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ASYMPTOTIC BEHAVIOR OF POSITIVELY CURVED STEADY RICCI SOLITONS

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ABSTRACT. In this paper, we analyze the asymptotic behavior of κ-noncollapsed and positively curved steady Ricci solitons and prove that any n-dimensional κ-noncollapsed steady Kähler–Ricci soliton with nonnegative sectional curvature must be flat.

1. Introduction

The classification of a positively curved steady soliton is an important problem in the study of Ricci flow. In his celebrated paper [20], Perelman conjectured that *all 3-dimensional κ-noncollapsed steady (gradient) Ricci solitons must be rotationally symmetric* (precisely, Perelman claims that the conjecture is true without giving any sketch of proof; see 11.9 of that paper). The conjecture was solved by Brendle in 2013 [1]. Brendle also proved that the same result holds for higher-dimensional κ-noncollapsed Ricci solitons with nonnegative sectional curvature if they are asymptotically cylindrical [2]. Under the condition of a locally conformally flat condition, Cao and Chen also proved the rotational symmetry of a gradient steady soliton in [7]. These rotationally symmetric metrics are usually called the Bryant steady Ricci solitons.

In the complex case, Cao constructed a family of $U(n)$-invariant steady Kähler–Ricci solitons with positive sectional curvature on $\mathbb{C}^n$ [5]. He also proposed the following open problem.

Problem 1.1. Is it true that any complete gradient steady Kähler–Ricci soliton with positive sectional curvature must be $U(n)$-invariant?

Unlike the Bryant solitons, one can check that Cao's solitons are all collapsed (cf. the Appendix in the present paper). Thus, it is interesting to ask the following question.

Problem 1.2. Does there exist a steady Kähler–Ricci soliton with positive sectional curvature which is κ-noncollapsed?

In this paper, we give a negative answer to Problem 1.2. Namely we prove the following theorem.

Theorem 1.3. *There is no κ-noncollapsed steady gradient Kähler–Ricci soliton with positive sectional curvature.*

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Theorem 1.3 gives a positive evidence to Problem 1.1. As an application of Theorem 1.3, we get the following rigidity result for the steady Kähler–Ricci solitons.

Theorem 1.4. *Any κ-noncollapsed steady Kähler–Ricci soliton with nonnegative sectional curvature must be flat. More generally, any κ-noncollapsed noncompact and eternal Kähler–Ricci flow with nonnegative sectional curvature and uniformly bounded curvature must be a flat flow.*

We use the induction argument to prove Theorem 1.3 and first prove it for Kähler surfaces. The main technique is to analyze the asymptotic behavior of positively curved steady Ricci solitons as used by many people, such as in [20], [18], [19], [1], etc. By the blow-down argument, we first generalize Perelman's compactness theorem for 3-dimensional κ-solution in [20] to higher dimensions (see Theorem 3.3). Then we apply the compactness theorem to the steady Ricci solitons and prove the following theorem.

Theorem 1.5. *Let (M, g, f) be a noncompact κ-noncollapsed steady Kähler–Ricci soliton with dimension n. Suppose that M has nonnegative sectional curvature and positive Ricci curvature. Then, for any $p_i \to \infty$, the sequence of rescaled flows $(M, R(p_i)g(R^{-1}(p_i)t), p_i)$ converges subsequently to a Kähler–Ricci flow $(N_1 \times N_2, \widetilde{g}(t))$ ($t \in (-\infty, 0]$) in the Cheeger–Gromov topology, where*

$$\widetilde{g}(t) = \mathrm{d}z \otimes \mathrm{d}\overline{z} + g_{N_2}(t),$$

N_1 is $\mathbb{C}^1$ or $\mathbb{R}^1 \times S^1$ with the flat metric $g_{N_1} = \mathrm{d}z \otimes \mathrm{d}\overline{z}$, and $(N_2, g_{N_2}(t))$ is a pseudo κ-solution (cf. Definition 3.2) of Kähler–Ricci flow on a complex manifold N_2 with dimension $n-1$. Furthermore, in case $\dim_{\mathbb{C}} M = 2$, $(N_2, g_{N_2}(t)) = (\mathbb{CP}^1, (1-t)g_{FS})$, where g_{FS} is the Fubini–Study metric of $\mathbb{C}P^1$.

Once Theorem 1.5 is available, we study integral curves generated by the Killing vector field $J\nabla f$ on (M, g, f). We show that there exists a sequence of closed integral curves whose lengths have a positive lower bound under suitable rescaled metrics of g. On the other hand, we can use the global Poincaré coordinates on M constructed by Bryant in [3] to prove that the length of those curves should tend to zero. This will lead to a contradiction!

We remark that the real version of Theorem 1.5 is also true.

Theorem 1.6. *Let (M, g, f) be a noncompact κ-noncollapsed steady Ricci soliton with dimension n. Suppose that M has a nonnegative curvature operator and positive Ricci curvature. We also assume that (M, g, f) has a unique equilibrium point. Then, for any $p_i \to \infty$, the sequence of rescaled flows $(M, R(p_i)g(R^{-1}(p_i)t), p_i)$ converges subsequently to a Ricci flow $(\mathbb{R} \times N, \widetilde{g}(t))$ ($t \in (-\infty, 0]$) in the Cheeger–Gromov topology, where*

$$\widetilde{g}(t) = \mathrm{d}s \otimes \mathrm{d}s + g_N(t),$$

and $(N, g_N(t))$ is a pseudo κ-solution on N with dimension $n-1$.

The proof of Theorem 1.6 is the same as Theorem 1.5. Theorem 1.6 gives an asymptotic behavior of κ-noncollapsed steady solitons with a nonnegative curvature operator in higher dimensions.

In a sequel of papers [13], we improve Theorem 1.3 in the sense of positive bisectional curvature as follows.

Theorem 1.7. *There is no κ-noncollapsed steady gradient Kähler–Ricci soliton with positive bisectional curvature.*

The paper is organized as follows. In section 2, we recall some facts on κ-solution. In section 3, we give a generalization of Perelman's compactness theorem for higher-dimensional κ-solutions. In section 4, we analyze the asymptotic geometry of steady solitons and prove Theorem 1.5. In section 5, we prove both of Theorem 1.3 and Theorem 1.4.

2. Preliminary on κ-solutions

A Riemannian metric (M, g) is called a (gradient) Ricci soliton if there exists a smooth function f on M such that

$$R_{ij} - \lambda g_{ij} = \nabla_i \nabla_j f, \tag{2.1}$$

where R_{ij} are components of Ricci curvature of g, ∇ is a co-derivative associated to g, and λ is a constant. (M, g) is called shrinking, steady, or expanding according to $\lambda >, =, < 0$, respectively. In the case that (M, J) is an n-dimensional complex manifold and g is a Kähler metric, then we call (M, g) a Kähler–Ricci soliton. It is easy to see that (2.1) is equivalent to

$$R_{i\bar{j}} - \lambda g_{i\bar{j}} = \nabla_i \nabla_{\bar{j}} f, \quad \nabla_{\bar{i}} \nabla_{\bar{j}} f = 0. \tag{2.2}$$

Let φ_t and ψ_t be the one parameter group generated by vector field ∇f and $J\nabla f$, respectively. Then φ_t, ψ_t are two families of biholomorphisms of M. Moreover ψ_t are isometric transformations since $J\nabla f$ is a Killing vector field (cf. [3]).

Recall that a complete n-dimensional Riemannian manifold (M^n, g) is called κ-noncollapsed if there exist some $\kappa > 0$ such that $\mathrm{vol}(B(p, r)) \geq \kappa r^n$ for any $r > 0$ whenever $|\mathrm{Rm}(q)| \leq r^{-2}$ for all $q \in B(p, r)$. For a solution of Ricci flow, Perelman introduced the following [20].

Definition 2.1. Let $g = g(t)$ be a solution of the Ricci flow on M,

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g), \ t \in (a, b]. \tag{2.3}$$

We say that $(M, g(t))$ is κ-noncollapsed on scales at most r_0 if there exist some $\kappa > 0$ such that

$$\mathrm{vol}(B(p, r, t)) \geq \kappa r^n,$$

whenever $|\mathrm{Rm}(q, t')| \leq r^{-2}$ for all $q \in B(p, r, t)$, $t' \in (t - r^2, t]$, $a \leq t - r^2$, and $0 \leq r \leq r_0$. We say that $(M, g(t))$ is κ-noncollapsed if it is κ-noncollapsed on all scales $r_0 \leq \infty$.

Definition 2.2. A complete solution $(M, g(t))$ of (2.3) is called ancient if it is defined on $(-\infty, 0]$ and the curvature operator of $g(t)$ is bounded and nonnegative for any $t \in (-\infty, 0]$. A complete Kähler–Ricci flow $(M, g(t))$ on $t \in (-\infty, 0]$ is called ancient if the bisectional curvature of $g(t)$ is bounded and nonnegative for any $t \in (-\infty, 0]$. Without confusion, we call a κ-noncollapsed, nonflat ancient solution of (2.3) a κ (Kähler) solution.

For a complete noncompact Riemannian manifold (M, g) with nonnegative Ricci curvature, we define the asymptotical volume by

$$\mathcal{V}(M, g) = \lim_{r \to \infty} \frac{\mathrm{vol}(B(p, r))}{r^n}.$$

Clearly, $\mathcal{V}(M, g)$ is independent of the choice of p. The following result says that it is always zero for an ancient solution $(M, g(t))$ (cf. [20], [18]).

Proposition 2.3. *Suppose that $(M, g(t))$ is a noncompact and nonflat ancient (Kähler) solution. Then $\mathcal{V}(M, g(t)) = 0$ for all $t \leq 0$.*

Next, we define the asymptotical scalar curvature of g by

$$\mathcal{R}(M, g) = \text{limsup}_{\rho(p,x)\to\infty} R(x)\rho^2(p, x),$$

where $\rho(p, \cdot)$ is a distance function from p on M. It is easy to see that $\mathcal{R}(M, g)$ is independent of the choice of p. By Proposition 2.3, we prove the following.

Corollary 2.4. *The asymptotical scalar curvature $\mathcal{R}(M, g(t))$ of a noncompact κ (Kähler) solution $(M, g(t))$ is infinite.*

Proof. We prove the corollary by contradiction. Suppose $\mathcal{R}(M, g(t_0)) < A$ for some positive constant $A > 1$ and $t_0 \leq 0$. For a fixed point $p \in M$, we have $R(x, t_0) \leq Ar^{-2}$ for all $x \in M \setminus B(p, r, t_0)$ when $r > r_0$. Fix any $q \in B(p, 3\sqrt{A}r, t_0) \setminus B(p, 2\sqrt{A}r, t_0)$. Then, replacing r by $\sqrt{A}r$ we have $R(x, t_0) \leq r^{-2}$ for all $x \in B(q, r, t_0)$. Since $(M, g(t_0))$ is κ-noncollapsed and has nonnegative curvature, we get $\text{vol}(B(q, r, t_0)) \geq \kappa r^n$. By the volume comparison theorem,

$$\begin{aligned}\text{vol}(B(p, (3\sqrt{A} + 1)r, t_0) &\geq \text{vol}B(q, r, t_0)\\ &\geq \kappa(3\sqrt{A} + 1)^{-n}(3\sqrt{A} + 1)r)^n, \ \forall\ r > r_0.\end{aligned}$$

It follows that

$$\mathcal{V}(M, g(t)) \geq \kappa(3\sqrt{A} + 1)^{-n}.$$

This is a contradiction to Proposition 2.3! ∎

3. Perelman's compactness theorem

In [20], Perelman proved the following compactness theorem for 3-dimensional κ-solutions.

Theorem 3.1. *Let $(M_k, g_k(t), p_k)$ be a sequence of 3-dimensional κ-solutions on a noncompact manifold M with $R(p_k, 0) = 1$. Then, $(M_k, g_k(t), p_k)$ subsequently converge to a κ-solution.*

To generalize Theorem 3.1 to higher-dimensional κ (Kähler) solutions, we introduce the following.

Definition 3.2. We call a κ-noncollapsed Ricci flow $(M, g(t))$ a pseudo κ (Kähler) solution if it is defined on $M \times (-\infty, 0]$ with a nonnegative curvature operator (nonnegative bisectional curvature) such that the following Harnack inequality holds along the flow:

$$\frac{\partial R}{\partial t} + 2\nabla_i R V^i + 2R_{ij}V^iV^j \geq 0, \ \forall\ V \in TM, \tag{3.1}$$

or in Kähler case,

$$\frac{\partial R}{\partial t} + \nabla_i R V^i + \nabla_{\bar{i}} R V^{\bar{i}} + R_{i\bar{j}}V^iV^{\bar{j}} \geq 0, \ \forall\ V \in T^{(1,0)}M. \tag{3.2}$$

(3.1) or (3.2) implies the Harnack inequality (cf. [14], [4]),

$$\frac{R(x_2, t_2)}{R(x_1, t_1)} \geq e^{-\frac{d^2_{t_1}(x_1,x_2)}{2(t_2-t_1)}}. \tag{3.3}$$

In this section, we prove the following.

Theorem 3.3. *Let $(M_k, g_k(t), p_k)$ be a sequence of n-dimensional κ (Kähler) solutions on a noncompact manifold with $R(p_k, 0) = 1$. Then $(M_k, g_k(t), p_k)$ subsequently converge to a pseudo κ (Kähler) solution of Ricci flow.*

It was mentioned by Morgan–Tian that Perelman's argument still works for higher-dimensional κ-solutions [17, p. 222] (also see [18]). In fact, our proof of Theorem 3.3 below is from Theorem 9.64 in [17], where Perelman's Theorem 3.1 is proved. The proof consists of several technical lemmas below, some of which will also be used in Sections 4 and 5. First we need an elementary lemma (cf. [17]).

Lemma 3.4. *Let (M, g) be a Riemannian manifold and let $p \in M$. Let f be a continuous and bounded function defined on $B(p, 2r) \to \mathbb{R}$ with $f(p) > 0$. Then there is a point $q \in B(p, 2r)$ such that $f(q) \geq f(p)$, $d(p, q) \leq 2r(1 - \alpha)$, and $f(q') < 2f(q)$ for all $q' \in B(q, \alpha r)$, where $\alpha = f(p)/f(q)$.*

By Proposition 2.3 and Lemma 3.4, we prove the following.

Lemma 3.5. *Let $(M_k, g_k(t), p_k)$ be a sequence of n-dimensional ancient solutions of flow (2.3). Let $\nu > 0$. Suppose that there are $p_k \in M_k$ and $r_k > 0$ such that $\mathrm{vol}(B(p_k, r_k, 0)) \geq \nu r_k^{2n}$. Then there is a $C(\nu)$ independent of k such that $r_k^2 R(q, 0) \leq C(\nu)$ for all $q \in B(p_k, r_k, 0)$.*

Proof. We argue by contradiction. Then there is a sequence of points $q_k \in B(p_k, r_k, 0)$ such that $r_k^2 R(q_k, 0) \to \infty$ as $k \to \infty$. Let $f(x, t) = \sqrt{R(x, t)}$. Applying Lemma 3.4 to $f(x, 0)$ defined on $B(q_k, 2r_k, 0)$, we see that there are $q_k' \in B(q_k, 2r_k, 0)$ such that $R(q_k', 0) \geq R(q_k, 0)$ and $R(q, 0) \leq 4R(q_k', 0)$ for all $q \in B(q_k', s_k, 0)$ with $s_k = r_k\sqrt{R(q_k, 0)/R(q_k', 0)}$. Since $\frac{\partial R}{\partial t} \geq 0$ by the Harnack inequality (3.1) (or (3.2)), we get

$$R(q, t) \leq 4R(q_k', 0), \ \forall\, t \leq 0, q \in B(q_k', s_k, 0). \tag{3.4}$$

On the the hand, by the relation

$$\rho_0(p_k, q_k') \leq \rho_0(p_k, q_k) + \rho_0(q_k, q_k') < 3r_k,$$

where $\rho_0(p_k, q_k)$ is a distance function between two points p_k, q_k in M_k with respect to $g_k(0)$, we have

$$\mathrm{vol}(B(q_k', 4r_k, 0)) \geq \mathrm{vol}(B(p_k, r_k, 0)) \geq (\nu/4^{2n})(4r_k)^{2n}.$$

It follows from the Bishop–Gromov volume comparison theorem,

$$\mathrm{vol}(B(q_k', s, 0)) \geq (\nu/4^{2n})s^{2n} \ \forall\ s \leq s_k \leq 3r_k. \tag{3.5}$$

Now we consider the rescaled flows $(M_k, Q_k g(Q_k^{-1}t), q_k')$ with $Q_k = R(q_k', 0)$. By (3.4) and (3.5), we see that the flows are all $(\nu/4^{2n})$-noncollapsed with the scalar curvature bounded by 4 on the geodesic balls of radii $s_k\sqrt{Q_k}$ centered at q_k'. Since $s_k\sqrt{Q_k} = r_k\sqrt{R(q_k, 0)} \to \infty$ as $k \to \infty$, by Hamilton's compactness theorem [15], $(M_k, Q_k g(Q_k^{-1}t), q_k')$ converge subsequently to an ancient solution $(M_\infty, g_\infty(t))$. Note that (3.5) implies that the limit $(M_\infty, g_\infty(0))$ has the maximal volume growth. This is a contradiction to Proposition 2.3. $\square$

Lemma 3.6. *Let $(M, g(t), p)$ be an n-dimensional κ-solution of Ricci flow. Suppose that there exists a point $q \in (M, g(0))$ such that*

$$\rho_0(p, q)^2 R(q, 0) = 1. \tag{3.6}$$

Then, there is a uniform constant $C>0$ independent of $g(t)$ such that $R(x,0)/R(q,0) \le C$ for all $x \in B(q,2d,0)$, where $d=\rho_0(p,q)$.

Proof. Suppose that the lemma is not true. Then there is a sequence of κ-solutions $(M_k, g_k(t), p_k)$ with points $q_k' \in B(q_k, 2d_k, 0)$ such that

$$\lim_{k\to\infty} (2d_k)^2 R(q_k', 0) = \infty,$$

where $d_k = \rho_0(p_k, q_k)$ and $\rho_0(p_k,q_k)^2 R(q,0) = 1$. By Lemma 3.5, it is easy to see that for any $\nu > 0$, there is an $N(\nu)$ such that

$$\mathrm{vol}(B(q_k, 2d_k, 0)) < \nu (2d_k)^{2n} \ \forall\ k > N(\nu).$$

Hence, by taking the diamond method, we may assume that

$$\lim_{k\to\infty} \mathrm{vol}(B(q_k,2d_k,0))/(2d_k)^{2n} = 0. \tag{3.7}$$

In particular,

$$\mathrm{vol}(B(q_k,2d_k,0)) < (\omega_{2n}/2)(2d_k)^{2n} \ \forall\ k \ge k_0,$$

where ω_{2n} is the volume of unit ball in $\mathbb{R}^{2n}$. Therefore, by the Bishop-Gromov volume comparison theorem, there exists an $r_k < 2d_k$ such that

$$\mathrm{vol}(B(q_k, r_k, 0)) = (\omega_{2n}/2) r_k^{2n}. \tag{3.8}$$

Note that by (3.7) and (3.8) we have $\lim_{k\to\infty} r_k/d_k = 0$.

Next we consider a sequence of rescaled ancient flows $(M_k, g_k'(t), q_k)$, where $g_k'(t) = r_k^{-2} g_k(r_k^2 t)$. Then by (3.8), we have

$$\mathrm{vol}(B(q_k, 1+A, g_k'(0))) \ge \mathrm{vol}(B(q_k,1,g_k'(0))) = \frac{\omega_{2n}}{2(1+A)^{2n}}(1+A)^{2n},$$

where $A > 0$ is any fixed constant. Thus by applying Lemma 3.5 to the ball $B(q_k, 1+A; g_k'(0))$, there is a constant $K(A)$ independent of k such that

$$(1+A)^2 R(q, g_k'(0)) \le K(A) \ \forall\ q \in B(q_k, 1+A; g_k'(0)).$$

Hence by the Harnack inequality, the scalar curvature on $B_{g_k'(0)}(q_k, A, 0) \times (-\infty, 0]$ is uniformly bounded by $K(A)$. By Hamilton's compactness theorem, $(M_k, g_k'(t), q_k)$ converges to a limit flow $(M_\infty, g_\infty(t), q_\infty)$. Note by (3.6) that

$$R(q_\infty, g_\infty(0)) = \lim_{k\to\infty} R(q_k, g_k'(0)) = \lim_{k\to\infty} \frac{(r_k)^2}{d_k^2} = 0.$$

Therefore, the strong maximum principle implies that $(M_\infty, g_\infty(t))$ is a flat flow.

At last, we prove that $(M_\infty, g_\infty(t))$ is isometric to the Euclidean space for any $t \le 0$. We need to consider at $t=0$. Fix any $r > 0$. Obviously,

$$\sup_{x\in B(q_\infty, r; g_k'(0))} |\mathrm{Rm}(x)| = 0 \le \varepsilon,$$

where ε can be chosen so that $\frac{\pi}{\sqrt{\varepsilon}} > 2r$. Note that $(M_\infty, g_\infty(t))$ is κ-noncollapsed for each $t \le 0$. Thus we have

$$\mathrm{vol}(B(q_\infty, r; g_\infty(0)) \ge \kappa r^{2n}.$$

It follows from the estimate of Cheeger–Taylor–Gromov [9],

$$\mathrm{inj}(q_\infty) \ge \frac{\pi}{2\sqrt{\varepsilon}} \frac{1}{1 + \frac{\omega_{2n}(r/4)^{2n}}{\mathrm{vol}(B(q_\infty, r/4; g_\infty(0)))}} \ge \frac{\kappa}{\kappa+\omega_{2n}} \cdot r.$$

Hence $B(q_\infty, \frac{\kappa}{\kappa+\omega_{2n}}\cdot r, g_\infty(0))$ is simply connected for all $r>0$. Therefore, M_∞ is simply connected, and consequently $g_\infty(t)$ are all isometric to the Euclidean metric.

The above implies that $\mathrm{vol}(B(q_\infty,1,g_\infty(0)))=\omega_{2n}$. On the other hand, by the convergence of $(M_k, g_k(t), p_k)$ and the relation (3.8), we get

$$\mathrm{vol}(B(q_\infty,1,g_\infty(0)))=\omega_{2n}/2.$$

This is a contradiction. The lemma is proved. □

Lemma 3.7. *Let $(M,g(t),p)$ be a κ-solution with $R(p,0)=1$. Then there exists a $\delta>0$ independent of $g(t)$ such that $R(q,0)\le\delta^{-2}$ for all $q\in B(p,\delta;0)$.*

Proof. By Corollary 2.4, there exists a point $q\in M$ such that

$$\rho_0(p,q)^2R(q,0)=1. \tag{3.9}$$

Applying Lemma 3.6, we get

$$R(x,0)/R(q,0)\le A \ \forall\ x\in B(q,2d,0), \tag{3.10}$$

where $d=\rho_0(p,q)$. It suffices to prove that $R(q,0)\le C_0$ for some $C_0>0$.

By the Harnack inequality, we have

$$R(x,t)\le R(x,0)\ \forall\ x\in B(q,2d,0). \tag{3.11}$$

Thus the Ricci curvature of $g(t)$ is uniformly bounded by $AR(q,0)$ on $B(q,\,2d,0)$ by (3.10). By the flow (2.3), it follows that

$$\frac{\mathrm{d}}{dt}L(t)\ge -AR(q,0)L(t),$$

where $L(t)$ is the length of $\gamma(s)$ with respect to $g(t)$ for any $t\le 0$ and $\gamma(s)$ is a minimal geodesic connecting p and q with respect to $g(0)$. Thus

$$d_t(p,q)\le L(t)\le e^{-AR(q,0)t}L(0)=e^{-AR(q,0)t}R(q,0)^{-1/2}.$$

Choose $t_c=-cR^{-1}(q,0)$, where $0<c<1$ is to be determined. By the Harnack inequality (3.3),

$$\frac{R(p,0)}{R(q,t)}\ge e^{\frac{d_t^2(p,q)}{2t}},$$

we obtain

$$R(q,t_c)\le \exp(e^{2cA}/2c)\le e^{C'/2c}. \tag{3.12}$$

Let $\widetilde{g}(t)=R(q,0)g(R(q,0)^{-1}t)$. By (3.11),

$$|\widetilde{R}(x,t)|\le A\ \forall x\in\ B(q,2d,0).$$

Since the Ricci curvature is nonnegative,

$$\widetilde{B}(q,2,t)\subseteq\widetilde{B}(q,2,0)=B(q,2d,0)\ \forall\ t\le 0.$$

By Shi's higher order estimates for curvature tensors [21], we have

$$|\widetilde{\Delta}\widetilde{R}|(x,t)\le C(A)\ \forall x\in\widetilde{B}(q,1,-1)\ ,t\in(-1,0].$$

It follows that

$$|\Delta R|(x,t)\le CR^2(q,0)\ \forall x\in\widetilde{B}(q,1,-1),\ t\in(-R(q,0)^{-1},0].$$

Hence

$$|\Delta R(q,t)|\le CR^2(q,0)\ \forall\ t\in(-R(q,0)^{-1},0]. \tag{3.13}$$

By (3.13) and the equation

$$\frac{\partial R}{\partial t} = \triangle R + 2|\mathrm{Ric}|^2,$$

we have

$$|\frac{\partial}{\partial t} R(q,t)| \le C' R^2(q,0).$$

By (3.12), it follows that

$$R(q,0) \le R(q,t_c) + C'|t_c| R^2(q,0)^2 \le e^{C/2c} + cC' R(q,0).$$

Thus by choosing $c = \frac{1}{2}(C')^{-1}$, we derive

$$R(q,0) \le C_0.$$

Let $\delta = \sqrt{(AC_0)^{-1}}$. Then the lemma immediately follows from (3.9) and (3.10). $\square$

Proof of Theorem 3.3. By Lemma 3.7, the κ-noncollapsed condition of $(M_k, g_k(t))$ implies

$$\mathrm{vol}(B(p_k,\delta,0)) \ge \kappa\delta^{2n},$$

where $\delta > 0$ is a uniform number. By the Bishop–Gromov volume comparison theorem, we have

$$\mathrm{vol}(B(p_k,\delta+r,0)) \ge \mathrm{vol}(B(p_k,\delta,0)) \ge \frac{\kappa}{(1+(r/\delta))^{2n}}(\delta+r)^{2n} \ \forall\ r > 0.$$

Applying Lemma 3.5 to each ball $B(p_k,\delta+r,0)$, we see that there is a $C(r)$ independent of k such that

$$R(q,0) \le C(r)(r+\delta)^{-2} \ \forall\ q \in B(p_k,\delta+r,0).$$

By the Harnack inequality, we also get

$$R(q,t) \le C(r)(r+\delta)^{-2} \ \forall\ q \in B(p_k,\delta+r,0).$$

Hence, Hamilton's compactness theorem implies that $(M_k, g_k(t), p_k)$ subsequently converges to a limit Ricci flow $(M_\infty, g_\infty(t))$ with a nonnegative curvature operator (or a nonnegative bisectional curvature) for any $t \le 0$ [15]. Moreover, $g_\infty(t)$ satisfies the Harnack inequality (3.1) or (3.2) since $g_k(t)$ satisfies the corresponding Harnack inequality (cf. [14], [4]). $\square$

By using the argument in the proof of Theorem 3.3, we have the following pointwisely estimate for the Laplace of scalar curvature.

Proposition 3.8. *Let $(M, g(t))$ be a κ-solution. Then there is a constant C independent of p, t such that*

$$\frac{|\Delta R(p,t)|}{R^2(p,t)} \le C \quad \forall\ (p,t) \in M \times (-\infty, 0].$$

Proof. On the contrary, we can find a sequence of p_i and t_i such that

$$\lim_{i\to\infty} \frac{|\Delta R(p_i,t_i)|}{R^2(p_i,t_i)} = \infty. \tag{3.14}$$

Consider a sequence of rescaled flows $(M, g_i(t), p_i)$ with

$$g_i(t) = R(p_i,t_i) g(R^{-1}(p_i,t_i)t + t_i).$$

Then $R(p_i, g_i(0)) = 1$. As in the proof of Theorem 3.3, we see that there is a constant C independent of i such that

$$R(q, g_i(0)) \leq C \ \forall \ q \in B(p_i, 1, g_i(0)).$$

Since the bisectional curvature is positive and the Harnack inequality holds on $(M, g_i(t))$, we have $\frac{\partial}{\partial t} R(q, g_i(t)) \geq 0$. Thus

$$|\mathrm{Rm}(q, g_i(t))| \leq C(n) R(q, g_i(t)) \leq C(n) R(q, g_i(0)) \leq C(n) C,$$

where $(q, t) \in B(p_i, 1, g_i(0)) \times [-1, 0]$.

Since the Ricci curvature $\mathrm{Ric}(q, g_i(t))$ is nonnegative along the flow, the metric $g_i(q, t)$ is decreasing along the flow. It follows that

$$B(p_i, 1, g_i(-1)) \subset B(p_i, 1, g_i(0)).$$

Hence,

$$|\mathrm{Rm}(q, g_i(t))| \leq C(n) C \ \forall \ (q, t) \in B(p_i, 1, g_i(-1)) \times [-1, 0].$$

By Shi's higher order estimate, it follows that

$$|\Delta R(q, g_i(t))| \leq C', \ \forall \ q \in B(p_i, \frac{1}{2}, g_i(-\frac{1}{2})) \times [-\frac{1}{2}, 0].$$

In particular,

$$\frac{|\Delta R(p_i, t_i)|}{R^2(p_i, t_i)} = |\Delta R(p_i, g_i(0))| \leq C'.$$

This is in contradiction with (3.14). $\square$

Proposition 3.8 will be used in the proof of Theorem 1.5 in next section.

4. Asymptotical geometry of solitons

In this section, we use Theorem 3.3 to prove Theorem 1.5. Let ϕ_t be a family of biholomorphisms generated by $-\nabla f$. Let $g(t) = \phi_t^*(g)$. Then $g(t)$ satisfies the Ricci flow (2.3). In [12], the authors proved that there exists a unique equilibrium point o such that $\nabla f(o) = 0$ for a steady gradient Kähler–Ricci soliton with nonnegative bisectional curvature and positive Ricci curvature. Thus for any $p \in M \setminus \{o\}$, it is easy to see that $\phi_t(p)$ converge to o as $t \to \infty$. In the following, we show that the growth order of $\rho(o, \phi_t(p))$ is actually equivalent to $|t|$ as $t \to -\infty$.

Lemma 4.1. *Let o be the equilibrium point defined above. Then for any $p \in M \setminus \{o\}$, there exist constants $C_1, C_2 > 0$ and $t_0 \leq 0$ such that*

$$C_1 |t| \leq \rho(o, \phi_t(p)) \leq C_2 |t| \ \forall \ t \leq t_0. \tag{4.1}$$

Proof. By the identity (cf. [16])

$$R + |\nabla f|^2 = A_0, \tag{4.2}$$

where A_0 is a constant, we have

$$|\nabla f|^2(x) + R(x) = R(o) \quad \forall x \ \in M.$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\phi_t(p)) = -|\nabla f|^2(\phi_t(p)) \quad \forall t \ \leq 0$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\phi_t(p)) = \frac{\mathrm{d}}{\mathrm{d}t} R(\phi_t(p)) = \mathrm{Ric}(\nabla f, \overline{\nabla f}) \geq 0 \quad \forall t \ \leq 0.$$

Thus

$$0 \leq R(\phi_t(p)) \leq R(p) \ \forall t \ \leq 0$$

and

$$R(o) - R(p) \leq -\frac{\mathrm{d}}{\mathrm{d}t} f(\phi_t(p)) \leq R(o) \ \forall t \ \leq 0.$$

It follows that

$$(R(o) - R(p))|t| \leq f(p) - f(\phi_t(p)) \leq R(o)|t| \quad \forall t \ \leq 0. \tag{4.3}$$

Consequently,

$$(R(o) - R(p))|t| + C(p) \leq f(o) - f(\phi_t(p)) \leq R(o)|t| + C(p) \quad \forall t \leq 0,$$

where $C(p) = f(o) - f(p)$. On the there hand, by Proposition 7 in [7], there are constants $C_1, C_2 > 0$ such that

$$C_1 \rho(o, \phi_t(p)) \leq f(o) - f(\phi_t(p)) \leq C_2 \rho(o, \phi_t(p)), \tag{4.4}$$

where $t \leq t_0$ and t_0 is small enough constant. Combining the above two inequalities, we get (4.1). □

Remark 4.2. Let $A(r) = \{p \in M : f(p) = r\}$ for any $r \in \mathbb{R}$. Then $A(r)$ is compact as $r \gg 1$ since f is strictly convex. Thus from the proof of Lemma 4.1, the constants C_1 and C_2 in (4.1) can be chosen uniformly for all $p \in A(r)$ so that both of them are independent of t.

Combining Lemma 4.1 and Proposition 3.8, we obtain a lower bound growth estimate for scalar curvature.

Proposition 4.3. *For a κ-noncollapsed steady Kähler–Ricci soliton (M, g) with nonnegative bisectional curvature and positive Ricci curvature, the scalar curvature satisfies*

$$\frac{C}{\rho(x)} \leq R(x), \text{ if } \rho(x) \geq r_0, \tag{4.5}$$

where $\rho(x) = \rho(o, x)$ and $C > 0$ is a uniform constant.

Proof. Sine the scalar curvature $R(p, t)$ of $g(p, t)$ satisfies

$$\frac{\partial}{\partial t} R(p, t) = \Delta R(p, t) + 2|\mathrm{Ric}(p, t)|^2,$$

by Proposition 3.8, there is a positive constant $C > 0$ such that

$$|\frac{\partial}{\partial t} R^{-1}(p, t)| \leq \frac{|\Delta R(p, t)|}{R^2(p, t)} + \frac{2|\mathrm{Ric}(p, t)|^2}{R^2(p, t)} \leq C + 2,$$

and consequently,

$$R(p, t)|t| \geq \frac{|t|}{(C+2)|t| + R(p, 0)^{-1}} \geq \frac{1}{2(C+2)} \tag{4.6}$$

as long as $|t|$ is large enough.

Next we show that (4.6) implies (4.5). We may assume $f(o) = 0$. For any x such that $f(x) \gg 1$, there exists $p_x \in \{q \in M | f(q) = 1\}$ and $t_x < 0$ such that $\phi_{t_x}(p_x) = x$. By (4.6) together with (4.3) and(4.4), we have

$$\begin{aligned} R(x) \geq & \frac{1}{|t_x|} \cdot \frac{1}{(C+2) + (R(p_x)|t_x|)^{-1}} \\ \geq & \frac{R(o) - R(p_x)}{f(x) - f(p_x)} \cdot \frac{1}{(C+2) + (R(p_x)|t_x|)^{-1}} \\ \geq & \frac{R(o) - R(p_x)}{2(f(x) - f(o))} \cdot \frac{1}{(C+2) + (R(p_x)|t_x|)^{-1}} \\ \geq & \frac{R(o) - M_1}{2C_2\rho(x)} \cdot \frac{1}{2(C+2)} \ \forall \ |t_x| \geq \frac{C+2}{R(p_x)}. \end{aligned}$$

Here $M_1 = \sup_{q \in \{f=1\}} R(q)$. On the other hand, by (4.3), we have

$$|t_x| \geq \frac{f(x) - f(p_x)}{R(o) - R(p_x)} = \frac{f(x) - 1}{R(o) - R(p_x)}.$$

Then it holds that

$$R(x) \geq \frac{R(o) - M_1}{4C_2(C+2)\rho(x)} \geq \frac{1}{C_3(C+2)\rho(x)},$$

as long as $f(x) \geq \frac{C+2}{m_1} \cdot (R(o) - m_1) + 1$, where $m_1 = \inf_{q \in \{f=1\}} R(q)$. Note that C, C_3, and m_1 are all independent of x, t. Hence, by (4.4), we get (4.5). □

Now we are ready to prove Theorem 1.5.

Proof Theorem 1.5. By Proposition 4.3, we have

$$\lim_{i \to \infty} \rho^2(o, p_i) R(p_i, 0) = \infty. \tag{4.7}$$

Let $\hat{g}_i(t) = R(p_i, 0) g(R^{-1}(p_i, 0)t)$ be a sequence of rescaled Ricci flows of $g(t)$. Clearly, $R(p_i; \hat{g}_i(0)) = 1$. Then applying Theorem 3.3 to $(M, \hat{g}_i(t), p_i)$, we see that $(M, \hat{g}_i(t), p_i)$ converges to a pseudo κ Kähler solution $(M_\infty, \tilde{g}(t), p_\infty)$ of (2.3). Moreover, by (4.7) and the nonnegative sectional curvature condition, we can construct a geodesic line through p_∞ in $(M_\infty, \tilde{g}(t), p_\infty)$ (cf. Theorem 5.35 in [17]). Thus by the Cheeger–Gromoll splitting theorem [8], $(M_\infty, \tilde{g}(0))$ must split off a line. Let X be the vector field tangent to the line with the norm equal to 1 and J_∞ the complex structure on M_∞. Then $J_\infty X$ generates a geodesic curve $\gamma(s)$ in M_∞. If $\gamma(s)$ is not closed, it is a geodesic line on M_∞. If $\gamma(s)$ is closed, it is a flat $\mathbb{S}^1$. Hence $(M_\infty, \tilde{g}(0))$ splits off a complex line $N_1 = \mathbb{C}^1$ or a cylinder $N_1 = \mathbb{R}^1 \times \mathbb{S}^1$. Namely, $M_\infty = N_1 \times N_2$ and $\tilde{g}(t) = \mathrm{d}z \otimes \mathrm{d}\overline{z} + g_{N_2}(t)$, where $g_{N_2}(t)$ is a pseudo κ Kähler solution of (2.3) on a complex manifold N_2 with dimension $n-1$.

In case $\dim_{\mathbb{C}}(M) = 2$, $(M_\infty, \tilde{g}(t)) = (N_1 \times N_2, \mathrm{d}z \otimes \mathrm{d}\overline{z} + g_{N_2}(t))$, where g_{N_2} is a pseudo κ Kähler solution of (2.3) on a surface N_2. In particular, the scalar curvture $\tilde{R}(\cdot, t)$ of $g_{N_2}(t))$ satisfies Harnack inequality

$$\frac{\partial}{\partial t} \tilde{R}(\cdot, t) \geq 0 \text{ in } N_2 \times (-\infty, 0]. \tag{4.8}$$

By Lemma 4.4 below, we see that $(N_2, g_{N_2}(t)) = (\mathbb{CP}^1, (1-t)g_{FS})$. □

Since Theorem 3.3 holds for κ-solutions and all lemmas in this section are true for all steady Ricci solitons, one can prove Theorem 1.6 by the same argument as in the proof of Theorem 1.5.

The following lemma is a generalization of Corollary 11.3 in [20] which says: Any oriented κ-solution on a surface is a shrinking round sphere.

Lemma 4.4. *Any oriented pseudo κ-solution $(M, g(\cdot, t))$ $(t \le 0)$ on a surface is a shrinking round sphere.*

Proof. By Corollary 11.3 in [20], it suffices to exclude the case that $(M, g(t))$ is noncompact and has unbounded curvature. In this case, we may assume that there is a sequence of points p_i such that $R(p_i, -1) \to \infty$ and $\rho_{g(-1)}(p_0, p_i) \to \infty$, where p_0 is a fixed point. In particular,

$$\rho^2_{g(-1)}(p_0, p_i) R(p_i, -1) \to \infty, \text{ as } i \to \infty. \tag{4.9}$$

By taking $f(x,t) = \sqrt{R(x,t)}$ and $r = r_i = \frac{1}{4}\rho_{g(-1)}(p_0, p_i)$ in Lemma 3.4, we can find a sequence of points q_i such that $R(q_i, -1) \ge R(p_i, -1)$ and

$$R(q, -1) \le 4R(q_i, -1) \ \forall\ q \in B(q_i, d_i, -1),$$

where $d_i \sqrt{R(q_i, -1)} = r_i \sqrt{R(p_i, -1)}$. Moreover,

$$\rho_{g(-1)}(p_i, q_i) \le 2r_i = \frac{1}{2}\rho_{g(-1)}(p_0, p_i).$$

Hence

$$\rho_{g(-1)}(p_0, q_i) \ge \rho_{g(-1)}(p_0, p_i) - \rho_{g(-1)}(p_i, q_i) \ge \frac{1}{2}\rho_{g(-1)}(p_0, p_i).$$

It follows that

$$\lim_{i \to \infty} \rho^2_{g(-1)}(p_0, q_i) R(q_i, -1) = \infty. \tag{4.10}$$

Now, we consider a sequence of rescaled Ricci flows $(M_i, g_i'(t), q_i)$, where $g_i'(t) = R(q_i, -1) g(R^{-1}(q_i, -1)(t+1) - 1)$. Since $\frac{\partial}{\partial t} R \ge 0$, we have

$$R_{g_i'}(q, t) \le 4 \ \forall\ q \in B(q_i, r_i \sqrt{R(p_i, -1)}, g_i'), t \le -1.$$

Note that $r_i \sqrt{R(p_i, -1)}$ go to infinity as $i \to \infty$ by (4.9). This means that the curvature of flows are locally uniformly bounded. Together with the κ-noncollapsed condition, $(M_i, g_i'(t), q_i)$ converge to a limit Ricci flow $(M_\infty, g_\infty(t), q_\infty)$ for $t \le -1$. Moreover it is a pseudo κ Kähler solution. On the other hand, by (4.10) and the nonnegative sectional curvature condition, one can construct a geodesic line through q_∞ in $(M_\infty, g_\infty, q_\infty)$ (cf. Theorem 5.35 in [17]). Thus $(M_\infty, g_\infty(-1)))$ splits off a line. As a consequence, it is isometric to $\mathbb{C}^1$ or $\mathbb{R}^1 \times \mathbb{S}^1$ with the flat metric. But this is impossible since $R(q_\infty, -1) = 1$. The lemma is proved. $\square$

Remark 4.5. A complete classification of ancient flow has been obtained by Daskalopoulos–Hamilton–Sesum (cf. [11], [10]). They need to assume that the curvature of the flow is uniformly bounded. In Lemma 4.4, the curvature of the flow can be unbounded.

As an application of Theorem 1.5, we get the following precise estimate for scalar curvature of steady Ricci solitons on a complex surface.

Corollary 4.6. *Let (M, g, f) be a 2-dimensional κ-noncollapsed steady Kähler–Ricci soliton with positive sectional curvature. Let $o \in M$ be the unique equilibrium point such that $\nabla f(o) = 0$ and $p \neq o$. Then*

$$R(p,t)|t| \to 1, \ \text{as } t \to -\infty. \tag{4.11}$$

As a consequence, there are constants C_1 and C_2 such that

$$\frac{C_1}{\rho(x)} \leq R(x) \leq \frac{C_2}{\rho(x)}. \tag{4.12}$$

Proof. We first prove the following claim.

Claim 4.7.

$$\lim_{t\to\infty} \frac{\partial}{\partial t} R^{-1}(p,-t) = 1. \tag{4.13}$$

Moreover, the convergence is uniform for all $p \in A(1)$, where $A(1) = \{q \in M | f(q) = 1\}$.

Proof of the claim. We prove the claim by contradiction. On the contrary, we can find $\delta > 0$, $p_{(i)} \in A(1)$, and $t_i \to \infty$ such that

$$|\frac{\partial}{\partial t} R^{-1}(p_{(i)}, -t_i) - 1| \geq \delta > 0. \tag{4.14}$$

Let ϕ_t be the group of biholomorphisms generated by $-\nabla f$ and let $g(t)$ be the corresponding Kähler–Ricci flow. Let $p_i = \phi_{t_i}(p_{(i)})$. Consider a sequence of rescaled Ricci flows $(M, \hat{g}_i(t), p_i)$ as in Theorem 1.5, where $\hat{g}_i(t) = R(p_i, 0)\ g(R^{-1}(p_i, 0)t)$. Then $(M, \hat{g}_i(t), p_i)$ subsequently converge to a limit Ricci flow $(M_\infty, \tilde{g}(t), p_\infty)$, while $(M_\infty, \tilde{g}(0), p_\infty)$ are isometric to $(N_1 \times \mathbb{CP}^1, dz \otimes d\bar{z} + g_{FS})$. Moreover, by the flow equation for scalar curvature $\tilde{R}(\cdot, t)$ of $\tilde{g}(t)$ at $(p_\infty, 0)$,

$$\frac{\partial}{\partial t}\tilde{R}(p_\infty, 0) = \Delta \tilde{R}(p_\infty, 0) + 2|\tilde{\mathrm{Ric}}|^2(p_\infty, 0),$$

we get

$$\frac{\partial}{\partial t}\tilde{R}(p_\infty, 0) = 1.$$

On the other hand, by the convergence of $(R(p_i, 0)g(R^{-1}(p_i, 0)t, p_i)$, we have

$$\frac{\partial}{\partial t}\tilde{R}(p_\infty, 0) = \lim_{i\to\infty} \frac{1}{R^2(p_i, 0)} \frac{\partial}{\partial t} R(p_i, 0) = \lim_{i\to\infty} \frac{1}{R^2(p_{(i)}, -t_i)} \frac{\partial}{\partial t} R(p_{(i)}, -t_i).$$

Thus

$$\lim_{i\to\infty} G(p_{(i)}, t_i) = 1,$$

where $G(p,t) = \frac{\partial}{\partial t} R^{-1}(p, -t)$. This is a contradiction to (4.14). Hence the claim is true. ☐

By Claim 4.7, for any $\epsilon > 0$, there exists a $t(\epsilon) < 0$ such that

$$R(p,t)|t| \leq \frac{1}{1-\epsilon}\ \forall\ p \in A(1),\ t \leq t(\epsilon). \tag{4.15}$$

We may assume $f(o) = 0$. For any x such that $f(x) \gg 1$, we can find $p_x \in \{q \in M | f(q) = 1\}$ and $t_x < 0$ such that $\phi_{t_x}(p_x) = x$. By (4.15) together with (4.3) and (4.4), we have

$$\begin{aligned} R(x) \leq & \frac{R(o)}{f(x) - f(p_x)} \cdot \frac{1}{1-\epsilon} \\ \leq & \frac{2R(o)}{f(x) - f(o)} \cdot \frac{1}{1-\epsilon} \\ \leq & \frac{2R(o)}{C\rho(x)} \cdot \frac{1}{1-\epsilon} \ \forall\ |t_x| \geq |t(\epsilon)|. \end{aligned}$$

Note that by (4.3) we have

$$|t_x| \geq \frac{f(x) - f(p_x)}{R(o) - R(p_x)} = \frac{f(x) - 1}{R(o) - R(p_x)}.$$

Thus as long as $f(x) \geq |t(\epsilon)| \cdot (R(o) - m_1) + 1$, where $m_1 = \inf_{q \in \{f=1\}} R(q)$, we obtain

$$R(x) \leq \frac{2R(o)}{C\rho(x)} \cdot \frac{1}{1-\epsilon}.$$

The proof is finished. □

5. Nonexistence of the noncollapsed steady Kähler–Ricci soliton

In this section, we prove Theorem 1.3 and Theorem 1.4. First we recall a result of Bryant about the existence of global Poincaré coordinates on a steady Kähler–Ricci soliton [3].

Theorem 5.1. *Let (M, g, f) be a steady Kähler–Ricci soliton with positive Ricci curvature, which admits an equilibrium point on M. Let $Z = \frac{\nabla f - \sqrt{-1} J \nabla f}{2}$. Then there exist global holomorphic coordinates (Poincaré coordinates) $z : M \to \mathbb{C}^n$ which linearize Z. Namely, there are positive constants $h_1, \ldots, h_n$ such that*

$$Z = \sum_{i=1}^{n} h_i z_i \frac{\partial}{\partial z_i}. \tag{5.1}$$

Corollary 5.2. *Let (M, g, f) be a steady Kähler–Ricci soliton with nonnegative bisectional curvature and positive Ricci curvature. Then, there exists a sequence of points $p_k \to \infty$ such that every integral curve $\gamma_k(s)$ of $J\nabla f$ starting from p_k is closed with the same period time. Moreover, the length of $\gamma_k(s)$ is uniformly bounded from above.*

Proof. By Theorem 1.1 in [12], there exists a unique equilibrium point on M. According to Theorem 5.1, we see that there exist global Poincaré coordinates $(z_1, \ldots, z_n)$ on M such that $Z = \frac{\nabla f - \sqrt{-1} J \nabla f}{2}$ satisfies (5.1).

Let $z_i = x_i + \sqrt{-1} y_i$. Then

$$J\nabla f = \sum_{i=1}^{n} h_i (x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}).$$

Choose points $p_k = (k, 0, \ldots, 0, 0 \ldots, 0) \in M$. Then the integral curves of $J\nabla f$ starting from p_k are given by

$$\gamma_k(s) = (k\cos(h_1 s), k\sin(h_1 s), 0, \ldots, 0, 0 \ldots, 0).$$

Clearly, these curves are all closed with period time $\frac{2\pi}{h_1}$. By the identity (4.2),

$$|\gamma_k'(s)| = |\nabla f|(\gamma_k(s)) \le A_0^{\frac{1}{2}}, \text{ as } k \to \infty.$$

Hence the length l_k of $\gamma_k(s)$ has a uniformly upper bound:

$$l_k = \int_0^{\frac{2\pi}{h_1}} |\gamma_k'(s)| \mathrm{d}s \le A_0^{\frac{1}{2}} \frac{2\pi}{h_1}. \tag{5.2}$$

□

In the remainder of this section, we use the estimates in Section 4 to get a lower bound of l_k to derive a contradiction. First, we need the following fundamental lemma.

Lemma 5.3. *Let $B(p,r)$ be a geodesic ball with radius r centered at p in a Riemannian manifold (M,g), and let X be a smooth vector field such that $|X|_g(x) \ge C_0$ and $|\nabla X|(x) \le C$ for any $x \in B(p,r)$, where C is a positive constant independent of $x \in B(p,r)$. Let $\gamma(s)$ be the integral curve of X starting from p and we assume that $\gamma(s)$ stays in $B(p,r)$ for all $s \in [0,\infty)$. Then there exists $c_0 > 0$, which depends only on r, C, C_0, and the metric g on $B(p,r)$, such that $\gamma(s)$ is away from p for all $s \in (0, c_0]$ and*

$$\text{Length}(\gamma(s)) \ge c_0 C_0. \tag{5.3}$$

Proof. Suppose that r_p is the injective radius at $p \in M$. Set $r_0 = \min\{r_p, \frac{r}{2}\}$. By the exponential map, we can choose a normal coordinate $(x_1, \ldots, x_n)$ on $B(p, r_0)$. Let $X(p) = (X_1(p), X_2(p), \ldots, X_n(p))$. We may assume that $|X_k(p)| = \max_{1\le i\le n} |X_i(p)|$. Then $|X_k(p)| \ge \frac{C_0}{\sqrt{n}}$. Note that

$$\begin{aligned} x_k(\gamma(s)) - x_k(\gamma(0)) =& \frac{dx_k(\gamma(0))}{ds} \cdot s + \frac{d^2 x_k(\gamma(\theta s))}{ds^2} \cdot s^2 \\ =& s(X_k(p) + \frac{d^2 x_k(\gamma(\theta s))}{ds^2} \cdot s) \end{aligned}$$

and

$$\begin{aligned} |\frac{d^2 x_k(\gamma(s))}{ds^2}| =& \left| \nabla_X X - \Gamma_{ij}^k \frac{dx_i(\gamma(s))}{ds} \frac{dx_j(\gamma(s))}{ds} \right| \\ \le& C_1 |\nabla_X X|_g + C_2 |X(\gamma(s))|_g \cdot \max_{1\le i,j,k\le n, x\in B(x,r)} |\Gamma_{ij}^k(x)| \\ \le& C_3, \end{aligned}$$

where C_3 is independent of $s \in [0, r_0]$. Choose $c_0 = \min\{r_0, \frac{C_0}{2\sqrt{n}C_3}\}$. Then

$$|X_k(p) + \frac{d^2 x_k(\gamma(\theta s))}{ds^2} \cdot s| \ge \frac{1}{2}|X_k(p)| > 0 \ \forall\ s \in (0, c_0]. \tag{5.4}$$

It follows that

$$|x_k(\gamma(s)) - x_k(\gamma(0))| \ge \frac{1}{2} s |X_k(p)| > 0 \ \forall\ s \in (0, c_0].$$

Thus, (5.3) is true. Hence, the lemma is proved. □

By Lemma 5.3, we prove the following lemma.

Lemma 5.4. *Let (M, g, f) be an n-dimensional κ-noncollapsed steady Kähler–Ricci soliton with nonnegative sectional curvature and positive Ricci curvature. Let p_k be the sequence of points constructed in Corollary 5.2. Then, there exists a positive constant C such that $R(p_k) > C$, where C is independent of p_k.*

Proof. We use the contradiction argument and suppose that $R(p_k) \to 0$ as $k \to \infty$. Let $g_k(t) = R(p_k)g(R^{-1}(p_k)t)$. Then by Theorem 1.5, the sequence of Ricci flows $(M, g_k(t), p_k)$ converge subsequently to a limit flow $(M_\infty, g_\infty(\cdot, t), p_\infty)$. Fix $r > A_0^{\frac{1}{2}} \frac{2\pi}{h_1}$ (cf. Corollary 5.2). Applying Lemma 3.7 to flows $(M, g_k(t), p_k)$, there is a positive constant $C = C(r)$ independent of k such that

$$\frac{R(x)}{R(p_k)} \leq C \ \forall \ x \in B_{g_k(0)}(p_k, r). \tag{5.5}$$

Thus

$$R(x) \to 0 \ \forall \ x \in B_{g_k(0)}(p_k, r). \tag{5.6}$$

Moreover, the convergence is uniform for $x \in B_{g_k(0)}(p_k, r)$.

Let $X_{(k)} = R(p_k)^{-\frac{1}{2}} J\nabla f$. Then

$$|X_{(k)}|^2_{g_k(0)}(x) = |\nabla f|^2(x) = A_0 - R(x).$$

By identity (4.2) together with condition (5.6), it follows

$$\lim_{k\to\infty} \sup_{B_{g_k(0)}(p_k,r)} ||X_{(k)}|_{g_k(0)} - \sqrt{A_0}| = 0.$$

By Shi's higher order estimate [21] and soliton equation (2.2), we also get

$$\sup_{B_{g_k(0)}(p_k,r)} |\tilde{\nabla}^m_{(g_k(0))} X_{(k)}|_{g_k(0)} \leq C(n) \sup_{B_{g_k(0)}(p_k,r)} |\tilde{\nabla}^{m-1}_{(g_k(0))} \mathrm{Ric}(g_k(0))|_{g_k(0)} \leq C_1,$$

where $\tilde{\nabla}$ denotes the connection with respect to the rescaled metric $g_k(0)$. As a consequence, the restricted vector field X_k on $B_{g_k(0)}(p_k, r)$ converges to a smooth vector field X_∞ on $B_{g_\infty(0)}(p_\infty, r) \subset M_\infty$ in C^∞-topology. On the other hand,

$$\tilde{\nabla}_{(g_k(0))J\nabla f}(J\nabla f) = \nabla_{J\nabla f}(J\nabla f) = -\nabla_{\nabla f}(\nabla f) = \nabla R. \tag{5.7}$$

Then

$$|\tilde{\nabla}_{(g_k(0))X_{(k)}} X_{(k)}|_{g_k(0)} = \frac{|\nabla R|(x)}{R^{\frac{1}{2}}(p_k)}.$$

Note that by (5.5) and Shi's higher order estimate,

$$\frac{|\nabla R|(x)}{R^{\frac{3}{2}}(p_k)} \leq C' \ \forall \ x \in B_{g_k(0)}(p_k, r). \tag{5.8}$$

Thus we get

$$|\tilde{\nabla}_{(g_\infty(0))X_{(\infty)}} X_{(\infty)}|_{g_\infty(0)} = \lim_{k\to\infty} |\tilde{\nabla}_{(g_k(0))X_{(k)}} X_{(k)}|_{g_k(0)} = 0, \tag{5.9}$$

where the convergence is uniform on $B_{g_k(0)}(p_k, r)$.

By the convergence, there are diffeomorphisms $\Phi_k : B_{g_k(0)}(p_k, r) \to M_\infty$ such that $\Phi_k(p_k) = p_\infty$, $\Phi_k(g_k(0)) \to g_\infty(0)$, and

$$(\Phi_k)_*(X_k) \to X_\infty, \ \text{as } k \to \infty.$$

By (5.9), it follows that

$$|\tilde{\nabla}_{(g_\infty(0))\overline{X}_{(k)}}\overline{X}_{(k)}|_{g_\infty(0)} \to 0, \text{ as } k \to \infty,$$

where $\overline{X}_{(k)} = (\Phi_k)_*(X_k)$. Let $\overline{\gamma}_k = \Phi_k(\gamma_k)$. Clearly $\overline{\gamma}_k \subset B_{g_\infty(0)}(p_\infty, r)$ as long as k is sufficiently large, since $\gamma_k \subset B_{g_k(0)}(p_k, r)$ by the choice of r. Then we can apply Lemma 5.3 to $\overline{\gamma}_k$ to see that there are constants $c_0, A > 0$, which depend only on the metric $g_\infty(0)$ on $B_{g_\infty(0)}(p_\infty, r)$ such that

$$\text{Length}(\overline{\gamma}_k, g_\infty(0)) \geq A$$

and $d(\overline{\gamma}_k(s), p_\infty) > 0$ for all $s \in (0, c_0]$. It follows that

$$\text{Length}(\gamma_k, g_k(0)) \geq \frac{1}{2}\text{Length}(\overline{\gamma}_k, g_\infty(0)) \geq \frac{1}{2}A$$

and $d(\gamma_k(s), p_k) > 0$ for all $s \in (0, c_0]$, as long as k is sufficiently large. On the other hand, by (5.2), we have

$$\text{Length}(\gamma_k, g_k(0)) \leq \frac{2\pi}{h_1}A_0^{\frac{1}{2}}R(p_k)^{\frac{1}{2}} \to 0, \text{ as } k \to \infty.$$

Hence we get a contradiction! The lemma is proved. □

Combining Lemma 5.4 and Corollary 4.6 in Section 4, we prove Theorem 1.4 in the surfaces case.

Proposition 5.5. *Let (M, g, f) be a 2-dimensional κ-noncollapsed steady Kähler–Ricci soliton with nonnegative sectional curvature. Then (M, g) is flat.*

Proof. If (M, g) is compact, then applying the maximum principle to the identity

$$\Delta f + |\nabla f|^2 = A_0,$$

it is easy to see that f is constant of g and so (M, g) is flat. If the soliton is not flat, then we may assume that (M, g) is a κ-noncollapsed, noncompact steady Kähler–Ricci soliton with positive Ricci curvature by the Cao dimension reduction theorem in [6].

Let $(z_1, \ldots, z_n)$ be the Poincaré coordinates as in Theorem 5.1 and let $\phi(t)$ be a family of diffeomorphisms generated by $-2\text{Re}(Z) = -\nabla f$. Let $p = (1, 0, 0, \ldots, 0)$. Then one can check $z_i(\phi_t(p)) = e^{-h_i t}z_i(p)$ (cf. Theorem 3 in [3]). Namely, $Z(\phi_t(p)) = (e^{-h_1 t}, 0, \ldots, 0)$. For $p_k = (k, 0, \ldots, 0)$ in Corollary 5.2, we see that $p_k = \phi_{t_k}(p)$ and $t_k = -\frac{\ln k}{h_1}$. By Lemma 5.4, we have $R(p_k) > C$ for some positive constant C independent of p_k. On the other hand, by (4.11) in Corollary 4.6 we have

$$R(p_k)\frac{\ln k}{h_1} = R(p, t_k)|t_k| \to 1 \text{ as } k \to \infty.$$

Hence, we get a contradiction. The proposition is proved. □

Now, we prove Theorem 1.3.

Proof of Theorem 1.3. We prove it by induction on the complex dimension of M. By Proposition 5.5, we suppose that there is no l-dimensional κ-noncollapsed steady Kähler–Ricci soliton with nonnegative sectional curvature and positive Ricci curvature for all $l < n$. To generalize the argument in the proof of Proposition 5.5 to higher dimensions, we only need to find a sequence of $R(p_k)$ as in Lemma 5.4 such that $\lim_{k\to\infty} R(p_k) \to 0$. In fact, we prove the following claim.

Claim 5.6. Let o be the unique equilibrium of M. Then, under the induction hypothesis, for any fixed $p \in M \setminus \{o\}$, $R(p,-t) \to 0$ as $t \to \infty$.

Proof. By the Harnack inequality, we have $\frac{\partial}{\partial t} R(p,t) \geq 0$. Note that $R(p,t) \geq 0$. So, $\lim_{t\to-\infty} R(p,t)$ exists. Then there exists a point $p \in M$ such that

$$\lim_{t\to-\infty} R(p,t) = C > 0, \tag{5.10}$$

if the claim is not true. Consider the sequence $(M, g_\tau(t), p_\tau)$, where $g_\tau = R(p,\tau)g(R^{-1}(p,\tau)t)$ and $p_\tau = \phi_\tau(p)$. Then the curvature of $(M, g_\tau(t))$ is uniformly bounded. Note that $(M, g_\tau(t))$ is also κ-noncollapsed. Thus there is a subsequence $(M, g_{\tau_i}(t), p_{\tau_i})$ which converges to a geometric limit $(M_\infty, g_\infty(t), p_\infty)$, where $t \in (-\infty, \infty)$. For any fixed $t \in (-\infty, +\infty)$, by (5.10),

$$\lim_{\tau_i \to -\infty} (\tau_i + R^{-1}(p,\tau_i)t) = -\infty.$$

Therefore,

$$\lim_{\tau_i\to-\infty} R(p_{\tau_i}, R^{-1}(p,\tau_i)t) = \lim_{\tau_i\to-\infty} R(p, \tau_i + R^{-1}(p,\tau_i)t) = C.$$

Hence

$$R_\infty(p_\infty, t) = \lim_{\tau_i\to-\infty} \frac{R(p_{\tau_i}, R^{-1}(p,\tau_i)t)}{R(p,\tau_i)} = 1, \tag{5.11}$$

and consequently,

$$\frac{\partial}{\partial t} R_\infty(p_\infty, t) \equiv 0. \tag{5.12}$$

By (5.11), $(M_\infty, g_\infty(t); p_\infty)$ is not flat. Then by Cao's dimension reduction theorem [6], we may assume that $(M_\infty, g_\infty(t))$ has positive Ricci curvature. Since $(M_\infty, g_\infty(t), p_\infty)$ satisfies the Harnack inequality (3.2) and there exists a point $p_\infty \in M_\infty$ which satisfies (5.12), following the argument in the proof of Theorem 4.1 in [4], we can further prove that $(M_\infty, g_\infty(t), p_\infty)$ is in fact a steady Kähler–Ricci soliton, which is κ-noncollapsed and has nonnegative sectional curvature and positive Ricci curvature.

On the other hand, it follows from (5.10) that

$$R(p,\tau_i)d^2(o, p_{\tau_i}) \to \infty, \text{ as } \tau_i \to \infty.$$

Then as in the proof of Theorem 1.5, $(M_\infty, g_\infty(0))$ splits off $M_\infty = N_1 \times N_2$ with $g_\infty(0) = g_{N_1} + g_{N_2}$, where $g_{N_1} = \mathrm{d}z \otimes \mathrm{d}\overline{z}$ is a flat metric on N_1 and g_{N_2} is a Riemannian metric on N_2. Consequently, g_{N_2} is an $(n-1)$-dimension κ-noncollapsed steady Kähler–Ricci soliton with nonnegative sectional curvature and positive Ricci curvature. It contradicts the induction hypothesis. The claim is proved. □

Let $p = (1, 0, 0, \ldots, 0)$. Then $p_t = \phi_t(p) = (e^{-h_1 t}, 0, \ldots, 0)$. By Claim 5.6, $R(p_t) \to 0$ as $t \to -\infty$. On the other hand, $R(p_t) = R(p,t)$ is increasing for $t \in (-\infty, +\infty)$ by the Harnack inequality. By Lemma 5.4, we see that there is a positive constant $C > 0$ such that $R(p_t) \geq C$ as long as $-t$ is sufficiently large. Therefore, we get a contradiction. The proof of Theorem 1.3 is complete. □

By Theorem 1.3 together with Cao's dimension reduction theorem [6], we immediately get the following corollary.

Corollary 5.7. *Any n-dimensional κ-noncollapsed steady Kähler–Ricci soliton with non-negative sectional curvature must be flat.*

In the end, we apply Corollary 5.7 to prove Theorem 1.4.

Proof of Theorem 1.4. We only need to prove that $R(p,t) \equiv 0$ for all $p \in M$ and $t \in (-\infty, +\infty)$. Suppose not. Fix any $p \in M$ such that $R(p,t') > 0$ for some $t' \in (-\infty, +\infty)$. Let $\{t_k\}$ be a sequence of numbers which tends to infinity and let $g_k(t) = g(t+t_k)$. Since each flow $(M, g_k(t), p)$ is κ-noncollapsed and has uniformly bounded curvature, $(M, g_k(t), p)$ converges to $(M_\infty, g_\infty(t), p_\infty)$ in the Cheeger–Gromov topology. Note that the Harnack inequality (3.2) holds along flow $(M, g(t))$ and that $(M, g(t))$ has uniformly bounded curvature. Thus $\frac{\partial}{\partial t}R(p,t) \geq 0$ and $R(p,t)$ is uniformly bounded. It follows that

$$R_\infty(p_\infty, t_1) = \lim_{t\to\infty} R(p, t+t_1) = \lim_{t\to\infty} R(p, t+t_2) = R_\infty(p_\infty, t_2),$$

and consequently,

$$\frac{\partial}{\partial t} R_\infty(p_\infty, t) \equiv 0. \tag{5.13}$$

Since $R_\infty(p_\infty, t) \geq R(p,t') > 0$, $(M_\infty, g_\infty(t))$ is nonflat. By Cao's dimension reduction theorem in [6], we may assume that $(M_\infty, g_\infty(t))$ has positive Ricci curvature. Since $(M_\infty, g_\infty(t); p_\infty)$ satisfies Harnack inequality (3.2) and there exists a point $p_\infty \in M_\infty$ which satisfies (5.13), following the argument in the proof of Theorem 4.1 in [4], we can prove that $(M_\infty, g_\infty(t), p_\infty)$ is in fact a (gradient) steady Kähler–Ricci soliton which is κ-noncollapsed and has nonnegative sectional curvature. By Corollary 5.7, $(M_\infty, g_\infty(t), p_\infty)$ is a flat metric flow. This is impossible because $R_\infty(p_\infty, t) \geq R(p,t') > 0$. Hence, we complete the proof. ☐

6. Appendix

In this appendix, we compute the curvature decay of the steady gradient Käher–Ricci solion on $\mathbb{C}^n$ constructed by Cao in [5] and we show that these steady solitons are collapsed.

We first recall Cao's construction. Let $(z_1, z_2, \ldots, z_n)$ be the standard holomorphic coordinates on $\mathbb{C}^n$. Assume that $g = (g_{i\bar{j}})$ is an U(n)-invariant metric on $\mathbb{C}^n$ and the corresponding Kähler potential is given by $u(s)$, where $u(s)$ is a strictly increasing and convex function on $(-\infty, \infty)$ and $s = \ln|z|^2 = \ln r^2$. By a direct computation, we have

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u(s) = e^{-s}u'(s)\delta_{ij} + e^{-2s}\bar{z}_i z_j (u''(s) - u'(s)),$$

$$g^{i\bar{j}} = \partial_i \partial_{\bar{j}} u(s) = e^{s}u'(s)^{-1}\delta_{ij} + z_i \bar{z}_j (u''(s) - u'(s)),$$

and

$$f(s) \triangleq -\ln\det(g_{i\bar{j}}) = ns - (n-1)\ln u'(s) - \ln u''(s). \tag{6.1}$$

Then

$$R_{i\bar{j}} = \partial_i \partial_{\bar{j}} f(s) = e^{-s}f'(s)\delta_{ij} + e^{-2s}\bar{z}_i z_j (f''(s) - f'(s)). \tag{6.2}$$

Thus $g_{i\bar{j}}$ is a steady gradient soliton if and only if

$$v^i \frac{\partial}{\partial z_i} = g^{i\bar{j}} \partial_{\bar{j}} f \frac{\partial}{\partial z_i} = (z_i \frac{f'}{u''}) \frac{\partial}{\partial z_i}$$

is a holomorphic vector field, which is equivalent to

$$f' = \lambda u'', \tag{6.3}$$

for some constant λ.

Let $\phi = u'$. Then by (6.3) and (6.1), we get an equation for ϕ,

$$\phi^{n-1}\phi' e^{\alpha\phi} = \beta e^{ns}. \tag{6.4}$$

After rescaling, we may choose $\alpha = \beta = 1$. Cao solved (6.4) by

$$\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{n!}{k!}\phi^k e^{\phi} = e^{ns} + (-1)^{n-1}n!. \tag{6.5}$$

Cao has observed the following properties of ϕ:

$$\begin{aligned} &\phi(s) > 0,\ \phi'(s) > 0\ \forall\ s \in (-\infty, +\infty),\\ &\lim_{s\to\infty}\frac{\phi(s)}{s} = n,\ \lim_{s\to\infty}\phi'(s) = n. \end{aligned} \tag{6.6}$$

He also proved that these solitons have positive sectional curvature.

The curvature asymptotic behavior can also be computed in the following. Let $o = (0, 0, \ldots, 0)$ and $p = (z_1, 0, \ldots, 0)$. Then by (6.2), we have

$$\begin{aligned} R(p) &= -\frac{1}{\phi'}\Big((n-1)(\frac{\phi'}{\phi})' + (\frac{\phi''}{\phi'})'\Big) + \frac{n-1}{\phi}\Big(n - (n-1)\frac{\phi'}{\phi} - \frac{\phi''}{\phi'}\Big)\\ &= n - \phi'. \end{aligned} \tag{6.7}$$

On the other hand, by differentiating (6.5), it follows that

$$\phi' = \frac{e^{ns}}{e^{ns} + (-1)^{n-1}n!}\sum_{k=0}^{n-1}\Big((-1)^{n-k-1}\frac{n!}{k!}\phi^{k-n+1}\Big).$$

Thus

$$\begin{aligned} R(p) =& \frac{(-1)^{n-1}n!\cdot n}{e^{ns} + (-1)^{n-1}n!}\\ &+ \frac{e^{ns}}{e^{ns} + (-1)^{n-1}n!}\Big(\frac{n(n-1)}{\phi} + \frac{1}{\phi^2}\sum_{k=0}^{n-3}(-1)^{n-k-1}\frac{n!}{k!}\phi^{k-n+3}\Big)\\ &\to (n-1),\ \text{as}\ |z_1| \to \infty. \end{aligned} \tag{6.8}$$

Let $\rho(x)$ be a distance function from the original point $o \in \mathbb{C}^n$. Then by (6.6), it is easy to see that

$$\rho(x) = \frac{\sqrt{n}}{2}s(x) + o(s(x)),\ \text{as}\ s \to \infty. \tag{6.9}$$

Hence, using the U(n)-symmetry of g, we obtain from (6.8) the following lemma.

Lemma 6.1. *The metric g satisfies the curvature condition*

$$R(x)\rho(x) \to \frac{1}{2}\sqrt{n}(n-1),\ \text{as}\ |x| \to \infty. \tag{6.10}$$

By Lemma 6.1, we prove the following proposition.

Proposition 6.2. *Any $U(n)$-symmetric steady gradient soliton on $\mathbb{C}^n$ is collapsed.*

Proof. Let $z_i = x_i + \sqrt{-1}y_i$ for $1 \le i \le n$. We introduce new coordinates $(r, \theta, x_2', y_2', \ldots, x_n', y_n')$ such that

$$\begin{cases} x_1 = \cos\theta\sqrt{r^2 - \Sigma_{i=2}^n(x_i^2 + y_i^2)}, \\ y_1 = \sin\theta\sqrt{r^2 - \Sigma_{i=2}^n(x_i^2 + y_i^2)}, \\ x_2 = rx_2', \\ y_2 = ry_2', \\ \cdots \\ x_n = rx_n', \\ y_n = ry_n'. \end{cases}$$

Then under the new coordinates the metric g has an expression,

$$\begin{aligned} g &= r^{-2}\phi'(s)(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) + \phi(s)\pi^* g_{FS} \\ &= \frac{\phi'(s)}{4}\mathrm{d}s^2 + \phi'(s)\mathrm{d}\theta^2 + \phi(s)\pi^* g_{FS} \\ &= \phi'(\tau^2)\tau^2\mathrm{d}\tau^2 + \phi'(\tau^2)\mathrm{d}\theta^2 + \phi(\tau^2)\pi^* g_{FS}, \end{aligned} \tag{6.11}$$

where $\pi : S^{2n-1} \to \mathbb{CP}^{n-1}$ is the S^1-Hopf fibration. Let $p_k \in M$ such that $|p_k|^2 = e^{k^2}$ and let $r_k = \frac{k}{2\sqrt{n-1}}$. By the choice of p_k, we have $s(p_k) = k^2$.

Let $N_k = \{x \in M : k^2 - k \le s(x) \le k^2 + k\}$ and $g_k = \phi(p_k)^{-1}g$. We consider open manifolds (N_k, g_k). By the asymptotic behavior of $\phi(s)$ and (6.11), it is easy to see that (N_k, g_k) converge to $(\mathbb{R} \times \mathbb{CP}^{n-1}, ds^2 \otimes g_{FS})$ in C^∞ topology. Note that $B(p_k, r_k) \subset N_k$. By the convergence, for any $x \in B(p_k, r_k)$, $s(x) \in [k^2 - 2r_k, k^2 + 2r_k]$ and $(x_2'(x), y_2'(x), \ldots, x_n'(x), y_n'(x)) \subset B_{FS}(p_k, 2\phi(p_k)^{-1/2}r_k)$, where $B_{FS}(p_k, r)$ is the geodesic ball of the submanifold $\{(r(p_k), \theta(p_k), x_2', y_2', \ldots, x_n', y_n') \in M\}$ with the metric $\pi^* g_{FS}$. Hence, the volume of $B(p_k, r_k)$ satisfies the following estimate for sufficiently large k:

$$\begin{aligned} &\mathrm{vol}(B(p_k, r_k)) \\ &\le \int_{k^2-2r_k}^{k^2+2r_k} ds \int_0^{2\pi} d\theta \int_{B_{FS}(p_k, 2\phi(p_k)^{-1/2}r_k)} \phi'(s)\phi(s)^{n-1}\mathrm{dvol}_{g_{FS}} \\ &= 2\pi(\phi(p_k))^{n-1} \int_{k^2-2r_k}^{k^2+2r_k} ds \int_{B_{FS}(p_k, 2\phi(p_k)^{-1/2}r_k)} \phi'(s)\Big(\frac{\phi(s)}{\phi(p_k)}\Big)^{n-1} \mathrm{dvol}_{g_{FS}} \\ &\le 2\pi(\phi(p_k))^{n-1} \int_{k^2-2r_k}^{k^2+2r_k} ds \int_{\mathbb{CP}^{n-1}} 2^{n-1} n \mathrm{dvol}_{g_{FS}} \\ &\le (32)^{n+1} n(n-1)^{n-1}\pi\omega_{2n-2} r_k^{2n-1}. \end{aligned} \tag{6.12}$$

It follows that

$$\lim_{k\to\infty} \frac{\mathrm{vol}(B(p_k, r_k))}{r_k^{2n}} = 0.$$

On the other hand, by Lemma 6.1,

$$R(x) \le \frac{2(n-1)}{k^2 - k} \le \frac{4(n-1)}{k^2} = \frac{1}{r_k^2} \ \forall\, x \in B(p_k, r_k),$$

when k is large enough. Hence g is collapsed. $\square$

From the computation in (6.12), it is easy to get the volume growth of $B(p,r)$,

$$\mathrm{vol}(B(p,r)) = O(r^n), \text{ as } r \to \infty.$$

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