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DARMON CYCLES AND THE KOHNEN-SHINTANI LIFTING

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Abstract. Let $\mathbf{f}(q)$ be a *Coleman family* of cusp forms of tame level N. Let k_0 be the classical weight at which the specialization of $\mathbf{f}(q)$ is new. By the Kohnen-Shintani correspondence, we associate to every even classical weight k, a half-integral weight form (for $k \neq k_0$) $g_k = \sum_{D>0} c(D,k)q^D \in S_{\frac{k+1}{2}}(\Gamma_0(4N))$ and $g_{k_0} = \sum_{D>0} c(D,k)q^D \in S_{\frac{k+1}{2}}(\Gamma_0(4Np))$.

We first prove that the Fourier coefficients $c(D,k)$ for $k \in 2\mathbb{Z}_{>0}$ can be interpolated by a p-adic analytic function $\tilde{c}(D,\kappa)$ with κ varying in a neighbourhood of k_0 in the p-adic weight space. For discriminants D such that $\tilde{c}(D,k_0) = 0$, which we call Type II, we show that $\frac{d}{d\kappa}[\tilde{c}(D,\kappa)]_{k=k_0}$ is related to certain algebraic cycles associated to the motive $\mathcal{M}_{k_0}$ attached to the space of cusp forms of weight $S_{k_0}(\Gamma_0(Np))$. These algebraic cycles appear in the theory of Darmon cycles.

1. Introduction

One of the major themes in the study of automorphic forms is Langlands' *principle of functoriality*, which describes the existence of correspondence between automorphic forms on different reductive groups. The Shimura and Shintani correspondences between integral weight modular forms (automorphic forms on $GL_2(\mathbb{Q})$) and half integral weight modular forms (automorphic forms on the metaplectic cover of $SL_2(\mathbb{Q})$) is one of the earliest examples of Langlands' functoriality. Shimura initiated the study of half integral weight modular forms in [44], in which he defined suitable Hecke operators and constructed a Hecke-equivariant correspondence between integral weight and half integral weight modular forms. Later, in [47], Shintani constructed the inverse correspondence using *theta lifts*. He showed the existence of a Hecke-equivariant $\mathbb{C}$-linear isomorphism

$$\theta_k : S_k(\Gamma_0(N)) \to S_{\frac{k+1}{2}}(\Gamma_0(4N))$$

for $k \geq 2$ even.

When N is odd square free, W. Kohnen showed the existence of a Hecke equivariant isomorphism (denoted as D-th Shintani liftings) in [25]:

$$\theta_{N,k} : S_k(\Gamma_0(N))^{\mathrm{new}} \to S^{\mathrm{new}}_{\frac{k+1}{2}}(\Gamma_0(4N))^{+},$$

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where + denotes the *Kohnen plus space* of newforms of weight $\frac{k+1}{2}$; i.e., if $g = \theta_{D,k}(f) \in S^{\mathrm{new}}_{\frac{k+1}{2}}(\Gamma_0(4N))^+$, then $g(z)$ admits a Fourier expansion $g(z) = \sum_{D>0} c(D)q^D$, where $c(D) = 0$ unless $D^* := (-1)^{k/2}D \equiv 0, 1 (\mathrm{mod}\ 4)$. The plus space was first introduced by W. Kohnen in [26]. We will be particularly interested in the Fourier coefficients $c(D)$ when $D^* \equiv 1 (\mathrm{mod}\ 4)$.

The arithmetic significance of the Kohnen-Shintani lifting is given by the following Waldspurger type formula (see Theorem 1 of [49] and Corollary 1 of [25]).

Let D be a fundamental discriminant such that $(D, N) = 1$. Then

$$c(D)^2 = \lambda_g D^{\frac{k-1}{2}} L(f, D^*, k/2) \qquad \text{if} \qquad \left(\frac{D^*}{\ell}\right) = w_l \ \forall\ l | N$$

where

- $L(f, D^*, s) := \sum_n a(n)\chi_{D^*}(n)n^{-s}$ is the twisted L-function attached to $f(z) = \sum a(n)q^n$ and the Dirichlet character $\chi_{D^*}(n) := \left(\frac{D^*}{n}\right)$.
- λ_g is a non-zero complex number which depends only on the choice of g.
- $w_l \in (\pm 1)$ are the eigenvalues of the Atkin-Lehner involution W_l acting on f.

The twisted L-function admits a functional equation relating the values at s and $k - s$. The sign that appears in this functional equation is given by

$$w(f, D^*) := (-1)^{k/2}\chi_{D^*}(-N)w_N,$$

where $w_N := \prod_{\ell | N} w_l$ and $f \in S_k(\Gamma_0(N))$.

In particular, the central critical value $L(f, D^*, k/2)$ vanishes when the sign $w(f, D^*)$ is -1. As a generalization of the *Birch and Swinnerton-Dyer conjecture*, the conjectures of Bloch and Beilinson (cf. [33, §4]) predict that the order of vanishing of the central critical L-value is the same as the rank of the appropriate Bloch-Kato Selmer group (see Definition 8).

In the weight 2 case, i.e., when f corresponds to a modular elliptic curve $E/\mathbb{Q}$, Gross and Zagier in [20] describe the Néron-Tate height of a Heegner point $P_{\mathbb{Q}(\sqrt{d})} \in E(\mathbb{Q}(\sqrt{d}))$ (for $d < 0$) in terms of the derivative $L'(E/\mathbb{Q}(\sqrt{d}), 1)$ from which they conclude that $L'(E/\mathbb{Q}(\sqrt{d}), 1) \neq 0$ (i.e., when $E/\mathbb{Q}(\sqrt{d})$ has analytic rank one) if and only if $P_{\mathbb{Q}(\sqrt{d})}$ is a $\mathbb{Q}(\sqrt{d})$-rational point of infinite order. Further, Gross, Kohnen, and Zagier in [21] show that these heights are given by the coefficients of the weight 3/2 form attached to E under the Shimura-Shintani correspondence. This has been generalized to the higher weight case by Hui Xue in [50] and Shaul Zemel in [51] by studying the heights of Heegner cycles in Kuga-Sato varieties of appropriate dimension.

Our main theorem (Theorem 1) can be viewed as a real quadratic p-adic variant of the higher weight Gross-Kohnen-Zagier formula since we relate *Darmon cycles* to Fourier coefficients of a p-adic modular form. Such a relation was shown by Henri Darmon and Gonzalo Tornaria in the weight 2 ordinary case [14] using the theory of *Stark-Heegner points.*

Let p be an odd prime integer such that $p \nmid N$. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and embeddings $\sigma_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\sigma_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Let U be an open affinoid of the p-adic weight space $\mathcal{X}$ (see Section 4.1). A p-adic analytic family of cuspidal eigenforms (*Coleman family*) over U is a formal q-expansion

$$\mathbf{f}(q) := \sum_{n\geq 1} \mathbf{a}_n q^n \in \mathcal{O}(U)[[q]]$$

such that for all $k \in U^{\mathrm{cl}} := \{n \in 2\mathbb{Z} : n \geq 0\} \cap U$,

$$f_k(q) = \sum_{n\geq 1} \mathbf{a}_n(k) q^n \in S_k(\Gamma_0(Np), \overline{\mathbb{Q}}).$$

The p-adic valuation of $\mathbf{a}_p(k)$ is a constant called the slope of $\mathbf{f}(q)$. We will assume that we are in the finite non-ordinary case (i.e., $\mathbf{a}_p(k) \neq 0$ and $v_p(\mathbf{a}_p(k)) > 0$) and also that f_k is N-new for all $k \in U^{\mathrm{cl}}$. Since the slope of $\mathbf{f}$ is constant, there is at most one $k_0 \in U^{\mathrm{cl}}$ such that f_{k_0} is new for the full level Np. This happens exactly when $\mathbf{a}_p(k_0) = \pm p^{k_0/2-1}$.

For every other $k \neq k_0 \in U^{\mathrm{cl}}$, there is a newform $f_k^{\#} \in S_k(\Gamma_0(N))^{\mathrm{new}}$ such that f_k is the p-stabilization of $f_k^{\#}$, i.e.,

$$f_k(q) = f_k^{\#}(q) - \frac{p^{k-1}}{\mathbf{a}_p(k)} f_k^{\#}(q^p).$$

In particular, the eigenvalues of the Hecke operators, T_l for all $l \nmid N$, of f_k and $f_k^{\#}$ coincide.

The quantity

$$w_N := (-1)^{k/2} w_{N,k},$$

where $w_{N,k} := \prod_{\ell | N} w_{\ell,k}$ is the product Atkin-Lehner eigenvalues of W_ℓ acting on $f_k^{\#}$, is independent of $k \in U^{cl}$. This quantity is called the *root number* of the *Coleman family.*

Let $g_k = \sum_{D>0} c(D,k) q^D \in S_{\frac{k+1}{2}}(\Gamma_0(4N))$ be the Shintani lift of $f_k^{\#}$ for all $k \neq k_0 \in U^{\mathrm{cl}}$ and let $g_{k_0} = \sum_{D>0} c(D,k_0) q^D \in S_{\frac{k_0+1}{2}}(\Gamma_0(4Np))$ correspond to the lift of f_{k_0}.

The values of D for which $c(D,k)$ need not necessarily vanish for $k \neq k_0 \in U^{\mathrm{cl}}$ can be classified in two types:

(I) All $D > 0$ such that $w_N \chi_{D^*}(-N) = 1$ and $\chi_{D^*}(p) = w_p$.

(II) All $D > 0$ such that $w_N \chi_{D^*}(-N) = 1$ but $\chi_{D^*}(p) = -w_p$.

Note that for Type II discriminants D, we have $w(f_{k_0}, D^*) = -1$ and hence $L(f_{k_0}, D^*, k_0/2) = 0$. This forces the vanishing of $c(D, k_0)$. However, since p divides the level of the newform f_{k_0} but not the level of $f_k^{\#}$ for any other classical $k \neq k_0$, the function $c(D,k)$ need not identically vanish in a neighborhood of k_0. In fact, by making a suitable normalization of the Fourier coefficients $c(D,k)$, we show that (see Proposition 4) the function $k \to c(D,k)$ extends to a non-zero rigid analytic function $\widetilde{c}(D,\kappa)$ in a neighbourhood of k_0. This motivates the study of the p-adic derivative of $\widetilde{c}(D,k)$ at $k = k_0$.

For D_2 a Type II discriminant and D_1 a Type I discriminant such that $c(D_1,k_0) \neq 0$, let $D := D_1^*.D_2^*$ and let K be the real quadratic field $\mathbb{Q}(\sqrt{D})$ (note that $D > 0$).

Let $\mathcal{M}_k$ be the motive over $\mathbb{Q}$ associated to $S_k(\Gamma_0(Np))$ constructed in [40]. For L any number field, let $CH_0^{k/2}(\mathcal{M}_k \otimes L)$ be the Chow group of algebraic cycles of codimension $k/2$ on $\mathcal{M}_k$ base change to L that are homologous to the null cycle. Let

V^{Np} be the p-adic étale realization of $\mathcal{M}_k$ viewed as a *p-adic Galois representation* of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and let V_p^{Np} denote the restriction of V^{Np} to a decomposition group at p. We have a *global p-adic Abel-Jacobi* map

$$\mathrm{cl}_{0,\mathrm{L}}^{\mathrm{k}/2} : \mathrm{CH}_0^{k/2}(\mathcal{M}_k \otimes L) \to \mathrm{Sel}_{st}(L, V_p^{Np}(k/2)).$$

See Sections 1 - 4 of [33] for a detailed discussion on the global Abel-Jacobi map.

The main theorem we prove is

Theorem 1. *There exist a global cycle*

$$d_{k_0}^{\chi_{D_2^*}} \in CH_0^{k_0/2}(\mathcal{M}_{k_0} \otimes \mathbb{Q}(\sqrt{D_2^*}))^{\chi_{D_2^*}} \subset (\mathcal{M}_{k_0} \otimes \mathbb{Q}(\sqrt{D_2^*}, \sqrt{D_1^*}))$$

and a constant $s_f \in K_{f_{k_0}}^{\times}$ such that

$$\frac{d}{dk}[\widetilde{c}(D_2, k)]_{k=k_0} = \frac{|D_2|^{\frac{k_0-2}{4}}}{|D_1|^{\frac{k_0-2}{4}}}.s_f.\log_{\mathrm{BK}}(\mathrm{res}_p(\mathrm{cl}_{0,H_K^+}^{k_0/2}(d_{k_0}^{\chi_{D_2^*}})))(\phi_{k_0}),$$

where $CH_0^{k_0/2}(\mathcal{M}_{k_0} \otimes \mathbb{Q}(\sqrt{D_2^*}))^{\chi_{D_2^*}}$ denotes the $\chi_{D_2^*}$-eigenspace of

$$CH_0^{k_0/2}(\mathcal{M}_{k_0} \otimes \mathbb{Q}(\sqrt{D_2^*})), \phi_{k_0}$$

is the modular symbol attached to f_{k_0}, $\log_{\mathrm{BK}}$ *is the Bloch-Kato logarithm map, and* $\mathrm{res_p} : Sel_{st}(H_K^+, V_p^{Np}(k_0/2)) \to H^1_{st}(K_p, V_p^{Np}(k_0/2))$ *is the restriction at p.*

2. The Kohnen-Shintani lifting

Fix $f \in S_k(\Gamma_0(N))$, a cusp form of weight k on $\Gamma_0(N)$. Let $Q(x,y) = ax^2+bxy+cy^2$ be a primitive integral binary quadratic form with a square-free discriminant. The group $SL_2(\mathbb{Z})$ acts on the right on the space of integral quadratic forms by

$$(Q \mid \epsilon)(x,y) := Q(\delta x - \gamma y, -\beta x + \alpha y)$$

for $\epsilon = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Let $D > 0$ be an integer such that $D^* := (-1)^{k/2}D$ is congruent to $0, 1 (\mathrm{mod}\ 4)$ and D^* divides $\Delta = b^2 - 4ac$, the discriminant of $Q(x,y)$. Let D' be the integer such that $\Delta = D^* D'^*$. Note that D'^* is also congruent to $0, 1 (\mathrm{mod}\ 4)$ as Δ is always congruent to $0, 1 (\mathrm{mod}\ 4)$.

Define

$$\omega_{D^*,D'^*}(Q) := \begin{cases} \left(\frac{D'^*}{Q(m,n)}\right) & \text{when } \gcd(D'^*, Q(m,n)) = 1, \\ \left(\frac{D^*}{Q(m,n)}\right) & \text{when } \gcd(D^*, Q(m,n)) = 1. \end{cases}$$

For any other pair of integers $(r,s) \in \mathbb{Z}$ such that $(D^*, Q(r,s)) = 1$, we have

$$\left(\frac{D^*}{Q(r,s)}\right) = \left(\frac{D^*}{Q(m,n)}\right).$$

Hence ω_{D^*,D'^*} is well-defined. Genus theory shows that ω_{D^*,D'^*} is a quadratic character of the class group of integral binary quadratic forms of discriminant Δ. This character cuts out the bi-quadratic extension $\mathbb{Q}(\sqrt{D^*}, \sqrt{D'^*})$.

Let δ be a positive integer such that $\delta^2 \equiv \Delta(\mathrm{mod} : 4N)$. We will call a primitive binary quadratic form $Q(x,y) = ax^2 + bxy + cy^2$ a *Heegner form* of level N if

$$N \mid a \quad \text{and} \quad b \equiv \delta(\mathrm{mod} N).$$

Denote the set of Heegner forms of discriminant Δ by $\mathcal{F}_\Delta$. Assume Δ is square free and let $r + s\sqrt{\Delta}$ be the totally positive (i.e., $r, s > 0$) fundamental unit in the order $\mathcal{O}_\Delta := \mathbb{Z}[\frac{\Delta+\sqrt{\Delta}}{2}]$. Let

$$\gamma_Q := \begin{pmatrix} r+sb & 2cs \\ -2as & r-sb \end{pmatrix} \in \Gamma_0(N)$$

be the generator of the cyclic subgroup Γ_Q, the stabilizer of Q in $\Gamma_0(N)$. For any point $\tau \in \mathcal{H}$, let C_Q be the image in $\Gamma_0(N)/\mathcal{H}$ of the geodesic in $\mathcal{H}$ of complex numbers $z = x + iy$ such that

$$a|z|^2 + bx + c = 0.$$

Let $\tau \in \mathcal{H}$ be any base point. In our case (i.e., Δ is not a perfect square), C_Q is equivalent to the geodesic joining τ and $\gamma_Q\tau$. To each $Q \in \mathcal{F}_\Delta$, we associate the Shintani period given by

$$r(f, Q) := \int_{C_Q} f(z)Q(z,1)^{\frac{k-2}{2}}dz.$$

Let $\Delta = D^*.D'^*$ be the factorization such that $D, D' > 0$ and $D^*, D'^* = (-1)^{k/2}D_i \equiv 0, 1 (\bmod\ 4)$ for $i = 1, 2$. Consider the liner combination

$$r_{k,N}(f, D^*, D'^*) = \sum_{Q\in\mathcal{F}_\Delta/\Gamma_0(N)} \omega_{D^*,D'^*}(Q)r(f,Q).$$

Let $\mu(n)$ denote the Mobius function which is defined as the sum of the primitive n-th roots of unity. Then $\mu(n) \in \{-1, 0, 1\}$. Let $S^+_{\frac{k+1}{2}}(\Gamma_0(4N))$ denote the Kohnen '+' space of half integral weight cusp forms, i.e., forms that have a Fourier expansion of the form

$$g(\tau) = \sum_{\substack{D\geq 1 \\ D^*\equiv 0,1 (\bmod\ 4)}} c(D)q^D \in S_{k+1/2}(\Gamma_0(4N)).$$

The '+' space was defined by Kohnen in [26]. For m a fundamental discriminant such that $m^* = (-1)^{k/2}m > 0$, the m-th Shintani lifting of $f \in S_k(\Gamma_0(N))$ is defined as

$$\Theta_{k,N,m}(f)(q) := \sum_{\substack{D\geq 1 \\ D^*\equiv 0,1 (\bmod\ 4)}} \Big(\sum_{t|N} \mu(t)\Big(\frac{m}{t}\Big)t^{\frac{k-1}{2}} r_{k,Nt}(f, m, (-1)^{k/2}Dt^2)\Big)q^D.$$

Theorem 2. *For every m as above, $\Theta_{k,N,m} : S_k(\Gamma_0(N)) \to S^+_{\frac{k+1}{2}}(\Gamma_0(4N))$ is an isomorphism. Further if N is odd square free, then $\Theta_{k,N,m}$ maps $S_k^{new}(\Gamma_0(N))$ isomorphically onto $S^{+,new}_{\frac{k+1}{2}}(\Gamma_0(4N))$.*

Proof. See Theorem 2 of [25]. □

We will now recall a formula of Kohnen which relates the Fourier coefficients of a Shintani lifting to the Shintani periods.

For any m as above, let the Fourier expansion of $\Theta_{k,N,m}(f)(z)$ be

$$\sum_{\substack{D\geq 1 \\ D^*\equiv 0,1 (\bmod\ 4)}} c(D)q^D,$$

i.e., $c(D) = \sum_{t|N} \mu(t)\Big(\frac{m}{t}\Big)t^{\frac{k-1}{2}} r_{k,Nt}(f, m, (-1)^{k/2}Dt^2))$.

Theorem 3. *Then we have*

$$\frac{c(D_1)\overline{c(D_2)}}{\langle g, g\rangle} = \frac{(-2i)^{k/2}2^{\nu(N)}}{\langle f, f\rangle} r_{k,N}(f, D_1^*, D_2^*),$$

where $\nu(N)$ is the number of distinct prime divisors of N.

Proof. See Theorem 3 of [27]. □

Remark 1. $r_{k,N}(f, D_1^*, D_2^*)$ is defined through a sum of oriented optimal embeddings in [37] as compared to non-oriented optimal embeddings in [27]. This difference in definition of $r_{k,N}(f, D_1^*, D_2^*)$ contributes an extra factor of $2^{\nu(N)}$ in the constant term between the above statement and Theorem 3 of [27].

Let $D > 0$ be an integer such that $D^* \equiv 0, 1 \pmod 4$. Recall the twisted L-series of f:

$$L(f, D^*, s) := \sum_{n\geq 1} \Big(\frac{D^*}{n}\Big) a(n) n^{-s}; \qquad \mathrm{Re}(s) \gg 0,$$

where $f(z) = \sum\limits_{n\geq 1} a(n)q^n \in S_k(\Gamma_0(N))$ and $\Big(\frac{D^*}{\cdot}\Big)$ is the quadratic Dirichlet character. This twisted L-function admits a holomorphic continuation to $\mathbb{C}$ given by

$$\Lambda(f, D^*, s) = (2\pi)^{-s}(ND^{*,2})^{s/2}\Gamma(s)L(f, D^*, s)$$

and admits a functional equation

$$\Lambda(f, D^*, s) = (-1)^{k/2}\Big(\frac{D^*}{-N}\Big) w_N \Lambda(f, D^*, k - s),$$

where $w_N := \prod\limits_{\ell | N} w_\ell \in \{\pm 1\}$ is the product of the Atkin-Lehner eigenvalues indexed by the primes dividing the level.

The following result follows from Theorem 3 above.

Corollary 1. *We have*

$$\frac{|c(|D|)|^2}{\langle g, g\rangle} = 2^{\nu(N)} \frac{(k/2-1)!}{\pi^{k/2}} |D|^{\frac{k-1}{2}} \frac{L(f, D^*, k/2)}{\langle f, f\rangle}$$

with $\nu(N)$ being the number of distinct prime divisors of N.

Proof. See Corollary 1 of [25]. □

Remark 2. *By the above lemma, we know that the vanishing of $c(D)$ is equivalent to the vanishing of $L(f, D^*, k/2)$.*

3. Darmon cycles

In this section, we will briefly recall the theory of Darmon cycles developed by Marco Seveso and Victor Rotger in [38]. All the results discussed in this section can be found in [38], [19], and [41].

Let p be an odd prime and let N be an odd square free integer such that $p \nmid N$. We fix a real quadratic extension $K/\mathbb{Q}$ such that

- all primes dividing N are split in K,
- p is inert in K.

Let D_K be the discriminant of K. Recall from the Introduction the fixed embeddings

$$\sigma_\infty : \overline{\mathbb{Q}} \to \mathbb{C}, \qquad \sigma_p : \overline{\mathbb{Q}} \to \mathbb{C}_p.$$

Let $\Gamma_0 := \Gamma_0(Np)$ and $\Gamma := \Gamma_0(N)$ be the congruence subgroups of level Np and N respectively. Let $\widetilde{\Gamma} := \Gamma[\frac{1}{p}]$ and $W := \mathbb{Q}_p^2 - (0,0)$. For any field E, denote by $P_{k-2}(E)$ the set of homogeneous polynomials of degree $k-2$ in two variables over E. Let $V_{k-2}(E)$ be the E-dual of $P_{k-2}(E)$. The group $GL_2(E)$ acts on the left on $P_{k-2}(E)$ naturally by the rule

$$(\gamma.P)(X,Y) := P(aX+cY, bX+dY),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Consequently, this induces the right dual action of $GL_2(E)$ on $V_{k-2}(E)$.

Let $\mathcal{T}$ denote the Bruhat-Tits tree of $\mathbb{Q}_p$ whose vertices are given by homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. We denote the set of vertices (resp. edges) of $\mathcal{T}$ by $\mathcal{V}$ (resp. $\mathcal{E}$). We will denote a vertex v by $[L]$, where $[L]$ stands for the homothety class of lattices equivalent to some lattice $L \subset \mathbb{Q}_p^2$. There is an edge e between two vertices v_1 and $v_2 \in \mathcal{V}$ if for some lattices $L_1, L_2 \subset \mathbb{Q}_p^2$ such that $v_1 = [L_1]$ and $v_2 = [L_2]$,

$$L_1 \supset L_2 \supset pL_1.$$

$\mathcal{T}$ is a tree with each vertex $v \in \mathcal{V}$ having degree $p+1$ (see Proposition 1.3.2 of [15]). We have a natural left $GL_2(\mathbb{Q}_p)$-action on $\mathcal{T}$ as follows: for $\gamma \in GL_2(\mathbb{Q}_p)$ and $[L] \in \mathcal{T}$ such that L is generated by the $\mathbb{Z}_p$-span of $\langle v_1, v_2 \rangle$, define

$$\gamma.[L] := [\gamma L],$$

where γL denotes the lattice generated by the $\mathbb{Z}_p$-span of $\langle \gamma v_1, \gamma v_2 \rangle$. Denote the distinguished vertex $v_* := [L_*]$, where $L_* := \mathbb{Z}_p^2$, and by $\mathcal{V}^+$ (respectively $\mathcal{V}^-$) the set of vertices at even (respectively odd) distance from v_*.

We can define an orientation on $\mathcal{T}$ as follows: for every $e \in \mathcal{E}(\mathcal{T})$, denote by $s(e)$ the source vertex of e and by $t(e)$ the target vertex of e. This assigns a direction to each edge, thus making $\mathcal{T}$ into a directed graph. Denote by $\bar{e}$ the edge such that $s(\bar{e}) = t(e)$ and $t(\bar{e}) = s(e)$.

Definition 1. The p-adic upper half plane $\mathcal{H}_p$ is the rigid analytic variety over $\mathbb{Q}_p$ whose E-rational points, for E a finite extension of $\mathbb{Q}_p$, are given by $\mathcal{H}_p(E) := \mathbb{P}^1(E) - \mathbb{P}^1(\mathbb{Q}_p)$.

We will denote $\mathcal{H}_p^{ur}$ for $\mathcal{H}_p(\mathbb{Q}_p^{ur}) = \mathbb{P}^1(\mathbb{Q}_p^{ur}) - \mathbb{P}^1(\mathbb{Q}_p)$. $\mathcal{H}_p^{ur}$ has a natural left action of $GL_2(\mathbb{Q}_p)$ via fractional linear transformation. There exists a unique $GL_2(\mathbb{Q}_p)$-equivariant reduction map (see Proposition 5.1 of [13])

$$\mathrm{red}_p : \mathcal{H}_p^{ur} \longrightarrow \mathcal{T}.$$

Let $\mathcal{H}_{p,\pm} := r^{-1}(\mathcal{V}^{\pm})$ and $\mathcal{H}_{p,v} := r^{-1}(v)$ for $v \in \mathcal{V}$.

Definition 2. Let $([L_1], [L_2], \ldots, [L_i], \ldots)$ be an infinite, non-retracing sequence of adjacent vertices in $\mathcal{T}$. By non-retracing, we mean that $\nexists n \in \mathbb{N}$ such that $[L_i] = [L_{i+n}]$ for all $i \in \mathbb{N}$. We interpret such a sequence as a ray starting from the vertex $v = [L_1]$ and heading off to ∞. We introduce an equivalence relation on the set of all such sequences given by

$$([L_1], [L_2], \ldots, [L_i], \ldots) \sim ([L_1'], [L_2'], \ldots, [L_i'], \ldots)$$

if there exists a fixed $m \in \mathbb{Z}$ such that $[L_n] = [L'_{n+m}]$ for all $n \in \mathbb{N}$. We call such an equivalence class an *end* in $\mathcal{T}$.

The compact open subsets of $\mathbb{P}^1(\mathbb{Q}_p)$ are in one-one correspondence with the ends in $\mathcal{E}$ (see Theorem 5.9 of [13, Chapter 5]). For $e \in \mathcal{E}$, we denote by U_e the compact open subset under this correspondence.

3.1. **p-adic Abel-Jacobi maps: Darmon's setting.** Let us denote by K_p the completion of the image of the embedding $\sigma_p : K \hookrightarrow \mathbb{C}_p$. By the hypothesis that p is inert in K, we know that K_p is isomorphic to the unramified quadratic extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$. For $*$ either empty, $\pm$, or $v \in \mathcal{V}$, denote $\Delta_* := (\mathrm{Div}(\mathcal{H}_{p,*}^{ur}))^{G_{K_p^{ur}/K_p}}$ and $\Delta_*^0 := (\mathrm{Div}^0(\mathcal{H}_{p,*}^{ur}))^{G_{K_p^{ur}/K_p}}$, where $G_{K_p^{ur}/K_p} = \mathrm{Gal}(K_p^{ur}/K_p)$ and $\mathrm{Div}(\mathcal{H}_{p,*}^{ur})$ (respectively $\mathrm{Div}^0(\mathcal{H}_{p,*}^{ur})$) denotes the set of divisors (respectively set of zero divisors) on $\mathcal{H}_{p,*}^{ur}$.

We can consider $\Delta_*(P_{k-2}) := \Delta_* \otimes_{\mathbb{Z}} P_{k-2}$ and $\Delta_*^0(P_{k-2}) := \Delta_*^0 \otimes_{\mathbb{Z}} P_{k-2}$ as left $GL_2(\mathbb{Q}_p)$-modules (resp. left $GL(L)$-modules) when $*$ is empty (resp. $* = v = [L]$) via the usual tensor product action. We have the following exact sequence:

$$0 \to \Delta_*^0(P_{k-2}) \to \Delta_*(P_{k-2}) \xrightarrow{\deg} P_{k-2} \to 0. \tag{1}$$

Recall the set of vertices $\mathcal{V}$ and edges $\mathcal{E}$ of the Bruhat-Tits tree $\mathcal{T}$. Denote by $\mathcal{C}(\mathcal{E}, V_{k-2})$ the set of all maps $c : \mathcal{E} \to V_{k-2}$.

Definition 3. A harmonic cocycle is an element in $\mathcal{C}(\mathcal{E}, V_{k-2})$ such that $c(\bar{e}) = -c(e)$ for all $e \in \mathcal{E}$ and $\sum_{s(e)=v} c(e) = 0$ for every $v \in \mathcal{V}$. The space of harmonic cocycles is denoted by $\mathcal{C}_{har}(\mathcal{E}, V_{k-2}) \subseteq \mathcal{C}(\mathcal{E}, V_{k-2})$.

For $W := \mathbb{Q}_p^2 - (0,0)$, let $\mathcal{A}(W)_{k-2}$ be the space of K_p-valued locally analytic functions on W that are homogeneous of degree $k-2$. Let $\mathcal{D}(W)_{k-2}$ be the continuous K_p-dual of $\mathcal{A}(W)_{k-2}$ equipped with the strong topology. Note that $P_{k-2}(K_p) \subset \mathcal{A}(W)_{k-2}$. Further, denote by $\mathcal{D}(W)_{k-2}^0$ the subspace of distributions that are zero on $P_{k-2}(K_p)$. Consider

$$\theta_\ell^{\tau_2-\tau_1,P} : W \to \mathbb{C}_p, \qquad \theta_\ell^{\tau_2-\tau_1,P}(x,y) := \ell\left(\frac{y+\tau_2 x}{y+\tau_1 x}\right) P(x,y),$$

where $\ell = \log\langle . \rangle$ is the Iwasawa logarithm (see §4.4.11 of [8]) or ord_p, $\tau_1, \tau_2 \in \mathcal{H}_{p,*}^{ur}$, and $P \in P_{k-2}(K_p)$. Since any $d \in \Delta_*^0$ is a linear combination of divisors of the form $\tau_2 - \tau_1$, we can extend by linearity to define $\theta_l^{d,P}$ for any $d \in \Delta_*^0$. For every $t \in \mathbb{Q}_p^\times$, $\theta_l^{d,P}(t(x,y)) = t^{k-2}\theta_l^{d,P}(x,y)$, and hence we have $\theta_l^{d,P} \in \mathcal{A}(W)_{k-2}$.

Lemma 1 ([19], Lemma 6.1)**.** *The pairing*

$$I_l^0 : \mathcal{D}(W)_{k-2}^0 \times \Delta_*^0(P_{k-2}) \to K_p,$$

where $I_l^0(\mu, d \otimes P) := \mu(\theta_l^{d,P})$, is invariant for the $GL_2(\mathbb{Q}_p)$-action (resp. $GL(L)$-action) when $*$ is empty (resp. $* = v = [L]$).

Let $\pi : W \to \mathbb{P}^1(\mathbb{Q}_p)$ be the projection $\pi(x,y) := y/x$. Note that π is well-defined since x and y cannot both be simultaneously 0.

Lemma 2. *The image of the $GL_2(\mathbb{Q}_p)$-equivariant map*

$$R : \mathcal{D}(W)_{k-2}^0 \to \mathcal{C}(\mathcal{E}, V_{k-2})$$

given by $R(\mu)(e)(P) := \mu(P.\chi_{W_e})$ is contained in $\mathcal{C}_{har}(\mathcal{E}, V_{k-2})$.

Proof. Note that $W = W_e \cup W_{\bar{e}}$. Hence we can write $P(x,y) = P.\chi_{W_e} + P.\chi_{W_{\bar{e}}}$. Since $\mu \in \mathcal{D}(W)^0_{k-2}$, we have $\mu(P(x,y)) = 0$ and hence $R(\mu)(\bar{e})(P) = -R(\mu)(e)(P)$. Now, for every $v \in \mathcal{V}$ we have $\bigcup_{s(e)=v} W_e = W$. We write $P(x,y) = \sum_{s(e)=v} P.\chi_{W_e}$ and hence $\mu(\sum_{s(e)=v} P.\chi_{W_e}) = 0$, which implies that $\sum_{s(e)=v} R(\mu)(e) = 0$. Hence $R(\mu) \in \mathcal{C}_{har}(\mathcal{E}, V_{k-2})$. □

For $e \in \mathcal{E}$, denote by $\rho_e : \mathcal{C}_{har}(\mathcal{E}, V_{k-2}) \to V_{k-2}$ the evaluation map. By Lemma 2, we have

$$R_e : \mathcal{D}(W)^0_{k-2} \xrightarrow{R} \mathcal{C}_{har}(\mathcal{E}, V_{k-2}) \xrightarrow{\rho_e} V_{k-2}.$$

The action of $GL_2(\mathbb{Q}_p)$ on the vertices $\mathcal{V}(\mathcal{T})$ induces an action on the edges $\mathcal{E}(\mathcal{T})$.

Definition 4. We say that a distribution $\mu \in \mathcal{D}(W)^0_{k-2}$ is h-admissible if for all $j \to \infty$, $i \geq 0$, and all $a \in \mathbb{Z}_p$, we have

$$|\mu((x-a)^i|, a + p^j\mathbb{Z}_p)| = o(p^{j(h-i)}),$$

for all $i = 0, 1, \ldots, h-1$.

Denote by $\mathcal{D}(W)^{0,h}_{k-2} \subset \mathcal{D}(W)^0_{k-2}$ the set of such *h-admissible* distributions. The definition of $H^1(\Gamma_0(Np), V_{k-2})^{p\text{-new}}$, the p-new part of the cohomology of $\Gamma_0(Np)$ with coefficients in V_{k-2}, is given in Definition 2.7 of [38].

Let $\widehat{e} \in \mathcal{E}$ be an edge such that $U_{\widehat{e}} = \mathbb{Z}_p$ and the stabilizer of $\widehat{e}$ in $\tilde{\Gamma}$ is $\Gamma_0(Np)$.

Lemma 3 ([19, Lemma 6.2])**.** *The map $R_{\widehat{e}}$ induces on cohomology groups an isomorphism*

$$R_{\widehat{e}} : H^1\Big(\tilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}\Big) \cong H^1(\Gamma_0(Np), V_{k-2})^{p-new}.$$

Denote by $\mathbb{T}^p_{Np}$ the Hecke algebra over $\mathbb{Q}_p$ generated by the Hecke operators T_ℓ for $\ell \nmid Np$ and U_ℓ for $\ell \mid Np$.

Definition 5. A module M over the Hecke algebra $\mathbb{T}^p_{Np}$ admits an Eisenstein/cuspidal decomposition if we can write $M = M_e \oplus M_c$ and there exists a Hecke operator T_l for $l \nmid Np$ such that $t_l := T_l - l^{k-1} - 1$ is nilpotent on M_e and is invertible on M_c. We call M_e (resp. M_c) the Eisenstein (resp. cuspidal) part of M.

Let V be a $\Gamma_0(Np)$-module and denote by $\Gamma_{0,c}$ the stabilizer in $\Gamma_0(Np)$ for c a $\Gamma_0(Np)$-equivalence class of cusps. We can then define the parabolic cohomology group to be

$$H^1_{\text{par}}(\Gamma_0(Np), V) := \ker\Big(H^1(\Gamma_0(Np), V) \xrightarrow{res} \bigoplus_{\text{cusps c}} H^1(\Gamma_{0,c}, V)\Big).$$

See the Appendix in [22] for the definition and properties of *parabolic cohomology*.

The Hecke module $H^1(\Gamma_0(Np), V_{k-2})^{p-new}$ admits an Eisenstein/cuspidal decomposition with the cuspidal part given by $H^1_{\text{par}}(\Gamma_0(Np), V_{k-2})^{p\text{-new}}$, which for brevity we denote by $\mathbb{H}_k$.

Remark 3. By the *Eichler-Shimura isomorphism*, the space

$$H^1(\Gamma_0(Np), V_{k-2}(\mathbb{C}))^{p\text{-new}}$$

is isomorphic as Hecke modules to the space of modular forms, $M_k(\Gamma_0(Np), \mathbb{C})$, which admits an Eisenstein/cuspidal decomposition with the Eisenstein part (resp. cuspidal part) given by the subspace of Eisenstein series of weight k (resp. the subspace of cusp forms of weight k). This induces the Eisenstein/cuspidal decomposition of $H^1(\Gamma_0(Np), V_{k-2})^{p-\mathrm{new}}$. See §2.4 of [38].

The isomorphism of Lemma 3 induces

$$R_{\widehat{e},c} : H^1\Big(\widetilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}\Big)_c \cong \mathbb{H}_k.$$

By taking the $\widetilde{\Gamma}$-homology of (1), we get

$$\ldots \to H_2(\widetilde{\Gamma}, P_{k-2}) \xrightarrow{\delta} H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2})) \xrightarrow{i} H_1(\widetilde{\Gamma}, \Delta(P_{k-2})) \to H_1(\widetilde{\Gamma}, P_{k-2}) \to \ldots.$$

We know that $H_1(\widetilde{\Gamma}, P_{k-2}) = 0$ by Lemma 3.10 of [38] and hence we have the isomorphism

$$\bar{i} : H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2}))/\mathrm{im}(\delta) \cong H_1(\widetilde{\Gamma}, \Delta(P_{k-2})).$$

Consider the cap product

$$H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2})) \times H^1(\widetilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}) \to H_0\Big(\widetilde{\Gamma}, \Delta^0(P_{k-2}) \otimes \mathcal{D}(W)^{0,h}_{k-2}\Big).$$

We know that $H_0\Big(\widetilde{\Gamma}, \Delta^0(P_{k-2}) \otimes \mathcal{D}(W)^{0,h}_{k-2}\Big) = (\Delta^0(P_{k-2}) \otimes \mathcal{D}(W)^{0,h}_{k-2})_{\widetilde{\Gamma}}$ (the set of $\widetilde{\Gamma}$ co-invariants), which is obtained from $\Delta^0(P_{k-2}) \otimes \mathcal{D}(W)^{0,h}_{k-2}$ by introducing the relations

$$\gamma.(\tau \otimes P(x,y)) \otimes \mu = (\tau \otimes P(x,y)) \otimes \mu.\gamma$$

for $\tau \in \Delta^0, P(x,y) \in P_{k-2}$ and $\mu \in \mathcal{D}(W)^{0,h}_{k-2}$. By Lemma 1, the pairing I^0_l is in particular $\widetilde{\Gamma}$-invariant and hence it extends to a pairing on the cap product; i.e., we have

$$\tilde{I}^0_l : H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2})) \times H^1(\widetilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}) \to K_p.$$

We now define

$$\mathrm{AJ}^0_l : H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2})) \xrightarrow{\tilde{I}^0_l} H^1\Big(\widetilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}\Big)^\vee \xrightarrow{pr_c \circ R^{-1}_{\widehat{e}}} \mathbb{H}^{\pm,\vee}_k,$$

where pr_c denotes the projection onto the cuspidal part, $\mathbb{H}^\pm_k$ denotes the direct summand of $\mathbb{H}_k$ on which $W_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts with eigenvalue ± 1, and '$\vee$' denotes K_p-dual.

By the isomorphism of Lemma 3, $H^1\Big(\widetilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}\Big)$ inherits an action of $\mathbb{T}^p_{Np}$. We have

Theorem 4 ([38, Corollary 3.13])**.** *There exists a unique $\mathcal{L} \in \mathbb{T}^p_{Np}$ such that $\tilde{I}^0_{log} - \mathcal{L}\tilde{I}^0_{ord}$ annihilates* $\mathrm{im}(\delta)$*.*

Remark 4. The unique element $\mathcal{L} \in \mathbb{T}^p_{Np}$ is called the $\mathcal{L}$-invariant associated to $S_k(\Gamma_0(Np))$, the space of weight k cusp forms on $\Gamma_0(Np)$.

Let $\log \mathrm{AJ}^0 := \mathrm{AJ}^0_{\log} - \mathcal{L}\mathrm{AJ}^0_{\mathrm{ord}}$. By the above theorem, we know that $\log \mathrm{AJ}^0$ factors through $H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2}))/\mathrm{im}(\delta)$. We then define the *cohomological Abel-Jacobi map* to be

$$\log \mathrm{AJ} : H_1(\widetilde{\Gamma}, \Delta(P_{k-2})) \xrightarrow{\bar{i}^{-1}} H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2}))/\mathrm{im}(\delta) \xrightarrow{\log \mathrm{AJ}^0} \mathbb{H}^{\pm,\vee}_k.$$

3.2. **Darmon cycles.** We can view K as a sub-field of both $\mathbb{R}$ and $\mathbb{C}_p$ via the fixed embeddings σ and σ_p respectively. For $\tau \in K$, we denote by $\overline{\tau}$ the image of the non-trivial automorphism $\gamma \in \mathrm{Gal}(K/\mathbb{Q})$. We can view the positive square root $\sqrt{D_K}$ as an element in K_p. Consider the set of all $\mathbb{Q}$-algebra embeddings of K into $M_2(\mathbb{Q})$, denoted by $\mathrm{Emb} := \mathrm{Emb}(K, M_2(\mathbb{Q}))$. Let $\mathcal{R}$ be the $\mathbb{Z}[\frac{1}{p}]$-order in $M_2(\mathbb{Q})$ which consists of matrices that are upper triangular modulo N. Note that $\widetilde{\Gamma} = \mathcal{R}_1^\times$ (the set of invertible matrices of $\mathcal{R}$ with determinant 1). For $\mathcal{O}$ a $\mathbb{Z}[\frac{1}{p}]$-order of conductor c such that $(c, D_K Np) = 1$, denote by

$$\mathrm{Emb}(\mathcal{O}, \mathcal{R}) := \{\psi : \mathcal{O} \hookrightarrow \mathcal{R} \in \mathrm{Emb} : \psi(K) \cap \mathcal{R} = \psi(\mathcal{O})\}$$

the set of $\mathbb{Z}[\frac{1}{p}]$-embeddings of $\mathcal{O}$ into $\mathcal{R}$. We can attach the following data to every $\psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$:

- the fixed points τ_ψ and $\overline{\tau}_\psi \in \mathcal{H}_p$ for the action of $\psi(K^\times) \subseteq M_2(\mathbb{Q})$ on $\mathcal{H}_p(K)$ by fractional linear transformation;
- the fixed vertex $v_\psi \in \mathcal{V}$ in the Bruhat-Tits tree for the action of $\psi(K^\times)$ on $\mathcal{V}$;
- the unique quadratic form

$$P_\psi(x, y) := cx^2 + (d - a)xy + by^2 \in P_2(K),$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \psi(\sqrt{D_K})$. In fact, the points τ_ψ and $\overline{\tau}_\psi$ are the roots of $P_\psi(z, 1)$;
- for $u \in \mathcal{O}^\times$, the fundamental unit (i.e., $\sigma(u) > 1$) of K, let $\gamma_\psi := \psi(u)$ and let Γ_ψ be the cyclic group generated by γ_ψ which is also the stabilizer of ψ in $\widetilde{\Gamma}$. In particular Γ_ψ also fixes $P_\psi(x, y)$. Note that $\Gamma_\psi = \psi(K^\times) \cap \widetilde{\Gamma}$.

We say that $\tau \in \mathcal{H}_p$ has positive orientation if $\mathrm{red}_p(\tau) \in \mathcal{V}^+$. Denote by $\mathcal{H}_p^+$ the set of elements of $\mathcal{H}_p$ with positive orientation. Say $\psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$ has positive orientation if $\tau_\psi, \overline{\tau}_\psi \in \mathcal{H}_p^+$. Since $\mathcal{V} = \mathcal{V}^+ \sqcup \mathcal{V}^-$, we have

$$\mathrm{Emb}(\mathcal{O}, \mathcal{R}) = \mathrm{Emb}_+(\mathcal{O}, \mathcal{R}) \sqcup \mathrm{Emb}_-(\mathcal{O}, \mathcal{R}).$$

The group $\widetilde{\Gamma}$ acts on $\mathrm{Emb}(\mathcal{O}, \mathcal{R})$ by conjugation. Since Γ_ψ is infinite cyclic, we have

$$H_1(\Gamma_\psi, \Delta(P_{k-2})) = H^0(\Gamma_\psi, \Delta(P_{k-2})) := (\Delta(P_{k-2}))^{\Gamma_\psi}$$

(see Example 1, Chapter 3, page 58 of [7]). Since Γ_ψ acts trivially on $\tau_\psi \otimes D_K^{\frac{k-2}{4}} P_\psi^{\frac{k-2}{2}}(x, y)$, we have

$$D_{\psi,k} := \psi(u) \otimes (\tau_\psi \otimes D_K^{\frac{k-2}{4}} P_\psi^{\frac{k-2}{2}}(x, y)) \in H_1(\Gamma_\psi, \Delta(P_{k-2})).$$

The inclusion $\Gamma_\psi \subset \widetilde{\Gamma}$ induces a co-restriction

$$H_1(\Gamma_\psi, \Delta(P_{k-2})) \to H_1(\widetilde{\Gamma}, \Delta(P_{k-2})).$$

Hence, we can consider $D_{\psi,k}$ as an element in $H_1(\widetilde{\Gamma}, \Delta(P_{k-2}))$.

Lemma 4. *The cycle $D_{\psi,k}$ does not depend on the representative of the conjugacy class of $\psi \in Emb(\mathcal{O}, \mathcal{R})$. Hence we have a well-defined map*

$$D_k : \widetilde{\Gamma} \setminus Emb(\mathcal{O}, \mathcal{R}) \to H_1(\widetilde{\Gamma}, \Delta(P_{k-2}))$$

given by $D_k([\psi]) := D_{k,[\psi]}$, where $[\psi]$ denotes the conjugacy class of ψ.

Proof. See Lemma 2.19 of [41]. □

Definition 6. The Darmon cycle associated to the $\widetilde{\Gamma}$-conjugacy class $[\psi]$ is the element

$$D_{k,[\psi]} := \psi(u) \otimes (\tau_\psi \otimes D_K^{\frac{k}{4}} P_\psi^{\frac{k-2}{2}}(x,y)) \in H_1(\widetilde{\Gamma}, \Delta(P_{k-2})).$$

Given an embedding $\psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$, we define $\overline{\psi}$ to be the embedding given by $\overline{\psi}(\tau) := \psi(\overline{\tau})$ for $\tau \in K$.

Lemma 5. *We have*

$$(\tau_{\overline{\psi}}, P_{\overline{\psi}}, \gamma_{\overline{\psi}}) = (\overline{\tau}_\psi, -P_\psi, \gamma_\psi^{-1}).$$

Proof. By the definition of $\overline{\psi}$, we have $\tau_{\overline{\psi}} = \overline{\tau}_\psi$. Now $\overline{\psi}(\sqrt{D_K}) = \psi(-\sqrt{D_K})$. Hence $\overline{\psi}(\sqrt{D_K}) = \left(\begin{smallmatrix} -a & -b \\ -c & -d \end{smallmatrix}\right)$ and we get that $P_{\overline{\psi}}(x,y) = -P_\psi(x,y)$. Now since u is a fundamental unit, we have that $\overline{u} = u^{-1}$. Thus $\gamma_{\overline{\psi}} = \overline{\psi}(u) = \psi(u^{-1}) = \gamma_\psi^{-1}$. $\square$

Recall the *cohomological Abel-Jacobi map* we defined earlier:

$$\log \mathrm{AJ} : H_1(\widetilde{\Gamma}, \Delta(P_{k-2})) \to \mathbb{H}_k^{\pm,\vee}.$$

Definition 7. The *Darmon cohomology class* associated to $[\psi] \in \widetilde{\Gamma}/\mathrm{Emb}(\mathcal{O}, \mathcal{R})$ is $[j_\psi] := \log \mathrm{AJ}(D_{k,[\psi]}) \in \mathbb{H}_k^{\pm,\vee}$.

Denote by $\mathrm{Pic}^+(\mathcal{O})$ the *narrow Picard group* of strict equivalence class of fractional $\mathcal{O}$-ideals. By class field theory (see Theorem 4.2 of [34]), we have the reciprocity isomorphism

$$rec : \mathrm{Pic}^+(\mathcal{O}) \cong \mathrm{Gal}(H_{\mathcal{O}}^+/K),$$

where $H_{\mathcal{O}}^+$ is the narrow ring class field of K associated to the order $\mathcal{O}$.

Proposition 1 ([12, Proposition 5.8])**.** *The sets $\widetilde{\Gamma}/Emb(\mathcal{O}, \mathcal{R})$ and $Pic^+(\mathcal{O})$ are in bijection.*

To each narrow ideal class $\mathfrak{c} \in \mathrm{Pic}^+(\mathcal{O})$, let us denote by $\Psi_{\mathfrak{c}} \in \widetilde{\Gamma}/\mathrm{Emb}(\mathcal{O}, \mathcal{R})$ the class of embedding associated to it by the bijection of Proposition 1. We have a left action of $\mathrm{Pic}^+(\mathcal{O})$ on $\widetilde{\Gamma}/\mathrm{Emb}(\mathcal{O}, \mathcal{R})$ as follows:

$$\mathfrak{c}'.\Psi_{\mathfrak{c}} := \Psi_{\mathfrak{c}.\mathfrak{c}'},$$

where $\Psi_{\mathfrak{c}.\mathfrak{c}'}$ is the $\widetilde{\Gamma}$-equivalent class of embedding associated to the product $\mathfrak{c}.\mathfrak{c}' \in \mathrm{Pic}^+(\mathcal{O})$. By the reciprocity isomorphism, we have an action of $\mathrm{Gal}(H_{\mathcal{O}}^+/K)$ on $\widetilde{\Gamma}/\mathrm{Emb}(\mathcal{O}, \mathcal{R})$.

Let $\chi : \mathrm{Gal}(H_{\mathcal{O}}^+/K) \to \mathbb{C}^\times$ be a character. We will consider the following liner combination:

$$D_k^\chi := \sum_{\sigma \in \mathrm{Gal}(H_{\mathcal{O}}^+/K)} \chi^{-1}(\sigma) D_{\sigma.[\psi],k} \in H_1(\widetilde{\Gamma}, \Delta(P_{k-2}))^\chi.$$

3.3. **Rationality of Darmon cycles.** In this section, we discuss the rationality conjecture of Darmon cycles. Recall from the Introduction V_p^{Np}, the *local p-adic Galois representation* of $\mathrm{Gal}(\bar{\mathbb{Q}}_{\scriptscriptstyle |}/\mathbb{Q}_{\scriptscriptstyle |})$.

Proposition 2. *The local p-adic Galois representation V_p^{Np} is semistable but not crystalline.*

Proof. This is Corollary 7.5 of [23]. $\square$

Fontaine and Mazur attach to V_p^{Np} the admissible monodromy module

$$D^{\mathrm{FM}} := D_{st}(V_p^{Np}),$$

where $D_{st}(V_p^{Np}) := (V_p^{Np} \otimes B_{st})^{G_{\mathbb{Q}_p}}$. Here B_{st} is Fontaine's semistable ring of periods defined in [17]. Let

$$\mathbb{D}_k := \mathbb{H}_k^{\pm,\vee} \oplus \mathbb{H}_k^{\pm,\vee},$$

which is free of rank two over $\mathbb{T}_p := \mathbb{T}^{\mathrm{new}}_{\Gamma_0(Np)} \otimes \mathbb{Q}_p$. We define a filtration $F_{\mathbb{D}_k}$ on $\mathbb{D}_k$ as follows:

$$0 = F^k \subset F^{k-1} = \cdots = F^1 \subset F^0 = \mathbb{D}_k,$$

where $F^i = \{(-\mathcal{L}_{\mathrm{FM}}x, x) : x \in \mathbb{H}_k^{\pm,\vee}\}$ for all $1 \leq i \leq k-1$ for $\mathcal{L}_{\mathrm{FM}}$ the $\mathcal{L}$-invariant of D^{FM}. $\mathbb{D}_k$ along with the filtration $F_{\mathbb{D}_k}$ is a $\mathbb{T}_p$-monodromy module over $\mathbb{Q}_p$.

Theorem 5. *We have a $\mathbb{T}_p$-monodromy module isomorphism*

$$D^{FM} \cong \mathbb{D}_k.$$

Further the isomorphism is stable under base change to K_p, and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{H}_k^{\pm}(K_p)^\vee \oplus \mathbb{H}_k^{\pm}(K_p)^\vee & \longrightarrow & D^{FM} \otimes K_p \\ \downarrow & & \downarrow \\ \mathbb{H}_k^{\pm}(K_p)^\vee & \xrightarrow{\simeq} & \frac{D^{FM} \otimes K_p}{F^{k/2}(D^{FM} \otimes K_p)} \end{array}$$

where the vertical arrow is $(x, y) \mapsto x + \mathcal{L}_{FM}y$.

Proof. See Proposition 4.6 and Theorem 4.7 of [38]. $\square$

By composing with the isomorphism of Theorem 5 above, we can consider the cohomological Abel Jacobi map as

$$\log \mathrm{AJ} : H_1(\widetilde{\Gamma}, \Delta(P_{k-2})) \to \mathbb{H}_k^{\pm}(K_p)^\vee \xrightarrow{\sim} \frac{D^{\mathrm{FM}} \otimes K_p}{F^{k/2}(D^{\mathrm{FM}} \otimes K_p)}.$$

We will also simultaneously refer to the above map as the p-adic Abel-Jacobi map.

Definition 8. Let L be a number field in which p is unramified. For every place v of L, we define

$$H^1_{st}(L_v, V_p^{Np}) := \ker\left(H^1(L_v, V_p^{Np}) \to \begin{cases} H^1(L_v^{unr}, V_p^{Np}) & \text{if } v \nmid p \\ H^1(L_v, B_{st} \otimes_{\mathbb{Q}_p} V_p^{Np}) & \text{if } v \mid p \end{cases} \right).$$

The semistable *Selmer* group associated to V_p^{Np} is then defined as

$$\mathrm{Sel}_{st}(L, V^{Np}) := \ker\left(H^1(L, V^{Np}) \xrightarrow{\prod res_v} \prod_v \frac{H^1(L_v, V_p^{Np})}{H^1_{st}(L_v, V_p^{Np})} \right).$$

Since the local Galois representation V_p^{Np} is semi-stable, the Bloch-Kato isomorphism (see Section 3 of [6]) induces

$$\exp_{\mathrm{BK}} : \frac{D^{FM} \otimes_{\mathbb{Q}_p} K_p}{F^{k/2}(D^{FM} \otimes_{\mathbb{Q}_p} K_p)} \xrightarrow{\cong} H^1_{st}(K_p, V_p^{Np}(k/2)).$$

Composing the p-adic Abel-Jacobi map with the above Bloch-Kato exponential along with the isomorphism of Theorem 5 we have

$$\mathrm{AJ} : H_1(\widetilde{\Gamma}, \Delta(P_{k-2})) \to H^1_{st}(K_p, V_p^{Np}(k/2)),$$

where $\mathrm{AJ} := \log \mathrm{AJ} \circ \exp_{BK}$.

We consider the image of Darmon cycles under this map as cohomology classes

$$s_\psi \in H^1_{st}(K_p, V_p^{Np}(k/2)), \qquad s_\chi \in H^1_{st}(K_p(\chi), V_p^{Np}(k/2)).$$

Since p is inert in K, it splits completely in the narrow Hilbert class field H_K^+. Hence the embedding σ_p induces an inclusion $\sigma_p : H_K^+ \hookrightarrow K_p$. Therefore we have

$$\mathrm{res_p} : \mathrm{Sel}_{st}(H_K^+, V_p^{Np}(k/2)) \to H^1_{st}(K_p, V_p^{Np}(k/2)).$$

Conjecture 1 ([38, Conjecture 5.7]).

(i) *For* $[\psi] \in \widetilde{\Gamma}/Emb(\mathcal{O}, \mathcal{R})$*, there exists a global cycle* $S_\psi \in Sel_{st}(H_K^+, V_p^{Np}(k/2))$ *such that*

$$res_p(S_\psi) = AJ(D_{[\psi],k}).$$

As an immediate consequence, we have

(ii) *For* $\chi : \mathrm{Gal}(H_K^+/K) \to \mathbb{C}^\times$ *a character, there exists* $S_\chi \in Sel_{st}(H_\chi, V_p^{Np}(k/2))^\chi$*, where* $H_\chi \subseteq H_K^+$ *is the extension of* K *cut out by* χ*, such that*

$$res_p(S_\chi) = AJ(D_k^\chi).$$

Remark 5. The conjecture is known to be true when χ is a *genus character* of K. See Theorem 13 in Section 5 for the precise formulation.

4. p-adic L-functions

In this section, we will recall the construction and interpolation properties of certain p-adic L-functions attached to real quadratic fields owing to their relevance in the proof of the main theorem.

4.1. **Overconvergent modular symbols.**

Let $\mathcal{X}$ denote the rigid analytic p-adic weight space over $\mathbb{Q}_p$. For a finite extension $E/\mathbb{Q}_p$, the rational points are given by $\mathcal{X}(E) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, E^\times)$. We have a natural inclusion, $\mathbb{Z} \subset \mathcal{X}$, given by $k \to [t \mapsto t^{k-2}]$. We can write every $t \in \mathbb{Z}_p^\times$ in the form $t = [t]\langle t \rangle$ where $[t] \in (\mathbb{Z}/p)^\times$ and $\langle t \rangle \in 1 + p\mathbb{Z}_p$. Let $U \subset \mathcal{X}$ be an open affinoid defined over E. Every $\kappa \in U(E)$ can be written uniquely in the form

$$\kappa(t) = \epsilon(t)\chi(t)\langle t \rangle^s$$

for $\epsilon, \chi : \mathbb{Z}_p^\times \to E^\times$ characters of order $p-1$ and p respectively and $s \in \mathcal{O}_E$. An integer k corresponds to the character $k(t) = [t]^{k-2}\langle t \rangle^{k-2}$. We will restrict to a neighbourhood U of k_0 such that $\epsilon(t) = [t]^{k_0-2}$ and $\chi = 1$ for all $\kappa \in U(K)$. Note that for all $k \in U$, $[t]^{k-2} = [t]^{k_0-2} \implies k \equiv k_0 \bmod (p-1)$.

Denote the set of non-zero vectors in $\mathbb{Q}_p^2$ by W and consider the natural projection to $\mathbb{P}^1(\mathbb{Q}_p)$ which is continuous for the p-adic topology

$$\pi : W = \mathbb{Q}_p^2 - (0,0) \to \mathbb{P}^1(\mathbb{Q}_p),$$

$$\pi(x,y) := \frac{x}{y}.$$

Let $L \subset \mathbb{Q}_p^2$ be a $\mathbb{Z}_p$-lattice. Denote by $L' := L - pL$ its set of primitive (not divisible by p) vectors and denote $|L| := p^{\mathrm{ord}_p(\det\gamma)}$, where γ is any $\mathbb{Z}_p$-basis of L written as a 2 x 2 matrix. As before, let $L_* := \mathbb{Z}_p \oplus \mathbb{Z}_p$ and let $L_\infty := \mathbb{Z}_p \oplus p\mathbb{Z}_p$ be its neighbour in the Bruhat-Tits tree $\mathcal{T}$. Recall that to each edge $e \in \mathcal{E}(\mathcal{T})$ we can associate open compact subsets in W and $\mathbb{P}^1(\mathbb{Q}_p)$ as follows:

$$W_e := L'_{s(e)} \cap L'_{t(e)} \text{ and } U_e := \pi(W_e).$$

Let e_∞ be the edge between $v_* = [L_*]$ and $v_\infty = [L_\infty]$. Further denote by W_∞ the set W_{e_∞}.

Let Y be an open compact subset of either W or $\mathbb{P}^1(\mathbb{Q}_p)$ and denote by $\mathcal{A}(Y)$ the space of $\mathbb{Q}_p$-valued locally analytic functions on Y. Let $\mathcal{D}(Y)$ be the continuous $\mathbb{Q}_p$-dual of $\mathcal{A}(Y)$ which will be called the space of locally analytic distributions on Y. For $\mu \in \mathcal{D}(Y)$ and $F \in \mathcal{A}(Y)$, we use the measure theoretic definition, $\int_Y F d\mu$, to denote $\mu(F)$. Further, for any $X \subset Y$ compact open, write $\int_X F d\mu$ to denote $\mu(F.\chi_X)$, where χ_X is the characteristic function on X.

Recall from Section 3 the action of $GL_2(\mathbb{Q}_p)$ on the set of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. This induces an action of $GL_2(\mathbb{Z}_p)$ on L' for any lattice L. Further let $\mathbb{Z}_p^*$ act on the left on L' by multiplication $(t.(x,y) := (tx, ty))$. $\mathcal{D}(\mathbb{Z}_p^*)$ acts on $\mathcal{D}(Y)$ as follows:

$$\mathcal{D}(\mathbb{Z}_p^*) \times \mathcal{D}(Y) \to \mathcal{D}(Y) \quad (r, \mu) \mapsto r\mu,$$

where $r\mu$ is defined as the distribution

$$\int_{L'_*} F(x,y) d(r\mu)(x,y) := \int_{\mathbb{Z}_p^\times} \Big(\int_{L'_*} F(tx, ty) d\mu(x,y) \Big) dr(t).$$

Let $k \in \mathbb{Z}^{\geq 0}$ and let $U_k \subset \mathcal{X}$ be an affinoid neighbourhood of k. The associated affinoid algebra $A(U_k)$ has a natural $\mathcal{D}(\mathbb{Z}_p^\times)$-algebra as follows:

$$\mu \mapsto \left[\kappa \mapsto \int_{\mathbb{Z}_p^\times} \kappa(t) d\mu(t) \right].$$

Hence we can consider the completed tensor product over $\mathcal{D}(\mathbb{Z}_p^\times)$,

$$\mathcal{D}_{U_k} := A(U_k) \widehat{\bigotimes}_{\mathcal{D}(\mathbb{Z}_p^\times)} \mathcal{D}(L'_*).$$

Let $P_{k-2}(E)$ denote the space of homogeneous polynomials of degree $k-2$ in two variables over a field E. The group $SL_2(\mathbb{Z})$ acts on the right on $P_{k-2}(E)$ by

$$(P \mid \gamma)(x,y) := P((x,y).\gamma^{-1}) = P(dx - cy, -bx + ay)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. The dual space $V_{k-2}(E) := \mathrm{Hom}_E(P_{k-2}(E), E)$ is endowed with the natural dual left action.

Let $\Delta := \mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$ (resp. $\Delta^0 := \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$) denote the space of divisors (resp. divisors of degree zero) over $\mathbb{P}^1(\mathbb{Q})$. Δ and Δ^0 are endowed with a natural left action of $SL_2(\mathbb{Z})$ acting via fractional linear transformations. The space $\mathrm{Hom}(\Delta^0, V_{k-2})$ has an induced right action of $SL_2(\mathbb{Z})$ given by

$$\phi \mid \gamma(D) := \phi(\gamma.D) \mid \gamma,$$

where $\gamma \in GL_2(\mathbb{Q})$ and $\phi : \Delta^0 \to V_{k-2}$. For Γ a congruence subgroup of $SL_2(\mathbb{Z})$ (usually $\Gamma_0(N)$ or $\Gamma_1(N)$), let $\mathrm{Symb}_\Gamma(V_{k-2}) \subset \mathrm{Hom}(\Delta^0, V_{k-2})$ be the sub-module invariant under the action of Γ. We call $\mathrm{Symb}_\Gamma(V_{k-2})$ the space of *modular symbols* on Γ. The matrix $W_\infty := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts as an involution on $V_{k-2}(E)$ (assuming $2 \nmid \mathrm{char}(E)$). We will denote by $V_{k-2}^{w_\infty}$ the direct summand of $V_{k-2} = V_{k-2}^{+} \oplus V_{k-2}^{-}$ on which W_∞ acts by $w_\infty \in \{\pm 1\}$.

Consider the $GL_2^+(\mathbb{Q})$-equivariant map

$$\widetilde{\phi} : S_k(\Gamma_0(N), \mathbb{C}) \to \mathrm{Symb}_{\Gamma_0(N)}(V_{k-2}(\mathbb{C}))$$

$$\widetilde{\phi_f}\{x - y\}(P) := 2\pi i \int_x^y f(z) P(z, 1) dz \in \mathbb{C}$$

for $P(x, y) \in P_{k-2}(\mathbb{C})$. Since Δ^0 is generated by divisors of the form $\{x - y\}$ for $x, y \in \mathbb{P}^1(\mathbb{Q})$, we extend the map ϕ by linearity to all of Δ^0.

We can write $\widetilde{\phi_f} = \widetilde{\phi_f^+} + \widetilde{\phi_f^-}$ where $\widetilde{\phi_f}^{\pm} \in \mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C})^{\pm})$.

Theorem 6 (Shimura)**.** *If f is a newform on $\Gamma_0(N)$, then there exist complex periods $\Omega^\pm \in \mathbb{C}$ such that*

$$\phi_f^\pm := \frac{\widetilde{\phi_f^\pm}}{\Omega^\pm} \in Symb_{\Gamma_0(N)}(V_{k-2}(K_f)^\pm).$$

The periods $\Omega^\pm$ can be chosen to satisfy

$$\Omega^+ \Omega^- = \langle f, f \rangle.$$

Proof. This is Theorem 1(ii) of [45]. $\square$

Recall the *Coleman family* defined over U:

$$\mathbf{f}(q) = \sum_{n \geq 1} \mathbf{a}_n(\kappa) q^n \in \mathcal{O}(U)[[q]]$$

and the classical forms associated to it:

$$f_k^\# \in S_k(\Gamma_0(N))^{\mathrm{new}} \quad and f_{k_0} \in S_{k_0}(\Gamma_0(Np))^{\mathrm{new}}.$$

To each $f_k^\#$ (resp. f_k) ($k \neq k_0$) we can attach a modular symbol, $\widetilde{\phi}_k^\# \in \mathrm{Symb}_{\Gamma_0(N)}(\mathbb{C})$ (resp. $\widetilde{\phi}_k \in \mathrm{Symb}_{\Gamma_0(Np)}(\mathbb{C})$) as above. Further, by Theorem 6, there exist complex Shimura periods $\Omega_k^{\#,\pm} \in \mathbb{C}$ (resp. $\Omega_k^\pm \in \mathbb{C}$) such that

$$\phi_k^{\#,\pm} := \frac{\widetilde{\phi_k^{\#,\pm}}}{\Omega_k^{\#,\pm}} \in \mathrm{Symb}_{\Gamma_0(N)}(V_{k-2}(K_{f_k^\#})^\pm)$$

and

$$\phi_k^\pm := \frac{\widetilde{\phi_k^\pm}}{\Omega_k^\pm} \in \mathrm{Symb}_{\Gamma_0(Np)}(V_{k_0-2}(K_{f_k})^\pm).$$

We will choose from now on a sign $w_\infty \in \{\pm 1\}$ compatible with the same choice we made for $\mathbb{H}_k^\pm$ and set

$$\phi_k^? := \phi_k^{?, w_\infty},$$

where $? \in \{\emptyset, \#\}$.

We have the following relation between ϕ_k and $\phi_k^\#$:

$$\phi_k\{r \to s\}(P) = \phi_k^\#\{r \to s\}(P) - \frac{p^{k/2-1}}{a_p(k)} \phi_k^\#\{r/p \to s/p\}(P(x, y/p)). \tag{2}$$

Definition 9. The space of overconvergent modular symbols for Γ is defined as the space of modular symbols with coefficients in $\mathcal{D}_U = A(U)\widehat{\bigotimes}_{\mathcal{D}(\mathbb{Z}_p^\times)}\mathcal{D}(L'_*)$. We will denote it by $\mathrm{Symb}_\Gamma(\mathcal{D}_U)$.

For every $k \in U^{\mathrm{cl}}$, we have a weight k-specialization map

$$\rho_k : \mathrm{Symb}_{\Gamma_0(N)}(\mathcal{D}_U) \to \mathrm{Symb}_{\Gamma_0(Np)}(V_{k-2}(\overline{\mathbb{Q}_p}))$$

$$\rho_k(I)\{r \to s\}(P) := \int_{W_\infty} P(x,y)dI\{r \to s\}(x,y).$$

Theorem 7 (G. Stevens)**.** *There exists $\Phi_* \in Symb_{\Gamma_0(N)}(\mathcal{D}_U)$ such that*
- *for any $k \in U^{\mathrm{cl}}$, the weight k-specialization, $\rho_k(\Phi_*) = \lambda(k)\phi_k$ for some constant $\lambda(k) \in \overline{\mathbb{Q}_p}^\times$,*
- *$\rho_{k_0}(\Phi_*) = \phi_{k_0}$.*

Proof. See Theorem 6.4.1 of [2]. □

We can define a family of distributions $\{\Phi_L\} \in \mathrm{Symb}_{\Gamma_0(N)}(\mathcal{D}_U)$, indexed by lattices $L \subset \mathbb{Q}_p^2$, as follows: for all $F \in \mathcal{A}(L')$,

$$\Phi_{L_*} := \Phi_*, \qquad \Phi_L\{r \to s\}(F) := \Phi_{L_*}\{\gamma r \to \gamma s\}(F \mid \gamma^{-1}),$$

where $\gamma.L = L_*$. This will be used in defining the p-adic L-functions associated to real quadratic fields (see Section 4.3).

4.2. **The Stevens-Mazur-Kitagawa p-adic L-function.** In this section, we recall the construction and interpolation property of the two variable Stevens-Mazur-Kitagawa p-adic L-function. The original construction of Mazur and Kitagawa ([24]) was only for the ordinary case (Hida families) and was extended by G. Stevens to the finite slope case using overconvergent modular symbols in [48].

Let $g \in S_k(\Gamma_0(N))$ be a cusp form and let $\phi_g \in \mathrm{Symb}_{\Gamma_0(N)}(K_g)$ be the modular symbol attached to g. For $n \in \mathbb{Z}^{>0}$, define

$$\phi_{g,n} : P_{k-2}(K_g) \times \mathbb{Z}/n\mathbb{Z} \to K_g$$

$$\phi_{g,n}(P,a) := \phi_g\{\infty \to a/n\}(P).$$

Since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & n \end{pmatrix} = \begin{pmatrix} 1 & -a+n \\ 0 & n \end{pmatrix}$$

and ϕ_g is invariant under the action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N), \phi_{g,n}$ depends only on the class of $a \in \mathbb{Z}/n$.

Let

$$\tau(\chi) := \sum_{a \in \mathbb{Z}/n\mathbb{Z}} \chi(a)e^{2\pi i a/n}$$

be the Gauss sum associated to χ, a primitive Dirichlet character mod n. For

$$\tilde{L}(g,\chi,j) := \frac{(j-1)!\tau(\chi)}{(-2\pi i)^{j-1}\Omega_g}L(g,\chi,j)$$

the 'algebraic part' of $L(g,\chi,j)$ and $P_{j,a} := \left(x - \frac{a}{n}y\right)^{j-1} y^{k-j-1} \in P_{k-2}(K_g)$, we have the following results:

Proposition 3. *For every integer* $1 \leq j \leq k-1$ *such that* $\chi(-1) = (-1)^{j-1}w_\infty$, *we have*

$$\sum_{a \in \mathbb{Z}/n\mathbb{Z}} \chi(a)\phi_{g,n}(P_{j,a}, a) = \tilde{L}(g,\chi,j).$$

Proof. This is a straightforward calculation relating L-values and modular symbols. This has been shown in [31, §7]. The relevant calculation for the twisted L-values is in [31, §8]. □

Remark 6. The proposition shows that $\tilde{L}(g,\chi,j)$ belongs to $K_g(\chi)$. In particular these quantities are algebraic and hence can be seen as p-adic numbers, thus making it possible to interpolate them p-adically. In the thesis, we deal with square free level and quadratic twists, which implies that K_g is a totally real field and that $\tilde{L}(g,\chi,j) \in \mathbb{R}$.

Let $(x,y) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ and let $p \nmid n$. Then we have

$$x - \frac{pa}{n}y \in \mathbb{Z}_p^\times + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times.$$

Hence, for $\kappa \in U$, we have the *locally analytic function*

$$F_{s,pa} := \left(x - \frac{pa}{n}y\right)^{s-1} y^{\kappa-s-1} \in \mathcal{A}_U(L'_*).$$

Definition 10. Let $\mathbf{f}(\mathbf{q})$ be the Coleman family of tame level N and let $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ be a Dirichlet character of conductor n such that $p \nmid n$. We define the Stevens-Mazur-Kitagawa p-adic L-function as follows:

$$\mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f},\chi,\kappa,s) : U \times \mathbb{Z}_p \to \mathbb{C}_p$$

$$(\kappa,s) \mapsto \mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f},\chi,\kappa,s)$$

$$\mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f},\chi,\kappa,s) := \sum_{a \in \mathbb{Z}/n\mathbb{Z}} \chi(ap) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} F_{s,pa} d\Phi_*\{\infty \to \frac{pa}{n}\},$$

where Φ_* is the big modular symbol from Theorem 7.

Interpolation of special values. We now recall an important result about the interpolation of classical L-values by the Stevens-Mazur-Kitagawa p-adic L-function $\mathcal{L}_p^{\mathrm{SMK}}$.

Theorem 8. *Let* $k \in U^{cl}$ *be a classical weight and let* χ *be a primitive character. For all integers* $1 \leq j \leq k-1$ *such that* $\chi(-1) = (-1)^{j-1}w_\infty$ *we have*

$$\mathcal{L}_p^{SMK}(\mathbf{f},\chi,k,j) = \lambda(k)\left(1 - \chi(p)\frac{p^{j-1}}{a_p(k)}\right)\tilde{L}(f_k,\chi,j).$$

Proof. The interpolation property of the two-variable p-adic L-function follows from Proposition 3.23 in [4] along with Theorem 7. □

We will be particularly interested in the specialization of $\mathcal{L}_p^{\mathrm{SMK}}$ to the central line $j = k/2$. Since we are working with the newforms $f_k^{\#}$, the following relation between the L-values of f_k and $f_k^{\#}$ will be useful:

$$\tilde{L}(f_k, \chi, j) = \left(1 - \chi(p)\frac{p^{k-j-1}}{a_p(k)}\right)\tilde{L}(f_k^{\#}, \chi, j). \tag{3}$$

Corollary 2. *For $k \neq k_0 \in U^{cl}$, suppose $\chi(-1) = (-1)^{\frac{k-2}{2}} w_\infty$. Then we have*

$$\mathcal{L}_p^{SMK}(\mathbf{f}, \chi, k, k/2) = \lambda(k)\left(1 - \chi(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\right)^2 \tilde{L}(f_k^{\#}, \chi, k/2).$$

For $k = k_0$,

$$\mathcal{L}_p^{SMK}(\mathbf{f}, \chi, k_0, k_0/2) = \left(1 - \chi(p)\frac{p^{\frac{k_0-2}{2}}}{a_p(k)}\right)\tilde{L}(f_{k_0}^{\#}, \chi, k_0/2),$$

where $f_{k_0}^{\#} = f_{k_0}$.

Proof. This follows from Theorem 7 and equation (3). $\square$

4.3. **p-adic L-functions attached to real quadratic fields.**

4.3.1. *p is inert in K.* In this section we will recall the construction and interpolation properties of a p-adic L-function attached to real quadratic fields due to M. Seveso in [41]. Recall that $K/\mathbb{Q}$ is a real quadratic field such that

- all the primes dividing N split in K while
- p is inert in K.

Recall the set $\mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$ introduced in Section 3.2. Let $\Psi \in \mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$ have conductor prime to D_K and Np (i.e., the order $\mathcal{O}$ has conductor prime to D_K and Np). Consider the triple $(\tau_\Psi, P_\Psi, \gamma_\Psi)$ associated to Ψ and a lattice $L_\Psi \subset \mathbb{Q}_p^2$ such that $v_\Psi = [L_\Psi]$.

Definition 11. Let $s \in \mathbb{P}^1(\mathbb{Q})$ be an arbitrary base point. For $\mathbf{f}(\mathbf{q})$ the Coleman family and $\Psi \in \mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$, we define the partial p-adic L-function $L_p^{\mathrm{Sev}}(\mathbf{f}/K, \Psi, -) : U \to \mathbb{C}_p$ as follows:

$$L_p^{\mathrm{Sev}}(\mathbf{f}/K, \Psi, \kappa) := |L_\psi|^{-\frac{k_0-2}{2}} \int_{L'_\Psi} \langle P_\Psi(x, y)\rangle^{\frac{\kappa-k_0}{2}} P_\Psi^{\frac{k_0-2}{2}}(x, y) d\Phi_{L_\Psi}\{s \to \gamma_\Psi s\}.$$

For $\chi : \mathrm{Gal}(H_\mathcal{O}^+/K) \to \mathbb{C}^\times$ a character, we define

$$L_p^{\mathrm{Sev}}(\mathbf{f}/K, \chi, \kappa) := \sum_{\sigma \in \mathrm{Gal}(H_\mathcal{O}^+/K)} \chi^{-1}(\sigma) L_p^{\mathrm{Sev}}(\mathbf{f}/K, \sigma\Psi, \kappa).$$

The p-adic L-function $\mathcal{L}_p^{\mathrm{Sev}}(\mathbf{f}/K, \chi, -) : U \to \mathbb{C}_p$ is then defined as

$$\mathcal{L}_p^{\mathrm{Sev}}(\mathbf{f}/K, \chi, \kappa) = L_p^{\mathrm{Sev}}(\mathbf{f}/K, \chi, \kappa)^2.$$

Remark 7. Unlike the Stevens-Mazur-Kitagawa p-adic L-function, the definition of $\mathcal{L}_p^{\mathrm{Sev}}$ depends on the class of the embedding $\Psi \in \widetilde{\Gamma}/\mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$. We make a suitable choice for L_Ψ as follows: choose $\gamma \in \widetilde{\Gamma}$ such that $\gamma v_\Psi = v_*$. This is possible since $\widetilde{\Gamma}$ acts transitively on $\mathcal{V}^+$. Thus $v_* = v_{\gamma\Psi\gamma^{-1}}$ and $L_* = L_{\gamma\Psi\gamma^{-1}}$ are associated to the embedding $\gamma\Psi\gamma^{-1} \in [\Psi]$, and the choice of the modular symbols Φ_{L_Ψ} can be taken to be the big modular symbol Φ_*. This will allow us to compare $\mathcal{L}_p^{\mathrm{Sev}}$ with the Stevens-Mazur-Kitagawa p-adic L-function.

Definition 12. A *genus character* of K is a *quadratic unramified* character $\chi : \mathrm{Gal}(H_K^+/K) \to \mathbb{C}^\times$ which cuts out a bi-quadratic extension $\mathbb{Q}(\sqrt{D_1}, \sqrt{D_2})$ of $\mathbb{Q}$ where $D_K = D = D_1.D_2$ is a factorization into co-prime factors of the discriminant of $K = \mathbb{Q}(\sqrt{D})$. In particular, the genus characters of K are in bijection with the factorization of the D into a product of two relatively prime fundamental discriminants.

Let χ_{D_i} denote the Dirichlet character associated to $\mathbb{Q}(\sqrt{D_i})$. Then $\chi_D = \chi_{D_1}.\chi_{D_2}$ and since K is real quadratic, we have

$$1 = \chi_D(-1) = \chi_{D_1}(-1)\chi_{D_2}(-1).$$

Since p is inert in K, $D_K \in \mathbb{Z}_p^\times$, and $D_K^{\frac{k-2}{2}}$ extends to the analytic function on U, $\langle D_K\rangle^{\frac{\kappa-2}{2}}$. We now state the result about the factorization of $\mathcal{L}^{\mathrm{Sev}}(\mathbf{f}/K, \chi, \kappa)$.

Theorem 9. *Suppose* $\chi(-1) = (-1)^{\frac{k_0-2}{2}} w_\infty$. *Then*

$$\mathcal{L}_p^{Sev}(\mathbf{f}/K, \chi, \kappa) = D_K^{\frac{\kappa-2}{2}} \mathcal{L}_p^{SMK}(\mathbf{f}, \chi_{D_1}, \kappa, \kappa/2)\mathcal{L}_p^{SMK}(\mathbf{f}, \chi_{D_2}, \kappa, \kappa/2).$$

Proof. This is Theorem 5.9 of [41]. □

Interpolation properties of $\mathcal{L}_p^{\mathbf{Sev}}$. For $g \in S_k(\Gamma_0(N))$ and ψ any character of the narrow class group of K, let $L(g/K, \psi, k/2)$ denote the central L-value of the (completed) Rankin L-series attached to $\pi_g \times \pi_\psi$ where π_g (resp. π_ψ) denotes the cuspidal automorphic form of $GL_2(\mathbb{A}_\mathbb{Q})$ attached to g (resp. ψ via the Jacquet-Langlands correspondence).

Let

$$L^*(g/K, \psi, k/2) := \frac{\left(\frac{k-2}{2}\right)!^2\sqrt{D_K}}{(2\pi i)^{k-2}(\Omega_g)^2} L(g/K, \psi, k/2).$$

We can now state a result about the interpolation properties of Seveso's p-adic L-function.

Theorem 10. *For all* $k \neq k_0 \in U^{cl}$,

$$\mathcal{L}_p^{Sev}(\mathbf{f}/K, \psi, k) = \lambda(k)^2\big(1 - p^{k-2}a_p(k)^{-2}\big)^2 D_K^{\frac{k-2}{2}} L^*(f_k^\#/K, \psi, k/2)$$

and

$$\mathcal{L}_p^{Sev}(\mathbf{f}/K, \psi, k_0) = 0.$$

Proof. See Theorem 5.8 of [41]. □

4.3.2. *p is split in K.* Only in this section we assume that K is a real quadratic field that satisfies the *Heegner hypothesis*; i.e., all the primes dividing Np are split in K. Let us denote by $\mathfrak{p}$ and $\mathfrak{p}'$ the primes above p.

Let $\Psi \in \mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$ be as above and let

$$e_1 := \begin{pmatrix} \tau_\Psi \\ 1 \end{pmatrix}, \qquad e_2 := \begin{pmatrix} \overline{\tau}_\Psi \\ 1 \end{pmatrix},$$

where τ_Ψ and $\overline{\tau_\Psi}$ are the fixed points in $\mathcal{H}_p$ associated to Ψ. Note that (e_1, e_2) is a $\mathbb{Z}_p$-basis for L_Ψ. Let $L''_\Psi \subset \mathbb{Q}_p^2$ be the set $\mathbb{Z}_p^\times e_1 \oplus \mathbb{Z}_p^\times e_2$. We will now recall the construction of a p-adic L-function due to Greenberg-Seveso-Shahabi in this setting (see Section 5.1 of [19]).

Definition 13. For $s \in \mathbb{P}^1(\mathbb{Q})$ an arbitrary base point and $\Psi \in \mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$, we define the partial p-adic L-function $L_p^{\mathrm{GSS}}(\mathbf{f}/K, \Psi, -) : U \to \mathbb{C}_p$ as follows:

$$L_p^{\mathrm{GSS}}(\mathbf{f}/K, \Psi, \kappa) := |L_\psi|^{-\frac{k_0-2}{2}} \int\limits_{L''_\Psi} \langle P_\Psi(x,y)\rangle^{\frac{\kappa-k_0}{2}} P_\Psi^{\frac{k_0-2}{2}}(x,y) d\Phi_{L_\Psi}\{s \to \gamma_\Psi s\}.$$

For $\chi : \mathrm{Gal}(H^+_{\mathcal{O}}/K) \to \mathbb{C}^\times$ a character, we define

$$L_p^{\mathrm{GSS}}(\mathbf{f}/K, \chi, \kappa) := \sum_{\sigma \in \mathrm{Gal}(H^+_{\mathcal{O}}/K)} \chi^{-1}(\sigma) L_p^{\mathrm{GSS}}(\mathbf{f}/K, \sigma\Psi, \kappa).$$

We then define the p-adic L-function $\mathcal{L}_p^{\mathrm{GSS}}(\mathbf{f}/K, \chi, -) : U \to \mathbb{C}_p$ to be

$$\mathcal{L}_p^{\mathrm{GSS}}(\mathbf{f}/K, \chi, \kappa) = L_p^{\mathrm{GSS}}(\mathbf{f}/K, \chi, \kappa)^2.$$

We can now state a result about the interpolation of $\mathcal{L}_p^{\mathrm{GSS}}$.

Theorem 11. *We have*

$$\mathcal{L}_p^{GSS}(\mathbf{f}/K, \psi, \kappa) = \Big(1 - \frac{\psi(\mathfrak{p})p^{\frac{k-2}{2}}}{a_p(k)}\Big)^2 \Big(1 - \frac{\psi(\mathfrak{p}')p^{\frac{k-2}{2}}}{a_p(k)}\Big)^2 L^*(f_k^{\#}/K, \psi, k/2).$$

Proof. See Theorem 1.6 and Proposition 5.5 of [19]. □

4.4. **Derivative of $L_p^{\mathbf{Sev}}$ and Darmon cycles.** For the rest of the thesis, we will assume that K is a real quadratic field in which p is split, unless otherwise stated. Recall the *cohomological Abel-Jacobi* map from Section 3:

$$\log \mathrm{AJ} : H_1(\tilde{\Gamma}, \Delta(P_{k-2})) \to \mathbb{H}_k^{w_\infty, \vee}.$$

Let $D_{[\Psi],k}$ be the *Darmon cycle* associated to the class of the embedding $[\Psi]$. By the Eichler-Shimura isomorphism, $\log \mathrm{AJ}(D_{[\Psi],k})$ can be considered an element in $\mathrm{Symb}_{\Gamma_0(Np)}(V_{k-2})^\vee$. We can consider the derivative of the partial p-adic L-function along the weight direction, i.e. $\frac{d}{d\kappa}[L_p^{\mathrm{Sev}}(\mathbf{f}/K, \Psi, \kappa)]$. We have the following result of Seveso:

Theorem 12. *We have*

$$\begin{aligned}&\frac{d}{d\kappa}[L_p^{Sev}(\mathbf{f}/K, \Psi, \kappa)]_{\kappa=k_0} \\ &\qquad = \frac{1}{2} D_K^{\frac{k_0-2}{4}} \Big(\log : \mathrm{AJ}(D_{[\Psi],k_0})(\phi_{k_0}) + (-1)^{k_0/2} \log : \mathrm{AJ}(D_{[\overline{\Psi}],k_0})(\phi_{k_0})\Big).\end{aligned}$$

Let $\chi : \mathrm{Gal}(H_K^+/K) \to \pm 1$ be the genus character corresponding to a factorization $D = D_1.D_2$. Since all primes dividing N split in K, we have

$$1 = \chi_D(N) = \chi_{D_1}(N)\chi_{D_2}(N).$$

Since $\chi_{D_1}(-1) = \chi_{D_2}(-1)$, we have $\chi_{D_1}(-N) = \chi_{D_2}(-N)$, which we will simply write as $\chi_{D_i}(-N)$.

Corollary 3. *We have*

$$\frac{d}{d\kappa}[L_p^{Sev}(\mathbf{f}/K, \chi, \kappa)]_{\kappa=k_0} = \frac{1}{2} D_K^{\frac{k_0-2}{4}} \Big(1 + (-1)^{k_0/2} w_N \chi_{D_i}(-N)\Big) \log : \mathrm{AJ}(D_{k_0}^{\chi})(\phi_{k_0}).$$

Remark 8. Since p is inert in K, we have that $\chi_D(p) = -1$. This implies that $\chi_{D_1}(-p) = -\chi_{D_2}(-p)$.

5. Main theorem

5.1. **Rationality of Darmon cycles - II.** Recall that in Section 3, we stated the rationality conjecture of Darmon cycles. Here we will state some known results towards this conjecture which will be used in the main theorem. We have the *global p-adic Abel-Jacobi* map

$$\mathrm{cl}_{0,L}^{k_0/2} : \mathrm{CH}_0^{k_0/2}(\mathcal{M}_{k_0} \otimes L) \to \mathrm{Sel}_{st}(L, V_p^{Np}(k_0/2))$$

(see Introduction).

Let $\chi : \mathrm{Gal}(H_K^+/K) \to \mathbb{C}^\times$ be the genus character of K corresponding to the factorization $D = D_1 D_2$ as above. Note that we have $\chi_{D_1^*}(-N) = \chi_{D_2^*}(-N)$ while $\chi_{D_1^*}(-p) = -\chi_{D_2^*}(-p)$. We now recall a theorem about the rationality of Darmon cycles:

Theorem 13. *Assume* $(-1)^{k_0/2} w_N \chi_{D_i^*}(-N) = 1$ *for* $i \in \{1, 2\}$*. Then, there exist a global cycle*

$$d_{k_0}^{\chi_{D_2^*}} \in CH_0^{k_0/2}(\mathcal{M}_{k_0} \otimes \mathbb{Q}(\sqrt{D_2^*}))^{\chi_{D_2^*}} \subset (\mathcal{M}_{k_0} \otimes \mathbb{Q}(\sqrt{D_2^*}, \sqrt{D_1^*}))$$

and a constant $s_f \in K_{f_{k_0}}^\times$ *such that*

$$res_p(cl_{0,H_K^+}^{k_0/2}(d_{k_0}^{\chi_{D_2^*}})) = s_f AJ(D_{k_0}^\chi).$$

Proof. This is Theorem 6.2 of [41]. The notation used here is different from [41], but it is similar to Theorem 6.11 of [19], which is a generalization of the above theorem to the quaternionic setting. ☐

Remark 9. Note that the global algebraic cycle $d_{k_0}^{\chi_{D_2^*}}$ depends only on D_2 and not on D_1.

5.2. **Normalized Fourier coefficients** $\widetilde{c}(D, k)$**.** Recall the *Coleman family* of cusp forms of tame level $\Gamma_0(N)$ over U:

$$\mathbf{f}(q) = \sum_{n \geq 1} \mathbf{a}_n(\kappa) q^n \in \mathcal{O}(U)[[q]]$$

and the classical cusp forms $f_k^\#$, for $k \in U^{\mathrm{cl}}/k_0$. Note that $f_k^\# \in S_k(\Gamma_0(N))^{\mathrm{new}}$ for $k \neq k_0$ while $f_{k_0}^\# = f_{k_0} \in S_{k_0}(\Gamma_0(Np))^{\mathrm{new}}$.

For all $k \neq k_0 \in U^{\mathrm{cl}}$, let

$$g_k := \sum_{D>0} c(D, k) q^D \in S_{\frac{k+1}{2}}^{+,\mathrm{new}}(\Gamma_0(4N))$$

be the Shintani lifting of $f_k^\#$ and let

$$g_{k_0} := \sum_{D>0} c(D, k_0) q^D \in S_{\frac{k_0+1}{2}}^{+,\mathrm{new}}(\Gamma_0(4Np))$$

be the Shintani lifting of $f_{k_0} = f_{k_0}^\#$.

Remark 10. Recall that for D a Type II discriminant, $L(f_{k_0}, D^*, k_0/2) = 0$ since the sign in the function equation $w(f_{k_0}, D^*) = -1$ and hence $c(D, k_0) = 0$. On the other hand, by the non-vanishing results for quadratic twists of modular L-functions due to Waldspurger (see Theorem 1.1 of [32]), there exist infinitely many fundamental discriminants D_1 of Type I such that $L(f, \chi_{D_1^*}, k_0/2) \neq 0$ and consequently $c(D_1, k_0) \neq 0$.

We fix a Type I discriminant D_1 such that $c(D_1, k_0) \neq 0$.

Lemma 6. *Up to further shrinking of U, $c(D_1, k)$ is nonvanishing for all $k \in U^{cl}$.*

Proof. The proof is similar to Lemma 3.2 of [29]. By Corollary 1,

$$c(D_1, k) \neq 0 \Leftrightarrow L(f_k^{\#}, \chi_{D_1^*}, k/2) \neq 0.$$

Recall the algebraic part of the central L-value:

$$\tilde{L}(f_k^{\#}, \chi_{D_1^*}, k/2) := \frac{(k/2-1)!\tau(\chi_{D_1^*})}{(-2\pi i)^{k/2-1}\Omega_{f_k^{\#}}} L(f_k^{\#}, \chi, k/2).$$

By the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we can look at $\tilde{L}(f_k^{\#}, \chi_{D_1^*}, k/2)$ as p-adic numbers. It suffices to show the non-vanishing of $\tilde{L}(f_k^{\#}, \chi_{D_1^*}, k/2)$ in a neighbourhood of k_0. Fix the choice of w_∞ such that $\chi_{D_1^*}(-1) = (-1)^{\frac{k_0-2}{2}} w_\infty$. By the interpolation property of the Stevens-Mazur-Kitagawa p-adic L-function attached to $\mathbf{f}(q)$, we have

$$\mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f}, \chi_{D_1^*}, k_0, k_0/2) = \Big(1 - \chi_{D_1^*}(p)\frac{p^{\frac{k_0-2}{2}}}{a_p(k_0)}\Big)\tilde{L}(f_{k_0}^{\#}, \chi_{D_1^*}, k_0/2).$$

Note that $a_p(k_0) = -w_p p^{\frac{k_0-2}{2}}$. Since D_1 is a Type I discriminant, $\chi_{D_1^*}(p) = w_p$. Hence the Euler-like factor $\Big(1 - \chi_{D_1^*}(p)\frac{p^{\frac{k_0-2}{2}}}{a_p(k_0)}\Big)$ is non-zero. This establishes the non-vanishing of $\mathcal{L}_p^{\mathrm{SMK}}$ at $(k_0, k_0/2)$. Since the Stevens-Mazur-Kitagawa p-adic L-function is a non-zero p-adic analytic function, up to shrinking U, we have the non-vanishing of $\mathcal{L}_p^{\mathrm{SMK}}$. Thus the non-vanishing result for $c(D_1, k)$ follows. $\square$

Let us also fix a Type II discriminant D_2 relatively prime to D_1 and let $D_K := D_1^* D_2^*$. Let $K := \mathbb{Q}(\sqrt{D_K})$ be the real quadratic field with discriminant D_K. We have a bijection between $\mathcal{F}_{D_K}/\Gamma_0(N)$, G_{D_K}, and $\mathrm{Gal}(H_K^+/K)$, where $\mathcal{F}_{D_K}$ is the set of Heegner forms of level N of discriminant D_K, G_{D_K} is the $SL_2(\mathbb{Z})$-equivalence class of primitive integral binary quadratic forms of discriminant D_K, and H_K^+ is the narrow Hilbert class field of K.

Let $\Psi_* \in \mathrm{Emb}_+(\mathcal{O}_K, \mathcal{R})$ be the optimal embedding such that $v_{\Psi_*} = v_* = [\mathbb{Z}_p^2]$. Note that

$$r_{k,N}(f_k^{\#}, D_1^*, D_2^*) = \sum_{Q \in \mathcal{F}_D/\Gamma_0(N)} \omega_{D_1^*, D_2^*}(Q) r(f_k^{\#}, Q), \tag{4}$$

where $\omega_{D_1^*, D_2^*}$ is the genus character corresponding to the bi-quadratic extension $\mathbb{Q}(\sqrt{D_1^*}, \sqrt{D_2^*})$ of K. Let $\chi_{D_1^*, D_2^*} : \mathrm{Gal}(H_K^+/K) \to \{\pm 1\}$ be the character obtained by composing $\omega_{D_1^*, D_2^*}$ with the isomorphism between G_{D_K} and $\mathrm{Gal}(H_K^+/K)$. Hence we can re-write (4) as

$$r_{k,N}(f_k^{\#}, D_1^*, D_2^*) = \sum_{\sigma \in \mathrm{Gal}(H_K^+/K)} \chi_{D_1^*, D_2^*}(\sigma) r(f_k^{\#}, P_{\sigma.[\Psi_*]}(x,y)).$$

Since D is not a perfect square, for all $\Psi \in \mathrm{Emb}_+(\mathcal{O}_K, \mathcal{R})$, we have

$$\frac{(2\pi i) r(f_k^{\#}, P_\Psi(x,y))}{\Omega_k^{\#}} = \phi_k^{\#}\{r \to \gamma_\Psi . r\}(P_\Psi^{\frac{k-1}{2}}(x,y)). \tag{5}$$

Lemma 7. *For all $k \in U^{cl}$ and $P(x,y) \in P_{k-2}$,*

$$\int\limits_{(\mathbb{Z}_p^2)'} P(x,y) d\Phi_*\{r \to s\} = \lambda(k)(1 - a_p(k)^{-2}p^{k-2})\phi_{f_k^\#}\{r \to s\}(P(x,y)).$$

Proof. See Corollary 4.6 of [41] and Proposition 2.4 of [5]. □

Now since $P_{\Psi_*}(x,y) \in \mathbb{Z}_p^\times$ for all $(x,y) \in (\mathbb{Z}_p^2)'$, we have by Lemma 4.1 of [41]

$$P_{\Psi_*}^{\frac{k-2}{2}}(x,y) = P_{\Psi_*}^{\frac{k-k_0}{2}}(x,y).P_{\Psi_*}^{\frac{k_0-2}{2}}(x,y) = \langle P_{\Psi_*}(x,y)\rangle^{\frac{k-k_0}{2}} P_{\Psi_*}^{\frac{k_0-2}{2}}.$$

Therefore, combining equations (4) and (5) with Lemma 7, we get

$$\begin{aligned} (6) \quad & \frac{\lambda(k)(1 - a_p(k)^{-2}p^{k-2})(2\pi i) r_{k,N}(f_k^\#, D_1^*, D_2^*)}{\Omega_k^\#} \\ &= \sum_{\sigma \in \mathrm{Gal}(H_K^+/K)} \chi_{D_1^*,D_2^*}(\sigma) \int\limits_{(\mathbb{Z}_p^2)'} \langle P_{\Psi_*}(x,y)\rangle^{\frac{k-k_0}{2}} P_{\Psi_*}(x,y)^{\frac{k_0-2}{2}} d\Phi_*\{r \to \gamma_{\Psi_*}.r\}. \end{aligned}$$

Since $\Phi_{L_{\Psi_*}} = \Phi_*$, the integral above is the value of the partial p-adic L-function L_p^{Sev} at $k \in U^{\mathrm{cl}}$ and (6) simplifies as

$$(7) \quad \frac{\lambda(k)(1 - a_p(k)^{-2}p^{k-2})(2\pi i) r_{k,N}(f_k^\#, D_1^*, D_2^*)}{\Omega_k^\#} = L_p^{\mathrm{Sev}}(\mathbf{f}/K, \chi_{D_1^*,D_2^*}, k).$$

We would like to interpolate the Fourier coefficients $c(D,k)$, for $k \in U^{\mathrm{cl}}$, by a p-adic analytic function over U. We now introduce a normalization of the Fourier coefficients $c(D,k)$ (see Proposition 1.3 of [14] and Proposition 3.3 of [29]). For D^* a fundamental discriminant of either type and for every $k \in U^{\mathrm{cl}}$, define the normalized Fourier coefficient as follows:

$$\widetilde{c}(D,k) := \frac{\left(1 - \chi_{D^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\right)c(D,k)}{\left(1 - \chi_{D_1^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\right)c(D_1,k)}.$$

Proposition 4. *Up to shrinking, the normalized coefficients $\widetilde{c}(D,k)$ extend to a p-adic analytic function in a neighbourhood of k_0.*

Proof. The proof is a higher weight analogue of Proposition 3.3 of [29]. We write

$$\frac{c(D,k)}{c(D_1,k)} = \frac{c(D,k)\overline{c(D_1,k)}}{\mid c(D_1,k) \mid^2}.$$

Assuming D is relatively prime to D_1, from Theorem 3 and Corollary 1, we can interpret the right hand side as

$$\frac{\pi^{k/2}(-2i)^{k/2}2^{\nu(N)} r_{k,N}(f_k^\#, D_1^*, D^*)}{2^{\nu(N)}(k/2-1)!|D_1|^{\frac{k-1}{2}} L(f_k^\#, D_1^*, k/2)},$$

which simplifies as

$$\frac{(-2\pi i)^{k/2} r_{k,N}(f_k^\#, D_1^*, D^*)}{(k/2-1)!|D_1|^{\frac{k-1}{2}} L(f_k^\#, D_1^*, k/2)}.$$

Expressing the central L-value in terms of its 'algebraic part'

$$L(f_k^{\#}, \chi_{D_1^*}, k/2) = \frac{(-2\pi i)^{\frac{k-2}{2}} \Omega_{f_k^{\#}}}{(k/2-1)! \tau(\chi_{D_1})} \tilde{L}(f_k^{\#}, \chi_{D_1^*}, k/2)$$

we have

$$\frac{c(D,k)}{c(D_1,k)} = \frac{-\tau(\chi_{D_1^*})(2\pi i) r_{k,N}(f_k^{\#}, D^*, D_1^*)}{|D_1|^{\frac{k-1}{2}} \tilde{L}(f_k^{\#}, D_1^*, k/2)\Omega_{f_k^{\#}}}.$$

Using the interpolation formula for the Stevens-Mazur-Kitagawa p-adic L-function and that $|\tau(\chi_{D_1^*})| = D_1^{1/2}$, we have

$$\widetilde{c}(D,k) = \frac{\lambda(k)(2\pi i)\big(1-\chi_{D^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\big)\big(1-\chi_{D_1^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\big) r_{k,N}(f_k^{\#}, D^*, D_1^*)}{|D_1|^{\frac{k-2}{2}} \mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f}, \chi_{D_1^*}, k, k/2)\Omega_{f_k^{\#}}}.$$

Now, suppose D is of Type II. Then, $\chi_{D^*}(p) = -\chi_{D_1^*}(p)$. Hence

$$\Big(1-\chi_{D^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\Big).\Big(1-\chi_{D_1^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\Big) = 1 - \frac{p^{k-2}}{a_p(k)^2}.$$

Since the primes dividing N split in the real quadratic field $\mathbb{Q}(\sqrt{D^* D_1^*})$ while p is inert, we have by (7),

$$\frac{\lambda(k).\big(1-\frac{p^{k-2}}{a_p(k)^2}\big).(2\pi i).r_{k,N}(f_k^{\#}, D^*, D_1^*)}{\Omega_{f_k^{\#}}} = L_p^{\mathrm{Sev}}(\mathbf{f}/\mathbb{Q}(\sqrt{D^* D_1^*}), \chi_{D^* D_1^*}, k/2).$$

Hence

$$\widetilde{c}(D,k) = \frac{L_p^{\mathrm{Sev}}(\mathbf{f}/\mathbb{Q}(\sqrt{D^* D_1^*}), \chi_{D^* D_1^*}, k/2)}{|D_1|^{\frac{k-2}{2}} \mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f}, \chi_{D_1^*}, k, k/2)}$$

up to some constant. Since $\widetilde{c}(D,k)$ is the ratio of p-adic analytic functions on some neighbourhood of k_0, we conclude the same about the normalized coefficients.

Now suppose D is also of Type I and $(D, D_1) = 1$. Note that in this case all the primes dividing Np split in the real quadratic field $\mathbb{Q}(\sqrt{D^* D_1^*})$. This is the *Heegner hypothesis*. Also

$$\Big(1-\chi_{D^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\Big).\Big(1-\chi_{D_1^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\Big) = \Big(1-\chi_{D_1^*}(p)\frac{p^{k-2}}{a_p(k)^2}\Big)^2.$$

In this case,

$$\text{(8)} \quad \frac{\lambda(k).\big(1-\chi_{D_1^*}(p)\frac{p^{k-2}}{a_p(k)^2}\big)^2.(2\pi i) r_{k,N}(f_k^{\#}, D^*, D_1^*)}{\Omega_{f_k^{\#}}} = L_p^{\mathrm{GSS}}(\mathbf{f}/\mathbb{Q}(\sqrt{D^* D_1^*}), \chi_{D^* D_1^*}, k/2).$$

We have skipped the detailed calculation, which is similar to the case when D is of Type II. Finally, when $(D, D_1) \neq 1$, choose a Type I discriminant D_1', prime to both D_1 and D, such that $c(D_1', k_0) \neq 0$. Then we can write

$$\frac{\widetilde{c}(D,k)}{\widetilde{c}(D_1,k)} = \Big(\frac{\widetilde{c}(D,k)}{\widetilde{c}(D_1',k)}\Big).\Big(\frac{\widetilde{c}(D_1',k)}{\widetilde{c}(D_1,k)}\Big)$$

and we repeat the same as above for each individual factor in the product. □

In particular, we have

$$\widetilde{c}(D_2,\kappa)=\frac{L_p^{\mathrm{Sev}}(\mathbf{f}/\mathbb{Q}(\sqrt{D_2^*D_1^*}),\chi_{D_2^*D_1^*},\kappa/2)}{|D_1|^{\frac{\kappa-2}{2}}\mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f},\chi_{D_1^*},\kappa,\kappa/2)}. \tag{9}$$

5.3. **Proof of the main theorem.** We will now compute the derivative of the analytic function $\widetilde{c}(D_2,\kappa)$ along the weight direction around a neighbourhood of k_0.

Theorem 14. *There exist a global cycle*

$$d_{k_0}^{\chi_{D_2^*}}\in CH_0^{k_0/2}(\mathcal{M}_{k_0}\otimes\mathbb{Q}(\sqrt{D_2^*}))^{\chi_{D_2^*}}\subset(\mathcal{M}_{k_0}\otimes\mathbb{Q}(\sqrt{D_2^*},\sqrt{D_1^*}))$$

and a constant $s_f\in K_{f_{k_0}}^{\times}$ such that

$$\frac{d}{dk}[\widetilde{c}(D_2,k)]_{k=k_0}=\frac{|D_2|^{\frac{k_0-2}{4}}}{|D_1|^{\frac{k_0-2}{4}}}.s_f.\log_{\mathrm{BK}}(\mathrm{res}_p(\mathrm{cl}_{0,H_K^+}^{k_0/2}(d_{k_0}^{\chi_{D_2^*}})))(\phi_{k_0}).$$

Proof. Taking the derivative w.r.t. κ on both sides of (9), we have

$$\begin{aligned}\frac{d}{d\kappa}\widetilde{c}(D_2,\kappa)=&\frac{|D_1|^{\kappa/2-1}\mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f},\chi_{D_1^*},\kappa,\kappa/2)\frac{d}{d\kappa}[L_p^{\mathrm{Sev}}(\mathbf{f}/K,\chi_{D_1^*,D_2^*},\kappa)]}{|D_1|^{\kappa-2}\mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f},\chi_{D_1^*},\kappa,\kappa/2)^2}\\&+\frac{L_p^{\mathrm{Sev}}(\mathbf{f}/K,\chi_{D_1^*,D_2^*},\kappa)\frac{d}{d\kappa}[|D_1|^{\kappa/2-1}\mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f},\chi_{D_1^*},\kappa,\kappa/2)]}{|D_1|^{\kappa-2}\mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f},\chi_{D_1^*},\kappa,\kappa/2)^2}.\end{aligned} \tag{10}$$

At $\kappa=k_0$, we know that $L_p^{\mathrm{Sev}}(\mathbf{f}/K,\chi_{D_1^*,D_2^*},k_0)=0$ (see Proposition 5.7 of [41]). Hence (9) simplifies as

$$\frac{d}{d\kappa}[\widetilde{c}(D_2,\kappa)]_{\kappa=k_0}=\frac{\frac{d}{d\kappa}[L_p^{\mathrm{Sev}}(\mathbf{f}/K,\chi_{D_1^*,D_2^*},\kappa)]_{\kappa=k_0}}{|D_1|^{\frac{k_0-2}{2}}\mathcal{L}_p^{\mathrm{SMK}}(\mathbf{f},\chi_{D_1^*},k_0,k_0/2)}. \tag{11}$$

By Corollary 2, we write (10) as

$$\frac{d}{d\kappa}[\widetilde{c}(D_2,\kappa)]_{\kappa=k_0}=\frac{D^{\frac{k_0-2}{4}}\log:\mathrm{AJ}(D_{k_0}^{\chi})(\phi_{k_0})}{|D_1|^{\frac{k_0-2}{2}}2\widetilde{L}(f_{k_0},\chi_{D_1^*},k_0,k_0/2)}.$$

Since $D=|D_1|.|D_2|$ and $2\widetilde{L}(f_{k_0},\chi_{D_1^*},k_0/2)\in K_{f_{k_0}}^{\times}$, the theorem follows from Theorem 13 on the rationality of Darmon cycles. □

Remark 11. The additional factor of $\frac{|D_2|}{|D_1|}^{\frac{k_0-2}{4}}$ is 1 in [14] since $k_0=2$ in their case.

Remark 12. In the statement of the main theorem, we have used $\log_{\mathrm{BK}}$ to denote the isomorphism:

$$\log_{\mathrm{BK}}:H^1_{st}(K_p,V_p^{Np}(k/2))\longrightarrow\frac{D^{\mathrm{FM}}\otimes K_p}{F^{k/2}(D^{\mathrm{FM}}\otimes K_p)}\longrightarrow\mathbb{H}_k^{\pm}(K_p)^{\vee}.$$

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