (i) and from (i) to (ii) are trivial. The implication from (ii) to (iii) follows from Proposition 3.4(i) with Remark 1.10. So we need to verify the implication from (iii) to (iv).

Keeping in mind Remark 1.11, we have a natural isomorphism

$$F(\mathbf{1}_{\mathcal{D}} \otimes A, (\mathcal{D} \otimes A)_{\infty}) \cong F(\mathbf{1}_{\mathcal{D}} \otimes A, ((\mathcal{D} \otimes A)^{\sim})_{\infty}).$$

Denote by $\pi : ((\mathcal{D} \otimes A)^{\sim})_{\infty} \cap (\mathbf{1}_{\mathcal{D}} \otimes A)' \rightarrow F(\mathbf{1}_{\mathcal{D}} \otimes A, (\mathcal{D} \otimes A)_{\infty})$ the canonical surjection. Note that by assumption, we have an equivariant, unital *-homomorphism from (\mathcal{D}, γ) to $F_{\infty, \alpha}(A)$. Consider the canonical inclusions

$$F_{\infty}(A), \mathcal{D} \subset F(\mathbf{1}_{\mathcal{D}} \otimes A, (\mathcal{D} \otimes A)_{\infty}),$$

which define commuting C^* -subalgebras. Since these inclusions are natural, they are equivariant with respect to the induced actions of α , γ and $\gamma \otimes \alpha$. By assumption, it follows that we have a unital and equivariant $*$ -homomorphism

$$\varphi : (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma) \rightarrow (F_{\gamma \otimes \alpha}(\mathbf{1}_{\mathcal{D}} \otimes A, (\mathcal{D} \otimes A)_{\infty}), (\gamma \otimes \alpha)_{\infty}^{\sim})$$

satisfying $\varphi(d \otimes \mathbf{1}_{\mathcal{D}}) \cdot (\mathbf{1}_{\mathcal{D}} \otimes a) = d \otimes a$ and $\varphi(\mathbf{1}_{\mathcal{D}} \otimes d) \cdot (\mathbf{1}_{\mathcal{D}} \otimes a) \in \mathbf{1}_{\mathcal{D}} \otimes A_{\infty}$ for all $a \in A$ and $d \in \mathcal{D}$. (Note that forming these products makes sense because of Remark 1.12.)

Now let $\varepsilon > 0$, $F_1 \subset \mathcal{D}$, $F_2 \subset A$ and $K \subset G$ compact. Without loss of generality, assume that F_1 and F_2 consist of contractions. Since (\mathcal{D}, γ) has approximately G -inner half-flip, choose a unitary $v \in \mathcal{D} \otimes \mathcal{D}$ with

$$\max_{g \in K} \|v - (\gamma \otimes \gamma)_g(v)\| \leq \varepsilon \quad \text{and} \quad v^*(d \otimes \mathbf{1}_{\mathcal{D}})v =_{\varepsilon} \mathbf{1}_{\mathcal{D}} \otimes d$$

for all $d \in F_1$. Moreover, because of Corollary 3.6, we may assume that v is homotopic to $\mathbf{1}_{\mathcal{D} \otimes \mathcal{D}}$ in $\mathcal{U}(\mathcal{D} \otimes \mathcal{D})$.

The unitary $u = \varphi(v) \in F_{\gamma \otimes \alpha}(\mathbf{1}_{\mathcal{D}} \otimes A, (\mathcal{D} \otimes A)_{\infty})$ then satisfies

$$u^*(d \otimes a)u = \varphi(v(d \otimes \mathbf{1}_{\mathcal{D}})v^*) \cdot (\mathbf{1}_{\mathcal{D}} \otimes a) =_{\varepsilon} \varphi(\mathbf{1}_{\mathcal{D}} \otimes d) \cdot (\mathbf{1}_{\mathcal{D}} \otimes a) \in \mathbf{1}_{\mathcal{D}} \otimes A_{\infty}$$

for all $a \in A$ with $\|a\| \leq 1$ and $d \in F_1$, and moreover

$$\|u - (\gamma \otimes \alpha)_{\infty, g}^{\sim}(u)\| \leq \|v - (\gamma \otimes \gamma)_g(v)\| \leq \varepsilon$$

for all $g \in K$.

Let $z \in ((\mathcal{D} \otimes A)_{\infty}^{\sim}) \cap (\mathbf{1}_{\mathcal{D}} \otimes A)'$ be an element with $u = \pi(z)$. Since we have chosen v to be homotopic to $\mathbf{1}_{\mathcal{D} \otimes \mathcal{D}}$, u is also homotopic to $\mathbf{1}$ in $F(\mathbf{1}_{\mathcal{D}} \otimes A, (\mathcal{D} \otimes A)_{\infty})$, and therefore we can choose the lift $z = [(z_n)_n]$ to be represented by a sequence of unitaries. Note that u is a continuous element with respect to $(\gamma \otimes \alpha)_{\infty}^{\sim}$, so it follows that $[g \mapsto (\mathbf{1}_{\mathcal{D}} \otimes a) \cdot (\gamma \otimes \alpha)_{\infty, g}^{\sim}(z)]$ is a continuous map on G for every $a \in A$. Moreover, because of the properties of u , we have

$$\text{dist}(z^*(d \otimes a)z, \mathbf{1}_{\mathcal{D}} \otimes A_{\infty}) \leq \varepsilon$$

for all $a \in A$ with $\|a\| \leq 1$ and $d \in F_1$, and moreover

$$\|(\mathbf{1}_{\mathcal{D}} \otimes a)(z - (\gamma \otimes \alpha)_{\infty, g}^{\sim}(z))\| \leq \varepsilon$$

for all $g \in K$ and $a \in A$ with $\|a\| \leq 1$.

Let $z_n \in (\mathcal{D} \otimes A)_{\infty}^{\sim}$ be a bounded sequence representing z . Using Lemma 2.3, there is some n with

- $\|[z_n, \mathbf{1}_{\mathcal{D}} \otimes a]\| \leq \varepsilon$ for all $a \in F_2$;
- $\text{dist}(z_n^*(d \otimes a)z_n, \mathbf{1}_{\mathcal{D}} \otimes A) \leq 2\varepsilon$ for all $d \in F_1$ and $a \in F_2$;
- $\max_{g \in K} \|z_n - (\gamma \otimes \alpha)_g(z_n)\| \leq 2\varepsilon$.

So we have met the conditions of (2.1a), (2.1b) and (2.1c) for the equivariant embedding $\mathbf{1}_{\mathcal{D}} \otimes \text{id}_A : (A, \alpha, u) \rightarrow (D \otimes A, \gamma \otimes \alpha, \mathbf{1}_{\mathcal{D}} \otimes u)$. This finishes the proof. \square

Corollary 3.8. *Let G be a second-countable, locally compact group. Let A be a separable C^* -algebra and $(\alpha, u) : G \curvearrowright A$ a cocycle action. Let \mathcal{D} be a strongly self-absorbing C^* -algebra. The following are equivalent:*

- (i) (A, α, u) is strongly cocycle conjugate to $(A \otimes \mathcal{D}, \alpha \otimes \text{id}_{\mathcal{D}}, u \otimes \mathbf{1}_{\mathcal{D}})$.
- (ii) (A, α, u) is cocycle conjugate to $(A \otimes \mathcal{D}, \alpha \otimes \text{id}_{\mathcal{D}}, u \otimes \mathbf{1}_{\mathcal{D}})$.
- (iii) There exists a unital, equivariant $*$ -homomorphism from \mathcal{D} to the fixed point algebra $F_{\infty, \alpha}(A)^{\alpha_{\infty}}$.

- (iv) *The first-factor embedding $\text{id}_A \otimes \mathbf{1}_{\mathcal{D}} : (A, \alpha, u) \rightarrow (A \otimes \mathcal{D}, \alpha \otimes \text{id}_{\mathcal{D}}, u \otimes \mathbf{1}_{\mathcal{D}})$ is approximately unitarily equivalent to an isomorphism inducing strong cocycle conjugacy.*

We finish this section with an example that illustrates how for non-compact acting groups, our main result, Theorem 3.7, cannot be strengthened to characterize tensorial absorption with respect to conjugacy:

Example 3.9. Let $\theta \in (0, 1)$ be an irrational number. Consider the Cuntz algebra \mathcal{O}_2 (see [6]) with the two canonical generators $s_1, s_2 \in \mathcal{O}_2$. Let the automorphism $\alpha \in \text{Aut}(\mathcal{O}_2)$ be given by $\alpha(s_j) = e^{2\pi i \theta} s_j$ for $j = 1, 2$. Then the associated \mathbb{Z} -action $\alpha : \mathbb{Z} \curvearrowright \mathcal{O}_2$ arises as the composition of the irrational rotation embedding $\mathbb{Z} \rightarrow \mathbb{T}$ as a dense subgroup, with the natural gauge action of \mathbb{T} on \mathcal{O}_2 given by $z \cdot s_j = z \cdot s_j$ for $j = 1, 2$. So we have $\mathcal{O}_2^\alpha \cong M_{2^\infty}$ because the fixed-point algebra of the gauge action is well-known to be the CAR algebra. On the other hand, α is quasi-free and faithful, so in particular pointwise outer; see [9, Section 2]. Since any endomorphism on \mathcal{O}_2 is asymptotically inner and $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$, it follows from Nakamura's uniqueness result [43, Theorem 9] that α is cocycle conjugate to $\alpha \otimes \text{id}_{\mathcal{O}_2}$. Since the fixed-point algebra of α is stably finite, these two actions clearly cannot be conjugate.

4. SEMI-STRONGLY SELF-ABSORBING ACTIONS

As we have seen in the previous section, strong cocycle conjugacy is in general a strictly weaker relation than conjugacy. In the non-equivariant context, this distinction naturally plays no role. So we see that this aspect adds some complexity to the equivariant situation. For instance, it follows from several results from the literature (see section 5 for details) that many actions on a strongly self-absorbing C*-algebra can be strongly cocycle conjugate (but not conjugate) to a strongly self-absorbing action. This prompts the question whether one can expand the tensorial absorption theorem from the previous section to a larger class of actions. In this section, we will therefore study C*-dynamical systems that are, in a sense, almost strongly self-absorbing:

Definition 4.1. Let G be a second-countable, locally compact group and \mathcal{D} a separable, unital C*-algebra. We call an action $\gamma : G \curvearrowright \mathcal{D}$ semi-strongly self-absorbing, if it is strongly cocycle conjugate to a strongly self-absorbing action.

We will study some properties that semi-strongly self-absorbing actions inherit from strongly self-absorbing actions. However, we first have to make a few general observations.

Lemma 4.2. *Let G be a second-countable, locally compact group and A and B two separable, unital C*-algebras. Suppose that $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ are two actions that are strongly cocycle conjugate. Then for every $\varepsilon > 0$ and compact set $K \subset G$, there exists an isomorphism $\varphi : A \rightarrow B$ with $\max_{g \in K} \|\beta_g \circ \varphi - \varphi \circ \alpha_g\| \leq \varepsilon$.*

Proof. Using that α and β are strongly cocycle conjugate, we find an isomorphism $\psi : A \rightarrow B$ and a sequence of unitaries $x_n \in \mathcal{U}(B)$ such that the functions $[g \mapsto x_n \beta_g(x_n^*)]$ on G converge uniformly on compact sets as n tends to infinity, such that with $w_g = \lim_{n \rightarrow \infty} x_n \beta_g(x_n^*)$, we have

$$\text{Ad}(w_g) \circ \beta_g \circ \psi = \psi \circ \alpha_g \quad \text{for all } g \in G.$$

Applying $\text{Ad}(x_n^*)$ to both sides of this equation, we obtain

$$\text{Ad}(x_n^*) \circ \psi \circ \alpha_g = \text{Ad}(x_n^* w_g) \circ \beta_g \circ \psi = \text{Ad}(x_n^* w_g \beta_g(x_n)) \circ \beta_g \circ \text{Ad}(x_n^*) \circ \psi.$$

Let $\varepsilon > 0$ and $K \subset G$ be given. By the definition of the cocycle w_g , there is some n such that $\max_{g \in K} \|x_n^* w_g \beta_g(x_n) - \mathbf{1}_B\| \leq \varepsilon/2$. Then the map $\varphi = \text{Ad}(x_n^*) \circ \psi$ has the properties that we are looking for. \square

Lemma 4.3. *Let G be a second-countable, locally compact group and A and B two separable, unital C^* -algebras. Suppose that $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ are two actions that are strongly cocycle conjugate. If β has approximately G -inner half-flip, then so does α .*

Proof. Let $\varepsilon > 0$, $F \subset A$ and a compact set $K \subset G$ be given. Applying Lemma 4.2, we find a sequence of isomorphisms $\varphi_n : A \rightarrow B$ with

$$\max_{g \in K} \|\beta_g \circ \varphi_n - \varphi_n \circ \alpha_g\| \xrightarrow{n \rightarrow \infty} 0.$$

It follows that

$$\max_{g \in K} \|(\beta \otimes \beta)_g \circ (\varphi_n \otimes \varphi_n) - (\varphi_n \otimes \varphi_n) \circ (\alpha \otimes \alpha)_g\| \leq \varepsilon$$

for some large enough n . Choose a unitary $v \in B \otimes B$ with

$$\max_{g \in K} \|v - (\beta \otimes \beta)_g(v)\| \leq \varepsilon$$

and

$$v(\varphi_n(a) \otimes \mathbf{1}_B)v^* =_\varepsilon \mathbf{1}_B \otimes \varphi_n(a) \quad \text{for all } a \in F.$$

For the unitary $u = (\varphi_n \otimes \varphi_n)^{-1}(v)$, it follows that

$$\begin{aligned} \max_{g \in K} \|u - (\alpha \otimes \alpha)_g(u)\| &= \max_{g \in K} \|v - (\varphi_n \otimes \varphi_n) \circ (\alpha \otimes \alpha)_g \circ (\varphi_n \otimes \varphi_n)^{-1}(v)\| \\ &\leq \varepsilon + \max_{g \in K} \|v - (\beta \otimes \beta)_g(v)\| \leq 2\varepsilon. \end{aligned}$$

Moreover, we have

$$u(a \otimes \mathbf{1}_A)u^* =_\varepsilon \mathbf{1}_A \otimes a \quad \text{for all } a \in F$$

by choice of v . This finishes the proof. \square

The following can be seen as a stronger variant of Lemma 2.3.

Lemma 4.4. *Let G be a second-countable, locally compact group and A a separable C^* -algebra. Suppose that $(\alpha, u) : G \curvearrowright A$ is a cocycle action. Let $x \in F_{\infty, \alpha}(A)$ be a continuous element with respect to α . Let $(x_n)_n \in \ell^\infty(\mathbb{N}, A)$ be a representing sequence of x . Then for every $\varepsilon > 0$, $a \in A_{1,+}$ and compact set $K \subset G$, there is $\ell \in \mathbb{N}$ such that*

$$\sup_{n \geq \ell} \|(\alpha_g(x_n) - x_n)a\| \leq \|\tilde{\alpha}_{\infty, g}(x) - x\| + \varepsilon$$

for all $g \in K$.

Proof. By definition, we have

$$(e4.1) \quad \|\tilde{\alpha}_{\infty, g}(x) - x\| = \sup_{a \in A_{1,+}} \limsup_{n \rightarrow \infty} \|(\alpha_g(x_n) - x_n)a\|$$

for all $g \in G$.

Let $\tilde{x} \in A_\infty \cap A'$ be an element with $x = \tilde{x} + \text{Ann}(A, A_\omega)$. Then since x is $\tilde{\alpha}_\infty$ -continuous, it follows that the assignment $g \mapsto \alpha_{\infty, g}(\tilde{x}) \cdot a$ is continuous on G .

Let $\varepsilon > 0$, let $K \subset G$ be a compact set, and let $a \in A_{1,+}$ be fixed. Let $g_0 \in K$. Applying Lemma 2.2, choose an open neighbourhood $U \ni g_0$ and $\ell \in \mathbb{N}$ with

$$(e4.2) \quad \sup_{n \geq \ell} \sup_{g \in U} \|(\alpha_g(x_n) - \alpha_{g_0}(x_n))a\| \leq \varepsilon$$

and

$$(e4.3) \quad \sup_{g \in U} \|\tilde{\alpha}_{\infty, g}(x) - \tilde{\alpha}_{\infty, g_0}(x)\| \leq \varepsilon.$$

Since $g_0 \in K$ is arbitrary, we thus get an open cover. By compactness of K , we find $g_1, \dots, g_N \in K$ and an open covering $K \subset \bigcup_{i=1}^N U_i$ and $\ell_1, \dots, \ell_N \in \mathbb{N}$, such that for every i , we have $g_i \in U_i$ and U_i satisfies (e4.2) with ℓ_i and (e4.3). Using (e4.1), we find $\ell_0 \geq \max(\ell_1, \dots, \ell_N)$ such that

$$(e4.4) \quad \sup_{n \geq \ell_0} \|(\alpha_{g_i}(x_n) - x_n)a\| \leq \|\tilde{\alpha}_{\infty, g_i}(x) - x\| + \varepsilon$$

for all $i = 1, \dots, N$. It follows for all $g \in K$ that

$$\begin{aligned} & \sup_{n \geq \ell_0} \|(\alpha_g(x_n) - x_n)a\| \\ & \leq \min_{1 \leq i \leq N} \sup_{n \geq \ell_0} \|(\alpha_g(x_n) - \alpha_{g_i}(x_n))a\| + \|(\alpha_{g_i}(x_n) - x_n)a\| \\ (e4.4) \quad & \leq \varepsilon + \min_{1 \leq i \leq N} \sup_{n \geq \ell_0} \|(\alpha_g(x_n) - \alpha_{g_i}(x_n))a\| + \|\tilde{\alpha}_{\infty, g_i}(x) - x\| \\ & \leq \varepsilon + \min_{1 \leq i \leq N} \sup_{n \geq \ell_0} \|(\alpha_g(x_n) - \alpha_{g_i}(x_n))a\| + \|\tilde{\alpha}_{\infty, g_i}(x) - \tilde{\alpha}_{\infty, g}(x)\| \\ & \quad + \|\tilde{\alpha}_{\infty, g}(x) - x\| \\ (e4.2), (e4.3) \quad & \leq 3\varepsilon + \|\tilde{\alpha}_{\infty, g}(x) - x\|. \end{aligned}$$

This finishes the proof. \square

Lemma 4.5. *Let G be a second-countable, locally compact group. Let A be a separable C*-algebra and B a separable, unital C*-algebra. Suppose that $(\alpha, u) : G \curvearrowright A$ is a cocycle action, and that $\beta, \gamma : G \curvearrowright B$ are two actions that are strongly cocycle conjugate. Suppose that there exists a unital and equivariant *-homomorphism from (B, β) to $(F_{\infty, \alpha}(A), \tilde{\alpha}_\infty)$. Then there exists a unital and equivariant *-homomorphism from (B, γ) to $(F_{\infty, \alpha}(A), \tilde{\alpha}_\infty)$.*

Proof. Let $\kappa : (B, \beta) \rightarrow (F_{\infty, \alpha}(A), \tilde{\alpha}_\infty)$ be a unital and equivariant *-homomorphism. Consider a lift $(\kappa_n)_n : B \rightarrow \ell^\infty(\mathbb{N}, A)$, where each κ_n is a *-linear map. Then this sequence satisfies:

- $\limsup_{n \rightarrow \infty} \|\kappa_n(b)a\| \leq \|b\|,$
- $\limsup_{n \rightarrow \infty} \|(\kappa_n(b_1)\kappa_n(b_2) - \kappa_n(b_1b_2))a\| = 0,$
- $\limsup_{n \rightarrow \infty} \|\kappa_n(\mathbf{1}_B)a - a\| = 0,$
- $\limsup_{n \rightarrow \infty} \|[\kappa_n(b), a]\| = 0,$
- $\limsup_{n \rightarrow \infty} \|((\alpha_g \circ \kappa_n)(b) - (\kappa_n \circ \beta_g)(b))a\| = 0,$
- $\limsup_{n \rightarrow \infty} \max_{h \in K} \left(\|\beta_h(b) - b\| - \|((\alpha_h \circ \kappa_n)(b) - (\kappa_n)(b))a\| \right) \geq 0$

for all $b, b_1, b_2 \in B$, $0 \leq a \leq \mathbf{1}$ in A , $g \in G$ and all compact sets $K \subset G$. Note that the last condition follows from Lemma 4.4.

Apply Lemma 4.2 and find a sequence of automorphisms $\varphi_n : B \rightarrow B$ with

$$\max_{g \in K} \|\beta_g \circ \varphi_n - \varphi_n \circ \gamma_g\| \rightarrow 0$$

for all compact $K \subset G$. Let $1_G \in M_n \subset G$ be an increasing sequence of finite sets whose union G_d is a countable, dense subgroup of G . Let $0 \in F_n \subset B$ be a sequence of finite subsets whose union is a countable, dense $\mathbb{Q}[i]$ -*-subalgebra B' of B , which is invariant under γ_g for every $g \in G_d$. Moreover, let $E_n \subset A_{1,+}$ be an increasing sequence of finite sets whose union is dense. Let $K_0 \subset G$ be a compact neighbourhood of 1_G .

For every $n \in \mathbb{N}$, we find a number $k_n \in \mathbb{N}$ satisfying:

- $\|\kappa_{k_n}(b)a\| \leq \|b\| + 1/n$,
- $\|(\kappa_{k_n}(b_1)\kappa_{k_n}(b_2) - \kappa_{k_n}(b_1b_2))a\| \leq 1/n$,
- $\|\kappa_{k_n}(\mathbf{1}_B)a - a\| \leq 1/n$,
- $\|[\kappa_{k_n}(b), a]\| \leq 1/n$,
- $\|((\alpha_g \circ \kappa_{k_n})(b) - (\kappa_{k_n} \circ \beta_g)(b))a\| \leq 1/n$,
- $\|((\alpha_h \circ \kappa_{k_n})(b) - \kappa_{k_n}(b))a\| \leq \|\beta_h(b) - b\| + 1/n$

for all $b_1, b_2 \in \varphi_n(F_n)$, $a \in E_n$, $h \in K_0$, $g \in M_n$ and

$$b \in \{\beta_g(\varphi_n(x)) - \varphi_n(\gamma_g(y)) \mid x, y \in F_n \text{ and } g \in M_n\}.$$

Now the last condition, together with the choice of φ_n , implies that for sufficiently large n , we have

$$\begin{aligned} & \|((\alpha_h \circ \kappa_{k_n} \circ \varphi_n)(b) - (\kappa_{k_n} \circ \varphi_n)(b))a\| \\ & \leq 1/n + \|(\beta_h \circ \varphi_n)(b) - (\varphi_n)(b)\| \\ & \leq 2/n + \|\gamma_h(b) - b\| \end{aligned}$$

for all $b \in F_n$, $a \in E_n$ and $h \in K_0$. Moreover, the first and fifth condition combined with the choice of the φ_n also implies

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \|((\alpha_g \circ \kappa_{k_n} \circ \varphi_n)(b) - (\kappa_{k_n} \circ \varphi_n \circ \gamma_g)(b))a\| \\ & \leq \limsup_{n \rightarrow \infty} \left(\|((\alpha_g \circ \kappa_{k_n} \circ \varphi_n)(b) - (\kappa_{k_n} \circ \beta_g \circ \varphi_n)(b))a\| \right. \\ & \quad \left. + \|((\kappa_{k_n} \circ \beta_g \circ \varphi_n)(b) - (\kappa_{k_n} \circ \varphi_n \circ \gamma_g)(b))a\| \right) \\ & \leq 0 + \limsup_{n \rightarrow \infty} \|\kappa_{k_n}((\beta_g \circ \varphi_n)(b) - (\varphi_n \circ \gamma_g)(b))a\| = 0 \end{aligned}$$

for every $g \in M_n$ and $b \in F_n$.

Combining all these observations, we see that the chosen sequence $k_n \in \mathbb{N}$ satisfies:

- $\limsup_{n \rightarrow \infty} \|(\kappa_{k_n} \circ \varphi_n)(b)a\| \leq \|b\|$,
- $\|((\kappa_{k_n} \circ \varphi_n)(b_1)(\kappa_{k_n} \circ \varphi_n)(b_2) - (\kappa_{k_n} \circ \varphi_n)(b_1b_2))a\| \rightarrow 0$,
- $\|((\kappa_{k_n} \circ \varphi_n)(\mathbf{1}_B)a - a)\| \rightarrow 0$,
- $\|[(\kappa_{k_n} \circ \varphi_n)(b), a]\| \rightarrow 0$,
- $\|((\alpha_g \circ \kappa_{k_n} \circ \varphi_n)(b) - (\kappa_{k_n} \circ \varphi_n \circ \gamma_g)(b))a\| \rightarrow 0$,
- $\limsup_{n \rightarrow \infty} \max_{h \in K_0} (\|\gamma_h(b) - b\| - \|((\alpha_h \circ \kappa_{k_n} \circ \varphi_n)(b) - (\kappa_{k_n} \circ \varphi_n)(b))a\|) \geq 0$

for all $b, b_1, b_2 \in B'$, $a \in \bigcup_n E_n$ and $g \in G_d$.

Thus we get a well-defined, unital and continuous *-homomorphism

$$\psi : B' \rightarrow F_\infty(A) \quad \text{via} \quad \psi(b) = [(\kappa_{k_n} \circ \varphi_n)(b)]_n.$$

The last condition above implies that

$$\|\tilde{\alpha}_{\infty, h}(\psi(b)) - \psi(b)\| \leq \|\gamma_h(b) - b\|$$

for all $b \in B'$ and $h \in K_0$. As K_0 is a neighbourhood of 1_G , it follows that the image of ψ is contained in the continuous part $F_{\infty,\alpha}(A)$.

Moreover, the fifth condition implies that we have $\alpha_{\infty,g} \circ \psi = \psi \circ \gamma_g$ for all $g \in G_d$. By continuity, ψ uniquely extends to a unital $*$ -homomorphism $\psi : B \rightarrow F_{\infty,\alpha}(A)$, and the equation $\alpha_{\infty,g} \circ \psi = \psi \circ \gamma_g$ must now hold for every $g \in G$ because of continuity and the fact that $G_d \subset G$ is dense. This finishes the proof. \square

With these intermediate technical results, we can now get a more intrinsic characterization of semi-strongly self-absorbing C^* -dynamical systems:

Theorem 4.6. *Let G be a second-countable, locally compact group and \mathcal{D} a separable, unital C^* -algebra. For an action $\beta : G \curvearrowright \mathcal{D}$, the following are equivalent:*

- (i) β is semi-strongly self-absorbing.
- (ii) β has approximately G -inner half-flip and there exists a unital, equivariant $*$ -homomorphism from (\mathcal{D}, β) to $(\mathcal{D}_{\infty,\beta} \cap \mathcal{D}', \beta_{\infty})$.
- (iii) β has approximately G -inner half-flip and is strongly cocycle conjugate to $(\bigotimes_{\mathbb{N}} \mathcal{D}, \bigotimes_{\mathbb{N}} \beta)$.

(i) \implies (ii). Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing action that is strongly cocycle conjugate to β . By Proposition 3.4(ii) and Lemma 4.3, it follows that β has approximately G -inner half-flip. By Proposition 3.4(i) and Theorem 3.7, there exists a unital and equivariant $*$ -homomorphism from (\mathcal{D}, γ) to $(\mathcal{D}_{\infty,\beta} \cap \mathcal{D}', \beta_{\infty})$. By Lemma 4.5, there also exists a unital and equivariant $*$ -homomorphism from (\mathcal{D}, β) to $(\mathcal{D}_{\infty,\beta} \cap \mathcal{D}', \beta_{\infty})$.

(ii) \implies (iii): Applying Proposition 3.3, we see that the infinite tensor power $(\bigotimes_{\mathbb{N}} \mathcal{D}, \bigotimes_{\mathbb{N}} \beta)$ is strongly self-absorbing. Since there exists a unital and equivariant $*$ -homomorphism from (\mathcal{D}, β) to $(\mathcal{D}_{\infty,\beta} \cap \mathcal{D}', \beta_{\infty})$, one can construct a unital and equivariant $*$ -homomorphism $(\bigotimes_{\mathbb{N}} \mathcal{D}, \bigotimes_{\mathbb{N}} \beta)$ to $(\mathcal{D}_{\infty,\beta} \cap \mathcal{D}', \beta_{\infty})$, by applying a standard reindexation argument (see also [26, 1.13]) and using Lemma 4.4 in the process. By Theorem 3.7, it follows that

$$(\mathcal{D}, \beta) \simeq_{\text{scc}} (\mathcal{D} \otimes \bigotimes_{\mathbb{N}} \mathcal{D}, \beta \otimes \bigotimes_{\mathbb{N}} \beta) = (\bigotimes_{\mathbb{N}} \mathcal{D}, \bigotimes_{\mathbb{N}} \beta).$$

(iii) \implies (i): This follows directly from Proposition 3.3. \square

We may in fact extend our equivariant McDuff-type main result Theorem 3.7 to cover semi-strongly self-absorbing actions:

Theorem 4.7. *Let G be a second-countable, locally compact group. Let A be a separable C^* -algebra and $(\alpha, u) : G \curvearrowright A$ a cocycle action. Let \mathcal{D} be a separable, unital C^* -algebra and $\beta : G \curvearrowright \mathcal{D}$ an action such that (\mathcal{D}, β) is semi-strongly self-absorbing. The following are equivalent:*

- (i) (A, α, u) is strongly cocycle conjugate to $(A \otimes \mathcal{D}, \alpha \otimes \beta, u \otimes \mathbf{1}_{\mathcal{D}})$.
- (ii) (A, α, u) is cocycle conjugate to $(A \otimes \mathcal{D}, \alpha \otimes \beta, u \otimes \mathbf{1}_{\mathcal{D}})$.
- (iii) There exists a unital, equivariant $*$ -homomorphism from (\mathcal{D}, β) to $(F_{\infty,\alpha}(A), \tilde{\alpha}_{\infty})$.

Proof. This follows directly from Theorem 3.7 and Lemma 4.5. \square

We conclude this section by considering the special case where the acting group is compact.

Proposition 4.8. *Let G be a second-countable, compact group. Let A be a separable, unital C^* -algebra and $\alpha, \beta : G \curvearrowright A$ two actions. Then α and β are strongly cocycle conjugate if and only if they are conjugate.*

Proof. Let $\varphi : A \rightarrow B$ be an isomorphism and $x_n \in \mathcal{U}(A)$ a sequence of unitaries such that $w_g = \lim_{n \rightarrow \infty} x_n \beta_g(x_n^*)$ converges uniformly and $\text{Ad}(w_g) \circ \beta_g \circ \varphi = \varphi \circ \alpha_g$ for all $g \in G$. For some large n , we may replace φ by $\text{Ad}(x_n^*) \circ \varphi$ and w_g by $x_n^* w_g \beta_g(x_n)$ (see the proof of Lemma 4.2), and assume

$$\text{Ad}(w_g) \circ \beta_g \circ \varphi = \varphi \circ \alpha_g \quad \text{for all } g \in G$$

and $\max_{g \in G} \|1 - w_g\| < 1$. Applying [15, 2.4], we deduce that $[g \mapsto w_g]$ is a coboundary, i.e., there exists $v \in \mathcal{U}(A)$ with $w_g = v \beta_g(v^*)$ for all $g \in G$. This implies

$$\beta_g \circ \text{Ad}(v^*) \circ \varphi = \text{Ad}(v^*) \circ \varphi \circ \alpha_g \quad \text{for all } g \in G.$$

So $\text{Ad}(v^*) \circ \varphi : (A, \alpha) \rightarrow (A, \beta)$ is an equivariant isomorphism. \square

Corollary 4.9. *Let G be a second-countable, compact group. Let \mathcal{D} be a separable, unital C^* -algebra and $\gamma : G \curvearrowright \mathcal{D}$ an action. Then γ is strongly self-absorbing if and only if γ is semi-strongly self-absorbing.*

Corollary 4.10. *If one restricts to compact G and genuine actions, then all instances of “strongly cocycle conjugate” in Theorem 2.6 can be replaced by “conjugate”.*

Question 4.11. Does Proposition 4.8 hold also in the case that A is non-unital? Does Proposition 4.8 hold, if $(\alpha, u), (\beta, w) : G \curvearrowright A$ are cocycle actions with non-trivial cocycles?

5. EXAMPLES

So far, we have only discussed a very straightforward class of examples of strongly self-absorbing actions, namely the trivial actions on strongly self-absorbing C^* -algebras. In this section, we shall discuss other examples of (semi-)strongly self-absorbing actions. Let us start by considering compact group actions on strongly self-absorbing C^* -algebras that have the Rokhlin property; see [14, 3.2] for the definition on unital C^* -algebras.

Example 5.1. Let G be a second-countable, compact group and \mathcal{D} a strongly self-absorbing C^* -algebra. Let $\gamma : G \curvearrowright \mathcal{D}$ be an action with the Rokhlin property. Then (\mathcal{D}, γ) is strongly self-absorbing.

Proof. Note that all unital $*$ -homomorphism of the form $\mathcal{D} \rightarrow \mathcal{D} \otimes A$ are mutually approximately unitarily equivalent by [50, 1.12]. In particular, this is the case for $A = \mathcal{C}(G)$. By [2] (see [15, 3.5] for the finite group case), the systems (\mathcal{D}, γ) and $(\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$ are conjugate. Let $\varphi : (\mathcal{D}, \gamma) \rightarrow (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$ be an equivariant isomorphism. Then φ is approximately unitarily equivalent to $\text{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}$ as an ordinary $*$ -homomorphism. Once again because of the Rokhlin property and [2] (see [11, Proposition 3.1] for the finite group case), it follows that these two equivariant $*$ -homomorphisms are indeed approximately G -unitarily equivalent. \square

For the next example, recall Kishimoto’s notion of the Rokhlin property for flows [29].

Example 5.2 (cf. [33]). A quasi-free Rokhlin flow on \mathcal{O}_2 is semi-strongly self-absorbing. In particular, if $s_1, s_2 \in \mathcal{O}_2$ are the canonical generators and $\lambda > 0$ is an irrational number, then the flow given by $\gamma_t(s_1) = e^{it}s_1$ and $\gamma_t(s_2) = e^{-i\lambda t}s_2$ is semi-strongly self-absorbing.

Proof. Let $\gamma : \mathbb{R} \curvearrowright \mathcal{O}_2$ be a quasi-free flow with the Rokhlin property. By [33, 3.5], γ has approximately G -inner flip. By [33, 3.6], there exists a unital and equivariant *-homomorphism from (\mathcal{O}_2, γ) to $((\mathcal{O}_2)_\infty \cap \mathcal{O}'_2, \gamma_\infty)$. By 4.6, it follows that γ is semi-strongly self-absorbing. \square

Let us now consider examples of discrete group actions that come from non-commutative Bernoulli shifts.

Example 5.3. Let G be a finite group. Let $n \in \mathbb{N}$ be a number and $\sigma : G \curvearrowright \{1, \dots, n\}$ an action. Let \mathfrak{p} be a supernatural number of infinite type. Then the induced tensorial shift action $\beta^\sigma : G \curvearrowright M_{\mathfrak{p}}^{\otimes n}$ given by

$$\beta_g^\sigma(a_1 \otimes \dots \otimes a_n) = a_{\sigma_g(1)} \otimes \dots \otimes a_{\sigma_g(n)}$$

is strongly self-absorbing.

Proof. We first note that since \mathfrak{p} is of infinite type, the UHF algebra $M_{\mathfrak{p}}$ is strongly self-absorbing. So choosing an isomorphism $M_{\mathfrak{p}} \cong M_{\mathfrak{p}}^{\otimes \infty}$, a straightforward rearrangement of the tensorial factors yields that $(M_{\mathfrak{p}}^{\otimes n}, \beta^\sigma) \cong \bigotimes_{\mathbb{N}} (M_{\mathfrak{p}}^{\otimes n}, \beta^\sigma)$. By Proposition 3.3, it suffices to show that β^σ has approximately G -inner flip. Now by definition, $(M_{\mathfrak{p}}^{\otimes n}, \beta^\sigma)$ is an equivariant inductive limit of C*-dynamical systems of the form $(M_p^{\otimes n}, \beta^\sigma)$ for some natural numbers $p \in \mathbb{N}$. So without loss of generality, it suffices to show that the tensorial shift action induced by σ on $M_p^{\otimes n}$ has G -inner flip.

If $\{e_{i,j}\}_{1 \leq i,j \leq p} \subset M_p$ are the standard matrix units, then

$$\{e_{\mathbf{i}, \mathbf{j}} = e_{i_1, j_1} \otimes \dots \otimes e_{i_n, j_n} \mid \mathbf{i} = (i_1, \dots, i_n), \mathbf{j} = (j_1, \dots, j_n) \in \{1, \dots, p\}^n\}$$

defines another set of matrix units for $M_p^{\otimes n}$. Consider the unitary in $M_p^{\otimes n} \otimes M_p^{\otimes n}$ given by

$$v = \sum_{\mathbf{i}, \mathbf{j} \in \{1, \dots, p\}^n} e_{\mathbf{i}, \mathbf{j}} \otimes e_{\mathbf{j}, \mathbf{i}}$$

Then v implements the flip automorphism. It is clear that β^σ restricts to a G -action on the matrix units $e_{\mathbf{i}, \mathbf{j}}$, just by acting on the index set. In particular, we see that v is fixed by the diagonal action $\beta^\sigma \otimes \beta^\sigma$ on $M_p^{\otimes n} \otimes M_p^{\otimes n}$. So indeed, $(M_p^{\otimes n}, \beta^\sigma)$ has G -inner flip. This finishes the proof. \square

Question 5.4. Is the action from the above example still strongly self-absorbing, if one replaces $M_{\mathfrak{p}}$ by some other strongly self-absorbing C*-algebra \mathcal{D} ? In particular, what about $\mathcal{D} = \mathcal{Z}$ or $\mathcal{D} = \mathcal{O}_\infty$?

The case $\mathcal{D} = \mathcal{O}_2$ indeed has an affirmative answer because a faithful, tensorial shift action of a finite group on \mathcal{O}_2 has the Rokhlin property by [15, 5.4], thus falling within the scope of Example 5.1.

More generally, let us ask:

Question 5.5. Let G be a countable, discrete group and $\sigma : G \curvearrowright \mathbb{N}$ an action. Let \mathcal{D} be a strongly self-absorbing C^* -algebra. Consider the induced tensorial shift action $\beta^\sigma : G \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{D} \cong \mathcal{D}$ given by

$$\beta_g^\sigma(a_1 \otimes a_2 \otimes a_3 \otimes \dots) = a_{\sigma_g(1)} \otimes a_{\sigma_g(2)} \otimes a_{\sigma_g(3)} \otimes \dots,$$

where all but finitely many of the $a_n \in \mathcal{D}$ are equal to $\mathbf{1}$. When is $(\mathcal{D}, \beta^\sigma)$ strongly self-absorbing?

At least a straightforward rearrangement of the tensorial factors can be done again in this case to deduce that one always has $(\bigotimes_{\mathbb{N}} \mathcal{D}, \beta^\sigma) = \bigotimes_{\mathbb{N}} (\bigotimes_{\mathbb{N}} \mathcal{D}, \beta^\sigma)$. So in view of Proposition 3.3, the question is in which cases β^σ has approximately G -inner half-flip.

The following trivial argument allows one to construct strongly self-absorbing actions of residually compact groups, with the help of examples of compact group actions.

Proposition 5.6. *Let G be a second-countable, locally compact group. Let $H \subset G$ be a closed, normal, cocompact subgroup. Let $\pi : G \rightarrow G/H$ be the quotient map. Let \mathcal{D} be a separable, unital C^* -algebra and $\gamma : G/H \curvearrowright \mathcal{D}$ a strongly self-absorbing G/H -action. Then $\gamma \circ \pi : G \curvearrowright \mathcal{D}$ is strongly self-absorbing.*

Remark 5.7. In particular, let $H_n \subset G$ be a sequence of closed, normal, cocompact subgroups with $\bigcap_{n \in \mathbb{N}} H_n = \{1_G\}$. Let $\pi_n : G \rightarrow G/H_n$ be the quotient maps, and let $\gamma^{(n)} : G/H_n \curvearrowright \mathcal{D}_n$ be a sequence of strongly self-absorbing actions on some separable, unital C^* -algebras. Then

$$\gamma = \bigotimes_{n \in \mathbb{N}} (\gamma^{(n)} \circ \pi_n) : G \curvearrowright \bigotimes_{n \in \mathbb{N}} \mathcal{D}_n$$

is a strongly self-absorbing action. If each $\gamma^{(n)}$ was faithful, then γ is faithful.

Corollary 5.8. *Let \mathfrak{p} be a supernatural number of infinite type. Then any countable, discrete, residually finite group admits a faithful, strongly self-absorbing action on $M_{\mathfrak{p}}$.*

Proof. Let G be a countable, discrete, residually finite group. Let $H_n \subset G$ be a sequence of normal subgroups with finite index and $\bigcap_{n \in \mathbb{N}} H_n = \{1_G\}$. By Example 5.3, the tensorial flip action of G/H_n on $M_{\mathfrak{p}}$, which is induced by the left translation action $G/H_n \curvearrowright G/H_n$, is faithful and strongly self-absorbing. By Remark 5.7, it follows that the infinite tensor product of the induced G -actions is a faithful, strongly self-absorbing action on $M_{\mathfrak{p}}$. \square

Theorem 5.9. *Let \mathfrak{p} be a supernatural number of infinite type. Every pointwise strongly outer action of \mathbb{Z}^d on $M_{\mathfrak{p}}$ is semi-strongly self-absorbing.*

Proof. By Corollary 5.8, there exists a faithful, strongly self-absorbing \mathbb{Z}^d -action γ on $M_{\mathfrak{p}}$. Because of 3.4(i), it is clearly pointwise strongly outer. By Matui’s uniqueness result [38, 5.4], every other pointwise strongly outer action of \mathbb{Z}^d on $M_{\mathfrak{p}}$ is strongly cocycle conjugate to γ , so the claim follows. \square

Example 5.10. Let G be a finite group. Then any quasi-free action of G on \mathcal{O}_{∞} is strongly self-absorbing.

Proof. Let $\alpha : G \curvearrowright \mathcal{O}_\infty$ be a quasi-free action. By [9, Section 2], every automorphism α_g is either trivial or outer. So by dividing out the kernel of α , if necessary, we may assume that α is pointwise outer. Now it follows from Goldstein-Izumi's absorption theorem [12, 5.1] that $(\mathcal{O}_\infty, \alpha) \cong \bigotimes_{\mathbb{N}}(\mathcal{O}_\infty, \alpha)$. On the other hand, [12, 4.1] implies that α has approximately G -inner half-flip. With Proposition 3.3, it follows that α is a strongly self-absorbing action. \square

With the argument from Remark 5.7, we again get more examples of strongly self-absorbing actions on \mathcal{O}_∞ :

Corollary 5.11. *Any countable, discrete, residually finite group admits a faithful, strongly self-absorbing action on \mathcal{O}_∞ .*

Theorem 5.12. *Let \mathcal{D} be a strongly self-absorbing Kirchberg algebra satisfying the UCT. (That is, let either $\mathcal{D} \cong \mathcal{O}_\infty$, $\mathcal{D} \cong \mathcal{O}_2$ or $\mathcal{D} \cong \mathcal{O}_\infty \otimes M_{\mathfrak{p}}$ for a supernatural number \mathfrak{p} of infinite type.) Then every pointwise outer \mathbb{Z}^d -action on \mathcal{D} is semi-strongly self-absorbing.*

Proof. By Corollary 5.11, there exists a faithful, strongly self-absorbing action $\gamma : \mathbb{Z}^d \curvearrowright \mathcal{D}$. This action must then be pointwise outer because of Proposition 3.4(i). By invoking either [36, 5.2] (for the case $\mathcal{D} \cong \mathcal{O}_2$) or [19, 6.18, 6.20] (for the other cases), we can deduce that every pointwise outer \mathbb{Z}^d -action on \mathcal{D} is strongly cocycle conjugate to γ . This finishes the proof. \square

Question 5.13. When is a quasi-free action of a countable, discrete group on \mathcal{O}_∞ semi-strongly self-absorbing?

Remark 5.14 (cf. [12, Section 6]). Let G be a countable, exact and discrete group. By Kirchberg and Phillips' \mathcal{O}_2 -embedding result, we can embed $C_r^*(G)$ into \mathcal{O}_2 unittally. In particular, there exists a faithful unitary representation $v : G \rightarrow \mathcal{U}(\mathcal{O}_2)$. Consider the induced infinite tensor product action $\delta : G \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{O}_2$ given by $\delta_g = \bigotimes_{\mathbb{N}} \text{Ad}(v(g))$ for all $g \in G$.

Choosing some non-zero (and necessarily non-unital) $*$ -homomorphism $\iota : \mathcal{O}_2 \rightarrow \mathcal{O}_\infty$, we also obtain a faithful unitary representation $u : G \rightarrow \mathcal{U}(\mathcal{O}_\infty)$ via $u(g) = \iota(v(g)) + \mathbf{1} - \iota(\mathbf{1})$ for all $g \in G$. We can then also consider the induced infinite tensor product action $\gamma : G \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{O}_\infty$ given by $\gamma_g = \bigotimes_{\mathbb{N}} \text{Ad}(u(g))$ for all $g \in G$.

Now assume that G is finite. In this case, it was shown by Goldstein and Izumi in the proof of [12, 6.2] that the action γ is conjugate to any faithful, quasi-free G -action on \mathcal{O}_∞ . In particular, it follows that γ is strongly self-absorbing. On the other hand, it is not hard to construct a unital $*$ -homomorphism from \mathcal{O}_2 into the fixed point algebra $((\mathcal{O}_2)_\infty \cap \mathcal{O}_2)^{\delta_\infty}$, so it follows from Theorem 2.6 and Proposition 4.8 that $(\mathcal{O}_2, \delta) \cong (\mathcal{O}_2 \otimes \mathcal{O}_2, \delta \otimes \text{id}_{\mathcal{O}_2})$. Since δ is pointwise outer, it follows from [15, 4.2] that δ has the Rokhlin property. (We note that this can also be seen more directly by using a unital and equivariant inclusion $(\mathcal{C}(G), G\text{-shift}) \hookrightarrow (\mathcal{O}_2, \text{Ad}(v))$.) By Example 5.1, it follows that δ is strongly self-absorbing. Moreover, the actions γ and δ turn out to fit into an equivariant Kirchberg-Phillips absorption result:

Theorem 5.15 (cf. [15, 4.3] and [12, 5.1]). *Let G be a finite group and A a separable, unital, nuclear, simple C*-algebra. Let $\alpha : G \curvearrowright A$ be an action. Then*

- (1) $(\mathcal{O}_2 \otimes A, \delta \otimes \alpha) \cong (\mathcal{O}_2, \delta)$.
- (2) $(A, \alpha) \cong (A \otimes \mathcal{O}_\infty, \alpha \otimes \gamma)$, if A is purely infinite and α is pointwise outer.

The following lists the main results of forthcoming work, which relies on the results and techniques of this paper. It is generalization of the above theorem to the case of countable, amenable groups.

Theorem 5.16 (see [49]). *Let G be countable, discrete, amenable group.*

- (1) *If A is a Kirchberg algebra and $(\alpha, u) : G \curvearrowright A$ is any cocycle action, then $(A, \alpha, u) \simeq_{\text{scc}} (A \otimes \mathcal{O}_\infty, \alpha \otimes \text{id}_{\mathcal{O}_\infty}, u \otimes \mathbf{1}_{\mathcal{O}_\infty})$.*
- (2) *The actions $\gamma : G \curvearrowright \mathcal{O}_\infty$ and $\delta : G \curvearrowright \mathcal{O}_2$ from Remark 5.14 are strongly self-absorbing.*
- (3) *If A is a Kirchberg algebra and $(\alpha, u) : G \curvearrowright A$ is a cocycle action such that α is pointwise outer, then $(A, \alpha, u) \simeq_{\text{scc}} (A \otimes \mathcal{O}_\infty, \alpha \otimes \gamma, u \otimes \mathbf{1}_{\mathcal{O}_\infty})$.*
- (4) *If A is a separable, unital, nuclear, simple C^* -algebra and $\alpha : G \curvearrowright A$ is an action, then $(\mathcal{O}_2, \delta) \simeq_{\text{scc}} (A \otimes \mathcal{O}_2, \alpha \otimes \delta)$. Moreover, if α is pointwise outer, then $(A \otimes \mathcal{O}_2, \alpha \otimes \text{id}_{\mathcal{O}_2}) \simeq_{\text{scc}} (\mathcal{O}_2, \delta)$.*

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NOTE ADDED IN PROOF

Previous preprint versions of this article contained a more general statement in Lemma 2.2 involving sequence algebras along arbitrary free filters ω on \mathbb{N} . However, the approach used in the proof has a gap for uncountably generated filters, so in particular for free ultrafilters, as was kindly pointed out by Martino Lupini to the author. For this reason, we have restricted our attention to ordinary sequence algebras and the Fréchet filter in this context.

There certainly exists a variant of our main results expressed in terms of ultrapowers, but with a stronger definition of the continuous (central) sequence algebras $A_{\omega, \alpha}$ and $F_{\omega, \alpha}(A)$ that we shall not pursue here; see [10].

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