TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 1, January 2018, Pages 749–754
http://dx.doi.org/10.1090/tran/7376
Article electronically published on September 29, 2017

CORRIGENDUM TO "DIVERGENCE IN LATTICES IN SEMISIMPLE LIE GROUPS AND GRAPHS OF GROUPS"

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Abstract. The authors correct an error in Proposition 3.24 of their work *Divergence in lattices in semisimple Lie groups and graphs of groups*, Trans. Amer. Math. Soc. **362** (2010), no. 5, 2451–2505.

The goal of this corrigendum is to correct [DMS, Proposition 3.24] and the proofs of [DMS, Theorems 4.4 and 4.9], relying on that proposition. We thank the authors of [ADT], who kindly pointed out the mistake in [DMS, Proposition 3.24] to us.

In what follows we assume that we work in a fixed metric space (X, dist).

We call a *quasi-geodesic segment in* X a quasi-isometric embedding $\mathfrak{q} : [a, b] \to X$, where $a < b$ are two finite real numbers. We call a *bi-infinite quasi-geodesic in* X (or simply a *quasi-geodesic in* X) a quasi-isometric embedding $\mathfrak{q} : \mathbb{R} \to X$. Given a bi-infinite quasi-geodesic $\mathfrak{q}$, its restriction to an interval (finite or infinite) is called a *sub-quasi-geodesic of* $\mathfrak{q}$.

A quasi-geodesic (segment) is *Morse* if for every $L \geq 1$ and $C \geq 0$, every (L, C)-quasi-geodesic $\mathfrak{p}$ with endpoints on the image of $\mathfrak{q}$ is contained in the M-tubular neighborhood of $\mathfrak{q}$, where M depends only on L, C.

The corrected version of [DMS, Proposition 3.24] is given below. The first four equivalent conditions there are taken verbatim from [DMS]. In particular the concepts of quasi-path and quasi-length are defined in [DMS] (we do not need these in this corrigendum, so we skip the definitions).

Proposition 1. *Let X be a metric space and for every pair of points $a, b \in X$ let $L(a, b)$ be a fixed set of (λ, κ)-quasi-geodesics (for some constants $\lambda \geq 1$ and $\kappa \geq 0$) with endpoints a and b. Let $L = \bigcup_{a,b\in X} L(a, b)$.*

Let $\mathfrak{q}$ be a bi-infinite quasi-geodesic in X, and for every two points x, y on $\mathfrak{q}$ denote by $\mathfrak{q}_{xy}$ the maximal sub-quasi-geodesic of $\mathfrak{q}$ with endpoints x and y.

The following conditions are equivalent:

(1) *In every asymptotic cone of X, the ultralimit of $\mathfrak{q}$ is either empty or contained in a transversal tree for some tree-graded structure.*
(2) *$\mathfrak{q}$ is a Morse quasi-geodesic.*
(3) *For every $C \geq 1$ there exists $D \geq 0$ such that every path of length $\leq Cn$ connecting two points a, b on $\mathfrak{q}$ at distance $\geq n$ crosses the D-neighborhood of the middle third of $\mathfrak{q}_{ab}$.*

Received by the editors April 21, 2017, and, in revised form, August 1, 2017.
2010 *Mathematics Subject Classification.* Primary 20F67; Secondary 20F65.

(4) *For every $C \geq 1$ and natural $k > 0$ there exists $D \geq 0$ such that every k-piecewise L quasi-path $\mathfrak{p}$ that:*
   – *connects two points $a, b \in \mathfrak{q}$,*
   – *has quasi-length $\leq C\mathrm{dist}(a, b)$,*

   crosses the D-neighborhood of the middle third of $\mathfrak{q}_{ab}$.

(5) *For every $C \geq 1$ and every $\epsilon > 0$ there exists $D \geq 0$ such that for every $a, b \in \mathfrak{q}$ with $\mathrm{dist}(a, b) \geq D$, and every path $\mathfrak{p}$ connecting a, b of length $\leq C\mathrm{dist}(a, b)$, the sub-quasi-geodesic $\mathfrak{q}_{ab}$ is contained in the $(\epsilon\mathrm{dist}(a, b))$-neighborhood of $\mathfrak{p}$.*

Remarks 2.

(1) Properties (1)–(4) are as in [DMS, Proposition 3.24]; property (5) is modified. The initial version of (5) stated that, given an arbitrary $C \geq 1$, for every $a, b \in \mathfrak{q}$, and every path $\mathfrak{p}$ with endpoints a, b and of length $\leq C\mathrm{dist}(a, b)$, the sub-quasi-geodesic $\mathfrak{q}_{ab}$ would be contained in the D-neighborhood of $\mathfrak{p}$, with D a constant depending only on C. This property is however strictly stronger than the property of being a Morse quasi-geodesic. Indeed, an example of a Morse geodesic that does not satisfy the property above, provided in [ADT], is an arbitrary bi-infinite geodesic $\mathfrak{q} : \mathbb{R} \to \mathbb{H}^2$ in the hyperbolic plane $\mathbb{H}^2$: for an arbitrarily large integer n, the path $\mathfrak{p}_n$ joining $\mathfrak{q}(-n)$ to $\mathfrak{q}(n)$ obtained as the concatenation of $\mathfrak{q}$ restricted to the interval $[-n, -\log n]$, with half of the hyperbolic circle centered in $\mathfrak{q}(0)$ and of radius $\log n$, and with $\mathfrak{q}$ restricted to the interval $[\log n, n]$, has length at most Cn, for some fixed constant C independent of n, yet $\mathfrak{q}(0)$ is between the endpoints of $\mathfrak{p}_n$ on $\mathfrak{q}$, and at distance $\log n$ of $\mathfrak{p}_n$.

   The mistake is in the proof of the implication (1) $\to$ (5) (see [DMS, Proposition 3.24]), where it is assumed that for a sequence of paths $\mathfrak{p}_n$ connecting pairs of points a_n, b_n on the bi-infinite quasi-geodesic $\mathfrak{q}$, the ω-limit of $\mathfrak{p}_n$ in any asymptotic cone is a rectifiable path. This is not true in general: in the example above, for observation points x_n coinciding with the midpoints of the half-circle, the ω-limit of $\mathfrak{p}_n$ in the asymptotic cone $\mathrm{Con}^\omega(\mathbb{H}^2, (x_n), (\log n))$ is not rectifiable.

   If, on the other hand, one considers sequences of paths $\mathfrak{p}_n$ of lengths $\ell_n \to \infty$ and their ω-limits only in asymptotic cones with scaling sequence $\lambda_n = \ell_n$, then these ω-limits, when non-empty, are rectifiable paths.

(2) The first paper to provide a correct proof of the equivalence (2) $\leftrightarrow$ (3) in Proposition 1 is thus [ADT], since our initial proof of this implication relied on the wrong version of (5).

Proposition 1 contains all known characterizations of Morse quasi-geodesics, with one exception that we explain below, to complete the list. Moreover, this characterization plays a central part in the proof of (2) $\leftrightarrow$ (3) in [ADT]. We begin with some terminology.

Definition 3. Let $\mathfrak{q}$ be a quasi-geodesic in a metric space X, and let $\eta > 0$. The *η–nearest point projection of a point $x \in X$ on* $\mathfrak{q}$, denoted by $\mathrm{proj}_{\mathfrak{q}}^{\eta}(x)$, is the set of points $x' \in \mathfrak{q}$ such that $\mathrm{dist}(x, x') \leq \mathrm{dist}(x, \mathfrak{q}) + \eta$.

Definition 4 (Sub-linear contraction [ACGH])**.** We say that a quasi-geodesic $\mathfrak{q}$ in a geodesic metric space X is *uniformly sub-linearly contracting* if there exists some constant $\eta > 0$ such that for every sub-quasi-geodesic $\mathfrak{q}'$ of $\mathfrak{q}$, the projection

$\mathrm{proj}^{\eta}_{\mathfrak{q}'}$ is uniformly sub-linearly contracting: for every $\epsilon > 0$ there exists $D = D(\epsilon)$ (independent of the specific sub-quasi-geodesic $\mathfrak{q}'$ of $\mathfrak{q}$) such that for every $D' \geq D$ and every $x \in X$ with $\mathrm{dist}(x, \mathfrak{q}') \geq 2D'$, the union of all nearest point projections $\mathrm{proj}^{\eta}_{\mathfrak{q}'}(y)$ of points y in the ball $\mathrm{B}(x, D')$ has diameter at most $\epsilon D'$.

In what follows, in all arguments using uniform sub-linearly contracting properties, we drop the parameter η from the notation.

Theorem 5 (Theorem 1.4, [ACGH])**.** *Let X be a geodesic metric space, and let $\mathfrak{q}$ be a quasi-geodesic in it. The following are equivalent:*

(1) $\mathfrak{q}$ *is Morse.*
(2) $\mathfrak{q}$ *is uniformly sub-linearly contracting.*

Remark 6. If X is δ-hyperbolic, then every bi-infinite geodesic $\mathfrak{g}$ of X is uniformly sub-linearly contracting, and an even stronger property holds: there exist constants $P = P(\delta)$ and $M = M(\delta)$ such that for every $\mathfrak{g}$-geodesic (segment, infinite or bi-infinite), every $D > 0$, and every $x \in X$ with $\mathrm{dist}(x, \mathfrak{g}) > D + M$, the union of nearest point projections $\mathrm{proj}_{\mathfrak{g}}$ of points y of the ball $\mathrm{B}(x, D)$ to $\mathfrak{g}$ has diameter at most P ([Gro87], [CDP, Proposition 2.1 in Chapter 10]).

This clearly extends to (λ, κ)-quasi-geodesics, modulo increasing the constants M and P with additive constants depending on (L, C).

Proof of Proposition 1. The equivalence of the five properties follows from the implications $(2) \to (3) \to (4) \to (1) \to (2)$ and the equivalence $(1) \leftrightarrow (5)$.

For $(2) \to (3)$ we refer to [ADT].

$(3) \to (4)$ is obvious. The proof of $(4) \to (1)$ is correct.

$(1) \to (2)$. Suppose that there exist $\mu \geq 1, \nu \geq 0$ such that for every $k > 1$ there exists a (μ, ν)-quasi-geodesic $\mathfrak{p}_k$ joining two points on $\mathfrak{q}$ and there exists $x_k \in \mathfrak{p}_k$ at distance $d_k > k$ from $\mathfrak{q}$. We can assume that d_k is the maximal distance from a point of $\mathfrak{p}_k$ to $\mathfrak{q}$. For any ultrafilter ω, in the asymptotic cone $\mathrm{Con}^{\omega}(X, (x_k), (d_k))$, the ω-limit $\mathfrak{q}_{\omega}$ of $\mathfrak{q}$ is a transversal geodesic, by (1), and the ω-limit $\mathfrak{p}_{\omega}$ of the sequence $(\mathfrak{p}_k)$ is either a μ-bi-Lipschitz path with endpoints on $\mathfrak{q}_{\omega}$, or a μ-bi-Lipschitz ray with origin on $\mathfrak{q}_{\omega}$, or a μ-bi-Lipschitz bi-infinite path. In all three cases $\mathfrak{p}_{\omega}$ stays 1-close to $\mathfrak{q}_{\omega}$, and has one point $x^{\omega} = (x_k)^{\omega}$ at distance 1 from $\mathfrak{q}_{\omega}$. In the latter two cases, we can obtain a simple path with endpoints on $\mathfrak{q}_{\omega}$ by choosing a point x on the ray far enough from the origin (respectively two points x, y far enough from each other), joining them by a geodesic $[x, x']$ (respectively by two geodesics $[x, x']$ and $[y, y']$) to nearest points on $\mathfrak{q}_{\omega}$, and by replacing x (respectively x, y) with the farthest from them intersection point between a geodesic and $\mathfrak{p}_{\omega}$. We then get a contradiction as in the end of the proof of $(1) \to (5)$ of Proposition 1.

$(1) \to (5)$. Suppose there exist constants $C > 1$ and $c > 0$, and a sequence of paths $\mathfrak{p}_k$ connecting pairs of points a_k, b_k on $\mathfrak{q}$ with $d_k = \mathrm{dist}(a_k, b_k) \to \infty$ such that the length of each $\mathfrak{p}_k$ is at most $C\mathrm{dist}(a_k, b_k)$ and $\mathfrak{q}_{a_k b_k}$ is not in the cd_k-neighborhood of $\mathfrak{p}_k$. Let x_k be a point on $\mathfrak{q}_{a_k b_k}$ such that $\mathrm{dist}(x_k, \mathfrak{p}_k)$ is within distance 1 of the maximal possible value, and consider the asymptotic cone $\mathcal{C} = \mathrm{Con}^{\omega}(X, (x_k), (d_k))$. Then the point (x_k) in $\mathcal{C}$ is at distance $\geq c$ from the ω-limit $\mathfrak{p}_{\omega}$ of the sequence $(\mathfrak{p}_n)$. The ω-limit $\mathfrak{p}_{\omega}$ is a path of length at most C, in particular, it is rectifiable (see the end of Remark 2(1)). Therefore the end of the proof of $(1) \to (5)$ of Proposition 1 works and we get a contradiction.

(5) $\to$ (1). Suppose that (5) is true, but that the ω-limit $\mathfrak{q}_\omega$ of $\mathfrak{q}$ in some asymptotic cone

$$\mathcal{C} = \mathrm{Con}^\omega(X, (x_k), (d_k))$$

is not a transverse geodesic. Hence there exists a path $\mathfrak{p}_\omega$ connecting two distinct points $a_\omega = (a_k), b_\omega = (b_k)$ on $\mathfrak{q}_\omega$ and having no other common points with $\mathfrak{q}_\omega$. By approximating $\mathfrak{p}_\omega$ well enough with piecewise geodesics (where the geodesic pieces are ultralimits of geodesics), and by eventually replacing a_ω and b_ω with two points that are nearer to each other, we can assume that $\mathfrak{p}_\omega$ is itself an ω-limit of a sequence $(\mathfrak{p}_k)$ of piecewise geodesics, with a uniformly bounded number of geodesic pieces, $\mathfrak{p}_k$ with endpoints a_k and b_k on $\mathfrak{q}$. Moreover, the hypothesis that $\mathfrak{p}_\omega$ intersects $\mathfrak{q}_\omega$ only in its distinct endpoints implies that there exists a point $x_\omega = (x_k)^\omega$ on $\mathfrak{q}_\omega$ situated between a_ω and b_ω, and with $c = \mathrm{dist}(x_\omega, \mathfrak{p}_\omega) > 0$. We have that the lengths of the paths $\mathfrak{p}_k$ are at most $C\mathrm{dist}(a_k, b_k)$ ω-almost surely, for some $C > 1$, and that $\mathrm{dist}(x_k, \mathfrak{p}_k) \geq \frac{c}{2} d_k$ ω-almost surely. This contradicts property (5). $\square$

As mentioned before, property (5) from [DMS, Proposition 3.24] is used in the proofs of [DMS, Theorems 4.4 and 4.9]. In the proof of [DMS, Theorem 4.9], property (5) is used only in the following paragraph:

> Since $\mathfrak{q}$ is a k-piecewise hierarchy path, by property (T2) it is shadowed by a k-piecewise tight geodesic $\mathrm{proj}(\mathfrak{q})$ in $\mathcal{G}$ of length $\leq K_1 n$ (for some constant K_1) connecting $g^{-3n} \cdot o$ and $g^{3n} \cdot o$. The fact that geodesics in a hyperbolic graph are Morse and part (5) of [DMS, Proposition 3.24] imply that the sub-arc $[g^{-3n} \cdot o, g^{3n} \cdot o]$ in $\mathfrak{g}$ is contained in the D-tubular neighborhood of $\mathrm{proj}(\mathfrak{q})$ for some constant D. In particular $[g^{-n} \cdot o, g^n \cdot o]$ has a sub-arc $\mathfrak{g}'$ of length $\geq K_2 n$ (for some constant K_2) contained in the D-tubular neighborhood of one of the tight geodesic sub-paths $\mathfrak{t}$ of $\mathrm{proj}(\mathfrak{q})$. Notice that the length $|\mathfrak{t}|$ is $\geq K_2 n - 2D \geq K_3 n$ for some constant K_3 (since $n \gg 1$).

It is easy to see that "D-neighborhood" can be replaced by "$o(n)$-neighborhood". Thus the new property (5) suffices to prove [DMS, Theorem 4.9].

The proof of [DMS, Theorem 4.4] requires more modifications. Here is its formulation.

Theorem 7 (See Theorem 4.4 in [DMS])**.** *Let G be an infinite finitely generated group acting on an infinite hyperbolic uniformly locally finite connected graph X. Suppose that for some $\ell > 0$ the stabilizer of any pair of points $x, y \in X$ with $\mathrm{dist}(x, y) \geq \ell$ is finite of uniformly bounded size. Let g be a loxodromic element of G. Then the sequence $(g^n)_{n \in \mathbb{Z}}$ is a Morse quasi-geodesic in G. In particular, every asymptotic cone of G has cut-points.*

Proof. In what follows n is a large enough natural number.

As in the proof of [DMS, Theorem 4.4], we can assume that g stabilizes a geodesic $\mathfrak{p}$ in X and acts on $\mathfrak{p}$ with translation length $\tau > 0$. Rescaling the metric in X if necessary, we can assume that $\tau = 1$.

Let us fix a point o in $\mathfrak{p}$. Let λ be the maximal distance between o and $a \cdot o$, where a is any of the generators in a finite generating set of G. Consider the map π from G to X defined by $\pi(h) = h \cdot o$.

Take a path $\mathfrak{g}$ from g^{-3n} to g^{3n} in the Cayley graph of G such that the length of $\mathfrak{g}$ is at most Cn for some $C \geq 1$. We need to show (by property (3) of Proposition 1) that $\mathfrak{g}$ passes boundedly close to one of g^i where $-n \leq i \leq n$.

Consider the image $\pi(\mathfrak{g})$ of $\mathfrak{g}$ in X. By connecting consecutive points on $\pi(\mathfrak{g})$ with geodesics we turn it into a path in X which we shall denote by $\mathfrak{g}'$. This path connects two points on $\mathfrak{p}$, $a_n = \pi(g^{-3n})$ and $b_n = \pi(g^{3n})$. The length of $\mathfrak{g}'$ is at most $Cn\lambda$.

By Remark 6, there exist constants c, D_0 such that for all $D > D_0$ if a path $\mathfrak{h}$ in X connecting x and x' is of length $\leq \frac{D}{2}$ and the distance $\mathrm{dist}(x, \mathfrak{p})$ is greater than D, then the diameter of $\mathrm{proj}_{\mathfrak{p}}(\mathfrak{h})$ is at most c. Let k be any integer greater than 200. We can also assume that $\frac{c}{D_0} < \frac{1}{kC\lambda}$ and $D_0 > 2l$, $D_0 > 2\lambda$.

Take the $4D_0$-neighborhood N_1 of the axis $\mathfrak{p}$. If $\mathfrak{g}' \subset N_1$, then we are done. So suppose that $\mathfrak{g}' \setminus N_1$ is not empty. Then $\mathfrak{g}' \setminus N_1$ is a union of sub-paths $\mathfrak{h}_i$ connecting points t_i, t_i' of $\mathfrak{g}'$ such that $\mathrm{dist}(t_i, \mathfrak{p}) = 4D_0 = \mathrm{dist}(t_i', \mathfrak{p})$. Dividing each $\mathfrak{h}_i$ into sub-paths of lengths between D_0 and $2D_0$, we conclude that the diameter of $\mathrm{proj}_{\mathfrak{p}}(\mathfrak{h}_i)$ is at most $\frac{c}{D_0}|h_i|$ provided $|h_i| \geq 2D_0$. Let H be the set of those sub-paths $\mathfrak{h}_i$ whose lengths are at least $2D_0$. Then the projection of the union of the paths $\mathfrak{h} \in H$ is covered by a union Z of intervals of total length at most

$$\frac{c}{D_0} \sum_{\mathfrak{h} \in H} |\mathfrak{h}| \leq \frac{c}{D_0} |\mathfrak{g}'| \leq \frac{1}{k\lambda C} Cn\lambda = \frac{n}{k\tau}. \tag{1}$$

Now let N_2 be the $5D_0$-neighborhood of $\mathfrak{p}$ and consider the set of maximal sub-paths $\mathfrak{h}_i'$ of $\mathfrak{g}' \setminus N_2$. Note that each $\mathfrak{h}_i'$ is inside some $\mathfrak{h}_j$; moreover this $\mathfrak{h}_j$ must have length at least $2D_0$, so it belongs to H. Therefore the set $\mathrm{proj}_{\mathfrak{p}}(\bigcup \mathfrak{h}_i')$ is covered by Z.

Note that the length of the piece p' of the axis $\mathfrak{p}$ between $a_n = \pi(g^{-3n})$ and $b_n = \pi(g^{3n})$ is $6n + 1$.

Let $m = \lfloor 20D_0 \rfloor + 1$. Consider the arithmetic progression $P = -n, -n + m, -n + 2m, \ldots$ of numbers between $-n$ and n. The distance between any two points $\pi(g^j)$, $\pi(g^k)$, $k \neq j \in P$, is $m|k - j| \geq m$. The size of the set P is $> \frac{n}{m}$.

For every $i \in P$ such that $\pi(g^i) \in Z$ let $\mathfrak{z}_i$ be the maximal sub-geodesic of $\mathfrak{p}$ containing $\pi(g^i)$, not containing any other $\pi(g^j), j \in P$, and contained in Z. Note that paths $\mathfrak{z}_i$ may overlap. So the sum of lengths of all $\mathfrak{z}_i$ is at most twice the measure of Z. If $i \in P$ is such that $\pi(g^i) \notin Z$, then let $\mathfrak{z}_i$ be just the point $\pi(g^i)$. Let u_i be the length of $\mathfrak{z}_i$. Then by (1)

$$\sum u_i \leq \frac{2n}{k}. \tag{2}$$

Take any positive $\alpha < 1 - \frac{100}{k}$ (note that $1 - \frac{100}{k} > 0$ since $k > 200$). If there are fewer than $\alpha|P|$ consecutive pairs $i, i + m \in P$ such that $u_i, u_{i+m} \leq D_0$, then there are at least $\frac{1}{2}|P|(1 - \alpha)$ numbers u_i which are bigger than D_0, hence $\sum_{i \in P} u_i > \frac{n}{2m}(1 - \alpha)D_0 > \frac{n}{2\lfloor 20D_0 \rfloor} \frac{100}{k} D_0 > \frac{2n}{k}$, a contradiction with (2). Hence there are at least $\alpha \frac{n}{m}$ pairs of consecutive numbers $i, j = i + m \in P$ such that $u_i, u_j \leq D_0$. Let M be the set of these pairs of numbers and $(i, i + m) \in M$. For $s = i, i + m$ let $B_s = \mathrm{B}(\pi(g^s), 6D_0)$ be the ball in X of radius $6D_0$ around $\pi(g^s)$. Let U_i, U_{i+m} be π-preimages of B_i, B_{i+m}, respectively. The path $\mathfrak{g}'$ must visit each of the balls B_i, B_{i+m}. Hence the path $\mathfrak{g}$ must intersect both sets U_i, U_{i+m} at points w_i, w_{i+m}, respectively.

Note that the distance between any point from B_i to any point in B_{i+m} is greater than l since $D_0 > 2l$.

Since the sum of distances $\sum_{(i,j)\in M} \operatorname{dist}(w_i, w_j)$ does not exceed Cn we have that at least one of the distances $\operatorname{dist}(w_i, w_j)$, $(i,j) \in M$, must be smaller than $R = \frac{Cn}{\alpha \frac{n}{m}} = \frac{Cm}{\alpha}$ which does not depend on n. Note that R is not smaller than m, the distance between $g^i \cdot o$ and $g^{i+m} \cdot o$, because $\alpha \leq 1$ and $C \geq 1$.

We need to show that $\operatorname{dist}(w_i, g^i) = |g^{-i}w_i|$ is bounded by a constant not depending on n. We have that $g^{-i}w_i \cdot o \in B = \mathrm{B}(o, 6D_0)$ and there exists $v \in G$ of length at most R such that $g^{-i}w_i v \cdot o \in B' = \mathrm{B}(g^m \cdot o, 6D_0)$.

Let V be the set of all $h \in G$ such that $h \cdot o \in B$ and for some v, $|v| \leq R$, $hv \cdot o \in B'$. Note that V does not depend on n, so it is enough to show that V is a finite set.

Recall that in [DMS, Lemma 4.2] we defined the sets $V_{a,b}$, $a, b \in G \cdot o$, as

$$V_{a,b} = \{h \in G \mid h \cdot o = a, \exists v \in G, |v| \leq R, hv \cdot o = b\}.$$

The proof of [DMS, Lemma 4.2] applies for every metric space X (not only a tree) with l-acylindrical action of a group G. It shows that if $\operatorname{dist}(a,b) > l$, then $V_{a,b}$ has uniformly bounded diameter (depending only on R).

If we denote $a_h = h \cdot o$ and $b_h = hv \cdot o$, then $h \in V_{a_h, b_h}$. Since $a_h \in B, b_h \in B'$, $\operatorname{dist}(a_h, b_h) > l$. Hence V_{a_h,b_h} is a finite set. The number of possible such pairs (a_h, b_h) does not exceed the size of the direct product $B \times B'$ which is a finite set because X is a locally finite graph. Hence V is finite as required. □

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