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AN EXAMPLE OF PET. COMPUTATION OF THE HAUSDORFF DIMENSION OF THE APERIODIC SET

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Abstract. We introduce a family of piecewise isometries. This family is similar to the ones studied by Hooper and Schwartz. We prove that a renormalization scheme exists inside this family and compute the Hausdorff dimension of the discontinuity set. The methods use some cocycles and a continued fraction algorithm.

1. Introduction

1.1. **Background.** A piecewise isometry in $\mathbb{R}^n$ is defined in the following way: consider a finite set of hyperplanes; the complement X of their union has several connected components. The piecewise isometry is a map T from X to $\mathbb{R}^n$ locally defined on each connected set as an isometry of $\mathbb{R}^n$. Now consider the pre-images of the union of the hyperplanes by T: it is a set of zero Lebesgue measure. Thus almost every point of X has an orbit under T, and we will study this dynamical system (X_0, T) where $X \setminus X_0$ is of zero measure. This class of maps has been well studied in dimension one with the example of the interval exchange maps; see [8]: the map is bijective, equal to the identity outside a compact interval, and the isometries which locally define T are translations. Remark that the case with nonoriented interval exchange is more difficult (and called interval exchange with flips). The strict case of the dimension two began ten years ago with the paper [1]. Since then, different examples have been examined in order to exhibit different behaviors; see for example [4]. The first general result has been obtained by Buzzi, proving that every piecewise isometry has zero entropy; see [7]. An important class of piecewise isometries is the outer billiard. This map has known a lot of developments in recent years with the work of Schwartz: [18], [19], [20] and [21]. He describes the first example of a piecewise isometry of the plane with an unbounded orbit. In his recent papers he defines a new type of piecewise isometry, called Polytope Exchange Transformation (PET for short) and shows that these maps describe the compactification of the outer billiard outside a kite. Independently, Hooper has studied renormalization in some piecewise isometries (see [11]) and in [10] shows some pictures of a discontinuity set which seems very close to Schwartz's work. Finally let us mention the work of [13] which is also close to our example.

In the present paper we describe a family of piecewise isometries. We prove that a renormalization scheme exists inside this family and compute the Hausdorff dimension of the discontinuity set.

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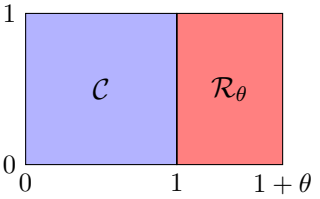
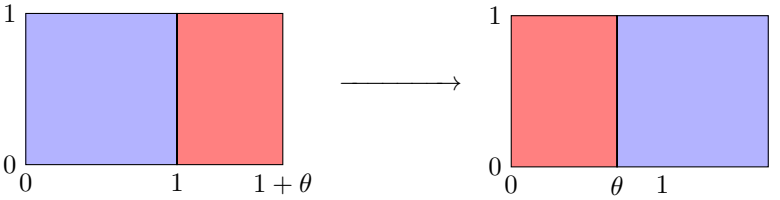
1.2. **An informal definition of our dynamical system.** Here we study a dynamical system closely related to a PET. We want to exchange one square and one rectangle. Let $\mathcal{C}$ be the square $[0,1]^2$, and let $\theta \in [0,1[$ be a real number. Consider the rectangle $\mathcal{R}_\theta = [1, 1+\theta] \times [0,1]$ and let $\mathcal{X}_\theta = \mathcal{C} \cup \mathcal{R}_\theta$. See the following figure:



Definition of $\mathcal{X}_\theta$.

We want to define a piecewise isometry which globally exchanges the square and the rectangle as described below:



For a fixed θ, there are exactly 24 transformations which isometrically exchange these two pieces, and some of them are conjugated via the orthogonal reflection through $y = 1/2$. In fact there are only two of them for which the dynamical behavior is interesting, and we will consider them in this paper. These two maps are parametrized by $\omega = (\theta, \varepsilon)$ with $\varepsilon = \pm 1$.

1.3. **Outline.** In Section 2 we give a precise definition of our dynamical system denoted $(\mathcal{X}_\omega, T_\omega)$. We define the coding of this map and the associated symbolic dynamical system. In Section 3, we introduce the renormalization and we study the map T_ω and show that an induction process exists: there exists a subset of $\mathcal{X}_\omega$ where the first return map of T_ω is conjugated to $T_{S(\omega)}$ for some map S. Then in Section 4 we study the map S. This map can be seen as a continued fraction algorithm. We study this continued fraction in Proposition 4.2 and compute an invariant measure. These results can be seen as an application of the theory developed by Arnoux-Schmidt in [3]. In Section 5, we describe the relation between the dynamics on the aperiodic set and a Sturmian subshift. The next step is to obtain a formula for the Hausdorff dimension of the set of aperiodic points in $(\mathcal{X}_\omega, T_\omega)$. The idea is to give a formula for the Hausdorff dimension in terms of a Lyapunov exponent of a cocycle. To obtain it, we need to use Oseledets' theorem. The problem is that the invariant measure of S does not have good properties. Thus we need to prove that an accelerated map is ergodic; see Section 4.4. Then in Section 6, we explain the relation between the Hausdorff dimension and a cocycle. Finally in Section 7 we deduce some approximations of the Hausdorff dimension. Some technical parts are left for the Appendix.

2. Definition of the dynamical system and first properties

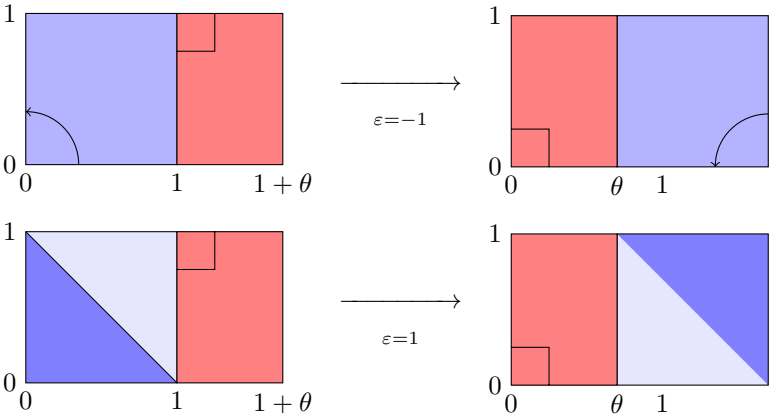
2.1. **Definition of the dynamical system.** Let us define $\omega = (\theta, \varepsilon) \in \Omega = [0,1[\times\{-1,1\}$ and denote the segments on the boundary of the square and the

rectangle by $\mathcal{D}_\theta = \partial\mathcal{C} \cup \partial\mathcal{R}_\theta$. Now let us define two maps f_ε from $[0,1]$ into itself by

$$\begin{array}{ccc} [0,1] & \to & [0,1] \\ x & \mapsto & f_\varepsilon(x) = \begin{cases} x & \text{if } \varepsilon = -1, \\ 1-x & \text{if } \varepsilon = 1. \end{cases} \end{array}$$

Consider the map T_ω defined by

$$\begin{array}{ccc} \mathring{\mathcal{C}} \cup \mathring{\mathcal{R}}_\theta & \to & \mathcal{C} \cup \mathcal{R}_\theta \\ z = (x,y) & \mapsto & \begin{cases} \big(1+\theta-y, f_\varepsilon(x)\big) & \text{if } (x,y) \in \mathring{\mathcal{C}}, \\ (x-1, 1-y) & \text{if } (x,y) \in \mathring{\mathcal{R}}_\theta. \end{cases} \end{array}$$



Lemma 2.1. *The map T_ω is the product of the following isometries:*

- *For $\varepsilon = -1$, the restriction of the map to the square is the product of the rotation of angle $\frac{\pi}{2}$ and center $(1/2, 1/2)$ and the translation by $(\theta, 0)$.*
- *For $\varepsilon = 1$, the restriction of the map to the square is the composition of the orthogonal reflection through $y + x = 1$ and the translation by $(\theta, 0)$.*
- *The restriction of the map to the rectangle does not depend on ε: it is the product of the translation by $(-1, 0)$ and the orthogonal reflection through $y = \frac{1}{2}$.*

The proof is left to the reader.

Definition 2.2. We define **the set of discontinuities** by $\mathcal{D}_\omega = \bigcup_{k\in\mathbb{N}} T_\omega^{-k}(\mathcal{D}_\theta)$. We also define $\mathcal{X}_\omega = \mathcal{X}_\theta \setminus \mathcal{D}_\omega$.

Lemma 2.3. *For every point z in $\mathcal{X}_\omega$ the orbit of z under T_ω is well defined. The set $\mathcal{X}_\omega$ is of Lebesgue measure* 1.

Proof. First of all, $\mathcal{D}_\omega$ is a countable union of segments; thus it is of zero Lebesgue measure. Now remark that the orbit of a point z under T_ω is not defined if and only if there exists an integer n such that $T_\omega^n(z) \in \mathcal{D}_\theta$. By definition $\mathcal{X}_\omega$ is the complement of this set. □

Remark 2.4. We can be more precise: let n be an integer; the set $T_\omega^{-n}\mathcal{D}_\theta$ is a finite union of horizontal or vertical segments. We deduce that $\mathcal{X}_\omega$ is the union of dense open sets.

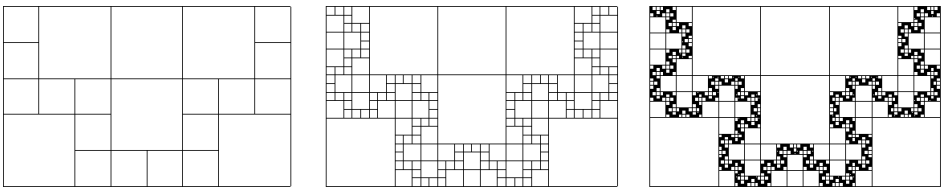


FIGURE 1. Representation of $\mathcal{D}_\omega$ for the following values for ω: $\left(\frac{3}{5},-1\right)$, $\left(\frac{13}{21},-1\right)$ and $\left(\frac{\sqrt{5}-1}{2}-1,-1\right)$.

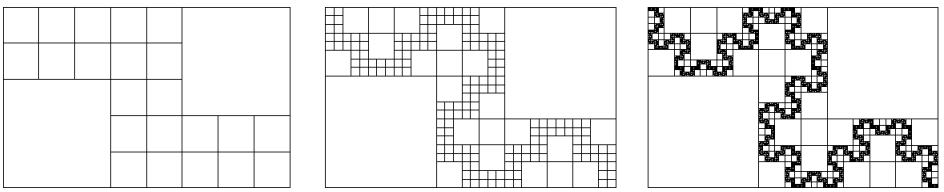


FIGURE 2. Representation of $\mathcal{D}_\omega$ of $\omega=\left(\frac{3}{5},1\right)$, $\omega=\left(\frac{13}{21},1\right)$ and for $\omega=\left(\frac{\sqrt{5}-1}{2}-1,1\right)$.

We introduce notation for the restriction of the defined sets to the square and the rectangle by:

- $\mathcal{D}_\omega^c=\mathcal{D}_\omega\cap\mathcal{C}$ and $\mathcal{D}_\omega^r=\mathcal{D}_\omega\cap\mathcal{R}_\theta$ (they are of zero Lebesgue measure);
- $\mathcal{C}_\omega=\mathcal{C}\cap\mathcal{X}_\omega$ and $\mathcal{R}_\omega=\mathcal{R}_\theta\cap\mathcal{X}_\omega$.

Proposition 2.5. *We have $T_\omega(\mathcal{X}_\omega)=\mathcal{X}_\omega$. Moreover the dynamical system $(\mathcal{X}_\omega,T_\omega)$ is a bijective piecewise isometry, with the Lebesgue measure as an invariant measure.*

Proof. Each isometry involved in the definition of T_ω is clearly a bijection. The following map from $T_\omega(\mathcal{X}_\omega)$ into $\mathcal{X}_\omega$ is thus well defined:

$$(x,y)\mapsto T_\omega^{-1}(x,y)=\begin{cases}\big(f_\varepsilon(y),1+\theta-x\big) & \text{if } (x,y)\in T_\omega\big(\mathcal{C}_\omega\big),\\ (x+1,1-y) & \text{if } (x,y)\in T_\omega\big(\mathcal{R}_\omega\big).\end{cases}$$

To prove the result we just have to analyze the discontinuity set of this map and prove that it coincides with $\mathcal{D}_\omega$. We define $\mathcal{D}_\theta^{-1}=\partial T_\omega(\mathcal{C})\cup\partial T_\omega(\mathcal{R}_\theta)$, $\mathcal{D}_\omega^{-1}=\bigcup_{n\in\mathbb{N}}T_\omega^{-n}(\mathcal{D}_\theta^{-1})$ and $\mathcal{F}_\omega$ as the union $\mathcal{D}_\theta\cup\mathcal{D}_\theta^{-1}$. It is clear that $z\in\mathcal{D}_\omega$ (resp. $z\in\mathcal{D}_\omega^{-1}$) if and only if there exists an integer n such that $T_\omega^n z\in\mathcal{D}_\theta$ (resp. $T_\omega^n z\in\mathcal{D}_\theta^{-1}$).

Remark that $\mathcal{F}_\omega$ is invariant by the map

$$\mathrm{Sym}_\omega:(x,y)\mapsto\big(1+\theta-x,f_\varepsilon(y)\big).$$

Since this map fulfills $T_\omega\circ Sym_\omega=Sym_\omega\circ T_\omega^{-1}$ we deduce that $T_\omega(\mathcal{X}_\omega)=\mathcal{X}_\omega$. The rest of the proof is easy since all the maps are isometries, and therefore the Lebesgue measure is an invariant measure. □

2.2. **Symbolic dynamics of a piecewise isometry.** We need to introduce some notions of symbolic dynamics; see [16]. Let $\mathcal{A}$ be a finite set called an alphabet, let **a word** be a finite string of elements in $\mathcal{A}$, and let its length be the number of elements in the string. The set of all finite words over $\mathcal{A}$ is denoted $\mathcal{A}^*$. A (one

sided) sequence of elements of $\mathcal{A}$, $u = (u_n)_{n\in\mathbb{N}}$, is called an infinite word. A word $v_0 \dots v_k$ appears in u if there exists an integer i such that $u_i \dots u_{i+k+1} = v_0 \dots v_k$. If $u = u_0 \dots u_n$ is a finite word, we denote by $\bar{u}$ the infinite word $u_0 \dots u_n u_0 \dots u_n \dots$. This word is **periodic** of period u. For an infinite word u, **the language** of u (respectively the language of length n) is the set of all words (respectively all words of length n) in $\mathcal{A}^*$ which appear in u. We denote it by $L(u)$ (respectively by $L_n(u)$). We endow the set of sequences $\mathcal{A}^{\mathbb{N}}$ with the product topology, then we define for a finite word v the **cylinder** $[v] = \{vu, u \in \mathcal{A}^{\mathbb{N}}\}$. The set of cylinders forms a basis of clopen sets for the topology.

A **substitution** σ is an application from an alphabet $\mathcal{A}$ to the set $\mathcal{A}^* \setminus \{\varepsilon\}$ of nonempty finite words on $\mathcal{A}$. It extends to a morphism of $\mathcal{A}^*$ by concatenation, that is, $\sigma(uv) = \sigma(u)\sigma(v)$.

Now we define an application $\phi : \mathbb{R}^2 \mapsto \{a; b\}$ by

$$\phi(x, y) = a \text{ if } x < 1 \quad \text{and} \quad \phi(x, y) = b \text{ otherwise.}$$

For $\omega = (\theta, \varepsilon) \in \Omega$ and $z \in \mathcal{X}_\omega$ we have

$$\phi(z) = a \Longleftrightarrow z \in \mathcal{C}_\omega \quad \text{and} \quad \phi(z) = b \Longleftrightarrow z \in \mathcal{R}_\omega.$$

Let $\Phi_\omega : \mathcal{X}_\omega \mapsto \{a; b\}^{\mathbb{N}}$ be **the coding map** defined by

$$\Phi_\omega(z) = (u_n)_{n\in\mathbb{N}} \quad \text{such that} \quad u_n = \phi\big(T_\omega^n(z)\big).$$

The image by the coding map of the points in $\mathcal{X}_\omega$ defines a language. For a finite (or infinite) word u in this language, a **cell** is the set of points which are coded by this word: $\mathcal{O}_u = \{z \in \mathcal{X}_\omega, \Phi_\omega(z) = u\}$. The cells $\mathcal{O}_{\bar{u}}$, where u is a finite word, are called **periodic cells**, and the period is defined as the period of the word $\bar{u}$.

Lemma 2.6. *If z is a periodic point of T_ω, then $\Phi_\omega(z)$ is a periodic word and the cell is a rectangle. The restriction of T_ω to a periodic cell is either a rotation of angle $-\frac{\pi}{2}, \pi$ or $\frac{\pi}{2}$ or an orthogonal reflexion. In all cases, every point of a periodic cell has a periodic orbit.*

The reader should take care not to confuse the period of a periodic cell and the period of the points. In Corollary 3.5 we will see that this result can be improved.

Proof. Let z be such that $T_\omega^n(z) = z$, and let $v = \Phi(z)$. By definition of z, the word v is periodic of period n. Let us denote by $u = u_0 \dots u_{n-1}$ its period; thus we have $v = u \dots u \dots$. We define $\mathcal{P}_a = \mathcal{C}_\omega$ and $\mathcal{P}_b = \mathcal{R}_\omega$. We deduce that $\mathcal{O}_{\bar{u}} = \bigcap_{0\le k\le n-1} T_\omega^{-k} P_{u_k}$. Remark that $T_\omega^{-k} P_{u_k}$ is a convex set for every integer k. Then $\mathcal{O}_{\bar{u}}$ is a decreasing intersection of convex sets; thus it is a convex set. Moreover $\mathcal{D}_\theta$ is the union of vertical and horizontal segments. Then its image by T_ω is a horizontal or vertical edge; thus every segment of $\mathcal{D}_\omega$ is horizontal or vertical: we conclude that $\mathcal{O}_{\bar{u}}$ is a rectangle. Moreover the restriction of T_ω to this rectangle has a periodic point. □

Definition 2.7. For $\omega \in \Omega$ and $n \in \mathbb{N}$, let us denote by $\mathcal{I}_\omega(n)$ the union of periodic cells of period less than n.

Consider the dynamical system $(\mathcal{X}_\omega, T_\omega)$. Denote the **periodic points** in this system by $\mathcal{I}_\omega$. A natural question seems to look at the complement of this subset. It is nonempty, but we do not know its size. Can we compute it? This question appears naturally in several papers such as [18]. Thus we define for each positive

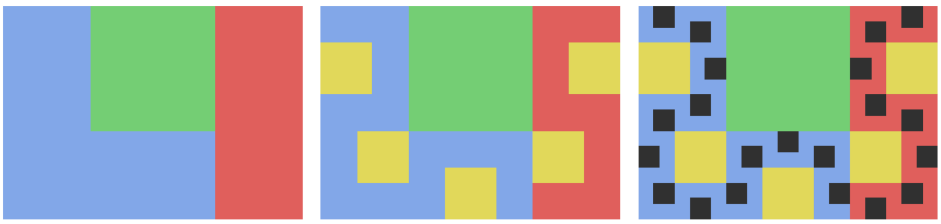


FIGURE 3. Example for $\omega = \left(\sqrt{2}-1,-1\right)$. The sets $\mathcal{K}_\omega^c(n)$ are included in the blue part, and $\mathcal{K}_\omega^r(n)$ is inside the red part. The periodic island of period one is in green, the islands of period five are in yellow, and in black are the periodic islands of period 21.

integer n:

$$\mathcal{K}_\omega(n) = \overline{\mathcal{X}_\omega \setminus \mathcal{I}_\omega(n)} \quad \text{and} \quad \mathcal{K}_\omega = \bigcap_{n\in\mathbb{N}} \mathcal{K}_\omega(n).$$

As above we also define

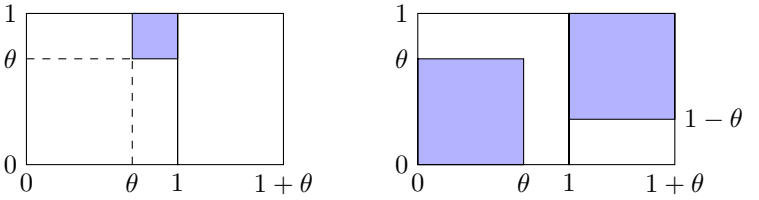
$$\mathcal{K}_\omega^r(n) = \mathcal{K}_\omega(n) \cap \mathcal{R}_\theta, \quad \mathcal{K}_\omega^c(n) = \mathcal{K}_\omega(n) \cap \mathcal{C},$$
$$\mathcal{K}_\omega^r = \bigcap_{n\in\mathbb{N}} \mathcal{K}_\omega^r(n) \quad \text{and} \quad \mathcal{K}_\omega^c = \bigcap_{n\in\mathbb{N}} \mathcal{K}_\omega(n)^c.$$

From Remark 2.4 we have $\mathcal{K}_\omega(n) = \mathcal{X}_\theta \setminus \mathring{\mathcal{I}}_\omega(n)$. The set $\mathcal{K}_\omega$ is called the **aperiodic set**.

Lemma 2.8.

- *For $\varepsilon = -1$, the set $\mathcal{I}_\omega(1)$ is a square: its Lebesgue measure is equal to $(1-\theta)^2$, and the coding associated to a point of $\mathcal{I}_\omega(1)$ is $\bar{a}$.*
- *For $\varepsilon = 1$, the set $\mathcal{I}_\omega(1)$ is empty, and $\mathcal{I}_\omega(2)$ is the union of two squares. Its area is equal to $2\theta^2$, and the coding associated to the points of $\mathcal{I}_\omega(2)$ is $\overline{ab}$ or $\overline{ba}$.*

Proof. The proof is left to the reader. See the following figure, which represents in blue $\mathcal{I}_\omega(1)$ for $\varepsilon = -1$ and $\mathcal{I}_\omega(2)$ for $\varepsilon = 1$.



□

Lemma 2.9. *Let $\omega \in \Omega$.*

(1) *The point z belongs to $\mathcal{K}_\omega$ if and only if z has a nonperiodic orbit under T_ω.*
(2) $\mathcal{K}_\omega \subset \bar{\mathcal{D}}_\omega$.

Proof. The first point can be deduced from Lemma 2.6.

Let $z \in \mathcal{K}_\omega \cap \mathcal{X}_\omega$. We will show that z belongs to $\mathcal{D}_\omega$. We argue by contradiction: suppose we have $\inf\{d(T^n z, \mathcal{D}_\theta) \mid n \in \mathbb{N}\} = \delta > 0$, and consider a square centered in

z of diameter $\delta/2$. The orbit of this square never intersects a segment of $\mathcal{D}_\omega$. Thus every point inside this square has the same coding, and this coding is nonperiodic by assumption. Thus the cell of a nonperiodic word has nonempty interior, which is impossible. □

To finish this section let us briefly sum up the notation used here:

$\mathcal{D}_\omega$	$\mathcal{I}_\omega$	$\mathcal{K}_\omega$
Discontinuities	Periodic points	Aperiodic points

The notation $\mathcal{D}_\omega(n)$ will be used for the same object restricted to the n first elements of an orbit.

2.3. **Results of the paper.** To start with, we prove the following:

- The map T_ω has a renormalization scheme (Section 3, Proposition 3.2).
- The dynamical system $(\mathcal{K}_\omega, T_\omega)$ is conjugated to a rotation of angle θ (Section 5, Proposition 5.3).

Moreover we compute the Hausdorff dimension of the aperiodic set and show that:

Theorem 2.10. *There exists a real number $\boldsymbol{s}$ such that for almost all ω, the Hausdorff dimension of $\mathcal{K}_\omega$ is equal to $\boldsymbol{s}$.*

Corollary 2.11.

- *We have* $1.07 \leq \boldsymbol{s} \leq 1.55$.
- *Moreover we obtain for* $n \in \mathbb{N}$ *:*

$$\dim_H(\mathcal{K}_\omega) = \begin{cases} -\dfrac{\ln(2n+\sqrt{4n^2+1})}{\ln\left(\sqrt{n^2+1}-n\right)} & \text{if } \omega = \left(\sqrt{n^2+1}-n, -1\right), \\ -\dfrac{\ln(2n+1+2\sqrt{n^2+n})}{\ln\left(n+1-\sqrt{n(n+2)}\right)} & \text{if } \omega = \left(\sqrt{n(n+2)}-n, 1\right). \end{cases}$$

This number $\boldsymbol{s}$ is obtained via the Lyapunov exponent of a cocycle introduced in Section 6. Remark that the computation of the Hausdorf dimension can be seen as a generalization of the classical case where the fractal set is the solution of an iterated function system.

3. Induction

3.1. **Notation.** Consider $\omega = (\theta, \varepsilon) \in]0,1[\times\{-1,1\}$. We denote $n_\omega = \left\lfloor \frac{1}{f_\varepsilon(\theta)} \right\rfloor$ where $\lfloor . \rfloor$ denotes the floor function. We define a map S by

$$\begin{array}{ccc}]0,1[\times\{-1,1\} & \to & \Omega \\ (\theta,\varepsilon) & \mapsto & S(\theta,\varepsilon) = \left(\frac{1}{f_\varepsilon(\theta)} - n_\omega, (-1)^{n_\omega+1}\right). \end{array} \tag{1}$$

Now let ψ_ω be a similitude defined by

$$\begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & \psi_{(\theta,-1)}(x,y) = \frac{1}{\theta}(y,x) \\ (x,y) & \mapsto & \psi_{(\theta,1)}(x,y) = \frac{1}{1-\theta}(x, y-\theta). \end{array}$$

Remark that this similitude has a ratio $r(\omega) = \frac{1}{f_\varepsilon(\theta)}$ and that the inverse of this map is given by

$$\psi^{-1}_{(\theta,-1)}(x,y) = \theta(y,x) \qquad \text{and} \qquad \psi^{-1}_{(\theta,1)}(x,y) = \big((1-\theta)x, \theta + y(1-\theta)\big).$$

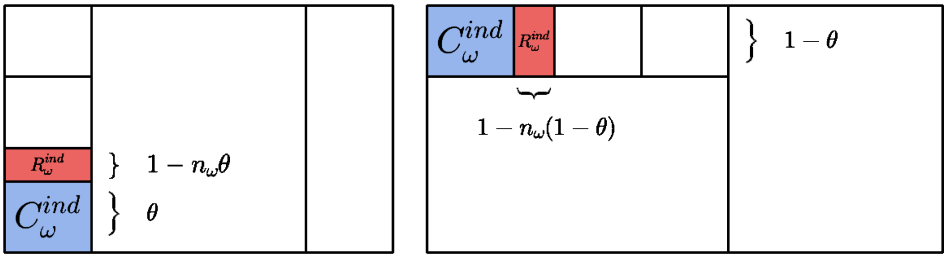


FIGURE 4. Induction zone for the parameters $(\theta,-1)$ and $(\theta,1)$.

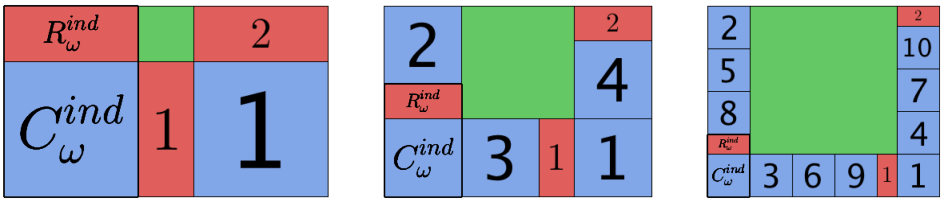


FIGURE 5. Induction scheme for $\varepsilon=-1$ with $\theta_n=\frac{1}{\sqrt{n(n+1)}}$ where $n=1,2,4$.

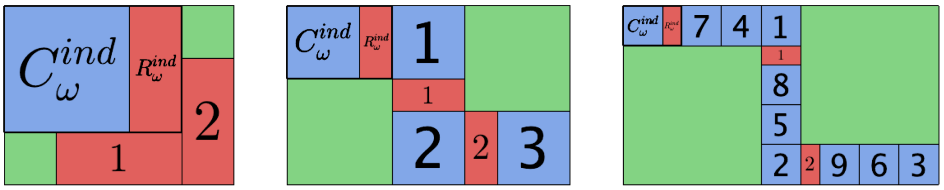


FIGURE 6. Induction scheme for $\varepsilon=1$ with $\theta_n=1-\frac{1}{\sqrt{n(n+1)}}$ where $n=1,2,4$.

The main objects of the paper are the following sets; the notation will be explained in Proposition 3.2.

Definition 3.1. For $\omega\in\Omega$ we define the **induction zones** by

$$\mathcal{C}_\omega^{\mathrm{ind}}=\psi_\omega^{-1}(\mathcal{C}_{S\omega}),\quad \mathcal{R}_\omega^{\mathrm{ind}}=\psi_\omega^{-1}(\mathcal{R}_{S\omega})\quad\text{and}\quad \mathcal{X}_\omega^{\mathrm{ind}}=\mathcal{C}_\omega^{\mathrm{ind}}\cup\mathcal{R}_\omega^{\mathrm{ind}}.$$

Remark that $\psi_\omega(\mathcal{X}_\omega^{\mathrm{ind}})=\mathcal{X}_{S\omega}$ by definition.

3.2. **First return map to the induction zone.** The first return time of $z\in\mathcal{X}_\omega^{\mathrm{ind}}=\mathcal{C}_\omega^{\mathrm{ind}}\cup\mathcal{R}_\omega^{\mathrm{ind}}$ is defined as $\mathsf{n}(z)=\min\{k\geq 1, T_\omega^k(z)\in\mathcal{X}_\omega^{\mathrm{ind}}\}$. Then we define T_ω^{ind} as the map of $\mathcal{X}_\omega^{\mathrm{ind}}$ into itself defined as $z\mapsto T^{\mathsf{n}(z)}(z)$. It is the first return map of T_ω^{ind} onto $\mathcal{X}_\omega^{\mathrm{ind}}$.

Proposition 3.2 (Induction)**.** *The maps* n *and* T_ω^{ind} *are well defined on* $\mathcal{X}_\omega^{ind}$*, and we have for all* $z\in\mathcal{X}_{S\omega}$*:*

$$\psi_\omega\circ T_\omega^{ind}\circ\psi_\omega^{-1}(z)=T_{S(\omega)}(z).$$

3.3. **Induction, proof of Proposition 3.2.**

• Let us start with $\varepsilon = -1$ for a parameter $\theta \in]0,1[$.

Let (x,y) be a point in the interior of $\mathcal{C}_\omega^{\mathrm{ind}}$. This means $0 < x < \theta$ and $0 < y < \theta$. Then we have

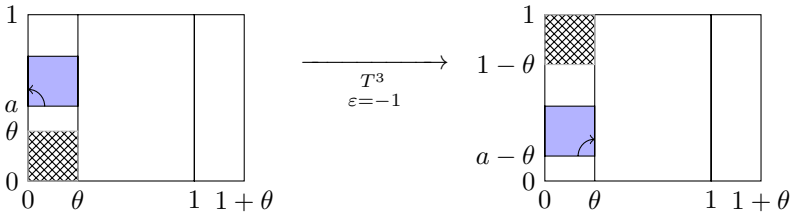
$$T^2(x,y) = T\big(1+\theta-y, x\big) = \big(\theta - y, 1-x\big).$$

Remark that we have $0 < \theta - y < \theta$ and $1-\theta < 1-x < 1$.

Now we compute $T^3(x,y)$ for some (x,y) such that $0 < x < \theta$ and $y > \theta$:

$$T^3(x,y) = T^2\big(1+\theta-y, x\big) = T\big(1+\theta-x, 1+\theta-y\big) = \big(\theta - x, y - \theta\big).$$

Thus we remark that a point inside $\{(x,y), y > \theta, x < \theta\}$ remains under the action of T^3 inside this set and its second coordinate decreases.



We deduce that

$$T_\omega^{\mathrm{ind}}(z) = \begin{cases} T^{3(n_\omega - 1)}T^2(z) & \text{if} \quad z \in \mathcal{C}_\omega^{\mathrm{ind}}, \\ T^3(z) & \text{if} \quad z \in \mathcal{R}_\omega^{\mathrm{ind}}. \end{cases} \tag{2}$$

Then, if we denote $s(x) = \theta - x$ we obtain

$$T_\omega^{\mathrm{ind}}(x,y) = \begin{cases} \big(s^{n_\omega - 1}(\theta - y), 1 - x - \theta(n_\omega - 1)\big) & \text{if} \quad z \in \mathcal{C}_\omega^{\mathrm{ind}}, \\ \big(s^{n_\omega}(y), 1 - x - \theta(n_\omega - 1)\big) & \text{if} \quad z \in \mathcal{R}_\omega^{\mathrm{ind}}. \end{cases} \tag{3}$$

To finish the proof we compute the conjugation of T_ω^{ind}:

$$\psi_\omega T_\omega^{\mathrm{ind}} \psi_\omega^{-1}(x,y) = \begin{cases} \big(1 - y + \frac{1}{\theta} - n_\omega, f_{(-1)^{n_\omega+1}}(x)\big), \\ (x-1, 1-y). \end{cases}$$

Finally we have proven

$$\psi_\omega T_\omega^{\mathrm{ind}} \psi_\omega^{-1}(x,y) = T_{S\omega}(x,y).$$

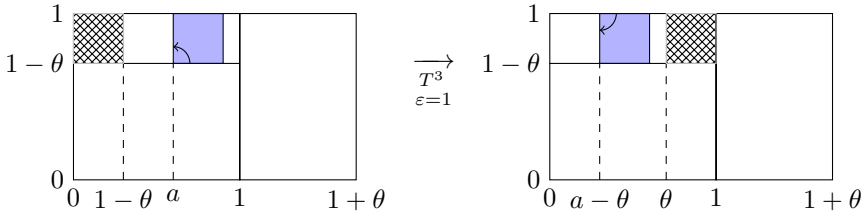
• Now we treat the case $\varepsilon = 1$. The computations are based on the same method; thus we just give the result:

$$T_\omega^{\mathrm{ind}}(z) = T^{3(n_\omega - 1)}T(z) \text{ if } z \in \mathcal{C}_\omega^{\mathrm{ind}} \quad \text{and} \quad T_\omega^{\mathrm{ind}}(z) = T^3(z) \text{ if } z \in \mathcal{R}_\omega^{\mathrm{ind}}.$$

Then, for $z = (x,y)$:

$$T_\omega^{\mathrm{ind}}(z) = \begin{cases} \big(1 + \theta - y - (1-\theta)(n_\omega - 1), s^{n_\omega - 1}(1-x)\big) & \text{if} \quad z \in \mathcal{C}_\omega^{\mathrm{ind}}, \\ \big(\theta + x - 1, 1 + \theta - y\big) & \text{if} \quad z \in \mathcal{R}_\omega^{\mathrm{ind}}. \end{cases} \tag{4}$$

Thus we obtain $T_{S\omega}(x,y) = \psi_\omega T_\omega^{\mathrm{ind}} \psi_\omega^{-1}(x,y)$.



3.4. **Some important corollaries of Proposition 3.2.** Let us define two substitutions

$$\sigma_{(\theta,-1)} : \begin{cases} a \to ab(aab)^{n_\omega - 1}, \\ b \to aab, \end{cases} \quad \text{and} \quad \sigma_{(\theta,1)} : \begin{cases} a \to a(aab)^{n_\omega - 1}, \\ b \to aab. \end{cases} \tag{5}$$

Corollary 3.3 (Language and substitution)**.** *For $z \in \mathcal{X}_\omega^{ind} = \mathcal{C}_\omega^{ind} \cup \mathcal{R}_\omega^{ind}$, we have*

$$\Phi_\omega(z) = \sigma_\omega \circ \Phi_{S(\omega)} \circ \psi_\omega(z).$$

Proof. By Proposition 3.2, we have

$$\psi_\omega \circ T_\omega^{\mathrm{ind}}(z) = T_{S(\omega)} \circ \psi_\omega(z).$$

If $z \in \mathcal{C}_\omega^{ind}$ we obtain by equation (2)

$$\psi_\omega \circ T^{3(n_\omega - 1)} T^2(z) = T_{S(\omega)} \circ \psi_\omega(z).$$

The infinite word $\Phi_{S(\omega)} \circ \psi_\omega(z)$ begins with a, and the proof of Proposition 3.2 shows that $\Phi_\omega(z)$ begins with $ab(ab)^{n_\omega - 1}$. Since the same argument works if z belongs to $\mathcal{R}_\omega^{ind}$ we deduce the result. □

These substitutions define linear maps by abelianization. These linear maps have matrices in $\mathcal{M}_2(\mathbb{Z})$, called the incidence matrices and denoted by

$$M(\theta,-1) = \begin{pmatrix} 2n_\omega - 1 & 2 \\ n_\omega & 1 \end{pmatrix} \quad \text{and} \quad M(\theta,1) = \begin{pmatrix} 2n_\omega - 1 & 2 \\ n_\omega - 1 & 1 \end{pmatrix}.$$

In Section B.3 we will return to the cocycle generated by the map $\omega \to M(\omega)$.

We denote these matrices and their coefficients as $M(\omega) = (m_{i,j}(\omega))_{1 \le i,j \le 2}$. The vector space $\mathbb{R}^2$ is equipped with norm $\|\cdot\|_1$. This defines a norm on $\mathcal{M}_2(\mathbb{R})$ by $\left\| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \right\|_1 = \max\{|a| + |c|, |b| + |d|\}$.

Definition 3.4. Let θ be an irrational number and $\omega = (\theta, \varepsilon)$. For each positive integer i we denote $\omega_i = (\theta_i, \varepsilon_i) = S^i(\theta, \varepsilon)$ (we will see in Proposition 4.2 that $S^i(\omega)$ exists for each integer i). Let us define also

$$M^{(k)}(\omega) = M(\omega) \times \cdots \times M(\omega_k) = \begin{pmatrix} m_{1,1}^{(k)}(\omega) & m_{1,2}^{(k)}(\omega) \\ m_{2,1}^{(k)}(\omega) & m_{2,2}^{(k)}(\omega) \end{pmatrix}.$$

Then we define a sequence of numbers $(p_i)_{i \in \mathbb{N}}$ by

$$p_i = \left\| M^{(i-1)}(\omega) \times k_i \right\|_1, \quad \text{where } k_i = \begin{pmatrix} 1 \\ \frac{1}{2}(1 + \varepsilon_i) \end{pmatrix}. \tag{6}$$

Corollary 3.5 (Periodic points)**.** *We deduce the following:*

(1) *Let $\mathcal{I}$ be a periodic cell for $T_{S(\omega)}$ associated to the periodic word u. Then $\psi_\omega^{-1}(\mathcal{I})$ is a periodic cell for T_ω associated to the word $\sigma_\omega(u)$. Moreover its period is given by*

$$\left\| M(\omega) \times \begin{pmatrix} |u|_a \\ |u|_b \end{pmatrix} \right\|_1 \tag{7}$$

where by definition u contains $|u|_a$ letters a and $|u|_b$ letters b.

(2) *Conversely if $\mathcal{I}$ is a periodic cell of period n for T_ω ($n \geq 2$ for $T_{(\theta,-1)}$ and $n \geq 3$ for $T_{(\theta,1)}$), then there exists an integer $0 \leq k \leq n-1$ such that $\psi_\omega\big(T_\omega^k(\mathcal{I})\big)$ is a periodic cell for $T_{S(\omega)}$.*

(3) *If $\mathcal{I}$ is a periodic cell, there exists an integer n such that its period is p_n, and its area is equal to*

$$\prod_{k=1}^{n} \frac{1}{r(\omega_k)}.$$

(4) *Each periodic cell is a square.*

(5) *The set $\mathcal{K}_{(\theta,\varepsilon)}$ is nonempty if and only if $\theta \notin \mathbb{Q}$.*

Proof.

(1) We use the relation $\psi_\omega \circ T_\omega^{\mathrm{ind}} \circ \psi_\omega^{-1} = T_{S(\omega)}$. If m is a periodic point for $T_{S\omega}$ of period k, we obtain that $\psi_\omega^{-1}(m)$ is periodic for T_ω^{ind}. Now equation (2) gives that T_ω^{ind} is some power of T_ω. The periodic word associated to this cell is obtained with the first point of Corollary 3.3.

(2) Consider a periodic cell $\mathcal{I}$ of T_ω of period n. We claim that there exists $k \leq n-1$ such that $T_\omega^k \mathcal{I} \in \mathcal{C}_\omega^{\mathrm{ind}} \cup \mathcal{R}_\omega^{\mathrm{ind}}$. This fact is proven by remarking that

$$\text{for} \quad \varepsilon = -1 \ : \quad \big(\mathcal{C}_\omega^{\mathrm{ind}} \cup T_\omega \mathcal{C}_\omega^{\mathrm{ind}} \cup \cdots \cup T_\omega^{3n_\omega - 2} \mathcal{C}_\omega^{\mathrm{ind}}\big) \cup \big(\mathcal{R}_\omega^{\mathrm{ind}} \cup T_\omega \mathcal{R}_\omega^{\mathrm{ind}} \cup T_\omega^2 \mathcal{R}_\omega^{\mathrm{ind}}\big),$$

$$\text{for} \quad \varepsilon = 1 \ : \quad \big(\mathcal{C}_\omega^{\mathrm{ind}} \cup T_\omega \mathcal{C}_\omega^{\mathrm{ind}} \cup \cdots \cup T_\omega^{3n_\omega - 3} \mathcal{C}_\omega^{\mathrm{ind}}\big) \cup \big(\mathcal{R}_\omega^{\mathrm{ind}} \cup T_\omega \mathcal{R}_\omega^{\mathrm{ind}} \cup T_\omega^2 \mathcal{R}_\omega^{\mathrm{ind}}\big),$$

cover $\mathcal{X}_\omega$ except the cells of period one ($\varepsilon = -1$) or two ($\varepsilon = 1$). By the proof of Proposition 3.2, we know that these sets are disjoint. Then we compute the area:

- If $\varepsilon = -1$ we obtain

$$(3n-1)|\mathcal{C}_\omega^{\mathrm{ind}}| + 3|\mathcal{R}_\omega^{\mathrm{ind}}| = (3n-1)\theta^2 + 3\theta(1-n\theta) = 3\theta - \theta^2.$$

  Now we compute the area of $\mathcal{X}_\theta \setminus \mathcal{I}_\omega(2)$. We obtain $1 + \theta - (1-\theta)^2 = 3\theta - \theta^2$.

- If $\varepsilon = 1$ we obtain:

$$(3n-2)|\mathcal{C}_\omega^{\mathrm{ind}}| + 3|\mathcal{R}_\omega^{\mathrm{ind}}| = (3n-2)(1-\theta)^2 + 3(1-\theta)\big(1-n(1-\theta)\big) = (1-\theta)(1+2\theta).$$

  Now we compute the area of $\mathcal{X}_\theta \setminus \mathcal{I}_\omega(1)$. We obtain $1 + \theta - 2\theta^2$.

Thus in all the cases we have shown that the complement of the cells of period at most two is equal to the first return sets of the induction zone.

Then by the claim consider the cell $T_\omega^k \mathcal{I}$. It is clearly a periodic cell for T_ω^{ind}. By the first point of the corollary we deduce that their iterations by ψ_ω form a periodic orbit for $T_{S\omega}$.

(3) Let $\mathcal{I}$ be a periodic cell of T_ω of period p. We apply the preceding result and deduce that for some $k \leq p-1$ the cell $\psi_\omega(T_\omega^k \mathcal{I})$ is a periodic cell for $T_{S\omega}$. The period of this cell is strictly less than p since the orbit of $\mathcal{I}$ under T_ω does not stay inside $\mathcal{X}_\omega^{\mathrm{ind}}$. We apply this argument recursively and at some step n we obtain a cell of period 1 or 2. The first point of the corollary allows us to deduce that $p = p_n$. The cell is thus given as the image of a square by the composition of the similitudes $\psi_\omega, \psi_{\omega_1}, \dots, \psi_{\omega_n}$. The formula of the area follows.
(4) By the previous point, the cell is the image by a similitude of a square.
(5) We use Proposition 4.2. □

3.5. **Aperiodic set.** Consider an irrational number θ. Here we describe a partition of the aperiodic set which will be used in Section 5.

Lemma 3.6. *For every integer ℓ, the set $\mathcal{K}_\omega(p_\ell)$ has a partition (up to a set of zero measure) defined by*

$$\bigcup_{1 \leq k \leq \left\| M^{(\ell)} \binom{1}{1} \right\|_1} \mathcal{P}_k^{(l)}$$

such that:

- *Each set $\mathcal{P}_k^{(l)}$ for $1 \leq k \leq \left\| M^{(\ell)} \binom{1}{0} \right\|_1$ is the image by a similitude of $\mathcal{C}_{\omega_\ell}$.*
- *Each set $\mathcal{P}_k^{(l)}$ for $\left\| M^{(\ell)} \binom{1}{0} \right\|_1 < k \leq \left\| M^{(\ell)} \binom{1}{0} \right\|_1 + \left\| M^{(\ell)} \binom{0}{1} \right\|_1$ is the image by a similitude of $\mathcal{R}_{\omega_\ell}$.*
- *Each similitude has a ratio equal to $\prod_{k=1}^{\ell} \frac{1}{r(\omega_k)}$.*

Proof. We fix $\omega = (\theta, \varepsilon) \in \Omega$ with θ an irrational number and for each integer ℓ, $S^\ell \omega = \omega_\ell = (\theta_\ell, \varepsilon_\ell)$. By definition $\mathcal{K}_\omega(p_1)$ is the closure of the complement of $\mathcal{I}_\omega(p_1)$. The proof of Proposition 3.2 shows that the orbit of $\mathcal{X}_\omega^{\mathrm{ind}}$ is equal to $X \setminus \mathcal{I}_\omega(p_1)$. We deduce that $\mathcal{K}_\omega(p_1)$ is the closure of the orbit of $\mathcal{X}_\omega^{\mathrm{ind}}$. By equality (2) we deduce that

$$\begin{aligned}
\text{for} \quad \varepsilon = -1: \qquad \mathcal{K}_\omega(p_1) &= \big(\mathcal{C}_\omega^{\mathrm{ind}} \cup T_\omega \mathcal{C}_\omega^{\mathrm{ind}} \cup \cdots \cup T_\omega^{3n_\omega - 2} \mathcal{C}_\omega^{\mathrm{ind}}\big) \\
&\quad \cup \big(\mathcal{R}_\omega^{\mathrm{ind}} \cup T_\omega \mathcal{R}_\omega^{\mathrm{ind}} \cup T_\omega^2 \mathcal{R}_\omega^{\mathrm{ind}}\big), \\
\text{for} \quad \varepsilon = 1: \qquad \mathcal{K}_\omega(p_1) &= \big(\mathcal{C}_\omega^{\mathrm{ind}} \cup T_\omega \mathcal{C}_\omega^{\mathrm{ind}} \cup \cdots \cup T_\omega^{3n_\omega - 3} \mathcal{C}_\omega^{\mathrm{ind}}\big) \\
&\quad \cup \big(\mathcal{R}_\omega^{\mathrm{ind}} \cup T_\omega \mathcal{R}_\omega^{\mathrm{ind}} \cup T_\omega^2 \mathcal{R}_\omega^{\mathrm{ind}}\big).
\end{aligned}$$

We recall that $\mathcal{C}_\omega^{\mathrm{ind}} = \psi_\omega^{-1}(\mathcal{C}_{S\omega})$ and $\mathcal{R}_\omega^{\mathrm{ind}} = \psi_\omega^{-1}(\mathcal{R}_{S\omega})$. In each case,

$$\mathcal{K}_\omega(p_1) = \bigcup_{i=0}^{m_{1,1}(\omega)+m_{2,1}(\omega)-1} T^i \circ \psi_\omega^{-1} \mathcal{C} \bigcup \bigcup_{i=0}^{m_{1,2}(\omega)+m_{2,2}(\omega)-1} T^i \circ \psi_\omega^{-1} \mathcal{R}_{\theta_n}.$$

The formula is proven for $\ell = 1$. We can repeat this to find the formula for each integer ℓ by using Corollary 3.5:

$$\begin{aligned}
\mathcal{K}_\omega(p_\ell) = &\bigcup_{i=0}^{m_{1,1}^{(\ell)}(\omega)+m_{2,1}^{(\ell)}(\omega)-1} T^i \circ \psi_{\omega_\ell}^{-1} \circ \cdots \circ \psi_\omega^{-1} \mathcal{C} \\
&\cup \bigcup_{i=0}^{m_{1,2}^{(\ell)}(\omega)+m_{2,2}^{(\ell)}(\omega)-1} T^i \circ \psi_{\omega_\ell}^{-1} \circ \cdots \circ \psi_\omega^{-1} \mathcal{R}_{\theta_\ell}.
\end{aligned} \tag{8}$$

And the union is disjoint by Corollary 3.5. □

Remark 3.7. We prefer to work with the set $\mathcal{K}_\omega$ rather than $\mathcal{D}_\omega$, in particular because we should add a finite union of segments in equation (8).

4. The renormalization map

The definition of the map S in Section 3.1, equation (3.1), invites us to study the continued fraction expansion generated by this map. We will study the invariant measures for S and will see that we should accelerate this map to get a nice dynamical system. We use an action by homography of $GL_2(\mathbb{R})$ on the real line $\mathbb{R}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.x = \frac{ax+b}{cx+d}.$$

4.1. **Periodic points of S and continued fractions.** In order to do this we begin with the following remark: The map S is clearly nonbijective, but it defines a continued fraction algorithm based on the fact that the equality

$$\begin{cases} S(\theta,1) = \left(\{\frac{1}{1-\theta}\}, (-1)^{n+1}\right), \\ S(\theta,-1) = \left(\{\frac{1}{\theta}\}, (-1)^{n+1}\right) \end{cases}$$

yields

$$\begin{cases} \theta = 1 - \frac{1}{n+\theta_1} = \frac{n-1+\theta_1}{n+\theta_1}, \\ \theta = \frac{1}{n+\theta_1}, \end{cases} \qquad \text{where} \quad S(\theta,\varepsilon) = (\theta_1, \varepsilon_1) \text{ and } n = n_\theta.$$

Thus we will speak about **S-continued fraction expansion**. The sequence $(\theta_n, \varepsilon_n)_{n\in\mathbb{N}}$ defined by $(\theta_n, \varepsilon_n) = S^n(\theta,\varepsilon)$ is called the **S-expansion** of (θ,ε). A point (θ,ε) is called **an ultimately periodic point** for the continued fraction algorithm if there exists an integer n such that $S^{n+m}(\theta,\varepsilon) = S^n(\theta,\varepsilon)$ for every integer m.

Example 4.1.

$$S^3\left(\frac{3}{8},-1\right) = S^2\left(\frac{2}{3},-1\right) = S\left(\frac{1}{2},1\right) = (0,-1),$$

$$S^3\left(\frac{\sqrt{2}}{2},-1\right) = S^2\left(\sqrt{2}-1,1\right) = S\left(\frac{\sqrt{2}}{2},1\right) = (\sqrt{2}-1,1).$$

In the first case we say that the S-continued fraction expansion is finite, and in the second we have an ultimately periodic S-continued fraction expansion:

$$\sqrt{2} = \frac{1}{\sqrt{2}/2} = \frac{1}{\frac{1}{1+\sqrt{2}-1}} = \frac{1}{\frac{1}{1+1-\frac{1}{1+\sqrt{2}/2}}}.$$

Proposition 4.2. *Let $\omega = (\theta,\varepsilon)$ be an element of Ω.*

- *The point $\omega \in \Omega$ has a finite S-expansion if and only if $\theta \in \mathbb{Q} \cap [0,1]$.*
- *The point $\omega \in \Omega$ has an ultimately periodic S-expansion if and only if θ is a quadratic number.*

4.2. **Proof of Proposition 4.2.** First of all remark that we have

$$\frac{1}{1-\begin{pmatrix}1 & n-1\\ 1 & n\end{pmatrix}.x} = x+n.$$

• If θ is a rational number it is clear that its expansion is finite. Now consider $r \in \mathbb{Q}\cap]0,1[$, and denote $S(r,\varepsilon) = (r_1,\varepsilon_1)$. Then r_1 is a rational number with a denominator strictly less than that of r. Thus if (r,ε) has an infinite S expansion we obtain an infinite strictly decreasing sequence of integers, a contradiction.

• Assume (θ,ε) has an ultimately periodic S expansion. Our algorithm can be written

$$\begin{cases} S(\theta,1) = \left(\begin{pmatrix} n & 1-n\\ -1 & 1\end{pmatrix}.\theta, (-1)^{n+1}\right),\\ S(\theta,-1) = \left(\begin{pmatrix} -n & 1\\ 1 & 0\end{pmatrix}.\theta, (-1)^{n+1}\right). \end{cases}$$

Now we remark that

$$\begin{pmatrix} n & 1-n\\ -1 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & n-1\\ 1 & n\end{pmatrix}, \begin{pmatrix} -n & 1\\ 1 & 0\end{pmatrix}^{-1} = \begin{pmatrix}0 & 1\\ 1 & n\end{pmatrix},$$

$$(\theta,\varepsilon) = \begin{cases} \left(\begin{pmatrix}1 & n-1\\ 1 & n\end{pmatrix}.\theta_1,\varepsilon_1\right),\\ \left(\begin{pmatrix}0 & 1\\ 1 & n\end{pmatrix}.\theta_1,\varepsilon_1\right). \end{cases}$$

This formulation is better to obtain a left action of $GL_2(\mathbb{Z})$.

By assumption $S^n(\theta,\varepsilon) = S^{n+m}(\theta,\varepsilon)$; thus there exist two integer matrices M,N such that

$$MN.\theta = N\theta.$$

We obtain a quadratic polynomial equation. Thus θ is a quadratic number.

• Assume that θ is a quadratic number. Now remark that

$$\begin{pmatrix}1 & n-1\\ 1 & n\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & n-1\end{pmatrix}.$$

Thus we can write

$$(\theta,\varepsilon) = \begin{cases} \left(\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & n-1\end{pmatrix}.\theta_1,\varepsilon_1\right),\\ \left(\begin{pmatrix}0 & 1\\ 1 & n\end{pmatrix}.\theta_1,\varepsilon_1\right). \end{cases}$$

This means that our algorithm can be seen as the usual algorithm where we add the number 0 as a digit (if $n=1$). Consider the classical expansion of θ. By Lagrange's theorem, θ has an ultimately periodic expansion for the classical continued fraction algorithm. If we add zero, we also have an ultimately periodic expansion.

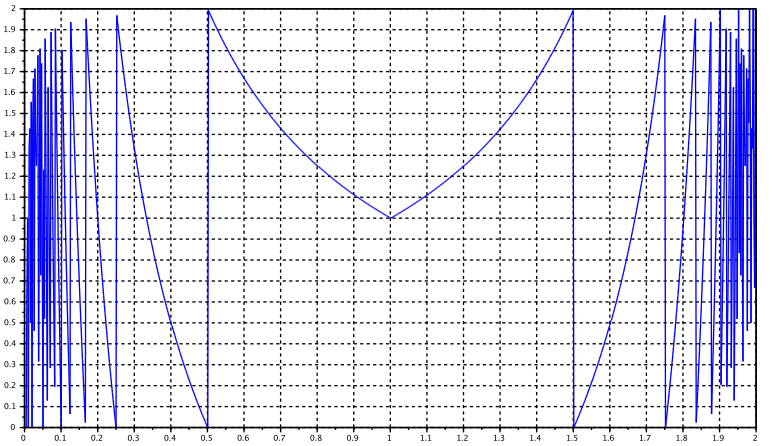


FIGURE 7. Graph of the map S.

4.3. **Dynamical properties of the renormalization map.** First we define a bijection from Ω to $]0,2[$ by $(\theta,\varepsilon)\to x=\theta+\frac{1}{2}(\varepsilon+1)$. This allows us to pass from the system (Ω,S) to the new system defined on $]0,2[$ and we keep the notation S for simplicity.

Remark 4.3. In what follows we will denote by M_x or M_ω the same class of objects, depending on the previous bijection.

On $]0,2[$, S can be expressed as

$$S(x)=\begin{cases}\left\{\dfrac{1}{x}\right\}+\dfrac{(-1)^{n_x-1}+1}{2} & \text{if}\quad 0<x<1,\\[2ex] \left\{\dfrac{1}{2-x}\right\}+\dfrac{(-1)^{n_x-1}+1}{2} & \text{if}\quad 1\le x<2.\end{cases}$$

S can also be expressed as $S(x)=A(x)\cdot x$ where $A(x)$ is defined for $n\ge 1$ by

$$A(x)=\begin{cases}\begin{pmatrix}n'-n & 1\\ 1 & 0\end{pmatrix} & \text{if}\quad x\in\left]\dfrac{1}{n+1},\dfrac{1}{n}\right],\\[2ex] \begin{pmatrix}n-n' & 1+2(n'-n)\\ -1 & 2\end{pmatrix} & \text{if}\quad x\in\left]2-\dfrac{1}{n},2-\dfrac{1}{n+1}\right],\end{cases}$$

where $n'=n \mod 2$. Thus S is a piecewise Moebius map; see Appendix A.1.

Remark 4.4. In Proposition A.2 we describe a method in order to determine an invariant measure for this map. We obtain for the density function:

$$\nu(x)=\begin{cases}\dfrac{1}{x+1} & \text{on } [0,1],\\[2ex] \dfrac{1}{x(x-1)} & \text{on }]1,2].\end{cases}$$

We do not develop this part further because this measure is not of finite volume and some important functions will not be L^1 integrable with respect to ν. This explains why we will consider the accelerated dynamical system described in the next section.

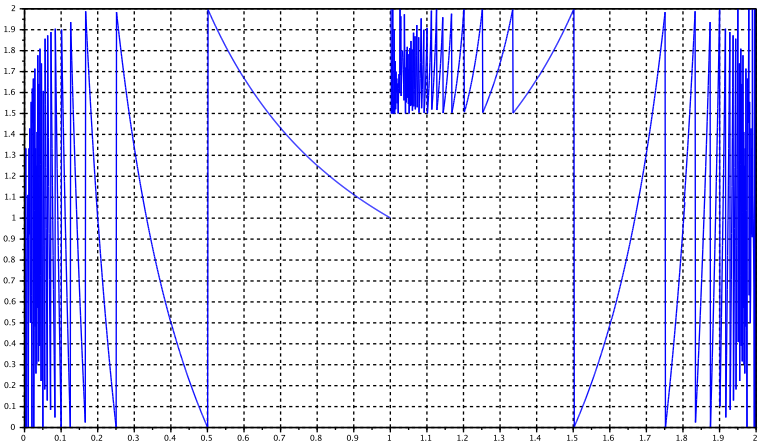


FIGURE 8. Graph of the map $\boldsymbol{S}$.

4.4. **Acceleration of the renormalization map and ergodic properties.** The point 1 is a parabolic repulsive fixed-point for S; that is why we decided to accelerate the map in a neighborhood of this point. See [3] for a complete reference.

Remark 4.5. In order to simplify the notation, we will write in **bold** all the objects which concern the accelerated map.

The acceleration of S is denoted by $\boldsymbol{S}$ and is given by $\boldsymbol{S} = S^{\mathsf{m}(x)}$ with

$$\mathsf{m}(x) = \begin{cases} 1 & \text{if} \quad x \in [0,1] \cup [3/2,2], \\ \min\{n \in \mathbb{N}; S^n x \notin [1,3/2]\} & \text{otherwise.} \end{cases}$$

We can compute m and get

$$\mathsf{m}(x) = \begin{cases} 1 & \text{if} \quad x \in [0,1] \cup [3/2;2], \\ k-1 & \text{if} \quad x \in \left[1+\frac{1}{k+1}, 1+\frac{1}{k}\right] \text{ with } k \geq 2. \end{cases}$$

We obtain $\boldsymbol{S}(x) = \boldsymbol{A}(x) \cdot x$ with

$$\boldsymbol{A}(x) = \begin{cases} A(x) & \text{if} \quad x \in [0,1] \cup [3/2;2], \\ A(x)^{k-1} = \begin{pmatrix} 2-k & k-1 \\ 1-k & k \end{pmatrix} & \text{if} \quad x \in \left[1+\frac{1}{k+1}, 1+\frac{1}{k}\right] \text{ with } k \geq 2. \end{cases}$$

We will show in the Appendix that the method described in Proposition A.2 gives the following formula for the density of an invariant measure:

$$\boldsymbol{\nu}(x) = \begin{cases} \dfrac{1}{1+x} & \text{on} \quad [0,1], \\ \dfrac{1}{x} & \text{on} \quad [1,3/2], \\ \dfrac{1}{x-1} & \text{on} \quad [3/2,2]. \end{cases}$$

Remark 4.6. In Corollary 1 of [3] the authors define the notion of a map of **first return type** and prove: "If T is of first return type and $\Gamma_{\boldsymbol{A}}$ is of finite covolume, then T is ergodic with respect to the measure $\boldsymbol{\nu}$." Unfortunately, the map $\boldsymbol{A}$ is not of first return type; thus we cannot apply this result.

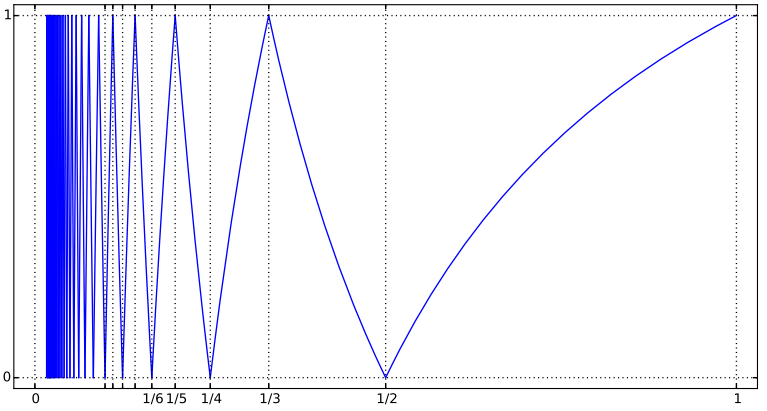


FIGURE 9. Graph of the map Q.

Thus we need to introduce another map in order to have some ergodic properties.

4.5. **Another map.** Consider the map defined on $]0,1[$ by

$$Q(x) = \begin{cases} \left\{\frac{1}{x}\right\}, & x \in \left[\frac{1}{2n+1}, \frac{1}{2n}\right], \quad n \in \mathbb{N}^*, \\ 1 - \left\{\frac{1}{x}\right\}, & x \in \left[\frac{1}{2n+2}, \frac{1}{2n+1}\right], \quad n \in \mathbb{N}. \end{cases}$$

Now let us define the map p:

$$\begin{array}{ccc} [0,2] & \to & [0,1] \\ x & \mapsto & \begin{cases} x & \text{if } \ x \in [0,1], \\ 2-x & \text{if } x \in [1,2]. \end{cases} \end{array}$$

Remark that $S(1-x) = S(1+x)$ for any $x \in [0,1[$. Thus we have a commutative diagram:

$$\begin{array}{ccc} [0,2[& \xrightarrow{S} & [0,2[\\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ ]0,1[& \xrightarrow[Q]{} &]0,1[\end{array}$$

We define the measure ν_1 on $[0,1[$ by the formula $\nu_1(A) = \nu(p^{-1}(A))$. By definition this measure is Q-invariant. We will use this application Q to show the following result.

Proposition 4.7. *The dynamical system* $(\Omega, \boldsymbol{S}, \boldsymbol{\nu})$ *is ergodic.*

4.6. **Proof of Proposition 4.7.** The map Q has the following properties:

- It is defined on a countable union of intervals I_n, with value on an interval I.
- On each interval I_n the map Q is a diffeomorphism.
- The map Q has **bounded distortion**: there exists a constant $K > 0$ such that

$$\sup_{I_n} \sup_{x,y \in I_n} \frac{|Q''(x)|}{|Q'(y)|^2} < K.$$

A classical result says that such a map is ergodic for the Lebesgue measure; see [12].

Lemma 4.8. *We have:*

- *The map Q is ergodic for the measure ν_1.*
- *If (I, Q, ν_1) is ergodic, then (Ω, S, ν) is ergodic.*
- *If S is ergodic, then $(\Omega, \boldsymbol{S}, \boldsymbol{\nu})$ is ergodic.*

Proof.

- It is easy to see that the map Q has the bounded distortion property. Indeed, the classical Gauss map $G : x \mapsto \left\{\frac{1}{x}\right\}$ has the bounded distortion property and for $x \in [0,1]$, $|G'(x)| = |Q'(x)|$. Thus it is ergodic with respect to the Lebesgue measure. Now the measure ν is absolutely continuous with respect to the Lebesgue measure; see Subsection 4.3. Thus the system (I, Q) is ergodic for some measure. This measure is in the same class as the Lebesgue measure; thus the system (I, Q, ν_1) is ergodic.
- Assume by contradiction that S is not ergodic. Then there exists a set A with $\nu(A) > 0$ such that $S^{-1}A = A$. By symmetry, the set A is symmetric with respect to $x = 1$. Then there exists a set $A' \in [0,1)$ such that $Q^{-1}A' = A'$. By definition we have $\nu_1(A') = \frac{\nu(A)}{2} > 0$, a contradiction.
- The last part is a classical result. □

4.7. **Acceleration of the renormalization map as first return.**

Definition 4.9. We consider the following substitutions $\boldsymbol{\sigma}_x = \sigma_x \circ \cdots \circ \sigma_{S^{\mathsf{m}(x)}x}$ associated to the matrices $\boldsymbol{M}(x) = M(x) \times \cdots \times M(S^{\mathsf{m}(x)}x)$. We recall that $x \in]0,2[$ is in bijection with $\omega \in \Omega$. Thus σ_x denotes the same object as σ_ω; see equation (5).

- For $n \geq 1$, on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$, $\mathsf{m}(x) = 1$ and

$$\boldsymbol{\sigma}_x : \begin{cases} a \to ab(aab)^{n-1} \\ b \to aab \end{cases} \quad \text{and} \quad \boldsymbol{M}(x) = \begin{pmatrix} 2n-1 & 2 \\ n & 1 \end{pmatrix}.$$

- For $n \geq 2$, on $\left[1 + \frac{1}{n+1}, 1 + \frac{1}{n}\right]$, $\mathsf{m}(x) = n - 1$ and

$$\boldsymbol{\sigma}_x : \begin{cases} a \to a \\ b \to a^{2(n-1)}b \end{cases} \quad \text{and} \quad \boldsymbol{M}(x) = \begin{pmatrix} 1 & 2(n-1) \\ 0 & 1 \end{pmatrix}.$$

- For $n \geq 2$, on $\left[2 - \frac{1}{n}, 2 - \frac{1}{n+1}\right]$, $\mathsf{m}(x) = 1$ and

$$\boldsymbol{\sigma}_x : \begin{cases} a \to a(aab)^{n-1} \\ b \to aab \end{cases} \quad \text{and} \quad \boldsymbol{M}(x) = \begin{pmatrix} 2n-1 & 2 \\ n-1 & 1 \end{pmatrix}.$$

Let us also define

$$\boldsymbol{M}^{(k)}(x) = \boldsymbol{M}(x) \times \cdots \times \boldsymbol{M}(\boldsymbol{S}^k x) = \begin{pmatrix} \boldsymbol{m}_{1,1}^{(k)}(x) & \boldsymbol{m}_{1,2}^{(k)}(x) \\ \boldsymbol{m}_{2,1}^{(k)}(x) & \boldsymbol{m}_{2,2}^{(k)}(x) \end{pmatrix}.$$

Now let us define the normalization factors with the help of Subsection 3.1:

$$\boldsymbol{r}(x) = r(x) \times \cdots \times r(S^{\mathsf{m}(x)}x).$$

A simple calculation gives

$$\boldsymbol{r}(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \in]0,1], \\ \dfrac{1}{n-(n-1)x} & \text{if } x \in [1,3/2[\text{ where } n = \left\lfloor \dfrac{1}{x-1} \right\rfloor, \\ \dfrac{1}{2-x} & \text{if } x \in]3/2,2]. \end{cases}$$

5. Dynamics on the aperiodic set

5.1. **Background on Sturmian substitutions.**

Definition 5.1 ([16])**.** An infinite word $u = u_0 \cdots u_n \cdots$ over the alphabet $\{a, b\}$ is a **Sturmian word** if one of the following conditions holds :

(1) there exist an irrational number $\alpha \in [0,1]$ called **the angle** of u and $\beta \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N},\ u_n = a \Leftrightarrow n\alpha+\beta-\lfloor n\alpha+\beta \rfloor \leq 1 \text{ or } \forall n \in \mathbb{N},\ u_n = a \Leftrightarrow n\alpha+\beta-\lceil n\alpha+\beta \rceil \leq 1,$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are respectively the floor and the ceiling functions,

(2) the symbolical dynamical system associated to u is measurably conjugated to a rotation on the circle by an irrational number,

(3) for each integer n, $\operatorname{card}\big(L_n(u)\big) = n+1$.

A substitution σ is said to be **Sturmian** if the image of every Sturmian word by σ is a Sturmian word.

5.2. **One technical lemma.** We have the following result.

Lemma 5.2.

(1) *σ is a Sturmian substitution if and only if it is a composition of the basic substitutions:*

$$s_1 : \begin{cases} a \to ab \\ b \to b, \end{cases} \quad s_2 : \begin{cases} a \to ba \\ b \to b, \end{cases} \quad s_3 : \begin{cases} a \to a \\ b \to ba, \end{cases}$$

$$s_4 : \begin{cases} a \to a \\ b \to ab \end{cases} \quad \textit{and} \quad s_5 : \begin{cases} a \to b \\ b \to a. \end{cases}$$

(2) *If $(\sigma_i)_{i\in\mathbb{N}}$ is a sequence of Sturmian substitutions such that $\sigma_1 \circ \cdots \circ \sigma_\ell(a)$ converge to an infinite word u, then u is a Sturmian word if and only if the sequence σ_i is ultimately constant equal to some s_i for $i \in \{1,2,3,4\}$.*

Proof. The first point is a consequence of [16]. Let us prove the second point: we fix an integer n and a word $u = \lim_{\ell\to+\infty} \sigma_1 \circ \cdots \circ \sigma_\ell(a)$. We want to count the number of words of length n factors of u. First by minimality we can find an integer N such that every word of length n appears in $u_1 \dots u_N$. Then there exists an integer m such that $u_1 \dots u_N$ is a factor of $\sigma_1 \circ \cdots \circ \sigma_m(a)$ by definition of u. Now consider a Sturmian word which begins with a. The word $\sigma_i(a \dots)$ is Sturmian by definition of σ_i for every integer $i \leq \ell$. We deduce that the number of factors of length n in $\sigma_1 \circ \cdots \circ \sigma_m(a)$ is bounded by $n+1$.

Now we use the fact that the basic substitutions are joined to classical Gauss continued fractions. Then the word is periodic if and only if the frequency of letters a and b are rational if and only if the expansion in continued fractions is finite. □

5.3. **Result.** We proved in Proposition 4.2 that $S^n(\theta,\varepsilon)$ is well defined for each integer n if and only if θ is an irrational number. We fix for this section an irrational number θ in $[0,1]$, $\varepsilon \in \{-1,1\}$ and $\omega = (\theta,\varepsilon)$. By definition of the substitutions σ_x, we remark that a is a prefix of $\sigma_x(a)$ for each substitution. This means that the following word is well defined:

$$u_\omega = \lim_{\ell\to+\infty} \boldsymbol{\sigma}_\omega \circ \cdots \circ \boldsymbol{\sigma}_{S^\ell\omega}(a).$$

It is clearly an infinite word, because for each $\omega \in]0,1[\times\{-1\}\cup]1/2,1[\times\{1\}$ we have $|\boldsymbol{\sigma}_\omega(u)| > |u|$ and $]0,1/2[\times\{1\}$ is not stable by S. Now we define $\Sigma_\omega := \overline{\{t^n u_\omega, n\in\mathbb{N}\}}$, where t is the shift map.

We can now state the main result of this section:

Proposition 5.3. *The dynamical system $(\mathcal{K}_\omega, T_\omega)$ is conjugate to an irrational rotation of the circle S^1.*

The proof is a consequence of Lemma 5.2 and the two following lemmas.

Lemma 5.4. *The sequence u_ω is a Sturmian sequence.*

Proof. With the previous lemma, we only have to verify that for each $\omega = (\theta,\varepsilon) \in \Omega$ such that θ is irrational, σ_ω is a Sturmian substitution.

Let n be an integer. We define

$$p : \begin{cases} a \to ab \\ b \to aab, \end{cases} \qquad q : \begin{cases} a \to a \\ b \to aab \end{cases} \qquad \text{and} \quad r_n : \begin{cases} a \to ab^n \\ b \to b. \end{cases}$$

It is clear that they are the composition of basic Sturmian substitutions and that each substitution σ_ω defined in (5) is of the form

$$p\circ r_n \ : \ \begin{cases} a \to ab(aab)^n \\ b \to aab \end{cases} \qquad \text{and} \quad q\circ r_n \ : \ \begin{cases} a \to a(aab)^n \\ b \to aab. \end{cases} \qquad \square$$

Remark 5.5. The relation between ω and the angle of the word u_ω is not obvious, as seen on Figure 10. We refer to Theorem B.6 in Section B.2. The angle of u_ω is the first coordinate of the vector z_ω where the sum of coordinates is equal to 1. It is also equal to the frequency of one letter in the word u_ω. Thus we can express it as $\lim_\ell \frac{\left|(1,0)\times \boldsymbol{M}^{(\ell)}(\omega)\binom{1}{0}\right|}{\left|(1,1)\times \boldsymbol{M}^{(\ell)}(\omega)\binom{1}{0}\right|}$.

Lemma 5.6. *The dynamical system $(\mathcal{K}_\omega, T_\omega)$ is conjugated to (Σ_ω, t).*

Proof. We will construct an explicit conjugation in Subsection 5.4. $\square$

We also denote the following quantities:

$$\begin{aligned} \boldsymbol{N}_a^{(\ell)} = \boldsymbol{N}_a^{(\ell)}(\omega) = \left\| \boldsymbol{M}^{(\ell)}(\omega)\tbinom{1}{0} \right\|_1, \quad \boldsymbol{N}_b^{(\ell)} = \boldsymbol{N}_b^{(\ell)}(\omega) = \left\| \boldsymbol{M}^{(\ell)}(\omega)\tbinom{0}{1} \right\|_1 \\ \text{and} \quad \boldsymbol{N}^{(\ell)} = \boldsymbol{N}^{(\ell)}(\omega) = \left\| \boldsymbol{M}^{(\ell)}(\omega)\tbinom{1}{1} \right\|_1. \end{aligned} \tag{9}$$

By Lemma 5.4 the system (Σ_ω, t) is uniquely ergodic. We denote by $\tilde{\mu}_\omega$ the unique invariant probability measure of (Σ_ω, t).

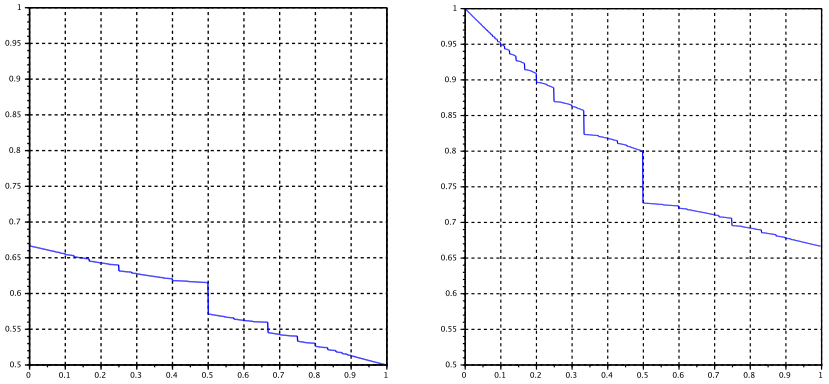


FIGURE 10. Numerical representation of the map $\omega \to \text{angle}(u_\omega)$ respectively for $\varepsilon = -1$ and $\varepsilon = 1$.

Lemma 5.7. *For every positive integer ℓ there exists a partition $\mathcal{P}_\ell$ of Σ_ω by the following sets such that*

$$\mathcal{P}_\ell = \{\boldsymbol{P}_1^{(\ell)}, \dots, \boldsymbol{P}_{\boldsymbol{N}_a^{(\ell)}(\omega)}^{(\ell)}, \boldsymbol{P}_{\boldsymbol{N}_a^{(\ell)}(\omega)+1}^{(\ell)}, \dots, \boldsymbol{P}_{\boldsymbol{N}^{(\ell)}(\omega)}^{(\ell)}\}.$$

- *The partition $\mathcal{P}_{\ell+1}$ is a refinement of the partition $\mathcal{P}_\ell$.*
- *The function $n \mapsto \tilde{\mu}_\omega(\boldsymbol{P}_n^{(\ell)})$ is constant on the intervals $\{1, \dots, \boldsymbol{N}_a^{(\ell)}\}$ and $\{\boldsymbol{N}_a^{(\ell)} + 1, \dots, \boldsymbol{N}^{(\ell)}\}$.*

Proof. Let $v = v_0 \dots v_n \dots$ be an element of Σ_ω and let ℓ be an integer.

We denote by $\boldsymbol{\sigma}_\omega \circ \cdots \circ \boldsymbol{\sigma}_{\boldsymbol{S}^\ell\omega} = \boldsymbol{\sigma}_\omega^{(\ell)}$. By unique desubstitution, there exists an integer n and a unique word $t_0 \dots t_{n-1}$ such that

$$v = t_0 \dots t_{n-1} \boldsymbol{\sigma}_\omega^{(\ell)}(v_\ell) \text{ where } v_\ell \in \Sigma_{\boldsymbol{S}^\ell\omega} \tag{10}$$

and $t_0 \dots t_{n-1}$ is a suffix of $\boldsymbol{\sigma}_\omega^{(\ell)}(a)$ or $\boldsymbol{\sigma}_\omega^{(\ell)}(b)$.

The word $\boldsymbol{\sigma}_\omega^{(\ell)}(a)$ has length equal to $\boldsymbol{N}_a^{(\ell)}(\omega)$. Thus its number of suffixes is equal to $\boldsymbol{N}_a^{(\ell)}(\omega)$. Let $n \in \{1, \dots, \boldsymbol{N}_a^{(\ell)}(\omega)\}$. We denote by $\boldsymbol{P}_n^{(\ell)}$ the set of words of the form (10) where $t_0 \dots t_{n-1}$ is the suffix of $\boldsymbol{\sigma}_\omega^{(\ell)}(a)$ of length n. Similarly we define for $n \in \{1 + \boldsymbol{N}_a^{(\ell)}(\omega), \boldsymbol{N}^{(\ell)}(\omega)\}$ $\boldsymbol{P}_n^{(\ell)}$ as the set of words of the form (10) where $t_0 \dots t_{n-1}$ is the suffix of $\boldsymbol{\sigma}_\omega^{(\ell)}(b)$ of length n.

These sets are disjoint up to a measure zero set. Indeed the result is clear for two sets P_i, P_j with $i, j \leq \boldsymbol{N}_a^{(\ell)}(\omega)$. If v belongs to the intersection of some $\boldsymbol{P}_{\boldsymbol{N}_a^{(\ell)}(\omega)+k}^{(\ell)}, \boldsymbol{P}_i^{(\ell)}$ it means that the infinite word v_ℓ can be extended to the left by two letters a, b. The set of these words is of zero measure.

Thus we deduce that

$$\Sigma_\omega = \bigcup_{1 \leq n \leq \boldsymbol{N}_a^{(\ell)}(\omega)} \boldsymbol{P}_n^{(\ell)} \cup \bigcup_{1 \leq k \leq \boldsymbol{N}_b^{(\ell)}(\omega)} \boldsymbol{P}_{\boldsymbol{N}_a^{(\ell)}(\omega)+k}^{(\ell)}.$$

Remark that $\boldsymbol{\sigma}_\omega^{(\ell+1)}(a) = \boldsymbol{\sigma}_\omega^{(\ell)}(\boldsymbol{\sigma}_{S^\ell\omega}(a))$. Thus in the sequences of such partitions, each partition refines the previous one. Moreover, notice that the image of $\boldsymbol{P}_n^{(\ell)}$ by the shift map is equal to $P_{n-1}^{(\ell)}$ (analogously for $Q_n^{(\ell)}$). Since the measure $\tilde{\mu}_\omega$ is shift invariant, the result follows. □

Definition 5.8. Let us denote these measures by α_ℓ and β_ℓ. Remark that $\boldsymbol{N}_a^{(\ell)}\alpha_\ell + \boldsymbol{N}_b^{(\ell)}\beta_\ell = 1$.

5.4. **Correspondance between $(T_\omega, \mathcal{K}_\omega)$ and the symbolic dynamical system.** Consider an irrational number ω. First we define a metric $\bar{d}$ on Σ_ω. Two infinite words u, v fulfill $\bar{d}(u,v) = \boldsymbol{R}^{-i}$ if $i = \max\{k \in \mathbb{N}, \exists n \in \mathbb{N}, u, v \in P_n^{(k)}\}$. We leave it to the reader to check, with the help of Lemma 5.7, that $\bar{d}$ is a metric, compatible with the topology on Σ_ω. Remark that the ball of center u and radius $\boldsymbol{R}^i$ is equal to $P_n^{(i)}$ for some integer n.

We prove Lemma 5.6 in the following form.

Proposition 5.9. *There exists a map $h : \Sigma_\omega \to \mathcal{K}_\omega$ which is*

- *continuous,*
- *almost everywhere injective and onto,*
- *such that $T_\omega \circ h = h \circ t$.*

Proof. We use the preceding lemma and equation (10).

Let ℓ be an integer and $v \in \Sigma_\omega$. Then we define the set B_ℓ by

$$B_\ell(v) = \begin{cases} T^n \circ \psi_\omega^{-1} \circ \cdots \circ \psi_{\boldsymbol{S}^{\ell}\omega}^{-1}(\mathcal{C}) & \text{if } \boldsymbol{\sigma}_\omega^{(\ell)}(v_\ell) = a \dots, \\ T^n \circ \psi_\omega^{-1} \circ \cdots \circ \psi_{\boldsymbol{S}^{\ell}\omega}^{-1}(\mathcal{R}) & \text{if } \boldsymbol{\sigma}_\omega^{(\ell)}(v_\ell) = b \dots. \end{cases}$$

By Proposition 3.2, the restriction of the map T^n to the set $\psi_\omega^{-1} \circ \cdots \circ \psi_{\boldsymbol{S}^{\ell}\omega}^{-1}(\mathcal{C})$ is an isometry. The same property is true for the restriction to the set $\psi_\omega^{-1} \circ \cdots \circ \psi_{\boldsymbol{S}^{\ell}\omega}^{-1}(\mathcal{R})$. Each map $\psi_{\boldsymbol{S}^k\omega}$ is a similitude of ratio $\boldsymbol{r}(\boldsymbol{S}^k\omega)$. We deduce that each set $B_\ell(v)$ is included in a square of size $\boldsymbol{R}^{(\ell)} = \frac{1}{\boldsymbol{r}(\omega)} \cdots \frac{1}{\boldsymbol{r}(\omega_\ell)}$.

By Proposition 4.2, the sequence $(\omega_\ell)_{\ell \in \mathbb{N}}$ is not periodic. By Lemma 6.4 and equation (B.8) we deduce that $\lim_{+\infty} \boldsymbol{R}^{(\ell)} = 0$ for almost all ω. The sequence $(B_\ell(v))_{\mathbb{N}}$ is thus a decreasing sequence of compact sets whose diameters converge to zero, and therefore there is a unique element in the intersection. We denote it by

$$h(v) = \bigcap_\ell B_\ell(v).$$

• We claim that h is a continuous function: consider $\eta > 0$ and u, v infinite words in Σ_ω such that $\bar{d}(u,v) \leq \eta$. By the preceding lemma this means that there exists an integer ℓ such that u and v are in the same atom $\mathcal{P}_i$ of the partition for $i \leq \ell$. Then $h(u)$ and $h(v)$ belong to the same ball B_ℓ. Thus the distance between $h(u)$ and $h(v)$ is bounded by the diameter of $B_\ell(u)$, which is bounded by $\boldsymbol{R}^{(\ell)}$. Since this number decreases to zero when ℓ goes to infinity, we deduce the continuity of h.

• The map is onto: remark that Lemma 3.6 implies that

$$\mathcal{K}_\omega(p_\ell) = \bigcup_{n \leq N^{(l)}} \bigcup_{v \in P_n^{(l)}} B_l(v).$$

Since $\mathcal{K}_\omega$ is obtained as the intersection of these sets, the surjectivity of the map is clear.

The injectivity comes from the definition and the fact that there is only one point which has a prescribed symbolic aperiodic coding.

• The relation between the maps h, T_ω and t is clear from the definition of h. □

Definition 5.10. As a by-product of the result, this map defines a measure μ on $\mathcal{K}_\omega$ by the formula $\mu(A) = \tilde{\mu}_\omega\big(h^{-1}(A)\big)$.

6. Hausdorff dimension of the aperiodic set

6.1. **Background on Hausdorff dimension.** We recall some usual facts about Hausdorff dimensions; see [15].

Definition 6.1. Let $\mathcal{F}$ be a compact set of $\mathbb{R}^n$ and s a positive real number:

$$H^s_\delta(\mathcal{F}) = \inf\left\{\sum_{i=1}^{+\infty} \operatorname{diam}(U_i)^s \mid \operatorname{diam}(U_i) \le \delta \quad \text{and} \quad \mathcal{F} \subset \bigcup_{i=1}^{+\infty} U_i\right\}.$$

We introduce $H^s(\mathcal{F}) = \lim\limits_{\delta\to 0} H^s_\delta(\mathcal{F}) \in \mathbb{R}^+ \cup \{+\infty\}$. Then there exists a unique positive real number s_0 such that $H^s(\mathcal{F}) = +\infty$ if $s > s_0$ and $H^s(\mathcal{F}) < +\infty$ if $s < s_0$. This real number s_0 is called the **Hausdorff dimension** of $\mathcal{F}$ and is denoted $\dim_H(\mathcal{F})$.

Definition 6.2. Let $\mathcal{F}$ be a compact set of $\mathbb{R}^n$. For $r > 0$, we denote by $\operatorname{Numb}(r)$ the minimal number of balls of radius r needed to cover $\mathcal{F}$. Then the **lower and upper box-counting dimensions** are respectively defined by

$$\underline{\dim}_B(\mathcal{F}) = \liminf_{r\to 0} -\frac{\ln \operatorname{Numb}(r)}{\ln r} \quad \text{and} \quad \overline{\dim}_B(\mathcal{F}) = \limsup_{r\to 0} -\frac{\ln \operatorname{Numb}(r)}{\ln r}.$$

If these numbers are equal we speak about **box-counting dimension**.

We recall a classical result also called the Frostman Lemma.

Lemma 6.3 ([15], Thm 4.4)**.** *Assume there exists a measure $\tilde{\mu}$ such that $\tilde{\mu}(\mathcal{F}_\infty) > 0$ and such that for almost every x in $\mathcal{F}_\infty$,*

$$\liminf \frac{\ln\Big(\tilde{\mu}\big(\mathcal{B}(x,r)\big)\Big)}{\ln(r)} \ge a.$$

Then we have $\dim_H(\mathcal{F}_\infty) \ge a$.

6.2. **Technical lemmas.** Let $\omega = (\theta, \varepsilon) \in \Omega$ with θ an irrational number. By an abuse of notation, we will use the symbol ω_ℓ to denote $\boldsymbol{S}^\ell\omega$ and $\boldsymbol{R}^{(\ell)} = \frac{1}{\boldsymbol{r}(\omega)} \cdots \frac{1}{\boldsymbol{r}(\omega_\ell)}$.

Lemma 6.4. *With the notation (9), for $\boldsymbol{\nu}$-almost all ω, there exist $\boldsymbol{\lambda}(\omega)$ and $\boldsymbol{R}(\omega)$ such that*

$$\lim_{\ell\to+\infty} \frac{1}{\ell} \ln \boldsymbol{N}^{(\ell)} = \boldsymbol{\lambda}(\omega), \tag{11}$$

$$\lim_{\ell\to+\infty} \frac{1}{\ell} \ln \boldsymbol{N}^{(\ell)}_a = \lim_{\ell\to+\infty} \frac{1}{\ell} \ln \boldsymbol{N}^{(\ell)}_b = \lim_{\ell\to+\infty} \frac{1}{\ell} \ln \boldsymbol{N}^{(\ell)} = \boldsymbol{\lambda}(\omega), \tag{12}$$

$$\lim_{\ell\to+\infty} \frac{1}{\ell} \ln \boldsymbol{R}^{(\ell)}(\omega) = \ln \boldsymbol{R}(\omega). \tag{13}$$

Consider the number $\boldsymbol{s}(\omega)$ defined $\boldsymbol{\nu}$-almost everywhere by

$$\boldsymbol{s}(\omega) = \frac{\boldsymbol{\lambda}(\omega)}{\ln \boldsymbol{R}(\omega)}.$$

Then the functions $\boldsymbol{\lambda}(\omega)$, $\ln \boldsymbol{R}(\omega)$ and $\boldsymbol{s}(\omega)$ are almost everywhere constant for the measure $\boldsymbol{\nu}$.

In this proof we will use Definition B.1 and Theorem B.2. Consider the cocycle defined by

$$\begin{array}{rcl} \Omega \times \mathbb{N} & \to & GL_2(\mathbb{Z}) \cap \mathcal{M}_2(\mathbb{N}) \\ (\omega, n) & \mapsto & {}^t\boldsymbol{M}(\boldsymbol{S}^n\omega) \times \cdots \times {}^t\boldsymbol{M}(\omega) = {}^t\boldsymbol{M}^{(n)}(\omega). \end{array}$$

Remark 6.5. By Lemma B.7, the map $\omega \to \ln \|M(\omega)\|$ does not belong to $L^1(\nu)$. Thus we cannot apply Oseledets' theorem; see Theorem B.2. This explains the necessity to define the acceleration of the map S.

Proof. By Lemma B.8 the functions $\omega \to \ln \|\boldsymbol{M}(\omega)\|$, $\omega \to \ln \|\boldsymbol{M}^{-1}(\omega)\|$ and $\ln \circ \boldsymbol{r}$ belong to $L^1(\boldsymbol{\nu})$. Thus the same is true for $\omega \to \ln \| {}^t\boldsymbol{M}(\omega)\|$ and $\omega \to \ln \| {}^t\boldsymbol{M}^{-1}(\omega)\|$.

By Theorem B.2 of Oseledets, we obtain the convergence for almost every ω of

$$\frac{1}{\ell} \ln \left\| {}^t\boldsymbol{M}^{(\ell)}(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_1 .$$

Now an easy computation shows that for each integer ℓ:

$$\begin{aligned} \left\| {}^t\boldsymbol{M}^{(\ell)}(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_1 &= \left\| (1,1) \times {}^t\boldsymbol{M}^{(\ell)}(\omega) \right\|_1 \\ &= \left\| {}^t \left(\boldsymbol{M}^{(\ell)}(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right\|_1 = \left\| \boldsymbol{M}^{(\ell)}(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_1 . \end{aligned}$$

This expression is equal to $\boldsymbol{m}^{(\ell)}_{1,1}(\omega) + \boldsymbol{m}^{(\ell)}_{1,2}(\omega) + \boldsymbol{m}^{(\ell)}_{2,1}(\omega) + \boldsymbol{m}^{(\ell)}_{2,2}(\omega)$.

By Proposition 4.7 the dynamical system is ergodic. From (18) and from the Furstenberg-Kesten Theorem and its Corollary B.4, the sequences of equality (11) converge for almost every ω and

$$\begin{aligned} \lim_{\ell \to +\infty} \frac{1}{\ell} \ln \boldsymbol{N}^{(\ell)} &= \lim_{\ell \to +\infty} \frac{1}{\ell} \ln \left\| \boldsymbol{M}^{(n)}(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| \\ &= \lim_{\ell} \frac{1}{\ell} \int \ln \| {}^t\boldsymbol{M}^{(\ell)}\| d\boldsymbol{\nu} = \lim_{\ell} \frac{1}{\ell} \int \ln \|\boldsymbol{M}^{(\ell)}\| d\boldsymbol{\nu}. \end{aligned}$$

The convergence of expression (13) is a direct application of the Birkhoff Theorem.

It remains to prove the convergence of the terms in equation (12). Let us equip $\mathbb{R}^2$ with the partial order defined by $z = (x, y) \leq z' = (x', y')$ if $x \leq x'$ and $y \leq y'$. By Definition 4.9 the matrices $\boldsymbol{M}(\omega)$ have positive coefficients. Thus if $0 \leq z \leq z'$, we obtain for all ω: $\|\boldsymbol{M}(\omega)z\|_1 \leq \|\boldsymbol{M}(\omega)(z')\|_1$. We deduce immediately that for each ℓ:

$$\begin{aligned} &\left\| \boldsymbol{M}^{(\ell)}(\omega) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_1 \leq \left\| \boldsymbol{M}^{(\ell)}(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_1 \quad \text{and} \\ &\left\| \boldsymbol{M}^{(\ell)}(\omega) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_1 \leq \left\| \boldsymbol{M}^{(\ell)}(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_1 . \end{aligned} \tag{14}$$

Now we use the correspondence between $x \in [0, 2]$ and $\omega \in \Omega$. Let us consider three cases:

- If $x \in (0, 1]$, then with $n = \left\lfloor \frac{1}{x} \right\rfloor \geq 1$:

$$\boldsymbol{M}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2n-1 \\ n \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{M}(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- If $x \in (1, 3/2)$, then with $n = \left\lfloor \frac{1}{x-1} \right\rfloor \geq 2$:

$$\boldsymbol{M}(x)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2(n-1)\\1\end{pmatrix} \geq \begin{pmatrix}1\\1\end{pmatrix}, \quad \text{but we also have} \quad \boldsymbol{M}(x)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

- If $x \in (3/2, 2]$, then with $n = \left\lfloor \frac{1}{2-x} \right\rfloor \geq 2$:

$$\boldsymbol{M}(x)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2n-1\\n-1\end{pmatrix} \geq \begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \boldsymbol{M}(x)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \geq \begin{pmatrix}1\\1\end{pmatrix}.$$

By definition of $\boldsymbol{S}$ we remark that if $x \in [1, 3/2]$, then $\boldsymbol{S}(x) \notin (1, 3/2)$. We deduce from the two preceding inequalities that

$$\boldsymbol{M}(\boldsymbol{S}x)\boldsymbol{M}(x)\begin{pmatrix}1\\0\end{pmatrix} \geq \boldsymbol{M}(\boldsymbol{S}x)\begin{pmatrix}1\\0\end{pmatrix} \geq \begin{pmatrix}1\\1\end{pmatrix}.$$

With equation (14), we deduce for all integers $\ell \geq 2$ that

$$\begin{aligned}\left\|\boldsymbol{M}^{(\ell-2)}(\omega)\tbinom{1}{1}\right\|_1 &\leq \left\|\boldsymbol{M}^{(\ell)}(\omega)\tbinom{1}{0}\right\|_1 \leq \left\|\boldsymbol{M}^{(\ell)}(\omega)\tbinom{1}{1}\right\|_1,\\ \left\|\boldsymbol{M}^{(\ell-2)}(\omega)\tbinom{1}{1}\right\|_1 &\leq \left\|\boldsymbol{M}^{(\ell)}(\omega)\tbinom{0}{1}\right\|_1 \leq \left\|\boldsymbol{M}^{(\ell)}(\omega)\tbinom{1}{1}\right\|_1.\end{aligned}$$

Finally we have

$$\frac{1}{\ell}\ln\left\|\boldsymbol{M}^{(\ell-2)}(\omega)\begin{pmatrix}1\\1\end{pmatrix}\right\|_1 \leq \frac{1}{\ell}\ln\left\|\boldsymbol{M}^{(\ell)}(\omega)\begin{pmatrix}1\\0\end{pmatrix}\right\|_1 \leq \frac{1}{\ell}\ln\left\|\boldsymbol{M}^{(\ell)}(\omega)\begin{pmatrix}1\\1\end{pmatrix}\right\|_1, \tag{15}$$

$$\frac{1}{\ell}\ln\left\|\boldsymbol{M}^{(\ell-2)}(\omega)\begin{pmatrix}1\\1\end{pmatrix}\right\|_1 \leq \frac{1}{\ell}\ln\left\|\boldsymbol{M}^{(\ell)}(\omega)\begin{pmatrix}0\\1\end{pmatrix}\right\|_1 \leq \frac{1}{\ell}\ln\left\|\boldsymbol{M}^{(\ell)}(\omega)\begin{pmatrix}1\\1\end{pmatrix}\right\|_1. \tag{16}$$

Since the left and right terms converge to $\boldsymbol{\lambda}(\omega)$, we deduce the convergence of the terms of equation (12).

The ergodicity of the map, by Proposition 4.7, allows to conclude that the functions $\boldsymbol{\lambda}(\omega)$, $\ln \boldsymbol{R}(\omega)$, and $\boldsymbol{s}(\omega)$ are almost everywhere constant. □

Definition 6.6. We denote respectively by $\boldsymbol{\lambda}$, $\ln \boldsymbol{R}$ the constant functions associated to $\boldsymbol{\lambda}(\omega)$, $\ln \boldsymbol{R}(\omega)$. We also denote $\boldsymbol{s} = \frac{\boldsymbol{\lambda}}{\ln \boldsymbol{R}}$. We finally define Ω' as the subset of Ω for which the previous expressions converge.

Remark that it is straightforward to check that the expressions converge if ω has an ultimately periodic S expansion.

The main objective of the next two subsections is the computation of the Hausdorff dimension of $\mathcal{K}_\omega$. We shall prove it is equal to $\boldsymbol{s}$.

6.3. **Minoration of the Hausdorff dimension in Theorem 2.10.** We use the measure μ on $\mathcal{K}_\omega$ of Definition 5.10.

Lemma 6.7. *For $\omega \in \Omega'$, we have for each $x \in \mathcal{K}_\omega$:*

$$\liminf_{r \to 0} \frac{\ln \mu\big(\mathcal{B}(x, r) \cap \mathcal{K}_\omega\big)}{\ln r} \geq \boldsymbol{s}. \tag{17}$$

Proof. Let x be an element of $\mathcal{K}_\omega$ and $r > 0$. The map $h : \Sigma_\omega \mapsto \mathcal{K}_\omega$ is onto by Proposition 5.9; thus there exists an element $u \in \Sigma_\omega$ such that $x = h(u)$. Now consider the integer ℓ such that $\boldsymbol{R}^{(\ell)} \leq \sqrt{2} \cdot r \leq \boldsymbol{R}^{(\ell-1)}$. By Lemma 5.7 there exists an element $C \in \mathcal{P}_\ell$ such that $u \in C$. Moreover the set $h(C)$ is included either in a rectangle of sides $(\boldsymbol{R}^{(\ell)}, \theta_\ell \boldsymbol{R}^{(\ell)})$ or in a square of side $\boldsymbol{R}^{(\ell)}$. This polygon

contains x. Thus, in any case, it is included inside $\mathcal{B}(x,\sqrt{2}r)$. We deduce $C \subset h^{-1}\big(\mathcal{B}(x,\sqrt{2}r)\cap\mathcal{K}_\omega\big)$ and

$$\mu\big(\mathcal{B}(x,\sqrt{2}r)\cap\mathcal{K}_\omega\big) = \tilde{\mu}_\omega\Big(h^{-1}\big(\mathcal{B}(x,\sqrt{2}r)\cap\mathcal{K}_\omega\big)\Big) \geq \tilde{\mu}_\omega(C).$$

By definition of C, the result of Lemma 5.7 implies that $\tilde{\mu}_\omega(C)$ is equal to α_ℓ or β_ℓ. We will consider two cases.

Assume first that $\tilde{\mu}_\omega(C) = \alpha_\ell$. Then we have

$$\begin{aligned}\frac{\ln\Big(\mu\big(\mathcal{B}(x,\sqrt{2}r)\cap\mathcal{K}_\omega\big)\Big)}{\ln(\sqrt{2}r)} &\geq \frac{\ln\alpha_\ell}{\ln \boldsymbol{R}^{(\ell-1)}} = \frac{\ln \alpha_\ell \boldsymbol{N}_a^{(\ell)}}{\ln \boldsymbol{R}^{(\ell-1)}} - \frac{\ln \boldsymbol{N}_a^{(\ell)}}{\ln \boldsymbol{R}^{(\ell-1)}} \\ &= \frac{\ln\big(1-\beta_\ell \boldsymbol{N}_b^{(\ell)}\big)}{\ln \boldsymbol{R}^{(\ell-1)}} - \frac{\ln \boldsymbol{N}_a^{(\ell)}}{\ln \boldsymbol{R}^{(\ell-1)}}.\end{aligned}$$

Since $1-\beta_\ell \boldsymbol{N}_b^{(\ell)} < 1$ and $\boldsymbol{R}^{(\ell-1)} < 1$ we deduce that

$$\frac{1}{\ln(\sqrt{2}r)}\ln\Big(\mu\big(\mathcal{B}(x,\sqrt{2}r)\cap\mathcal{K}_\omega\big)\Big) \geq -\frac{\ln \boldsymbol{N}_a^{(\ell)}}{\ln \boldsymbol{R}^{(\ell-1)}}.$$

The same proof works if $\tilde{\mu}_\omega(C) = \beta_\ell$, and we obtain

$$\frac{1}{\ln(\sqrt{2}r)}\ln\Big(\mu\big(\mathcal{B}(x,\sqrt{2}r)\cap\mathcal{K}_\omega\big)\Big) \geq -\frac{\min(\ln \boldsymbol{N}_a^{(\ell)}, \ln \boldsymbol{N}_b^{(\ell)})}{\ln \boldsymbol{R}^{(\ell-1)}}.$$

Finally we deduce that

$$\liminf_{r\to 0} \frac{\ln \mu\big(\mathcal{B}(x,r)\cap\mathcal{K}_\omega\big)}{\ln r} \geq \liminf_{\ell} -\frac{\min(\ln \boldsymbol{N}_a^{(\ell)}, \ln \boldsymbol{N}_b^{(\ell)})}{\ln \boldsymbol{R}^{(\ell-1)}}.$$

Due to Lemma 6.4, if $\omega \in \Omega'$, the expressions converge to the same value $\boldsymbol{s}(\omega)$. □

Then by Lemma 6.3 we obtain:

Corollary 6.8. *For $\omega \in \Omega'$ we have $\dim_H(\mathcal{K}_\omega) \geq \boldsymbol{s}$.*

6.4. **Majoration of the Hausdorff dimension in Theorem 2.10.** We refer to Appendix 6.1 for a quick background on the different notions and notation.

Lemma 6.9. *For $\omega \in \Omega'$, $\dim_H(\mathcal{K}_\omega) \leq \underline{\dim}_B(\mathcal{K}_\omega) \leq \overline{\dim}_B(\mathcal{K}_\omega) \leq \boldsymbol{s}$.*

Proof. We want to obtain a majoration of $-\liminf_{r\to 0} \dfrac{\ln \mathrm{Numb}(r)}{\ln r}$.

Remark that if $0 < r < s$ and $\mathcal{K}_\omega \subset \bigcup \mathcal{B}(x_i, r)$, then $\mathcal{K}_\omega \subset \bigcup \mathcal{B}(x_i, s)$, and thus we have $\mathrm{Numb}(r) \geq \mathrm{Numb}(s)$. Let us fix $r > 0$ and denote by ℓ the integer such that $\boldsymbol{R}^{(\ell)} \leq r \leq \boldsymbol{R}^{(\ell-1)}$. Now we claim the following fact (proved at the end):

$$\mathrm{Numb}\,(\boldsymbol{R}^{(\ell)}) \leq \ln\left\|\boldsymbol{M}(\boldsymbol{S}^\ell\omega)\times\cdots\times\boldsymbol{M}(\omega)\begin{pmatrix}1\\1\end{pmatrix}\right\|_1 = \ln \boldsymbol{N}^{(\ell)}(\omega).$$

We deduce for $\boldsymbol{R}^{(\ell)} \leq r \leq \boldsymbol{R}^{(\ell-1)}$:

$$-\frac{\ln \mathrm{Numb}(r)}{\ln r} \leq -\frac{\ln \boldsymbol{N}^{(\ell)}}{\ln \boldsymbol{R}^{(\ell-1)}},$$

obtaining

$$\overline{\dim}_B(\mathcal{K}_\omega) \leq -\limsup \frac{\ln \boldsymbol{N}^{(\ell)}}{\ln \boldsymbol{R}^{(\ell-1)}}.$$

We finish with a proof of the claim. By Lemma 3.6 the set $\mathcal{K}_\omega(\boldsymbol{p}_\ell)$ can be covered by $\boldsymbol{m}_{1,1}^{(\ell)}(\omega)+\boldsymbol{m}_{2,1}^{(\ell)}(\omega)$ images of $\mathcal{C}$ and $\boldsymbol{m}_{1,2}^{(\ell)}(\omega)+\boldsymbol{m}_{2,2}^{(\ell)}(\omega)$ images of $\mathcal{R}_{\theta_\ell}$ by some similitudes of ratio $\boldsymbol{R}^{(\ell)}$. Since $\mathcal{R}_{\theta_\ell}$ can be covered by one square of size 1, we deduce that $\mathcal{K}_\omega(\boldsymbol{p}_\ell)$ can be covered by $\boldsymbol{N}^{(\ell)}(\omega)$ squares of size $\boldsymbol{R}^\ell$. Each such square is covered by one ball of the same radius, and we have $\mathcal{K}_\omega \subset \mathcal{K}_\omega(\boldsymbol{p}_\ell)$. This finishes the proof of the claim. □

6.5. **Conclusion.**

Proposition 6.10. *For all $\omega \in \Omega'$, $\dim_H(\mathcal{K}_\omega) = \boldsymbol{s}$.*

Proof. We have proved the lower bound in Corollary 6.8 and the upper bound in Lemma 6.9. □

7. Numerical values

7.1. **For $\boldsymbol{\nu}$-almost parameter.**

Lemma 7.1. *For $\boldsymbol{\nu}$-almost $\omega \in \Omega$,*

$$\boldsymbol{\lambda} \le \int \ln \|\boldsymbol{M}(\omega)\|_\infty d\boldsymbol{\nu}(\omega) \le 3.8.$$

Proof. Let $u \in (\mathbb{R}_+)^2$ and $\omega \in \Omega'$ such that

$$\frac{1}{\ell}\sum_{k=0}^{\ell} \ln \|\boldsymbol{M}(\boldsymbol{S}^k\omega)\|_\infty \to \int \ln \|\boldsymbol{M}(\omega)\|_\infty d\boldsymbol{\nu}(\omega) \text{ and } \frac{1}{\ell}\ln \|\boldsymbol{M}^{(\ell)}(\omega)u\|_\infty \to \boldsymbol{\lambda}.$$

Then, for each ℓ:

$$\frac{1}{\ell}\ln \|\boldsymbol{M}^{(\ell)}(\omega)u\|_\infty \le \frac{1}{\ell}\ln\|u\|_\infty + \frac{1}{\ell}\sum_{k=0}^{\ell}\ln\|\boldsymbol{M}(\boldsymbol{S}^k\omega)\|_\infty.$$

The limit of the different terms when ℓ tends to infinity gives the result. We refer to Lemma B.8 for the computation of the integrals. □

First we recall a result of [23] and [24] (Proposition 1.11, p. 25):

Lemma 7.2. *Consider the function*

$$f : \begin{array}{ccc} \Omega & \to & \mathbb{R} \\ \omega & \mapsto & \sqrt{\boldsymbol{m}_{1,1}(\omega)\boldsymbol{m}_{2,2}(\omega)} + \sqrt{\boldsymbol{m}_{1,2}(\omega)\boldsymbol{m}_{2,1}(\omega)}. \end{array}$$

Then we have

$$\boldsymbol{\lambda} = \lim_\ell \frac{1}{\ell}\int \ln\|\boldsymbol{M}^{(\ell)}(\omega)\| d\boldsymbol{\nu}(\omega) \ge \int \ln f(\omega) d\boldsymbol{\nu}(\omega).$$

Corollary 7.3. *For $\boldsymbol{\nu}$-almost $\omega \in \Omega$, we obtain*

$$2.66 \le \int \ln f(\omega) d\boldsymbol{\nu}(\omega) \le \boldsymbol{\lambda}.$$

Proof. Due to Proposition 4.7 the system is ergodic. Then we apply the Furstenberg-Kesten Theorem (Theorem B.3 and Corollary B.4) in order to get

$$\boldsymbol{\lambda}(\omega) = \lim_\ell \frac{1}{\ell}\int \ln\|\boldsymbol{M}^{(\ell)}(\omega)\| d\boldsymbol{\nu}(\omega).$$

Now we use Lemma 7.2. The rest of the proof is a simple computation. □

n	1	2	3	4	5
Hausdorff dimension of $\mathcal{K}_{(\alpha_{2n},-1)}$	1.637 938	1.450 998	1.370 279	1.325 467	1.296 563
Hausdorff dimension of $\mathcal{K}_{(\beta_{2n+1},1)}$	1.338 499	1.300 488	1.276 470	1.259 479	1.246 613

FIGURE 11. Approximation of the Hausdorff dimensions.

Corollary 7.4. *We obtain for $\boldsymbol{\nu}$ almost all ω:*

$$1.07 \leq \boldsymbol{s} \leq 1.55.$$

Proof. Recall that by Definition 6.6 we have $\boldsymbol{s} = \frac{\boldsymbol{\lambda}}{\ln \boldsymbol{R}}$. We use Lemma 7.1; the numerical values are obtained from equation (20) in Lemma B.8. We deduce that

$$\boldsymbol{s} \leq \frac{\int \ln \|\boldsymbol{M}(\omega)\|_\infty d\boldsymbol{\nu}(\omega)}{\int \ln\big(\boldsymbol{R}(\omega)\big) d\boldsymbol{\nu}(\omega)} \leq \frac{3.8}{2.46} \leq 1.55.$$

Then Lemma 7.3 gives

$$1.07 \leq \frac{2.66}{2.47} \leq \frac{\int \ln f(\omega) d\boldsymbol{\nu}(\omega)}{\int \ln\big(\boldsymbol{R}(\omega)\big) d\boldsymbol{\nu}(\omega)} \leq \boldsymbol{s}.$$

□

7.2. **Self-similar points.** A straightforward computation shows that the fixed points of S and $\boldsymbol{S}$ are of the following forms:

$$\Big(\sqrt{n^2+1}-n,-1\Big) \quad \text{and} \quad \Big(\sqrt{n(n+2)}-n,1\Big) \quad \text{for} \quad n \in \mathbb{N}.$$

Let us define the sequences $\alpha_n = \frac{\sqrt{n^2+4}-n}{2}, \beta_n = \frac{1-n+\sqrt{(n-1)(n+3)}}{2}$. This allows us to denote these fixed points as

$$(\alpha_{2n},-1) \quad \text{and} \quad (\beta_{2n+1},1).$$

Let us compute the Hausdorff dimension for the first family. We apply Theorem 2.10:

$$\dim_H(\mathcal{K}_{(\alpha_{2n},-1)}) = \lim_\ell -\frac{\ln\big\|M^{(\ell)}(\alpha_{2n},-1)\big\|}{\ell \ln(\alpha_{2n})} = \lim_\ell -\frac{\ln\big\|M^{\ell}(\alpha_{2n},-1)\big\|}{\ell \ln(\alpha_{2n})}.$$

If λ_ω is the dominant eigenvalue of the matrix $M(\omega)$ we obtain

$$\dim_H(\mathcal{K}_\omega) = -\frac{\ln|\lambda_\omega|}{\ln(\alpha_{2n})}.$$

We conclude by the computation of the dominant eigenvalue of $\left(\begin{smallmatrix} 2n-1 & 2 \\ n & 1 \end{smallmatrix}\right)$.

Figure 11 shows the numerical values of the dimension. We can compare with the second statement of Theorem 2.10.

APPENDIX A. THE RENORMALIZATION MAP

A.1. **Invariant measure for a continued fraction algorithm.** We recall the method introduced by Arnoux and Nogueira [2] and developed in Arnoux-Schmidt; see [3]. We consider a measure-preserving dynamical system (T, I, B, ν), where $T : I \to I$ is a measurable map on the measurable space (I, B) which preserves

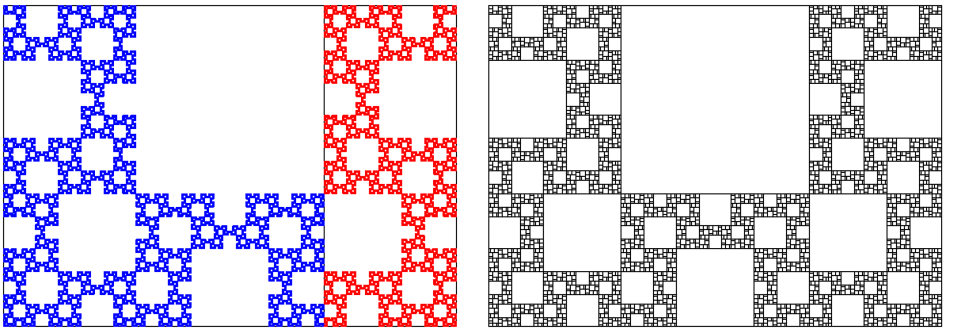


FIGURE 12. The sets $\mathcal{K}_\omega$ and $\mathcal{D}_\omega$ for $\omega = (\alpha_2, -1)$.

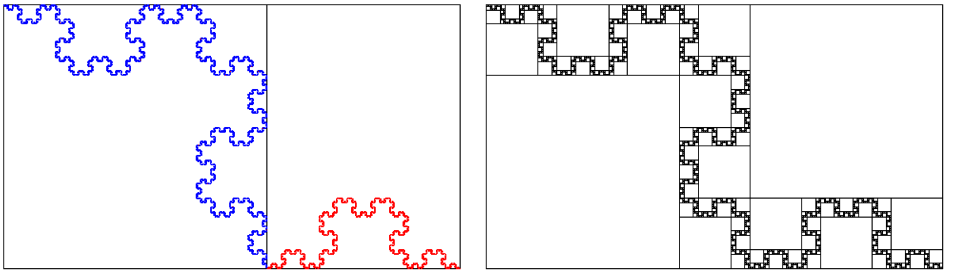


FIGURE 13. The sets $\mathcal{K}_\omega$ and $\mathcal{D}_\omega$ for $\omega = (\beta_3, 1)$.

the measure ν (ν is usually a probability measure). A natural extension of the dynamical system (T, I, B, ν) is an invertible system $(\tilde{T}, \Theta, B', \mu)$ with a surjective projection $\pi : \Theta \to I$ making (T, I, B, ν) a factor of $(\tilde{T}, \Theta, B', \mu)$, and such that any other invertible system with this property has its projection factoring through $(\tilde{T}, \Theta, B', \mu)$. The natural extension of a dynamical system exists always and is unique up to measurable isomorphism; see [17]. Informally, the natural extension is given by appropriately giving to (forward) T-orbits an infinite (in general) past; an abstract model of the natural extension is easily built using inverse limits. There is an efficient heuristic method for explicitly determining a geometric model of the natural extension of an interval map when this map is given (piecewise) by Mobius transformations. If the map is given by generators of a Fuchsian group of finite covolume, then one can hope to realize the natural extension as a factor of a section of the geodesic flow on the unit tangent bundle of the hyperbolic surface uniformized by the group.

Definition A.1. An interval map $f : I \to I$ is called a **piecewise Mobius map** if there is a partition of I into intervals $I = \bigcup I_\alpha$ and a set $M := \{M(\alpha)\}$ of elements $PSL(2, \mathbb{R})$ such that the restriction of f to I_α is exactly given by $x \in I_\alpha \to M(\alpha).x$. We call the subgroup of $PSL(2, \mathbb{R})$ generated by M the group generated by f and denote it by Γ_f.

We will use the following classical result; see [2].

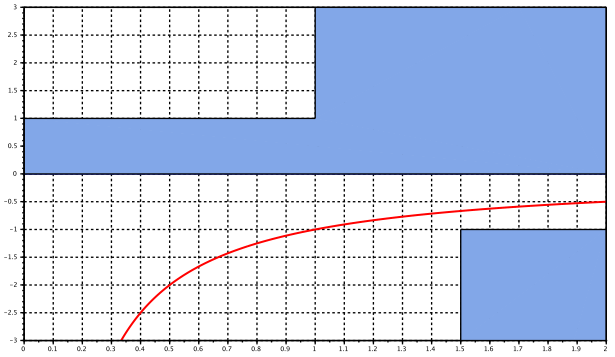


FIGURE 14. A part of the domain $\boldsymbol{\Theta}$ and the red curve $y = -1/x$.

Proposition A.2. *Assume $d\nu$ is a finite nonzero measure, S invariant, such that the measure of a set is equal to the measure of its closure. There exists $\Theta \subset \mathbb{R}^2$ such that dynamical system $(\tilde{S}, \Theta, d\mu)$ is a natural extension of the dynamical system $(S, [0,2], d\nu)$ where:*

- *The map $\tilde{S}$ is piecewise defined by the following formula where $A \in SL_2(\mathbb{Z})$:*

$$\begin{array}{rccc} \tilde{S}: & \Theta & \to & \Theta \\ & (x,y) & \mapsto & \left(A(x).x, \dfrac{-1}{A(x).(-1/y)}\right) \end{array}.$$

- *The map $\tilde{S}$ has an invariant measure given by $d\mu = \dfrac{dxdy}{(1+xy)^2}$.*

A.2. **Invariant measure for the accelerated renormalization map.** It remains to find a domain $\boldsymbol{\Theta}$ where this map is bijective. The following figure describes this domain:

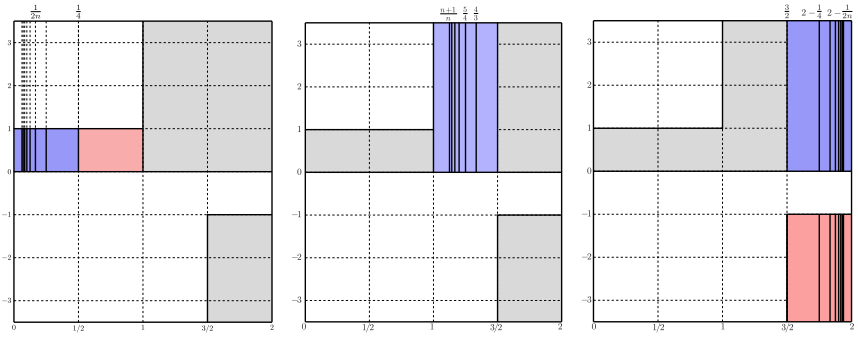
$$\begin{array}{ccc} \Theta & \xrightarrow{\quad\tilde{\boldsymbol{S}}\quad} & \Theta \\ \pi \big\downarrow & & \big\downarrow \pi \\ [0,2] & \xrightarrow[\quad\boldsymbol{S}\quad]{} & [0,2] \end{array}$$

Lemma A.3. *The domain $\boldsymbol{\Theta} = [0,1] \times [0,1] \cup [1,2] \times \mathbb{R}^+ \cup [3/2,2] \times [-1,+\infty)$ is invariant for the application $\tilde{\boldsymbol{S}}$ defined by*

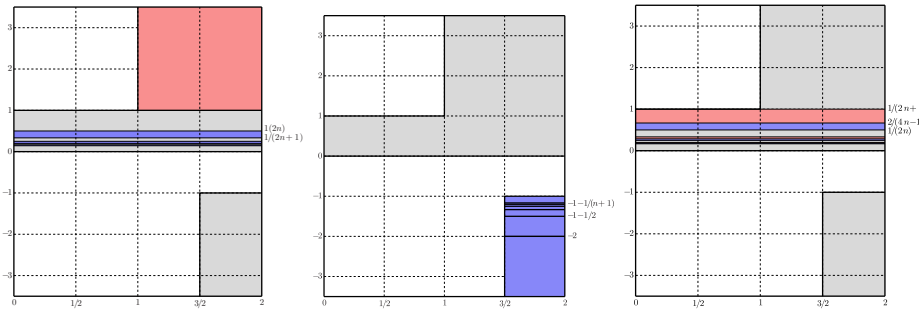
$$(x,y) \mapsto \left(\boldsymbol{A}(x).x, \frac{-1}{\boldsymbol{A}(x).(-\frac{1}{y})}\right).$$

Moreover the map $\tilde{\boldsymbol{S}} : \boldsymbol{\Theta} \mapsto \boldsymbol{\Theta}$ is bijective.

Proof. The proof is based on the following diagrams. The three following images form a partition of Θ.



We represent the corresponding images by $\tilde{\boldsymbol{S}}$ in the following images. They also form a partition of Θ. This proves that $\tilde{\boldsymbol{S}}$ is a bijection on Θ.



□

Appendix B. Lyapunov exponent and cocycles

B.1. **Background on the Lyapunov exponent.** We refer to [6].

Definition B.1. A cocycle of the dynamical system (X, T) is a map $M : X \times \mathbb{N} \to GL_2(\mathbb{R})$ such that

- $M(x, 0) = Id$ for all $x \in X$,
- $M(x, n + m) = M(T^n(x), m)M(x, n)$ for all $x \in X$ and $n, m \in \mathbb{N}$.

Theorem B.2 (Oseledets)**.** *Let (X, T) be a dynamical system and μ be an invariant probability measure for this system. Let M be a cocycle over T such that for each $n \in \mathbb{N}$ the maps $x \mapsto \ln ||M(x, 1)||, x \mapsto \ln ||M(x, -1)||$ are L^1 integrable with respect to μ.*

Then there exist a measurable map Z from X to $\mathbb{R}^2 \setminus \{0\}$ and two functions λ_+ and λ_- from X to $\mathbb{R}$ which are T-invariant, such that $\lambda_+ \geq \lambda_-$ and for μ almost all x and for every nonzero vector $z \in \mathbb{R}^2$:

$$\begin{cases} \lim \frac{1}{n} \ln ||M(x, n)z|| \underset{n \to +\infty}{\longrightarrow} \lambda_-(x) & \text{if } z \in \ vect\Big(Z(x)\Big), \\ \lim \frac{1}{n} \ln ||M(x, n)z|| \underset{n \to +\infty}{\longrightarrow} \lambda_+(x) & \text{if } z \notin \ vect\Big(Z(x)\Big). \end{cases}$$

The numbers $\lambda_\pm(x)$ are called **Lyapunov exponents** of the cocycle; see [14] and [9].

Theorem B.3 (Furstenberg-Kesten)**.** *Let (X, μ, T) be a dynamical system and μ be an invariant probability measure for this system. Suppose that* $\log ||M(x,1)||$ *and* $\log ||M(x,-1)||$ *are $L^1(\mu)$ integrable. Then, for μ almost every x:*

$$\lambda_+(x) = \lim_n \frac{1}{n} \ln \|M(x,n)\| \quad \text{and} \quad \lambda_-(x) = \frac{1}{n} \lim_n \ln \|M(x,n)^{-1}\|^{-1}.$$

The functions λ_+ and λ_- are T-invariant and

$$\int \lambda_+ d\mu = \lim_n \frac{1}{n} \int \ln \|M(x,n)\| d\mu(x)$$

and

$$\int \lambda_- d\mu = \lim_n \frac{1}{n} \int \ln \|M(x,n)^{-1}\|^{-1} d\mu(x).$$

Corollary B.4. *If μ is an ergodic measure for (X, T), then λ_+ and λ_- are constant μ almost everywhere and*

$$\lambda_+ = \lim_n \frac{1}{n} \int \ln \|M(x,n)\| d\mu(x) \quad \text{and} \quad \lambda_- = \lim_n \frac{1}{n} \int \ln \|M(x,n)^{-1}\|^{-1} d\mu(x).$$

B.2. **Special case of cocyles in positive matrices in $GL_2(\mathbb{Z})$.** Assume now that each matrix has nonnegative coefficients. Then it is clear that for almost every x:

$$\text{vect}\Big(Z(x)\Big) \cap \Big(\mathbb{R}_+^*\Big)^2 = \emptyset. \tag{18}$$

Recall the metric on $\mathbb{PR}^2$ given by

$$d\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix}\right) = \log \frac{\max(x/z, y/t)}{\min(x/z, y/t)}.$$

Now if M is a positive matrix, then the **Lipschitz constant** of M is defined by the following formula and is less than 1; see [5, p. 220]:

$$L(M) = \max \frac{d(Mu, Mv)}{d(u,v)} = \tanh\left(\frac{1}{4} \ln\left(\frac{ad}{bc}\right)\right) \leq 1.$$

Thus we deduce (see also [22]):

Lemma B.5. *Let $(M_n)_n$ be a sequence of positive matrices. If* $\lim_n L(M_1 \dots M_n) = 0$*, then there exists Z such that for every $z \in \mathbb{R}_+^2$ the sequence $M_1 \dots M_n z$ converges to Z.*

Theorem B.6. *Let (X, μ, T) be a dynamical system and μ be an ergodic probability measure for this system. Let $MX \to GL_2(\mathbb{Z}) \cap \mathcal{M}_2(\mathbb{N})$ be a measurable map.*

We suppose that for almost each $x \in X$, $M(x)$ is hyperbolic.

Then, there exists a measurable map $x \to z_x$ such that for almost each $z \in \mathbb{R}_+^2$,

$$\lim_{n \to +\infty} M(x) \times \dots \times M(T^n x)(\mathbb{R}_+^2) = \mathbb{R}_+ \cdot z_x.$$

We recall that a matrix is hyperbolic if $|tr(M)| > 2$.

Remark that a compact subgroup of $GL_2(\mathbb{R})$ is included up to conjugation in $O(2)$. Thus we can replace one hypothesis by for each integer n, M_n is a positive matrix and

$$\mu\Big(x \in X; M(x) \text{ is hyperbolic}\Big) > 0. \tag{19}$$

B.3. **Cocycles over the system (Ω, S, ν) and over $(\Omega, \boldsymbol{S}, \boldsymbol{\nu})$.** The next lemma explains why we do not work with the system (Ω, S, ν).

Lemma B.7. *The maps $\omega \mapsto \ln \|M(\omega)\|$ and $\omega \mapsto \ln \left\|M(\omega)^{-1}\right\|$ are not L^1 integrable with respect to ν.*

Proof. The matrices $M(\omega)$ are in $\mathcal{M}_2(\mathbb{N})$ and then $\ln \|M_\omega\|_1$ is positive.

We consider the identification between Ω and $[0,2]$. Then $\ln \|M(\omega)\|_1 = \ln(2)$ if $\omega \in [1, 3/2]$ and a density of ν is $\frac{1}{x(x-1)}$ on $[1, 3/2]$. □

Lemma B.8. *We have:*

1. *The function $\omega \to \ln \boldsymbol{r}$ is in $L^1(\boldsymbol{\nu})$.*
2. *The functions $\omega \to \ln \|\boldsymbol{M}(\omega)\|$ and $\omega \to \ln \|\boldsymbol{M}(\omega)^{-1}\|$ are in $L^1(\boldsymbol{\nu})$.*

Moreover

$$\int \ln \|\boldsymbol{M}(\omega)\|_\infty d\boldsymbol{\nu}(\omega) \leq 3.8 \quad \text{and} \quad 2.46 \leq \int \ln \boldsymbol{r}(\omega) d\boldsymbol{\nu}(\omega) \leq 2.47. \tag{20}$$

Proof. We find that

$$\begin{aligned}
\int \ln \|\boldsymbol{M}(x)\|_\infty d\boldsymbol{\nu}(x) &= \int \ln \circ \max\Big(\boldsymbol{m}_{1,1}(x) + \boldsymbol{m}_{1,2}(x), \boldsymbol{m}_{2,1}(x) + \boldsymbol{m}_{2,2}(x)\Big) d\boldsymbol{\nu}(x) \\
&= \sum_{n\geq 1} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \ln \circ \max\Big(2n+1, n+1\Big) \frac{1}{x+1} dx \\
&\qquad + \sum_{n\geq 2} \int_{1+\frac{1}{n+1}}^{1+\frac{1}{n}} \ln \circ \max\Big(2n-1, 1\Big) \frac{1}{x} dx \\
&\qquad + \sum_{n\geq 2} \int_{2-\frac{1}{n}}^{2-\frac{1}{n+1}} \ln \circ \max\Big(2n+1, n\Big) \frac{1}{x-1} dx \\
&= \sum_{n\geq 1} \ln(2n+1) \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1}{x+1} dx + \sum_{n\geq 2} \ln(2n-1) \int_{1+\frac{1}{n+1}}^{1+\frac{1}{n}} \frac{1}{x} dx \\
&\qquad + \sum_{n\geq 2} \ln(2n+1) \int_{2-\frac{1}{n}}^{2-\frac{1}{n+1}} \frac{1}{x-1} dx
\end{aligned}$$

and

$$\begin{aligned}
\int \ln \boldsymbol{r}(x) d\boldsymbol{\nu}(x) &= \int_0^1 \ln \boldsymbol{r}(x) \frac{1}{x+1} dx + \int_1^{3/2} \ln \boldsymbol{r}(x) \frac{1}{x} dx + \int_{3/2}^2 \ln \boldsymbol{r}(x) \frac{1}{x-1} dx \\
&= -\int_0^1 \frac{\ln(x)}{x+1} dx + \sum_{n\geq 2} \int_{1+\frac{1}{n+1}}^{1+\frac{1}{n}} \ln \boldsymbol{r}(x) \frac{1}{x} dx - \int_{3/2}^2 \frac{\ln(2-x)}{x-1} dx \\
&= -\int_0^1 \frac{\ln(x)}{x+1} dx - \sum_{n\geq 2} \int_{1+\frac{1}{n+1}}^{1+\frac{1}{n}} \frac{\ln\big(n-(n-1)x\big)}{x} dx \\
&\quad - \textstyle\int_{3/2}^2 \frac{\ln(2-x)}{x-1} dx.
\end{aligned}$$

□

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