# SEQUENCES OF POWERS WITH SECOND DIFFERENCES EQUAL TO TWO AND HYPERBOLICITY 

NATALIA GARCIA-FRITZ


#### Abstract

By explicitly finding the complete set of curves of genus 0 or 1 in some surfaces of general type, we prove that under the Bombieri-Lang conjecture for surfaces, there exists an absolute bound $M>0$ such that there are only finitely many sequences of length $M$ formed by $k$-th rational powers with second differences equal to 2 . Moreover, we prove the unconditional analogue of this result for function fields, with $M$ depending only on the genus of the function field. We also find new examples of Brody-hyperbolic surfaces arising from the previous arithmetic problem. Finally, under the Bombieri-Lang conjecture and the ABC-conjecture for four terms, we prove analogous results for sequences of integer powers with possibly different exponents, in which case some exceptional sequences occur.


## 1. Introduction and main results

A sequence $a_{1}, \ldots, a_{n}$ is said to have second differences equal to 2 if for all $3 \leq i \leq n$ one has

$$
\left(a_{i}-a_{i-1}\right)-\left(a_{i-1}-a_{i-2}\right)=2,
$$

that is,

$$
a_{i}-2 a_{i-1}+a_{i-2}=2 .
$$

The main problem that motivates our work is
Problem 1. Let $k>2$. Does there exist a positive integer $M$ such that there are no sequences of length $M$ formed by $k$-th integer powers having second differences equal to 2 ?

An example of a sequence of this type is the sequence of cubes $64,-1,-64,-125$; hence for $k=3$ the bound $M$ has to be at least 5 .

Let us discuss some context for the above problem. If instead we consider integer squares, then no such $M$ can exist, because any sequence of the form $(x+i)^{2}$, with $x$ an integer and $i=1, \ldots, M$, has second differences equal to 2 , and these are called trivial sequences. The following problem was proposed by Büchi in 1970 (see Lip90, Maz94), and it has been extensively studied due to its implications in undecidability aspects in number theory (cf. [PPV10).

[^0]Problem 2 ( $n$-Squares Problem). Does there exist a positive integer $n$ such that every sequence of $n$ or more integers, whose squares have second differences equal to 2 , is necessarily a trivial sequence?

The $n$-Squares Problem was solved positively under the Bombieri-Lang conjecture in 2000 by Vojta Voj00a, inspired by the work of Bogomolov (cf. Bog10, [Des79] ) on curves of low genus on surfaces. Vojta's approach consists of explicitly finding all the curves of genus 0 or 1 on certain surfaces of general type associated to this problem. From this, and using the Bombieri-Lang conjecture, it is proved that there are finitely many non-trivial integer sequences of length 8 formed by integer squares having second differences equal to 2 , and hence there exists an $n$ (possibly greater than 8 ) such that the only sequences of length $n$ formed by integer squares with second differences equal to 2 are the trivial sequences.

The purpose of this work is to investigate a higher degree version of the $n$ Squares Problem, namely Problem 1 which as we will see leads to the construction of algebraic surfaces with some interesting geometric features, to new results in arithmetic, and to some undecidability results in number theory. To some extent, we use this arithmetic problem as an opportunity to spell out and exemplify a method implicit in Vojta's work Voj00a. This can be of independent interest.

In our work, for any $k \geq 3$ and $n \geq 2$, we will consider the smooth surfaces $X_{n, k} \subseteq \mathbb{P}^{n}$ defined by the equations

$$
X_{n, k}:\left\{\begin{array}{l}
2 x_{0}^{k}=x_{1}^{k}-2 x_{2}^{k}+x_{3}^{k}  \tag{1}\\
\vdots \\
2 x_{0}^{k}=x_{n-2}^{k}-2 x_{n-1}^{k}+x_{n}^{k}
\end{array}\right.
$$

Note that rational points on $X_{n, k}$ with $x_{0} \neq 0$ correspond to sequences of length $n$ formed by rational $k$-th powers having second differences equal to 2 .

As in Voj00a, we will study the curves of genus 0 or 1 in $X_{n, k}$, for $n$ large enough. Moreover, in Section 3 below we spell out and extend the method implicit in Vojta's work, which can be used in further applications. Let us now state our main results.

Theorem 3. Let $k \geq 3$, let $g \geq 1$, and let $n>\frac{4 g}{k-1}+3$. If $C$ is an irreducible curve in $X_{n, k}$, then the geometric genus of $C$ satisfies $g(C)>g$.

Note that in the statement we do not require that the curve $C$ be smooth. Also note that after fixing $g$, the bound for $n$ can be made independent of $k$. For example, there are no curves of geometric genus less than or equal to $g$ on $X_{n, k}$ when $n \geq \frac{g}{2}+3$. This good dependence on $k$ does not follow from a direct adaptation of Vojta's work Voj00a, but instead, we need finer control on ramification. See Theorem 24 and Proposition 25 in Section 3 for details.

Theorem [3] specializes as follows for $g=0,1$ :
Corollary 4. There are no curves of genus 0 or 1 on $X_{n, k}$ if
(a) $k=3$ and $n \geq 6$;
(b) $k=4,5$ and $n \geq 5$;
(c) $k \geq 6$ and $n \geq 4$.

The case $k=2$ is considered by Vojta in Voj00a, and for every $n \geq 2$ the surface $X_{n, 2}$ indeed has rational curves, which are associated to the trivial sequences discussed above.

Actually, the intermediate results that we obtain in order to prove Theorem 3 together with Proposition 43, also give the following result, which is stronger than Corollary 4

Theorem 5. The surface $X_{n, k}$ is Brody-hyperbolic if
(a) $k=3$ and $n \geq 6$;
(b) $k=4,5$ and $n \geq 5$;
(c) $k \geq 6$ and $n \geq 4$.

Observe that Brody-hyperbolicity does not hold for the surfaces studied by Vojta in Voj00a, due to the presence of trivial lines.

Corollary 4 gives us examples of regular surfaces (i.e., $q:=\operatorname{dim} H^{0}\left(X, \Omega_{X / \mathbb{C}}^{1}\right)=$ 0 ) without rational or elliptic curves. While the approach of using symmetric differentials to study curves in surfaces is by now classical, there is a crucial difference between the approach of Bogomolov (and others) and the approach used in this work (originated in Voj00a): Bogomolov uses a numerical condition to ensure existence of symmetric differentials by means of the Hirzebruch-Riemann-Roch theorem to count dimensions, while here we explicitly construct the symmetric differentials using ramified coverings. In fact, we obtain examples of surfaces which do not satisfy Bogomolov's condition on Chern numbers $c_{1}^{2}>c_{2}$ (see Des79, Bog10), but which have no curves of geometric genus 0 or 1 and, in fact, are Brody-hyperbolic. The following examples of numerical invariants for $X_{n, k}$ were computed using Magma:

| $n$ | $k$ | $c_{1}^{2}$ | $c_{2}$ | $c_{1}^{2}-c_{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| 4 | 6 | 1764 | 2088 | -324 |
| 4 | 7 | 3969 | 4263 | -294 |
| 4 | 8 | 7744 | 7808 | -64 |
| 6 | 3 | 2025 | 2187 | -162 |

In the case that $k=2$, Vojta notes that Bogomolov's condition holds for $n \geq 10$, although he is able to explicitly find all the curves of genus 0 or 1 on $X_{n, 2}$ for $n \geq 8$. However, as mentioned before, in Vojta's work trivial rational curves do exist; thus those examples are not Brody-hyperbolic.

From Theorem 3 we obtain the following unconditional result on the arithmetic of function fields.

Theorem 6. Let $K$ be a function field of genus $g \geq 0$ with constant field $\mathbb{C}$, let $k \geq 3$, and let $n>\frac{4 \max \{g, 1\}}{k-1}+3$. Let $f_{1}, \ldots, f_{n} \in K$ be such that the $k$-th powers of this sequence have second differences equal to 2 . Then the sequence $\left(f_{1}, \ldots, f_{n}\right)$ is a sequence of complex numbers.

Let us now discuss arithmetic applications of the previous geometric results. A common topic in arithmetic geometry is the study of how the geometry of a variety implies some control on its rational points. For instance, one expects that "complicated" varieties should have a sparse set of rational points. The following conjectures are concrete instances of this expectation. The first one is a conjecture due to Bombieri (see Nog81) and Lang (see Lan86, Corollary 5.7).

Conjecture 7 (Bombieri-Lang). If $X$ is a smooth projective algebraic variety of general type defined over a number field $K_{0}$, then there exists a proper Zariski-closed subset $Z$ of $X$ such that for all number fields $K$ containing $K_{0}$, the set $X(K) \backslash Z(K)$ is finite.

The second one is due to Lang; see Lan91, VIII.1.2].
Conjecture 8 (Lang). The following conditions are equivalent for a projective variety $X$, defined over a subfield of the complex numbers finitely generated over the rationals:

- $X(\mathbb{C})$ is hyperbolic,
- $X$ is Mordellic,
- every subvariety of $X$ is pseudocanonical.

From Theorem 5 and Corollary 4, these two conjectures are equivalent for the surfaces $X_{n, k}$ for $n>\frac{4 g}{k-1}+3$ (they are of general type for $n \geq \frac{4}{k-1}+2$; see Proposition 17 below), so we will refer to them as the Bombieri-Lang conjecture. In this case, we will prove

Theorem 9. Assume the Bombieri-Lang conjecture for the surfaces $X_{n, k}$. Let $L$ be a number field. There are only finitely many sequences of $N$ elements of $L$ whose $k$-th powers have second differences equal to 2, provided that

- $N=6$ if $k=3$;
- $N=5$ if $k=4$ or $k=5$;
- $N=4$ if $k \geq 6$.

Moreover, for any $k \geq 3$, there exists $M_{k, L}>0$ such that there are no sequences of $M_{k, L}$ elements of $L$ whose $k$-th powers have second differences equal to 2 .

We can also ask the question of finding a bound for the length of sequences with second differences equal to 2 formed by powers of possibly different exponents. We also obtain a result in this direction by using the 4 -term ABC conjecture over $\mathbb{Z}$ from [BB94], as proposed by Browkin and Brzezinski, and the Bombieri-Lang conjecture.

Theorem 10. Assume the Bombieri-Lang conjecture for the surfaces $X_{n, k}$ with $n>\frac{4}{k-1}+3$ and the 4 -term ABC conjecture. There exists an $M>0$ such that there are no non-trivial sequences of length $M$ consisting of integer powers (of possibly different exponents greater than or equal to 2) which have constant differences equal to 2 .

To prove this result, we will (partially) reduce to the case of powers of equal exponents by a combinatorial argument involving Szemerédi's theorem. Theorem 10 was motivated by a question of M. R. Murty about the $n$-term ABC conjecture in our context. I thank him for asking that question. An unconditional result for the analogue of Theorem [10 for powers in the ring $K[x]$ of polynomials over any field was studied by the author in [Gar13] by a different method.

The problem of finiteness of non-trivial sequences of length $n$ formed by powers with second differences equal to 2 in number fields was conditionally solved by Pasten Pas13 under Vojta's general ABC conjecture for algebraic numbers of bounded degree Voj98. He also gives a solution for the analogue of Problem 1 for function fields over fields of characteristic zero using completely different methods
(Nevanlinna theory), but his bounds on $n$ are much weaker than ours due to the methods that he uses.

Let us briefly compare the conjectural assumptions in our work and those in Pasten's work Pas13. The Bombieri-Lang conjecture and the 4 -term ABC conjecture that we use concern a fixed number field, while Vojta's ABC conjecture used in Pas13] concerns algebraic points of bounded degree in infinitely many number fields. This is necessary even if one is only interested in $\mathbb{Q}$. On the other hand, the Bombieri-Lang conjecture for surfaces and the 4 -term ABC conjecture can be seen as diophantine statements in dimension 2, while Vojta's ABC conjecture is in dimension 1.

Pasten also obtains results in logic in Pas13. He proves an undecidability result for systems of linear equations when some prescribed unknowns are required to be powers, again under Vojta's general ABC conjecture for algebraic numbers of bounded degree. In our case the undecidability result for powers is obtained under the Bombieri-Lang conjecture and the 4 -term ABC conjecture and is as follows:

Theorem 11. Assume the Bombieri-Lang conjecture for surfaces and the 4 -term ABC conjecture. Then there is no algorithm to decide whether a system of linear equations with integer coefficients has an integer solution or not, with some prescribed unknowns required to be powers.

The proof is similar to the proofs in Section 5 of Pas13, but in our case we use Theorem 10 as our arithmetic input. Note that the analogous undecidability result for squares (rather than powers) is established under the Bombieri-Lang conjecture in Voj00a. In all these cases, the undecidability results are deduced from the arithmetic results generalizing standard ideas from Büchi's work in 1970 (see Lip90, Maz94).

Finally, let us briefly mention that there is another generalization for the $n$ Squares Problem in the literature, proposed by Pheidas and Vidaux PV05, considering $k$-th differences of $k$-th powers. Although we will not consider this generalization, let us briefly comment on the existing results for it. Pasten Pas13 also solves this under Vojta's general ABC conjecture for algebraic numbers and unconditionally for function fields over fields of characteristic zero. Moreover, An, Huang, and Wang AHW13 give an alternative solution for function fields of characteristic zero following the methods of PV06, PV10. In all the works just mentioned, the methods are very different from the approach in Voj00a and in this work. Also, let us remark that the problem of $k$-th differences of $k$-th powers does not lead to results on Brody-hyperbolicity and non-existence of low genus curves, because in fact the associated varieties contain "trivial lines" (corresponding to sequences $\left.(x+i)^{k}, i=1,2, \ldots\right)$.

## 2. The geometry of the surfaces $X_{n, k}$

Let $n \geq 2$ and $k>2$. By convention, we define $X_{2, k}:=\mathbb{P}^{2}$. Recall that for each $n$, the scheme $X_{n, k}$ defined by equation (11) is in $\mathbb{P}_{\mathbb{C}}^{n}$. For $3 \leq i \leq n$, let

$$
f_{i}=2 x_{0}^{k}-x_{i-2}^{k}+2 x_{i-1}^{k}-x_{i}^{k}
$$

be the generators of the ideal defining $X_{n, k}$, and let

$$
g_{i}=(i-1)(i-2) x_{0}^{k}-(i-2) x_{1}^{k}+(i-1) x_{2}^{k}-x_{i}^{k} .
$$

The equality of ideals $\left(f_{3}, \ldots, f_{n}\right)=\left(g_{3}, \ldots, g_{n}\right)$ in $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ can be proved.

Lemma 12. If $\left[x_{0}: \cdots: x_{n}\right]$ is a point on $X_{n, k}$, then no three of $x_{0}, \ldots, x_{n}$ are zero.

Proof. If $x_{0}=x_{1}=x_{2}=0$, then all $x_{i}$ are zero, because $X_{n, k}$ is defined by the equations $g_{i}$. From this we know that at least one of $x_{0}, x_{1}, x_{2}$ is different from zero.

Now view $(j-1)(j-2) x_{0}^{k}-(j-2) x_{1}^{k}+(j-1) x_{2}^{k}=0$ as an equation in $j$. It can be written in the form

$$
\begin{equation*}
x_{0}^{k} j^{2}+\left(-3 x_{0}^{k}-x_{1}^{k}+x_{2}^{k}\right) j+2 x_{0}^{k}+2 x_{1}^{k}-x_{2}^{k}=0 . \tag{2}
\end{equation*}
$$

If $x_{0} \neq 0$, then equation (2) has at most two solutions; hence there are at most two values of $j$ for which $x_{j}=0$. If $x_{0}=0$, then equation (2) becomes a linear equation in $j$. If also both $-3 x_{0}^{k}-x_{1}^{k}+x_{2}^{k}=0$ and $2 x_{1}^{k}-x_{2}^{k}=0$, we get $x_{0}=x_{1}=x_{2}=0$, which is not possible. Hence equation (2) has at most one solution in $j$ when $x_{0}=0$.

For each $n \geq 3$, let $\pi_{n}: X_{n, k} \rightarrow X_{n-1, k}$ be the restriction to $X_{n, k}$ of the morphism

$$
\begin{aligned}
\tilde{\pi}_{n}: \mathbb{P}^{n} \backslash\{[0: \cdots: 0: 1]\} & \rightarrow \mathbb{P}^{n-1} \\
{\left[x_{0}: \cdots: x_{n}\right] } & \mapsto
\end{aligned}\left[x_{0}: \cdots: x_{n-1}\right] .
$$

The rational map $\tilde{\pi}_{n}$ corresponds to $\mathbb{C}\left[x_{0}, \ldots, x_{n-1}\right] \rightarrow \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ (which respects the grading) in the sense of Har77, Ex. II.2.14(b)], and the morphism $\pi_{n}$ corresponds to the induced map

$$
\mathbb{C}\left[x_{0}, \ldots, x_{n-1}\right] /\left(f_{3}, \ldots, f_{n-1}\right) \rightarrow \mathbb{C}\left[x_{0}, \ldots, x_{n}\right] /\left(f_{3}, \ldots, f_{n}\right),
$$

which exists because

$$
\left(f_{3}, \ldots, f_{n-1}\right) \subseteq \mathbb{C}\left[x_{0}, \ldots, x_{n-1}\right] \cap\left(f_{3}, \ldots, f_{n}\right)
$$

Lemma 13. For each $n \geq 3$, the map $\pi_{n}: X_{n, k} \rightarrow X_{n-1, k}$ is finite and surjective.
Proof. Let $P=\left[x_{0}: \cdots: x_{n-1}\right]$ be in $X_{n-1, k}$. Then for any $x_{n} \in \mathbb{C}$, the point $\tilde{P}=\left[x_{0}: \cdots: x_{n}\right]$ is a preimage of $P$ under $\tilde{\pi}_{n}$. We have that $\tilde{P}$ lies on $X_{n, k}$ if and only if

$$
x_{n}^{k}=(n-1)(n-2) x_{0}^{k}-(n-2) x_{1}^{k}+(n-1) x_{2}^{k} .
$$

Since this equation always has a solution $x_{n} \in \mathbb{C}$, we see that $\pi_{n}$ is surjective. Moreover, we see that $\#\left(\pi_{n}^{-1}(P)\right) \leq k$, so the map is quasi-finite, hence finite, by [Har77, Ex. III.11.2] because $\pi_{n}$ is projective.

Since $X_{n, k}$ is the intersection of $n-2$ hypersurfaces in $\mathbb{P}^{n}$, we have by Har77, Theorem I.7.2] that each irreducible component of $X_{n, k}$ has dimension greater than or equal to 2. By Lemma 13 the morphism $\rho_{n, k}=\pi_{3} \circ \cdots \circ \pi_{n}: X_{n, k} \rightarrow \mathbb{P}^{2}$ is finite and surjective; hence any irreducible component $Y$ of $X_{n, k}$ must satisfy $\operatorname{dim} Y \leq \operatorname{dim} \mathbb{P}^{2}$. We thus obtain

Lemma 14. Each irreducible component of $X_{n, k}$ has dimension 2.
The following observation will be useful for several subsequent lemmas.
Observation 15. Let $\alpha, \beta \neq 1,2$, with $\alpha \neq \beta$. The matrix

$$
\left(\begin{array}{ll}
-(\alpha-2)(\alpha-1) x_{0}^{k-1} & (\alpha-2) x_{1}^{k-1} \\
-(\beta-2)(\beta-1) x_{0}^{k-1} & (\beta-2) x_{1}^{k-1}
\end{array}\right)
$$

has determinant $x_{0}^{k-1} x_{1}^{k-1}(\alpha-2)(\beta-2)(\beta-\alpha) \neq 0$ when $x_{0} x_{1} \neq 0$. The matrix

$$
\left(\begin{array}{cc}
-(\alpha-2)(\alpha-1) x_{0}^{k-1} & -(\alpha-1) x_{2}^{k-1} \\
-(\beta-2)(\beta-1) x_{0}^{k-1} & -(\beta-1) x_{2}^{k-1}
\end{array}\right)
$$

has determinant $x_{0}^{k-1} x_{2}^{k-1}(\alpha-1)(\beta-1)(\alpha-\beta) \neq 0$ when $x_{0} x_{2} \neq 0$. The matrix

$$
\left(\begin{array}{ll}
(\alpha-2) x_{1}^{k-1} & -(\alpha-1) x_{2}^{k-1} \\
(\beta-2) x_{1}^{k-1} & -(\beta-1) x_{2}^{k-1}
\end{array}\right)
$$

has determinant $x_{1}^{k-1} x_{2}^{k-1}(\beta-\alpha) \neq 0$ when $x_{1} x_{2} \neq 0$.
Lemma 16. For each $n \geq 2$, the scheme $X_{n, k}$ is smooth.
Proof. Note that $X_{2, k} \cong \mathbb{P}^{2}$; thus it is smooth and irreducible. Let $\left[x_{0}: \cdots: x_{n}\right]$ be a point in $X_{n, k}$ with $n \geq 3$. We need to check that the Jacobian matrix of the homogeneous equations defining $X_{n, k}$ evaluated at $\left[x_{0}: \cdots: x_{n}\right] \in X_{n, k}$ has rank $n-2$, because $X_{n, k}$ is equidimensional of dimension 2 . Since the ideal $\left(g_{1}, \ldots, g_{n}\right)$ defines $X_{n, k}$, we get the $(n-2) \times(n+1)$ matrix

$$
k\left(\begin{array}{ccccccc}
-2 x_{0}^{k-1} & x_{1}^{k-1} & -2 x_{2}^{k-1} & x_{3}^{k-1} & 0 & \cdots & 0 \\
-6 x_{0}^{k-1} & 2 x_{1}^{k-1} & -3 x_{2}^{k-1} & 0 & x_{4}^{k-1} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
-(n-2)(n-1) x_{0}^{k-1} & (n-2) x_{1}^{k-1} & -(n-1) x_{2}^{k-1} & 0 & \cdots & 0 & x_{n}^{k-1}
\end{array}\right) .
$$

We will prove by induction on $i$ that this matrix has rank $n-2$ at every point of $X_{n, k}$. We know from Lemma 12 that no three of $x_{0}, \ldots, x_{n}$ are zero; hence

$$
k\left(\begin{array}{llll}
-2 x_{0}^{k-1} & x_{1}^{k-1} & -2 x_{2}^{k-1} & x_{3}^{k-1}
\end{array}\right)
$$

is not the zero vector. Let $3 \leq i \leq n-1$ and suppose by the induction hypothesis that the following $(i-2) \times(i+1)$ matrix has rank $i-2$ :

$$
M_{i}=k\left(\begin{array}{ccccccc}
-2 x_{0}^{k-1} & x_{1}^{k-1} & -2 x_{2}^{k-1} & x_{3}^{k-1} & 0 & \cdots & 0 \\
-6 x_{0}^{k-1} & 2 x_{1}^{k-1} & -3 x_{2}^{k-1} & 0 & x_{4}^{k-1} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
-(i-2)(i-1) x_{0}^{k-1} & (i-2) x_{1}^{k-1} & -(i-1) x_{2}^{k-1} & 0 & \cdots & 0 & x_{i}^{k-1}
\end{array}\right)
$$

and consider the $(i-1) \times(i+2)$ matrix

$$
M_{i+1}=k\left(\begin{array}{ccccccc}
-2 x_{0}^{k-1} & x_{1}^{k-1} & -2 x_{2}^{k-1} & x_{3}^{k-1} & 0 & \cdots & 0 \\
-6 x_{0}^{k-1} & 2 x_{1}^{k-1} & -3 x_{2}^{k-1} & 0 & x_{4}^{k-1} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
-(i-1)(i) x_{0}^{k-1} & (i-1) x_{1}^{k-1} & -i x_{2}^{k-1} & 0 & \cdots & 0 & x_{i+1}^{k-1}
\end{array}\right)
$$

If $x_{i+1} \neq 0$, then the matrix $M_{i+1}$ has maximal $\operatorname{rank} \operatorname{rk}\left(M_{i+1}\right)=\operatorname{rk}\left(M_{i}\right)+1=i-1$.
Now suppose that $x_{i+1}=0$. If none of $x_{3}, \ldots, x_{i}$ are zero, then $M_{i+1}$ has maximal rank. By Lemma 12, at most one among $x_{0}, \ldots, x_{i}$ can be zero. If $x_{j}=0$, then we only have to prove that the $j$-th row is not a multiple of the $(i+1)$-st row. By Lemma 12 we have that at least two of $x_{0}, x_{1}, x_{2}$ are different from zero. Then by Observation 15, we obtain that the $j$-th row is not a multiple of the $(i+1)$-st row, so the matrix $M_{i+1}$ has maximal rank. Therefore the Jacobian matrix has rank $n-2$, and thus the surface $X_{n, k}$ is smooth.

Proposition 17. The projective surface $X_{n, k}$ is smooth and irreducible. It is a regular complete intersection. Its canonical sheaf is $\mathcal{O}(k(n-2)-n-1)$. The surface $X_{n, k}$ is of general type for $n \geq \frac{2 k+2}{k-1}=2+\frac{4}{k-1}$.
Proof. The surface $X_{n, k}$ is a complete intersection because it is defined by $n-2$ equations in $\mathbb{P}^{n}$. Since $X_{n, k}$ is a complete intersection, from Har77, Ex. II.8.4(c)], we obtain that $X_{n, k}$ is connected, and since it is smooth we have that it is irreducible. Since $X_{n, k}$ is a complete intersection in $\mathbb{P}^{n}$, we deduce from Har77, Ex. III.5.5(c)] that $\operatorname{dim} H^{0}\left(X_{n, k}, \Omega_{X_{n, k} / \mathbb{C}}^{1}\right)=0$; that is, $X_{n, k}$ is regular.

From Har77, Ex.II.8.4(e)], we know that the canonical sheaf of $X_{n, k}$ is $\mathcal{O}(k(n-2)-n-1)$. We have that $k(n-2)-n-1 \geq 1$ when $n \geq \frac{2 k+2}{k-1}$. Thus $X_{n, k}$ is of general type when $n \geq \frac{2 k+2}{k-1}$.

## 3. Explanation of the method

This section is an account of the method implicit in Vojta's work Voj00a, inspired by work of Bogomolov. For analytic proofs of these facts in the case studied by Vojta, see Voj00a. For the general case treated in an algebraic setting, see Chapter 3 of Gar15. In particular Theorem 24 (cf. Theorem 3.87 in Gar15) is an improvement of Lemma 2.10 in Voj00a and permits us to obtain better numerical results. We outline this method in some generality beyond what we need here because it can be useful in other applications.

We work with the notion of $\omega$-integral curve from Voj00a:
Definition 18. Let $X$ be a smooth variety over a field of characteristic zero, let $\mathcal{L}$ be an invertible sheaf on $X$, and let $\omega \in H^{0}\left(X, \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right)$, where $r$ is an integer. An irreducible curve $C$ on $X$ is said to be $\omega$-integral if the image of the section $\varphi_{C}^{*} \omega$ in $H^{0}\left(\tilde{C}, \varphi_{C}^{*} \mathcal{L} \otimes S^{r} \Omega_{\tilde{C}}^{1}\right)$ is zero, where $\varphi_{C}: \tilde{C} \rightarrow X$ is the normalization of $C \subset X$.

From this point on, we restrict to the case of surfaces. One can check if an irreducible curve $C$ on $X$ is $\omega$-integral by looking at its equations, thanks to the following result (see Corollary 3.72 in Gar15):
Theorem 19. Let $X$ be a smooth complex surface, and let $\omega \in H^{0}\left(X, \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right)$. Let $C$ be an irreducible curve in $X$, let $U=\operatorname{Spec}(A)$ be an affine open set in $X$ such that $C \cap U$ is non-empty and principal, and let $\mathcal{L}_{\mid U} \cong \mathcal{O}_{U}$. Let $I=(g)$ be an ideal in $A$ such that $C \cap U$ is defined by $I$. Let $\omega_{0}$ be the image of $\omega$ under the maps

$$
H^{0}\left(X, \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right) \rightarrow H^{0}\left(U, \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right) \rightarrow H^{0}\left(U, S^{r} \Omega_{U / \mathbb{C}}^{1}\right)=S^{r} \Omega_{A / \mathbb{C}}^{1}
$$

If $\omega_{0} \in S^{r} \Omega_{A / \mathbb{C}}^{1}$ lies in $g S^{r}\left(\Omega_{A / \mathbb{C}}^{1}\right)+d g S^{r-1} \Omega_{A / \mathbb{C}}^{1}$, then $C$ is $\omega$-integral.
From this, we can verify for any $C \subset X$ whether it is an $\omega$-integral curve.
Once we find a list of $\omega$-integral curves on $X$, we want to check if this list consists of all $\omega$-integral curves of $X$. We can do this by defining the discriminant of $\omega$, which permits us to locally count the number of $\omega$-integral curves passing through a point. Fix a non-empty affine open subset $V$ on $X$ such that there are regular functions $u, v \in \mathcal{O}_{X}(V)$ with the property that $d u, d v$ are a basis of $\Omega_{X / \mathbb{C}}^{1}(V)$ as an $\mathcal{O}_{X}(V)$-module. Let $U \subset V$ be a non-empty basic affine open set such that $\mathcal{L}_{\mid U} \cong \mathcal{O}_{U}$. Under the isomorphism $H^{0}\left(U, \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right) \cong H^{0}\left(U, S^{r} \Omega_{X / \mathbb{C}}^{1}\right)$ we have that the image of $\omega_{\mid U}$ in $H^{0}\left(U, S^{r} \Omega_{X / \mathbb{C}}^{1}\right)$ can be written as $\sum_{i=0}^{r} A_{i}(d u)^{r-i}(d v)^{i}$ with $A_{i} \in \mathcal{O}_{U}(U)$.

Definition 20. Let $\delta \in K:=k(X)$ be the discriminant of the monic polynomial $\sum_{i=0}^{r} \frac{A_{i}}{A_{0}} T^{r-i} \in K[T]$. We define the discriminant of $\omega$ to be the Zariski closed set

$$
\Delta_{U}:=(X / U) \cup \mathbb{V}_{U}\left(A_{0}\right) \cup \mathbb{V}_{U \backslash \mathbb{V}_{U}\left(A_{0}\right)}(\delta) \subset X
$$

(where $\mathbb{V}_{W}(f)$ denotes the zero locus of $f \in \mathcal{O}_{W}(W)$ on the open set $W$ ).
One can count the number of $\omega$-integral curves passing through any point $P$ outside $\Delta_{U}$ by the following result (see Theorem 3.76 in Gar15):
Theorem 21. Let $\omega \in H^{0}\left(X, \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right)$, and let $\Delta_{U}$ be defined as before. For any given point $P \in X \backslash \Delta_{U}$ there are at most $r \omega$-integral curves passing through $P$. More precisely, the sum of the multiplicities $\mu_{P}(C)$ for all $\omega$-integral curves $C$ passing through $P$ is at most $r$.

Hence, if we find $r \omega$-integral curves for a point $P \in X \backslash \Delta_{U}$ we know that these are all the $\omega$-integral curves passing through $P$. Verifying that we have found $r$ $\omega$-integral curves passing through each point of $X \backslash U$ and checking if the component curves of $\Delta_{U}$ are $\omega$-integral (using Theorem (19) leads us to know that our list of $\omega$-integral curves of $X$ is complete.

In applications we would like to use an $\omega \in H^{0}\left(X, \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right)$, for which it is easy to find all $\omega$-integral curves.

Once we find all $\omega$-integral curves in $X$, we can use a morphism $\pi: X^{\prime} \rightarrow X$ to find all $\omega^{\prime}$-integral curves on $X^{\prime}$, for $\omega^{\prime}$ a suitable differential in $X^{\prime}$ which depends on $\omega$. Here we consider $\omega^{\prime}=\pi^{\bullet} \omega$, where given any morphism $f: Y \rightarrow Z$ of varieties, the homomorphism $f^{\bullet}$ is the induced map on global sections from the $\mathcal{O}_{Z}$-homomorphism

$$
\begin{aligned}
\mathcal{L} \otimes S^{r} \Omega_{Z / \mathbb{C}}^{1} & \rightarrow f_{*} f^{*}\left(\mathcal{L} \otimes S^{r} \Omega_{Z / \mathbb{C}}^{1}\right) \\
& \rightarrow f_{*}\left(f^{*} \mathcal{L} \otimes f^{*} S^{r} \Omega_{Z / \mathbb{C}}^{1}\right) \\
& \rightarrow f_{*}\left(f^{*} \mathcal{L} \otimes S^{r} f^{*} \Omega_{Z / \mathbb{C}}^{1}\right) \\
& \rightarrow f_{*}\left(f^{*} \mathcal{L} \otimes S^{r} \Omega_{Y / \mathbb{C}}^{1}\right)
\end{aligned}
$$

where the last map is induced by the morphism

$$
f_{Y / Z / \mathbb{C}}: f^{*} \Omega_{Z / \mathbb{C}}^{1} \rightarrow \Omega_{Y / \mathbb{C}}^{1}
$$

from [EGA, IV.16.4.19.1]. (Note that translating Definition 18 into this language, we say that a curve $C$ on a surface $X$ is $\omega$-integral if and only if $\varphi_{C}^{\bullet} \omega=0$.)

The following result will allow us to find all $\pi^{\bullet} \omega$-integral curves on $X^{\prime}$ (which corresponds to Theorem 3.35 in Gar15).
Theorem 22. Let $\pi: X^{\prime} \rightarrow X$ be a dominant morphism of smooth surfaces. Let $C^{\prime} \subset X^{\prime}$ be an irreducible curve and let $C=\pi\left(C^{\prime}\right)$ be an irreducible curve on $X$. Let $\mathcal{L}$ be an invertible sheaf on $X$ and let $\omega \in H^{0}\left(X, \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right)$. The following are equivalent:

- the curve $C$ is $\omega$-integral;
- the curve $C^{\prime}$ is $\pi^{\bullet} \omega$-integral.

The last step is to have a criterion to check that any irreducible curve $C \subset X^{\prime}$ of genus less than or equal to $g$ is in the list of $\pi^{\bullet} \omega$-integral curves. This is done by showing that the degree of the sheaf $\varphi_{C}^{*} \mathcal{L} \otimes S^{r} \Omega_{\tilde{C} / \mathbb{C}}^{1}$ (from Definition 18) over the normalization $\tilde{C}$ of a curve of genus less than or equal to $g$ on $X^{\prime}$ is negative, so the section $\varphi_{C}^{\bullet}\left(\pi^{\bullet} \omega\right)$ is forced to be zero.

If this degree is not negative for the sheaf $\mathcal{L}$ under consideration, we can find a "better sheaf $\mathcal{L}$ " when the branch curves of the morphism $\pi: X^{\prime} \rightarrow X$ are $\omega$-integral. This requires the following definition:

Definition 23. Given a smooth irreducible surface $X$, an effective Cartier divisor $D$ of $X$ with associated subscheme $Y=Y_{D}$, a locally free sheaf $\mathcal{F}$, and a section $s \in H^{0}(X, \mathcal{F})$, we say that $s$ vanishes identically along $D$ if the image of $s$ under the map $H^{0}(X, \mathcal{F}) \rightarrow H^{0}\left(X, \mathcal{F} \otimes i_{Y *} \mathcal{O}_{Y}\right)$ is zero.

In this context, one can see that if $D$ and $D^{\prime}$ are effective divisors with no common component, then $s$ vanishes along $D$ and $D^{\prime}$ if and only if $s$ vanishes along $D+D^{\prime}$ (cf. Proposition 3.79 in Gar15).

The following result is a generalization of Lemma 2.10 in Voj00a to morphisms with higher order ramification. For a proof, note that Vojta's argument already covers this case using analytic methods. Alternatively, see Theorem 3.87 in Gar15] for a purely algebraic proof.

Theorem 24. Let $X$ and $X^{\prime}$ be smooth integral surfaces defined over $\mathbb{C}$. Let $\pi: X^{\prime} \rightarrow X$ be a dominant morphism and let $D \subseteq X^{\prime}$ be a prime divisor such that $C=\pi(D)$ is a curve. Suppose that $\pi$ has ramification index $e=e_{D / C}(\pi)>1$ at $D$. Let $\mathcal{L}$ be an invertible sheaf on $X$, let $r$ be a positive integer, and let $\omega \in$ $H^{0}\left(X, \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right)$. If $C$ is $\omega$-integral, then $\pi^{\bullet} \omega \in H^{0}\left(X^{\prime}, \pi^{*} \mathcal{L} \otimes S^{r} \Omega_{X^{\prime} / \mathbb{C}}^{1}\right)$ vanishes identically along $(e-1) D$.

Tensoring the exact sequence

$$
0 \rightarrow \mathcal{O}_{X}(-D) \rightarrow \mathcal{O}_{X} \rightarrow i_{D *} \mathcal{O}_{D} \rightarrow 0
$$

from [Mum66, p. 63] by the locally free sheaf $\mathcal{F}:=\mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}$ and then taking global sections, we obtain the exact sequence

$$
0 \rightarrow H^{0}\left(X, \mathcal{O}_{X}(-D) \otimes \mathcal{F}\right) \rightarrow H^{0}(X, \mathcal{F}) \rightarrow H^{0}\left(X, i_{D *} \mathcal{O}_{D} \otimes \mathcal{F}\right)
$$

which leads us to the following result (corresponding to Proposition 3.88 in Gar15]; see also Corollary 2.11 in Voj00a):

Proposition 25. Let $X$ be a smooth integral surface and let

$$
\omega \in H^{0}\left(X, \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right)
$$

Let $D$ be an effective divisor on $X$. Suppose that $\omega$ vanishes identically along $D$. Then there is a symmetric differential $\omega^{\prime} \in H^{0}\left(X, \mathcal{O}(-D) \otimes \mathcal{L} \otimes S^{r} \Omega_{X / \mathbb{C}}^{1}\right)$ such that all $\omega^{\prime}$-integral curves are among the $\omega$-integral curves.

This proposition combined with Theorem 24]allows us to prove that the invertible sheaf $\mathcal{L}^{\prime}=\mathcal{O}(-(e-1) D) \otimes \pi^{*} \mathcal{L}$ has a section $\bar{\omega}$ making all $\bar{\omega}$-integral curves in $X^{\prime}$ to be $\pi^{\bullet} \omega$-integral.

## 4. Finding all $\omega_{2, k}$-Integral curves in $X_{2, k}$

Let us prove the following lemma, which will be useful later.
Lemma 26. Consider the complex polynomial

$$
P\left(x_{1}, x_{2}\right):=1+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{1}^{k}-2 x_{2}^{k}-2 x_{1}^{k} x_{2}^{k}
$$

- If $k$ is an even integer, then P factors in irreducible factors as follows:

$$
P=\left(-x_{1}^{\frac{k}{2}}-x_{2}^{\frac{k}{2}}-1\right)\left(x_{1}^{\frac{k}{2}}+x_{2}^{\frac{k}{2}}-1\right)\left(-x_{1}^{\frac{k}{2}}+x_{2}^{\frac{k}{2}}-1\right)\left(x_{1}^{\frac{k}{2}}-x_{2}^{\frac{k}{2}}-1\right)
$$

- If $k$ is an odd integer, then $P\left(x_{1}, x_{2}\right)$ is irreducible.

Proof. That $P\left(x_{1}, x_{2}\right)$ factors in that form for $k$ even is easily checked. When $k$ is an odd integer, we want to prove that $P\left(x_{1}, x_{2}\right)$ is irreducible. Let $F: k\left[x_{1}, x_{2}\right] \rightarrow$ $k[u, v]$ be the homomorphism of $k$-algebras defined by $F\left(x_{1}\right)=u^{2}, F\left(x_{2}\right)=v^{2}$. If $P\left(x_{1}, x_{2}\right)=Q\left(x_{1}, x_{2}\right) R\left(x_{1}, x_{2}\right)$, then $P\left(u^{2}, v^{2}\right)=Q\left(u^{2}, v^{2}\right) R\left(u^{2}, v^{2}\right)$. Note that

$$
P\left(u^{2}, v^{2}\right)=\left(-u^{k}-v^{k}-1\right)\left(u^{k}+v^{k}-1\right)\left(-u^{k}+v^{k}-1\right)\left(u^{k}-v^{k}-1\right)
$$

by the previous assertion, and this factorization into irreducible factors is unique (up to constants) because $k[u, v]$ is a unique factorization domain.

Therefore (without loss of generality, because of the symmetry of $Q$ and $R$ ) we have, from the above factorization of $P\left(u^{2}, v^{2}\right)$, that $Q\left(u^{2}, v^{2}\right)$ is either irreducible or a product of two irreducible factors.

In the first case $Q\left(u^{2}, v^{2}\right)=\epsilon_{1} u^{k}-\epsilon_{2}^{k}-1$, with $\epsilon_{i} \in\{ \pm 1\}$, and in the second case we have that

$$
\begin{aligned}
Q\left(u^{2}, v^{2}\right)= & \left(\epsilon_{1} u^{k}+\epsilon_{2} v^{k}-1\right)\left(\epsilon_{3} u^{k}+\epsilon_{4} v^{k}-1\right) \\
= & \epsilon_{1} \epsilon_{3} u^{2 k}+\epsilon_{1} \epsilon_{4} u^{k} v^{k}-\epsilon_{1} u^{k}+\epsilon_{2} \epsilon_{3} u^{k} v^{k}+\epsilon_{2} \epsilon_{4} v^{2 k} \\
& -\epsilon_{2} v^{k}-\epsilon_{3} u^{k}-\epsilon_{4} v^{k}+1,
\end{aligned}
$$

but neither of these two polynomials is in the image of $F$, because $k$ is odd and in both cases we obtain some exponent equal to $k$. Therefore $P\left(x_{1}, x_{2}\right)$ is irreducible.

Let $\left\{U_{i}\right\} \subseteq \mathbb{P}^{2}$ be the usual affine open cover of $\mathbb{P}^{2}$. In $U_{0}$ with affine coordinates $x_{1}=\frac{X_{1}}{X_{0}}, x_{2}=\frac{X_{2}}{X_{0}}$, consider the following symmetric differential form:

$$
x_{1}^{k-1} x_{2} d x_{1} d x_{1}+\left(1-x_{1}^{k}-x_{2}^{k}\right) d x_{1} d x_{2}+x_{1} x_{2}^{k-1} d x_{2} d x_{2}
$$

Proposition 27. This differential form in $U_{0}$ can be extended to a global section

$$
\omega_{2, k} \in H^{0}\left(\mathbb{P}^{2}, \mathcal{O}(k+3) \otimes S^{2} \Omega_{\mathbb{P}^{2}}^{1}\right) .
$$

Proof. In the open set $U_{1}$ with affine coordinates $x_{0}=\frac{X_{0}}{X_{1}}, x_{2}=\frac{X_{2}}{X_{1}}$, this form becomes

$$
\begin{aligned}
\omega_{2, k \mid U_{1}} & =\frac{x_{2}}{x_{0}^{k}} d \frac{1}{x_{0}} d \frac{1}{x_{0}}+\left(1-\frac{1}{x_{0}^{k}}-\frac{x_{2}^{k}}{x_{0}^{k}}\right) d \frac{1}{x_{0}} d \frac{x_{2}}{x_{0}}+\frac{x_{2}^{k-1}}{x_{0}^{k}} d \frac{x_{2}}{x_{0}} d \frac{x_{2}}{x_{0}} \\
& =\frac{1}{x_{0}^{k+3}}\left(x_{0}^{k-1} x_{2} d x_{0} d x_{0}+\left(1-x_{0}^{k+1}-x_{2}^{k}\right) d x_{0} d x_{2}+x_{0} x_{2}^{k-1} d x_{2} d x_{2}\right)
\end{aligned}
$$

Similarly on $U_{2}$ we have (with affine coordinates $x_{0}=\frac{X_{0}}{X_{2}}, x_{1}=\frac{X_{1}}{X_{2}}$ )

$$
\omega_{2, k \mid U_{2}}=\frac{1}{x_{0}^{k+3}}\left(x_{0}^{k-1} x_{1} d x_{0} d x_{0}+\left(-x_{1}^{k}-x_{0}^{k}+1\right) d x_{0} d x_{1}+x_{0} x_{1}^{k-1} d x_{1} d x_{1}\right)
$$

Remark 28. Choosing this particularly convenient symmetric differential is a nontrivial key step in the argument. In view of the method outlined in Section 3. this $\omega_{2, k}$ must make all the irreducible components of the branch curves of the morphisms $\rho_{n, k}$ to be $\omega$-integral. In Appendix A in Voj00a, Vojta explains how
he found a suitable differential by using a computer search in positive characteristic. In this work instead, we impose the condition that branch curves of the tower of morphisms of surfaces be $\omega_{2, k}$-integral. This is a strong constraint on which $\omega_{2, k}$ can be taken, which led to our choice of $\omega_{2, k}$.
Lemma 29. For a natural number $k \geq 1$, the following irreducible curves are $\omega_{2, k}$-integral curves on $X_{2, k}=\mathbb{P}^{2}$ :
(i) $x_{0}=0, x_{1}=0, x_{2}=0$;
(ii) $D_{c}: c(c+1) x_{0}^{k}=c x_{1}^{k}-(c+1) x_{2}^{k}$, with $c \in \mathbb{C} \backslash\{-1,0\}$.

If $k$ is an even natural number, the following are also $\omega_{2, k}$-integral curves on $X_{2, k}$ :
(iii) $x_{0}^{\frac{k}{2}} \pm x_{1}^{\frac{k}{2}}= \pm x_{2}^{\frac{k}{2}}$.

If $k$ is an odd natural number, the following (irreducible) curve is also $\omega_{2, k}$-integral:
(iv) $x_{0}^{2 k}+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{0}^{k} x_{1}^{k}-2 x_{0}^{k} x_{2}^{k}-2 x_{1}^{k} x_{2}^{k}=0$.

Proof. For curves of type (i), the curve $x_{i}=0$ satisfies $d x_{i}=0$ (writing $\omega_{2, k}$ in appropriate coordinates for each case). Hence by Theorem 19 one gets that curves of type (i) are $\omega_{2, k}$-integral.

For a curve $D_{c}$ of type (ii), we have $c(c+1)=c x_{1}^{k}-(c+1) x_{2}^{k}$ in $U_{0}$. Taking differentials we obtain $d x_{1}=\left(\frac{c+1}{c}\right)\left(\frac{x_{2}}{x_{1}}\right)^{k-1} d x_{2}$. Hence on that curve $\omega_{2, k}$ is

$$
\begin{aligned}
\left(x_{1}^{k-1}\right. & \left.x_{2}\left(\frac{c+1}{c}\right)^{2}\left(\frac{x_{2}}{x_{1}}\right)^{2 k-2}+\left(1-x_{1}^{k}-x_{2}^{k}\right) \frac{c+1}{c}\left(\frac{x_{2}}{x_{1}}\right)^{k-1}+x_{1} x_{2}^{k-1}\right) d x_{2} d x_{2} \\
& =\left(x_{2}^{k}\left(\frac{c+1}{c}\right)^{2}+\frac{c+1}{c}\left(-c-\frac{2 c+1}{c} x_{2}^{k}\right)+c+1+\frac{c+1}{c} x_{2}^{k}\right) \frac{x_{2}^{k-1}}{x_{1}^{k-1}} d x_{2} d x_{2} \\
& =0 .
\end{aligned}
$$

By Theorem 19 we get that $D_{c}$ is $\omega_{2, k}$-integral.
Let $C_{\epsilon_{1}, \epsilon_{2}}^{k}: x_{0}^{\frac{k}{2}}+\epsilon_{1} x_{1}^{\frac{k}{2}}=\epsilon_{2} x_{2}^{\frac{k}{2}}$ with $\epsilon_{1}, \epsilon_{2} \in\{ \pm 1\}$ be a curve of type (iii). Then we have $1+\epsilon_{1} x_{1}^{\frac{k}{2}}=\epsilon_{2} x_{2}^{\frac{k}{2}}$ in $U_{0}$. Taking differentials we obtain $\epsilon_{1} x_{1}^{\frac{k}{2}-1} d x_{1}=\epsilon_{2} x_{2}^{\frac{k}{2}-1} d x_{2}$, and doing a similar computation, we obtain that $C_{\epsilon_{1}, \epsilon_{2}}^{k}$ is $\omega_{2, k}$-integral.

Now we consider the case $k$ odd and the curve of type (iv): Note that this curve is irreducible by Lemma 26. Taking differentials of $1+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{1}^{k}-2 x_{2}^{k}-2 x_{1}^{k} x_{2}^{k}=0$ we obtain

$$
\left(x_{1}^{2 k-1}-x_{1}^{k-1}-x_{1}^{k-1} x_{2}^{k}\right) d x_{1}-\left(x_{1}^{k} x_{2}^{k-1}+x_{2}^{k-1}-x_{2}^{2 k-1}\right) d x_{2}=0 .
$$

Hence, by a similar computation, $\omega_{2, k}$ restricted to the curve of type (iv) on the open set $U_{3} \cap D_{+}\left(x_{1}^{k}-x_{2}^{k}-1\right)$ has equation

$$
-\frac{x_{1}^{1-k} x_{2}^{k-1}\left(1+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{1}^{k}-2 x_{2}^{k}-2 x_{1}^{k} x_{2}^{k}\right)}{\left(x_{1}^{k}-x_{2}^{k}-1\right)^{2}} d x_{2} d x_{2}=0 .
$$

We have that $U_{3} \cap D_{+}\left(x_{1}^{k}-x_{2}^{k}-1\right)$ intersects the irreducible curve

$$
1+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{1}^{k}-2 x_{2}^{k}-2 x_{1}^{k} x_{2}^{k}=0
$$

because $x_{1}^{k}-x_{2}^{k}-1$ is not a multiple of this curve. Therefore applying Theorem 19 to this open set, we have that the curve

$$
1+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{1}^{k}-2 x_{2}^{k}-2 x_{1}^{k} x_{2}^{k}=0
$$

is $\omega_{2, k}$-integral when $k$ is odd.

Lemma 30. If $k$ is an odd natural number, the curves of types (i), (ii), and (iv) are the only $\omega_{2, k}$-integral curves on $X_{2, k}=\mathbb{P}^{2}$. If $k$ is even, the curves of types (i), (ii), and (iii) are the only $\omega_{2, k}$-integral curves on $X_{2, k}$.

Proof. The restriction of $\omega_{2, k}$ to $U_{0}$ has equation

$$
\omega_{2, k}=x_{1}^{k-1} x_{2} d x_{1} d x_{1}+\left(1-x_{1}^{k}-x_{2}^{k}\right) d x_{1} d x_{2}+x_{1} x_{2}^{k-1} d x_{2} d x_{2} .
$$

We have from Definition 20,

$$
\Delta=\left(\mathbb{P}^{2} \backslash U_{0}\right) \cup\left\{P \in U_{0}: A_{0}(P)=0 \text { or } A_{1}^{2}(P)-4 A_{0}(P) A_{2}(P)=0\right\} \subseteq \mathbb{P}^{2} .
$$

In our case (for $P=\left(x_{1}, x_{2}\right)$ )

$$
\begin{aligned}
& A_{0}=x_{1}^{k-1} x_{2}, \\
& A_{1}=1-x_{1}^{k}-x_{2}^{k}, \\
& A_{2}=x_{1} x_{2}^{k-1}
\end{aligned}
$$

so the last condition becomes

$$
\begin{aligned}
A_{2}^{2}(P)-4 A_{1}(P) A_{3}(P) & =\left(1-x_{1}^{k}-x_{2}^{k}\right)^{2}-4\left(x_{1}^{k-1} x_{2}\right)\left(x_{1} x_{2}^{k-1}\right) \\
& =1+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{1}^{k}-2 x_{2}^{k}-2 x_{1}^{k} x_{2}^{k} .
\end{aligned}
$$

Therefore

$$
\Delta=\left\{\left[x_{0}: x_{1}: x_{2}\right]: x_{0} x_{1} x_{2}\left(1+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{1}^{k}-2 x_{2}^{k}-2 x_{1}^{k} x_{2}^{k}\right)=0\right\} .
$$

Note that by Lemma 29, when $k$ is odd, $\Delta$ is the union of the curves of types (i) and (iv), and when $k$ is even, $\Delta$ is the union of the curves of types (i) and (iii).

Now we want to prove that the $\omega_{2, k}$-integral curves not contained in $\Delta$ are exactly the curves of type (ii). Let $P=\left[x_{1}: x_{2}: x_{3}\right]$ be a point outside $\Delta$. From Theorem 21] we only need to prove that there are at least two $\omega_{2, k}$-integral curves of type (ii) passing through $P$. The point $P$ lies on $D_{c}$ if and only if $c(c+1) x_{0}^{k}=c x_{1}^{k}-(c+1) x_{2}^{k}$. The discriminant of the equation $c x_{1}^{k}-(c+1) x_{2}^{k}=c(c+1)$ (with $c$ the variable) is

$$
1+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{1}^{k}-2 x_{2}^{k}-2 x_{1}^{k} x_{2}^{k},
$$

which is different from zero because $P$ is outside $\Delta$. Therefore there are two values of $c$ for which $D_{c}$ passes through $P$.

## 5. Pullbacks of $\omega_{2, k}$-Integral curves

Now that we have the complete list of $\omega_{2, k}$-integral curves of $X_{2, k} \cong \mathbb{P}^{2}$ for any $k$, we will use them to find integral curves on the other surfaces $X_{n, k}$.

We have an infinite chain of finite surjective morphisms:

$$
X_{2, k} \stackrel{\pi_{3}}{\leftarrow} X_{3, k} \stackrel{\pi_{4}}{\leftarrow} X_{4, k} \stackrel{\pi_{5}}{\stackrel{ }{c} \cdots .}
$$

Recall that for each $n \geq 3$ we denoted by $\rho_{n, k}$ the composition $\pi_{3} \circ \pi_{4} \circ \cdots \circ \pi_{n}$.
Lemma 31. The morphisms $\pi_{n}: X_{n, k} \rightarrow X_{n-1, k}$ are finite of degree $k$. Therefore the morphisms $\rho_{n, k}: X_{n, k} \rightarrow X_{2, k}$ are finite of degree $k^{n-2}$.

Proof. From the proof of Lemma 13, the preimage of a point in $X_{n-1, k}$ under the finite morphism $\pi_{n}$ generically consists of $k$ points. Since the surfaces $X_{i, k}$ are irreducible, we obtain that $\operatorname{deg}\left(\pi_{n}\right)=k$.

Lemma 32. The pullbacks under $\rho_{n, k}$ of the curves of types (i) and (ii) with $c \neq-1, \ldots, n-2$ of Lemma 29 are smooth complete intersection curves.

Proof. The pullback of a curve of type (ii) with $c \neq-1, \ldots, n-2$ is given by the equations

$$
\begin{aligned}
c(c+1) x_{0}^{k}-c x_{1}^{k}+(c+1) x_{2}^{k} & =0 \\
2 x_{0}^{k}-x_{1}^{k}+2 x_{2}^{k} & =x_{3}^{k} \\
& \vdots \\
(n-1)(n-2) x_{0}^{k}-(n-2) x_{1}^{k}+(n-1) x_{2}^{k} & =x_{n}^{k} .
\end{aligned}
$$

The pullback of a curve of type (i) is given by the equations

$$
\begin{aligned}
x_{j} & =0, \\
2 x_{0}^{k}-x_{1}^{k}+2 x_{2}^{k} & =x_{3}^{k} \\
& \vdots \\
(n-1)(n-2) x_{0}^{k}-(n-2) x_{1}^{k}+(n-1) x_{2}^{k} & =x_{n}^{k}
\end{aligned}
$$

with $j=1,2,3$.
These curves are clearly complete intersections. They are smooth by a computation similar to the proof of Lemma 16 .

In the discussion below, we will consider the following divisor of $X_{j, k}$ :

$$
C_{j}=\operatorname{div}_{X_{j, k}}\left(x_{j}\right)
$$

Lemma 33. Let $n \geq 3$. The morphism $\pi_{n}: X_{n, k} \rightarrow X_{n-1, k}$ is ramified only at the components of the divisor $C_{n}$. Moreover, we have

$$
\rho_{n, k}\left(C_{n}\right)=D_{n-2},
$$

with $D_{n-2}$ a curve of type (ii) as defined in Lemma 29, In addition,

$$
\rho_{n, k}^{*} D_{n-2}=k C_{n} .
$$

Proof. From the proof of Lemma [13, we see that $\#\left(\pi_{n}^{-1}(P)\right)=k$ for all $P \in X_{n-1, k}$ except when $P=\left[x_{0}: \ldots: x_{n-1}\right]$ lies on the curve $\tilde{D}_{n-2}=\rho_{n-1, k}^{*}\left(D_{n-2}\right)$, which is defined on $X_{n-1, k}$ by the equation

$$
(n-1) x_{2}^{k}-(n-2) x_{1}^{k}-(n-1)(n-2) x_{0}^{k}=0 .
$$

Thus $\pi_{n}$ is unramified at any curve $C \not \subset \tilde{D}_{n-2}$. Moreover, since $\#\left(\pi_{n}^{-1}(P)\right)=1$ for each $P \in \operatorname{supp}\left(\tilde{D}_{n-2}\right)$, we see that $\pi_{n}$ is totally branched of degree $k$ at each component of $\tilde{D}_{n-2}$. Now

$$
\begin{aligned}
\pi_{n}^{*} \tilde{D}_{n-2} & =\operatorname{div}_{X_{n, k}}\left((n-1) x_{2}^{k}-(n-2) x_{1}^{k}-(n-1)(n-2) x_{0}^{k}\right) \\
& =\operatorname{div}_{X_{n, k}}\left(x_{n}^{k}\right)=k C_{n}
\end{aligned}
$$

Thus, $\pi_{n}$ is ramified precisely in the components of $C_{n}$. Finally

$$
\rho_{n, k}^{*} D_{n-2}=\pi_{n}^{*} \rho_{n-1, k}^{*} D_{n-2}=\pi_{n}^{*} \tilde{D}_{n-2}=k C_{n} .
$$

Lemma 34. Let $3 \leq j \leq n$, and let $C_{n, j}=\left(\pi_{j+1} \circ \cdots \circ \pi_{n}\right)^{*} C_{j}$ in $X_{n, k}$. The curves $C_{n, j}$ are smooth complete intersection curves. In particular, the curve $C_{n}=C_{n, n}$ is a smooth complete intersection.

Proof. Let $C_{n, j}=\left(\pi_{j+1} \circ \cdots \circ \pi_{n}\right)^{*} C_{j}$. It is given by the following equations:

$$
\begin{aligned}
x_{j} & =0 \\
2 x_{0}^{k}-x_{1}^{k}+2 x_{2}^{k} & =x_{3}^{k} \\
& \vdots \\
(n-1)(n-2) x_{0}^{k}-(n-2) x_{1}^{k}+(n-1) x_{2}^{k} & =x_{n}^{k}
\end{aligned}
$$

These curves are complete intersections, and they are smooth by a proof similar to Lemma 16

For $k$ an even integer and $\epsilon_{1}, \ldots, \epsilon_{n} \in\{ \pm 1\}$, define the schemes $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k} \subseteq \mathbb{P}^{n}$ by the equations

$$
\begin{align*}
x_{0}^{\frac{k}{2}}+\epsilon_{1} x_{1}^{\frac{k}{2}} & =\epsilon_{2} x_{2}^{\frac{k}{2}} \\
& \vdots  \tag{3}\\
(n-1) x_{0}^{\frac{k}{2}}+\epsilon_{1} x_{1}^{\frac{k}{2}} & =\epsilon_{n} x_{n}^{\frac{k}{2}} .
\end{align*}
$$

Lemma 35. For $k$ even, the schemes $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ are smooth irreducible curves in $X_{n, k}$.
Proof. Let $C_{\epsilon_{1}, \epsilon_{2}}^{k}$ be of type (iii). Squaring the first equation of $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ we have $2 \epsilon_{1} x_{0}^{\frac{k}{2}} x_{1}^{\frac{k}{2}}=-x_{0}^{k}-x_{1}^{k}+x_{2}^{k}$. Squaring $(i-1) x_{0}^{\frac{k}{2}}+\epsilon_{1} x_{1}^{\frac{k}{2}}=\epsilon_{i} x_{i}^{\frac{k}{2}}$ and replacing $2 \epsilon_{1} x_{0}^{\frac{k}{2}} x_{1}^{\frac{k}{2}}$ by $-x_{0}^{k}-x_{1}^{k}+x_{2}^{k}$ one gets

$$
(i-1)(i-2) x_{0}^{k}-(i-2) x_{1}^{k}+(i-1) x_{2}^{k}=x_{i}^{k} .
$$

Since this holds for every $i \geq 3$ we obtain that $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k} \subseteq X_{n, k}$.
We have $\pi_{n}\left(C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}\right) \subseteq C_{\epsilon_{1}, \ldots, \epsilon_{n-1}}^{k} \subseteq X_{n, k}$. Since for any

$$
P=\left[x_{0}: \cdots: x_{n-1}\right] \in C_{\epsilon_{1}, \ldots, \epsilon_{n-1}}^{k} \subseteq X_{n, k}
$$

we have that $Q=\left[x_{0}: \cdots: x_{n}\right] \in C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ with $\epsilon_{n} x_{n}^{k / 2}=\epsilon_{1} x_{1}^{k / 2}+(n-1) x_{0}^{k / 2}$ is a preimage of $P$, we obtain that $\pi_{n}\left(C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}\right) \supseteq C_{\epsilon_{1}, \ldots, \epsilon_{n-1}}^{k}$; thus $\pi_{n}\left(C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}\right)=$ $C_{\epsilon_{1}, \ldots, \epsilon_{n-1}}^{k}$. From this we also get that every component of $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ has dimension less than or equal to 1 since $\pi_{n}\left(C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}\right) \neq \pi_{n}\left(X_{n, k}\right)=X_{n-1, k}$ and $\pi_{n}$ is finite. On the other hand, every component of $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ has dimension at least 1 because it is defined by $n-1$ equations in $\mathbb{P}^{n}$, and we conclude by Theorem I.7.2 in Har77. Therefore, $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ has all its irreducible components of dimension exactly 1 (it is equidimensional).

Since $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ is equidimensional of dimension 1 in $\mathbb{P}^{n}$ and is defined by $n-1$ equations, it is a complete intersection. The Jacobian matrix of $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ evaluated at $\left[x_{0}: \cdots: x_{n}\right]$ is the following $(n-1) \times(n+1)$ matrix:

$$
\frac{k}{2}\left(\begin{array}{cccccc}
x_{0}^{\frac{k}{2}-1} & \epsilon_{1} x_{1}^{\frac{k}{2}-1} & \epsilon_{2} x_{2}^{\frac{k}{2}-1} & 0 & \cdots & 0 \\
2 x_{0}^{\frac{k}{2}-1} & \epsilon_{1} x_{1}^{\frac{k}{2}-1} & 0 & \epsilon_{3} x_{3}^{\frac{k}{2}-1} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \cdots \\
(n-1) x_{0}^{\frac{k}{2}-1} & \epsilon_{1} x_{1}^{\frac{k}{2}-1} & 0 & \cdots & 0 & \epsilon_{n} x_{n}^{\frac{k}{2}-1}
\end{array}\right)
$$

If none of $x_{2}, \ldots, x_{n}$ are zero, then this matrix has maximal rank. If one of $x_{2}, \ldots, x_{n}$ is zero, then at least one of $x_{0}, x_{1}$ is not zero; hence the matrix has
maximal rank. If two of $x_{2}, \ldots, x_{n}$ are zero, then we have $x_{0} x_{1} \neq 0$. Noting that for any $2 \leq i \neq j \leq n$, the determinant

$$
\left(\begin{array}{cc}
(i-1) x_{0}^{\frac{k}{2}-1} & \epsilon_{1} x_{1}^{\frac{k}{2}-1} \\
(j-1) x_{0}^{\frac{k}{2}-1} & \epsilon_{1} x_{1}^{\frac{k}{2}-1}
\end{array}\right)
$$

is non-zero, we obtain that the $(n-1) \times(n+1)$ matrix has maximal rank. Since $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ is a smooth complete intersection, we obtain that it is irreducible.
Lemma 36. If $\epsilon_{i} \neq \epsilon_{i}^{\prime}$ for some $1 \leq i \leq n$, then the curves $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ and $C_{\epsilon_{1}^{\prime}, \ldots, \epsilon_{n}^{\prime}}^{k}$ are distinct.

Proof. Let $i^{\prime}=i$ if $i \neq 1$, and $i^{\prime}=2$ if $i=1$. The linear projection $\mathbb{P}^{n} \rightarrow \mathbb{P}^{2}$ $\left[x_{0}: \cdots: x_{n}\right] \mapsto\left[x_{0}: x_{1}: x_{i^{\prime}}\right]$ maps $C_{\epsilon}^{k}$ and $C_{\epsilon^{\prime}}^{k}$ to two different Fermat type curves.

Lemma 37. Let $k$ be an even integer. Then

$$
\rho_{n, k}^{*} C_{\epsilon_{1}, \epsilon_{2}}^{k}=\sum_{\epsilon_{1}, \ldots, \epsilon_{n} \in\{ \pm 1\}} C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k} .
$$

Proof. For fixed $\epsilon_{1}, \epsilon_{2}$ the image of any $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ under $\rho_{n, k}$ is $C_{\epsilon_{1}, \epsilon_{2}}^{k}$. Now we want to prove that these irreducible curves are all the preimages of $C_{\epsilon_{1}, \epsilon_{2}}^{k}$. The restriction of the morphism $\mathbb{P}^{n} \backslash[0: \cdots: 0: 1] \rightarrow \mathbb{P}^{n-1}$ to $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ gives a morphism $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k} \rightarrow C_{\epsilon_{1}, \ldots, \epsilon_{n-1}}^{k}$ which has degree $\frac{k}{2}$. The composition of these restrictions has degree $\left(\frac{k}{2}\right)^{n-2}$, hence $\operatorname{deg}\left(\rho_{n, k \mid C_{\epsilon_{1}}^{k}, \ldots, \epsilon_{n}}\right)=\left(\frac{k}{2}\right)^{n-2}$. From Lemma 36, all the components of the preimage of $C_{\epsilon_{1}, \epsilon_{2}}^{k}$ under $\rho_{n, k}$ are distinct.

Since there are $2^{n-2}$ distinct curves $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ in the preimage of $C_{\epsilon_{1}, \epsilon_{2}}^{k}$, we get $\left(\frac{k}{2}\right)^{n-2} 2^{n-2}=k^{n-2}$. Hence the curves $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$ are all the components of the pullback of $C_{\epsilon_{1}, \epsilon_{2}}^{k}$ counting multiplicities.
Lemma 38. Let $k$ be odd. If $C_{n, k}^{(i v)}$ is defined as the pullback to $X_{n, k}$ of the curve of type (iv) of $X_{2, k}$, then $C_{n, k}^{(i v)}$ is reduced, irreducible, and given by the equations

$$
\begin{aligned}
x_{0}^{2 k}+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{0}^{k} x_{1}^{k}-2 x_{0}^{k} x_{2}^{k}-2 x_{1}^{k} x_{2}^{k} & =0 \\
2 x_{0}^{k}-x_{1}^{k}+2 x_{2}^{k} & =x_{3}^{k} \\
& \vdots \\
(n-1)(n-2) x_{0}^{k}-(n-2) x_{1}^{k}+(n-1) x_{2}^{k} & =x_{n}^{k} .
\end{aligned}
$$

Moreover, the $2^{n}$ curves $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{2 k} \subseteq X_{n, 2 k}$ are isomorphic to each other and are birational to $C_{n, k}^{(i v)}$.
Proof. The rule $\left[x_{0}: \cdots: x_{n}\right] \mapsto\left[x_{0}^{2}: \cdots: x_{n}^{2}\right]$ defines a surjective morphism $F_{n}: X_{n, 2 k} \rightarrow X_{n, k}$. We have the commutative diagram (as can be seen from the equations)


Note that $C_{2, k}^{(i v)}=\operatorname{div}_{\mathbb{P}^{2}}\left(\bar{P}_{k}\right)$ with $\bar{P}_{k}=x_{0}^{2 k}+x_{1}^{2 k}+x_{2}^{2 k}-2 x_{0}^{k} x_{1}^{k}-2 x_{0}^{k} x_{2}^{k}-2 x_{1}^{k} x_{2}^{k}$, which is the homogenization of the polynomial $P$ from Lemma 26. By that lemma, we have $F_{2}^{*} C_{2, k}^{(i v)}=\sum_{\epsilon_{1}, \epsilon_{2}} C_{\epsilon_{1}, \epsilon_{2}}^{2 k}$.

By Lemma 37 we obtain $\rho_{n, 2 k}^{*} \sum_{\epsilon_{1}, \epsilon_{2}} C_{\epsilon_{1}, \epsilon_{2}}^{2 k}=\sum_{\bar{\epsilon} \in G} C_{\bar{\epsilon}}^{2 k}$, where $G=\{ \pm 1\}^{n}$. By definition, we have $C_{n, k}^{(i v)}:=\rho_{n, k}^{*} C_{2, k}^{(i v)}$, so we get

$$
F_{n}^{*} C_{n, k}^{(i v)}=F_{n}^{*} \rho_{n, k}^{*} C_{2, k}^{(i v)}=\rho_{n, 2 k}^{*} F_{2}^{*} C_{2, k}^{(i v)}=\sum_{\bar{\epsilon} \in G} C_{\bar{\epsilon}}^{2 k} .
$$

In particular $C_{n, k}^{(i v)}$ is reduced because the curves $C_{\bar{\epsilon}}^{2 k}$ are reduced by Lemma 37 .
Let $G=\{ \pm 1\}^{n} \cong(\mathbb{Z} / 2 \mathbb{Z})^{n}$ act on $\mathbb{P}^{n}$ via

$$
\tau\left(\left[x_{0}: \cdots: x_{n}\right]\right)=\left[x_{0}: \tau_{1} x_{1}: \cdots: \tau_{n} x_{n}\right] .
$$

Then $\tau\left(X_{n, 2 k}\right)=X_{n, 2 k}$, for all $\tau \in G$ and $F_{n} \circ \tau=F_{n}$. Moreover $\tau C_{\bar{\epsilon}}^{2 k}=C_{\tau \bar{\epsilon}}^{2 k}$ (since $k$ is odd), where $\bar{\epsilon}=\left(\epsilon_{1}, \ldots, \epsilon_{n}\right) \in G$, and so $C_{\bar{\epsilon}}^{2 k}=\bar{\epsilon} C_{(1, \ldots, 1)}^{2 k}$ are all isomorphic to each other. Thus $F_{n}\left(C_{\bar{\epsilon}}^{2 k}\right)=F_{n}\left(C_{(1, \ldots, 1)}^{2 k}\right)$, for all $\bar{\epsilon} \in G$.

Thus, since $F_{n}$ is surjective, we have

$$
C_{n, k}^{(i v)}=F_{n} F_{n}^{-1}\left(C_{n, k}^{(i v)}\right)=F_{n}\left(\bigcup C_{\bar{\epsilon}}^{2 k}\right)=F_{n}\left(\tau C_{(1, \ldots, 1)}^{2 k}\right)=F_{n}\left(C_{(1, \ldots, 1)}^{2 k}\right) .
$$

Since $C_{(1, \ldots, 1)}^{2 k}$ is irreducible, so is $C_{n, k}^{(i v)}=F_{n}\left(C_{\bar{\epsilon}}^{2 k}\right)$. Thus $C_{n, k}^{(i v)}$ is a reduced and irreducible curve, and $F_{n}\left(C_{\bar{\epsilon}}^{2 k}\right)=C_{n, k}^{(i v)}$, for all $\bar{\epsilon}$. Since $\operatorname{deg}\left(F_{n}\right)=2^{n}$ and we have $2^{n}$ curves $C_{\epsilon_{1}, \ldots, \epsilon_{n}}^{k}$, we have that the residue degrees $f_{C_{\epsilon}^{2 k} / C_{n, k}^{(i v)}}=1$ for all $\bar{\epsilon}$, and hence $F_{n \mid C_{\epsilon}^{2 k}}: C_{\bar{\epsilon}}^{2 k} \rightarrow C_{n, k}^{(i v)}$ is birational.

Finally, we obtain
Lemma 39. The pullbacks under $\rho_{n, k}$ of curves of type (i) and the curves of type (ii) with $c \neq-1, \ldots, n-2$ are smooth, irreducible, and reduced. If $k$ is even, then the pullbacks of the curves of type (iii) comprise the sum of the curves $C_{\bar{\epsilon}}^{k}$ with $\bar{\epsilon} \in G$ which are smooth, irreducible, and reduced. If $k$ is odd, the pullback of the curve of type (iv) is irreducible and reduced. The pullback of curves of type (ii) with $c=1, \ldots, n-2$ is $\rho_{n, k}^{*}\left(D_{c}\right)=k C_{n, c+2}$, where $C_{n, j}$ are smooth, irreducible, and reduced.

Proof. From Lemma 38 we know that the curves of type (iv) are irreducible. The pullbacks of the curves of types (i), (iii), and (ii) with $c \neq-1, \ldots, n-2$ are smooth and complete intersection curves by Lemmas 32 and 37. From Lemma 34 we know that the curves $C_{i}$ are smooth complete intersections. From Har77, Ex. II.8.4(c)] we get that all these curves are connected, hence irreducible. The curves of type (ii) with $c=1, \ldots, n-2$ are irreducible (but not reduced) because they are $k$ times a curve $C_{n, c+2}$ (see Lemma 34).

## 6. $\omega_{n, k}$-INTEGRAL CURVES ON $X_{n, k}$

Since the morphisms $\pi_{n}$ are linear projections, we have that $\rho_{n, k}^{*} \mathcal{O}_{\mathbb{P}^{2}}(1)=$ $\mathcal{O}_{X_{n, k}}(1)$. Thus, we can define

$$
\omega_{n, k}=\rho_{n, k}^{\bullet} \omega_{2, k} \in H^{0}\left(X_{n, k}, \mathcal{O}(k+3) \otimes S^{2} \Omega_{X_{n, k}}^{1}\right) .
$$

Proposition 40. For $k \geq 2$, the $\omega_{n, k}$-integral curves in $X_{n, k}$ are the following:
(a) The pullbacks under $\rho_{n, k}$ of the coordinate axes of $X_{2, k}$. These curves are smooth and irreducible with genus $\frac{k^{n-2}}{2}(k(n-2)-n)+1$.
(b) $C_{n, c+2}=\left(\rho_{n, k}^{*}\left(D_{c}\right)\right)_{\text {red }}$ with $c \neq\{-1,0\}$. These curves are smooth and irreducible. When $c \notin\{1, \ldots, n-2\}$, they have genus

$$
\frac{k^{n-1}}{2}(k(n-1)-n-1)+1,
$$

and when $c \in\{1, \ldots, n-2\}$, they have genus $\frac{k^{n-2}}{2}(k(n-2)-n)+1$.
Moreover, the following curves are also $\omega_{n, k}$-integral:
(c) If $k$ is odd, the pullback of the curve of type (iii). It is reduced, irreducible, and has geometric genus $\frac{k^{n-1}}{2}(k(n-1)-n-1)+1$.
(d) If $k$ is even, the $2^{n}$ curves $C_{\epsilon_{1}, \ldots, \epsilon_{n}}$ :

$$
\begin{aligned}
\epsilon_{1} x_{1}^{\frac{k}{2}} & =\epsilon_{2} x_{2}^{\frac{k}{2}}-x_{0}^{\frac{k}{2}} \\
& \vdots \\
\epsilon_{1} x_{1}^{\frac{k}{2}} & =\epsilon_{n} x_{n}^{\frac{k}{2}}-(n-1) x_{0}^{\frac{k}{2}} .
\end{aligned}
$$

They are smooth and irreducible of genus $\frac{1}{2}\left(\frac{k}{2}\right)^{n-1}\left(\frac{k}{2}(n-1)-n-1\right)+1$.
Proof. Let $C \subseteq X_{n, k}$ be an $\omega_{n, k}$-integral curve. By Theorem 22 its image $D=$ $\rho_{n, k}(C)$ must be $\omega_{2, k}$-integral. Therefore $C$ is a component of $\rho_{n, k}^{*}(D)$. Hence by Lemma 29 and Lemma 39, $C$ is a curve described in this proposition.

Now we compute the genus of these curves. Let $C$ be a curve of type (a). We have from Ex. IV.3.3.2 in Har77 that $\operatorname{deg}_{C} i^{*} \mathcal{O}_{\mathbb{P}^{n}}(1)=\operatorname{deg}(C)$, which is equal to $1 \cdot k^{n-2}$ by [EH00, Theorem III-71]. From Lemma 32 we know that they are a complete intersection, so by [Har77, Ex. II.8.4(e)] we have $K_{C}=\mathcal{O}(k(n-2)-n)$; hence the genus of $C$ is $\frac{k^{n-2}}{2}(k(n-2)-n)+1$.

Let $C$ be a curve of type (b) with $c \notin\{1, \ldots, n-2\}$. Then $\operatorname{deg}(C)=k^{n-1}$, and from Lemma 32 we have $K_{C}=\mathcal{O}(k(n-1)-n-1)$. Hence the genus of $C$ is $\frac{k^{n-1}}{2}(k(n-1)-n-1)+1$. If $C$ is a curve of type (b) with $c \in\{1, \ldots, n-2\}$, we have $\operatorname{deg}(C)=k^{n-2}$ and $K_{C}=\mathcal{O}(k(n-2)-n)$ by Lemmas 34 and 39, hence the genus of $C$ is $\frac{k^{n-2}}{2}(k(n-2)-n)+1$.

Let $k$ be even and let $C$ be a curve of type (d). By Lemma 37 we have $\operatorname{deg}(C)=\left(\frac{k}{2}\right)^{n-1}$ and $K_{C}=\mathcal{O}\left((n-1) \frac{k}{2}-n-1\right)$, and we obtain that $C$ has genus $\frac{1}{2}\left(\frac{k}{2}\right)^{n-1}\left(\frac{k}{2}(n-1)-n-1\right)+1$.

Since the genus of a curve of type (d) in $X_{n, 2 k}$ is $\frac{1}{2}\left(\frac{2 k}{2}\right)^{n-1}\left(\frac{2 k}{2}(n-1)-n-1\right)+1$, we obtain by Lemma 38 that the geometric genus of the curve of type (c) in $X_{n, k}$ is $\frac{k^{n-1}}{2}(k(n-1)-n-1)+1$.

## 7. Curves of low genus on $X_{n, k}$

Now we will show that the curves of bounded genus (for a suitable bound depending on $n$ and $k$ ) on $X_{n, k}$ are $\omega_{n, k}$-integral.

Lemma 41. The section $\omega_{n, k}$ defines a unique section

$$
\omega_{n, k}^{\prime} \in H^{0}\left(X_{n, k}, \mathcal{O}(k+3-(k-1)(n-2)) \otimes S^{2} \Omega_{X_{n, k}}^{1}\right) .
$$

Moreover, every $\omega_{n, k}^{\prime}$-integral curve is $\omega_{n, k}$-integral.

Proof. By Lemma 39, we have that $\rho_{i, k}^{*}\left(D_{i-2}\right)=k C_{n, i}$ for $i=3, \ldots, n$. Since the curves $D_{c}$ with $c=1, \ldots, n-2$ are $\omega_{2, k}$-integral, Theorem 24]shows that the section $\omega_{n, k}$ vanishes along $(k-1) \sum_{i=3}^{n} C_{n, i}$. Thus by Proposition 255, we get that for each $n$, the section $\omega_{n, k} \in H^{0}\left(X_{n, k}, \mathcal{O}(k+3) \otimes S^{2} \Omega_{X_{n, k}}^{1}\right)$ determines a unique section $\omega_{n, k}^{\prime} \in H^{0}\left(X_{n, k}, \mathcal{O}(k+3-(k-1)(n-2)) \otimes S^{2} \Omega_{X_{n, k}}^{1}\right)$ which makes the $\omega_{n, k}^{\prime}$-integral curves be $\omega_{n, k}$-integral. Here we used $\mathcal{O}\left(C_{n, i}\right)=\mathcal{O}_{X_{n, k}}(1)$ as $C_{n, i}=\operatorname{div}_{X_{n, k}}\left(x_{i}\right)$; cf. the proof of Lemma 34
Proposition 42. Let $k \geq 3$, let $g \geq 1$, and let $n>\frac{4 g}{k-1}+3$. If $C$ is an irreducible curve of geometric genus $g(C) \leq g$ in $X_{n, k}$, then $C$ is $\omega_{n, k}^{\prime}$-integral. In particular, it is $\omega_{n, k}$-integral.
Proof. Recall that $\varphi_{C}: \tilde{C} \rightarrow X_{n, k}$ is the normalization of $C$. We know from Example IV.3.3.2 in Har77] that $\operatorname{deg}_{\tilde{C}} \varphi_{C}^{*} \mathcal{O}(1)=\operatorname{deg}(C) \geq 1$. Thus, since $n>$ $\frac{4}{k-1}+3$, we have that

$$
\begin{aligned}
\operatorname{deg}_{\tilde{C}}\left(\varphi_{C}^{*} \mathcal{O}(k+3-(k-1)(n-2))\right. & \left.\otimes S^{2} \Omega_{\tilde{\tilde{C}} / \mathbb{C}}^{1}\right) \\
& \leq \operatorname{deg}_{\tilde{C}}\left(\varphi_{C}^{*} \mathcal{O}(k+3-(k-1)(n-2))\right)+2(2 g-2) \\
& =(k+3-(k-1)(n-2)) \operatorname{deg}_{\tilde{C}} \varphi_{C}^{*} \mathcal{O}(1)+4 g-4 \\
& <k-(k-1)(n-2)+4 g-1 .
\end{aligned}
$$

Hence $H^{0}\left(\tilde{C}, \varphi_{C}^{*} \mathcal{O}(k+3-(k-1)(n-2)) \otimes S^{2} \Omega_{\tilde{C} / \mathbb{C}}^{1}\right)=0$ because $n>\frac{4 g}{k-1}+3$, and therefore in this case the curve $C$ is $\omega_{n, k}^{\prime}$-integral. The last statement holds by Lemma 41

## 8. Proof of the main results

Proof of Theorem 3. Let $g \geq 1$ be fixed and let $n>\frac{4 g}{k-1}+3$. Recall that $k \geq 3$. By Proposition 40, we know the genus of all the $\omega_{n, k}$-integral curves in $X_{n, k}$. We will now show that the genus of any of these curves is strictly greater than $g$.

The curves of type (a) and the curves of type (b) with $c \in\{1, \ldots, n-2\}$ have genus

$$
\begin{aligned}
\frac{k^{n-2}}{2}(k(n-2)-n)+1 & =\frac{k^{n-2}}{2}(n(k-1)-2 k)+1 \\
& >\frac{k^{n-2}}{2}(4 g+3(k-1)-2 k)+1 \\
& =\frac{k^{n-2}}{2}(4 g+k-3)+1>4 g+2-3+1 \geq g
\end{aligned}
$$

Hence curves of type (a) and curves of type (b) with $c \in\{1, \ldots, n-2\}$ have genus strictly greater than $g$.

Curves of type (b) with $c \notin\{1, \ldots, n-2\}$ and curves of type (c) have genus

$$
\begin{aligned}
\frac{k^{n-1}}{2}(k(n-1)-n-1)+1 & =\frac{k^{n-1}}{2}(n(k-1)-k-1)+1 \\
& >\frac{k^{n-1}}{2}(4 g+3(k-1)-k-1)+1 \\
& =\frac{k^{n-1}}{2}(4 g+2 k-4)+1>4 g+4-4 \geq g
\end{aligned}
$$

Thus, curves of type (b) with $c \notin\{1, \ldots, n-2\}$ and curves of type (c) have genus strictly greater than $g$.

Now suppose that $k \geq 4$ is even. Curves of type (d) have genus

$$
\begin{aligned}
& \frac{1}{2}\left(\frac{k}{2}\right)^{n-1}\left(\frac{k}{2}(n-1)-n-1\right)+1=\frac{1}{2}\left(\frac{k}{2}\right)^{n-1}\left(n\left(\frac{k}{2}-1\right)-\frac{k}{2}-1\right)+1 \\
& \quad>\frac{1}{2}\left(\frac{k}{2}\right)^{n-1}\left(\left(\frac{4 g}{k-1}+3\right)\left(\frac{k}{2}-1\right)-\frac{k}{2}-1\right)+1 \\
& \quad=\frac{1}{2}\left(\frac{k}{2}\right)^{n-1}\left(2 g \frac{k-2}{k-1}+k-4\right)+1 \\
& \quad>\left(2 g \frac{2}{3}+4-4\right)+1 \geq g
\end{aligned}
$$

Hence for even $k \geq 4$, curves of type (d) have genus greater than $g$.
From Proposition 42 we get that all curves with geometric genus $g(C) \leq g$ are $\omega_{n, k}$-integral. Since for $k \geq 3$ and $n>\frac{4 g}{k-1}+3$ the $\omega_{n, k}$-integral curves have geometric genus strictly greater than $g$, we get that there are no curves of geometric genus $g(C) \leq g$ in $X_{n, k}$.
Proof of Corollary 4. If $k \geq 3$ and $n \geq 6$, then we have $n>\frac{4 \cdot 1}{k-1}+3$. If $k=4,5$ and $n \geq 5$, then we have $n>\frac{4 \cdot 1}{k-1}+3$. If $k \geq 6$ and $n \geq 4$, then we also have $n>\frac{4 \cdot 1}{k-1}+3$. Therefore by Theorem 3 we get that there are no curves of genus $g(C) \leq 1$ on $X_{n, k}$ in these cases.
Proof of Theorem 6. Let $K$ be a function field of genus $g \geq 0$, let $k \geq 3$, and let $n>\frac{4 \max \{g, 1\}}{k-1}+3$. The solutions over $K$ (up to scaling) of the system of equations (11) are in bijection with the morphisms $\left\{f: C_{K} \rightarrow X / \mathbb{C}\right\}$, with $C_{K}$ the curve (up to isomorphism) with function field $K$. By Riemann-Hurwitz, these morphisms are either constant or must map the curve $C_{K}$ to curves in $X$ with genus less than or equal to $g$. By Theorem 3, there are no curves of genus less than or equal to $g$ in $X_{n, k}$. Therefore there are no non-constant solutions in $K$ of the system of equations (1), so after dehomogenizing we see that there are no sequences of length $n$ of elements in $K$ not all constant whose $k$-th powers have second differences equal to 2.

To prove Theorem 5 we will need the following result from Voj00b (see also McQ98). We will be using standard notation from Nevanlinna theory; cf. Voj11.
Proposition 43. Let $X$ be a non-singular complex projective variety, let $f: \mathbb{C} \rightarrow X$ be a holomorphic curve, let $d$ be a positive integer, let $\mathcal{L}$ be a line sheaf on $X$, let $\omega$ be a global section of $\mathcal{L}^{\vee} \otimes S^{d} \Omega_{X / \mathbb{C}}^{1}$, and let $\mathcal{A}$ be an ample line sheaf on $X$. If $f^{*} \omega \neq 0$, then

$$
T_{\mathcal{L}, f}(r) \leq_{e x c} O\left(\log ^{+} T_{\mathcal{A}, f}(r)\right)+o(\log r),
$$

where the notation $\leq_{\text {exc }}$ means that the inequality holds for all $r>0$ outside a set of finite Lebesgue measure.

Proof of Theorem [5. Let $n>\frac{4}{k-1}+3$ and let $f: \mathbb{C} \rightarrow X_{n, k}$ be a holomorphic map. We will show that $f$ must be constant.

Suppose that $f$ is a non-constant map. Then we can consider the holomorphic symmetric differential $f^{*} \omega_{n, k}^{\prime}$ on $\mathbb{C}$, with $\omega_{n, k}^{\prime}$ as in Lemma 41 Write $M=$ $(k-1)(n-2)-(k-3)$ and note that $M \geq 1$ by our assumption on $n$, so $\mathcal{L}:=\mathcal{O}(M)$ is ample and $\omega_{n, k}^{\prime} \in H^{0}\left(X_{n, k}, \mathcal{L}^{\vee} \otimes S^{2} \Omega_{X_{n, k}}^{1}\right)$.

If $f^{*} \omega_{n, k}^{\prime} \neq 0$, then we can apply Proposition 43 with $\mathcal{A}=\mathcal{L}$ as it is ample. Since $f$ is non-constant and $\mathcal{L}$ is ample, there is a constant $c>0$ such that

$$
\begin{equation*}
T_{\mathcal{L}, f}(r)>c \cdot \log (r) \tag{4}
\end{equation*}
$$

for $r$ large enough. Note that

$$
T_{\mathcal{L}, f}(r)+O\left(\log ^{+} T_{\mathcal{A}, f}(r)\right) \geq \frac{1}{2} T_{\mathcal{L}, f}(r)+O(1)
$$

and from Proposition 43 we have

$$
T_{\mathcal{L}, f}(r) \leq_{\text {exc }} o(\log r),
$$

which contradicts equation (4). Hence in this case $f$ must be constant.
On the other hand, if $f^{*} \omega_{n, k}^{\prime}=0$, then $f$ is locally a solution to the algebraic differential equations defined by $\omega_{n, k}^{\prime}$, and the image of $f$ is contained in an $\omega_{n, k^{-}}^{\prime}$ integral curve; hence it is contained in an algebraic curve.

Let $C \subseteq X_{n, k}$ be the irreducible algebraic curve containing the image of $f$. Then we get a map $f: \mathbb{C} \rightarrow C$ which lifts to a map $\tilde{f}: \mathbb{C} \rightarrow \tilde{C}$ satisfying $\varphi_{C} \circ \tilde{f}=f$, where $\varphi_{C}: \tilde{C} \rightarrow C$ is the normalization map. The holomorphic map $\tilde{f}$ is non-constant, so by Picard's theorem $g(\tilde{C}) \leq 1$. By our assumption on $n$ we can apply Corollary 4 to deduce that there is no curve of geometric genus 0 or 1 in $X_{n, k}$; hence $C$ cannot exist. Therefore $f$ is constant. Hence, $X_{n, k}$ is Brody-hyperbolic.

Remark 44. The proof of Theorem 9 uses the fact that a sequence $a_{1}, \ldots, a_{n}$ in a number field has second differences equal to 2 if and only if for all $1 \leq j \leq n$ we have

$$
a_{j}=-(j-2) a_{1}+(j-1) a_{2}+(j-1)(j-2),
$$

which can be easily checked.
Proof of Theorem 9. Let $a_{1}, \ldots, a_{n}$ be a sequence of $n$ elements of $L$ whose $k$-th powers have second differences equal to 2 . Then $\left[1: a_{1}: \cdots: a_{n}\right]$ is an $L$-rational point on $X_{n, k}$ by the previous remark and because the ideal of $X_{n, k}$ is $\left(g_{1}, \ldots, g_{n}\right)$; cf. Section 2. If we have infinitely many sequences of length $n$ satisfying these conditions, then we obtain infinitely many $L$-rational points on $X_{n, k}$. There are only finitely many $L$-rational points on $X_{n, k}$ which are not in the curves of genus 0 or 1 of $X_{n, k}$ by the Bombieri-Lang conjecture, since $X_{n, k}$ is of general type for $n \geq 4$ by Proposition [17. By Corollary 4. we get that there are finitely many sequences of this form for $n \geq 6$ when $k>2$ and for $n \geq 4$ when $k \geq 6$. This finite number depends only on $k$ and $L$. (Independently, $X_{n, k}(L)$ is finite by Conjecture 8 and Theorem 5)

Let $k>2$, and suppose that there are $N$ sequences of length 6 whose $k$-th powers have second differences equal to 2 (for $k \geq 6$ we can replace 6 by 4 in this argument). Let $a_{1}, \ldots, a_{N+7}$ be a sequence of elements of $L$ whose $k$-th powers have second differences equal to 2 . By a proof similar to the one of Lemma 12, we have that no term appears three times in the sequence. The $N+2$ sequences $a_{i}, \ldots, a_{i+5}$ (for $1 \leq i \leq N+2$ ) contain at least $N+1$ distinct sequences of length 6 whose $k$-th powers have second differences equal to 2 . This contradicts the fact that there are only $N$ sequences satisfying this condition.

Now we will prove Theorem 10. The following conjecture is due to Browkin and Brzezinski [BB94]:

Conjecture 45 ( $n$-term ABC conjecture). Given any integer $n>2$ and any $\epsilon>0$, there exists a constant $C_{n, \epsilon}$ such that for all integers $a_{1}, \ldots, a_{n}$ with $a_{1}+\cdots+a_{n}=0$, $\operatorname{gcd}\left(a_{1}, \ldots, a_{n}\right)=1$ and no proper zero subsum, we have

$$
\max \left(\left|a_{1}\right|, \ldots,\left|a_{n}\right|\right) \leq C_{n, \epsilon} \operatorname{rad}\left(a_{1} \cdots a_{n}\right)^{2 n-5+\epsilon}
$$

We will need the following very important theorem:
Theorem 46 (Szemerédi's theorem). Let $k$ be a positive integer and let $0<\delta<1$. There exists a positive integer $N=N(k, \delta)$ such that every subset of $\{1, \ldots, N\}$ of size at least $\delta N$ contains an arithmetic progression of length $k$.

Lemma 47. For $k \geq 2$, define the sets

$$
S_{k}=\{n \in \mathbb{Z}: n \text { is a } k \text {-th power }\}=\left\{m^{k}: m \in \mathbb{Z}\right\}
$$

and also define

$$
S_{\infty}=\{n \in \mathbb{Z}: n \text { is a } k \text {-th power, with } k \geq 13\}=\bigcup_{k \geq 13} S_{k} \text {. }
$$

There exists an $N$ such that for any sequence $a_{1}, \ldots, a_{N}$ formed by integer powers, there is an arithmetic progression

$$
m, m+n, \ldots, m+20 n
$$

(of length 21) in $\{1, \ldots, N\}$ such that for all $0 \leq j \leq 20$ we have $a_{m+j n} \in S_{k}$, for a fixed $k \in\{2, \ldots, 12, \infty\}$. Moreover, $n \leq(N-1) / 20$.
Proof. Let $N=N(21,1 / 13)$ be the integer obtained by Szemerédi's theorem. There exists $k \in\{2, \ldots, 12, \infty\}$ such that at least $\frac{1}{13}$ of the elements of $\left\{a_{1}, \ldots, a_{N}\right\}$ are in $S_{k}$ (since the sequence consists of integer powers with exponent at least 2). Hence by Theorem 46 there is an arithmetic progression $m, m+n, \ldots, m+20 n$ such that $a_{m}, \ldots, a_{m+20 n}$ consists of elements of the same $S_{k}$.

Notation 48. If $\alpha$ is an algebraic number over $\mathbb{Q}$, then we denote the number field $\mathbb{Q}(\alpha)$ by $L_{\alpha}$.
Lemma 49. Fix $N$ as in the previous lemma. Assume the Bombieri-Lang conjecture. Then there is a finite collection $\mathcal{F}$ of integer sequences of length 21, depending only on the choice of $N$, with the following property:

Let $a_{1}, \ldots, a_{N}$ be a sequence of powers with second differences equal to 2. Suppose that for some $k \in\{2,3, \ldots, 12\}$ and some $n \in\left\{1, \ldots, \frac{N-1}{20}\right\}$ there is a subsequence $a_{m}, \ldots, a_{m+20 n}$ consisting of $k$-th powers.
(i) If $k>2$, then this subsequence belongs to $\mathcal{F}$.
(ii) If $k=2$, then either the subsequence $a_{m}, \ldots, a_{m+20 n}$ belongs to $\mathcal{F}$ or the sequence $a_{1}, \ldots, a_{N}$ is a trivial sequence.
Proof. Since $a_{1}, \ldots, a_{N}$ have second differences equal to 2 , there exists a monic polynomial $P(x)=x^{2}+b x+c \in \mathbb{Q}[x]$ such that $P(i)=a_{i}$ (cf. Remark 44).

Suppose that $P(m), P(m+n), \ldots, P(m+20 n)$ are all $k$-th powers. Then the monic polynomial

$$
Q(z):=\frac{1}{n^{2}} P(m+z n)=z^{2}+\frac{1}{n}(2 m+b) z+\frac{1}{n^{2}}\left(m^{2}+b m+c\right)
$$

(which is an element in $\mathbb{Q}[z]$ ) satisfies that $Q(0), \ldots, Q(20)$ are $k$-th powers in $L_{n^{2 / k}}$. As $Q(z)$ is monic of degree 2, the sequence $Q(0), \ldots, Q(20)$ has second differences equal to 2 .

Let us prove the two items separately. For item (i), Theorem 9 and the BombieriLang conjecture (for the number field $L_{n^{2 / k}}$ ) give us that there are finitely many sequences of length 21 formed by $k$-th powers (in $L_{n^{2 / k}}$ ) whose second differences are equal to two. Let $\mathcal{S}_{n, k}$ be this finite set of sequences. Then $a_{m}, \ldots, a_{m+20 n}$ belongs to $n^{2} \mathcal{S}_{n, k}$, where we multiply each term of each element of $\mathcal{S}_{n, k}$ by $n^{2}$. Therefore, every subsequence considered in item (i) belongs to the finite set

$$
\bigcup_{k=3}^{12} \bigcup_{n=1}^{(N-1) / 20} n^{2} \mathcal{S}_{n, k}
$$

For item (ii), note that $L_{n^{2 / k}}=\mathbb{Q}$. From Theorem 0.5 in Voj00a, the sequence $Q(0), \ldots, Q(20)$ is either trivial or it belongs to a finite set $\mathcal{S}$. In the case that the subsequence belongs to $\mathcal{S}$, we conclude the proof amplifying by $n^{2}$ and taking the union of these finite sets for $n=1, \ldots, \frac{N-1}{2}$. Suppose now that $Q(0), \ldots, Q(20)$ is a trivial sequence. This means that $Q(z)=(z+h)^{2}$ for some $h \in \mathbb{Q}$, and therefore

$$
(z+h)^{2}=\frac{1}{n^{2}} P(m+z n)
$$

from which we deduce that $P(x)=(x-m+n h)^{2}$, and we conclude that the sequence $a_{1}, \ldots, a_{N}$ is trivial.

The following observation will help us to prove finiteness for sequences in the remaining cases.
Observation 50. Let $\mathcal{G} \subset \mathbb{Q}$ be a finite set, and fix $N$. Let $\mathcal{T}_{\mathcal{G}, N}$ be the set of all sequences $s=\left(a_{1}, \ldots, a_{N}\right)$ which satisfy:

- The sequence $s$ has second differences equal to 2 .
- At least 2 terms of $s$ belong to $\mathcal{G}$.

Then $\mathcal{T}_{\mathcal{G}, N}$ is finite.
Lemma 51. Assume the 4-term ABC conjecture. There is a finite collection $\mathcal{F}^{\prime}$ of integer sequences of length 21, depending only on the choice of $N$, with the following property:

Let $n \in\left\{1, \ldots, \frac{N-1}{20}\right\}$ and $a_{1}, \ldots, a_{N}$ be a sequence with second differences equal to 2 . If the subsequence $a_{m}, \ldots, a_{m+20 n}$ consists of elements in $S_{\infty}$, then it belongs to $\mathcal{F}^{\prime}$.

Proof. We know that there are at most 2 values of $j$ for which $a_{m+j n}=0$, that there are at most 2 values of $j$ for which $a_{m+j n}=2 n^{2}$, and that there are at most 2 values of $j$ for which $a_{m+j n}=-n^{2}$. Since our subsequence consists of 21 elements, there are three consecutive elements such that they all are different from $0,2 n^{2},-n^{2}$. The elements $a_{m+j n}$ (in our subsequence) satisfy (for $0 \leq j \leq 21$ ) the relation

$$
\begin{equation*}
a_{m+(j+2) n}-2 a_{m+(j+1) n}+a_{m+j n}-2 n^{2}=0, \tag{5}
\end{equation*}
$$

because the sequence $a_{1}, a_{2}, \ldots, a_{n}$ has second differences equal to 2 . If a subsum of three terms in equation (5) is equal to zero, then the fourth term has to be equal to zero, but this cannot hold since all terms are different from zero. We cannot
have $a_{m+j n}=2 n^{2}, a_{m+(j+2) n}=2 n^{2}$, or $a_{m+(j+1) n}=-n^{2}$; hence no proper subsum consisting of two terms is zero. Therefore no proper subzero sum of

$$
a_{m+(j+2) n}-2 a_{m+(j+1) n}+a_{m+j n}-2 n^{2}
$$

is zero, and $a_{m+(j+2) n}-2 a_{m+(j+1) n}+a_{m+j n}-2 n^{2}=0$. If

$$
\operatorname{gcd}\left(a_{m+(l+2) n}, 2 a_{m+(l+1) n}, a_{m+l n}, 2 n^{2}\right) \neq 1
$$

we divide by the common factor. From Conjecture 45 with $\epsilon=\frac{1}{5}$, there exists $C_{4, \epsilon}>0$ such that

$$
\begin{aligned}
& \max \left(\left|a_{m+(l+2) n}\right|,\left|2 a_{m+(l+1) n}\right|,\left|a_{m+l n}\right|,\left|2 n^{2}\right|\right) \\
& \quad \leq C_{4, \epsilon} \operatorname{rad}\left(a_{m+(l+2) n} a_{m+(l+1) n} a_{m+l n}\right)^{\frac{16}{5}} .
\end{aligned}
$$

(The gcd condition can be omitted because the relevant gcd is at most $2 n^{2}<N^{2}$, which can be absorbed in $C_{4, \epsilon}$.) Hence we have

$$
\begin{aligned}
\left|a_{m+(l+2) n} a_{m+(l+1) n} a_{m+l n}\right|^{1 / 3} & \leq \max \left(\left|a_{m+(l+2) n}\right|,\left|2 a_{m+(l+1) n}\right|,\left|a_{m+l n}\right|,\left|2 n^{2}\right|\right) \\
& \leq C_{4, \epsilon} \operatorname{rad}\left(a_{m+(l+2) n} a_{m+(l+1) n} a_{m+l n}\right)^{\frac{16}{5}} \\
& \leq C^{\prime}\left(\left|a_{m+(l+2) n} a_{m+(l+1) n} a_{m+l n}\right|^{1 / 13}\right)^{16 / 5}
\end{aligned}
$$

for an absolute constant $C^{\prime}$, because $a_{m+(l+2) n}, 2 a_{m+(l+1) n}, a_{m+l n}$ are powers of exponent at least 13 (they are in $S_{\infty}$ ). As $1 / 3>(1 / 13)(16 / 5)$ we conclude that $a_{m+(l+2) n}, 2 a_{m+(l+1) n}, a_{m+l n}$ are bounded by an absolute constant; hence there are only finitely many possibilities for these three integers. Thus there are only finitely many $a_{m+l n}, a_{m+(l+1) n}, a_{m+(l+2) n}$ in $\mathcal{S}_{\infty}$ satisfying equation (5). We conclude by Observation 50.

Proof of Theorem 10. The fact that the sets $\mathcal{F}$ and $\mathcal{F}^{\prime}$ from the previous lemmas are finite, together with Observation 50 gives the result up to finitely many sequences. We conclude by the same combinatorial argument as in the proof of Theorem 9 .

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Department of Mathematics, University of Toronto, 40 St. George Street BA6103, Toronto, Ontario, Canada, M5S 2E4

Current address: Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Campus San Joaquín, Avenida Vicuña Mackenna 4860, Santiago, Chile

Email address: natalia.garcia@mat.uc.cl


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