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# THE REGULARITY OF DIOPHANTINE QUADRUPLES

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Abstract. A set of positive integers is called a Diophantine tuple if the product of any two elements in the set increased by the unity is a perfect square. A conjecture on the regularity of Diophantine quadruples asserts that any Diophantine triple can be uniquely extended to a Diophantine quadruple by joining an element exceeding the maximal element of the triple. The problem is reduced to studying an equation expressed as the coincidence of two linear recurrence sequences with initial terms composed of the fundamental solutions of some Pellian equations. In this paper, we determine the values of those initial terms completely and obtain finiteness results on the number of solutions of the equation. As one of the applications to the problem on the regularity of Diophantine quadruples, we show in general that the number of ways of extending any given Diophantine triple is at most 11.

## 1. Introduction

Diophantus of Alexandria posed the problem of finding four numbers with the property that the product of any two of them increased by the unity is a perfect square. While he also gave an answer $\{1/16, 33/16, 17/4, 105/16\}$ composed of rational numbers, we shall be concerned with sets of positive integers. The first set $\{1, 3, 8, 120\}$ of four positive integers having the above property was found by Fermat. A set of positive integers is called *a Diophantine tuple* if the product of any two elements in the set increased by the unity is a perfect square.

Any Diophantine pair $\{a, b\}$ can be extended to a Diophantine triple $\{a, b, c\}$. For example, Euler found that $\{a, b, a + b + 2r\}$ is a Diophantine triple, where $r = \sqrt{ab + 1}$. Such a triple is called *a regular Diophantine triple*, and the largest element is known to be minimal among the $c$'s such that $\{a, b, c\}$ is a Diophantine triple with $c > \max\{a, b\}$.

Euler further extended a regular Diophantine triple to the Diophantine quadruple $\{a, b, a+b+2r, 4r(a+r)(b+r)\}$, which is specialized to Fermat's quadruple by putting $a = 1$ and $b = 3$. A more general construction of Diophantine quadruples was established by Arkin, Hoggatt, and Strauss ([1]), and by Gibbs ([24]) independently. They observed that for a Diophantine triple $\{a, b, c\}$ with $r = \sqrt{ab + 1}$, $s = \sqrt{ac + 1}$, $t = \sqrt{bc + 1}$, the set $\{a, b, c, d_+\}$ with $d_+ = a+b+c+2abc+2rst$ forms a Diophantine quadruple. Indeed,

$$ad_+ + 1 = (at + rs)^2, \quad bd_+ + 1 = (bs + rt)^2, \quad cd_+ + 1 = (cr + st)^2.$$

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Such a quadruple is called *a regular Diophantine quadruple*, and the largest element $d_+$ is known to be minimal among the $d$'s such that $\{a,b,c,d\}$ is a Diophantine quadruple with $a<b<c<d$. In case $c>a+b+2r$, a Diophantine triple $\{a,b,c\}$ has an extension to another regular Diophantine quadruple $\{a,b,c,d_-\}$, where $d_-=a+b+c+2abc-2rst$. Indeed,

$$ad_-+1=(at-rs)^2, \quad bd_-+1=(bs-rt)^2, \quad cd_-+1=(cr-st)^2.$$

Note that $0\le d_-<c$ in general and that $d_->0$ if and only if $c>a+b+2r$.

Arkin, Hoggatt, and Strauss (and independently Gibbs) also raised the following problem (cf. [1, 17, 24]).

**Conjecture 1.1.** *Any Diophantine quadruple is regular.*

All the Diophantine quadruples that have been so far known are regular, including the quadruples found by Euler. The first non-trivial result supporting Conjecture 1.1 was given by Baker and Davenport ([2]), which asserts that if $\{1,3,8,d\}$ is a Diophantine quadruple, then $d=120(=d_+)$. Since this work was published, active research on Diophantine tuples has been carried out by several researchers, including Dujella, and the result above is generalized to show the validity of Conjecture 1.1 for the quadruples $\{a,b,c,d\}$ with $a<b<c<d$ containing the following triples $\{a,b,c\}$ or pairs $\{a,b\}$:

(i) $\{k-1,k+1,4k\}$ with an integer $k\ge 2$ ([12]);
(ii) $\{1,3\}$ ([17]);
(iii) $\{k-1,k+1\}$ ([5, 21]);
(iv) $\{k,A^2k+2A,(A+1)^2k+2(A+1)\}$ with positive integers $k$ and $A$ satisfying $A\le 10$ or $A\ge 52330$ ([25–27]);
(v) $\{a,b\}$ with $b<a+4\sqrt{a}$ ([20]).

Recently, Cipu, Mignotte, and the first author showed that any Diophantine quadruple containing the regular Diophantine triple $\{k,A^2k+2A,(A+1)^2k+2(A+1)\}$ with any positive integers $k$ and $A$, which appears in (iv), is always regular ([9]).

Conjecture 1.1 immediately implies that there exists no Diophantine quintuple, which is a folklore conjecture in this field. While the finiteness of irregular Diophantine quadruples has not been shown yet, it is known that there exist only finitely many Diophantine quintuples by Dujella in [14], where he also proved that there exists no Diophantine sextuple. More explicitly, he ([15]) gave the upper bound $10^{1930}$ for the number of Diophantine quintuples. This bound was first reduced to $10^{276}$ in [23] by showing the following:

> Any fixed Diophantine triple can only be extended to
> a Diophantine quintuple in at most four ways by joining fourth
> and fifth elements exceeding the maximal element in the triple,

the proof of which relies largely on Okazaki's lemma (see [4, Lemma 2.2] or Lemma 7.1) asserting that there are "large" gaps between solutions to a system of Pellian equations. The record of the bound has been updated to be $10^{31}$ by Cipu ([6]), and $2.4\cdot 10^{29}$ by Trudgian ([33]). Very recently, Cipu and Trudgian together set a new record of $5.441\cdot 10^{26}$ ([11]). A pertinent overview and a great deal of references on Diophantine tuples can be found on the web page [16].

The primary purpose of the present paper is to give an absolute upper bound for the number of Diophantine quadruples $\{a,b,c,d\}$ with $d>\max\{a,b,c\}$ for a fixed Diophantine triple $\{a,b,c\}$.

**Theorem 1.2.** *Any fixed Diophantine triple can only be extended to a Diophantine quadruple in at most* 11 *ways by joining a fourth element exceeding the maximal element in the triple.*

For a fixed Diophantine triple $\{a, b, c\}$, one can find a fourth element $d$ such that $\{a, b, c, d\}$ is a Diophantine quadruple by solving the following system of Pellian equations:

$$\begin{cases} az^2 - cx^2 = a - c, \\ bz^2 - cy^2 \ = b - c. \end{cases} \tag{1.1}$$

It is well-known that there exist only finitely many fundamental solutions $(z_0, x_0)$ and $(z_1, y_1)$ of the Pellian equations above, using which one can express any solution to the system as $z = v_m = w_n$ with non-negative integers $m$ and $n$, where $\{v_m\}_{m\geq 0}$ and $\{w_n\}_{n\geq 0}$ are recurrence sequences defined by

$$v_0 = z_0,\ v_1 = sz_0 + cx_0,\ v_{m+2} = 2sv_{m+1} - v_m,$$
$$w_0 = z_1,\ w_1 = tz_1 + cy_1,\ w_{n+2} = 2tw_{n+1} - w_n$$

(see Section 2). Dujella determined the initial terms of sequences $\{v_m\}$ and $\{w_n\}$ almost explicitly ([14, Lemma 8]), where there is some ambiguity in the case where both $m$ and $n$ are even. What we first have to do for the proof of Theorem 1.2 is to disambiguate the statement, which, in itself, is one of the motivations of this paper.

**Theorem 1.3.** *Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d$ and that $z = v_m = w_n$ has a solution for some integers $m$ and $n$. Then, one of the following four cases holds*:

(1) *Both $m$ and $n$ are even with $z_0 = z_1$ and $|z_0| \in \{1, cr - st\}$.*
(2) *$m$ is odd and $n$ is even with $|z_0| = t$, $|z_1| = cr - st$, and $z_0z_1 < 0$.*
(3) *$m$ is even and $n$ is odd with $|z_0| = cr - st$, $|z_1| = s$, and $z_0z_1 < 0$.*
(4) *Both $m$ and $n$ are odd with $|z_0| = t$, $|z_1| = s$, and $z_0z_1 > 0$.*

*Moreover, if $d > d_+$, then case* (2) *cannot occur.*

Thanks to Theorem 1.3, we are left with ten equations of form (1.1) to be studied. In order to obtain finiteness results on the number of solutions to each of such equations, we will rely on the methods used in the study of the following system of Pellian equations:

$$\begin{cases} X^2 - AZ^2 = 1, \\ Y^2 - BZ^2 = 1, \end{cases} \tag{1.2}$$

where $A$ and $B$ are given distinct positive integers. The number of solutions to system (1.2) has been actively studied by several authors in the literature. We only refer to the papers [3, 4, 10, 30, 34] and the references given therein. The final achievement on the number of solutions to (1.2) is due to Bennett, Cipu, Mignotte, and Okazaki ([4]), who showed that system (1.2) has at most two solutions $(X, Y, Z)$ in positive integers. Similarly to our situation, solving (1.2) is reduced to examining when certain two linear recurrence sequences coincide, where the initial terms of the sequences are uniquely determined (as the constant terms are 1). Although the initial terms of $\{v_{m'}\}$ and $\{w_{n'}\}$ (i.e., the fundamental solutions to system (1.1)) have not been uniquely determined, we may apply Okazaki's lemma, [4, Lemma 2.2], to system (1.1) in view of Theorem 1.3, as well as [23, Theorem 1.2] mentioned above (note that the fundamental solutions corresponding to Diophantine quintuples have already been determined completely; see [23, Lemma 2.2]). As a result, we obtain the following.

**Theorem 1.4.** *Let $N(z_0, z_1)$ be the number of positive integers $d$ with $d > d_+$ such that $\{a, b, c, d\}$ forms a Diophantine quadruple and $d$ corresponds to a fundamental solution $(z_0, z_1)$. Then $N(z_0, z_1) \le 2$. Moreover, if*

$$(z_0, z_1) \in \{(st - cr, st - cr), (st - cr, s), (t, s)\},$$

*then $N(z_0, z_1) \le 1$.*

Theorem 1.2 now follows from Theorem 1.4 together with a simple but thorough investigation (Lemma 2.8) on the bound for the second largest element $c$ in the quadruple (see Section 5).

An outline of the proof of Theorem 1.4 goes as follows. Now that the possible fundamental solutions are explicitly described in Theorem 1.3, Okazaki's lemma (Lemma 7.1) on gaps between solutions, together with an upper bound for solutions obtained by Baker's method on linear forms in three logarithms, shows the first part of Theorem 1.4, where Lemma 7.1 is in fact applied to the case where there exist three solutions $d = d_0, d_1, d_2$ with $d_+ < d_0 < d_1 < d_2$ coming from the same fundamental solution. Combining this with Theorem 1.2, one can deduce that the number of possible extensions of a fixed Diophantine triple $\{a, b, c\}$ ($a < b < c$) to a Diophantine quadruple $\{a, b, c, d\}$ with $d > c$ is at most 13. In order to lessen the number of extensions, we have to apply Lemma 7.1 to the case where $d_0 = d_+$, and we therefore need a better upper bound for solutions. To do this, we consider the difference $\Gamma := \Lambda_2 - \Lambda_1$ between linear forms $\Lambda_1$ and $\Lambda_2$ in three logarithms corresponding respectively to the solutions $d_1$ and $d_2$, which is a linear form in two logarithms. Then, Baker's method on linear forms in two logarithms can work well if $d_1$ and $d_2$ are not so far apart, which is ensured by application of Rickert's theorem ([32]) on simultaneous rational approximation of irrationals, as is the case with [3, Corollary 3.3]. In the cases of $(z_0, z_1) \in \{(st-cr, st-cr), (st-cr, s), (t, s)\}$, where $d = d_+$ can be attained, we thus obtain a contradiction.

As another application of Theorem 1.3, one may increase lower bounds for solutions (compare Lemma 3.1 with Lemma 4.1). Such bounds, combined with a version (Theorem 3.3) of Rickert's theorem, show the following, which generalizes and improves [19, Theorem 1.2] in view of $b > 4000$ (see Lemma 2.1).

**Theorem 1.5.** *Let $\{a, b, c, d\}$ be a Diophantine quadruple with $a < b < c < d$. If $b < 2a$ and $c \ge 9.864b^4$, $2a \le b \le 12a$ and $c \ge 4.321b^4$, or $b > 12a$ and $c \ge 721.8b^4$, then $d = d_+$.*

It is known by [22, Theorem 2] that if $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$, then $d = d_+$. Since, then, any of $\{a, b, d, e\}$, $\{a, c, d, e\}$, and $\{b, c, d, e\}$ is irregular, the following theorem generalizes [22, Theorem 2].

**Theorem 1.6.** *Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d_+ < d$. Then, any Diophantine quadruple $\{e, a, b, c\}$ with $e < c$ must be regular.*

Note that taking contraposition of the statement of Theorem 1.6 we see that if $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d_+ < d$, then any

Diophantine quadruple $\{a_1, a_2, d, e\}$ with $\{a_1, a_2\} \subset \{a, b, c\}$ and $d < e$ must be regular.

Consider now the regular Diophantine triple $\{a, b, c\}$ with $c = a + b + 2r$. By [28, Theorem 8], if $d > d_+$, then only case (1) in Theorem 1.3 can occur with $|z_0| = 1$. Since the relation $cr - st = 1$ holds, the following is an immediate consequence of Theorem 1.4.

**Corollary 1.7.** *Any fixed regular Diophantine triple can only be extended to a Diophantine quadruple in at most* 4 *ways by joining a fourth element exceeding the maximal element in the triple.*

The organization of this paper is as follows. In Section 2, we recall several known facts ameliorated by using Lemma 2.1. In particular, the bound for $c$ attached to each pair of possible values of $|z_0|$, $|z_1|$ is precisely examined in Lemma 2.8. In Section 3, we give lower bounds for solutions and prove Theorem 1.3 using the lower bounds with a version of Rickert's theorem. Section 4 is devoted to proving Theorems 1.5 and 1.6. In Section 5, it is shown that Theorem 1.2 is a consequence of Theorem 1.4 with the help of Theorem 1.3 and Lemma 2.8. The proof of Theorem 1.4 will be the goal of Sections 6 through 10. In Section 6, we give an upper bound for solutions using Matveev's theorem ([31]) on linear forms in three logarithms. In Section 7, we give a "large" gap between the solutions using Okazaki's lemma (Lemma 7.1), which together with the result obtained in the preceding section shows an absolute upper bound for $c$, on the assumption that the relevant Pellian equations have three solutions belonging to the same class of solutions. In Section 8, another type of application of Rickert's theorem shows that there is a "small" gap between the solutions (Proposition 8.4). In Section 9, we transform the two linear forms in three logarithms corresponding to two solutions into a linear form in two logarithms and apply the theorem ([29, Theorem 2]) of Laurent on Baker's method. In Section 10, we complete the proof of Theorem 1.4. Finally in Section 11, we conclude this paper with some remarks on further progress.

## 2. Preliminary lemmas

Let $\{a, b, c\}$ be a Diophantine triple with $a < b < c$ such that

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2,$$

where $r, s, t$ are positive integers. Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $c < d$. Then there exist integers $x, y, z$ such that

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2,$$

from which, eliminating $d$, we obtain the following system of Pellian equations:

$$az^2 - cx^2 = a - c, \tag{2.1}$$

$$bz^2 - cy^2 = b - c. \tag{2.2}$$

Throughout this paper, the following lemma, which is a product of the reduction method (cf. [2, Lemma], [17, Lemma 5]) and a computer, will be frequently used without referring to it in order to reduce the amount of computation in the proofs.

**Lemma 2.1** ([7, Lemma 3.4]). *Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d_+ < d$.*

(1) *If $b < 2a$, then $b > 21000$.*
(2) *If $2a \leq b \leq 12a$, then $b > 130000$.*
(3) *If $b > 12a$, then $b > 4000$.*

In the rest of this section, we will assume that $d > d_+$, unless otherwise specified.

**Lemma 2.2.** *Let $(z, x)$, $(z, y)$ be positive solutions of (2.1), (2.2), respectively. Then there exist solutions $(z_0, x_0)$ of (2.1) and $(z_1, y_1)$ of (2.2) in the ranges*

$$1 \leq x_0 < \sqrt{\frac{s+1}{2}} < 0.7128\sqrt[4]{ac},$$

$$1 \leq |z_0| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} < 0.089c,$$

$$1 \leq y_1 < \sqrt{\frac{t+1}{2}} < 0.7072\sqrt[4]{bc},$$

$$1 \leq |z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}} < 0.0112c$$

*such that*

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m, \tag{2.3}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n \tag{2.4}$$

*for some non-negative integers $m$ and $n$.*

*Proof.* This can be shown by applying [14, Lemma 1] with $b > 4000$ obtained from Lemma 2.1. □

By (2.3) and (2.4), we may write $z = v_m = w_n$, where

$$v_0 = z_0,\ v_1 = sz_0 + cx_0,\ v_{m+2} = 2sv_{m+1} - v_m,$$
$$w_0 = z_1,\ w_1 = tz_1 + cy_1,\ w_{n+2} = 2tw_{n+1} - w_n.$$

**Lemma 2.3.** *We have the following:*

(1) *If $z \geq w_4$ (resp. $w_5$, $w_6$), then $d > 20b^{2.5}c^{3.5}$ (resp. $81b^{3.5}c^{4.5}$, $327b^{4.5}c^{5.5}$).*
(2) *If $z \geq v_6$, then $d > 256a^{4.5}c^{5.5}$.*

*Proof.* The proof proceeds along the same lines as those of [22, Lemma 15] or [14, Proposition 1] (note that Lemma 2.2 implies $w_1 > 0.5655b^{-1/4}c^{3/4}$ and $v_1 > 0.5205a^{-1/4}c^{3/4}$). □

**Lemma 2.4.** *Assume that $z = v_m = w_n$ has a solution for some integers $m$ and $n$. If $c > b^{3.5}$ (resp. $b^4$, $b^{5.5}$), then*

$$m \leq \frac{9}{7}n + \frac{5}{7} \quad \left(\text{resp. } \frac{5}{4}n + \frac{3}{4},\ \frac{13}{11}n + \frac{9}{11}\right).$$

*Proof.* This can be shown in the same way as [14, Lemma 4]. □

**Lemma 2.5.** *If $z = v_m = w_n$ has a solution for some integers $m$ and $n$, then* $\min\{m, n\} \geq 4$.

*Proof.* The assertion follows from [22, Lemmas 7–11] (note that we are assuming $a < b < c < d_+ < d$). $\square$

**Lemma 2.6.** *Assume that $c \le 20a^{2.5}b^{3.5}$. If $z = v_m = w_n$ has a solution for some even integers $m$ and $n$, then $d > 256a^{4.5}c^{5.5}$. Moreover, if $b < 2a$, then $d > 327b^{4.5}c^{5.5}$.*

*Proof.* The assumption $c \le 20a^{2.5}b^{3.5}$, together with Lemma 2.3, leads us to show $v_4 \ne w_4$ in the same way as [22, Lemma 13]. It follows from Lemma 2.5 with $m$, $n$ even that $\max\{m, n\} \ge 6$, which together with Lemma 2.3 implies the first assertion.

If $b < 2a$, then one can show that $v_{m'} \ne w_4$ for all even integers $m'$, following the argument in the proof of [22, Lemma 14(1)]. Hence we obtain $z \ge w_6$ and $d > 327b^{4.5}c^{5.5}$ by Lemma 2.3. $\square$

**Lemma 2.7.** *If $z = v_m = w_n$ has a solution for some even integers $m$ and $n$ with $|z_0| \notin \{1, cr - st\}$, then $|z_0| < 0.653b^{-5/14}c^{9/14}$.*

*Proof.* One can prove this lemma in a similar fashion to [22, Lemma 18] (or [14, Lemma 8(1)]). Indeed, putting $d_0 = (z_0^2 - 1)/c$, we have the irregular Diophantine quadruple $\{a, b, d_0, c\}$ with $d_0 < c$, and Lemma 2.3, together with $n \ge 4$ by Lemma 2.5, implies that $c > 20d_0^{3.5}b^{2.5}$. Hence, we see from $b > 4000$ that $d_0 > 0.999z_0^2/c$ and $|z_0| < 0.653b^{-5/14}c^{9/14}$. $\square$

**Lemma 2.8.** *Set*

$$\tau = \frac{\sqrt{ab}}{r}\left(1 - \frac{a + b + 1/c}{c}\right) \quad (< 1).$$

*Then we have*

(i) *If $|z_0| = cr - st$, then $c < 4\tau^{-4}ab^2$.*
(ii) *If $|z_1| = cr - st$, then $c < 4\tau^{-4}a^2b$.*
(iii) *If $|z_0| = t$, then $c > 4ab^2$.*
(iv) *If $|z_1| = s$, then $c > 4a^2b$.*

*Moreover,* (ii) *and* (iii) *cannot occur simultaneously.*

*Proof.* We use Lemma 2.2. Thus

$$|z_0| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}, \quad |z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}.$$

(i), (ii) Observe that

$$cr - st = \frac{c^2 - ac - bc - 1}{cr + st} = \frac{c}{2\sqrt{ab}}\,\frac{2\sqrt{ab}\,c}{cr + st}\left(1 - \frac{a + b + 1/c}{c}\right) > \frac{c}{2\sqrt{ab}}\,\tau.$$

If $|z_0| = cr - st$, then

$$\frac{c}{2\sqrt{ab}}\,\tau < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}.$$

Raising this to the fourth power yields $c < 4\tau^{-4}ab^2$. Similarly, the inequality

$$\frac{c}{2\sqrt{ab}}\,\tau < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}$$

implies $c < 4\tau^{-4}a^2b$.

(iii), (iv) If $|z_0| = t$, then

$$\sqrt{bc} < t < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}.$$

Raising this to the fourth power yields $c > 4ab^2$. Similarly, the inequality

$$s < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}$$

implies $c > 4a^2 b$.

The last assertion follows from the following stronger one:

$$\text{if } c > 4ab^2, \text{ then } b > \tau^{-4} a. \tag{2.5}$$

Indeed,

$$1 - \tau < 1 - \frac{\sqrt{ab}}{r} + \frac{a+b+1/c}{c} < \frac{1}{2ab} + \frac{a+b+1/(4ab^2)}{4ab^2} < \frac{1}{ab}.$$

It follows from $b > 4000$ that $\tau > 1 - 1/4000 > 0.99$. Hence we obtain

$$\tau^{-4} = 1 + \tau^{-1}(\tau^{-2}+1)(\tau^{-1}+1)(1-\tau) < 1 + \frac{4.2}{ab} < \frac{b}{a}.$$

□

## 3. Proof of Theorem 1.3

**Lemma 3.1.** *Assume that $c > 20a^2 b^{3.5}$. Suppose that $z = v_m = w_n$ has a solution for some integers $m$ and $n$ with $n \geq 4$. Then, $m \equiv n \pmod 2$ and the following hold:*

1. *In the case where both $m$ and $n$ are even, if $b \geq 2a$, then $n > 2b^{-9/28}c^{5/28}$; otherwise*

$$n > \min\left\{1.999a^{-1/2}b^{-1/8}c^{1/8}, 1.298b^{-11/28}c^{3/28}\right\}.$$

2. *In the case where both $m$ and $n$ are odd, we have $n > 1.39b^{-3/4}c^{1/4}$.*

*In any case, we have $n \geq 6$ and $d > 327b^{4.5}c^{5.5}$.*

*Proof.* The first assertion follows from [14, Lemma 8] with Lemma 2.8(i), (ii).

(1) By [14, Lemma 8] and Lemma 2.7 we have $z_0 = z_1$ and either $|z_0| \in \{1, cr - st\}$ or $|z_0| < 0.653b^{-5/14}c^{9/14}$.

If $|z_0| = 1$, then the argument in the proof of [6, Lemma 2.4] is still valid, and we have $m > b^{-1/2}c^{1/2}$, which, noting Lemma 2.4, satisfies the desired inequality. If $|z_0| = cr - st$, then we will arrive at a contradiction in the same way as the proof of [14, Lemma 10 (case (1.2))].

Assume that $|z_0| \notin \{1, cr - st\}$, that is, $|z_0| < 0.653b^{-5/14}c^{9/14}$. Since $(z_0, x_0)$ and $(z_1, y_1)$ are solutions to (2.1) and (2.2), respectively, we see from $b > 4000$ that

$$x_0 < 0.683a^{1/2}b^{-5/14}c^{1/7} \text{ and } y_1 < 0.654b^{1/7}c^{1/7}.$$

Considering $v_m \equiv w_n \pmod{8c^2}$, we have

$$az_0m^2 - bz_0n^2 \equiv 2ty_1n - 2sx_0m \pmod{16c} \tag{3.1}$$

(see [13, Lemma 4]).

Suppose now that $n \le 2b^{-9/28}c^{5/28}$. Since $a|z_0|m^2 < 14c$, $b|z_0|n^2 < 14c$, $ty_1n < c$, and $sx_0m < c$, congruence (3.1) is in fact an equality:

$$az_0m^2 + 2sx_0m = bz_0n^2 + 2ty_1n. \tag{3.2}$$

In the case where $b \ge 2a$, we see from $z_0^2 \ge \max\{c+1, 3c/a\}$ that

$$0 \le \frac{sx_0}{a|z_0|} - 1 < 0.1669, \quad 0 \le \frac{ty_1}{b|z_0|} - 1 < 0.0418,$$

which are exactly the same as the inequalities in the proof (p. 199) of [14, Lemma 10]. If $z_0 > 1$, then $am(m+2.3338) \ge bn(n+2)$, which together with $b \ge 2a$ and Lemma 2.4 yields $17n^2 - 41.0294n - 106.683 < 0$, and hence $n < 3.99$, which contradicts $n \ge 4$. If $z_0 < -1$, then $am(m-1) \ge bn(n-2.0836)$. In case $b \ge 3a$, similarly to the above we have $264n^2 - 1081.1568n + 180 < 0$, which gives $n < 3.94$, a contradiction. In case $2a \le b < 3a$, we have

$$2n(n-2.0836) < m(m-2). \tag{3.3}$$

Then, since $c > 20a^2b^{3.5} > 20 \cdot 3^{-2}b^{5.5} > b^{5.5}$, Lemma 2.4 implies that $292n^2 - 1808.9248n + 468 < 0$. Hence, $n < 5.94$ and we obtain $n = 4$ and $m = 4$. However, then inequality (3.3) implies $8 > 8(4-2.0836) > 8$, a contradiction. This has shown the assertion for $b \ge 2a$.

In the case where $b < 2a$, squaring both sides of (3.1) gives

$$(am^2 - bn^2)^2 \equiv 4x_0^2m^2 + 4y_1^2n^2 - 8stx_0y_1mn \pmod{16c},$$

which, multiplied by $s$ and by $t$ respectively, shows that

$$Cs \equiv -8tx_0y_1mn, \quad Ct \equiv -8sx_0y_1mn \pmod{16c}, \tag{3.4}$$

where $C = (am^2 - bn^2)^2 - 4(x_0^2m^2 + y_1^2n^2)$. Suppose that

$$n \le \min\{1.999a^{-1/2}b^{-1/8}c^{1/8}, 1.299b^{-11/28}c^{3/28}\}.$$

Then, we have equation (3.2). Since $x_0^2 < y_1^2 < 0.427b^{2/7}c^{2/7}$ and $m \le 61n/44$ by Lemma 2.4 with $c > b^{5.5}$ and $n \ge 4$, we have

$$|Cs| < |Ct| < \max\{a^2n^4\sqrt{bc+1}, 4.9908b^{2/7}c^{2/7}n^2\sqrt{bc+1}\} < 16c,$$

$$\begin{aligned} 8tx_0y_1mn &< 8ty_1^2 \cdot \frac{61}{44}n^2 \\ &< 4.736b^{11/14}c^{11/14}\min\{1.999^2a^{-1}b^{-1/4}c^{1/4}, 1.299^2b^{-11/14}c^{3/14}\} \\ &< 8c. \end{aligned} \tag{3.5}$$

Hence, from (3.4) we obtain either $Cs = -8tx_0y_1mn$, $Ct = -8sx_0y_1mn$ or $Cs = 16c - 8tx_0y_1mn$, $Ct = 16c - 8sx_0y_1mn$. The former case cannot happen because of $t > s$, while the latter implies $2c = (t+s)x_0y_1mn$, which contradicts (3.5). Therefore, the assertion for $b < 2a$ holds.

(2) By [14, Lemma 8] and [13, Lemma 4], we have

$$\pm st\{a(m^2-1) - b(n^2-1)\} \equiv 2r(n-m) \pmod{8c}.$$

Multiplying both sides by $s$ and by $t$ respectively, we obtain

$$\pm t\{a(m^2-1) - b(n^2-1)\} \equiv 2rs(n-m) \pmod{8c}, \tag{3.6}$$

$$\pm s\{a(m^2-1) - b(n^2-1)\} \equiv 2rt(n-m) \pmod{8c}. \tag{3.7}$$

Suppose that $n \leq 1.39b^{-3/4}c^{1/4}$. Then, by Lemma 2.4 with $c > b^{3.5}$ and $n \geq 5$ we have

$$\begin{aligned} at(m^2-1) &< (10/7)^2\sqrt{1+1/(bc)}ab^{1/2}c^{1/2}n^2 < 4c, \\ bt(n^2-1) &< \sqrt{1+1/(bc)}b^{3/2}c^{1/2}n^2 < 4c, \\ 2rtm &< 2\sqrt{1+1/(ab)}\sqrt{1+1/(bc)}a^{1/2}bc^{1/2}n < 4c, \end{aligned}$$

which mean that (3.6) and (3.7) are in fact equalities. Hence we obtain $m = n$ and $a(m^2-1) = b(n^2-1)$, which contradict $a < b$ and $m > 0$.

Now it is not difficult to show the last assertion. In the case where both $m$ and $n$ are even, if $b \geq 2a$, then $n > 2 \cdot 20^{5/28}b^{17/56} > 42$; if $b < 2a$, then $b > 21000$ and

$$\begin{aligned} n &> \min\{1.999 \cdot 20^{1/8}a^{-1/4}b^{5/16}, 1.298 \cdot 20^{3/28}a^{3/14}b^{-1/56}\} \\ &> \min\{1.999 \cdot 20^{1/8}b^{1/16}, 1.298 \cdot 20^{3/28} \cdot 2^{-3/14}b^{11/56}\} > 5.4. \end{aligned}$$

Hence, $n \geq 6$. In the case where both $m$ and $n$ are odd, we have $n > 1.39 \cdot 20^{1/4}b^{1/8} > 8.2$, which yields $n \geq 9$. It follows that $n \geq 6$ and from Lemma 2.3(1) that $d > 327b^{4.5}c^{5.5}$. This completes the proof of Lemma 3.1. $\square$

**Lemma 3.2.** *Assume that $c > b^4$. If $z = v_m = w_n$ has a solution for some integers $m$ and $n$ with $n \geq 4$, then $\log z > n\log(4bc)$.*

*Proof.* One can show this lemma along the same lines as the proof of [22, Lemma 25]. $\square$

**Theorem 3.3** ([8, Theorem 2.2])**.** *Let $a$, $b$, and $N$ be integers with $0 < a \leq b-5$, $b > 2000$, and $N \geq 3.706a'b^2(b-a)^2$, where $a' = \max\{b-a, a\}$. Assume that $N$ is divisible by $ab$. Then the numbers $\theta_1 = \sqrt{1+b/N}$ and $\theta_2 = \sqrt{1+a/N}$ satisfy*

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > \left(\frac{1.413 \cdot 10^{28}a'bN}{a}\right)^{-1} q^{-\lambda}$$

*for all integers $p_1$, $p_2$, $q$ with $q > 0$, where*

$$\lambda = 1 + \frac{\log(10a^{-1}a'bN)}{\log(2.699a^{-1}b^{-1}(b-a)^{-2}N^2)} < 2.$$

**Lemma 3.4.** *Assume that $c \geq 3.706b^4$. If $z = v_m = w_n$ has a solution for some integers $m$ and $n$ with $n \geq 4$, then $n < 8\varphi(a,b,c)$, where*

$$\varphi(a,b,c) = \frac{\log(8.406 \cdot 10^{13}a^{1/2}(a')^{1/2}b^2c)\log(1.643a^{1/2}b^{1/2}(b-a)^{-1}c)}{\log(4bc)\log(0.2699a(a')^{-1}b^{-1}(b-a)^{-2}c)}.$$

*Proof.* This can be shown by Lemma 3.2 and Theorem 3.3 in exactly the same way as [8, Lemma 3.3]. $\square$

**Proposition 3.5.** *Suppose that $\{a,b,c,d\}$ is a Diophantine quadruple with $a < b < c < d$ and that $z = v_m = w_n$ has a solution for some integers $m$ and $n$. If either $b \geq 2a$ and $c > \max\{20a^2b^{3.5}, b^5\}$ or $b < 2a$ and $c > b^{10}$, then $d = d_+$.*

*Proof.* Suppose that $d > d_+$. By Lemma 3.4 we have

$$n < 8\varphi(a,b,c). \tag{3.8}$$

We may assume by Lemma 3.1 that $z = v_m = w_n$ with $m \equiv n \pmod 2$.

(1) Consider first the case where both $m$ and $n$ are even. Assume that $b \geq 2a$ and $c > \max\{20a^2b^{3.5}, b^5\}$. By Lemma 3.1 and (3.8) we have $b^{-9/28}c^{5/28} < 4\varphi(a,b,c)$. From $a^{1/2}(b-a)^{1/2} \leq b/2$ and $a^{1/2}b^{1/2}(b-a)^{-1} \leq \sqrt{2}$, we see that

$$\varphi(a,b,c) < \frac{\log(4.203 \cdot 10^{13}b^3c)\log(2.324c)}{\log(4bc)\log(0.2699b^{-4}c)}.$$

Since the right-hand side is a decreasing function of $c$, the assumption $c > b^5$ shows that

$$b^{4/7} < \frac{4\log(4.203 \cdot 10^{13}b^8)\log(2.324b^5)}{\log(4b^6)\log(0.2699b)} < \frac{80\log(50.46b)\log(1.184b)}{3\log(1.259b)\log(0.2699b)}.$$

Therefore we obtain $b < 1000$, which contradicts $b > 4000$.

Assume that $b < 2a$ and $c > b^{10}$. By Lemmas 2.5 and 3.1, we have either $n > 1.999b^{-5/8}c^{1/8}$ or $n > 1.298b^{-11/28}c^{3/28}$. If $n > 1.999b^{-5/8}c^{1/8}$, then by (3.8) we have $1.999b^{-5/8}c^{1/8} < 8\varphi(a,b,c)$. Since $a^{1/2}(b-a)^{-1} < 0.5$ (by [8, Lemma 3.5]) and $b(b-a)^2 < b^3/4$, we find that

$$\varphi(a,b,c) < \frac{\log(8.406 \cdot 10^{13}b^3c)\log(0.8215b^{1/2}c)}{\log(4bc)\log(1.0796b^{-3}c)},$$

which together with $c > b^{10}$ implies that

$$\begin{aligned}1.999b^{5/8} &< \frac{8\log(8.406 \cdot 10^{13}b^{13})\log(0.8215b^{10.5})}{\log(4b^{11})\log(1.0796b^7)} \\ &< \frac{156\log(11.78b)\log(0.982b)}{11\log(1.134b)\log(1.011b)} < \frac{156\log(11.78b)}{11\log(1.134b)}.\end{aligned}$$

It follows that $b < 50$, which contradicts Lemma 2.1. If $n > 1.298b^{-11/28}c^{3/28}$, then (3.8) and $c > b^{10}$ together imply that

$$1.298b^{19/28} < \frac{156\log(11.78b)}{11\log(1.134b)},$$

which yields $b < 70$, a contradiction.

(2) Consider secondly the case where both $m$ and $n$ are odd. If $b \geq 2a$ and $c > \max\{20a^2b^{3.5}, b^5\}$, then Lemma 3.1 and (3.8) together imply

$$1.39b^{1/2} < \frac{80\log(50.46b)\log(1.184b)}{3\log(1.259b)\log(0.2699b)},$$

which yields $b < 1300$, a contradiction. If $b < 2a$ and $c > b^{10}$, then we similarly have

$$1.39b^{7/4} < \frac{78\log(11.78b)}{11\log(1.134b)},$$

which yields $b < 10$, a contradiction. This completes the proof of Proposition 3.5. □

*Proof of Theorem* 1.3. Assume first that $d = d_+$. If $n \geq 3$, then by [14, (17), p. 188] we have

$$z \geq w_3 > \frac{c}{3.132\sqrt[4]{bc}}(2t-1)^2 > b^{3/4}c^{7/4} > 4c\sqrt{ab},$$

which contradicts $z = cr + st < 2cr < 3c\sqrt{ab}$. Hence $n \leq 2$. Then, the proof of [14, Lemma 5] shows that $(m,n) \in \{(1,1),(1,2),(2,1),(2,2)\}$. More precisely, if $(m,n) = (1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, then $(z_0, z_1) = (t,s)$, $(t, st-cr)$, $(st-cr, s)$, $(st-cr, st-cr)$, respectively.

Assume secondly that $d > d_+$. The second assertion immediately follows from the last one in Lemma 2.8. Note that if $c = a+b+2r$, then $|z_1| = 1$ by [28, Theorem 8], together with [22, Lemma 6] (see also [14, Lemma 8]). Since parts (2) to (4) have already been proven in [14, Lemma 8], we only have to show that if $v_m = w_n$ for some even integers $m$ and $n$, then $|z_0| \in \{1, cr-st\}$. Suppose, on the contrary, that $|z_0| \notin \{1, cr-st\}$. Putting $d_0 = (z_0^2-1)/c$, we obtain the irregular Diophantine quadruple $\{a, b, d_0, c\}$ with $d_0 < c$. Let $\{a_1, a_2, a_3\} = \{a, b, d_0\}$ with $a_1 < a_2 < a_3$.

If $a_3 \le 20a_1^{2.5}a_2^{3.5}$, then Lemma 2.6 implies that $c > 327a_2^{4.5}a_3^{5.5} \ge 327a^{4.5}b^{5.5} > 14b^{10}$ if $b < 2a$ and that $c > 256a_1^{4.5}a_3^{5.5} \ge 256b^{5.5}$ if $b \ge 2a$. Thus, in any case, Proposition 3.5 shows that $d = d_+$, which contradicts the assumption.

Suppose now that $a_3 > 20a_1^{2.5}a_2^{3.5}$. If $b = a_2$, then we see from Lemmas 2.3 and 2.5 that

$$c > 20a_2^{2.5}a_3^{3.5} > 20b^{2.5}(20b^{3.5})^{3.5} > 7 \cdot 10^5 b^{14.75},$$

which together with Proposition 3.5 (or even [22, Proposition 16]) yields $d = d_+$, a contradiction. If $b = a_3$, then $b > 20a_1^{2.5}a_2^{3.5}$, and Lemma 3.1 shows that $c > 327a^{4.5}b^{5.5}$, which again contradicts Proposition 3.5. □

## 4. Proofs of Theorems 1.5 and 1.6

**Lemma 4.1.** *Assume that $c > b^4$. Suppose that $z = v_m = w_n$ has a solution for some integers $m$ and $n$. Then, $m \equiv n \pmod 2$ and $n > 2.778b^{-3/4}c^{1/4}$.*

*Proof.* The first assertion follows from [14, Lemma 8] with Lemma 2.8(i), (ii).

In case both $m$ and $n$ are even, by Theorem 1.3 we have $|z_0| \in \{1, cr-st\}$, and it is easy to see from Lemma 2.8(i) with $c > b^4$ that $|z_0| = cr - st$ cannot happen. Hence $|z_0| = 1$, which together with the argument in the proof of [6, Lemma 2.4] implies that $m > b^{-1/2}c^{1/2}$. It follows from Lemma 2.4 that $n > 0.8b^{-1/2}c^{1/2} - 0.3 > 2.778b^{-3/4}c^{1/4}$.

In case both $m$ and $n$ are odd, we have congruences (3.6) and (3.7). Suppose that $n \le 2.778b^{-3/4}c^{1/4}$. Since Lemma 2.4 with $c > b^4$ and $n \ge 5$ shows that

$$t|a(m^2-1) - b(n^2-1)| \le btn^2 < 1.0001b^{3/2}c^{1/2}n^2 < 7.719c,$$
$$2rt(m-n) < 0.8001b^{3/2}c^{1/2}n < 0.28c,$$

(3.6) and (3.7) are in fact equalities. Hence we obtain a contradiction as in the proof of Lemma 3.1(2). This completes the proof of Lemma 4.1. □

*Proof of Theorem* 1.5. We will derive a contradiction assuming $d > d_+$. Suppose first that $b < 2a$ and $c \ge 9.864b^4$. Since $a^{1/2}(b-a)^{-1} < 0.5$ and $b(b-a)^2 < b^3/4$, by Lemmas 3.4 and 4.1 we have

$$4.9231b^{1/4} < n < \frac{8\log(8.1931 \cdot 10^{14}b^7)\log(8.1033b^{4.5})}{\log(39.456b^5)\log(10.649b)}.$$

Then, we obtain $b < 21000$, which contradicts Lemma 2.1.

Suppose secondly that $2a \le b \le 12a$ and $c \ge 4.321b^4$. Since $a^{1/2}(b-a)^{1/2} \le b/2$, $a^{1/2}b^{1/2}(b-a)^{-1} \le \sqrt{2}$, and $ab^{-1}(b-a)^{-3} \ge 12^2/(11b)^3$, we have

$$4.0052b^{1/4} < n < \frac{8\log(1.8162 \cdot 10^{14}b^7)\log(10.04b^4)}{\log(17.284b^5)\log(0.12617b)},$$

which contradicts $b > 130000$ by Lemma 2.1.

Suppose finally that $b > 12a$ and $c \geq 721.8b^4$. Since $a^{1/2}(b-a)^{1/2} \leq \sqrt{11}b/12$ and $a^{1/2}b^{1/2}(b-a)^{-1} \leq \sqrt{12}/11$, we have

$$14.399b^{1/4} < n < \frac{8\log(1.677\cdot 10^{16}b^7)\log(373.47b^4)}{\log(2887.2b^5)\log(194.81)},$$

which contradicts $b > 4000$. This completes the proof of Theorem 1.5. □

**Lemma 4.2.** *If $\{a,b,c,d\}$ is a Diophantine quadruple with $a < b < c < d_+ < d$, then $d > \min\{81b^{3.5}c^{4.5}, 256a^{4.5}c^{5.5}\}$.*

*Proof.* Since we know by Theorem 1.3 that if $v_m = w_n$ with $m \equiv n \equiv 0 \pmod 2$, then $|z_0| \in \{1, cr-st\}$, the proof of [22, Lemma 13] implies that $v_4 \neq w_4$. It follows from Lemma 2.5 and [22, Lemma 9] that either $m \geq 6$ or $n \geq 5$. Therefore, the assertion is obtained from Lemma 2.3. □

*Proof of Theorem* 1.6. If $\{e,a,b,c\}$ is an irregular Diophantine quadruple with $e < c$, then Lemma 4.2 shows that $c > 81b^{4.5}$, which contradicts Theorem 1.5. □

## 5. Preparations for Theorem 1.2

Let $\{a,b,c\}$ be a fixed Diophantine triple with $a < b < c$. Suppose that $d$ is a positive integer such that $\{a,b,c,d\}$ forms a Diophantine quadruple. By Theorem 1.3, any fundamental solution $(z_0, z_1)$ of simultaneous Pellian equations (2.1) and (2.2) satisfies

$$(z_0,z_1) \in \{(st-cr, st-cr), (t, st-cr), (st-cr, s), (t,s)\} \tag{5.1}$$

or

$$(z_0,z_1) \in \{(1,1), (-1,-1), (cr-st, cr-st), (-t, cr-st), (cr-st, -s), (-t,-s)\}.$$

Put $z = \sqrt{cd+1}$. Then, as seen in Section 1, there exist two recurrence sequences $\{v\}, \{w\}$ for which

$$z = v_m = w_n \tag{5.2}$$

holds with some positive integers $m, n$.

As shown at the beginning of the proof of Theorem 1.3, $d = d_+$ (which is equivalent to $z = cr + st$) is attained exactly in each of the four cases of (5.1), which are expressed in more detail as follows:

$$\begin{cases} (z_0, x_0; z_1, y_1) = (st-cr, rs-at; st-cr, rt-bs); & (m,n) = (2,2), \\ (z_0, x_0; z_1, y_1) = (t, r; st-cr, rt-bs); & (m,n) = (1,2), \\ (z_0, x_0; z_1, y_1) = (st-cr, rs-at; s, r); & (m,n) = (2,1), \\ (z_0, x_0; z_1, y_1) = (t, r; s, r); & (m,n) = (1,1). \end{cases}$$

Note that by the last assertion of Theorem 1.3 any extension of $\{a,b,c\}$ to a quadruple $\{a,b,c,d\}$ with $d > \max\{a,b,c\}$ is regular in the case where $m$ is odd and $n$ is even. The latter part of Theorem 1.4 implies that there exists at most one extension of $\{a,b,c\}$ to a quadruple $\{a,b,c,d\}$ with $d > d_+$ in each of the remaining three cases in (5.1).

The proof of Theorem 1.4 will be given in Section 10. Before that, we confirm that Theorem 1.4 together with Theorem 1.3 and Lemma 2.8 implies Theorem 1.2.

*Proof of Theorem* 1.2. We denote by $N = N(a, b, c)$ the number of positive integers $d > d_+$ such that $\{a, b, c, d\}$ forms a Diophantine quadruple. It suffices to show that $N \le 10$. Put

$$\begin{aligned}
N_{ee} &= N(1,1) + N(-1,-1) + N(cr-st, cr-st) + N(st-cr, st-cr),\\
N_{oe} &= N(t, st-cr) + N(-t, cr-st),\\
N_{eo} &= N(st-cr, s) + N(cr-st, -s),\\
N_{oo} &= N(t,s) + N(-t,-s).
\end{aligned}$$

Then, we know by Lemma 2.8 that $N_{oe} = 0$ and by Theorems 1.3 and 1.4 that

$$N_{ee} \le 7, \quad N_{eo} \le 3, \quad N_{oo} \le 3,$$

$$N \le N_{ee} + N_{eo} + N_{oo}.$$

We shall consider several cases separately.

In case $c > 4\tau^{-4}ab^2$, by Lemma 2.8 (i), (ii), we have $|z_0|, |z_1| \ne cr - st$. Then

$$N_{eo} = 0, \quad N_{ee} = N(1,1) + N(-1,-1),$$

and so

$$N \le N_{ee} + N_{oo} \le 4 + 3 = 7.$$

In case $4\tau^{-4}ab^2 > c > 4ab^2 (> 4\tau^{-4}a^2b)$, Lemma 2.8(ii) shows that $|z_1| \ne cr-st$. Then

$$N_{ee} = N(1,1) + N(-1,-1),$$

and so

$$N \le N_{ee} + N_{eo} + N_{oo} \le 4 + 3 + 3 = 10.$$

In case $4ab^2 > c > 4\tau^{-4}a^2b$, by Lemma 2.8 (ii), (iii), we have $|z_1| \ne cr - st$ and $|z_0| \ne t$. Then

$$N_{oo} = 0, \quad N_{ee} = N(1,1) + N(-1,-1),$$

and so

$$N \le N_{ee} + N_{eo} \le 4 + 3 = 7.$$

In case $4\tau^{-4}a^2b > c > 4a^2b$, note that $c > 4a^2b$ implies $4ab^2 > 4\tau^{-4}a^2b$ by (2.5). By Lemma 2.8 (iii), we have $|z_0| \ne t$. Then,

$$N_{oo} = 0, \quad N \le N_{ee} + N_{eo} \le 7 + 3 = 10.$$

In case $4a^2b > c$, by Lemma 2.8 (iii), (iv), we have $|z_0| \ne t$ and $|z_1| \ne s$. Then,

$$N_{eo} = N_{oo} = 0, \quad N \le N_{ee} \le 7.$$

This completes the proof of Theorem 1.2. □

## 6. Linear form in three logarithms

An extension of a Diophantine triple $\{a, b, c\}$ leads us to examine the following linear form in three logarithms:

$$\Lambda = m \log \xi - n \log \eta + \log \mu$$

with non-negative integers $m$ and $n$, where

$$\xi = s + \sqrt{ac}, \quad \eta = t + \sqrt{bc}, \quad \mu = \frac{\sqrt{b}\,(x_0\sqrt{c} + z_0\sqrt{a}\,)}{\sqrt{a}\,(y_1\sqrt{c} + z_1\sqrt{b}\,)}.$$

In the following lemma, we estimate the value of $\Lambda$.

**Lemma 6.1.** *Let* $(m,n)$ *be a solution of equation* (5.2). *Assume that* $m>0$ *and* $n>0$. *Then we have*

$$0<\Lambda<\kappa\,\xi^{-2m},$$

*where we may take* $\kappa$ *as*

$$\kappa=\begin{cases}6\sqrt{ac} & \text{if } m\ge 4,\\ 6 & \text{if } |z_0|=1,\\ 2.001c/b & \text{if } z_0=st-cr,\\ 1/(2ab) & \text{if } z_0=t.\end{cases}$$

*Proof.* As shown in [13, Lemma 5], we have

$$0<\Lambda<\kappa_0\,\xi^{-2m},$$

where

$$\kappa_0=\frac{2(c-a)}{(z_0\sqrt{a}+x_0\sqrt{c}\,)^2}=\frac{2(x_0\sqrt{c}-z_0\sqrt{a}\,)^2}{c-a}.$$

If $m\ge 4$, then $d>d_+$, and from Lemma 2.2 we find that

$$(0<)\quad x_0\sqrt{c}-z_0\sqrt{a}<2x_0\sqrt{c}<1.5\sqrt{c}\sqrt[4]{ac}.$$

This together with the fact that $c>4a$ gives us the first desired upper bound for $\kappa$. If $|z_0|=1$, then

$$\kappa_0\le\frac{2(\sqrt{c}+\sqrt{a}\,)}{\sqrt{c}-\sqrt{a}}<2\left(1+\frac{2\sqrt{a}}{2\sqrt{a}-\sqrt{a}}\right)=6.$$

Furthermore, since

$$\begin{aligned}(cr-st)\sqrt{a}+(rs-at)\sqrt{c}&=\frac{c^2-ac-bc-1}{cr+st}\sqrt{a}+\frac{ab+ac-a^2+1}{rs+at}\sqrt{c}\\&<\frac{a(c^2-ac-bc-1)}{2st\sqrt{a}}+\frac{c(ab+ac-a^2+1)}{2at\sqrt{c}}\\&<\frac{(2ac+1)(c-a)}{2a\sqrt{bc}}=\left(1+\frac{1}{2ac}\right)\frac{c-a}{\sqrt{b}},\end{aligned}$$

$$t\sqrt{a}+r\sqrt{c}>2\sqrt{abc},$$

it follows that $\kappa_0$ is at most

$$\begin{aligned}&\frac{2(x_0\sqrt{c}-z_0\sqrt{a})^2}{c-a}<\frac{2.001(c-a)}{b}<\frac{2.001c}{b} && \text{if } z_0=st-cr,\\&\frac{2(c-a)}{(z_0\sqrt{a}+x_0\sqrt{c}\,)^2}<\frac{2c}{4abc}=\frac{1}{2ab} && \text{if } z_0=t.\end{aligned}$$

□

Next, in order to give an upper bound for $\Lambda$, we appeal to a result on linear forms in (three) logarithms due to Matveev [31].

For an algebraic number $\alpha$ of degree $d$ over $\mathbb{Q}$, we define the *absolute logarithmic height* of $\alpha$ by the following formula:

$$\mathrm{h}(\alpha)=\frac{1}{d}\left(\log|a_0|+\sum_{i=1}^{d}\log\max\{1,|\alpha^{(i)}|\}\right),$$

where $a_0$ is the leading coefficient of the minimal polynomial of $\alpha$ over $\mathbb{Z}$ and $\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(d)}$ are the conjugates of $\alpha$ in the field of complex numbers.

**Proposition 6.2.** *Let $\alpha_1, \alpha_2, \alpha_3$ be positive, totally real algebraic numbers such that they are multiplicatively independent. Let $b_1, b_2, b_3$ be rational integers with $b_3 \neq 0$. Consider the following linear form $\Lambda$ in the three logarithms:*

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.$$

*Define real numbers $A_1, A_2, A_3$ by*

$$A_j = \max\{\, D \cdot \mathrm{h}(\alpha_j),\ |\log \alpha_j|\,\} \quad (j = 1, 2, 3),$$

*where*

$$D = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}].$$

*Put*

$$B = \max\Big\{1,\ \max\{\,(A_j/A_3)\,|b_j| : j = 1, 2, 3\,\}\Big\}.$$

*Then we have*

$$\log|\Lambda| > -C(D) A_1 A_2 A_3 \log\big(1.5\,\mathrm{e}\,D \log(\mathrm{e}\,D) \cdot B\big)$$

*with*

$$C(D) = 11796480\,\mathrm{e}^4\, D^2 \log\big(3^{5.5}\,\mathrm{e}^{20.2}\, D^2 \log(\mathrm{e}\,D)\big).$$

**Proposition 6.3.** *Let $(m, n)$ be a solution of equation (5.2) with $m \geq 4$. Then we have*

$$\frac{m}{\log\big(38.92 B(m)\big)} < A(c) C(4) \log \eta,$$

*where $B(m) = \max\{1.001m,\ m + 1\}$ and*

$$A(c) = \begin{cases} 8.1 \log c & \text{if } c = a + b + 2r, \\ 8.6 \log(0.632c) & \text{if } c > a + b + 2r. \end{cases}$$

*Proof.* Set

$$(\alpha_1, \alpha_2, \alpha_3) = (\xi, \mu, \eta), \quad (b_1, b_2, b_3) = (m, 1, -n).$$

Since none of $ab, ac, bc$ are a square, we have $D = 4$. Then we easily see that

$$A_1 = 2 \log \xi, \quad A_3 = 2 \log \eta, \tag{6.1}$$

and

$$\frac{A_1}{A_3} < \frac{\log\big(2.001\sqrt{ac}\,\big)}{\log\big(2\sqrt{bc}\,\big)} < 1.001.$$

In order to estimate $A_2$, denote the conjugates of $\mu$ by

$$\mu_+^+,\ \mu_-^+,\ \mu_+^-,\ \mu_-^-,$$

where the signs are the ones appearing in the numerator and denominator, respectively; for example,

$$\mu_-^+ = \frac{\sqrt{b}\,(x_0\sqrt{c} + |z_0|\sqrt{a}\,)}{\sqrt{a}\,(y_1\sqrt{c} - |z_1|\sqrt{b}\,)}.$$

Note that $\mu_-^+ > \max\{\mu_+^+,\ \mu_-^-\}$ and $\min\{\mu_+^+,\ \mu_-^-\} > \mu_+^-$. Since

$$\mu_+^+ \mu_-^- = \frac{b(c-a)}{a(c-b)} > 1,$$

at least one of $\mu_+^+$ and $\mu_-^-$ is greater than one, and hence $\mu_-^+ > 1$. If $\mu_+^- > 1$, since the minimal polynomial of $\mu$ is

$$a^2(c-b)^2 T^4 + 4a^2 b(c-b) T^3 + 2ab(3ab - ac - bc - c^2) T^2 + 4ab^2(c-a) T + b^2(c-a)^2$$

divided by the greatest common divisor of the coefficients, we have

$$4\mathrm{h}(\mu) \le \log(b^2(c-a)^2).$$

If $\mu_+^- < 1$, since

$$\begin{aligned} \mu_-^+\mu_+^+ &= \frac{b(x_0\sqrt{c}+|z_0|\sqrt{a}\,)^2}{a(c-b)} < \frac{4x_0^2bc}{a(c-b)}, \\ \mu_-^+\mu_-^- &= \frac{b(c-a)(y_1\sqrt{c}+|z_1|\sqrt{b}\,)^2}{a(c-b)^2} < \frac{4y_1^2b(c-a)}{a(c-b)^2}, \\ \mu_-^+\mu_+^+\mu_-^- &= \frac{b\sqrt{b}(c-a)(x_0\sqrt{c}+|z_0|\sqrt{a}\,)}{a\sqrt{a}(c-b)(y_1\sqrt{c}-|z_1|\sqrt{b}\,)} < \frac{4x_0y_1b^{3/2}c(c-a)}{a^{3/2}(c-b)^2}, \end{aligned}$$

we have $4\mathrm{h}(\mu) < \log\bigl(\max\{4x_0^2abc^2,\ 4y_1^2abc^2,\ 4x_0y_1a^{1/2}b^{3/2}c^2\}\bigr)$. Hence, we obtain

$$\mathrm{h}(\mu) < \frac{1}{4}\log P_2,$$

where

$$P_2 = \max\bigl\{\, b^2c^2, 4x_0^2abc^2, 4y_1^2abc^2, 4x_0y_1a^{1/2}b^{3/2}c^2 \,\bigr\}.$$

In the case where $c = a+b+2r$, since

$$c > \begin{cases} 9b/4 & \text{if } b < 4a, \\ 25b/16 > 25a/4 & \text{if } 4a < b < 16a, \\ 25a & \text{if } b > 16a, \end{cases}$$

we see from $x_0 = y_1 = 1$ (by [28, Theorem 8]) that $P_2 < c^4$. In the case where $c > a+b+2r$, Lemma 2.2 together with $c > 4ab+b > 20000$ implies

$$\begin{aligned} 4x_0^2ab &< 4\cdot 0.7097^2 \cdot \left(\frac{\sqrt{c}}{2}\right)^{1/2} \cdot \frac{c}{4}\cdot c^{1/2} < 0.357c^{7/4}, \\ 4y_1^2ab &< 4\cdot 0.7072^2 \cdot \frac{c}{4}\cdot\left(\frac{c}{5}\right)^{1/2}\cdot c^{1/2} < 0.224c^2, \\ 4x_0y_1a^{1/2}b^{3/2} &< 4\cdot 0.7097\cdot 0.7072\cdot\left(\frac{c}{4}\right)^{3/4}\cdot\frac{c}{5}\cdot c^{1/2} < 0.142c^{9/4}, \end{aligned}$$

which yield $P_2 < 0.142c^{17/4}$. Therefore, we obtain

$$A_2 = \max\bigl\{4\,\mathrm{h}(\mu),\ |\log\mu|\bigr\} < \begin{cases} 4\log c & \text{if } c = a+b+2r, \\ (17/4)\log(0.632c) & \text{if } c > a+b+2r. \end{cases} \tag{6.2}$$

Furthermore, since $A_3 > 2\log(\sqrt{bc}\,) > (6/5)\log c$ by Theorem 1.5, we have

$$\begin{aligned} B &= \max\bigl\{(A_1/A_3)\,m, A_2/A_3, n\bigr\} \\ &\le \max\bigl\{1.001\,m,\ 85/24,\ m+1\bigr\} = \max\bigl\{1.001m,\ m+1\bigr\}. \end{aligned} \tag{6.3}$$

Now Proposition 6.2 shows that

$$\log\Lambda > -A_1A_2A_3C(4)\log(38.92B),$$

which together with (6.1), (6.2), (6.3) and Lemma 6.1 gives us the desired inequality. $\square$

## 7. An exponential gap principle

In this section, we consider three solutions of equation (5.2) belonging to the same class of solutions. We write them as $(m_0, n_0)$, $(m_1, n_1)$, $(m_2, n_2)$ with $m_0 < m_1 < m_2$. Assume that

$$m_1 \geq 4.$$

For $0 \leq i \leq 2$, we put

$$\Lambda_i = m_i \log \xi - n_i \log \eta + \log \mu.$$

First, we make use of an idea of Okazaki (cf. [4, Lemma 2.2]) to find a very sharp lower bound for $m_2 - m_1$ in terms of $m_0$.

**Lemma 7.1.** *Assume that $v_{m_0}$ is positive. Then we have*

$$m_2 - m_1 > \Lambda_0^{-1} \Delta \log \eta,$$

*where*

$$\Delta = \begin{vmatrix} n_1 - n_0 & n_2 - n_1 \\ m_1 - m_0 & m_2 - m_1 \end{vmatrix} > 0.$$

*In particular, if $m_0 > 0$ and $n_0 > 0$, then*

$$m_2 - m_1 > \kappa^{-1} (4ac)^{m_0} \Delta \log \eta.$$

*Proof.* Firstly, we note that the second inequality follows from Lemma 6.1 with the fact that $\xi > 2\sqrt{ac}$. Although the proof of the first inequality proceeds along similar lines to that of [23, Lemma 5.1], we give it briefly for the sake of completeness.

The equation $v_m = w_n$ is expressed as

$$\frac{X_+ \xi^m - X_- \xi^{-m}}{\sqrt{a}} = \frac{Y_+ \eta^n - Y_- \eta^{-n}}{\sqrt{b}} \quad (> 0),$$

where $X_+, X_-, Y_+, Y_-$ are positive numbers given by

$$X_+ = x_0\sqrt{c} + z_0\sqrt{a}, \quad X_- = x_0\sqrt{c} - z_0\sqrt{a},$$
$$Y_+ = y_1\sqrt{c} + z_1\sqrt{b}, \quad Y_- = y_1\sqrt{c} - z_1\sqrt{b}.$$

Now, we consider the curve defined by $F = F(p, q) = 0$ in two variables $p$ and $q$, where

$$F = Y_+ \,\mathrm{e}^q - Y_- \,\mathrm{e}^{-q} - (X_+ \,\mathrm{e}^p - X_- \,\mathrm{e}^{-p}) \sqrt{\frac{b}{a}}.$$

Note that the three points $(p, q) = (p_i, q_i)$ with $i = 0, 1, 2$ are on the curve, where

$$(p_i, q_i) = (m_i \log \xi,\ n_i \log \eta).$$

Since

$$\frac{\partial F}{\partial q} = Y_+ \,\mathrm{e}^q + Y_- \,\mathrm{e}^{-q} > 0,$$

we may implicitly differentiate $F(p, q) = 0$ to find

$$(Y_+ \,\mathrm{e}^q + Y_- \,\mathrm{e}^{-q}) \cdot \frac{\mathrm{d}q}{\mathrm{d}p} = (X_+ \,\mathrm{e}^p + X_- \,\mathrm{e}^{-p}) \sqrt{\frac{b}{a}}, \tag{7.1}$$

which yields

$$\frac{\mathrm{d}q}{\mathrm{d}p} = \sqrt{\frac{b}{a}} \, \frac{X_+ \,\mathrm{e}^p + X_- \,\mathrm{e}^{-p}}{Y_+ \,\mathrm{e}^q + Y_- \,\mathrm{e}^{-q}} = \sqrt{\frac{(X_+ \,\mathrm{e}^p - X_- \,\mathrm{e}^{-p})^2 + 4(c - a)}{(X_+ \,\mathrm{e}^p - X_- \,\mathrm{e}^{-p})^2 + 4(c \cdot a/b - a)}} > 1. \tag{7.2}$$

Moreover, implicitly differentiating (7.1) we have

$$(Y_+ \mathrm{e}^q + Y_- \mathrm{e}^{-q}) \cdot \frac{\mathrm{d}^2 q}{\mathrm{d}p^2} + (Y_+ \mathrm{e}^q - Y_- \mathrm{e}^{-q}) \left(\frac{\mathrm{d}q}{\mathrm{d}p}\right)^2 = (X_+ \mathrm{e}^p - X_- \mathrm{e}^{-p})\sqrt{\frac{b}{a}}.$$

Since $Y_+ e^q - Y_- e^{-q} = 2\sqrt{b}\, w_n = 2\sqrt{b}\, v_m$ is positive for $m \in \{m_0, m_1, m_2\}$ by assumption, (7.1) and (7.2) together imply that

$$\frac{\mathrm{d}^2 q}{\mathrm{d}p^2} = \left(1 - \left(\frac{\mathrm{d}q}{\mathrm{d}p}\right)^2\right) \frac{Y_+ \mathrm{e}^q - Y_- \mathrm{e}^{-q}}{Y_+ \mathrm{e}^q + Y_- \mathrm{e}^{-q}} < 0. \tag{7.3}$$

Combining (7.2) with (7.3) shows that

$$0 < \frac{q_1 - q_0}{p_1 - p_0} - \frac{q_2 - q_1}{p_2 - p_1} < \frac{q_1 - q_0}{p_1 - p_0} - 1 < \frac{\Lambda_0}{p_1 - p_0},$$

from which the first desired inequality follows. □

We can combine Lemma 7.1 with Proposition 6.3 to show that the $d$ corresponding to the solution $(m_0, n_0)$ is nothing but $d_+$. In particular, the first assertion of Theorem 1.4 is proved.

**Proposition 7.2.** *Suppose that there exist three positive solutions* $(x_{(i)}, y_{(i)}, z_{(i)})$ $(i \in \{0, 1, 2\})$ *to the system of Pellian equations* (2.1) *and* (2.2) *with* $z_{(0)} < z_{(1)} < z_{(2)}$ *belonging to the same class of solutions. Put* $z_{(i)} = v_{m_i} = w_{n_i}$ $(i \in \{0, 1, 2\})$.

(1) $m_0 \le 2$.
(2) *If* $m_0 = 2$ *with* $z_0 = st - cr$, *then* $c > a + b + 2r$.

*Proof.* Suppose $m_0 > 2$. Then, $m_0 \ge 4$ by Lemma 2.5. Since the left-hand side of the inequality in Proposition 6.3 is an increasing function of $m$ and $\eta < 2c - 1$, Proposition 6.3 and Lemma 7.1 together imply the following (note that $B(m_2) = 1.001 m_2$ by $m_2 > \kappa^{-1}(4ac)^4 \log \eta > 1000$):

If $c = a + b + 2r$, then

$$\frac{\frac{128}{3} a^4 c^4}{\log(38.96 \cdot \frac{128}{3} a^4 c^4 \log(2c - 1))} < 8.1 C(4) \log c,$$

which yields $c < 2100$, contradicting Lemma 2.1.

If $c > a + b + 2r$, then

$$\frac{\frac{128}{3} a^{3.5} c^{3.5}}{\log(38.96 \cdot \frac{128}{3} a^{3.5} c^{3.5} \log(2c - 1))} < 8.6 C(4) \log(0.632c),$$

which yields $c < 6400$, contradicting Lemma 2.1 with $c > 5b$.

(2) If $c = a + b + 2r$, then in a fashion similar to (1) we find that

$$\frac{\frac{128}{3} a^2 c^2}{\log\bigl(38.96 \cdot \frac{128}{3} a^2 c^2 \log(2c - 1)\bigr)} < 8.1 C(4) \log c.$$

Observe that the left-hand side of this inequality is regarded as an increasing function of $c$. Thus, we have $c < 6.16 \cdot 10^6$. This together with the above inequality yields $ac < 6.16 \cdot 10^6$. Therefore $c < 2500$, which contradicts $c > b > 4000$. Then, one can easily reduce the upper bound $M = 10^{17}$ for $m_2$ obtained from Propositon (6.3), using the reduction method (cf. [2, Lemma], [17, Lemma(5a)]), and get $m_2 \le 7$ in each case, which contradicts Lemma (7.1) with $m_0 = 2$. □

## 8. A gap principle via Padé approximation

Here, we give a gap principle which ensures that $m_2$ is not too large with respect to $m_1$. A similar principle appears in [3, Corollary 3.3].

**Theorem 8.1.** *Let* $a$, $b$, $c$ *be integers with* $0 < a < b < c$ *and let* $a_1 = a(c-b)$, $a_2 = b(c-a)$, $N = abz^2$, *where* $z$ *is a solution to the system of Pellian equations* (2.1) *and* (2.2). *Put* $u = c-b$, $v = c-a$, *and* $w = b-a$. *Assume that* $N \geq 10^5 a_2$. *Then the numbers*

$$\theta_1 = \sqrt{1 + a_1/N}, \quad \theta_2 = \sqrt{1 + a_2/N}$$

*satisfy*

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > \left(\frac{32.01 a_1' a_2 u N}{a_1}\right)^{-1} q^{-\lambda}$$

*for all integers* $p_1$, $p_2$, $q$ *with* $q > 0$, *where* $a_1' = \max\{a_1, a_2 - a_1\}$ *and*

$$\lambda = 1 + \frac{\log\left(\frac{16 a_1' a_2 u N}{a_1}\right)}{\log\left(\frac{1.6874 N^2}{a_1 a_2 (a_2 - a_1) u v w}\right)}.$$

*Proof.* The proof needs a general lemma stated as follows.

**Lemma 8.2** (cf. [3, Lemma 3.1])**.** *Let* $\theta_1, \theta_2$ *be arbitrary real numbers and* $\theta_0 = 1$. *Assume that there exist positive real numbers* $l$, $p$, $L$, *and* $P$ *with* $L > 1$ *such that for each positive integer* $k$, *we can find integers* $p_{ijk}$ $(0 \leq i, j \leq 2)$ *with non-zero determinant,*

$$|p_{ijk}| \leq pP^k \ \ (0 \leq i, j \leq 2)$$

*and*

$$\left|\sum_{j=0}^{2} p_{ijk}\theta_j\right| \leq lL^{-k} \ \ (0 \leq i \leq 2).$$

*Then*

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \ \left|\theta_2 - \frac{p_2}{q}\right|\right\} > cq^{-\lambda}$$

*holds for all integers* $p_1, p_2, q$ *with* $q > 0$, *where*

$$\lambda = 1 + \frac{\log P}{\log L} \quad \text{and} \quad c^{-1} = 4pP\left(\max\{1, 2l\}\right)^{\lambda - 1}.$$

For $0 \leq i$, $j \leq 2$, let $p_{ij}(x)$ be the polynomial defined by

$$p_{ij}(x) = \sum_{ij} \binom{k+1/2}{h_j} (1 + a_j x)^{k - h_j} x^{h_j} \prod_{l \neq j} \binom{-k_{il}}{h_l} (a_j - a_l)^{-k_{il} - h_l},$$

where $k_{il} = k + \delta_{il}$ with $\delta_{il}$ the Kronecker delta, $\sum_{ij}$ denotes the sum over all non-negative integers $h_0, h, h_2$ satisfying $h_0 + h + h_2 = k_{ij} - 1$, and $\prod_{l \neq j}$ denotes the product from $l = 0$ to $l = 2$ omitting $l = j$. Then we have

$$p_{ij}(1/N) = \sum_{ij} \binom{k+1/2}{h_j} C_{ij}^{-1} \prod_{l \neq j} \binom{-k_{il}}{h_l},$$

where

$$C_{ij} = \frac{N^k}{(N + a_j)^{k - h_j}} \prod_{l \neq j} (a_j - a_l)^{k_{il} + h_l}.$$

Recall that $(a_0, a_1, a_2) = (0, au, bv)$ and $N = abz^2$. If $j = 0$, then

$$|C_{i0}| = \frac{a^{k_{i1}+h_0+h_1-k} b^{k_{i2}+h_0+h_2-k} u^{k_{i1}+h_1} v^{k_{i2}+h_2} N^k}{z^{2k-2h_0}}.$$

Since $k_{il} + h_0 + h_1 - k \le k_{il} + k_{i0} - 1 - k \le k$ and $k_{il} + h_l \le k_{il} + k_{i0} - 1 \le 2k$ for $l = 1, 2$, we have $a_1^k a_2^k u^k v^k N^k C_{i0}^{-1} = a^k b^k u^{2k} v^{2k} N^k C_{i0}^{-1} \in \mathbb{Z}$. If $j = 1$, then noting that

$$N + a_1 = a(bz^2 + c - b) = acx^2,$$

we have

$$|C_{i1}| = \frac{a^{k_{i0}+h_0+h_1-k} u^{k_{i0}+h_0} (b-a)^{k_{i2}+h_2} c^{k_{i2}+h_1+h_2-k} N^k}{x^{2k-2h_1}},$$

which implies that $a_1^k (a_2 - a_1)^k u^k w^k N^k C_{i1}^{-1} \in \mathbb{Z}$. Similarly, if $j = 2$, then $a_2^k (a_2 - a_1)^k v^k w^k N^k C_{i2}^{-1} \in \mathbb{Z}$. To sum up, we obtain

$$\{a_1 a_2 (a_2 - a_1) uvwN\}^k C_{ij}^{-1} \in \mathbb{Z}$$

for all $i$, $j$. It follows from the proof of [18, Theorem 2.5] that

$$p_{ijk} := 2^{-1} \{4a_1 a_2 (a_2 - a_1) uvwN\}^k p_{ij}(1/N) \in \mathbb{Z}.$$

Therefore, as in the proof of [22, Theorem 21], we see from the assumption $N \ge 10^5 a_2$ that

$$|p_{ijk}| < pP^k \quad \text{and} \quad \left| \sum_{j=0}^{2} p_{ijk} \theta_j \right| < lL^{-k},$$

where

$$\begin{aligned}
p &= \frac{1}{2}\left(1 + \frac{a_1'}{2N}\right)^{1/2} < 0.5001, \\
P &= \frac{32\left(1 + \frac{3a_2 - a_1}{2N}\right) a_1 a_2 (a_2 - a_1)^2 uN}{\zeta} < \frac{16 a_1' a_2 uN}{a_1}, \\
&\left(\zeta = \begin{cases} a_1^2(2a_2 - a_1) & \text{if } a_2 - a_1 \ge a_1, \\ (a_2 - a_1)^2 (a_1 + a_2) & \text{if } a_2 - a_1 < a_1 \end{cases}\right), \\
l &= \frac{27}{64}\left(1 - \frac{a_2}{N}\right)^{-1} < 0.4219, \\
L &= \frac{1}{4a_1 a_2 (a_2 - a_1) uvwN} \frac{27}{4} \left(1 - \frac{a_2}{N}\right)^2 N^3 > \frac{1.6874 N^2}{a_1 a_2 (a_2 - a_1) uvw}.
\end{aligned}$$

Now the assertion follows from Lemma 8.2. □

**Lemma 8.3.** *Let $(x_{(i)}, y_{(i)}, z_{(i)})$ be positive solutions to the system of Pellian equations* (2.1) *and* (2.2) *for $i \in \{1, 2\}$, and let $\theta_1$, $\theta_2$ be as in Theorem* 8.1 *with $z = z_{(1)}$. Then we have*

$$\max\left\{ \left|\theta_1 - \frac{acy_{(1)}y_{(2)}}{abz_{(1)}z_{(2)}}\right|, \ \left|\theta_2 - \frac{bcx_{(1)}x_{(2)}}{abz_{(1)}z_{(2)}}\right| \right\} < \frac{c^{3/2}}{2a^{3/2}} z_{(2)}^{-2}.$$

*Proof.*
$$\left|\sqrt{1+\frac{a_1}{N}}-\frac{p_1}{q}\right| = \frac{y_{(1)}\sqrt{c}}{bz_{(1)}z_{(2)}}\left|z_{(2)}\sqrt{b}-y_{(2)}\sqrt{c}\right| < \frac{(c-b)\sqrt{c}\,y_{(1)}}{2b\sqrt{b}\,z_{(1)}z_{(2)}^2} < \frac{c^{3/2}}{2b^{3/2}}z_{(2)}^{-2},$$
$$\left|\sqrt{1+\frac{a_2}{N}}-\frac{p_2}{q}\right| = \frac{x_{(1)}\sqrt{c}}{az_{(1)}z_{(2)}}\left|z_{(2)}\sqrt{a}-x_{(2)}\sqrt{c}\right| < \frac{(c-a)\sqrt{c}\,x_{(1)}}{2a\sqrt{a}\,z_{(1)}z_{(2)}^2} < \frac{c^{3/2}}{2a^{3/2}}z_{(2)}^{-2}.$$
□

**Proposition 8.4.** *Suppose that $\{a,b,c,d_i\}$ for $i\in\{1,2\}$ are Diophantine quadruples with $a<b<c<d_1<d_2$ and $x_{(i)}$, $y_{(i)}$, $z_{(i)}$ are positive integers such that $ad_i+1=x_{(i)}^2$, $bd_i+1=y_{(i)}^2$, $cd_i+1=z_{(i)}^2$ for $i\in\{1,2\}$. Denote by $z_{(i)}=v_{m_i}=w_{n_i}$ the corresponding sequences to $z_{(i)}$ for $i\in\{1,2\}$.*

(1) *If $n_1\ge 8$, then*
$$n_2 < \frac{(n_1+1.1)(3.5001n_1+4.2502)}{0.4999n_1-3.7501}-1.1 < 148n_1.$$

(2) *If $n_1=7$ with $z_1=s$, then $n_2\le 462(\le 66n_1)$.*

*Proof.* Taking $N=abz_{(1)}^2$, $p_1=acy_{(1)}y_{(2)}$, $p_2=bcx_{(1)}x_{(2)}$, $q=abz_{(1)}z_{(2)}$, we see from Theorem 8.1 and Lemma 8.3 that
$$z_{(2)}^{2-\lambda} < \frac{c^{3/2}}{2a^{3/2}}a^{\lambda}b^{\lambda}z_{(1)}^{\lambda}\cdot 32.01b^3c^2z_{(1)}^2 < 16.005a^{\lambda-3/2}b^{\lambda+3}c^{7/2}z_{(1)}^{\lambda+2}. \tag{8.1}$$

(1) From $0.5655b^{-1/4}c^{3/4}<w_1<1.4144b^{1/4}c^{5/4}$ we know that
$$0.5655\cdot 1.99975^{n_1-1}b^{n_1/2-3/4}c^{n_1/2+1/4} < w_{n_1} < 1.4144\cdot 2.0001^{n_1-1}b^{n_1/2-1/4}c^{n_1/2+3/4}.$$

Since $z_{(1)}=w_{n_1}$,
$$\frac{16a_1'a_2uN}{a_1} < 16b^3c^2z_{(1)}^2 < 32.01\cdot 2.0001^{2n_1-2}b^{n_1+5/2}c^{n_1+7/2} < (4.001bc)^{n_1+3.5},$$
and
$$\frac{1.6874N^2}{a_1a_2(a_2-a_1)uvw} = \frac{1.6874abz_{(1)}^4}{c(b-a)^2(c-b)^2(c-a)^2} > 0.172\cdot 1.99975^{4n_1-4}b^{2n_1-4}c^{2n_1-4} > (3.999bc)^{2n_1-4},$$
we have
$$\lambda < 1+\frac{(n_1+3.5)\log(4.001bc)}{(2n_1-4)\log(3.999bc)} < 1+\frac{0.5001n_1+1.7501}{n_1-2}.$$
It follows from (8.1) that
$$z_{(2)}^{0.4999n_1-3.7501} < 16.005^{n_1-2}a^{4.7501-0.9999n_1}b^{5.0001n_1-6.2499}c^{3.5n_1-7}z_{(1)}^{3.5001n_1-4.2499}.$$
The lower estimate $z_{(1)}=w_{n_1}>1.99975^{n_1-1.82255}(bc)^{0.5n_1-0.25}$ shows that
$$16.005^{n_1-2}b^{5.0001n_1-6.2499}c^{3.5n_1-7} < 16.005^{n_1-2}(bc)^{4.25005n_1-6} < z_{(1)}^{8.5001}.$$
Hence, we obtain
$$z_{(2)} < z_{(1)}^{\sigma} \quad \text{with } \sigma=\frac{3.5001n_1+4.2502}{0.4999n_1-3.7501}. \tag{8.2}$$

Note that $n_1 \geq 8$ implies $\sigma > 1$. If $n_2 \geq n_1\sigma + 1.1(\sigma - 1)$, then

$$\frac{z_{(2)}}{z_{(1)}^{\sigma}} \geq \left(\frac{2\sqrt{b}}{y_1\sqrt{c} + z_1\sqrt{b}}\right)^{\sigma-1} (t+\sqrt{bc})^{n_2-n_1\sigma} \frac{1 - A(t+\sqrt{bc})^{-2n_1\sigma}}{(1 - A(t+\sqrt{bc})^{-2n_1})^{\sigma}},$$

where $A = (y_1\sqrt{c} - z_1\sqrt{b})/(y_1\sqrt{c} + z_1\sqrt{b})$. Since by Theorem 1.5 and Lemma 2.2 we have

$$\begin{aligned}
&\left(\frac{2\sqrt{b}}{y_1\sqrt{c} + z_1\sqrt{b}}\right)^{\sigma-1} (t+\sqrt{bc})^{n_2-n_1\sigma} \\
&\qquad > \left(\frac{1}{0.7072 \cdot 18.757^{1/5}}\right)^{\sigma-1} \cdot 2^{n_2-n_1\sigma}(bc)^{(n_2-n_1\sigma)/2-11(\sigma-1)/20} \\
&\qquad \geq \left(\frac{2^{1.1}}{0.7072 \cdot 18.757^{1/5}}\right)^{\sigma-1} > 1,
\end{aligned}$$

we see that

$$\frac{z_{(2)}}{z_{(1)}^{\sigma}} > \frac{1 - A(t+\sqrt{bc})^{-2n_1\sigma}}{(1 - A(t+\sqrt{bc})^{-2n_1})^{\sigma}} > 1,$$

where the last inequality follows from

$$1 - AX^{\sigma} > (1 - AX)^{\sigma},$$

which holds for $0 < X < 1/(A+1)$. Therefore, the desired inequality is derived from (8.2).

(2) Since

$$w_7 = (64b^3c^3 + 80b^2c^2 + 24bc + 1)(cr + st) + 2(16b^2c^2 + 16bc + 3)cr$$

we have

$$128a^{1/2}b^{7/2}c^4 < w_7 < 128.02a^{1/2}b^{7/2}c^4.$$

Hence, if $z_{(1)} = w_7$, then

$$\begin{aligned}
\frac{16a_1'a_2uN}{a_1} &< 2.6223 \cdot 10^5 ab^{10}c^{10}, \\
\frac{1.6874N^2}{a_1a_2(a_2 - a_1)uvw} &> 4.5295 \cdot 10^8 a^3b^{13}c^{11},
\end{aligned}$$

and

$$\lambda < 1 + \frac{\log(2.6223 \cdot 10^5 ab^{10}c^{10})}{\log(4.5295 \cdot 10^8 a^3b^{13}c^{11})} < \frac{21}{11}.$$

It follows from (8.1) that

$$z_{(2)} < 7.2404 \cdot 10^{103} a^{26} b^{409/2} c^{421/2},$$

which together with $z_{(2)} = w_{n_2} > 0.5655 \cdot 1.99975^{n_2-1} b^{n_2/2-3/4} c^{n_2/2+1/4}$ yields

$$1.99975^{n_2} b^{n_2/2} c^{n_2/2} < 2.5604 \cdot 10^{104} b^{883/4} c^{883/4}.$$

Therefore we obtain $n_2 < 925/2$, that is, $n_2 \leq 462 (\leq 66n_1)$. □

## 9. Linear form in two logarithms

In this section, we consider the following linear form in two logarithms:

$$\Gamma = \Lambda_2 - \Lambda_1 = j \log \xi - k \log \eta,$$

where $j, k$ are positive integers given by

$$j = m_2 - m_1, \quad k = n_2 - n_1.$$

Note that $\Gamma$ is non-zero as $\xi, \eta$ are multiplicatively independent. Since, by Lemma 6.1, both linear forms $\Lambda_1, \Lambda_2$ are in the range $(0, \kappa\, \xi^{-2m_1})$, we have

$$0 < |\Gamma| < \kappa\, \xi^{-2m_1}. \tag{9.1}$$

In order to obtain a lower bound for $|\Gamma|$, we use the following result due to Laurent.

**Proposition 9.1** ([29, Theorem 2]). *Let $\gamma_1$ and $\gamma_2$ be multiplicatively independent algebraic numbers with $|\gamma_1| \geq 1$ and $|\gamma_2| \geq 1$. Let $b_1$ and $b_2$ be positive integers. Consider the linear form in two logarithms:*

$$\Gamma = b_2 \log \gamma_2 - b_1 \log \gamma_1,$$

*where $\log \gamma_1, \log \gamma_2$ are any determinations of the logarithms of $\gamma_1, \gamma_2$ respectively. Let $\rho$ and $\mu$ be real numbers with $\rho > 1$ and $1/3 \leq \mu \leq 1$. Set*

$$\sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad \lambda = \sigma \log \rho.$$

*Let $a_1, a_2$ be real numbers such that*

$$a_i \geq \max\{\, 1,\ \rho\, |\log \gamma_i| - \log |\gamma_i| + 2D\, \mathrm{h}(\gamma_i)\, \} \quad (i = 1, 2),$$
$$a_1 a_2 \geq \lambda^2,$$

*where*

$$D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}] \,/\, [\mathbb{R}(\gamma_1, \gamma_2) : \mathbb{R}]\,.$$

*Let $h$ be a real number such that*

$$h \geq \max\left\{ D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.75\right) + 0.06,\ \lambda,\ \frac{D \log 2}{2} \right\} + \log\rho.$$

*Put*

$$H = \frac{h}{\lambda}, \quad \omega = 2 + 2\sqrt{1 + \frac{1}{4H^2}}, \quad \theta = \sqrt{1 + \frac{1}{4H^2}} + \frac{1}{2H}.$$

*Then we have*

$$\log |\Gamma| \geq -C a_1 a_2 h^2 - \sqrt{\omega\theta}\, h - \log\left(C' a_1 a_2 h^2\right)$$

*with $C = C_0 \mu / (\lambda^3 \sigma)$ and $C' = \sqrt{C_0 \omega\theta / \lambda^6}$, where*

$$C_0 = \left( \frac{\omega}{6} + \frac{1}{2}\sqrt{ \frac{\omega^2}{9} + \frac{8\lambda \omega^{5/4}\theta^{1/4}}{3\sqrt{a_1 a_2}\, H^{1/2}} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{\lambda\omega}{H} } \right)^2.$$

**Proposition 9.2.** *If $z = v_{m_i} = w_{n_i}$ $(i \in \{1, 2\})$ has a solution with $m_1 < m_2$, then*

$$\frac{2m_1}{\log \eta} < \frac{\mathcal{C}\mu(\rho+3)^2}{\lambda^3 \sigma} h^2 + \frac{4\sqrt{\omega\theta}h + 8\log h + 4\log\left(\sqrt{\mathcal{C}\omega\theta}\lambda^{-3}(\rho+3)^2\,\right)}{\log^2 16000} + 1,$$

*where* $\rho = 7.9$, $\mu = 0.62$,

$$h = 4\log\left(\frac{2j}{\log\eta} + 1\right) + 4\log\left(\frac{\lambda}{\rho+3}\right) + 7.06 + \log\rho,$$

$$\mathcal{C} = \left(\frac{\omega}{6} + \frac{1}{2}\sqrt{\frac{\omega^2}{9} + \frac{16\lambda\omega^{5/4}\theta^{1/4}}{3(\rho+3)H^{1/2}\log(16000)} + \frac{16\lambda\omega}{3(\rho+3)H\log(16000)}}\right)^2$$

*and* $\sigma$, $\lambda$, $H$, $\omega$, $\theta$ *are as in Proposition* 9.1.

*Proof.* We apply Proposition 9.1 with

$$(\gamma_1, \gamma_2) = (\eta, \xi), \quad (b_1, b_2) = (k, j).$$

Set $\rho = 7.9$ and $\mu = 0.62$. Since $\Gamma$ is small and $D = 4$, we may take

$$a_1 = (\rho+3)\log\eta, \quad a_2 = (\rho+3)\log\xi,$$

and

$$h = 4\log\left(\frac{2j}{\log\eta} + 1\right) + 4\log\left(\frac{\lambda}{\rho+3}\right) + 7.06 + \log\rho.$$

We easily check that all the inequality assumptions on $a_1, a_2, h$ hold. Since $a_1 > a_2 > (\rho+3)\log(2\sqrt{ac})$, it follows from Proposition 9.1 that

$$\log|\Gamma| > -\frac{\mathcal{C}\mu}{\lambda^3\sigma}(\rho+3)^2(\log\xi)(\log\eta)h^2 - \sqrt{\omega\theta}h$$
$$- \log\left(\sqrt{\mathcal{C}\omega\theta}\lambda^{-3}(\rho+3)^2(\log\xi)(\log\eta)h^2\right).$$

Now the desired inequality can be deduced from this inequality with (9.1). □

## 10. Proof of Theorem 1.4

**Lemma 10.1.** *Suppose that* $\{a, b, c, d\}$ *is a Diophantine quadruple with* $a < b < c < d_+ < d$. *Assume that* $z = v_m = w_n$ *has a solution with* $z_1 \in \{s, st - cr\}$. *Then,* $m \geq 7$ *and* $n \geq 7$.

*Proof.* By Lemma 2.5 we have $\min\{m, n\} \geq 4$. Note that

$$v_{\alpha_1} = (16a^2c^2 + 12ac + 1)(cr + st) + 4(2ac + 1)cr \tag{10.1}$$

for $(\alpha_1, z_0) \in \{(5, t), (6, st - cr)\}$. It is easy to see that $v_{\alpha_1} > w_{\beta_1}$ for $(\beta_1, z_1) \in \{(3, s), (4, st - cr)\}$, where

$$w_{\beta_1} = (4bc + 1)(cr + st) + 2cr.$$

We know from the proof of [22, Lemma 13] that $v_4 \neq w_4$. Equalities (10.1) and

$$w_{\beta_2} = (16b^2c^2 + 12bc + 1)(cr + st) + 4(2bc + 1)cr$$

for $(\beta_2, z_1) \in \{(5, s), (6, st - cr)\}$ immediately imply that $v_{\alpha_1} < w_{\beta_2}$. Thus, we obtain $m \geq 7$. It is also easy to check from

$$v_{\alpha_3} = (256a^4c^4 + 448a^3c^3 + 240a^2c^2 + 40ac + 1)(cr + st)$$
$$+ 8(16a^3c^3 + 24a^2c^2 + 10ac + 1)cr$$

that $v_{\alpha_3} > w_{\beta_2}$ for $(\alpha_3, z_0) \in \{(9, t), (10, st - cr)\}$. Hence, we only have to prove that $v_{\alpha_2} \neq w_{\beta_2}$ for $(\alpha_2, z_0) \in \{(7, t), (8, st - cr)\}$, where

$$v_{\alpha_2} = (64a^3c^3 + 80a^2c^2 + 24ac + 1)(cr + st) + 2(16a^2c^2 + 16ac + 3)cr.$$

Suppose on the contrary that $v_{\alpha_2} = w_{\beta_2}$. Then, it is clear that $b > 2a$. Since we know by Proposition 7.2(2) that $c > a + b + 2r$, we see from [28, Lemma 4] that $c > 4ab + b > 20000$, which implies that $v_{\alpha_2} < 64.01a^3c^3(cr + st) + 5cr + st$, while $w_{\beta_2} > 16b^2c^2(cr + st) + 5cr + st$. Hence we obtain

$$4.001a^3c > b^2. \tag{10.2}$$

Considering $v_{\alpha_2} \equiv w_{\beta_2} \pmod{4c^2}$ we have $6(b - 2a)st \equiv r \pmod{2c}$; in other words,

$$6(b - 2a) \equiv r(st - cr) \pmod{c}. \tag{10.3}$$

If $6(b - 2a) \geq c$, then $c > 4ab$ shows that $(3 - 2a)b > 6a$; that is, $a = 1$, which is incompatible with (10.2), $c \leq 6b - 12a$, and $b > 4000$. Hence $6(b - 2a) < c$. Since

$$|2r(st - cr)| = 2r(cr - st) < \frac{2rc^2}{2st} < 2c,$$

we obtain from (10.3) the equality $6(b - 2a) = c - r(cr - st)$, where the right-hand side is at least $c - cr/(2\sqrt{ab}) > 0.499c$. It follows from $c > 4ab$ that $a \leq 3$. Moreover, inequality (10.2) shows that

$$b < \frac{4.001 \cdot 6a^3}{0.499} < 1400,$$

which contradicts $b > 4000$. This proves $v_{\alpha_2} \neq w_{\beta_2}$. □

*Proof of Theorem* 1.4. As mentioned just before Proposition 7.2, it suffices to show the second assertion, in other words, $N(z_0, z_1) \leq 1$ for $z_0 \in \{t, st - cr\}$. Suppose on the contrary that $N(z_0, z_1) \geq 2$ for $z_0 \in \{t, st - cr\}$. We observe that this will lead to a contradiction. Now, as observed in Section 5, we have three solutions of equation (5.2), say $(m_i, n_i)$ $(i \in \{0, 1, 2\})$, satisfying $m_0 < m_1 < m_2$, $m_1 \geq 7$, and $n_1 \geq 7$ (by Lemma 10.1), where $m_0 = 1$ or $2$ for $z_0 = t$ or $st - cr$, respectively.

Since $\varDelta \geq 4$ by $m_i \equiv m_j \pmod 2$ and $n_i \equiv n_j \pmod 2$ for $i, j \in \{0, 1, 2\}$, Lemmas 6.1 and 7.1 together show that

$$\frac{j}{\log \eta} > 7.99a^2bc \cdot 4 > 5.2 \cdot 10^8, \tag{10.4}$$

where $j = m_2 - m_1$. On the other hand, from Proposition 8.4 and Lemma 10.1 we see that

$$m_2 \leq 2n_2 + 1 \leq 296n_1 - 1 \leq 296m_1 + 295 \leq 338.2m_1,$$

and so

$$j = m_2 - m_1 < 338m_1.$$

It follows from Proposition 9.2 that

$$\frac{j}{169 log \eta} < \frac{\mathcal{C}\mu(\rho + 3)^2}{\lambda^3\sigma}h^2 + \frac{4\sqrt{\omega\theta}h + 8\log h + 4\log\left(\sqrt{\mathcal{C}\omega\theta}\lambda^{-3}(\rho + 3)^2\right)}{\log^2 16000} + 1.$$

Therefore we obtain $j/\log \eta < 1.8 \cdot 10^7$, which contradicts (10.4). This completes the proof of Theorem 1.4. □

## 11. Concluding remarks

One of the most possible ways of improving Theorem 1.2 is to show that $N(z_0, z_1) \leq 1$ for $(z_0, z_1) = (-t, -s)$, $(cr - st, -s)$, or $(cr - st, cr - st)$, where $d = d_-$ is attained when $(m, n) = (1, 1)$, $(0, 1)$, or $(0, 0)$, respectively. In the latter two cases, it is hard even to bound $b$ or $c$ from above, since as good an estimate for the linear form $\Lambda$ as the one in Lemma 6.1 cannot be obtained because of $m = 0$. In case $(z_0, z_1) = (-t, -s)$, we know $m_2 \leq 3n_2/2 + 1/2$ by a version of Lemma 2.4, which together with Proposition 8.4 and Lemma 10.1 implies that $m_2 \leq 222m_1$ and $j = m_2 - m_1 \leq 221m_1$. Applying Proposition 9.2 yields $j/\log\eta < 1.11 \cdot 10^7$, whereas Lemmas 6.1 and 7.1 together imply $j/\log\eta > 4ac \cdot 4/(8.1ab) > 7.9ab$ (note that one may take $\kappa = 8.1ab$ in Lemma 6.1). Therefore we obtain $ab < 1.41 \cdot 10^6$, $a \leq 352$, and $c < 5.62 \cdot 10^6 b$. The remaining ranges for $a$, $b$, $c$ are so large that we could not complete the reduction method ([17, Lemma 5a)]) based on [2, Lemma] by Baker and Davenport.

We may have a similar possibility to show that $N(z_0, z_1) \leq 1$ for $z_0 = z_1 = \pm 1$. In the specific triple $\{a, b, c\}$ with $c = a+b+2r$, we in fact showed that $N(-1, -1) \leq 1$ in Theorem 1.4 (note that $st - cr = -1$ in this case). Since the initial terms of sequences attached to the largest element in a Diophantine quintuple are 1 or $-1$, the estimate $N(z_0, z_1) \leq 1$ for $z_0 = z_1 = \pm 1$ would improve the current bound for the number of Diophantine quintuples.

Although Theorem 1.2 may bring us a little closer to settling Conjecture 1.1, there is still a longer way to do it than to give a solution to the folklore conjecture asserting that there exists no Diophantine quintuple, in view of the present state that it has not even been known that there exist only finitely many irregular Diophantine quadruples. The difficulty in proving the finiteness is that the congruence $v_m \equiv w_n \pmod{4c^2}$, used in Lemmas 3.1 and 4.1, does not work well when $b$ and $c$ are very close to each other to get lower bounds for solutions. Even if we restrict only to a regular Diophantine triple $\{a, b, c\}$, an absolute upper bound for $d$ such that $\{a, b, c, d\}$ forms an irregular Diophantine quadruple has not been found yet. In such cases, it seems we have to search for a strategy other than the congruence to bound solutions from below.

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