

NUCLEAR DIMENSION AND CLASSIFICATION OF C^* -ALGEBRAS ASSOCIATED TO SMALE SPACES

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ABSTRACT. We show that the homoclinic C^* -algebras of mixing Smale spaces are classifiable by the Elliott invariant. To obtain this result, we prove that the stable, unstable, and homoclinic C^* -algebras associated to such Smale spaces have finite nuclear dimension. Our proof of finite nuclear dimension relies on Guentner, Willett, and Yu’s notion of dynamic asymptotic dimension.

0. INTRODUCTION

The classification programme for separable simple unital nuclear C^* -algebras has made remarkable progress in recent years. The programme seeks to classify such C^* -algebras by an invariant, known as the Elliott invariant, consisting of K -theory, the tracial state space, and a pairing between these objects. Since the early days of classification, there have been many interactions with the field of topological dynamical systems. Recent research has focused on proving the presence (or absence) of certain regularity properties of a given class of C^* -algebras which might make it amenable to classification. In particular, a certain dimensionality condition—finiteness of the nuclear dimension—is required to avoid certain pathologies which might make a C^* -algebra “unclassifiable” by Elliott invariants, that is, one may find a second C^* -algebra with the same invariant which is nonisomorphic to the original. For a good discussion on the nuclear dimension and its relation to classification, see [51], [36], or [43].

Minimal dynamical systems on a compact metric space X (systems in which the orbit of every point under the homeomorphism φ is dense in X) have provided a wealth of examples of simple separable unital nuclear C^* -algebras via the crossed product construction. The question of their classification has seen a lot of interest, for example [8, 21, 27, 39, 44, 45] to name but a few. The case of a system with mean dimension zero (see [22]) was recently fully resolved by Huaxin Lin [18]. Going beyond minimal homeomorphisms, various results relating a dynamical system to an associated C^* -algebra have been established, often focusing on crossed product C^* -algebras [13, 32, 40], but also for other C^* -algebras stemming from dynamics, for example, the purely infinite Cuntz–Krieger algebras associated to subshifts of finite type.

In this paper we focus on C^* -algebras associated to certain hyperbolic dynamical systems. A Smale space is a dynamical system (X, φ) where X is a compact metric

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space and φ is a homeomorphism, with a particularly tractable local structure: at each point $x \in X$ there is a small neighbourhood which splits into two sets, which we think of as local coordinates. Along one coordinate, the system is expanding; along the other it is contracting. Ruelle defined Smale spaces in [34] to model the restriction of an Axiom A diffeomorphism to its nonwandering set or one of its basic sets. As such this class of dynamical systems is quite diverse: it includes subshifts of finite type, William's attractors (for example solenoids), Anosov diffeomorphisms (for example hyperbolic toral automorphisms), among many others.

The construction of C^* -algebras associated to a Smale space is considered in [28, 30, 33]. We briefly review these constructions. Irreducible (see Definition 1.2) Smale spaces have dense periodic points. As such, the crossed product construction, which can be seen as coming from the orbit equivalence relation (an étale groupoid) of a dynamical system, will be far from simple. However, when considering a Smale space, we are rather more interested in the behaviour of points at infinity (in either direction), that is, the expanding and contracting behaviour of our system. From this point of view, there are three naturally associated groupoids called the stable, unstable, and homoclinic groupoids. Suitably defined (see the preamble of Section 1.3) these are all étale groupoids and using the groupoid C^* -algebra construction as in [31], result in separable, nuclear C^* -algebras. Each of these algebras is stably finite and the homoclinic C^* -algebra is unital. Furthermore, in the case that the original Smale space is mixing (see Definition 1.2), each of these three C^* -algebras is simple, thus providing an excellent class of examples to examine from the point of view of the classification programme.

In this paper we show that, given a mixing Smale space (X, φ) the associated stable, unstable and homoclinic C^* -algebras have finite nuclear dimension. Our proof that the Smale space C^* -algebras have finite nuclear dimension uses the notion of dynamic asymptotic dimension for groupoids, introduced in [9]. In fact, using Smale's decomposition theorem, these results also hold under the weaker condition that the Smale space is irreducible. For the homoclinic C^* -algebra, which is unital and has a unique tracial state, we then can deduce that in the mixing case such C^* -algebras are distinguished by their Elliott invariants. A number of the special cases of Smale space C^* -algebras have been studied in the context of C^* -classification. For example, Thomsen has studied inductive limit structures [41, 42], in particular see [41, Section 4.5] and [42, Introduction]. Tikuisis, White, and Winter have published a paper on quasidiagonality of nuclear C^* -algebras [43] which implies that once we show the homoclinic algebra has finite nuclear dimension, quasidiagonality is automatic. Nevertheless, we give a proof which is independent of their result.

The stable, unstable, and homoclinic algebras each have a natural action induced from the original homeomorphism φ . While not the subject of this paper, the associated crossed product algebras, called the Ruelle algebras, are also of great interest. When the Smale space is mixing, the stable and unstable Ruelle algebras are simple and purely infinite [30, Theorems 1.4 and 1.5]. Thus, the Kirchberg–Phillips classification theorem implies they are classified by K -theory. When the Smale space in question is a shift of finite type, these algebras are (stably) isomorphic to the well-known Cuntz–Krieger algebras [5]; in this sense the Ruelle algebras are generalisations of Cuntz–Krieger algebras. Furthermore, the Smale space C^* -algebras formed from the stable and unstable relations play a role analogous to that of the

AF-cores (studied independently by Krieger [17]) in the theory of Cuntz–Krieger algebras.

The paper is structured as follows. In Section 1 we recall some definitions about topological dynamical systems and topological groupoids and then introduce Smale spaces and their associated groupoids. In Section 2 we review the notion of dynamic asymptotic dimension. We prove that the groupoids associated to an irreducible Smale space have finite dynamic asymptotic dimension in Section 3, and hence their C^* -algebras have finite nuclear dimension. We then prove that the homoclinic algebra is quasidiagonal in Section 4. Also in Section 4, we give further results for the homoclinic algebras of mixing Smale spaces, in particular, we show this class of C^* -algebras is classified by the Elliott invariant. We also obtain results about the stable and unstable algebras when they contain nonzero projections.

1. PRELIMINARIES

We begin by recalling a few definitions about topological dynamical systems. Throughout, X is an infinite compact metric space and $\varphi : X \rightarrow X$ is a homeomorphism.

Definition 1.1. Let (X, φ) be a dynamical system. A point $x \in X$ is called a periodic point if there exists $n \in \mathbb{Z}_{>0}$ such that $\varphi^n(x) = x$. A set P of periodic points is φ -invariant if for any $x \in P$ we have $\varphi(x) \in P$.

Definition 1.2. Let (X, φ) be a dynamical system.

- (i) (X, φ) is called irreducible if for every ordered pair of nonempty open sets $U, V \subset X$ there exists $n \in \mathbb{Z}_{>0}$ such that $\varphi^n(U) \cap V$ is nonempty.
- (ii) (X, φ) is called mixing if for every ordered pair of nonempty open sets $U, V \subset X$ there exists $N \in \mathbb{Z}_{>0}$ such that $\varphi^n(U) \cap V$ is nonempty for every $n \geq N$.

Note that a mixing dynamical system is irreducible, but the converse need not be true.

1.1. Smale spaces. Smale spaces are hyperbolic dynamical systems with a particularly nice local structure. They were defined by Ruelle [34] based on Smale’s work on Axiom A diffeomorphisms [37]. Around every point we can find a small neighbourhood which can be decomposed into stable and unstable coordinates. The precise definition, given below, may appear abstruse, so for a good heuristic description we encourage the reader to see [28, Section 2] or [29, Section 2.1].

Definition 1.3 ([34, Section 7.1]). Let (X, d) be a compact metric space and let $\varphi : X \rightarrow X$ be a homeomorphism. The dynamical system (X, φ) is called a Smale space if there are two constants $\epsilon_X > 0$ and $0 < \lambda_X < 1$ and a map, called the bracket map,

$$[\cdot, \cdot] : X \times X \rightarrow X$$

which is defined for $x, y \in X$ such that $d(x, y) < \epsilon_X$. The bracket map is required to satisfy the following axioms:

- B1. $[x, x] = x$,
- B2. $[x, [y, z]] = [x, z]$,
- B3. $[[x, y], z] = [x, z]$,
- B4. $\varphi[x, y] = [\varphi(x), \varphi(y)]$;

for x, y , and z whenever both sides in the above equations are well-defined. Furthermore, the system also satisfies

- C1. For $x, y \in X$ such that $[x, y] = y$, we have $d(\varphi(x), \varphi(y)) \leq \lambda_X d(x, y)$ and
 C2. for $x, y \in X$ such that $[x, y] = x$, we have $d(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \lambda_X d(x, y)$.

We note that if the bracket exists, it is unique (see for example [28, page 181]).

Definition 1.4. Suppose (X, φ) is a Smale space, $x \in X$, and $0 < \epsilon \leq \epsilon_X$ and $Y, Z \subset X$ a subset of points. Then we define the following sets:

- (i) $X^s(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon, [y, x] = x\}$,
- (ii) $X^u(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon, [x, y] = x\}$,
- (iii) $X^s(x) := \{y \in X \mid \lim_{n \rightarrow +\infty} d(\varphi^n(x), \varphi^n(y)) = 0\}$,
- (iv) $X^u(x) := \{y \in X \mid \lim_{n \rightarrow -\infty} d(\varphi^n(x), \varphi^n(y)) = 0\}$,
- (v) $X^s(Z) := \bigcup_{x \in Z} X^s(x)$ and $X^u(Z) := \bigcup_{x \in Z} X^u(x)$, and
- (vi) $X^h(Y, Z) := X^s(Y) \cap X^u(Z)$.

We say x and y are stably equivalent and write $x \sim_s y$ if $y \in X^s(x)$. Similarly, we say x and y are unstably equivalent, written $x \sim_u y$, if $y \in X^u(x)$. Points $x, y \in X$ are homoclinic if $y \in X^s(x) \cap X^u(x)$, meaning $\lim_{|n| \rightarrow \infty} d(\varphi^n(x), \varphi^n(y)) = 0$.

The basic properties of (X, φ) will not be discussed in detail here. However, we will need the following result, which is a special case of the main result of [23]:

Proposition 1.5. *If (X, φ) is a Smale space, then X has finite covering dimension.*

The reader can find more on the basic properties of Smale spaces in various places, for example [28, 29, 34].

1.2. Topological groupoids. Let G be a groupoid. We denote its unit space by $G^{(0)}$ and its range and source maps $r, s : G \rightarrow G^{(0)}$. The ordered pair $g, h \in G$ is composable if $s(g) = r(h)$ and their composition is denoted gh . The inverse of $g \in G$ is denoted g^{-1} . In this paper, all groupoids are assumed to be locally compact and Hausdorff with locally compact unit space. Moreover, we will require that they are étale, meaning that r and s are local homeomorphisms. In this case $G^{(0)}$ is an open subset of G and the Haar system is given by counting measures.

To such a groupoid G , we associate its reduced groupoid C^* -algebra as follows. Let $C_c(G)$ denote the (vector space of) compactly supported continuous functions on G . For $f_1, f_2, f \in C_c(G)$ we define multiplication and involution by

$$(f_1 \cdot f_2)(g) = \sum_{h_1 h_2 = g} f_1(h_1) f_2(h_2) \text{ for all } g \in G$$

and

$$f^*(g) = \overline{f(g^{-1})}.$$

With these operations $C_c(G)$ is a $*$ -algebra. For every $x \in G^{(0)}$, let $\ell^2(s^{-1}(x))$ denote the Hilbert space of square-summable functions on $s^{-1}(x)$. From this we can define a $*$ -representation

$$\pi_x : C_c(G) \rightarrow \mathcal{B}(\ell^2(s^{-1}(x)))$$

by

$$(\pi_x(f)\xi)(g) = \sum_{h_1 h_2 = g} f(h_1) \xi(h_2),$$

for $f \in C_c(G)$, $\xi \in \ell^2(s^{-1}(x))$, $g \in s^{-1}(x)$. The reduced groupoid C*-algebra, denoted $C_r^*(G)$ is the completion of $C_c(G)$ with respect to the norm

$$\|f\| = \sup_{x \in G^{(0)}} \|\pi_x(f)\|.$$

One may also define a full groupoid C*-algebra $C^*(G)$. In the case that the groupoid is amenable, as is the case for the groupoids associated to a Smale space, the two coincide.

1.3. Groupoids associated to Smale spaces. Following [33] (also see [28]), the homoclinic groupoid of a Smale space is defined directly using the homoclinic equivalence relation. We denote this groupoid by G_H ; it is étale.

One could also consider the groupoids given by the stable and unstable equivalence relations, but they are not étale. However, following the construction in [30], one can (by restricting in each case to an abstract transversal) obtain groupoids which are equivalent to the groupoids of stable and unstable equivalence (in the sense of [25]) but which are in fact étale.

Let (X, φ) be an irreducible Smale space. We define the following groupoids associated to its stable and unstable equivalence relations [30] (see also [11, 15, 47]).

Definition 1.6. Let P and Q be finite φ -invariant sets of periodic points of (X, φ) . Define

$$G_S(P) := \{(x, y) \in X \times X \mid x \sim_s y \text{ and } x, y \in X^u(P)\}$$

and

$$G_U(Q) := \{(x, y) \in X \times X \mid x \sim_u y \text{ and } x, y \in X^s(Q)\}.$$

1.7. To define a topology on $G_S(P)$, we follow [30] (also see [14, Section 2.2.4]).

Let $(x, y) \in G_S(P)$. Then, $x \sim_s y$ and hence there exists $N > 0$ such that $d(\varphi^N(v), \varphi^N(w)) < \epsilon_X/2$. Now by continuity there is $0 < \delta < \epsilon_X/2$ such that, for $0 \leq n \leq N$ we have

$$\varphi^n(X^u(w, \delta)) \subseteq X^u(\varphi^n(w), \epsilon_X/2)$$

and

$$\varphi^n(X^u(v, \delta)) \subset X^u(\varphi^n(v), \epsilon_X/2).$$

From this we define a map $h : X^u(w, \delta) \rightarrow X^u(v, \delta)$ via

$$z \mapsto \varphi^{-N}([\varphi^N(z), \varphi^N(v)]).$$

By [30], $h : X^u(w, \delta) \rightarrow X^u(v, \delta)$ is a local homeomorphism, mapping $X^u(w, \delta)$ homeomorphically to a neighbourhood of v .

For such a 5-tuple v, w, δ, h, N as above, we define an open set

$$V(v, w, \delta, h, N) := \{(h(z), z) \mid z \in X^u(w, \delta)\} \subset G_S(P).$$

Such sets are the basic sets generating a topology for $G_S(P)$.

Theorem 1.8 (See for example [14, Theorem 2.17]). *We have the following properties of $G_S(P)$ and the basic sets introduced in the previous paragraph:*

- (1) *The map h is a local homeomorphism;*
- (2) *$V(v, w, \delta, h, N)$ gives a neighbourhood base for a topology on $G_S(P)$;*
- (3) *$G_S(P)$ is an étale groupoid when we use this topology;*
- (4) *the unit space of $G_S(P)$ is $X^u(P)$; it is locally compact, but not compact.*

The topology on $G_U(Q)$ is completely analogous and hence the details are omitted.

It is clear that $\varphi \times \varphi : X \times X \rightarrow X \times X$ defines an automorphism of the groupoid G_H . Moreover, since we have chosen φ -invariant sets P and Q , it also defines an automorphism on each of $G_S(P)$ and $G_S(Q)$.

1.9. It is shown in [30, Theorem 1.1] that each of these groupoids is amenable. In this case, the completion of any faithful $*$ -representation of the compactly supported functions on the groupoid will be $*$ -isomorphic to the reduced and full groupoid C^* -algebras. In particular, if P and Q are two sets of periodic points, we may represent both $C_c(G_S(Q))$ and $C_c(G_U(P))$ on the Hilbert space

$$\mathcal{H} = \ell^2(X^h(P, Q))$$

where $X^h(P, Q)$ is the countable set of points in X which are both stably equivalent to a point in P and unstably equivalent to a point in Q . For further details on this construction see [11, Sections 3 and 6] or [15, Section 1.3]. The choice of the set P (or Q) only affects the C^* -algebra up to stable isomorphism.

Definition. Let P and Q be finite φ -invariant sets of periodic points of (X, φ) . The stable C^* -algebra, $C^*(G_S(P))$, is defined to be the groupoid C^* -algebra associated to $G_S(P)$ and the unstable algebra, $C^*(G_U(Q))$, is defined to be the groupoid C^* -algebra associated to $G_U(Q)$. The homoclinic C^* -algebra of (X, φ) is the C^* -algebra of the groupoid G_H ; it is denoted by $C^*(G_H)$.

2. NUCLEAR DIMENSION AND DYNAMIC ASYMPTOTIC DIMENSION

We review the definitions of nuclear dimension and dynamic asymptotic dimension [9] as well as their connection.

2.1. Let A and B be C^* -algebras. A completely positive (c.p.) map $\phi : A \rightarrow B$ is said to be order zero if it preserves orthogonality, that is, if $a, b \in A$ satisfy $ab = 0$, then $\phi(a)\phi(b) = 0$.

Definition ([51, Definition 2.1]). Let A be a C^* -algebra. We say that A has nuclear dimension at most n , written $\dim_{\text{nuc}}(A) \leq n$, if there exists a net $(F_\lambda, \psi_\lambda, \phi_\lambda)_{\lambda \in \Lambda}$ where F_λ are finite-dimensional C^* -algebras, $\psi_\lambda : A \rightarrow F_\lambda$ and $\phi_\lambda : F_\lambda \rightarrow A$ are c.p. maps satisfying the following:

- (i) ψ_λ is contractive,
- (ii) for each $\lambda \in \Lambda$, F_λ decomposes into $n+1$ ideals $F_\lambda = F_\lambda^{(0)} \oplus \cdots \oplus F_\lambda^{(n)}$ such that $\phi_\lambda|_{F_\lambda^{(i)}}$ is c.p.c. order zero for $i \in \{0, \dots, n\}$,
- (iii) $\phi_\lambda \circ \psi_\lambda(a) \rightarrow a$ uniformly on finite subsets of A .

If no such n exists, then A is said to have infinite nuclear dimension, $\dim_{\text{nuc}}(A) = \infty$.

The nuclear dimension should be thought of as a noncommutative analogue of topological covering dimension. Indeed, in the case of a commutative C^* -algebra $C_0(X)$ where X is a locally compact Hausdorff space, the nuclear dimension is equal to the covering dimension of X [51, Proposition 2.4]. In the noncommutative case of course we no longer have such a correspondence. Nevertheless, as C^* -algebras are often obtained by underlying structures such as dynamical systems or groupoids where one can ask if a notion of dimension, again in analogy to covering dimension,

can be defined and related to the nuclear dimension of the associated C*-algebra. Here we deal with groupoid C*-algebras and the appropriate notion of dimension is the dynamic asymptotic dimension defined in [9].

Let G be an étale groupoid with locally compact unit space $G^{(0)}$. For any set $K \subset G$, we set

$$s(K) := \{s(g) \mid g \in K\} \subset G^{(0)},$$

and similarly

$$r(K) := \{r(g) \mid g \in K\} \subset G^{(0)}.$$

Definition 2.1 ([9]). Let G be an étale groupoid with locally compact unit space $G^{(0)}$. We say that G has dynamic asymptotic dimension less than or equal to d if, for every open relatively compact $K \subset G$, there exist open sets $U_0, \dots, U_d \subset G^{(0)}$ satisfying the following:

- (i) $\{U_0, \dots, U_d\}$ covers $s(K) \cup r(K)$ and
- (ii) for every $i = 0, \dots, d$, the groupoid generated by $\{g \in K \mid s(g), r(g) \in U_i\}$ is a relatively compact subgroupoid of G .

The groupoids associated to a Smale space are second countable and are also principal (or free, in the language of [9]). Recall that a groupoid G is principal if for every $x \in G^{(0)}$ the isotropy group

$$G_x^x := \{g \in G \mid s(g) = r(g) = x\}$$

is trivial. The groupoid associated to an equivalence relation is always principal [31, Example 1.2 c].

The following is a special case of [9, Theorem 8.6].

Theorem 2.2. *Let G be a second countable principal étale groupoid with locally compact unit space $G^{(0)}$. Furthermore assume that G has dynamic asymptotic dimension at most d . Then*

$$\dim_{\text{nuc}}(C_r^*(G)) \leq (d+1)(\dim G^{(0)} + 1) - 1,$$

where $\dim G^{(0)}$ is the topological covering dimension of $G^{(0)}$.

3. DYNAMIC ASYMPTOTIC DIMENSION OF SMALE SPACES AND NUCLEAR DIMENSION OF THEIR C*-ALGEBRAS

Throughout this section we let (X, φ) be an irreducible Smale space where the space X has $\dim X = d < \infty$ and let P be a finite φ -invariant set of periodic points. The stable groupoid is denoted $G_S(P)$, as given in Definition 1.6.

Lemma 3.1. *Let $K \subset G_S(P)$ be an open relatively compact subset and let $U \subset G_S(P)^{(0)} = X^u(P)$ be an open subset. Suppose that there is $N \in \mathbb{N}$ such that the subset $\{(x, y) \in (\varphi^N \times \varphi^N)(K) \mid x, y \in U\}$ generates a relatively compact subgroupoid of G . Then so does $\{(x, y) \in K \mid x, y \in \varphi^{-N}(U)\}$.*

Proof. Since $\varphi^N \times \varphi^N$ is a homeomorphism of the groupoid, by mapping generators to generators, we see that the closures of the groupoids generated are in fact homeomorphic as subspaces of $G_S(P)$. \square

Lemma 3.2. *Let $K \subset G_S(P)$ be an open relatively compact subset. Then for any $0 < \gamma < \frac{\varepsilon_X}{2}$ there exists $N_K \in \mathbb{N}$ such that, for each $n \geq N_K$, we have $x \in X^s(y, \gamma)$ whenever $(x, y) \in (\varphi^n \times \varphi^n)(K)$. In particular, if $n \geq N_K$, then $d(x, y) < \gamma$ for every $(x, y) \in (\varphi^n \times \varphi^n)(K)$.*

Proof. Since \overline{K} is compact there exists a finite cover of \overline{K} and hence K by basic sets. Let $V(v_i, w_i, h_i, \delta_i, N_i)$, $i = 0, \dots, m$ be such a finite open cover. By taking the maximum over $i = 0, \dots, m$, we need only prove the result for one of the basic sets.

So let $V(v, w, h, \delta, N)$ be a basic set and $(x, y) \in V(v, w, h, \delta, N)$. By the definition of h , we have that $\varphi^N(x) \in X^s(\varphi^N(y), \epsilon_X)$. Using the fact that if $0 < \epsilon \leq \epsilon_X$ and $z \in X^s(t, \epsilon)$, then $\varphi(z) \in X^s(\varphi(t), \lambda_X \epsilon)$ (this fact follows from B4 and C1 in Definition 1.3), we have that, for any $n \geq N$,

$$\varphi^n(x) \in X^s(\varphi^n(y), \lambda_X^{n-N} \epsilon_X).$$

Since $0 < \lambda_X < 1$, the result follows by taking n large enough so that $\lambda_X^{n-N} \epsilon_X < \gamma$. We emphasise that n is independent of the choice of $(x, y) \in V(v, w, h, \delta, N)$; it only depends on N and the global constants of the Smale space. \square

Lemma 3.3. *Let $0 < \gamma' \leq \gamma < \frac{\epsilon_X}{2}$ and let $K \subset G_S(P)$ be an open relatively compact subset. For any cover of K by basic sets $\{V(v_i, w_i, h_i, \delta_i, N_i)\}_{i=0}^m$, $m > 0$ with $\delta_i < \frac{\epsilon_X}{4}$, there is $N \in \mathbb{N}$ and a cover \mathcal{U}' of K by basic sets $\{V(v'_i, w'_i, h'_i, \delta'_i, N)\}_{i=0}^n$, $n \geq m$, satisfying*

$$\varphi^N(X^u(v'_i, \delta'_i)) \subset X^u(\varphi^N(v'_i), \gamma) \quad \text{and} \quad \varphi^N(X^u(w'_i, \delta'_i)) \subseteq X^u(\varphi^N(w'_i), \gamma)$$

for each $0 \leq i \leq n$ and such that

$$d(x, y) < \gamma' \text{ for all } (x, y) \in (\varphi^N \times \varphi^N)(K).$$

Proof. By the previous lemma, for the given γ' , we may find $N_K \in \mathbb{N}$ sufficiently large so that $d(x, y) < \gamma'$ whenever $(x, y) \in (\varphi^{N_K} \times \varphi^{N_K})(K)$. Let

$$N := \max\{N_0, \dots, N_m, N_K\}.$$

Fix i , $0 \leq i \leq n$. For each $x \in X^u(w_i, \delta_i)$, arguing as in Paragraph 1.7, there exists $0 < \delta_x \leq \delta_i$ such that

$$\varphi^N(X^u(x, \delta_x)) \subseteq X^u(\varphi^N(x), \gamma)$$

and

$$\varphi^N(X^u(h_i(x), \delta_x)) \subseteq X^u(\varphi^N(h_i(x)), \gamma).$$

Shrinking δ_x if necessary, we may further assume that

$$X^u(h_i(x), \delta_x) \subset X^u(v_i, \delta_i).$$

Let h_x denote the restriction $h_x := h_i|_{X^u(x, \delta_x)}$. Then $V(h_i(x), x, h_x, \delta_x, N)$ is a basic set.

Now let $(h_x(z), z) \in V(h_x(x), x, \delta_x, h_x N)$. Then

$$\begin{aligned} h_x(z) &= \varphi^{-N}([\varphi^N(z), \varphi^N(h_i(x))]) \\ &= \varphi^{-N_i-(N-N_i)}([\varphi^{-N_i-(N-N_i)}(z), \varphi^{-N_i-(N-N_i)}(h_i(x))]) \\ &= \varphi^{-N_i-(N-N_i)}(\varphi^{N-N_i}[\varphi^{N_i}(z), \varphi^{N_i}(h_i(x))]) \\ &= \varphi^{-N_i}([\varphi^{N_i}(z), \varphi^{N_i}(h_i(x))]), \end{aligned}$$

by the fact that the bracket in the above computations is everywhere defined, allowing us to apply B4 of Definition 1.3.

Furthermore,

$$\begin{aligned} [\varphi^{N_i}(z), \varphi^{N_i}(h_i(x))] &= [\varphi^{N_i}(z), [\varphi^{N_i}(h_i(x)), \varphi^{N_i}(v)]] \\ &= [\varphi^{N_i}(z), \varphi^{N_i}(v)], \end{aligned}$$

so $h_x(z) = \varphi^{-N_i}([\varphi^{N_i}(z), \varphi^{N_i}(v)]) = h_i(z)$ and we have that

$$V(h_i(x), x, h_x, \delta_x, N) \subseteq V(w_i, v_i, \delta_i, h_i, N_i).$$

Since we may do this for each i and x , the argument is completed using the compactness of \bar{K} . \square

For $0 < \epsilon < \frac{\epsilon_X}{2}$, we let $U(w, \epsilon)$ denote the image of $X^u(w, \epsilon) \times X^s(w, \epsilon)$ under the bracket map.

Lemma 3.4. *Let $0 < \gamma' \leq \gamma < \frac{\epsilon_X}{4}$ and let $K \subset G_S(P)$ be an open relatively compact subset. Then there is $N \in \mathbb{N}$ such that for every $w \in X$, the set*

$$\{(x, y) \in (\varphi^N \times \varphi^N)(K) \mid x, y \in U(w, \gamma)\}$$

generates an open relatively compact groupoid and such that

$$d(x, y) < \gamma' \text{ for every } (x, y) \in (\varphi^N \times \varphi^N)(K).$$

Proof. Since w and γ are fixed throughout the proof, we set

$$U := U(w, \gamma).$$

By Lemma 3.3, there is $N \in \mathbb{N}$ and a finite subcover of K by $\{V(v_i, w_i, h_i, \delta_i, N)\}_{i=0}^m$ such that

$$\varphi^N(X^u(v_i, \delta_i)) \subseteq X^u((\varphi^N(v_i), \gamma)) \quad \text{and} \quad \varphi^N(X^u(w_i, \delta_i)) \subseteq X^u((\varphi^N(w_i), \gamma))$$

for each $0 \leq i \leq m$ and such that $d(x, y) < \gamma'$ for every $(x, y) \in \varphi^N \times \varphi^N(K)$. It also follows from Lemma 3.3 that for each $0 \leq i \leq m$ there is some $\delta'_i < \frac{\epsilon_X}{4}$ such that

$$(\varphi^N \times \varphi^N)(V(v_i, w_i, h_i, \delta_i, N)) \subseteq V(\varphi^N(v_i), \varphi^N(w_i), [\cdot, \varphi^N(v_i)], \delta'_i, 0),$$

In particular, $\{V(\varphi^N(v_i), \varphi^N(w_i), [\cdot, \varphi^N(v_i)], \delta'_i, 0)\}_{i=0}^m$ is a cover of $(\varphi^N \times \varphi^N)(K)$.

We need only consider i such that

$$(3.1) \quad X^u(\varphi^N(w_i), \delta'_i) \cap U \text{ and } X^u(\varphi^N(v_i), \delta'_i) \cap U$$

are both nonempty. Suppose that there are $n \leq m$ such sets. By reordering, let $\{V(\varphi^N(v_i), \varphi^N(w_i), [\cdot, \varphi^N(v_i)], \delta'_i, 0)\}_{i=0}^n$ be the sets in the cover of $(\varphi^N \times \varphi^N)(K)$ with this property.

By (3.1), the triangle inequality and the choice of δ_i and γ , we may define the following points $r_0, \dots, r_{2n+1} \in X$ using the bracket:

$$r_l = \begin{cases} [w, \varphi^N(v_i)] & \text{for } 0 \leq l \leq n, \\ [w, \varphi^N(w_i)] & \text{for } n+1 \leq l \leq 2n+1. \end{cases}$$

Let

$$H = \bigcup_{1 \leq l, m \leq 2n} V(r_l, r_m, [\cdot, r_l], \gamma).$$

We first claim that H is an open relatively compact subgroupoid $G_S(P)$. Openness and relative compactness follow because the set is the union of finitely many open relatively compact subsets of $G_S(P)$. Thus we need only show that H is closed under inverses and, where defined, products. So let $(x, y) \in H$. Then $(x, y) \in V(r_l, r_m, [\cdot, r_l], \gamma)$ for some $0 \leq l, m \leq 2n+1$, so we have that

$$x \in X^u(r_m, \gamma) \text{ and } y \in X^u(r_l, \gamma)$$

and that

$$[y, r_m] = x.$$

But also

$$[x, r_l] = y,$$

so we have that

$$(y, x) = ([x, r_l], x) \in V(r_m, r_l, [\cdot, r_m], \gamma) \subset H,$$

whence we see that H is closed under inverses. Now suppose that $(x, y), (y, z) \in H$ and that they are composable as elements of $G_S(P)$. Then, as above, for some k, l, m , we have that

$$x \in X^u(r_k, \gamma), \quad y \in X^u(r_l, \gamma), \quad z \in X^u(r_m, \gamma),$$

and moreover that

$$[y, r_k] = x, \text{ and } [z, r_l] = y.$$

Since γ is sufficiently small, we may compose the bracket to get that

$$x = [[z, r_l], r_k] = [z, r_k],$$

thus

$$(x, z) = ([z, r_k], z) \in V_{r_m, r_k, [\cdot, r_k], \gamma} \subset H.$$

Next we claim the groupoid generated by $\{(x, y) \in (\varphi^N \times \varphi^N)(K) \mid x, y \in U\}$ is contained in H . To prove this claim, it is enough to show

$$((\varphi^N \times \varphi^N)(K)) \cap (U \times U) \subset H.$$

Let $(x, y) \in ((\varphi^N \times \varphi^N)(K)) \cap (U \times U)$. Then

$$(x, y) \in V(\varphi^N(v_i), \varphi^N(w_i), [\cdot, \varphi^N(v_i)], \delta'_i)$$

for some $0 \leq i \leq n'$. Thus $x = [y, \varphi^N(v_i)]$. Since $y \in X^u(w, \gamma)$, we have that $[y, w]$ is well-defined and so too is the composition $[y, [w, \varphi^N(v_i)]]$. Thus

$$[y, r_i] = [y, [w, \varphi^N(v_i)]] = [y, \varphi^N(v_i)] = x,$$

hence

$$(x, y) = ([y, r_i], y) \in H.$$

It now follows from the two claims that $\{(x, y) \in (\varphi^N \times \varphi^N)(K) \mid x, y \in U\}$ generates a relatively compact subgroupoid of $G_S(P)$. \square

Lemma 3.5. *Let (X, φ) be a Smale space (where X is an infinite compact metric space with $\dim X = d < \infty$) and $0 < \gamma < \frac{\varepsilon_X}{4}$. Then, there exist finite open covers of X , $\{U_j\}_{j=1}^M$ and $\{V_j\}_{j=1}^M$, with the following properties:*

- (i) *for each j , there exists $w \in X$ such that $U_j \subseteq U(w, \gamma)$;*
- (ii) *the cover $\{U_j\}_{j=1}^M$ has order d ;*
- (iii) *for each j , $V_j \subseteq U_j$;*
- (iv) *for each j_1, j_2 , $U_{j_1} \cap U_{j_2} = \emptyset$ implies $\overline{V}_{j_1} \cap \overline{V}_{j_2} = \emptyset$;*
- (v) *there exists $\gamma' > 0$, such that for any j_1, j_2 , $x \in V_{j_1}$, $y \in V_{j_2}$ with $U_{j_1} \cap U_{j_2} = \emptyset$, we have that $d(x, y) > \gamma'$. Furthermore, we can assume $\gamma' < \gamma$.*

Proof. By compactness, there is a finite open cover of X by sets of the form $U(w, \gamma)$, $w \in X$. The definition of covering dimension applied to X implies that there is a finite refinement of that cover to a cover of order d . We denote the refined cover by $\{U_i\}_{i=1}^M$. We construct the second cover inductively on $i = 1, \dots, M$.

For $i = 1$, we take $V_1 = U_1$.

Next suppose $\{V_1, \dots, V_i\}$ is a set of open subsets of X such that

- (i) $\{V_1, \dots, V_i\} \cup \{U_j\}_{j=i+1}^M$ is an open cover of X ;

(ii) $V_s \subseteq U_s$ for each $1 \leq s \leq i$;

(iii) for each j_1 and j_2 in $\{1, \dots, i\}$, whenever $U_{j_1} \cap U_{j_2} = \emptyset$, $\overline{V}_{j_1} \cap \overline{V}_{j_2} = \emptyset$.

Let $J_{i+1} \subseteq \{1, \dots, M\}$ be the set of indices k such that $U_{i+1} \cap U_k = \emptyset$, but $\overline{U}_{i+1} \cap \overline{U}_k \neq \emptyset$. Then $F_{i+1} = \bigcup_{k \in J_{i+1}} \overline{U}_{i+1} \cap \overline{U}_k$ is closed and hence compact. Furthermore, since $\{V_1, \dots, V_i\} \cup \{U_j\}_{j=i+1}^M$ covers X and $F_{i+1} \cap U_{i+1} = \emptyset$, we have that

$$F_{i+1} \subseteq \left(\bigcup_{j=1}^i V_j \right) \cup \left(\bigcup_{j=i+2}^M U_j \right).$$

Take an open set G_{i+1} such that

$$F_{i+1} \subseteq G_{i+1} \subseteq \overline{G}_{i+1} \subseteq \left(\bigcup_{j=1}^i V_j \right) \cup \left(\bigcup_{j=i+2}^M U_j \right).$$

Let $V_{i+1} = U_{i+1} \cap \overline{G}_{i+1}^c$, which is open and contained in U_{i+1} . One then checks that

(i) $\{V_1, \dots, V_{i+1}\} \cup \{U_j\}_{j=i+2}^M$ is an open cover of X ;

(ii) $V_s \subseteq U_s$ for each $1 \leq s \leq i+1$;

(iii) for each j_1 and j_2 in $\{1, \dots, i+1\}$, whenever $U_{j_1} \cap U_{j_2} = \emptyset$, $\overline{V}_{j_1} \cap \overline{V}_{j_2} = \emptyset$.

Thus, using induction, we have covers $\{U_i\}_{i=1}^M$ and $\{V_j\}_{j=1}^M$ that satisfy items (i) to (iv) in the statement of the lemma.

We show that we also have item (v). For $j, j' \in \{1, \dots, M\}$ with $U_j \cap U_{j'} = \emptyset$, let $\gamma_{j,j'}$ be a positive constant such that $d(x, y) > \gamma_{j,j'}$ whenever $x \in V_j$ and $y \in V_{j'}$. We note that $\gamma_{j,j'}$ can be taken to be positive because $\overline{V}_j \cap \overline{V}_{j'} = \emptyset$. Let

$$\gamma' = \min\{\gamma_{j,j'} \mid j, j' \text{ such that } U_j \cap U_{j'} = \emptyset\}.$$

By construction, γ' (and any positive constant less than or equal to it) has the required property. \square

Lemma 3.6. Suppose $0 < \gamma < \frac{\epsilon_X}{4}$ and $\{U_j\}_{j=1}^M$, $\{V_j\}_{j=1}^M$, and γ' are as in the statement of Lemma 3.5. Furthermore, suppose K is an open relatively compact subset of $G_S(P)$ and $N \in \mathbb{N}$ is as in the statement of Lemma 3.4 (it depends on $0 < \gamma' < \gamma < \frac{\epsilon_X}{4}$ along with K). Then, for each j , the groupoid generated by

$$\{(x, y) \in (\varphi^N \times \varphi^N)(K) \mid x, y \in V_j\}$$

is relatively compact.

Proof. By assumption, for each j , there exists $w \in X$ such that $V_j \subseteq U_j \subseteq U(w, \gamma)$. The required result then follows from Lemma 3.4. \square

Theorem 3.7. Let (X, φ) be an irreducible Smale space and let P be a finite φ -invariant set of points. Then both the stable groupoid $G_S(P)$ and the unstable groupoid $G_U(P)$ have dynamic asymptotic dimension at most $\dim X$.

Proof. By Proposition 1.5, X has finite covering dimension; set $d = \dim X$. Fix $0 < \gamma < \frac{\epsilon_X}{4}$ and take $\{U_j\}_{j=1}^M$, $\{V_j\}_{j=1}^M$, and γ' as in Lemma 3.5. Furthermore, let $K \subseteq G_S(P)$ be an open relatively compact subset.

Take $N \in \mathbb{N}$ as in the statement of Lemma 3.4 and let $\tilde{K} = (\varphi^N \times \varphi^N)(K)$. Note that, in particular, $d(x, y) < \gamma'$ whenever $x, y \in \tilde{K}$. It follows from Lemma 3.1 that

it is enough to show that there are open sets $\{\tilde{V}_0, \dots, \tilde{V}_d\}$ which form an open cover of $s(\tilde{K}) \cup r(\tilde{K})$ and such that, for each $0 \leq i \leq d$, the set

$$\{(x, y) \in \tilde{K} \mid x, y \in \tilde{V}_i\}$$

generates a relatively compact subgroupoid.

Let V be an open relatively compact subset of $X^u(P)$ that contains $s(\tilde{K}) \cup r(\tilde{K})$. Then, for each $1 \leq j \leq M$, $U_j \cap V$ and $V_j \cap V$ are open subsets of $X^u(P)$. Moreover, both $\{U_j \cap V\}_{j=1}^M$ and $\{V_j \cap V\}_{j=1}^M$ are covers of $s(\tilde{K}) \cup r(\tilde{K})$.

Since $\{U_j\}_{j=1}^M$ has order d , we may decompose $\{1, 2, \dots, M\}$ into subsets J_0, \dots, J_d such that, for every $i = 0, \dots, d$, the sets $U_j, j \in J_i$ are pairwise disjoint.

For every $i = 0, \dots, d$, let

$$\tilde{V}_i = \bigcup_{j \in J_i} V_j \cap V.$$

Then $\{\tilde{V}_0, \dots, \tilde{V}_d\}$ is an open cover of $s(\tilde{K}) \cup r(\tilde{K})$.

Using item (v) in the statement of Lemma 3.5 and the fact that the $U_j, j \in J_i$ are pairwise disjoint, we have that for $j \neq j' \in J_i$, $d(x, y) > \gamma'$ whenever $x \in V_j$ and $y \in V_{j'}$. It follows that, for each $i = 0, \dots, d$, the groupoid generated by

$$\{(x, y) \in \tilde{K} \mid x, y \in \tilde{V}_i\}$$

is simply the disjoint union (over $j \in J_i$) of the groupoids generated by

$$\{(x, y) \in \tilde{K} \mid x, y \in V_j\}.$$

Thus, using Lemma 3.6, we have that for each $i = 0, \dots, d$, the set $\{(x, y) \in (\varphi^N \times \varphi^N)(K) \mid x, y \in \tilde{V}_i\}$ generates a relatively compact groupoid. In other words, $\tilde{V}_0, \dots, \tilde{V}_d$ is the required cover of $s(\tilde{K}) \cup r(\tilde{K})$. This completes the proof for $G_S(P)$. To show that $G_U(P)$ has dynamic asymptotic dimension at most d , we simply observe that $G_U(P)$ is the stable groupoid of the irreducible Smale space (X, φ^{-1}) and the result follows. \square

Corollary 3.8. *Let (X, φ) be an irreducible Smale space and let P be a finite set of φ -invariant periodic points. Then the stable, unstable and homoclinic C^* -algebras each have finite nuclear dimension.*

Proof. Combining Theorem 3.7 with Theorem 2.2, we have that $\dim_{\text{nuc}}(C^*(G_S(P)))$ and $C^*(G_U(P))$ are bounded by $(\dim(X) - 1)(\dim(X^u(P)) - 1) + 1$ and $(\dim(X) - 1)(\dim(X^s(P)) - 1) + 1$ respectively. The result for $C^*(G_H)$ follows from the fact that it is Morita equivalent to $C^*(G_S(P)) \otimes C^*(G_U(P))$ [28, Theorem 3.1]. \square

4. STRUCTURE AND CLASSIFICATION

To begin, we will prove the following result:

Theorem 4.1. *Let (X, φ) be a mixing Smale space. Then the homoclinic C^* -algebra is quasidiagonal.*

To prove the theorem, we require the following lemmas. The first lemma is likely well-known and is implicit in [2], but the proof is short so we include it.

Lemma 4.2. *Suppose A is a separable, nuclear C^* -algebra and that*

$$A \subset \frac{\prod_{n \in \mathbb{N}} \mathcal{K}(\mathcal{H})}{\bigoplus_{n \in \mathbb{N}} \mathcal{K}(\mathcal{H})}.$$

Then A is quasidiagonal.

Proof. The compact operators are MF (see [2, Definition 3.2.1]), so by [2, Corollary 3.4.3] A is also MF. Since A is nuclear, this is equivalent to A being quasidiagonal by [2, Theorem 5.2.2]. \square

In the proof of the next lemma, we use basic properties of the representations of $C^*(G_S(P))$ and $C^*(G_U(Q))$ on the Hilbert space $\mathcal{H} = \ell^2(X^h(P, Q))$ (see Paragraph 1.9). These results can be found in [11, Section 6], [15, Theorem 1.3] and [47, Section 4]. The last of these references contains complete proofs of each of the results we require. Note that we do not make use of [47, Lemma 4.4.9] so we do not need to assume that the sets of periodic points used to define the stable and unstable algebras are disjoint.

Lemma 4.3. *Let P and Q be finite sets of φ -invariant periodic points. Suppose that $C^*(G_S(P))$ and $C^*(G_U(Q))$ are the stable and unstable C^* -algebras associated to a mixing Smale space. Then there exists an injective $*$ -homomorphism*

$$\Phi : C^*(G_S(P)) \otimes C^*(G_U(Q)) \rightarrow \frac{\prod_{n \in \mathbb{N}} \mathcal{K}(\mathcal{H})}{\bigoplus_{n \in \mathbb{N}} \mathcal{K}(\mathcal{H})}.$$

Proof. Let $\alpha_s : C^*(G_S(P)) \rightarrow C^*(G_S(P))$ be the $*$ -automorphism induced by φ . By [11, Lemma 6.1], for any $a \in C^*(G_S(P))$ and $b \in C^*(G_U(Q))$, $ab \in \mathcal{K}(\mathcal{H})$. Thus, the map on elementary tensors defined by $a \otimes b \mapsto (\alpha_s^n(a)b)_{n \in \mathbb{N}}$ extends to a linear map

$$\Phi : C^*(G_S(P)) \otimes C^*(G_U(Q)) \rightarrow \frac{\prod_{n \in \mathbb{N}} \mathcal{K}(\mathcal{H})}{\bigoplus_{n \in \mathbb{N}} \mathcal{K}(\mathcal{H})}.$$

We show that Φ is a $*$ -homomorphism. To do so, we must show that, for each $a, a' \in C^*(G_S(P))$ and $b, b' \in C^*(G_U(Q))$,

$$(4.1) \quad (\alpha_s^n(a)b\alpha_s^n(a')b' - \alpha_s^n(a)\alpha_s^n(a')bb')_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} \mathcal{K}(\mathcal{H}) \text{ and}$$

$$(4.2) \quad (\alpha_s^n(a^*)b^* - (\alpha_s^n(a)b)^*)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} \mathcal{K}(\mathcal{H}).$$

We prove only the former as the proofs are rather similar. Let $\epsilon > 0$. We must show that there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\|\alpha_s^n(a)b\alpha_s^n(a')b' - \alpha_s^n(a)\alpha_s^n(a')bb'\| < \epsilon.$$

By [11, Lemma 6.3], there exists $N \in \mathbb{N}$ such that, for any $n \geq N$,

$$\|b\alpha_s^n(a') - \alpha_s^n(a')b\| < \frac{\epsilon}{\|a\|\|b'\|}.$$

Hence

$$\|\alpha_s^n(a)b\alpha_s^n(a')b' - \alpha_s^n(a)\alpha_s^n(a')bb'\| \leq \|a\|\|b\alpha_s^n(a') - \alpha_s^n(a')b\|\|b'\| < \epsilon.$$

Finally, we must show that Φ is injective. Recall that $C^*(G_S(P))$ and $C^*(G_U(Q))$ are both simple so $C^*(G_S(P)) \otimes C^*(G_U(Q))$ is also simple. Thus, we need only show that Φ is nonzero on a single element in $C^*(G_S(P)) \otimes C^*(G_U(Q))$. The existence of such an element follows from the proof of Lemma 4.4.13 in [47]. In that proof, elements $a \in C^*(G_S(P))$ and $b \in C^*(G_U(Q))$ are constructed with the property that $\|\alpha_s^{n_k}(a)b\| \geq 1$ for a strictly increasing sequence n_k . \square

We can now prove the main result of the section.

Proof of Theorem 4.1. The previous two lemmas imply that the tensor product $C^*(G_S(P)) \otimes C^*(G_U(Q))$ is quasidiagonal. Since $C^*(G_H)$ is stably isomorphic to $C^*(G_S(P)) \otimes C^*(G_U(Q))$ it is also quasidiagonal [10, Theorem 20]. \square

Corollary 4.4. *Let (X, φ) be an irreducible Smale space. Then its homoclinic C^* -algebra is quasidiagonal.*

Proof. It follows from Smale's decomposition theorem [37, Theorem 6.2] that the Smale space C^* -algebras of an irreducible Smale space have a special form (see for example [14, Section 2.5]): they are each the direct sum of finitely many C^* -algebras each of which is the C^* -algebra of a mixing Smale space. The result then follows from Theorem 4.1 because quasidiagonality is preserved under direct sums. \square

If, in the definition of nuclear dimension (Definition 2.1, we in addition require the maps ϕ_λ to be contractive, then we have the definition for the decomposition rank, a precursor to nuclear dimension which first appeared in [16]. For the homoclinic algebra we obtain the following bound on both the decomposition rank as well as a smaller bound on the nuclear dimension.

Corollary 4.5. *Let (X, φ) be a mixing Smale space. Then the associated homoclinic C^* -algebra A has decomposition rank at most one.*

Proof. By [24, Theorem 1.1], since A is simple, separable, unital, nuclear and has unique tracial state, it has finite decomposition rank. By [3, Corollary 8.6] this is equivalent to having decomposition rank at most one. Hence A also has nuclear dimension at most one. \square

Corollary 4.6. *Let A be the homoclinic C^* -algebra of a mixing Smale space (X, φ) . Then $A \otimes \mathcal{U}$ is tracially approximately finite (TAF) for any UHF algebra \mathcal{U} of infinite type.*

Proof. The C^* -algebra A is separable, unital and nuclear and has a unique trace. Since (X, φ) is mixing, A is simple. By Theorem 4.1 it is also quasidiagonal. Thus it follows from [24, Theorem 6.1] that $A \otimes \mathcal{U}$ is TAF for any UHF algebra \mathcal{U} of infinite type. \square

Theorem 4.7. *The homoclinic algebras associated to mixing Smale spaces are contained in a class of C^* -algebras that is classified by the Elliott invariant. In particular,*

- (i) *the homoclinic algebra associated to a mixing Smale space is approximately subhomogeneous,*
- (ii) *if A and B are the homoclinic algebras associated to the mixing Smale spaces (X, φ) and (Y, ψ) , then an isomorphism*

$$\phi : (K_0(A), K_0(A)_+, [1_A], K_1(A)) \rightarrow (K_0(B), K_0(B)_+, [1_B], K_1(B))$$

lifts to a $$ -isomorphism*

$$\Phi : A \rightarrow B$$

inducing ϕ .

Proof. Once we know finite decomposition rank, this in fact follows directly from the main results in [7], since the homoclinic groupoid of (X, φ) is amenable and hence its C^* -algebra satisfies the UCT [46, Proposition 10.7]. However, we do not

need the full strength of this classification result and can appeal to earlier work. By Corollary 4.6, for any homoclinic algebra A corresponding to a mixing Smale space (X, ϕ) we have that $A \otimes \mathcal{U}$ is tracially approximately finite for any UHF algebra \mathcal{U} of infinite type. By applying [19, Theorem 5.4] the C*-algebra $A \otimes \mathcal{Z}$ belongs to a class classified by the Elliott invariant. But A also has finite nuclear dimension by Corollary 3.8, so since it is also separable, simple, unital and nonelementary, A tensorially absorbs \mathcal{Z} [49, Corollary 7.3]. The result follows. \square

In the case that the stable and unstable algebras associated to a mixing Smale space both have nonzero projections, we can also say something about their structure.

Theorem 4.8. *Suppose that (X, φ) is a mixing Smale space and that the associated unstable and stable algebras each contain a nontrivial projection. Then they are approximately subhomogeneous (ASH) algebras.*

Proof. Recall that the choice of the particular sets of periodic periods P and Q in the definition of the stable and unstable C*-algebras only affects the algebras up to stable isomorphism. We prove that the stable algebra $C^*(G_S(P))$ is ASH since the proof for $C^*(G_U(Q))$ is identical upon changing the roles of the two algebras.

Since $C^*(G_U(Q))$ contains a nonzero projection it is isomorphic to a separable subalgebra of $C^*(G_S(P)) \otimes C^*(G_U(Q))$. By (the proof of) Theorem 4.1, we have that $C^*(G_S(P)) \otimes C^*(G_U(Q))$ is quasidiagonal so it follows that $C^*(G_S(P))$ is also quasidiagonal. Thus the stable algebra is quasidiagonal, satisfies the UCT (again by [46, Proposition 10.7]) and, by Corollary 3.8, has finite nuclear dimension.

Let p be a nonzero projection in $C^*(G_S(P))$. Since $C^*(G_S(P))$ is simple, p is full so $C^*(G_S(P))$ is stably isomorphic to $p(C^*(G_S(P)))p$ [4, Corollary 2.6]. Thus $C^*(G_S(P))$ is stably isomorphic to a C*-algebra of rational generalised tracial rank at most one by [7, Theorem 4.3] and [43, Theorem A], and hence has an ASH model given by [6]. Since any stabilisation and any hereditary subalgebra of an ASH algebra is again ASH, it follows that $C^*(G_S(P))$ is ASH. \square

In [48] Wieler constructs many examples of Smale spaces with totally disconnected stable sets. It is not difficult to show that in this case the unstable algebra will have many projections; the unit space of the unstable groupoid will be totally disconnected in this case. It is natural to ask the following:

Question 4.9. When do the stable and unstable algebras of a mixing Smale space contain projections?

Based on examples, it is plausible that the answer to this question is “always”.

5. FURTHER REMARKS

Though our classification results only apply to the homoclinic C*-algebra, there are situations where our results for the stable and unstable algebras also have consequences for classification. In particular, the proof of Theorem 4.7 implies that the class of unital C*-algebras Morita equivalent to the unstable algebra of a Smale space is classifiable by the Elliott invariant.

An explicit example of this situation comes from certain substitution tilings. Let A be the C*-algebra considered in [12, Section 5]. Then A is unital and Morita equivalent to the Smale space constructed by Anderson and Putnam in [1, Section

3]. We should note that for this specific case, classification can also be proved using [50, Corollary 3.2] and the main result of [35] (also see [38, Remark 7.10]). Phillips has asked if these substitution tiling C^* -algebras are TAF (again see [38, Remark 7.10]). Our results do not imply this, but classification could be used to reduce this to a question about K -theory. For example, if the K -theory is torsion-free, then the relevant C^* -algebra is AT.

In general, our results indicate the importance of computing the K -theory of Smale space C^* -algebras. This is already an active area of research. Some calculations have been done in particular cases; see for example [26, 28, 41, 42]. Moreover, Putnam has constructed a homology theory for Smale spaces [29]. A link between Putnam's homology and the K -theory of the C^* -algebra has been conjectured in [29, Section 8.4]. Our results are relevant for the computation of the K -theory. This is because finite dynamic asymptotic dimension (see [9, Introduction] for details) should lead, in principal, to a computation of the K -theory. It will be interesting to see if this leads to an effective way to compute the K -theory (at least for special classes of Smale spaces).

Finally, we observe that in certain special cases, an inductive limit structure for the homoclinic C^* -algebra can be found directly from the dynamics. For example, in the case of a shift of finite type, the homoclinic (and also the stable and unstable) C^* -algebra is AF. One-dimensional solenoids are another example; see the work of Thomsen [42]. It follows from Theorem 4.7 that the homoclinic C^* -algebra has the inductive limit structure of an ASH C^* -algebra [20, Lemma 5.1]. However, it would be interesting to construct an inductive limit structure for the homoclinic C^* -algebra of a general Smale space directly from the dynamics.

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REFERENCES

- [1] Jared E. Anderson and Ian F. Putnam, *Topological invariants for substitution tilings and their associated C^* -algebras*, *Ergodic Theory Dynam. Systems* **18** (1998), no. 3, 509–537, DOI 10.1017/S0143385798100457. MR1631708
- [2] Bruce Blackadar and Eberhard Kirchberg, *Generalized inductive limits of finite-dimensional C^* -algebras*, *Math. Ann.* **307** (1997), no. 3, 343–380, DOI 10.1007/s002080050039. MR1437044
- [3] Joan Bosa, Nathaniel P. Brown, Yasuhiko Sato, Aaron Tikuisis, Stuart White, and Wilhelm Winter, *Covering dimension of C^* -algebras and 2-coloured classification*, arXiv preprint math.OA/1506.03874v2, to appear in *Mem. Amer. Math. Soc.*, 2015.

- [4] Lawrence G. Brown, *Stable isomorphism of hereditary subalgebras of C^* -algebras*, Pacific J. Math. **71** (1977), no. 2, 335–348. MR0454645
- [5] Joachim Cuntz and Wolfgang Krieger, *A class of C^* -algebras and topological Markov chains*, Invent. Math. **56** (1980), no. 3, 251–268, DOI 10.1007/BF01390048. MR561974
- [6] George A. Elliott, *An invariant for simple C^* -algebras* (English, with English and French summaries), Canadian Mathematical Society. 1945–1995, Vol. 3, Canadian Math. Soc., Ottawa, ON, 1996, pp. 61–90. MR1661611
- [7] George A. Elliott, Guhua Gong, Huaxin Lin, and Zhuang Niu, *On the classification of simple amenable C^* -algebras with finite decomposition rank II*, arXiv preprint math.OA/1507.03437v2, 2015.
- [8] Thierry Giordano, Ian F. Putnam, and Christian F. Skau, *Topological orbit equivalence and C^* -crossed products*, J. Reine Angew. Math. **469** (1995), 51–111. MR1363826
- [9] Erik Guentner, Rufus Willett, and Guoliang Yu, *Dynamic asymptotic dimension: relation to dynamics, topology, coarse geometry, and C^* -algebras*, Math. Ann. **367** (2017), no. 1–2, 785–829, DOI 10.1007/s00208-016-1395-0. MR3606454
- [10] Don Hadwin, *Strongly quasidiagonal C^* -algebras*, J. Operator Theory **18** (1987), no. 1, 3–18. With an appendix by Jonathan Rosenberg. MR912809
- [11] Jerome Kaminker, Ian F. Putnam, and Michael F. Whittaker, *K -theoretic duality for hyperbolic dynamical systems*, J. Reine Angew. Math. **730** (2017), 263–299. MR3692021
- [12] Johannes Kellendonk and Ian F. Putnam, *Tilings, C^* -algebras, and K -theory*, Directions in mathematical quasicrystals, CRM Monogr. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2000, pp. 177–206. MR1798993
- [13] David Kerr and Piotr W. Nowak, *Residually finite actions and crossed products*, Ergodic Theory Dynam. Systems **32** (2012), no. 5, 1585–1614, DOI 10.1017/S0143385711000575. MR2974211
- [14] D. Brady Killough, *Ring Structures on the K -Theory of C^* -Algebras Associated to Smale Spaces*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)—University of Victoria (Canada). MR2890168
- [15] D. B. Killough and I. F. Putnam, *Trace asymptotics for C^* -algebras from Smale spaces*, Proc. Amer. Math. Soc. **143** (2015), no. 1, 317–325, DOI 10.1090/S0002-9939-2014-12221-0. MR3272757
- [16] Eberhard Kirchberg and Wilhelm Winter, *Covering dimension and quasidiagonality*, Internat. J. Math. **15** (2004), no. 1, 63–85, DOI 10.1142/S0129167X04002119. MR2039212
- [17] Wolfgang Krieger, *On dimension functions and topological Markov chains*, Invent. Math. **56** (1980), no. 3, 239–250, DOI 10.1007/BF01390047. MR561973
- [18] Huaxin Lin, *Crossed products and minimal dynamical systems*, J. Topol. Anal. (2016), DOI 10.1142/S1793525318500140
- [19] Huaxin Lin and Zhuang Niu, *Lifting KK -elements, asymptotic unitary equivalence and classification of simple C^* -algebras*, Adv. Math. **219** (2008), no. 5, 1729–1769, DOI 10.1016/j.aim.2008.07.011. MR2458153
- [20] Huaxin Lin and Zhuang Niu, *The range of a class of classifiable separable simple amenable C^* -algebras*, J. Funct. Anal. **260** (2011), no. 1, 1–29, DOI 10.1016/j.jfa.2010.08.019. MR2733569
- [21] Huaxin Lin and N. Christopher Phillips, *Crossed products by minimal homeomorphisms*, J. Reine Angew. Math. **641** (2010), 95–122, DOI 10.1515/CRELLE.2010.029. MR2643926
- [22] Elon Lindenstrauss and Benjamin Weiss, *Mean topological dimension*, Israel J. Math. **115** (2000), 1–24, DOI 10.1007/BF02810577. MR1749670
- [23] Ricardo Mañé, *Expansive homeomorphisms and topological dimension*, Trans. Amer. Math. Soc. **252** (1979), 313–319, DOI 10.2307/1998091. MR534124
- [24] Hiroki Matui and Yasuhiko Sato, *Decomposition rank of UHF-absorbing C^* -algebras*, Duke Math. J. **163** (2014), no. 14, 2687–2708, DOI 10.1215/00127094-2826908. MR3273581
- [25] Paul S. Muhly, Jean N. Renault, and Dana P. Williams, *Equivalence and isomorphism for groupoid C^* -algebras*, J. Operator Theory **17** (1987), no. 1, 3–22. MR873460
- [26] Volodymyr Nekrashevych, *C^* -algebras and self-similar groups*, J. Reine Angew. Math. **630** (2009), 59–123, DOI 10.1515/CRELLE.2009.035. MR2526786
- [27] Ian F. Putnam, *On the topological stable rank of certain transformation group C^* -algebras*, Ergodic Theory Dynam. Systems **10** (1990), no. 1, 197–207, DOI 10.1017/S0143385700005484. MR1053808

- [28] Ian F. Putnam, *C*-algebras from Smale spaces*, Canad. J. Math. **48** (1996), no. 1, 175–195, DOI 10.4153/CJM-1996-008-2. MR1382481
- [29] Ian F. Putnam, *A homology theory for Smale spaces*, Mem. Amer. Math. Soc. **232** (2014), no. 1094, viii+122.
- [30] Ian F. Putnam and Jack Spielberg, *The structure of C*-algebras associated with hyperbolic dynamical systems*, J. Funct. Anal. **163** (1999), no. 2, 279–299, DOI 10.1006/jfan.1998.3379. MR1680475
- [31] Jean Renault, *A groupoid approach to C*-algebras*, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, 1980. MR584266
- [32] Mikael Rørdam and Adam Sierakowski, *Purely infinite C*-algebras arising from crossed products*, Ergodic Theory Dynam. Systems **32** (2012), no. 1, 273–293, DOI 10.1017/S0143385710000829. MR2873171
- [33] David Ruelle, *Noncommutative algebras for hyperbolic diffeomorphisms*, Invent. Math. **93** (1988), no. 1, 1–13, DOI 10.1007/BF01393685. MR943921
- [34] David Ruelle, *Thermodynamic formalism*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004. The mathematical structures of equilibrium statistical mechanics. MR2129258
- [35] Lorenzo Sadun and R. F. Williams, *Tiling spaces are Cantor set fiber bundles*, Ergodic Theory Dynam. Systems **23** (2003), no. 1, 307–316, DOI 10.1017/S0143385702000949. MR1971208
- [36] Yasuhiko Sato, Stuart White, and Wilhelm Winter, *Nuclear dimension and Z-stability*, Invent. Math. **202** (2015), no. 2, 893–921, DOI 10.1007/s00222-015-0580-1. MR3418247
- [37] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817, DOI 10.1090/S0002-9904-1967-11798-1. MR0228014
- [38] Charles Starling, *Finite symmetry group actions on substitution tiling C*-algebras*, Münster J. Math. **7** (2014), no. 2, 381–412. MR3426223
- [39] Karen R. Strung, *C*-algebras of minimal dynamical systems of the product of a Cantor set and an odd dimensional sphere*, J. Funct. Anal. **268** (2015), no. 3, 671–689, DOI 10.1016/j.jfa.2014.10.014. MR3292350
- [40] Gábor Szabó, *The Rokhlin dimension of topological \mathbb{Z}^m -actions*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 3, 673–694, DOI 10.1112/plms/pdu065. MR3342101
- [41] Klaus Thomsen, *C*-algebras of homoclinic and heteroclinic structure in expansive dynamics*, Mem. Amer. Math. Soc. **206** (2010), no. 970, x+122, DOI 10.1090/S0065-9266-10-00581-8. MR2667385
- [42] Klaus Thomsen, *The homoclinic and heteroclinic C*-algebras of a generalized one-dimensional solenoid*, Ergodic Theory Dynam. Systems **30** (2010), no. 1, 263–308, DOI 10.1017/S0143385709000042. MR2586354
- [43] Aaron Tikuisis, Stuart White, and Wilhelm Winter, *Quasidiagonality of nuclear C*-algebras*, Ann. of Math. (2) **185** (2017), no. 1, 229–284, DOI 10.4007/annals.2017.185.1.4. MR3583354
- [44] Andrew S. Toms and Wilhelm Winter, *Minimal dynamics and the classification of C*-algebras*, Proc. Natl. Acad. Sci. USA **106** (2009), no. 40, 16942–16943, DOI 10.1073/pnas.0903629106. MR2559393
- [45] Andrew S. Toms and Wilhelm Winter, *Minimal dynamics and K-theoretic rigidity: Elliott’s conjecture*, Geom. Funct. Anal. **23** (2013), no. 1, 467–481, DOI 10.1007/s00039-012-0208-1. MR3037905
- [46] Jean-Louis Tu, *La conjecture de Baum-Connes pour les feuilletages moyennables* (French, with English and French summaries), K-Theory **17** (1999), no. 3, 215–264, DOI 10.1023/A:1007744304422. MR1703305
- [47] Michael F. Whittaker, *Poincare Duality and Spectral Triples for Hyperbolic Dynamical Systems*, ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)—University of Victoria (Canada). MR2890169
- [48] Susana Wieler, *Smale spaces via inverse limits*, Ergodic Theory Dynam. Systems **34** (2014), no. 6, 2066–2092, DOI 10.1017/etds.2013.19. MR3272784
- [49] Wilhelm Winter, *Nuclear dimension and Z-stability of pure C*-algebras*, Invent. Math. **187** (2012), no. 2, 259–342, DOI 10.1007/s00222-011-0334-7. MR2885621
- [50] Wilhelm Winter, *Classifying crossed product C*-algebras*, Amer. J. Math. **138** (2016), no. 3, 793–820, DOI 10.1353/ajm.2016.0029. MR3506386
- [51] Wilhelm Winter and Joachim Zacharias, *The nuclear dimension of C*-algebras*, Adv. Math. **224** (2010), no. 2, 461–498, DOI 10.1016/j.aim.2009.12.005. MR2609012

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