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# LOCAL GEOMETRY OF THE $k$-CURVE GRAPH

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Abstract. Let $S$ be an orientable surface with negative Euler characteristic. For $k \in \mathbb{N}$, let $\mathcal{C}_k(S)$ denote the *k-curve graph*, whose vertices are isotopy classes of essential simple closed curves on $S$ and whose edges correspond to pairs of curves that can be realized to intersect at most $k$ times. The theme of this paper is that the geometry of Teichmüller space and of the mapping class group captures local combinatorial properties of $\mathcal{C}_k(S)$, for large $k$. Using techniques for measuring distance in Teichmüller space, we obtain upper bounds on the following three quantities for large $k$: the clique number of $\mathcal{C}_k(S)$ (exponential in $k$, which improves on previous bounds of Juvan, Malnič, and Mobar and Przytycki); the maximum size of the intersection, whenever it is finite, of a pair of links in $\mathcal{C}_k$ (quasi-polynomial in $k$); and the diameter in $\mathcal{C}_0(S)$ of a large clique of $\mathcal{C}_k(S)$ (uniformly bounded). As an application, we obtain quasi-polynomial upper bounds, depending only on the topology of $S$, on the number of short simple closed geodesics on any unit-square tiled surface homeomorphic to $S$.

## 1. Introduction

Let $S$ be an orientable surface with genus $g$ and with $p$ punctures or marked points. Define the *complexity* of $S$, denoted $\omega(S)$, to be equal to $3g + p - 4$. Then the *curve graph* of $S$, denoted $\mathcal{C}(S)$, is the graph whose vertices correspond to isotopy classes of essential simple closed curves on $S$ and such that there is an edge between isotopy classes that can be realized disjointly on $S$. The curve graph has deep connections to the geometry of Teichmüller space $\mathcal{T}(S)$ and to the mapping class group $\mathrm{Mod}(S)$ ([2], [10], [15], [16], [24], [26]). Indeed, as a metric space it is quasi-isometric to the *electrified Teichmüller space*, the space obtained from $\mathcal{T}(S)$ equipped with the Teichmüller metric by coning off, for each simple closed curve $\alpha$, the region associated to those hyperbolic surfaces on which $\alpha$ is very short ([15]). Moreover, the group of simplicial automorphisms of $\mathcal{C}(S)$ is isomorphic to $\mathrm{Mod}^{\pm}(S)$ ([8], [11], [13]), the extended mapping class group.

In this paper, we consider for each $k \in \mathbb{N}$, a variant of $\mathcal{C}(S)$ called the *k-curve graph*, denoted $\mathcal{C}_k(S)$: vertices are the same as $\mathcal{C}(S)$, and edges correspond to pairs of isotopy classes that can be realized with at most $k$ intersections. The large scale geometry of $\mathcal{C}_k(S)$ is well understood, because it is quasi-isometric to the standard curve graph $\mathcal{C}(S) = \mathcal{C}_0(S)$. However, the local combinatorics of $\mathcal{C}_k(S)$ and how they depend on $k$ remain largely unexplored. The theme of this paper is that large scale

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geometric features of $\mathcal{T}(S)$ can be translated into local geometric features of $\mathcal{C}_k(S)$ when $k$ is large.

1.1. **Cliques in $\mathcal{C}_k(S)$.** As motivation, we recall the following question, first popularized by Farb and Leininger:

**Question 1.** As a function of $\omega(S)$, what is the largest size of a collection $\Omega$ of pairwise non-homotopic, essential simple closed curves, such that no two curves in $\Omega$ intersect more than once?

Question 1 is surprisingly challenging and remains open, although progress has been made towards its resolution. Most recently, Przytycki has shown that any such $\Omega$ has size bounded above by an explicit function that grows as a cubic polynomial in $\omega(S)$ ([22]). On the other hand, it is not difficult to construct sequences of such collections whose cardinalities grow quadratically in $\omega(S)$ ([1], [14]); the exact growth rate remains an open problem. A natural generalization of Question 1 is to ask, as a function of $k \in \mathbb{N}$, for the largest size of a collection of (pairwise non-homotopic) simple closed curves, pairwise intersecting at most $k$ times. We call such a collection of curves a *k-system*. Replacing Question 1 with this generalization, we now consider a problem parametrized by two variables: $\omega(S)$ and $k$. That is, one can fix $k$ and ask for the size of the largest $k$-system as a function of complexity, one can fix the surface $S$ and vary $k$, or one can vary both simultaneously.

Przyticky's bounds apply to this generalization, and in particular his result states that the maximum size of a $k$-system on $S$ grows at most as a polynomial in $\omega(S)$ of degree $k^2 + k + 1$. Juvan-Malnič-Mohar have also considered this question, and when $k$ is very large compared to $\omega(S)$, they show that such a collection has size roughly at most $k^k$ ([9]). In this paper, we will focus on varying $k$ while keeping $S$ fixed.

Question 1 can be reinterpreted as asking for the largest size of a *clique*—a complete subgraph—in $\mathcal{C}_1(S)$, and thus its generalization asks for the largest clique size in $\mathcal{C}_k(S)$. The size of a largest clique in a graph is called the *clique number* of the graph. Our first result provides, for a fixed surface $S$, an upper bound for the clique number of $\mathcal{C}_k(S)$, which when $k$ is large, outperforms the bounds from [22] and from [9]:

**Theorem 3.1.** *Fix a surface $S$ with $\chi(S) < 0$, and let $N_S(k)$ denote the clique number of $\mathcal{C}_k(S)$. Then*

$$\log(N_S(k)) \prec k.$$

See subsection 2.1 below for the definition of $\prec$.

Thus, for a fixed surface $S$, $N_S(k)$ grows at most exponentially as a function of $k$. In [1] we showed, for each $g$, the existence of a complete subgraph of $\mathcal{C}_k(S_g)$ whose size was on the order of $g^{k/2}$, where $S_g$ is the closed surface of genus $g$. Combining this with Theorem 3.1, it follows that $N_S(k)$ grows exponentially in $k$, with base at least $\sqrt{\omega(S)}$.

1.2. **Intersections of links in $\mathcal{C}_k(S)$.** Given a simple closed curve $\alpha$ on $S$, the *k-link* of $\alpha$, denoted $\mathcal{L}_k(\alpha)$, is the sphere of radius 1 in $\mathcal{C}_k(S)$, centered at the vertex associated to $\alpha$. Let $\alpha, \beta$ be a pair of simple closed curves on $S$ in *minimal position*, meaning that $|\alpha \cap \beta|$ has been minimized over all possible choices of homotopic representatives for $\alpha$ or $\beta$. In this setting, we define $|\alpha \cap \beta|$ to be the *geometric intersection number* between $\alpha$ and $\beta$, and we write $|\alpha \cap \beta| := i(\alpha, \beta)$.

Then if $\alpha$ and $\beta$ *fill* $S$, that is, if $S \setminus (\alpha \cup \beta)$ is a disjoint union of topological disks, boundary parallel annuli, and once-punctured disks, then $|\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)|$ is finite, for all $k$. On the other hand, if $\alpha$ and $\beta$ do not fill, this intersection can be infinite. Our next result states that for fixed $k$, $|\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)|$ is *uniformly* bounded over all choices of filling pairs $\alpha, \beta$, and furthermore that this bound grows at most quasi-polynomially in $k$. A function $f : \mathbb{N} \to \mathbb{N}$ grows *at most quasi-polynomially* if there exists some positive $c, \lambda \geq 1$ so that

$$f(n) \leq 2^{(\lambda \cdot \log(n))^c}.$$

When $c = 1$ the left-hand side is bounded above by a polynomial of degree $\lceil \lambda \rceil$. We show:

**Theorem 4.1.** *There exists a function $r_S(k)$ depending only on the topology of $S$, which grows at most quasi-polynomially and which satisfies the following. Let $\alpha, \beta$ be simple closed curves on $S$ which fill $S$, and let $\mathcal{L}_k(\alpha)$ denote the set of all vertices in $\mathcal{C}_k(S)$ that are distance 1 from $\alpha$. Then $|\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)| \leq r_S(k)$.*

We remark that our methods also prove Theorem 4.1 when $\alpha$ and $\beta$ are allowed to be multi-curves, multi-arcs, or a combination of pairwise disjoint arcs and curves: there exists a uniform upper bound on the number of simple closed curves intersecting a filling multi-arc/curve pair at most $k$ times, and this bound grows at most quasi-polynomially in $k$.

In [1] we asked whether large $k$-systems project to small diameter subsets of the curve graph $\mathcal{C}_0(S)$. More concretely, a result of Hempel [5] (see also Lickorish [12]) implies that for any $\alpha, \beta$ simple closed curves with $i(\alpha, \beta) \leq k$,

$$d_{\mathcal{C}_0(S)}(\alpha, \beta) \leq 2 \log(k) + 2.$$

Thus, the diameter in $\mathcal{C}_0(S)$ of any $k$-system is at most roughly $\log(k)$, and we ask whether large $k$-systems can obtain this upper bound in diameter. Theorem 4.1 answers this in the negative; indeed, any large $k$-system projects to a diameter 2 subset of $\mathcal{C}_0(S)$:

**Corollary 4.2.** *Let $\Omega$ be a $k$-system on $S$ with $|\Omega| = N_k(S)$. Then for all sufficiently large $k$, $\Omega$ projects to a subset of the curve graph of diameter 2.*

1.3. **Unit-square tiled surfaces.** Assume $S$ is a closed surface. Another application of Theorem 4.1 is to bound the number of short simple closed curves on square-tiled surfaces homeomorphic to $S$. A *unit-square tiled surface* is a metric surface $\mathcal{S}$ obtained by gluing together finitely many copies of the unit square in $\mathbb{C}$ (see Section 4 for precise definitions). Such a surface admits a flat metric structure away from finitely many singularities, which occur at vertices around which more than four squares have been glued; the area of this metric is equal to the number of squares.

If $\mathcal{S}$ is a unit-square tiled surface, let $N_{\mathcal{S}}(L)$ denote the number of homotopy classes of simple closed curves admitting a representative on $\mathcal{S}$ with length at most

$L$. Finally, let $\mathcal{X}(S)$ denote the set of all unit-square tiled surfaces whose underlying topology is that of $S$. We show:

**Corollary 4.3.** *For $S$ a closed surface, there exists a function $P_S$ which grows at most quasi-polynomially, such that*

$$\sup\{N_{\mathcal{S}}(L) : \mathcal{S} \in \mathcal{X}(S)\} \leq P_S(L).$$

1.4. **The use of Teichmüller geometry.** As our main results are largely combinatorial, a natural question is to ask whether or not there exist purely combinatorial proofs. The author is unaware of such arguments or of any argument proving the main results that completely circumvent the use of Teichmüller geometry. The existence of such a combinatorial argument would be of independent interest and would follow a recurring theme in this research area: the first proof of an inherently combinatorial result relies heavily on Teichmüller theory, and much later a purely combinatorial argument is discovered. This is perhaps reflective of the fact that many of the most powerful and well-known tools for studying the combinatorics of $\mathcal{C}(S)$ and of $\mathrm{Mod}(S)$ reference Teichmüller geometry. For example,

(1) the original proof of the hyperbolicity of the complex of curves due to Masur-Minsky [15] and of the hyperbolicity of the arc graph by Masur-Schleimer [17] were reproved in a combinatorial fashion by Hensel-Przytycki-Webb [7];
(2) Choi-Rafi proved a purely combinatorial inequality (included below as (3.2)) relating subsurface projections to geometric intersection numbers using Teichmüller geometry, and this was recently reproved using only combinatorial methods by Watanabe [28].

While combinatorial proofs usually exist, Teichmüller geometry seems to capture well the intersection patterns of curves on surfaces, often in ways that make particularly efficient and concise arguments possible.

1.5. **Organization.** In section 2, we cover the necessary preliminaries. The material covered in this section is completely standard and well-known to experts, with the possible exception of subsection 2.5, in which we impose several conventions and prove a basic lemma that will simplify the proof of Theorem 3.1. In section 3, we prove Theorem 3.1. In section 4, we prove Theorem 4.1 and Corollary 4.3.

## 2. Preliminaries

2.1. **Notation and Coarseness.** Given two quantities (or functions) $f, g$, by $f \asymp_C g$ we mean

$$\frac{1}{C} f - C \leq g \leq C \cdot f + C. \tag{2.1}$$

In this paper, unless otherwise stated, the constant $C$ in relations as above will depend only on the topology of the underlying surface $S$. When the explicit constant $C$ is not of interest, we will suppress it by using the notation $f \asymp g$, meaning there exists some constant $C$ such that $f \asymp_C g$.

By $f \prec g$ (respectively $f \succ g$), we mean that there exists a constant $C$ such that the right-hand (resp. left-hand) inequality of (2.1) holds. We say $f$ is *coarsely less than* or *coarsely at most* $g$ to mean $f \prec g$, and that $f$ and $g$ are *coarsely equal* if $f \asymp g$.

By $f \prec^{+} g$ or $f \asymp^{+} g$, we mean that there exists some $C$ so that

$$f \leq g + C$$

or

$$f - C \leq g \leq f + C,$$

respectively.

Given metric spaces $X, Y$, and a map $f : X \to 2^Y$ from $X$ to the power set of $Y$, we say that $f$ is a *coarsely well-defined map* from $X$ to $Y$ if there exists a constant $K \geq 0$ so that for any $x \in X$,

$$\mathrm{diam}_Y(f(x)) < K.$$

Finally, we remark here that all logarithms in this paper will be base 2.

2.2. **Curves and arcs.** A *simple closed curve* on a surface $S$ is the image of an embedding $\phi : S^1 \to S$. All homotopies between pairs of curves or arcs are not required to fix boundary components pointwise. A curve is *essential* if it is not homotopically trivial and not homotopic into a neighborhood of a puncture or parallel to a boundary component. A *simple arc* on $S$ is a proper embedding of either a closed interval $[0, 1]$—and in this case the endpoints are mapped to boundary components—or an open interval. A simple arc is *essential* if it cannot be homotoped to lie within a neighborhood of a puncture or boundary component.

A multi-curve (or multi-arc) is a disjoint union of simple closed curves (respectively arcs).

Given two homotopy classes of curves or arcs $\alpha, \beta$, their *geometric intersection number*, denoted $i(\alpha, \beta)$, is defined as

$$i(\alpha, \beta) = \min_{x \sim \alpha, y \sim \beta} |x \cap y|,$$

where $\sim$ denotes homotopy. If curves $\alpha, \beta$ achieve the geometric intersection number associated to the corresponding pair of homotopy classes, we say they are in *minimal position*. A pair of simple closed curves $\alpha, \beta$ are in minimal position if and only if no connected component of $S \setminus (\alpha \cup \beta)$ is a bigon, which is a simply connected region bounded by one arc of $\alpha$ and one of $\beta$ (see section 1.2.4 of [4]).

A *curve system* is a collection of pairwise non-homotopic, pairwise in minimal position essential simple closed curves on $S$, and a *curve and arc system* is defined analogously. If $\Lambda, \Gamma$ are two curve systems, we say $\Lambda$ is homotopic to $\Gamma$ as curve collections if there is a bijection from $\Lambda$ to $\Gamma$ such that the image of each curve in $\Lambda$ is homotopic to it. Then we define the geometric intersection number $i(\Gamma, \Gamma')$ by

$$i(\Gamma, \Gamma') = \sum_{\gamma \in \Gamma, \gamma' \in \Gamma'} i(\gamma, \gamma').$$

A curve and arc system $\{\alpha_1, \ldots, \alpha_n\}$ in pairwise minimal position is said to *fill* $S$ if $S \setminus \bigcup_i \alpha_i$ is a disjoint union of topological disks, once-punctured disks, and boundary parallel annuli. Alternatively, the system fills if and only if any essential simple closed curve $\gamma$ has positive geometric intersection number with at least one element of the system.

2.3. **The mapping class group and the curve graph.** The *mapping class group* of $S$, denoted $\mathrm{Mod}(S)$, is the group of orientation preserving homeomorphisms of $S$ fixing the boundary pointwise, up to isotopy. The *extended mapping class group*, denoted $\mathrm{Mod}^{\pm}(S)$, is the group of all isotopy classes of homeomorphisms (orientation preserving or reversing) of $S$ fixing the boundary pointwise. In all that follows, suppose that we have equipped the surface $S$ with a fixed complete hyperbolic metric $\rho$ whenever $S$ admits such a metric.

If $\omega(S) > 0$, then the *curve graph* of $S$, denoted $\mathcal{C}(S)$, is the graph whose vertices correspond to isotopy classes of essential simple closed geodesics on $S$, and two vertices span an edge exactly when the corresponding geodesics are disjoint on $S$. If $S$ is a torus with 0 or 1 punctures, then vertices of $\mathcal{C}(S)$ are isotopy classes of essential simple closed curves, and adjacency corresponds to simple closed curves intersecting once. If $S$ is the 4-holed sphere, then adjacency corresponds to geodesics which intersect twice.

Finally, if $S$ is an annulus, then $S$ equipped with $\rho$ can be identified with the quotient of the hyperbolic plane by the action of an infinite cyclic subgroup $\Gamma$ of $PSL(2,\mathbb{R})$ generated by a hyperbolic matrix. Thus $S$ admits a compactification $\bar{S}$ compatible with the hyperbolic structure on $S$, obtained by adding in the domain of discontinuity for the action of $\Gamma$. Then vertices of $\mathcal{C}(S)$ correspond to geodesic simple arcs on $\bar{S}$ running from one boundary component to the other, and adjacency corresponds to disjointness. If $\alpha$ is the core curve of the annulus, we will sometimes refer to $\mathcal{C}(S)$ as $\mathcal{C}(\alpha)$.

The curve graph of any surface is made into a metric space by identifying each edge with $[0,1]$. Let $d_S(,)$ denote distance in $\mathcal{C}(S)$. The curve graph admits an isometric (but not properly discontinuous) action of $\mathrm{Mod}^{\pm}(S)$.

If $\omega(S) > 0$ or if $S$ is a punctured torus or 4-holed sphere, define $\mathcal{AC}(S)$, the *arc and curve graph of* $S$, to be the graph whose vertices correspond to isotopy classes of essential simple closed geodesics and geodesic arcs on $S$. As with $\mathcal{C}(S)$, two vertices are connected by an edge if and only if the corresponding geodesics can be realized disjointly. If $S$ is an annulus, set $\mathcal{AC}(S) := \mathcal{C}(S)$.

By a simple surgery argument, when $S$ is not an annulus, distance in $\mathcal{C}(S)$ is bounded above by a logarithmic function of intersection number ([5], [12]); given any two simple closed curves $\alpha, \beta$,

$$d_S(\alpha,\beta) \le 2\log(i(\alpha,\beta)) + 2. \tag{2.2}$$

When $S$ is an annulus, $\mathcal{C}(S)$ is quasi-isometric to $\mathbb{Z}$, and distance is coarsely measured by intersection number ([16]).

2.4. **Subsurface projections.** A *non-annular subsurface* $Y$ of $S$ is the closure of a complementary component of an essential multi-curve on $S$ which is not homeomorphic to a sphere with the sum of boundary components and punctures at most 3. An *annular subsurface* $Y \subseteq S$ is a closed neighborhood of an essential simple closed curve on $S$, homeomorphic to $[0,1] \times S^1$.

Let $Y \subseteq S$ be a subsurface of $S$, where $\omega(S) > 0$; as in the previous subsection, $S$ has been equipped with a complete hyperbolic metric $\rho$. Then there is a covering space $S^Y$, which is a non-compact hyperbolic surface after we lift $\rho$ to $S^Y$, associated to the inclusion $\pi_1(Y) < \pi_1(S)$. We can then compactify $S^Y$ as above, obtaining a compact surface $\overline{S^Y}$ by adding the domain of discontinuity for the action of $\pi_1(Y)$, and we note that $\overline{S^Y}$ is homeomorphic to $Y$. Via this homeomorphism, we identify

$\mathcal{AC}(Y)$ with $\mathcal{AC}\left(\overline{S^Y}\right)$. Then, given $\alpha \in \mathcal{AC}^0(S)$, we obtain a coarsely well-defined map $\pi_Y : \mathcal{AC}(S) \to \mathcal{AC}(Y)$ defined by setting $\pi_Y(\alpha)$ equal to the essential curves and arcs in its preimage under the covering map $S^Y \to S$.

When $Y$ is not an annulus, given an arc $a \in \mathcal{AC}(Y)$, there is a closely related simple closed curve $\tau(a) \in \mathcal{C}(Y)$ obtained from $a$ by surgering along the boundary components that $a$ meets. More concretely, let $\mathcal{N}(a)$ denote a regular neighborhood of the union of $a$ together with the (at most two) boundary components of $Y$ that $a$ meets, and define $\tau_Y(a) \in 2^{\mathcal{C}(S)}$ to be the essential components of $\partial(N(a))$; this is non-empty because $a$ is not boundary parallel and it is in minimal position with $\partial Y$ since it is a geodesic arc in the metric $\rho$. On the other hand, if $Y$ is an annulus, define $\tau_Y$ to be the identity map.

Thus we obtain a *subsurface projection*

$$\psi_Y := \tau \circ \pi_Y : \mathcal{C}(S) \to \mathcal{C}(Y)$$

for $Y \subseteq S$ any essential subsurface. In practice, to obtain $\psi_Y(\alpha)$, consider the intersection of $\alpha$ with $Y$. If $\alpha$ is contained completely within $Y$ and $Y$ is not an annulus, we define $\psi_Y(\alpha) = \alpha$; if $\alpha \cap Y = \emptyset$, then the projection $\psi_Y(\alpha)$ is undefined. Finally, if $\alpha \cap Y$ consists of a collection of arcs, define $\psi_Y(\alpha)$ to be the curves obtained by surgering those arcs via the process described in the previous paragraph. Note also that if $A$ is an annulus and $c$ is its core curve, the projection $\psi_A(c)$ is not defined. We will sometimes refer to $\psi_A$ as $\psi_\alpha$ where $\alpha$ is the core curve of $A$.

Given $\alpha, \beta \in \mathcal{C}(S)$, define $d_Y(\alpha, \beta)$ by

$$d_Y(\alpha, \beta) := \operatorname{diam}_{\mathcal{C}(Y)}(\psi_Y(\alpha) \cup \psi_Y(\beta)).$$

We note that $\psi_Y$ is coarsely Lipschitz (see Lemma 2.3 of [16]):

$$d_Y(\alpha, \beta) \leq 2 \cdot d_S(\alpha, \beta) + 2,$$

and furthermore, for any $\delta$ a component of $\psi_Y(\alpha)$ and $\eta$ a component of $\psi_Y(\beta)$,

$$i(\eta, \delta) \leq 4 \cdot i(\alpha, \beta) + 4. \tag{2.3}$$

Equation (2.3) follows from the fact that $\psi_Y(\alpha)$ traverses $\alpha \cap Y$ at most twice, one in each direction; the extra 4 intersections account for the portion of $\psi_Y(\alpha)$ that traverses part of $\partial Y$. See also Figure 7 of [16].

2.5. **The marking graph and the Masur-Minsky distance formula.** In [16], Masur-Minsky construct a locally finite connected graph, the *marking graph* $M(S)$ of a surface $S$, which they prove is quasi-isometric to a Cayley graph for the mapping class group $\operatorname{Mod}(S)$ with respect to any finite generating set. The vertices of $M(S)$ are so-called *complete clean markings*, which are certain curve systems that fill $S$, and edges correspond to *elementary moves* between such markings. As we will not need precise definitions of these terms here, we refer the reader to [16] for background on the marking graph.

Given $k \in \mathbb{N}$, a *k-marking* is a graph $\mu$ that fills $S$ (i.e., each complementary region of $\mu$ is simply connected or once-punctured) and such that $\mu$ has at most $k$ edges. Define the intersection between two graphs $i^{\mathcal{G}}(\Gamma_1, \Gamma_2)$ to be the minimum of $|\Gamma_1' \cap \Gamma_2'|$, where $\Gamma_i'$ is isotopic to $\Gamma_i$ (via an isotopy that is not required to fix vertices pointwise), and so that $\Gamma_1', \Gamma_2'$ share no vertices in common and all intersections are transverse.

A marking $\mu$, and in fact any curve system in pairwise minimal position, can be reinterpreted as a graph on the surface $S$ by defining the vertices to be intersections between curves in the system, and edges to be arcs of the curves running between intersection points. Choosing different isotopy representatives of the curves in the system will a priori result in different isomorphism types of graphs. We address this ambiguity by choosing once and for all a hyperbolic metric $\rho$ and considering and interpreting each curve system $\Gamma$ as a graph by considering the union of all geodesic representatives of the curves in $\Gamma$, after perhaps a slight modification described as follows. It will be useful to assume that $\Gamma$, when interpreted as a graph, has no triple points, meaning that each vertex is 4-valent. Thus, for any curve system having the property that three of its elements have geodesic representatives on $\rho$ which all meet in the same point, we slightly homotope one of these three curves to move it off the common intersection point. There is ambiguity in how to choose this homotopy, but for each curve system we make these choices as necessary and fix them once and for all.

The next lemma establishes that graph intersection number and geometric intersection number of curve systems are comparable:

**Lemma 2.1.** *If* $\Lambda = \{\alpha_1, \dots, \alpha_n\}, \Gamma = \{\beta_1, \dots, \beta_m\}$ *are two curve systems, then*

$$i(\Gamma, \Lambda) \leq i^{\mathcal{G}}(\Gamma, \Lambda) \leq \max(n, m) \cdot i(\Gamma, \Lambda).$$

*Proof.* That $i(\Gamma, \Lambda) \leq i^{\mathcal{G}}(\Gamma, \Lambda)$ follows from the fact that every isotopy of $\Gamma$ as a graph can be re-interpreted as an isotopy of each individual curve in $\Gamma$.

For the other inequality, consider the hyperbolic metric $\rho$ on $S$ described above, and realize the elements of $\Lambda$ and of $\Gamma$ by geodesics; let $\Gamma_\rho, \Lambda_\rho$ denote these geodesics, each interpreted as a graph with geodesic edges. By slightly homotoping the geodesic representatives so that they are not geodesic but are very nearly geodesic, we can assume that there are no triple points in $\Lambda_\rho \cup \Gamma_\rho$. This homotopy can be performed without altering the isotopy class of either graph. Then $i(\Gamma, \Lambda)$ will be equal to the set-theoretic intersection number $|\Gamma_\rho \cap \Lambda_\rho|$, which is also at least $i^{\mathcal{G}}(\Gamma, \Lambda)$ so long as $\Lambda$ and $\Gamma$ share no curves in common. Indeed, the computation of $i^{\mathcal{G}}$ requires that we choose representatives of each graph that share no edges in common; thus for each curve $c$ that $\Lambda$ and $\Gamma$ share, we must alter either $\Gamma_\rho$ or $\Lambda_\rho$ slightly by adding a parallel, nearly geodesic copy of $c$. This has the effect of doubling each intersection between $c$ and curves in either $\Gamma$ or $\Lambda$, and one has $i(c, \Gamma), i(c, \Lambda) \leq i(\Gamma, \Lambda)$. ☐

Now, define

$$j_1 = \max_{\mu, \mu'} |\mu| \cdot i(\mu, \mu'),$$

where the maximum is taken over all pairs of complete clean markings $\mu, \mu'$ connected by an edge in $M(S)$; note that this maximum is well-defined since the cardinality of any complete clean marking is a constant depending only on the topology of $S$, and there are only finitely many edges in $M(S)$ up to the action of $\mathrm{Mod}(S)$, and this action preserves geometric intersection number. Let $B_1$ be an upper bound on the number of edges in any complete clean marking $\mu$, interpreted as a graph on $S$, and let $B := \max(B_1, 4|\chi(S)|)$. Finally, let

$$j_2 = \max_{\Gamma} \min_{\mu} i^{\mathcal{G}}(\Gamma, \mu),$$

where the maximum is taken over all graphs $\Gamma$ that fill $S$ with at most $B$ edges (and there are finitely many such graphs up to the action of $\mathrm{Mod}(S)$), and the minimum is taken over all complete clean markings $\mu$.

Then for $j = \max(j_1, j_2)$, consider the graph $M_{j,B}(S)$, whose vertices are isotopy classes of $B$-markings and whose edges correspond to pairs $(\Gamma, \Gamma')$ with $i^{\mathcal{G}}(\Gamma, \Gamma') \leq j$. The graph $M_{j,B}(S)$ is connected by choice of $j, B$ and by the connectedness of $M(S)$, and the map $\iota : M(S) \to M_{j,B}(S)$ sending a complete clean marking to itself (interpreted as a $B$-marking) is a $\mathrm{Mod}(S)$-equivariant quasi-isometry. We note that a very similar construction of $M_{j,B}(S)$ appears in [29]. We also remark that choosing $B \geq 4|\chi(S)|$ has not been used yet; to obtain a connected graph that is quasi-isometric to $\mathrm{Mod}(S)$, it suffices to choose $B := B_1$. However our choice of $B$ will be used at the end of the proof of Theorem 3.1.

We can extend the subsurface projection operation to $B$-markings as follows. Given a subsurface $Y$, suppose that $\Gamma$ is a $B$-marking which is also a curve system $\{\alpha_1, \ldots, \alpha_n\}$, and let $j$ be such that $\alpha_j$ intersects $Y$. Since $\Gamma$ must fill $S$, there always exists such a $j$. Then we can define $\psi_Y(\Gamma) := \psi_Y(\alpha_j)$. Since $i(\alpha_j, \alpha_k) \leq B$ for each $k$, inequality (2.2) implies that the diameter of $\Gamma$ in the curve complex of $S$ is uniformly bounded, and thus by the fact that $\psi_Y$ is coarsely Lipschitz, it follows that $d_Y(\psi_Y(\alpha_j), \psi_Y(\alpha_k))$ is bounded solely in terms of $B$. Thus coarsely, the computation of distance between projections of curve systems does not depend on which curve in each system we choose to define each projection.

Next, if $\Gamma$ is an arbitrary $B$-marking, by construction of $M_{j,B}(S)$, it is distance 1 from at least one graph $\Lambda$ in $M_{j,B}(S)$, so that $\Lambda$ is a curve system (in fact a complete clean marking). In this case we define $\psi_Y(\Gamma) := \psi_Y(\Lambda)$. Again, using inequality (2.2) and the fact that subsurface projection is coarse Lipschitz, the distances between projections of $B$-markings will be coarsely independent of which curve system we choose when defining projections.

The following formula, due to Masur and Minsky, allows for the computation of distance in $M(S)$ via subsurface projections ([16]):

**Theorem 2.2.** *There exists $D = D(S)$ such that for any $T > D$, the following holds. There exists $N$ such that for any $\mu_1, \mu_2$ complete clean markings,*

$$d_{\mathcal{M}}(\mu_1, \mu_2) \asymp_N \sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_T,$$

*where $[[x]]_T = x$ for $x \geq T$ and $0$ otherwise.*

Our definitions of subsurface projections for $B$-markings have been set up so that the inclusion map $\iota : M(S) \hookrightarrow M_{j,B}(S)$ being a $\mathrm{Mod}(S)$-equivariant quasi-isometry implies that Theorem 2.2 applies as written to $M_{j,B}(S)$.

2.6. **Hierarchy paths.** As part of the proof of Theorem 2.2, given complete, clean markings $\mu, \mu'$ Masur and Minsky construct certain paths in $M(S)$, called *hierarchy paths*, from $\mu$ to $\mu'$ ([16]). A hierarchy path $H$ from $\mu$ to $\mu'$ can be identified with a collection $C_H$ of geodesics defined in curve graphs of various essential subsurfaces of $S$. Given a geodesic $h \in C_H$, let $D(h) \subseteq S$ denote the subsurface on which $h$ is defined. There exists a certain relation $\searrow^d$ (*direct forward subordinacy*) on the geodesics in $C_H$ whose transitive closure generates a partial order on $C_H$ (denoted

$\searrow$ and called *forward subordinacy*). We will not need the details of this identification, nor the exact definition of forward subordinacy, and therefore we only record some important properties below:

(1) Exactly one of the geodesics in $C_H$, called the *main geodesic*, lives in $\mathcal{C}(S)$, and thus every other one is a geodesic in the curve graph of some proper subsurface. Furthermore, the length of the hierarchy path is equal to the sum of the lengths over all geodesics in $C_H$ (condition 1 of Definition 4.4 of [16]).
(2) There exists $J = J(S)$ such that if $g$ is a geodesic in $C_H$ supported on some subsurface $Y \subseteq S$, then the length of $g$, which we denote by $|g|$, is within $J$ of $d_Y(\mu, \mu')$. Moreover, if $d_Y(\mu, \mu') > J$, then there exists a geodesic $h$ in the hierarchy with $D(h) = Y$ (Lemma 6.2 of [16]).
(3) If $h \searrow^d g$, then $|\chi(D(h))| < |\chi(D(g))|$. Furthermore, given $g \in C_H$, the number of geodesics $h$ satisfying $h \searrow^d g$ is at most $|g| + 4$ (equation (9.19) of [18]).
(4) Let $m \in C_H$ denote the main geodesic. Then for any $h \in C_H, h \neq m$, we have $h \searrow m$ (condition 3 of Definition 4.4 of [16]).

2.7. **Teichmüller space and Rafi's formula.** For this section, we assume $S$ has no boundary (but perhaps punctures). The *Teichmüller space* of $S$, denoted $\mathcal{T}(S)$, is the space of marked Riemann surfaces homeomorphic to $S$. Concretely, $\mathcal{T}(S)$, as a set, is the collection of pairs $(\phi, \sigma)$ modulo a certain equivalence relation, where $\sigma$ is a finite area complete hyperbolic surface homeomorphic to $S$ and $\phi : S \to \sigma$ is a homeomorphism. The equivalence relation is defined as follows: $(\phi, \sigma) \sim (\phi', \sigma')$ exactly when there exists an isometry $j : \sigma \to \sigma'$ such that $j \circ \phi$ is homotopic to $\phi'$. Given $x = (\phi, \sigma) \in \mathcal{T}(S)$, $\phi$ is called the *marking*, or *marking homeomorphism*, of $x$.

We will assume that $\mathcal{T}(S)$ is equipped with the metric topology coming from the *Teichmüller metric*, denoted by $d_{\mathcal{T}}(\cdot, \cdot)$. In this metric, the distance between two marked Riemann surfaces $x = (\phi_1, \sigma_1)$ and $y = (\phi_2, \sigma_2)$ is determined by the logarithm of the minimal dilatation associated to a quasiconformal map $\Phi : x \to y$ such that $\Phi \circ \phi_1$ is isotopic to $\phi_2$. $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6+2p}$, where $g$ is the genus of $S$ and $p$ is the number of punctures [3].

Fix $\epsilon > 0$. The *$\epsilon$-thick part* of $\mathcal{T}(S)$, denoted $\mathcal{T}_\epsilon$, is the set of all points in $\mathcal{T}(S)$ whose underlying hyperbolic metric has injectivity radius at least $\epsilon$; equivalently, it is the set of all marked hyperbolic surfaces on which every essential simple closed curve has length at least $2\epsilon$. Let $x, y \in \mathcal{T}_\epsilon$, and let $\mu_x, \mu_y$ be the shortest complete clean markings on $x, y$ respectively. Then the following formula due to Rafi relates the Teichmüller distance $d_{\mathcal{T}}(x, y)$ to subsurface projections ([24]):

**Theorem 2.3.** *For $x, y \in \mathcal{T}_\epsilon$, there exists $P > 1$ such that*

$$d_{\mathcal{T}}(x, y) \asymp \sum_{Y \subseteq S} [[d_Y(\mu_x, \mu_y)]]_P + \sum_{A \subset S} \log([[d_A(\mu_x, \mu_y)]]_P),$$

*where the first sum is over all non-annular essential subsurfaces $Y$, and the second is over all essential annuli. Moreover, we define* $\log([[w]]_P)$ *to be equal to* $0$ *if $w < P$, and to be* $\log(w)$ *otherwise.*

We will also have use for the following coarse equality due to Choi-Rafi, which relates distance in the $\epsilon$-thick part of Teichmüller space to the logarithm of intersection number ([23]):

$$\log(i(\mu_x, \mu_y)) \asymp^+ d_{\mathcal{T}}(x, y). \tag{2.4}$$

In equation (2.4), the additive constant depends on both the surface $S$ and the thickness constant $\epsilon$. Thus, when we use this equation, we will fix an $\epsilon > 0$ once and for all which depends only on the surface $S$, thereby converting (2.4) into an equation depending only on $S$.

## 3. Bounds on $K$-systems

In this section, we prove Theorem 3.1:

**Theorem 3.1.** *Fix a surface $S$ with $\chi(S) < 0$, and let $N_S(k)$ denote the clique number of $\mathcal{C}_k(S)$. Then*

$$\log(N_S(k)) \prec k.$$

Let $\Gamma$ be any curve system representing a clique in $\mathcal{C}_k(S)$. The strategy of the proof will be to produce a constant $W$ (depending only on $S$ and not on $k$) and a map $\Phi : \Gamma \to M_{j,B}(S)$ such that (1) the pre-image of any point has cardinality at most $W$ and (2) such that the image $\Phi(\Gamma)$ is contained in a ball of radius coarsely at most $k$ in $M_{j,B}(S)$. Since $M_{j,B}(S)$ is a locally finite graph with valence bounded solely in terms of the topology of the surface $S$, the number of vertices in a ball of radius $R$ in $M_{j,B}(S)$ grows at most exponentially in $R$, and thus the theorem follows.

*Proof.* We first note that it suffices to assume that $\Gamma$ fills $S$, for if not, we can decompose $S$ into a disjoint union of subsurfaces $S_1, S_2, \ldots$ such that each $\gamma$ is contained in one $S_i$ and such that each $S_i$ is filled by the subset $\Gamma_i$ of $\Gamma$ it contains. The number of such subsurfaces is bounded above solely in terms of the topology of $S$, and therefore the desired bound on $|\Gamma|$ in Theorem 3.1 follows by a bound on each $\Gamma_i$. Thus we assume that $\Gamma$ has one connected component. □

We first show that distance in $M_{j,B}(S)$ is coarsely bounded above by intersection number:

**Lemma 3.2.** *Given $\mu_1, \mu_2 \in M_{j,B}(S)$,*

$$d_{M_{j,B}(S)}(\mu_1, \mu_2) \prec i^{\mathcal{G}}(\mu_1, \mu_2).$$

*Remark* 3.3. We note that Lemma 3.2 is sharp: suppose $\mu_1$ is a curve system and $\mu_2$ is obtained from $\mu_1$ by applying $k \gg j$ Dehn twists about one of the curves in $\mu_1$. Then both $d_{M_{j,B}}(\mu_1, \mu_2)$ and $i^{\mathcal{G}}(\mu_1, \mu_2)$ will be coarsely equal to $k$.

*Remark* 3.4. The proof of Lemma 3.2 factors through Teichmüller distance, and in particular Theorem 2.3. However all that is actually used is equation (3.2), which is a purely combinatorial statement (we thank the referee for making this observation). In the spirit of subsection 1.4, this motivates the search for an even more combinatorial proof of the lemma. For instance, when $S$ is punctured we sketch an alternative, more combinatorial argument for Lemma 3.2:

The *flip graph* $\mathcal{F}(S)$ is the graph whose vertices consist of (isotopy classes of) ideal triangulations of $S$ (thus $\mathcal{F}(S)$ is non-empty if and only if $S$ has at least one puncture) and whose edges correspond to *diagonal swaps*: one triangulation is obtained from the other by deleting a diagonal of one quadrilateral in the triangulation and replacing it with the other diagonal.

Mosher [20] and independently Penner [21] proved that $d_{\mathcal{F}(S)}(\tau_1, \tau_2) \leq i(\tau_1, \tau_2)$, and this suggests an alternative and perhaps more combinatorial proof of Lemma 3.2 in the punctured setting: since the mapping class group acts properly and cocompactly on $\mathcal{F}(S)$, it is quasi-isometric to $M_{j,B}(S)$, and Lemma 3.2 would follow by exhibiting an explicit quasi-isometry that coarsely preserves intersection number. A natural candidate for such a quasi-isometry is described as follows: given a triangulation $\tau$, consider the marking obtained by taking a regular neighborhood of each arc of $\tau$. Any two of the resulting curves can intersect at most 4 times (twice for each end point of the arcs), and thus we obtain a $B$-marking. Moreover, each intersection between a pair of arcs contributes at most 4 intersections between resulting curves, and therefore

$$i^{\mathcal{G}}(\theta(\tau_1), \theta(\tau_2)) \prec i(\tau_1, \tau_2), \tag{3.1}$$

as desired. We thank the referee for suggesting this quasi-isometry.

*Proof of Lemma* 3.2. To prove Lemma 3.2 we use the following inequality, which follows from Rafi's distance formula and from (2.4) (see Corollary D of [23]): there exists some constant $P = P(S)$ such that

$$\log(i^{\mathcal{G}}(\mu_1, \mu_2)) \asymp \sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_P + \sum_{A \subset S} \log([[d_A(\mu_1, \mu_2)]]_P), \tag{3.2}$$

where the first sum is over all non-annular essential subsurfaces of $Y$, and the second is taken over all essential annuli. We remark that in [23] this formula is proven for $\mu_1, \mu_2$ complete clean markings; however by Lemma 2.1 and the way we have defined subsurface projection for $B$-markings, (3.2) holds for $B$-markings as well.

We also recall the Masur-Minsky distance formula ([16]), which asserts the existence of a constant $T = T(S)$ such that

$$d_{M_{j,B}(S)}(\mu_1, \mu_2) \asymp \sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_T, \tag{3.3}$$

where the sum is taken over all essential subsurfaces of $S$, including annuli. We first note that it suffices to assume that the threshold $P$ in (3.2) is equal to the threshold $T$ in (3.3). Indeed, assume first that $P < T$. However we are free to raise the threshold $P$ until it equals $T$, as we will only be using the direction of (3.2) which bounds $\log(i^{\mathcal{G}}(\mu_1, \mu_2))$ from below, and raising the threshold will only make the right-hand side of (3.2) smaller. Thus it suffices to assume that $T \leq P$. On the other hand, if $T < P$, we can also raise $T$ so that it coincides with $P$. This follows from the fact that the Masur-Minsky distance formula (3.3) holds for all sufficiently large thresholds (however different thresholds require different coarse equality constants). Thus henceforth, we can assume that $P = T$.

Moreover, we can set the threshold $T$ to be larger than the constant $J$ from property (2) of hierarchies listed in subsection 2.6; that is, any such subsurface with $d_Y(\mu_1, \mu_2) > J$ must appear in any hierarchy from $\mu_1$ to $\mu_2$.

Define $R_1, R_2$ by

$$R_1 := \sum_{Y \subseteq S, \text{non-annular}} [[d_Y(\mu_1, \mu_2)]]_T; R_2 := \sum_{A \subset S, A \text{ annular}} [[d_A(\mu_1, \mu_2)]]_T. \tag{3.4}$$

Then the sum on the right-hand side of the Masur-Minsky distance formula (3.3) is simply $R_1 + R_2$. Then since $P = T$, (3.2) implies that

$$\log(i^{\mathcal{G}}(\mu_1, \mu_2)) \succ R_1. \tag{3.5}$$

Thus, applying (3.3), we have reduced Lemma 3.2 to proving that

$$i^{\mathcal{G}}(\mu_1, \mu_2) \succ R_2. \tag{3.6}$$

To this end, we first claim that $R_2$ has coarsely at most $\log(i^{\mathcal{G}}(\mu_1, \mu_2))$ summands. To see this, note that by property (2) of hierarchies, any annulus $A$ for which $d_A(\mu_1, \mu_2) > T$ will appear in a hierarchy path from $\mu_1$ to $\mu_2$. Thus, each such annulus must be directly forward subordinate to one of the non-annular subsurfaces in the hierarchy, and for any non-annular subsurface $Y$ in the hierarchy, there are at most $|g_Y| + 4$ subsurfaces which are directly forward subordinate to $Y$, where $|g_Y|$ is the length of the geodesic $g_Y$ supported on $Y$. It follows that the number of annuli appearing in the hierarchy is coarsely at most the length of the hierarchy coming from geodesics supported on non-annular subsurfaces, which is coarsely at most $R_1$. Hence the desired bound follows from (3.5).

We next claim that there exists a constant $L$ depending only on $S$, such that there exists at most $L$ essential annuli $A \subset S$ satisfying

$$d_A(\mu_1, \mu_2) > i^{\mathcal{G}}(\mu_1, \mu_2)/\log(i^{\mathcal{G}}(\mu_1, \mu_2)). \tag{3.7}$$

Assuming (3.7) and using the fact that $d_A(\mu_1, \mu_2) \prec i^{\mathcal{G}}(\mu_1, \mu_2)$ for any annulus (this follows from (2.3) and the fact that distance in an annular complex is coarsely at most the intersection number), we have the bound

$$\begin{aligned} R_2 &\prec L \cdot i^{\mathcal{G}}(\mu_1, \mu_2) + \log(i^{\mathcal{G}}(\mu_1, \mu_2)) \left[ \frac{i^{\mathcal{G}}(\mu_1, \mu_2)}{\log(i^{\mathcal{G}}(\mu_1, \mu_2))} \right] \\ &= (L+1) i^{\mathcal{G}}(\mu_1, \mu_2), \end{aligned}$$

which implies the conclusion of Lemma 3.2. Therefore it remains to prove the existence of such an $L$; assume by way of contradiction that no such $L$ exists. Then there exists a sequence of pairs of $B$-markings

$$(\mu_1^{(j)}, \mu_2^{(j)})_{j=1}^{\infty}$$

satisfying the property that if $\mathcal{S}^{(j)}$ denotes the number of annuli onto which the projection of the pair $\mu_1^{(j)}, \mu_2^{(j)}$ has distance at least

$$i^{\mathcal{G}}(\mu_1^{(j)}, \mu_2^{(j)})/\log(i^{\mathcal{G}}(\mu_1^{(j)}, \mu_2^{(j)})),$$

then $\mathcal{S}^{(j)} \to \infty$.

Henceforth, let $i^{(j)}$ denote the intersection $i^{\mathcal{G}}(\mu_1^{(j)}, \mu_2^{(j)})$; note that $i^{(j)} \to \infty$. Indeed, the assumption that $S^{(j)} \to \infty$ and (3.3) implies that the distance

$$d_{M_{j,B}(S)}(\mu_1^{(j)}, \mu_2^{(j)}) \to \infty,$$

and given any finite $C > 0$, there are only finitely many pairs of $B$-markings, up to the action of $\mathrm{Mod}(S)$ (which acts isometrically on $M_{j,B}(S)$ and which preserves intersection number) which intersect each other at most $C$ times. Then (3.2) implies that for each $j$,

$$\log(i^{(j)}) \succ \sum_{A\subset S,\text{annular}} \log[[d_A(\mu_1^{(j)}, \mu_2^{(j)})]]_T =: R_2^{(j)}, \tag{3.8}$$

where the multiplicative and additive constants in the coarse equality depend only on $S$ and not on $j$. Exponentiating both sides of this inequality, we obtain

$$i^{(j)} > 2^{-Q} \prod_{A\subset S} ([[d_A(\mu_1^{(j)}, \mu_2^{(j)})]]_T)^{1/Q}, \tag{3.9}$$

for some $Q > 0$ depending only on $S$. However, since there are $\mathcal{S}^{(j)}$ summands of $R_2^{(j)}$ whose size is at least $i^{(j)}/\log(i^{(j)})$ (and since each summand is at least 1), it follows that the product on the right-hand side of (3.9) is at least on the order of

$$2^{-Q}\left[i^{(j)}/\log(i^{(j)})\right]^{\mathcal{S}^{(j)}/Q},$$

which grows super-polynomially in $i^{(j)}$ because $\mathcal{S}^{(j)} \to \infty$, and this contradicts (3.9) since $i^{(j)} \to \infty$. Hence $\mathcal{S}^{(j)}$ must be uniformly bounded. This completes the proof of Lemma 3.2. □

Lemma 3.2 implies that Theorem 3.1 follows so long as we can construct the aforementioned map $\Phi : \Gamma \to M_{j,B}(S)$, so that (1) the cardinality of each preimage of $\Phi$ is bounded solely in terms of the topology of $S$, and (2) there exists $\gamma \in \Gamma$ such that for any $\gamma' \in \Gamma$,

$$i^{\mathcal{G}}(\Phi(\gamma), \Phi(\gamma')) \prec k.$$

Thus, our goal is to associate a $B$-marking to each element of our connected $k$-system $\Gamma$. As in the proof of Lemma 2.1, using a hyperbolic metric we can realize $\Gamma$ in such a way so that there are no triple points.

Given $\gamma \in \Gamma$, we will build a $B$-marking by starting with $\gamma$ and adding additional edges that are subarcs of other elements in $\Gamma$. To start, choose some $\gamma' \in \Gamma$ such that $i(\gamma, \gamma') \neq 0$. Then there exists a subarc $e$ of $\gamma'$ with endpoints on $\gamma$; then extend $\gamma$ to the graph $\gamma \cup e$. As we are assuming that elements of $\Gamma$ are in pairwise minimal position, no complementary component of $\gamma \cup e$ is a bigon. Now, we simply iterate: extend $\gamma \cup e$ to a larger graph by adding an edge $e'$ associated to a subarc of another element of $\Gamma$ intersecting $\gamma \cup e$. At each stage, $e'$ is chosen so that the absolute value of the Euler characteristic of the subsurface filled by the extended graph grows monotonically. Moreover, the endpoints of $e'$ occur at the interior of pre-existing edges and not at vertices, since $\Gamma$ has no triple points.

Thus, after at most $|\chi(S)|$ iterations, we obtain a graph $\Phi(\gamma)$, built from $\gamma$ and from arcs of elements in $\Gamma$, that fills $S$. At each stage, the number of edges increases by at most 3: one new edge is added, and at most two previous edges are subdivided into two edges (or perhaps one pre-existing edge has been subdivided into three edges). As we have set $B$ to be larger than $4|\chi(S)|$, $\Phi(\gamma)$ is a $B$-marking. Furthermore, given $\gamma, \gamma' \in \Gamma$ it follows that

$$i^{\mathcal{G}}(\Phi(\gamma), \Phi(\gamma')) \prec k,$$

since each edge of both graphs is a subarc of some element of $\Gamma$, and therefore any edge of $\Phi(\gamma)$ can intersect an edge of $\Phi(\gamma')$ at most $k$ times. Thus the bound follows from the fact that both graphs are $B$-markings and have at most $B$ edges each by definition. Finally, we note that for each $\gamma \in \Gamma$, $\gamma$ is an embedded cycle in the graph $\Phi(\gamma)$; that is, $\Phi(\gamma)$ contains a graph-path homotopic to $\gamma$ which does not traverse any edge more than once. The number of such cycles is bounded above solely in terms of the number of edges of $\Phi(\gamma)$, and therefore the cardinality of any pre-image $\Phi^{-1}(\gamma)$ is bounded above solely in terms of $B$. This completes the proof of Theorem 3.1.

## 4. Intersections of links

In this section, we prove uniform bounds on the size of the intersection of $k$-links for a pair of filling curves $\alpha, \beta$:

**Theorem 4.1.** *There exists a function $r_S(k)$ depending only on the topology of $S$, which grows at most quasi-polynomially and which satisfies the following. Let $\alpha, \beta$ be simple closed curves on $S$ which fill $S$, and let $\mathcal{L}_k(\alpha)$ denote the set of all vertices in $\mathcal{C}_k(S)$ that are distance 1 from $\alpha$. Then $|\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)| \leq r_S(k)$.*

We will actually prove Theorem 4.1 when $\alpha, \beta$ are each simplices in $\mathcal{AC}(S)$. That is, both $\alpha$ and $\beta$ are curve and arc systems, each consisting of a finite collection of curves and arcs that are pairwise disjoint, and so that any essential simple closed curve has positive geometric intersection number with either $\alpha$ or $\beta$.

The strategy of the proof is as follows: first we reduce to the case where the intersection number $i(\alpha, \beta)$ is bounded above by a quasi-polynomial function of $k$. To do this, we use the technology of hierarchies to argue that if $i(\alpha, \beta)$ is very large, there must exist some subsurface $Y$ of $S$ on which a definite number of intersections between $\alpha$ and $\beta$ accumulate. It will then follow that no curve in $\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)$ can intersect $Y$, and thus we may restrict our attention to its complement. Then using Choi-Rafi's estimate for distance in the thick part of Teichmüller space in terms of intersection number ([23]), we find a hyperbolic surface $\sigma(\alpha, \beta)$ on which both $\alpha, \beta$ have bounded length (quasi-polynomial in $k$) representatives. It follows that the number of curves in $\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)$ is comparable to the number of geodesics on $\sigma$ with length bounded above by some explicit quasi-polynomial function of $k$. Finally we appeal to a result of Rivin ([25]) estimating the number of such bounded length curves on $\sigma$.

*Proof.*

*Step* 1 (Bound $i(\alpha, \beta)$). Suppose there exists a non-annular essential subsurface $Y \subseteq S$ such that

$$d_Y(\alpha, \beta) > 4\log(k) + 10.$$

Then if $\gamma \in \mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)$, $\gamma$ must be disjoint from $Y$ or homotopic into the boundary of $Y$. Indeed, if $\gamma$ projected to $Y$, then $\psi_Y(\gamma)$ would intersect both $\psi_Y(\alpha), \psi_Y(\beta)$ at most $4k+4$ times by (2.3), and thus by inequality (2.2),

$$d_Y(\gamma, \alpha), d_Y(\gamma, \beta) \leq 2\log(4k+4) + 2,$$

and we obtain a contradiction by applying the triangle inequality in $\mathcal{C}(Y)$. Similarly, if there exists an annulus $A$ with $d_A(\alpha, \beta) > 4k + 4$, no element of the intersection $\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)$ can cross $A$.

If such a non-annular subsurface $Y$ or an annulus $A$ exists, then consider its complement $S \setminus Y$ or $S \setminus A$. In either case, denote this complement by $S'$. Then any curve in $\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)$ must be homotopic into $S'$ (or boundary parallel in $S'$).

Note that this argument can be repeated indefinitely, since $\pi_Y(\alpha), \pi_Y(\beta)$ are curve and arc systems on $Y$ which necessarily fill $Y$. Indeed, there cannot exist an essential simple closed curve $\delta$ on $Y$ with $i(\delta, \pi_Y(\alpha)), i(\delta, \pi_Y(\beta)) = 0$, or else $\delta$ would be disjoint from $\alpha \cup \beta$ on $S$, and this contradicts the assumption that $\alpha$ and $\beta$ fill the full surface.

Thus, we can apply the above argument iteratively to remove any non-annular subsurfaces $Y$ with $d_Y(\alpha, \beta) > 4\log(k) + 10$ or annuli $A$ so that $d_A(\alpha, \beta) > 4k + 4$. Concretely, let $\mathcal{R}$ denote the collection of all such subsurfaces. We apply the above analysis to the subsurfaces in $\mathcal{R}$, removing them one by one until we obtain a subsurface $S'$ such that

(1) $d_Y(\pi_{S'}(\alpha), \pi_{S'}(\beta)) \leq 4\log(k) + 10$ for any essential non-annular subsurface $Y \subseteq S'$, and $d_A(\pi_{S'}(\alpha), \pi_{S'}(\beta)) \leq 4k+4$ for any essential annular subsurface of $S'$.
(2) Since the absolute value of the Euler characteristic decreases at each stage, the number of curves in $\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)$ which are boundary parallel at some stage of the process is coarsely at most $|\chi(S)|$, hence:
(3) all but coarsely at most $|\chi(S)|$ of the curves $\gamma \in \mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)$ satisfy $\max(i(\gamma, \pi_{S'}(\alpha)), i(\gamma, \pi_{S'}(\beta))) \leq k$.

We note that $S'$ may be disconnected, and it may also be empty. If indeed $S'$ is empty, this implies that $\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)$ is coarsely at most $|\chi(S)|$, and in particular it is bounded independently of $k$ and we are done.

Henceforth, we replace $S$ with $S'$ and $\alpha$ (resp. $\beta$) with its projection $\pi_{S'}(\alpha)$ (resp. $\pi_{S'}(\beta)$).

Now, using (1) above, we will bound the intersection number $i(\alpha, \beta)$, and this bound will later be used to bound $\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)$, which by (2) and (3) above will imply the desired bound for the original surface $S$ and filling pair $\alpha, \beta$.

To do this, we first augment $\alpha, \beta$ to $B$-markings $\mu_\alpha, \mu_\beta$ such that

$$i(\mu_\alpha, \mu_\beta) \asymp i(\alpha, \beta).$$

The marking $\mu_\alpha$ is obtained by projecting $\beta$ to the complement $S \setminus \alpha$. Concretely, $\mu_\alpha$ is the $B$-marking obtained by taking the union of $\alpha$ with a maximal collection of pairwise non-homotopic arcs of $\beta \cap (S \setminus \alpha)$, interpreted as a graph on $S$, and similarly for $\mu_\beta$. Then by choosing a sufficiently high cut-off as in the proof of Lemma 3.2, the Choi-Rafi formula relating intersection number to subsurface projections states that

$$\log(i^{\mathcal{G}}(\mu_\alpha, \mu_\beta)) \prec \sum_{Y \subseteq S} [[d_Y(\mu_\alpha, \mu_\beta)]]_P + \sum_{A \subset S} \log([[d_A(\mu_\alpha, \mu_\beta)]]_P). \tag{4.1}$$

Then the assumption that there are no large projections, together with the properties of hierarchy paths recorded in section 2, will allow us to bound the right-hand side from above in terms of $k$. Indeed, a hierarchy organizes the subsurfaces which support large projections between $\mu_\alpha, \mu_\beta$ into a *weighted tree*, which is a based, directed tree whose vertices are labeled by natural numbers. The *size* of a weighted tree is the sum, over each vertex, of the weight of that vertex. Adjacency in this tree will be encoded by forward subordinacy.

Concretely, by property (4) of hierarchies, every subsurface in the hierarchy is forward subordinate to the main geodesic. Thus, we will interpret the main geodesic $g$ as being the base vertex $v$ of a weighted tree. By property (3), the number of subsurfaces that are *directly* forward subordinate to $g$ is at most $|g| + 4$; thus, we assign $v$ a weight of $|g|+4$, which will also be an upper bound for the valence of $v$. We then add vertices for each of the subsurfaces directly forward subordinate to $g$ and connect each to $v$ by an edge directed away from $v$. The weight of each of these vertices will be equal to the number of subsurfaces directly forward subordinate to it, and we iterate in this fashion.

For each vertex $x$ in this weighted tree, let $Y(x)$ denote the associated subsurface. Then by property (3) of hierarchies, if $z$ is adjacent to $x$ by an edge directed towards $z$, $|\chi(Y(z))| < |\chi(Y(x))|$. It follows that the distance from $v$ to any vertex associated to an annulus is at most $|\chi(S)|$. Therefore, the size of the weighted tree is bounded above by an upper bound on the size of any geodesic in the hierarchy, raised to the power $|\chi(S)|$. By construction, the size of the tree is an upper bound on the size of the hierarchy, and by property (2) of hierarchies, each geodesic that appears has length coarsely equal to the projection distance of $\mu_\alpha, \mu_\beta$ to that subsurface. Thus, the weight of each vertex corresponding to a non-annular subsurface is coarsely bounded above by $\log(k)$, and the weight of each annular vertex is coarsely at most $k$. Then by choosing the cut-off to be sufficiently large as in the proof of Lemma 3.2, every summand appearing on the right-hand side of (4.1) must appear in the hierarchy, and thus also as a vertex in the weighted tree. Thus, the right-hand side of (4.1) is coarsely at most

$$[\log(k)]^{f(S)},$$

where

$$f(S) \asymp |\chi(S)|.$$

Thus, we have shown the existence of some constant $U$ so that

$$\log(i^{\mathcal{G}}(\mu_\alpha, \mu_\beta)) \leq U \cdot [\log(k)]^{f(S)} + U,$$

and exponentiating both sides of this yields

$$i^{\mathcal{G}}(\mu_\alpha, \mu_\beta) \leq 2^{U\cdot[\log(k)]^{f(S)}+U} = U' \cdot 2^{[\log(k)]^{U\cdot f(S)}} =: w(k),$$

which is quasi-polynomial in $k$.

*Step* 2. Find the hyperbolic surface $\sigma(\alpha, \beta)$. We note that there is $\epsilon > 0$ and $N > 0$, each depending only on $S$, so that each $x \in \mathcal{T}_\epsilon(S)$ admits a $B$-marking of length at most $N$, and every $B$-marking has length at most $N$ at some point in $\mathcal{T}_\epsilon(S)$ (see page 5 of [23]). Let $\sigma(\alpha)$ (resp. $\sigma(\beta)$) denote a hyperbolic surface in the $\epsilon$-thick part so that $\mu_\alpha$ (resp. $\mu_\beta$) has length at most $N$ on $\sigma(\alpha)$ (resp. $\sigma(\beta)$).

Thus $\sigma(\alpha), \sigma(\beta)$ are both finite area complete hyperbolic surfaces of the same topological type, potentially disconnected and with parabolic cusps but without boundary. In the case that $S'$ is disconnected (and therefore $\sigma(\alpha), \sigma(\beta)$ are also disconnected), we work with each connected component separately, and so henceforth we assume that $S'$ is connected. As there can only be coarsely at most $|\chi(S)|$ components, obtaining the desired bound for each component and then adding them all together suffices. Then the Choi-Rafi estimate (2.4), together with the bound obtained in the previous subsection, implies that

$$\log(w(k)) \succ d_{\mathcal{T}}(\sigma(\alpha), \sigma(\beta)), \tag{4.2}$$

where we emphasize that the coarse constants in the above inequality depend only on $S$, since our choice of $\epsilon$ depends only on $S$.

*Remark* 4.2. We remark here that (4.2) is part of Theorem 2.2 of [23] and that this estimates the Teichmüller distance between points $x, y$ in the thick part in terms of their *shortest* markings, not simply markings which are relatively short. However, as mentioned in the proof of Corollary D in [23], the result applies if one uses relatively short markings. For completeness, we include an argument for this here:

Let $N > 0$ be larger than the minimal length (taken over $\mathcal{T}(S)$) of any complete clean marking. Fix a complete clean marking $\mu$, and let $\mathcal{T}(\mu, N) \subset \mathcal{T}(S)$ be the set of all points on which $\mu$ has length at most $N$. We first note that there exists $\delta > 0$ depending only on $N$ and $S$ so that $\mathcal{T}(\mu, N) \subset \mathcal{T}_\delta(S)$. Indeed, since $\mu$ cuts up the surface into simply-connected and once punctured regions, for any surface on which there exists a very short essential simple closed curve, $\mu$ must be long by the collar lemma. Then Theorem B of [23] applies, which states that Teichmüller and Lipschitz distances are comparable in $\mathcal{T}_\delta(S)$, where the comparability constants depend only on $S$.

We claim that $\mathcal{T}(\mu, N)$ has bounded diameter in the Teichmüller metric, independent of $\mu$. Using Theorem B of [23], to prove this it suffices to show that $\mathrm{diam}(\mathcal{T}(\mu, N))$ is bounded in the Lipschitz metric, and for this one must show that $\ell_y(\alpha)/\ell_z(\alpha)$ is uniformly bounded over all essential simple closed curves $\alpha$ and any $y, z \in \mathcal{T}(\mu, N)$. This follows from the fact that for any $y \in \mathcal{T}(\mu, N)$, $\ell_y(\alpha)$ is coarsely equal to $i(\alpha, \mu)$ (see Proposition 3.5 of [19]). Thus $\mathrm{diam}(\mathcal{T}(\mu, N))$ is bounded in the Teichmüller metric, as desired.

Then as mentioned in the proof of Corollary D of [23] there exists a point $x_\mu \in \mathcal{T}(S)$ so that $\mu$ is a short marking on $x_\mu$; note that $x_\mu$ is necessarily contained in $\mathcal{T}(\mu, N)$. It follows that (4.2) applies when stated in terms of relatively short complete clean markings (for instance, at most length $N$), after perhaps accounting for the additional additive error $2 \cdot \mathrm{diam}(\mathcal{T}(\mu, N))$.

In particular, (4.2) implies that $\alpha$ admits a representative on $\sigma(\beta)$ whose length is coarsely at most $w(k)$. This follows, for instance, by Wolpert's inequality [30], which states that for any essential simple closed curve $\gamma$ and any points $x, y \in \mathcal{T}(S)$,

$$\frac{\ell_x(\gamma)}{\ell_y(\gamma)} \leq e^{d_{\mathcal{T}(S)}(x,y)}.$$

Thus the desired bound follows from the fact that $\alpha$ has length at most $N$ on $\sigma(\alpha)$, and $N$ depends only on $\epsilon$ and $S$. Since the length of $\beta$ is bounded on $\sigma(\beta)$ in terms of only the topology of the surface $S$, it follows that $\alpha \cup \beta$ has length bounded coarsely from above by $w(k)$ on $\sigma(\beta)$. We set $\sigma(\alpha, \beta) := \sigma(\beta)$.

*Step* 3 (Bounding the number of short curves on $\sigma(\alpha, \beta)$). Thus the geodesic representatives for $\alpha$ and $\beta$ (which by abuse of notation we also refer to as $\alpha$ and $\beta$) decompose $\sigma(\alpha, \beta)$ into hyperbolic polygons $P_1, \ldots, P_N$ and possibly also a finite number of once-punctured regions with piecewise geodesic boundaries, such that:

(1) for each $i = 1, \ldots, N$, each side of $P_i$ has hyperbolic length (coarsely) at most $w(k)$;
(2) for each $i = 1, \ldots, N$, $P_i$ has a uniformly bounded number of sides (in terms only of the topology of $S$).

Property (1) follows from the previous subsection. Property (2) follows from (4.3) below, and in particular it is true for any filling pair on $S$, regardless of intersection number. Indeed, let $N$ denote the number of simply-connected components of

$S \setminus (\alpha \cup \beta)$. Note that when interpreted as a graph on the surface, $\alpha \cup \beta$ is 4-valent, and therefore it has twice as many edges as vertices. Since the vertices are in correspondence with the intersections between $\alpha$ and $\beta$, one has that $\chi(S) = i(\alpha,\beta) - 2 \cdot i(\alpha,\beta) + N$

$$\Rightarrow \chi(S) = N - i(\alpha,\beta). \tag{4.3}$$

Since each of these regions has at least 4 sides, it follows that no region can have more than $4|\chi(S)| + 5$ sides. Note that the same bound applies for the number of sides of any of the once-punctured complementary regions. Indeed, if there exists a once-punctured region $R$ with at least 4 sides, then $\alpha, \beta$ will be in minimal position on the surface $\tilde{S}$ obtained by filling in that puncture as this does not create any bigons. Thus $R$ cannot have more than $4|\chi(S)| + 5$ sides by applying the same argument on $\tilde{S}$.

Therefore, each $P_i$ has diameter coarsely bounded above by $w(k)$, and this implies that the length of an arc of any geodesic contained within one of the $P_i$'s has length coarsely at most $w(k)$. Moreover, although the diameter of a once-punctured region is infinite, the same argument implies that the length of an arc of a geodesic contained within one of the once-punctured complementary regions is also coarsely at most $w(k)$. Hence if $\gamma \in \mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta)$, the geodesic representative for $\gamma$ on $\sigma(\alpha,\beta)$ has hyperbolic length at most $Z \cdot (k \cdot w(k)) + Z$, for some constant $Z > 0$. That is, if $N_\sigma(Z \cdot (k \cdot w(k)) + Z)$ denotes the collection of simple closed geodesics on $\sigma(\alpha,\beta)$ of length at most $Z \cdot (k \cdot w(k)) + Z$, then we have

$$\mathcal{L}_k(\alpha) \cap \mathcal{L}_k(\beta) \subseteq N_\sigma(Z \cdot (k \cdot w(k)) + Z).$$

Rivin ([25]) has shown that there exists some constant $V = V(\sigma)$ such that for any $L$,

$$N_\sigma(L) \leq V \cdot L^{|\dim(\mathcal{T}(S))|} + V.$$

The constant $V$ necessarily diverges as injectivity radius decays to 0; however we have chosen $\sigma(\alpha,\beta)$ to be uniformly thick, independent of the choice of filling pair $\alpha, \beta$, and in any thick part, $V$ is uniformly bounded. This follows from the compactness of the (any) thick part of Moduli space. This completes the proof of Theorem 4.1. ∎

4.1. **Unit-square tiled surfaces.** We conclude with a brief discussion of Corollary 4.3 and its proof.

Let $S$ be a closed surface. A *unit-square tiling* of $S$ is a surface $\mathcal{S}$ homeomorphic to $S$ obtained by gluing copies of the unit square in $\mathbb{C}$ together, so that

(1) vertical edges glue to vertical edges, and similarly horizontal edges glue to horizontal edges;
(2) each vertex is adjacent to at least 4 squares after the gluing.

In particular, we do not require that a left-hand vertical edge glue to a right-hand vertical edge or that a top horizontal edge glue to a bottom one. We also call $\mathcal{S}$ a *unit-square tiled surface*. A unit-square tiled surface admits a metric that is induced by identifying each square with the standard unit square in $\mathbb{C}$; this is a Euclidean metric away from finitely many singular points which correspond to locations around which more than 4 squares are glued.

In the special case that top edges glue to bottom edges and left edges glue to right ones, the resulting surface is a branched cover of the torus and is conformally equivalent to a so-called *origami*, a type of translation surface whose $SL(2,\mathbb{R})$ orbit has important dynamical and algebro-geometric properties ([6], [27]).

Let $\mathcal{X}(S)$ denote the set of all unit-square tilings of $S$, and for each $\mathcal{S} \in \mathcal{X}(S)$, let $N_{\mathcal{S}}(L)$ denote the number of homotopy classes of simple closed geodesics on $\mathcal{S}$ of length at most $L$. We prove:

**Corollary 4.3.** *There exists a function $P_S$ which grows at most quasi-polynomially, such that*

$$\sup\{N_{\mathcal{S}}(L) : \mathcal{S} \in \mathcal{X}(S)\} \leq P_S(L).$$

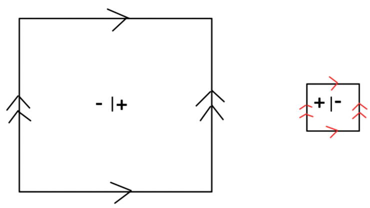


FIGURE 1. A genus 2 singular flat surface, consisting of two flat tori glued together via a small slit on the interior of each torus. The left hand side of the left slit is glued to the right hand side of the right slit. The area of the small torus is $\epsilon$ and the area of the larger is $1-\epsilon$. For $\epsilon$ very small, there will be many homotopy classes of simple closed curves with representatives on the smaller torus, all with short lengths.

*Remark* 4.4. That there exists a polynomial upper bound of degree $\dim(\mathcal{T}(S))$ for $N_{\mathcal{S}}(L)$, for a *fixed* unit-square tiled surface $\mathcal{S}$, follows from work of Rivin [25] and also Mirzakhani [19]. However, this does not necessarily imply a uniform quasi-polynomial bound over all $\mathcal{X}(S)$. Indeed, given a fixed hyperbolic metric $\sigma$ on $S$, the number $N_\sigma(L)$ of simple closed geodesics of length $\leq L$ also satisfies a polynomial upper bound of degree $\dim(\mathcal{T}(S))$ (again by [25] and [19]), but a *uniform* upper bound for $N_\sigma(L)$, taken over all hyperbolic metrics $\sigma$, necessarily grows exponentially in $L$ (see for instance inequality (3.14) of Proposition 3.6 of [19]). There is also a polynomial upper bound for the number of (homotopy classes of) simple closed geodesics of length $\leq L$ for any fixed unit-area singular flat metric on $S$. However there can be no uniform bound (subexponential or otherwise) over all unit-area singular flat metrics on $S$, as Figure 1 demonstrates.

We also highlight that in both the hyperbolic and unit-area flat setting, all of the metrics we consider have a fixed bounded area, and this is not the case for $\mathcal{X}(S)$. If we were to scale down the surfaces in $\mathcal{X}(S)$ so that each has unit area, we could expect any result like Corollary 4.3 to hold. It follows that the only real obstruction to Corollary 4.3 for unit-area flat surfaces is the presence of arbitrarily small squares. Finally, we remark that we are using the simple closed curves in an essential way: for any surface in $\mathcal{S}$, the number of homotopy classes of closed curves below a certain length $L$ grows exponentially in $L$.

*Proof of Corollary* 4.3. Given $\mathcal{S} \in \mathcal{X}(S)$, consider its *vertical and horizontal curves* $v$ and $h$: $v$ (resp. $h$) is the multi-curve obtained by concatenating all vertical (resp. horizontal) midsegments of squares. The requirement that no vertical edges glue to horizontal edges guarantees that $v$ and $h$ are both multi-curves (potentially with many parallel components) and are distinct from each other.

If $\alpha$ is a minimum length representative of a simple closed homotopy class on $\mathcal{S}$, then at the cost of increasing length by a factor of at most $\sqrt{2}$, we can homotope $\alpha$ so that it lies on the 1-skeleton of $\mathcal{S}$. Therefore the length of $\alpha$ is coarsely equal to the number of times it intersects both $v$ and $h$. We can assume that any two components of $v$ (resp. $h$) are not homotopic to each other, since deleting parallel components only reduces the intersection number with other curves. Hence $N_{\mathcal{S}}(L)$ is bounded above by

$$|\mathcal{L}_{\lceil\sqrt{2}L\rceil}(v) \cap \mathcal{L}_{\lceil\sqrt{2}L\rceil}(h)|,$$

and thus the corollary follows by applying Theorem 4.1. □

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