FRACTIONAL SMOOTHNESS OF DISTRIBUTIONS OF POLYNOMIALS AND A FRACTIONAL ANALOG OF THE HARDY–LANDAU–LITTLEWOOD INEQUALITY

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ABSTRACT. We prove that the distribution density of any non-constant polynomial $f(\xi_1, \xi_2, \ldots)$ of degree d in independent standard Gaussian random variables ξ_i (possibly, in infinitely many variables) always belongs to the Nikolskii–Besov space $B^{1/d}(\mathbb{R}^1)$ of fractional order $1/d$ (and this order is best possible), and an analogous result holds for polynomial mappings with values in $\mathbb{R}^k$.

Our second main result is an upper bound on the total variation distance between two probability measures on $\mathbb{R}^k$ via the Kantorovich distance between them and a suitable Nikolskii–Besov norm of their difference.

As an application we consider the total variation distance between the distributions of two random k-dimensional vectors composed of polynomials of degree d in Gaussian random variables and show that this distance is estimated by a fractional power of the Kantorovich distance with an exponent depending only on d and k, but not on the number of variables of the considered polynomials.

1. INTRODUCTION

This paper is concerned with distributions of polynomials in Gaussian random variables and estimates in the total variation distance between measures with densities from fractional Nikolskii–Besov classes.

Our first main result (presented in Section 4 and Section 5) states that the distribution of any non-constant polynomial of degree d (possibly, in infinitely many variables) with respect to a Gaussian measure always belongs to the Nikolskii–Besov space $B^{1/d}(\mathbb{R}^1)$ (so that the order of smoothness depends only on the degree of this polynomial and this order is best possible) and that an analogous result holds for multidimensional polynomial mappings. It is well-known that a non-constant polynomial in Gaussian random variables has a distribution density; however, in many cases this density is not locally bounded (which happens already for the square of the standard Gaussian random variable), hence does not belong to an integer order Sobolev class. The established fractional regularity is the first general result in this direction (so far only lower continuity of distribution densities has been known, but this is not specific for polynomials and holds for general non-degenerate Malliavin functions; see [5]).

Our second main result gives new lower bounds for the Kantorovich distance (all definitions are given in Section 2) between probability measures on $\mathbb{R}^k$; these bounds can also be viewed as upper bounds for the total variation distance. Our principal new result is a fractional multidimensional analog of the classical Hardy–Landau–Littlewood inequality. We obtain an upper bound on the total variation distance between two probability measures on $\mathbb{R}^k$ in terms of the Kantorovich distance between them and a suitable Nikolskii–Besov norm of their difference. A particular case of our inequality is the estimate of the total variation norm via the Kantorovich norm and the BV-norm established in [12], [13]. The classical Hardy–Landau–Littlewood result [18] states that

$$\|f'\|_{L^1}^2 \leq C\|f\|_{L^1}\|f''\|_{L^1}$$

for every integrable function f on the real line with two integrable derivatives. A multidimensional analog of this bound was obtained in [12], [13] (see also [21] and [33]) in the following form: for every k, there is a number $C(k)$ such that for every two probability measures μ and ν on $\mathbb{R}^k$ with densities ϱ_μ and ϱ_ν belonging to the class BV of functions of bounded variation one has

$$(1.1) \qquad d_{\mathrm{TV}}(\mu, \nu)^2 \leq C(k) d_{\mathrm{K}}(\mu, \nu)\|D\varrho_\mu - D\varrho_\nu\|_{\mathrm{TV}},$$

where d_{TV} is the total variation distance and d_{K} is the Kantorovich distance (see definitions below). In the one-dimensional case, this inequality is equivalent to the Hardy–Landau–Littlewood inequality (and can be obtained from the latter by passing to smooth compactly supported functions and taking for f the difference of the distribution functions of the given measures). However, this result does not directly apply to polynomial images of Gaussian measures, our second main object. For example, the distribution density of the χ^2-distribution with one degree of freedom is unbounded (it behaves like $t^{-1/2}$ near zero) and does not belong to the class BV. For this reason, having in mind applications to distributions of polynomials (treated in Sections 4 and 5), in Section 3 we first obtain a suitable extension of (1.1) that involves fractional derivatives in place of gradients. Namely, given two Borel probability measures ν, σ in the Nikolskii–Besov class $B^\alpha(\mathbb{R}^k)$, $\alpha \in (0, 1]$, we prove that

$$\|\sigma - \nu\|_{\mathrm{TV}} \leq C(k, \alpha)\|\sigma - \nu\|_{B^\alpha}^{1/(1+\alpha)} d_{\mathrm{K}}(\sigma, \nu)^{\alpha/(1+\alpha)}.$$

As an application (considered in Sections 4 and 5) we give upper bounds on the total variation distance via the Kantorovich distance between the distributions of two random k-dimensional vectors whose components are polynomials of degree d in Gaussian variables. The former distance is estimated by a certain fractional power of the latter with an exponent depending only on the degree d and dimension k of the vectors, but not on the number of variables of these polynomials, which yields an immediate infinite-dimensional extension. Our bounds improve the recent results of Nourdin, Nualart, and Poly [27]. This improvement is due to a new method based on the aforementioned fractional multidimensional analog of the Hardy–Landau–Littlewood inequality and also involves Nikolskii–Besov classes. In this relation recall that Nourdin and Poly [29, Theorem 3.1] proved the following interesting fact (the concepts involved in the formulation are defined in the next section). If $\{f_n\}$ is a sequence of polynomials of degree d on a space with a Gaussian measure γ such that their distributions $\gamma \circ f_n^{-1}$ (the images of γ under f_n) converge weakly to

an absolutely continuous measure, then there is a number C such that

$$d_{\mathrm{TV}}(\gamma \circ f_n^{-1}, \gamma \circ f_m^{-1}) \leq C d_{\mathrm{KR}}(\gamma \circ f_n^{-1}, \gamma \circ f_m^{-1})^\theta, \quad \theta = \frac{1}{2d+1},$$

where d_{KR} is the Kantorovich–Rubinstein distance (see below; the term "Fortet–Mourier distance" used in [29] is reserved in our paper for the equivalent metric d_{FM} from the original paper [17]). The proof in [29] implies that, for any two γ-measurable polynomials of degree d with variances σ_f, σ_g in a given interval (a, b) with $a > 0$, there is a number $C = C(a, b, d)$, depending only on a, b, d, such that

$$d_{\mathrm{TV}}(\gamma \circ f^{-1}, \gamma \circ g^{-1}) \leq C d_{\mathrm{KR}}(\gamma \circ f^{-1}, \gamma \circ g^{-1})^{\frac{1}{2d+1}}.$$

In the multidimensional case, it was shown in [27] that, given a sequence of k-dimensional random vectors f_n composed of γ-measurable polynomials of degree d such that their distributions $\gamma \circ f_n^{-1}$ converge weakly and the expectations of the determinants of their Malliavin matrices are separated from zero, for every

$$\theta < \frac{1}{(k+1)(4k(d-1)+3)+1}$$

there exists a number C such that

$$d_{\mathrm{TV}}(\gamma \circ f_n^{-1}, \gamma \circ f_m^{-1}) \leq C d_{\mathrm{KR}}(\gamma \circ f_n^{-1}, \gamma \circ f_m^{-1})^\theta.$$

Here we develop a different approach based on multidimensional analogs of the Hardy–Landau–Littlewood inequality, and in Section 4 we prove an estimate with a much better rate of convergence: given $d \in \mathbb{N}$, $a, b > 0$, for each positive number

$$\theta < \frac{1}{4k(d-1)+1},$$

there exists a number $C = C(d, a, b, \theta)$ such that, whenever f and g are k-dimensional polynomial mappings of degree d (in an arbitrary, possibly infinite, number of variables) with variances of components bounded by b and the expectations of the determinants of the Malliavin matrices separated from zero by a, one has

$$d_{\mathrm{TV}}(\gamma \circ f^{-1}, \gamma \circ g^{-1}) \leq C d_{\mathrm{KR}}(\gamma \circ f^{-1}, \gamma \circ g^{-1})^\theta.$$

In Section 5 we consider separately the case $k = 1$ and also improve the aforementioned bound from [29] from the power $\theta = (2d+1)^{-1}$ to nearly $(2d-1)^{-1}$. More precisely, we establish the foregoing bound with any power $\theta < 1/(2d-1)$. Moreover, with a worse constant we obtain a bound with the power $\theta = 1/(d+1)$, which is close to $1/d$, and the latter cannot be increased. Finally, in Section 6 we give two related estimates connected with results from [16] and [29]. The readers not interested in the infinite-dimensional case can just ignore the corresponding statements; the essence of the paper is in finite-dimensional results independent of the number of variables.

2. DEFINITIONS AND NOTATION

The standard Gaussian measure γ_n on $\mathbb{R}^n$ has density

$$(2\pi)^{-n/2} \exp(-|x|^2/2).$$

The image of a measure μ on a measurable space under a measurable mapping f with values in $\mathbb{R}^k$ is denoted by the symbol $\mu \circ f^{-1}$ and defined by the formula

$$\mu \circ f^{-1}(B) = \mu(f^{-1}(B)) \quad \text{for every Borel set } B \subset \mathbb{R}^k.$$

If $\xi_1, \ldots, \xi_n$ are independent standard Gaussian random variables, $f\colon \mathbb{R}^n \to \mathbb{R}^k$, then the law of $f(\xi_1, \ldots, \xi_n)$ is exactly $\gamma_n \circ f^{-1}$. If $k = 1$, then the distribution density of $\mu \circ f^{-1}$ (if it exists) is the derivative a.e. of the function $t \mapsto \mu(f < t)$.

We set $\|\varphi\|_\infty = \sup_x |\varphi(x)|$ for any bounded function φ on any set. The L^p-norm of a function f is occasionally denoted by $\|f\|_p$.

The total variation distance $d_{\mathrm{TV}}(\mu, \nu)$ between two Borel measures μ, ν on $\mathbb{R}^k$ is generated by the norm

$$\|\sigma\|_{\mathrm{TV}} := \sup\left\{ \int \varphi \, d\sigma, \ \varphi \in C_b^\infty(\mathbb{R}^k), \ \|\varphi\|_\infty \leq 1 \right\},$$

where $C_b^\infty(\mathbb{R}^k)$ is the class of all infinitely differentiable functions with bounded derivatives of all orders. The Kantorovich distance (or the Kantorovich–Rubinstein distance [19], [20], sometimes historically erroneously called the Wasserstein distance) between two Borel probability measures μ, ν on $\mathbb{R}^k$ with finite first moments is defined by the formula

$$d_{\mathrm{K}}(\mu, \nu) := \sup\left\{ \int \varphi \, d(\mu - \nu), \ \varphi \in C_b^\infty(\mathbb{R}^k), \ \|\nabla\varphi\|_\infty \leq 1 \right\}.$$

For measures without moments, the following Fortet–Mourier distance can be used (see [17, pp. 277–279]; other distances including d_{K} are considered there):

$$d_{\mathrm{FM}}(\mu, \nu) := \sup\left\{ \int \varphi \, d(\mu - \nu), \ \varphi \in C_b^\infty(\mathbb{R}^k), \ \|\varphi\|_\infty + \|\nabla\varphi\|_\infty \leq 1 \right\}.$$

An equivalent distance (also called the Kantorovich–Rubinstein distance, since it is a special case of a metric used in [20, Theorem 1']) which is generated by an equivalent norm is defined by

$$d_{\mathrm{KR}}(\mu, \nu) := \sup\left\{ \int \varphi \, d(\mu - \nu), \ \varphi \in C_b^\infty(\mathbb{R}^k), \ \|\varphi\|_\infty \leq 1, \ \|\nabla\varphi\|_\infty \leq 1 \right\}.$$

These distances can be defined on general metric spaces where in place of C_b^∞ one takes the class of all bounded Lipschitz functions. It is clear that $d_{\mathrm{FM}} \leq d_{\mathrm{KR}} \leq d_{\mathrm{K}}$.

Recall (see [6], [26]) that the Nikolskii–Besov class $B_{1,\infty}^\alpha(\mathbb{R}^k)$ of order $\alpha \in (0, 1)$ consists of all functions $\varrho \in L^1(\mathbb{R}^k)$ such that

$$\|\varrho(\cdot + h) - \varrho\|_{L^1} \leq C(\varrho)|h|^\alpha \quad \forall h \in \mathbb{R}^k$$

for some number $C(\varrho)$; it is also denoted by $H_1^\alpha(\mathbb{R}^k)$ in [26], by $B^{\alpha;1,\infty}(\mathbb{R}^k)$ in [1], and by $\Lambda_\alpha^{1,\infty}$ in [36]. This class is a particular case of the class $H_p^\alpha(\mathbb{R}^k)$ defined similarly with the L^p-norm in place of the L^1-norm. Throughout we use the shortened notation $B^\alpha(\mathbb{R}^k)$. Moreover, we use the symbol $B^1(\mathbb{R}^k)$ also for $\alpha = 1$, which corresponds to the class $BV(\mathbb{R}^d)$ of functions of bounded variation (which is smaller than the usual Nikolskii–Besov class with $\alpha = 1$ defined via symmetric differences $\varrho(\cdot + h) + \varrho(\cdot - h) - 2\varrho$). However, it will be more convenient to deal with measures possessing densities from these classic spaces rather than with functions.

Let ν be a bounded Borel measure on $\mathbb{R}^k$ and let ν_h denote its shift by the vector h:

$$\nu_h(A) = \nu(A - h).$$

Let $0 < \alpha \leq 1$. Then the class $B^\alpha(\mathbb{R}^k)$ coincides with the class of densities of bounded Borel measures ν on $\mathbb{R}^k$ such that, for some number C_ν, one has

$$\|\nu_h - \nu\|_{\mathrm{TV}} \leq C_\nu |h|^\alpha \quad \forall h \in \mathbb{R}^k.$$

We shall identify measures with their densities and speak of measures in the class $B^\alpha(\mathbb{R}^k)$ in this sense.

We need the following norm on the space $B^\alpha(\mathbb{R}^k)$:

$$\|\nu\|_{B^\alpha} := \inf\{C\colon \|\nu - \nu_h\|_{\mathrm{TV}} \le C|h|^\alpha\}.$$

It is readily seen that this is indeed a norm. However, the space $B^\alpha(\mathbb{R}^k)$ is not complete with this norm: its standard Banach norm is given by $\|\nu\|_{\mathrm{TV}} + \|\nu\|_{B^\alpha}$. The latter is larger than $\|\nu\|_{B^\alpha}$, and the two norms are not equivalent: indeed, letting $f_n(x) = 1$ on $[-n, n]$, $f_n(0) = 0$ outside $[-n-1, n+1]$, and $f_n(x) = n+1 - |x|$ if $n < |x| < n+1$, we have $\|f_n\|_{L^1} \to \infty$, $\sup_n \|f_n\|_{B^\alpha} < \infty$, where we identify f_n with the measure $f_n dx$. The situation is similar to Sobolev spaces once we use only the norm of the gradient.

The following embedding holds (see [26, Section 6.3]):

$$(2.1) \qquad B^\alpha(\mathbb{R}^k) \subset H_p^\beta(\mathbb{R}^k) \subset L^p(\mathbb{R}^k), \quad \beta = \kappa\alpha, \ \kappa = 1 - \frac{k(p-1)}{\alpha p}.$$

Hence all measures from $B^\alpha(\mathbb{R}^k)$ have densities in $L^p(\mathbb{R}^k)$ for all $p < k/(k - \alpha)$. These embeddings to L^p on balls (i.e., the restrictions $B^\alpha(\mathbb{R}^k)|_U \to L^p(U)$) are compact.

For infinite-dimensional extensions of our results we recall the corresponding concepts. A probability measure defined on the Borel σ-field of a locally convex space X is called Radon if its value on each Borel set is the supremum of measures of compact subsets of this set. A centered Radon Gaussian measure γ is a Radon probability measure on X such that every continuous linear functional f on X is a centered Gaussian random variable on (X, γ); in other words, $\gamma \circ f^{-1}$ is either Dirac's measure at zero or has a distribution density $(2\pi\sigma_f)^{-1/2}\exp(-x^2/(2\sigma_f))$, where $\sigma_f = \|f\|_{L^2(\gamma)}^2$. On complete separable metric spaces all Borel measures are automatically Radon. Typical examples of Gaussian measures are the countable power of the standard Gaussian measure on $\mathbb{R}$ (defined on the countable power $\mathbb{R}^\infty$ of $\mathbb{R}$) and the Wiener measure (see [8] and [10] about Gaussian measures).

Let $H \subset X$ be the Cameron–Martin space of the measure γ, i.e., the space of all vectors h such that $\gamma_h \sim \gamma$. If γ is the countable power of the standard Gaussian measure on the real line, then H is the usual Hilbert space l^2 (of course, for the standard Gaussian measure on $\mathbb{R}^d$ the Cameron–Martin space is $\mathbb{R}^d$ itself). The Cameron–Martin space of the Wiener measure on $C[0,1]$ is the space of absolutely continuous functions on $[0,1]$ vanishing at 0 and having derivatives in $L^2[0,1]$. For a general Radon Gaussian measure the Cameron–Martin space is also a separable Hilbert space (see [8, Theorem 3.2.7 and Proposition 2.4.6]) with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $|\cdot|_H$ defined by

$$|h|_H = \sup\left\{ l(h)\colon \int_X l^2\, d\gamma \le 1,\ l \in X^* \right\}.$$

Let $\mathcal{P}^d(\gamma)$ be the closure in $L^2(\gamma)$ of the linear space of all functions of the form

$$\varphi_d(l_1(x), \dots, l_m(x)),$$

where $\varphi_d(t_1, \dots, t_m)$ is a polynomial in m variables of degree d and $l_1, \dots, l_m$ are continuous linear functionals on X (m can be an arbitrary natural number). Functions from the class $\mathcal{P}^d(\gamma)$ will be called measurable polynomials of degree d.

The Wiener chaos $\mathcal{H}_d$ of order d is defined as the orthogonal complement of $\mathcal{P}^{d-1}(\gamma)$ in $\mathcal{P}^d(\gamma)$; $\mathcal{H}_0$ is the space of constants. It is well-known (see, e.g., [8, Section 2.9]) that $L^2(\gamma)$ is decomposed into the orthogonal sum $L^2(\gamma) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$.

It is clear that $\mathcal{P}^d(\gamma) = \bigoplus_{k=0}^{d} \mathcal{H}_k$. The subspaces $\mathcal{H}_k$ can also be defined by means of multiple Wiener–Itô stochastic integrals. This interpretation can be found in [31, Section 1.1.2] or in [8, Section 2.11].

Let us define Sobolev derivatives and gradients of measurable polynomials. Let $\{e_n\}$ be an orthonormal basis in H. One can assume that γ is the countable power of the standard Gaussian measure on $\mathbb{R}$ and $\{e_n\}$ is the usual basis in l^2. For any $f \in \mathcal{P}^d(\gamma)$ and $p \geq 1$, $r \in \mathbb{N}$, one can define the Sobolev norm

$$\|f\|_{p,r} = \sum_{k=0}^{r} \left(\int_X \left(\sum_{i_1,\dots,i_k} (\partial_{e_{i_1}} \dots \partial_{e_{i_k}} f)^2 \right)^{p/2} d\gamma \right)^{1/p}$$

and the Sobolev gradient

$$\nabla f(x) = \sum_{k=0}^{\infty} \partial_{e_k} f(x)\, e_k,$$

where ∂_{e_k} is the partial derivative along the vector e_k. One can pick a version of f such that these partial derivatives exist and $\nabla f(x) \in H$. Moreover, $\|f\|_{p,r} < \infty$ for all $p, r < \infty$. The Sobolev class $W^{p,r}(\gamma)$ is the completion of the union of $\mathcal{P}^d(\gamma)$ with respect to the norm $\|\cdot\|_{p,r}$. This class coincides also with the completion with respect to the Sobolev norm of the space of functions of the form $f(l_1(x), \dots, l_m(x))$, where $f \in C_b^{\infty}(\mathbb{R}^m)$. In the case of $X = \mathbb{R}^n$ and the standard Gaussian measure γ one has $H = X = \mathbb{R}^n$ and ∇f is the gradient of f in the usual sense.

As in the finite-dimensional case, all γ-measurable polynomials have derivatives of all orders, and the following estimate (the reverse Poincaré inequality) holds true:

$$(2.2) \qquad \int_X |\nabla f|^2 \, d\gamma \leq c(d) \int_X (f - m_f)^2 \, d\gamma, \quad m_f = \int_X f \, d\gamma.$$

This fact follows from the equivalence of all Sobolev norms and all L^p-norms on the space of measurable polynomials of degree d (see, e.g., Example 5.3.4 in [8]). This equivalence of L^p-norms gives a bound

$$\|f\|_q \leq \|f\|_p \leq C(p,q,d) \|f\|_q$$

for all measurable polynomials f of degree d and any $p > q \geq 1$.

For a detailed discussion of γ-measurable polynomials, see [8, Section 5.10].

We need the following inequality proved by Carbery and Wright [15] (and also by Nazarov, Sodin, Volberg [25]): there is an absolute constant c such that, for every Gaussian measure (more generally, for every convex measure) γ on $\mathbb{R}^n$ and for every polynomial f of degree d, one has

$$(2.3) \qquad \gamma(|f| \leq t) \left(\int_{\mathbb{R}^n} |f| \, d\gamma \right)^{1/d} \leq c d t^{1/d}, \quad t \geq 0.$$

Generalizations to the case of s-concave measures are considered in [7]; on measurable polynomials on infinite-dimensional locally convex spaces see also [4].

We also recall the following known fact about weakly convergent sequences of distributions of γ-measurable polynomials with the same γ as above, i.e., the measures $\gamma \circ f_n^{-1}$ (more generally, a sequence of polynomials of degree d possessing uniformly tight distributions is bounded in all L^p; see, e.g., [9, Exercise 9.8.19]).

We recall that a family of probability measures is uniformly tight if, for each $\varepsilon > 0$, there is a common compact set of measure at least $1 - \varepsilon$ for all these measures.

Lemma 2.1. *Let $\{f_n\}$ be a sequence of γ-measurable polynomials of degree d. Suppose that the distributions $\mu_n = \gamma \circ f_n^{-1}$ converge weakly to a measure μ on $\mathbb{R}$. Then, for any $p \geq 1$, one has convergence of moments*

$$\lim_{n \to \infty} \int_{\mathbb{R}^1} |x|^p \, d\mu_n = \int_{\mathbb{R}^1} |x|^p \, d\mu.$$

3. Fractional Hardy–Landau–Littlewood type estimates

Let us give a sufficient condition for membership in the class $B^\alpha(\mathbb{R}^k)$. Actually, this condition is also necessary, which will be shown in a forthcoming paper.

Proposition 3.1. *Let $\alpha \in (0, 1]$. Let ν be a Borel measure on $\mathbb{R}^k$. Suppose that for every function $\varphi \in C_b^\infty(\mathbb{R}^k)$ and every unit vector $e \in \mathbb{R}^k$ one has*

$$\int_{\mathbb{R}^k} \partial_e \varphi(x) \, \nu(dx) \leq C \|\varphi\|_\infty^\alpha \|\partial_e \varphi\|_\infty^{1-\alpha}.$$

Then

$$\|\nu_h - \nu\|_{\mathrm{TV}} \leq 2^{1-\alpha} C |h|^\alpha \quad \forall \, h \in \mathbb{R}^k,$$

that is, $\nu \in B^\alpha(\mathbb{R}^k)$ and $\|\nu\|_{B^\alpha} \leq 2^{1-\alpha} C$. In particular, the density of ν belongs to all $L^p(\mathbb{R}^k)$ with $p < k/(k - \alpha)$ according to (2.1).

Proof. Let $e = |h|^{-1} h$. It is easy to see that

$$\|\nu_h - \nu\|_{\mathrm{TV}} = \sup_{\varphi \in C_b^\infty(\mathbb{R}^k), \, \|\varphi\|_\infty \leq 1} \int_{\mathbb{R}^k} \varphi(x) \, (\nu_h - \nu)(dx)$$

$$= \sup_{\varphi \in C_b^\infty(\mathbb{R}^k), \, \|\varphi\|_\infty \leq 1} \int_{\mathbb{R}^k} [\varphi(x + h) - \varphi(x)] \, \nu(dx)$$

$$= \sup_{\varphi \in C_b^\infty(\mathbb{R}^k), \, \|\varphi\|_\infty \leq 1} \int_{\mathbb{R}^k} \int_0^{|h|} \partial_e \varphi(x + se) \, ds \, \nu(dx).$$

Let $\varphi \in C_b^\infty(\mathbb{R}^k)$ and $\|\varphi\|_\infty \leq 1$. Consider the function

$$\Phi(x) = \int_0^{|h|} \varphi(x + se) \, ds.$$

Note that $\sup_{x \in \mathbb{R}^k} |\Phi(x)| \leq |h|$ and

$$|\partial_e \Phi(x)| = \left| \int_0^{|h|} \partial_e \varphi(x + se) \, ds \right| = |\varphi(x + h) - \varphi(x)| \leq 2.$$

By the assumptions of the theorem (applied with Φ in place of φ) we have

$$\int_{\mathbb{R}^k} \partial_e \Phi(x) \, \nu(dx) \leq C |h|^\alpha 2^{1-\alpha},$$

which means that

$$\int_{\mathbb{R}^k} \int_0^{|h|} \partial_e \varphi(x + se) \, ds \, \nu(dx) \leq C 2^{1-\alpha} |h|^\alpha,$$

which completes the proof. $\qquad \square$

The next theorem gives a fractional analog of the multidimensional Hardy–Landau–Littlewood inequality established in [12] in the case $\alpha = 1$, which states that

$$\|\sigma - \nu\|_{\mathrm{TV}}^2 \le C(k)\|D\sigma - D\nu\|_{\mathrm{TV}} d_{\mathrm{K}}(\sigma, \nu) \tag{3.1}$$

for two probability measures σ and ν with densities of bounded variation, where $\|D\sigma - D\nu\|_{\mathrm{TV}}$ is the total variation norm of the difference of their generalized derivatives (in case of Sobolev class densities this is the L^1-norm of the gradient of the difference of these densities). We recall that the original Hardy–Landau–Littlewood inequality states that

$$\|f' - g'\|_{L^1}^2 \le 2\|f - g\|_{L^1}\|f'' - g''\|_{L^1},$$

which is equivalent to (3.1) when applied to the measures with densities f' and g' (in that case $d_{\mathrm{K}}(f'\,dx, g'\,dx)$ coincides with $\|f - g\|_{L^1}$).

Theorem 3.2. *Let $\nu, \sigma \in B^\alpha(\mathbb{R}^k)$ be two Borel probability measures on $\mathbb{R}^k$. Then*

$$\|\sigma - \nu\|_{\mathrm{TV}} \le C(k, \alpha)\|\sigma - \nu\|_{B^\alpha}^{1/(1+\alpha)} d_{\mathrm{K}}(\sigma, \nu)^{\alpha/(1+\alpha)}, \tag{3.2}$$

where

$$C(k, \alpha) = 1 + \int_{\mathbb{R}^k} |x|^\alpha \, \gamma_k(dx).$$

Proof. Let γ_k^ε be the centered Gaussian measure on $\mathbb{R}^k$ with the covariance matrix $\varepsilon^2 I$, i.e., with density $(2\pi\varepsilon^2)^{-k/2} \exp(-|x|^2/(2\varepsilon^2))$. By the triangle inequality we have

$$\|\sigma - \nu\|_{\mathrm{TV}} \le \|(\sigma - \nu) - (\sigma - \nu) * \gamma_k^\varepsilon\|_{\mathrm{TV}} + \|\sigma * \gamma_k^\varepsilon - \nu * \gamma_k^\varepsilon\|_{\mathrm{TV}}. \tag{3.3}$$

For any function $\varphi \in C_b^\infty(\mathbb{R}^k)$ with $\|\varphi\|_\infty \le 1$ the following equalities hold true, where all integrals in this proof are taken over $\mathbb{R}^k$:

$$\int \varphi \, d(\sigma * \gamma_k^\varepsilon - \nu * \gamma_k^\varepsilon) = \int \varphi(x) \int (2\pi\varepsilon^2)^{-k/2} \exp\left(-\frac{|y - x|^2}{2\varepsilon^2}\right) (\nu - \sigma)(dy)\, dx$$

$$= \int \left(\int \varphi(x)(2\pi\varepsilon^2)^{-k/2} \exp\left(-\frac{|y - x|^2}{2\varepsilon^2}\right) dx\right) (\nu - \sigma)(dy).$$

Let us consider the function

$$\Phi(y) := \int \varphi(x)(2\pi\varepsilon^2)^{-k/2} \exp\left(-\frac{|y - x|^2}{2\varepsilon^2}\right) dx.$$

We have

$$\nabla\Phi(y) = \varepsilon^{-1} \int \varphi(y + \varepsilon z)(2\pi)^{-k/2} z \exp\left(-\frac{|z|^2}{2}\right) dz,$$

hence $|\Phi(y)| \le 1$, $|\nabla\Phi(y)| \le \varepsilon^{-1}$. Therefore,

$$\|\sigma * \gamma_k^\varepsilon - \nu * \gamma_k^\varepsilon\|_{\mathrm{TV}} \le \varepsilon^{-1} d_{\mathrm{K}}(\sigma, \nu). \tag{3.4}$$

We now estimate the remaining term in the right-hand side of (3.3):

$$\|(\sigma - \nu) - (\sigma - \nu) * \gamma_k^\varepsilon\|_{\mathrm{TV}}$$

$$= \sup_{\|\varphi\|_\infty \leq 1} \int \left((2\pi\varepsilon^2)^{-k/2} \exp\left(-\frac{|y|^2}{2\varepsilon^2}\right) \int \varphi(x)\big((\sigma - \nu) - (\sigma_y - \nu_y)\big)(dx) \right) dy$$

$$\leq \|\sigma - \nu\|_{B^\alpha} \int (2\pi\varepsilon^2)^{-k/2} \exp\left(-\frac{|y|^2}{2\varepsilon^2}\right) |y|^\alpha \, dy$$

$$= \varepsilon^\alpha \|\sigma - \nu\|_{B^\alpha} (2\pi)^{-k/2} \int |y|^\alpha \exp\left(-\frac{|y|^2}{2}\right) dy.$$

Hence we have

$$\|\sigma - \nu\|_{\mathrm{TV}} \leq \varepsilon^{-1} d_{\mathrm{K}}(\sigma, \nu) + \varepsilon^\alpha \|\sigma - \nu\|_{B^\alpha} \int |x|^\alpha \gamma_k(dx).$$

Taking $\varepsilon = \big(\|\sigma - \nu\|_{\mathrm{K}}/\|\sigma - \nu\|_{B^\alpha}\big)^{1/(1+\alpha)}$, we obtain (3.2). $\qquad\square$

Remark 3.3.
(i) One can modify the previous proof to obtain the following estimate for probability measures $\nu, \sigma \in B^\alpha(\mathbb{R}^k)$ employing the Fortet–Mourier metric:

$$\|\sigma - \nu\|_{\mathrm{TV}} \leq C(k, \alpha) \|\sigma - \nu\|_{B^\alpha}^{1/(1+\alpha)} d_{\mathrm{FM}}(\sigma, \nu)^{\alpha/(1+\alpha)} + d_{\mathrm{FM}}(\sigma, \nu)$$

$$\leq \big(C(k, \alpha) \|\sigma - \nu\|_{B^\alpha}^{1/(1+\alpha)} + 2^{1/(1+\alpha)}\big) d_{\mathrm{FM}}(\sigma, \nu)^{\alpha/(1+\alpha)},$$

where $C(k, \alpha)$ is the same as above. To this end, in place of inequality (3.4) we write $\|\sigma * \gamma_k^\varepsilon - \nu * \gamma_k^\varepsilon\|_{\mathrm{TV}} \leq \big(\varepsilon^{-1} + 1\big) d_{\mathrm{FM}}(\sigma, \nu)$ and then proceed as in the proof above. The additional quantity $2^{1/(1+\alpha)}$ is not needed if we slightly decrease the power at d_{FM} as explained in (ii).

(ii) In relation to (i) we observe that the two distances d_{FM} and d_{K}, which in general admit only the one-sided estimate $d_{\mathrm{FM}} \leq d_{\mathrm{K}}$, are very close on the set of distributions of polynomials of degree d with variances not exceeding a fixed number b. More precisely, there is a number $L(d, b)$ such that

$$d_{\mathrm{K}}(\gamma \circ f^{-1}, \gamma \circ g^{-1}) \leq L(d, b) d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1})(|\log d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1})|^{d/2} + 1).$$

Indeed, it is known (see [8, Corollary 5.5.7]) that

$$\gamma(x\colon |f(x)| \geq t\|f\|_2) \leq c_r \exp(-rt^{2/d}), \quad r < \frac{d}{2e},$$

where c_r depends only on r. Let φ be a 1-Lipschitz function on $\mathbb{R}$. We can assume that $\varphi(0) = 0$, since $\varphi(f) - \varphi(g)$ does not change if we subtract $\varphi(0)$ from φ. Considering the bounded function $\varphi_R = \max(-R, \min(R, \varphi))$, we obtain

$$\int_{\mathbb{R}^k} [\varphi(f) - \varphi(g)] \, d\gamma$$

$$\leq (R + 1) d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1}) + \int_{\mathbb{R}^k} \big[|\varphi(f) - \varphi_R(f)| + |\varphi(g) - \varphi_R(g)|\big] \, d\gamma$$

$$\leq (R + 1) d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1}) + \int_{|f| > R} |f| \, d\gamma + \int_{|g| > R} |g| \, d\gamma$$

$$\leq (R + 1) d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1}) + C_1 \exp(-C_2 R^{2/d}).$$

Now we take
$$R = \Big(\frac{|\log d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1})|}{C_2} \Big)^{d/2}$$
and immediately get the desired estimate if $d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1}) \le 1$. Finally, we observe that if $d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1}) > 1$, then $\exp(-C_2 R^{2/d}) < d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1})$, and thus we obtain the estimate in the general case. However, we do not know whether the logarithmic factor is really needed.

Remark 3.4. Let $\nu \in B^\alpha(\mathbb{R}^k)$ be a Borel measure on $\mathbb{R}^k$. Then one can prove by a similar reasoning that for every Borel set A one has
$$\nu(A) \le C_1(k, \alpha) \|\nu\|_{B^\alpha}^{k/(\alpha+k)} \lambda_k(A)^{\alpha/(\alpha+k)},$$
where λ_k is the standard Lebesgue measure on $\mathbb{R}^k$,
$$C_1(k, \alpha) = (2\pi)^{-k/2} + (2\pi)^{-k/2} \int_{\mathbb{R}^k} \exp\Big(-\frac{|y|^2}{2}\Big) |y|^\alpha dy.$$
However, the embedding theorem for Nikolskii–Besov spaces (see (2.1)) gives a slightly better power: for any $r < \alpha/k$ there is $C_2(k, \alpha, r) > 0$ such that
$$\nu(A) \le C_2(k, \alpha, r)(\|\nu\|_{B^\alpha} + 1)\lambda_k(A)^r \quad \text{for every Borel set } A.$$

4. FRACTIONAL SMOOTHNESS OF POLYNOMIAL IMAGES OF GAUSSIAN MEASURES

Let us recall that the Ornstein–Uhlenbeck operator L associated with the standard Gaussian measure γ_n on $\mathbb{R}^n$ is defined by
$$L\varphi(x) = \Delta\varphi(x) - \langle x, \nabla\varphi(x) \rangle,$$
where Δ is the Laplace operator. The operator L is symmetric in $L^2(\gamma_n)$ (with domain $W^{2,2}(\gamma_n)$) and is frequently used in the integration by parts formula
$$\int_{\mathbb{R}^n} \varphi L\psi \, d\gamma_n = -\int_{\mathbb{R}^n} \langle \nabla\varphi, \nabla\psi \rangle \, d\gamma_n.$$
We employ this formula below.

Let $f \colon \mathbb{R}^n \to \mathbb{R}^k$ be a mapping such that its components $f_1, \ldots, f_k$ are polynomials of degree d. Let us introduce the Malliavin matrix of f by
$$M_f(x) = (m_{i,j}(x))_{i,j \le k}, \quad m_{i,j}(x) := \langle \nabla f_i(x), \nabla f_j(x) \rangle.$$
Let
$$A_f := (a_{i,j})_{i,j \le k}$$
be the adjugate matrix of M_f, i.e., $a_{i,j} = M^{j,i}$, where $M^{j,i}$ is the cofactor of $m_{j,i}$ in the matrix M_f. Note that $a^{i,j}$ is a polynomial of degree $k - 1$ in $m_{s,t}$. Set
$$\Delta_f := \det M_f.$$
It is a polynomial of degree $2k(d - 1)$. We observe that $\Delta_f \ge 0$ and
$$(4.1) \qquad\qquad\qquad \Delta_f \cdot M_f^{-1} = A_f.$$
Let $\sigma_{f_i}^2$ denote the variance of the random variable f_i on $(\mathbb{R}^n, \gamma_n)$:
$$\sigma_{f_i}^2 := \int_{\mathbb{R}^n} \Big(f_i - \int_{\mathbb{R}^n} f_i \, d\gamma_n \Big)^2 d\gamma_n.$$

The first main result of this section is the following theorem, which says that the distribution of a polynomial mapping f with respect to a Gaussian measure such that f is non-degenerate (in the sense that $\Delta_f > 0$ on a positive measure set or, equivalently, $\gamma_n \circ f^{-1}$ is absolutely continuous) always belongs to some Nikolskii–Besov class whose index depends only on the maximal degree of components and the number of components, but not on the number of variables.

Theorem 4.1. *Let $k, d \in \mathbb{N}$, $a > 0$, $b > 0$, $\tau > 0$. Then there exists a number $C(d, k, a, b, \tau) > 0$ such that, for every mapping $f = (f_1, \ldots, f_k) \colon \mathbb{R}^n \to \mathbb{R}^k$, where each f_i is a polynomial of degree d and*

$$\int_{\mathbb{R}^n} \Delta_f \, d\gamma_n \geq a, \quad \max_{i \leq k} \sigma_{f_i} \leq b,$$

for every function $\varphi \in C_b^\infty(\mathbb{R}^k)$ and every vector $e \in \mathbb{R}^k$ with $|e| = 1$, one has

$$\int_{\mathbb{R}^n} \partial_e \varphi(f(x)) \, \gamma_n(dx) \leq C(d, k, a, b, \tau) \|\varphi\|_\infty^\alpha \|\partial_e \varphi\|_\infty^{1-\alpha}, \quad \alpha = \frac{1}{4k(d-1) + \tau}.$$

Therefore, we have

$$\|\gamma_n \circ f^{-1} - (\gamma_n \circ f^{-1})_h\|_{\mathrm{TV}} \leq C(d, k, a, b, \tau) |h|^\alpha,$$

equivalently,

$$\gamma_n \circ f^{-1} \in B^\alpha(\mathbb{R}^k) \quad \text{for every } \alpha < \frac{1}{4k(d-1)}.$$

In particular, the density of $\gamma_n \circ f^{-1}$ belongs to all $L^p(\mathbb{R}^k)$ with $p < k/(k - \alpha)$.

Proof. We can assume that $\|\varphi\|_\infty \leq 1$. If $\|\partial_e \varphi\|_\infty \leq 1$, then for any $\alpha > 0$ we have (omitting indication of $\mathbb{R}^n$ in all integrations in this proof)

$$\int \partial_e \varphi(f(x)) \, \gamma_n(dx) \leq \|\partial_e \varphi\|_\infty \leq \|\partial_e \varphi\|_\infty^{1-\alpha}.$$

Suppose now that $\|\partial_e \varphi\|_\infty \geq 1$. It can be easily verified that

$$M_f(\partial_{x_1} \varphi(f), \ldots, \partial_{x_k} \varphi(f)) = \left(\langle \nabla(\varphi \circ f), \nabla f_1 \rangle, \ldots, \langle \nabla(\varphi \circ f), \nabla f_k \rangle\right).$$

Here the left-hand side is interpreted as the standard product of a matrix and a vector (with components $\partial_{x_i} \varphi(f)$) and ∇ denotes the gradient of a function of n variables. Then by equality (4.1) we obtain

$$(\partial_e \varphi)(f) \Delta_f = \langle v, A_f e \rangle, \quad v = \left(\langle \nabla(\varphi \circ f), \nabla f_1 \rangle, \ldots, \langle \nabla(\varphi \circ f), \nabla f_k \rangle\right).$$

Let $\varepsilon \in (0, 1)$ be a fixed number that will be chosen later. The integral that we want to estimate can be written as

$$(4.2) \qquad \int \partial_e \varphi(f) \, d\gamma_n = \int \partial_e \varphi(f) \frac{\Delta_f}{\Delta_f + \varepsilon} \, d\gamma_n + \varepsilon \int \frac{\partial_e \varphi(f)}{\Delta_f + \varepsilon} \, d\gamma_n.$$

We now estimate each term. By the reasoning above we can write

$$\int \partial_e \varphi(f) \frac{\Delta_f}{\Delta_f + \varepsilon} \, d\gamma_n = \int \frac{\langle \left(\langle \nabla \varphi \circ f, \nabla f_1 \rangle, \ldots, \langle \nabla \varphi \circ f, \nabla f_k \rangle\right), A_f e \rangle}{\Delta_f + \varepsilon} \, d\gamma_n.$$

Letting $h(x) = A_f(x)e$, we can integrate by parts and write the above term as

$$\int (\Delta_f + \varepsilon)^{-1} \sum_{i=1}^{k} \langle \nabla\varphi \circ f, \nabla f_i \rangle h_i \, d\gamma_n$$

(4.3)
$$= -\sum_{i=1}^{k} \int \varphi \circ f \Big(\frac{h_i L f_i}{\Delta_f + \varepsilon} - \frac{h_i \langle \nabla f_i, \nabla\Delta_f \rangle}{(\Delta_f + \varepsilon)^2} + \frac{\langle \nabla f_i, \nabla h_i \rangle}{\Delta_f + \varepsilon} \Big) d\gamma_n$$

$$\leq \int \Big| \sum_{i=1}^{k} h_i L f_i \Big| (\Delta_f + \varepsilon)^{-1} \, d\gamma_n + \int \Big| \sum_{i=1}^{k} h_i \langle \nabla f_i, \nabla\Delta_f \rangle \Big| (\Delta_f + \varepsilon)^{-2} \, d\gamma_n$$

$$+ \int \Big| \sum_{i=1}^{k} \langle \nabla f_i, \nabla h_i \rangle \Big| (\Delta_f + \varepsilon)^{-1} \, d\gamma_n.$$

We have to estimate each of the three terms. First of all, note that Δ_f is itself a polynomial of degree $2k(d-1)$. We set

$$\beta = \frac{1}{2k(d-1)}$$

and use the Carbery–Wright inequality (2.3) to obtain

(4.4)
$$\int (\Delta_f + \varepsilon)^{-p} \, d\gamma_n = p \int_0^{\varepsilon^{-1}} t^{p-1} \gamma_n \big((\Delta_f + \varepsilon)^{-1} \geq t \big) \, dt$$

$$= p \int_0^{\infty} (s + \varepsilon)^{-p-1} \gamma_n \big(\Delta_f \leq s \big) \, ds$$

$$\leq 2cpk(d-1) \Big(\int \Delta_f \, d\gamma_n \Big)^{-\beta} \int_0^{\infty} (s + \varepsilon)^{-p-1} s^{\beta} \, ds$$

$$= \varepsilon^{-p+\beta} 2cpk(d-1) \Big(\int \Delta_f \, d\gamma_n \Big)^{-\beta} \int_0^{\infty} (s + 1)^{-p-1} s^{\beta} \, ds.$$

Let

$$c(p, d) := \Big(2cpk(d-1) \int_0^{\infty} (s + 1)^{-p-1} s^{\beta} \, ds \Big)^{1/p}.$$

Let $\|A\|_{HS} = \big(\sum_{i,j} a_{ij}^2 \big)^{1/2}$ be the Hilbert–Schmidt norm of a matrix $A = (a_{ij})$. Then $\|A_f(x)\|_{HS}$ is estimated by a polynomial in the matrix elements $m_{i,j}(x)$. Hence its L^p-norms are bounded by powers of b (with some constants depending on d, k, and p). Let us estimate the first term in the right-hand side of (4.3):

$$\int \Big| \sum_{i=1}^{k} h_i L f_i \Big| (\Delta_f + \varepsilon)^{-1} \, d\gamma_n \leq \int (\Delta_f + \varepsilon)^{-1} \|A_f\|_{HS} \Big(\sum_{i=1}^{k} |L f_i|^2 \Big)^{1/2} d\gamma_n$$

$$\leq \varepsilon^{-1} \int \|A_f\|_{HS} \Big(\sum_{i=1}^{k} |L f_i|^2 \Big)^{1/2} d\gamma_n.$$

Next we estimate the second term in the right-hand side of (4.3):

$$\int \Big| \sum_{i=1}^{k} h_i \langle \nabla f_i, \nabla \Delta_f \rangle \Big| (\Delta_f + \varepsilon)^{-2} \, d\gamma_n$$

$$\leq \int (\Delta_f + \varepsilon)^{-2} \|A_f\|_{HS} \Big(\sum_{i=1}^{k} \langle \nabla f_i, \nabla \Delta_f \rangle^2 \Big)^{1/2} \, d\gamma_n$$

$$\leq \int (\Delta_f + \varepsilon)^{-2} \|A_f\|_{HS} |\nabla \Delta_f| \Big(\sum_{i=1}^{k} |\nabla f_i|^2 \Big)^{1/2} \, d\gamma_n$$

$$\leq \Big(\int (\Delta_f + \varepsilon)^{-2q} \, d\gamma_n \Big)^{1/q} \Big(\int \|A_f\|_{HS}^{q'} |\nabla \Delta_f|^{q'} \Big(\sum_{i=1}^{k} |\nabla f_i|^2 \Big)^{q'/2} \, d\gamma_n \Big)^{1/q'}$$

$$\leq c(2q, d)^2 \varepsilon^{-2+\beta/q} \Big(\int \Delta_f \, d\gamma_n \Big)^{-\beta/q}$$

$$\times \Big(\int \|A_f\|_{HS}^{q'} |\nabla \Delta_f|^{q'} \Big(\sum_{i=1}^{k} |\nabla f_i|^2 \Big)^{q'/2} \, d\gamma_n \Big)^{1/q'},$$

where $q' = q/(q-1)$ appears due to Hölder's inequality.

Finally, let us estimate the third term in the right-hand side of (4.3):

$$\int \Big| \sum_{i=1}^{k} \langle \nabla f_i, \nabla h_i \rangle \Big| (\Delta_f + \varepsilon)^{-1} \, d\gamma_n \leq \int (\Delta_f + \varepsilon)^{-1} \sum_{i=1}^{k} |\nabla f_i| \, |\nabla h_i| \, d\gamma_n$$

$$\leq \frac{1}{2\varepsilon} \int \sum_{i=1}^{k} \Big(|\nabla f_i|^2 + |\nabla h_i|^2 \Big) \, d\gamma_n.$$

Since $-2 + \beta/q < -1$ and $\varepsilon \leq 1$, we have $\varepsilon^{-1} \leq \varepsilon^{-2+\beta/q}$.

We now use (4.4) to estimate the second term in the right-hand side of (4.2):

$$\int \frac{\partial_e \varphi(f)}{\Delta_f + \varepsilon} \, d\gamma_n \leq \|\partial_e \varphi\|_\infty c(1, d) \varepsilon^{-1+\beta} \Big(\int \Delta_f \, d\gamma_n \Big)^{-\beta}.$$

Setting $\tau = \frac{q-1}{q}$ and taking

$$\varepsilon = \|\partial_e \varphi\|_\infty^\omega, \quad \omega = -\frac{1}{2 + \tau\beta} = -\frac{2k(d-1)}{4k(d-1) + \tau},$$

we arrive at the estimate

$$(4.5) \qquad \int \partial_e \varphi(f) \, d\gamma_n \leq C \|\partial_e \varphi\|_\infty^{1-\alpha}, \quad \alpha = \frac{1}{4k(d-1) + \tau},$$

where

$$C = \int \|A_f\|_{HS} \Big(\sum_{i=1}^{k} |Lf_i|^2 \Big)^{1/2} d\gamma_n$$

$$+ c(2q, d)^2 \Big(\int \Delta_f \, d\gamma_n \Big)^{-\beta/q} \Big(\int \|A_f\|_{HS}^{q'} |\nabla \Delta_f|^{q'} \Big(\sum_{i=1}^{k} |\nabla f_i|^2 \Big)^{q'/2} d\gamma_n \Big)^{1/q'}$$

$$+ \frac{1}{2} \int \sum_{i=1}^{k} \Big[|\nabla f_i|^2 + |\nabla h_i|^2 \Big] d\gamma_n + c(1, d) \Big(\int \Delta_f \, d\gamma_n \Big)^{-\beta}.$$

Using inequality (2.2) and the equivalence of the L^p-norms of measurable polynomials of degree d we can replace this number C by a number $C(d, k, a, b, \tau)$ that depends only on d, k, a, b, and τ. Recall that $\|A_f\|_{HS}$ is estimated by a polynomial in the matrix elements $m_{i,j}(x)$. Hence its L^p-norms are also bounded by powers of b. By choosing $q > 1$ sufficiently close to 1, we can make $\tau = \frac{q-1}{q}$ in (4.5) as small as we wish. It remains to take into account Proposition 3.1. □

It is worth noting that, according to [14] and [32], if $k = 2$ and we deal with a pair of functions from $\mathcal{H}_d$, then their Malliavin matrix is zero a.e. precisely when they are linearly dependent. Thus, if two functions f, g from $\mathcal{H}_d$ are not proportional, then the distribution of the vector (f, g) possesses a density from each Nikolskii–Besov class $B^\alpha(\mathbb{R}^2)$ with $\alpha < 1/(8d - 8)$.

By the aforementioned compact embedding (2.1) on balls, we immediately obtain convergence of densities in $L^p(\mathbb{R}^k)$ with $p < k/(k - \alpha)$ in case of weak convergence of distributions of mappings satisfying the assumptions of Theorem 4.1 (which sharpens a result from [27]).

Combining Theorem 4.1 and Theorem 3.2 we obtain the following result.

Theorem 4.2. *Let $k, d \in \mathbb{N}$, $a > 0$, $b > 0$, $\tau > 0$. Then there is $C = C(d, k, a, b, \tau)$ such that, whenever $f = (f_1, \ldots, f_k)$ and $g = (g_1, \ldots, g_k)$ are mappings from $\mathbb{R}^n$ to $\mathbb{R}^k$ such that their components f_i, g_i are polynomials of degree d with*

$$\int_{\mathbb{R}^n} \Delta_f \, d\gamma_n \geq a, \quad \int_{\mathbb{R}^n} \Delta_g \, d\gamma_n \geq a, \quad \max_{i \leq k} \sigma_{f_i} \leq b, \quad \max_{i \leq k} \sigma_{g_i} \leq b,$$

one has

$$d_{\mathrm{TV}}(\gamma_n \circ f^{-1}, \gamma_n \circ g^{-1}) \leq C d_{\mathrm{K}}(\gamma_n \circ f^{-1}, \gamma_n \circ g^{-1})^\theta, \quad \theta = \frac{1}{4k(d - 1) + 1 + \tau}.$$

Remark 4.3. Using Remark 3.3, one can replace d_K with d_{FM}; that is, under the assumptions of the theorem the following estimate is also true:

$$d_{\mathrm{TV}}(\gamma_n \circ f^{-1}, \gamma_n \circ g^{-1}) \leq C d_{\mathrm{FM}}(\gamma_n \circ f^{-1}, \gamma_n \circ g^{-1})^\theta, \quad \theta = \frac{1}{4k(d - 1) + 1 + \tau}$$

for every $\tau > 0$ and some other number $C = C(d, k, a, b, \tau)$.

We observe that the constants in Theorems 4.1 and 4.2 do not depend on the numbers of variables n of f_i. Hence these theorems hold true when $f_i \colon X \to \mathbb{R}$ are γ-measurable polynomials with respect to an arbitrary centered Radon Gaussian measure γ on a locally convex space X.

Corollary 4.4. *Let γ be a centered Radon Gaussian measure on a locally convex space X. Let $k, d \in \mathbb{N}$, $a > 0$, $b > 0$, $\tau > 0$. Then there is $C(d, k, a, b, \tau) > 0$ such that, for every mapping $f = (f_1, \ldots, f_k)\colon X \to \mathbb{R}^k$, where each f_i is a γ-measurable polynomial of degree d and*

$$\int_X \Delta_f \, d\gamma > a, \quad \max_{i \le k} \sigma_{f_i} \le b,$$

for every function $\varphi \in C_b^\infty(\mathbb{R}^k)$ and every vector $e \in \mathbb{R}^k$ with $|e| = 1$, one has

$$\int_X \partial_e \varphi(f(x)) \, \gamma(dx) \le C(d, k, a, b, \tau) \|\varphi\|_\infty^\alpha \|\partial_e \varphi\|_\infty^{1-\alpha}, \quad \alpha = \frac{1}{4k(d-1) + \tau}.$$

Therefore, if $\Delta_f > 0$ on a positive measure set, the induced measure $\gamma \circ f^{-1}$ belongs to the Nikolskii–Besov class $B^\alpha(\mathbb{R}^k)$ with α that depends only on d and k.

Proof. By the Tsirelson isomorphism theorem (see [8, Chapter 3]), we can assume that γ is the countable power of the standard Gaussian measure on the real line (i.e., γ is defined on $\mathbb{R}^\infty$). In that case we can approximate each polynomial f_i by the sequence of its finite-dimensional conditional expectations $f_{i,n}$ with respect to the σ-fields generated by the first n variables $x_1, \ldots, x_n$. Recall that

$$f_{i,n}(x_1, \ldots, x_n) = \int_X f_i(x_1, \ldots, x_n, y) \, \gamma(dy),$$

where we write vectors in $\mathbb{R}^\infty$ in the form $(x_1, \ldots, x_n, y)$, $y = (y_1, y_2, \ldots) \in \mathbb{R}^\infty$. It is well-known that each $f_{i,n}$ is a polynomial of degree d (see [8, Propositions 5.4.5 and 5.10.6]). Moreover, the polynomials $f_{i,n}$ converge to f_i almost everywhere and in all Sobolev norms (see [8, Corollary 3.5.2 and Proposition 5.4.5]). Therefore, for the corresponding mappings $f_n = (f_{1,n}, \ldots, f_{k,n})$ the integrals of Δ_{f_n} are not less than a for all n sufficiently large. In addition, $\sigma_{f_{i,n}} \le \sigma_{f_i} \le b$. This enables us to pass to the limit $n \to \infty$ in the inequality in Theorem 4.1. $\qquad\square$

Similarly we obtain the following result.

Corollary 4.5. *Let γ be a centered Radon Gaussian measure on a locally convex space X. Let $k, d \in \mathbb{N}$, $a > 0$, $b > 0$, $\tau > 0$ be fixed. Then there exists a number $C_1 = C_1(d, k, a, b, \tau)$ such that, whenever*

$$f = (f_1, \ldots, f_k) \quad and \quad g = (g_1, \ldots, g_k)$$

are mappings from X to $\mathbb{R}^k$ such that their components f_i, g_i are γ-measurable polynomials of degree d with

$$\int_X \Delta_f \, d\gamma \ge a, \quad \int_X \Delta_g \, d\gamma \ge a, \quad \sigma_{f_i} \le b, \quad \sigma_{g_i} \le b, \quad i = 1, \ldots, k,$$

one has

$$d_{\mathrm{TV}}(\gamma \circ f^{-1}, \gamma \circ g^{-1}) \le C_1 d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1})^{1/(4k(d-1)+1+\tau)}.$$

Along with Lemma 2.1 this yields the following fact.

Corollary 4.6. *Let γ be a Radon Gaussian measure on a locally convex space X. Let $f_n = (f_{1,n}, \ldots, f_{k,n})\colon X \to \mathbb{R}^k$ be a sequence of mappings such that each $f_{j,n}$ is a γ-measurable polynomial of degree d. Suppose that the distributions $\gamma \circ f_n^{-1}$ converge weakly on $\mathbb{R}^k$ and there is $a > 0$ such that for all $n \in \mathbb{N}$,*

$$\int_X \Delta_{f_n} \, d\gamma > a.$$

Then these measures also converge in variation and, for every $\tau > 0$, there exists a number C_2, depending on d, k, a, τ, and a common bound for the variances of the components of f_n, such that

$$d_{\mathrm{TV}}(\gamma \circ f_m^{-1}, \gamma \circ f_n^{-1}) \leq C_2 d_{\mathrm{FM}}(\gamma \circ f_m^{-1}, \gamma \circ f_n^{-1})^{1/(4k(d-1)+1+\tau)}.$$

This is a multidimensional generalization of [29, Theorem 3.1] and an improvement of the rate of convergence as compared to [27, Theorem 4.1].

It is worth noting that, as was shown in [27] extending a result from [23], a polynomial mapping f from an infinite-dimensional space with a Gaussian measure γ to $\mathbb{R}^k$ has an absolutely continuous distribution precisely when Δ_f is not zero a.e. (equivalently, $\Delta_f > 0$ on a positive measure set due to the $0 - 1$ law for polynomials; see [8, Proposition 5.10.10]). Moreover, $\gamma \circ f^{-1}$ is not absolutely continuous precisely when there is a polynomial Q on $\mathbb{R}^k$ such that $Q(f)$ is a constant a.e. Therefore, the assumed lower bound on the expectations of Δ_f and Δ_g is quite natural.

Combining Theorem 4.1 and Remark 3.4, one can obtain the following theorem, which in a sense generalizes the Carbery–Wright inequality (but the latter has been used in the proof).

Corollary 4.7. *Let $k, d \in \mathbb{N}$, $a > 0$, $b > 0$, $\tau > 0$. Then there is $C = C(d, k, a, b, \tau)$ such that if $f = (f_1, \ldots, f_k) \colon \mathbb{R}^n \to \mathbb{R}^k$, where each f_i is a polynomial of degree d, satisfies the conditions*

$$\int_{\mathbb{R}^n} \Delta_f \, d\gamma_n \geq a, \quad \max_{i \leq k} \sigma_{f_i} \leq b,$$

then

$$\gamma_n(f \in A) \leq C(d, k, a, b, \tau) \lambda_k(A)^\theta, \quad \theta = \frac{1}{4k^2(d-1)+\tau},$$

where λ_k is the standard Lebesgue measure on $\mathbb{R}^k$.

Let us mention a result from [14] on distributions of multidimensional random vectors the components of which are general functions belonging to the Sobolev classes $W^{p,2}(\gamma)$, where γ is a general centered Radon Gaussian measure on a locally convex space X. Suppose we are given a sequence of mappings

$$F_n = (F_n^1, \ldots, F_n^k) \colon X \to \mathbb{R}^k$$

such that $F_n^i \in W^{4k,2}(\gamma)$. Let $\mu_n = \gamma \circ F_n^{-1}$. The following theorem proved in [14] is based on a simple observation that by the compactness of the embedding of the space $BV(U)$ of functions of bounded variation on a ball $U \subset \mathbb{R}^k$ to the space $L^1(U)$, every weakly convergent sequence of non-negative measures μ_n on U with densities bounded in the norm of $BV(U)$ converges also in variation. In order to obtain from this convergence in variation on the whole space, it is necessary to add the uniform tightness of the measures μ_n, i.e., the condition $\lim_{R \to \infty} \sup_n \mu_n(\mathbb{R}^k \backslash U_R) = 0$, where U_R is the closed ball of radius R centered at the origin. In our situation the uniform tightness follows from the estimate $\sup_{n,i} \|F_n^i\|_{L^1(\gamma)} < \infty$, which gives the estimate

$$\sup_n \int_{\mathbb{R}^k} |x| \, \mu_n(dx) < \infty.$$

The assumption of the theorem is chosen in such a way that we are able to apply the indicated reasoning not to the original sequence of induced measures μ_n but to some sequence asymptotically approaching it. For the reader's convenience and also

taking into account that the condition in [14] contains a misprint (the considered norm in Theorem 2 and Corollary 1 in [14] should be $\|F_n^i\|_{4d,2}$, not $\|F_n^i\|_{2d,2}$), we include the proof that is not long. Set

$$\delta(\varepsilon) = \sup_n \ \gamma(\Delta_{F_n} \leq \varepsilon).$$

Theorem 4.8. *Suppose that*

$$\sup_n \|F_n^i\|_{4k,2} < \infty \quad and \quad \lim_{\varepsilon \to 0} \delta(\varepsilon) = 0.$$

Then the sequence of measures $\mu_n = \gamma \circ F_n^{-1}$ has a subsequence convergent in variation.

Proof. Let us consider the measures

$$\nu_{n,\varepsilon} = \left(\frac{\Delta_n}{\Delta_n + \varepsilon^2} \cdot \gamma\right) \circ F_n^{-1}, \quad \Delta_n := \Delta_{F_n}, \ \varepsilon > 0.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^k)$. Applying (4.1) and using the notation $m_{i,j}^n$ and $a_{i,j}^n$ for the elements of M_{F_n} and A_{F_n}, respectively, we obtain

$$\int_{\mathbb{R}^n} \partial_{x_i} \varphi \, d\nu_{n,\varepsilon} = \int_X (\partial_{x_i}\varphi(F_n)) \frac{\Delta_n}{\Delta_n + \varepsilon^2} \, d\gamma = \int_X \sum_{j,l} \frac{a_{i,j}^n}{\Delta_n + \varepsilon^2} \, m_{j,l}^n \cdot (\partial_{x_l}\varphi)(F_n) \, d\gamma$$

$$(4.6) \qquad\qquad = \int_X \sum_j \frac{a_{i,j}^n}{\Delta_n + \varepsilon^2} \, \langle \nabla(\varphi \circ F_n), \nabla F_n^j \rangle_H \, d\gamma.$$

It is known (see [8, Section 5.8] or [35, Section 4.2]) that for every function v in the second Sobolev class $W^{p,2}(\gamma)$, where $p > 1$, and for every function $g \in W^{p',1}(\gamma)$, where $p' = p/(p-1)$, one has the following integration by parts formula:

$$\int_X \langle \nabla g, \nabla v \rangle_H \, d\gamma = - \int_X g Lv \, d\gamma,$$

where $Lv \in L^p(\gamma)$ is the extension of the Ornstein–Uhlenbeck operator to $W^{p,2}(\gamma)$. Hence for all $g \in W^{qp',1}(\gamma)$ and $\psi \in W^{q'p',1}(\gamma)$ with $q > 1$, $q' = q/(q-1)$ we have (since $\psi g \in W^{p',1}(\gamma)$)

$$\int_X \langle \nabla g, \nabla v \rangle_H \psi \, d\gamma = - \int_X [g\psi Lv + g\langle \nabla \psi, \nabla v \rangle_H] \, d\gamma.$$

We are going to apply this formula to (4.6). The hypothesis of the theorem implies that

$$v = F_n^j \in W^{4k,2}(\gamma), \quad g = \varphi \circ F_n \in W^{4k,1}(\gamma).$$

To apply the integration by parts formula, we only need to ensure that

$$\psi = \frac{a_{i,j}^n}{\Delta_n + \varepsilon^2} \in W^{s,1}(\gamma), \quad s = \frac{4k}{4k-2}.$$

The $L^s(\gamma)$-norm of ψ is finite, since $|\psi| \leq |a_{i,j}^n|/\varepsilon^2$ and

$$\|a_{i,j}^n\|_{L^s} \leq C \sum_{l,r \neq i,j} \|\langle \nabla F_n^l, \nabla F_n^r \rangle_H\|_{L^{s(k-1)}} \, d\gamma \leq C \sum_l \|F_n^l\|_{s(2k-2),2}.$$

The right-hand side is finite, because

$$s \cdot (2k-2) = (4k(k-1))/(2k-1) < 4k \quad \text{and} \quad \sup_n \|F_n^i\|_{4k,2} < \infty$$

by the assumption of the theorem. Thus, $\psi \in L^s(\gamma)$.

Next, we show that $|\nabla \psi|_H \in L^s(\gamma)$. Using the cofactor expansion for the determinant Δ_n, we see that

$$\nabla \Delta_n = \nabla \det M_{F_n} = \sum_{i,j} \frac{\partial \det M_{F_n}}{\partial m_{i,j}^n} \nabla m_{i,j}^n = \sum_{i,j} a_{i,j}^n \nabla m_{i,j}^n$$

and thus

$$\nabla \psi = \frac{\nabla a_{i,j}^n}{\Delta_n + \varepsilon^2} - \frac{a_{i,j}^n}{(\Delta_n + \varepsilon^2)^2} \sum_{k,r} a_{k,r}^n \nabla m_{k,r}^n.$$

Similarly to the calculations above we prove that

$$\left\| |\nabla a_{i,j}^n|_H \right\|_{L^s} \leq C \sum_l \|F_n^l\|_{s(2k-2),2}$$

and

$$\left\| |a_{i,j}^n a_{k,r}^n \nabla m_{k,r}^n|_H \right\|_{L^s} \leq C \sum_t \|F_n^t\|_{s(4k-2),2} = C \sum_t \|F_n^t\|_{4k,2}.$$

Thus, $|\nabla \psi|_H \in L^s(\gamma)$ and $\psi \in W^{s,1}(\gamma)$, as announced.

Applying the integration by parts formula to (4.6), we obtain

$$\int_{\mathbb{R}^n} \partial_{x_i} \varphi \, d\nu_{n,\varepsilon} = \sum_j \int_X \frac{a_{i,j}^n}{\Delta_n + \varepsilon^2} \langle \nabla(\varphi \circ F_n), \nabla F_j \rangle_H \, d\gamma$$

$$= -\sum_j \int_X \varphi(F_n) \frac{a_{i,j}^n}{\Delta_n + \varepsilon^2} L F_j \, d\gamma - \sum_j \int_X \varphi(F_n) \left\langle \nabla \frac{a_{i,j}^n}{\Delta_n + \varepsilon^2}, \nabla F_j \right\rangle_H d\gamma.$$

Hence the generalized partial derivatives of the measure $\nu_{n,\varepsilon}$ are the measures

$$\left[\sum_j \left(\frac{a_{i,j}^n}{\Delta_n + \varepsilon^2} L F_j + \left\langle \nabla \frac{a_{i,j}^n}{\Delta_n + \varepsilon^2}, \nabla F_j \right\rangle_H \right) \gamma \right] \circ F_n^{-1}.$$

Therefore, the measure $\nu_{n,\varepsilon}$ has a density $\varrho_{n,\varepsilon}$ of class BV, and its BV-norm is dominated by

$$1 + \left\| \sum_j \left(\frac{a_{i,j}^n}{\Delta_n + \varepsilon^2} L F_j + \left\langle \nabla \frac{a_{i,j}^n}{\Delta_n + \varepsilon^2}, \nabla F_j \right\rangle_H \right) \right\|_{L^1(\gamma)} \leq M(\varepsilon).$$

It is known that the embedding $BV(U_R) \to L_1(U_R)$ is compact, where U_R is the ball of radius R centered at the origin in $\mathbb{R}^d$. Hence there exists a subsequence $\{i_n\}$ such that $\{\varrho_{i_n,1/m}\}$ converges in $L_1(U_m)$ for every $m \in \mathbb{N}$.

Let us estimate $\|\nu_{i,\varepsilon} - \mu_i\|_{TV}$ in the following way:

$$(4.7) \qquad \|\nu_{i,\varepsilon} - \mu_i\|_{TV} = \int_X \frac{\varepsilon^2}{\Delta_i + \varepsilon^2} \, d\gamma \leq \varepsilon + \gamma(\Delta_i \leq \varepsilon) \leq \varepsilon + \delta(\varepsilon).$$

We observe that the family of measures $\{\nu_{i,\varepsilon}\}$, where $i \geq 1$, $\varepsilon > 0$, is uniformly tight. This follows by the boundedness of $\{F_n\}$ in $L^1(\gamma)$ and the Chebyshev inequality.

Let us now show that the sequence of measures μ_{i_n} is Cauchy in variation. Let $\alpha > 0$. Using the uniform tightness and (4.7) we take M such that

$$\|\nu_{i,1/M} - \mu_i\|_{TV} \leq \alpha/5, \quad \nu_{i,\delta}(\mathbb{R}^d \backslash U_M) \leq \alpha/5 \quad \forall \delta > 0.$$

Next, we take N such that for all $n, m > N$ we obtain

$$\|\varrho_{i_n,1/M} - \varrho_{i_m,1/M}\|_{L_1(U_M)} \leq \alpha/5.$$

Then for all $n, m > N$ we have

$$\|\mu_{i_n} - \mu_{i_m}\|_{\mathrm{TV}} \leq \|\nu_{i_n,1/M} - \nu_{i_m,1/M}\|_{\mathrm{TV}} + \frac{2\alpha}{5} = \|\varrho_{i_n,1/M} - \varrho_{i_m,1/M}\|_{L_1(\mathbb{R}^d)} + \frac{2\alpha}{5}$$

$$\leq \|\varrho_{i_n,1/M} - \varrho_{i_m,1/M}\|_{L_1(U_M)} + \frac{4\alpha}{5} \leq \alpha.$$

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 4.9. *If a sequence $\{F_n^i\}$ is bounded in $W^{4k,2}(\gamma)$ and $\delta(\varepsilon) \to 0$ and the distributions of F_n converge weakly, then they converge in variation.*

This corollary provides another proof of the already known fact that if we have $F_n^i \in \mathcal{P}_d$ and $\|\Delta_n\|_1 \geq \beta > 0$ and the sequence of distributions of F_n converges weakly, then it converges in variation.

5. THE CASE $k = 1$

In the case of mappings with values in the real line (i.e., $k = 1$) one can obtain some better estimates. They are derived from the following theorem that replaces Theorem 4.1 in this case and a similar result in Theorem 5.7 that yields an even better fractional order at the Kantorovich norm (namely, $1/(d+1)$), but at the cost of a worse constant. As above, γ_n is the standard Gaussian measure on $\mathbb{R}^n$.

Theorem 5.1. *Let $d \in \mathbb{N}$, $\tau > 0$. Then there is a number $C(d,\tau) > 0$ such that, whenever $f \colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree d, for all $\varphi \in C_b^\infty(\mathbb{R}^1)$ one has*

$$\int_{\mathbb{R}^n} \varphi'(f(x))\,\gamma_n(dx) \leq C(d,\tau)\sigma_f^{-\alpha}\|\varphi\|_\infty^\alpha \|\varphi'\|_\infty^{1-\alpha}, \quad \alpha = \frac{1}{2d-2+\tau}.$$

Therefore, $\gamma_n \circ f^{-1}$ belongs to the Nikolskii–Besov class $B^\alpha(\mathbb{R})$ independent of n, provided that f is not a constant.

Proof. We can assume that $\|\varphi\|_\infty \leq 1$. Fix $\varepsilon > 0$ (which will be chosen later). The integral that we want to estimate equals (we again omit indication of $\mathbb{R}^n$ in the integrals below)

$$(5.1) \qquad \begin{aligned} \int \varphi'(f(x))\,\gamma_n(dx) &= \int \varphi'(f(x)) \frac{\langle \nabla f(x), \nabla f(x)\rangle}{\langle \nabla f(x), \nabla f(x)\rangle + \varepsilon}\,\gamma_n(dx) \\ &+ \varepsilon \int \frac{\varphi'(f(x))}{\langle \nabla f(x), \nabla f(x)\rangle + \varepsilon}\,\gamma_n(dx). \end{aligned}$$

Let us estimate every term. For the first term, integrating by parts, we have

$$(5.2) \qquad \begin{aligned} \int \varphi'(f) \frac{\langle \nabla f, \nabla f\rangle}{\langle \nabla f, \nabla f\rangle + \varepsilon}\,d\gamma_n &= \int \frac{\langle \nabla \varphi \circ f, \nabla f\rangle}{\langle \nabla f, \nabla f\rangle + \varepsilon}\,d\gamma_n \\ &= -\int \varphi(f)\Big(\frac{Lf}{\langle \nabla f, \nabla f\rangle + \varepsilon} - \frac{\langle D^2 f \cdot \nabla f, \nabla f\rangle}{(\langle \nabla f, \nabla f\rangle + \varepsilon)^2} \Big)\,d\gamma_n \\ &\leq \int \frac{|Lf|}{\langle \nabla f, \nabla f\rangle + \varepsilon}\,\gamma_n + \int \frac{\|D^2 f\|_{HS}}{\langle \nabla f, \nabla f\rangle + \varepsilon}\,d\gamma_n \\ &\leq \big(\|Lf\|_{L^{q'}(\gamma_n)} + \|D^2 f\|_{L^{q'}(\gamma_n)}\big) \Big(\int (\langle \nabla f, \nabla f\rangle + \varepsilon)^{-q}\,d\gamma_n \Big)^{1/q}, \end{aligned}$$

where $q > 1$, $q' = q/(q-1)$. Set

$$\beta = \frac{1}{2(d-1)}.$$

Using inequality (2.2) and the equivalence of the Sobolev and L^p-norms of polynomials of degree d, we obtain that

$$\|Lf\|_{L^{q'}(\gamma_n)} + \|D^2 f\|_{L^{q'}(\gamma_n)} \leq C(d, q)\sigma_f.$$

Using (4.4), we obtain that the last expression in (5.2) is not greater than

$$C(d, q)\sigma_f \varepsilon^{-1+\beta/q} \left(\int \langle \nabla f, \nabla f \rangle \, d\gamma_n \right)^{-\beta/q},$$

which by the Poincaré inequality is not greater than

$$c_1(d, q)\sigma_f^{1-2\beta/q} \varepsilon^{-1+\beta/q}.$$

Now let us estimate the second term in the right-hand side of (5.1). As above, using (4.4) and the Poincaré inequality, we obtain

$$\int (\langle \nabla f, \nabla f \rangle + \varepsilon)^{-1} \, d\gamma_n \leq c(d)\sigma_f^{-2\beta} \varepsilon^{-1+\beta}.$$

Therefore,

$$\varepsilon \int \frac{\varphi'(f)}{\langle \nabla f, \nabla f \rangle + \varepsilon} \, d\gamma_n \leq \|\varphi'\|_\infty c(d)\sigma_f^{-2\beta} \varepsilon^{\beta}.$$

Let

$$\tau = \frac{q-1}{q}, \quad \varepsilon = \|\varphi'\|_\infty^\omega, \quad \omega = -\frac{1}{1+\beta\tau}.$$

Then for (5.1) we have the bound

$$\int \varphi'(f(x)) \gamma_n(dx) \leq (c_1(d, q)\sigma_f^{1-2\beta/q} + c(d)\sigma_f^{-2\beta})\|\varphi'\|_\infty^{1-\alpha}, \quad \alpha = \frac{1}{2d - 2 + \tau}.$$

We now take the function $\psi(t) = \varphi(t\sigma_f^{-1})$. Applying the above inequality to the polynomial $f \cdot \sigma_f^{-1}$, we can write

$$\int \psi'(f(x)) \gamma_n(dx) = \sigma_f^{-1} \int \varphi'(f(x)\sigma_f^{-1}) \gamma_n(dx)$$
$$\leq \sigma_f^{-1}(c_1(d, q) + c(d))\|\varphi'\|_\infty^{1-\alpha} = \sigma_f^{-\alpha}(c_1(d, q) + c(d))\|\psi'\|_\infty^{1-\alpha}.$$

Since τ can be taken as small as we wish, the theorem is proved. $\qquad\square$

The last assertion about membership in Nikolskii–Besov classes is improved below in Theorem 5.7. Similarly to the multidimensional case, the following theorem is obtained on the basis of the previous theorem.

Theorem 5.2. *Let $d \in \mathbb{N}$, $a > 0$, $\tau > 0$. Then there is a number $C = C(d, a, \tau) > 0$ such that whenever f and g are real polynomials on $\mathbb{R}^n$ of degree d with $\sigma_f, \sigma_g \geq a$ one has*

$$d_{\mathrm{TV}}(\gamma_n \circ f^{-1}, \gamma_n \circ g^{-1}) \leq C d_{\mathrm{K}}(\gamma_n \circ f^{-1}, \gamma_n \circ g^{-1})^\theta, \quad \theta = \frac{1}{2d - 1 + \tau}.$$

As in the multidimensional case, we obtain the following infinite-dimensional extensions.

Corollary 5.3. *Let γ be a centered Radon Gaussian measure on a locally convex space X. Let $d \in \mathbb{N}$, $\tau > 0$. Then there is a number $C(d,\tau) > 0$ such that whenever $f \colon X \to \mathbb{R}$ is a γ-measurable polynomial of degree d, for all $\varphi \in C_b^\infty(\mathbb{R}^1)$ one has*

$$\int_X \varphi'(f(x))\,\gamma(dx) \leq C(d,\tau)\sigma_f^{-\alpha}\|\varphi\|_\infty^\alpha\|\varphi'\|_\infty^{1-\alpha}, \quad \alpha = \frac{1}{2d-2+\tau}.$$

Therefore, $\gamma \circ f^{-1}$ belongs to the Nikolskii–Besov class $B^\alpha(\mathbb{R})$, provided that f is not a constant a.e.

Corollary 5.4. *Let γ be a centered Radon Gaussian measure on a locally convex space X. Let $d \in \mathbb{N}$, $a > 0$, $\tau > 0$. Then there is a number $C_1 = C_1(d,a,\tau)$ such that whenever f and g are γ-measurable polynomials on X of degree d with $\sigma_f, \sigma_g \geq a$ one has*

$$d_{\mathrm{TV}}(\gamma \circ f^{-1}, \gamma \circ g^{-1}) \leq C_1 d_{\mathrm{FM}}(\gamma \circ f^{-1}, \gamma \circ g^{-1})^{1/(2d-1+\tau)}.$$

Corollary 5.5. *Let γ be a Radon Gaussian measure on a locally convex space. Let $\{f_n\}$ be a sequence of γ-measurable polynomials of degree d. Suppose that the distributions $\gamma \circ f_n^{-1}$ converge weakly to an absolutely continuous measure ν on $\mathbb{R}$. Then they also converge in variation, and for every $\tau > 0$ there exists a number $C_2 = C_2(d,\sigma_\nu,\tau)$, where σ_ν^2 is the variance of ν, such that for all n and m sufficiently large*

$$d_{\mathrm{TV}}(\gamma \circ f_m^{-1}, \gamma \circ f_n^{-1}) \leq C_2 d_{\mathrm{FM}}(\gamma \circ f_m^{-1}, \gamma \circ f_n^{-1})^{1/(2d-1+\tau)}.$$

The second result provides an estimate with a better rate of convergence than the one obtained in Theorem 3.1 in [29].

Remark 5.6. Note that in this case, unlike Corollary 4.6, there is no condition that the integrals of Δ_{f_n} are separated from zero. In the case $k = 1$, due to the Poincaré inequality, this condition is replaced by $\sigma_{f_n} \geq a > 0$ (see Corollaries 4.5 and 5.4), which is automatically satisfied for n large enough, because for the distributions of polynomials weak convergence implies convergence of all moments (see Lemma 2.1).

We now show that one can even achieve the exponent $\theta = 1/(d+1)$, however, with a worse constant than before (depending on some special norm of the gradient). Actually, by using a different approach in the one-dimensional case, it is still possible to prove this result with the same type of constant (depending on the variance), which has recently been done for general convex measures in [22]. We include a somewhat less sharp result below, because its proof is much simpler.

Let γ be a centered Radon Gaussian measure on a locally convex space X and let H be its Cameron–Martin space. For a function $f \in W^{2,1}(\gamma)$ we define $\|\nabla f\|_*$ by

$$(5.3) \qquad\qquad \|\nabla f\|_*^2 := \sup_{|e|_H=1} \int_X |\partial_e f|^2 d\gamma.$$

It is clear that $\|\nabla f\|_* > 0$ once f is not a constant and that $\|\nabla f\|_* \leq \| |\nabla f|_H \|_{L^2(\gamma)}$.

Theorem 5.7. *Let γ_n be the standard Gaussian measure on $\mathbb{R}^n$. Then, for every $d \in \mathbb{N}$, there is a number $C(d)$ depending only on d such that for every polynomial f of degree d on $\mathbb{R}^n$ and every function $\varphi \in C_b^\infty(\mathbb{R})$, we have*

$$\int_{\mathbb{R}^n} \varphi'(f)\,d\gamma_n \leq C(d)\|\nabla f\|_*^{-1/d}\|\varphi\|_\infty^{1/d}\|\varphi'\|_\infty^{1-1/d}.$$

Therefore, $\gamma_n \circ f^{-1}$ belongs to the Nikolskii–Besov class $B^{1/d}(\mathbb{R})$ provided that f is not a constant.

Proof. We can assume that $\|\varphi\|_\infty \leq 1$. Let $e \in \mathbb{R}^n$, $|e| = 1$. We have (again omitting the indication of $\mathbb{R}^n$ in integrations)

$$\int \varphi'(f)\, d\gamma = \int \Big[\frac{(\partial_e f)^2}{(\partial_e f)^2 + \varepsilon} \varphi'(f) \Big]\, d\gamma + \varepsilon \int \frac{\varphi'(f)}{(\partial_e f)^2 + \varepsilon}\, d\gamma.$$

Writing the first term as

$$\int \frac{(\partial_e f)^2}{(\partial_e f)^2 + \varepsilon} \varphi'(f)\, d\gamma = \int \partial_e(\varphi(f)) \frac{\partial_e f}{(\partial_e f)^2 + \varepsilon}\, d\gamma$$

and integrating by parts in the last expression, we obtain

$$-\int \varphi(f) \Big[\frac{\partial_e^2 f + \langle x, e\rangle \partial_e f}{(\partial_e f)^2 + \varepsilon} - 2\frac{(\partial_e f)^2 \partial_e^2 f}{((\partial_e f)^2 + \varepsilon)^2} \Big]\, d\gamma$$

$$\leq 3\int \Big| \frac{\partial_e^2 f}{(\partial_e f)^2 + \varepsilon} \Big|\, d\gamma + \int \Big| \frac{\partial_e f}{(\partial_e f)^2 + \varepsilon} \Big| |\langle x, e\rangle|\, d\gamma$$

$$= \varepsilon^{-1/2} \Big(3\int \Big| \frac{\partial_e^2 g}{(\partial_e g)^2 + 1} \Big| d\gamma + \int \Big| \frac{\partial_e g}{(\partial_e g)^2 + 1} \Big| |\langle x, e\rangle|\, d\gamma \Big)$$

$$\leq \varepsilon^{-1/2}(3d\sqrt{\pi/2} + 1),$$

where $g = f\varepsilon^{-1/2}$. By using the Carbery–Wright inequality (2.3) in the same manner as in the derivation of (4.4) we have

$$\int \frac{\varphi'(f)}{(\partial_e f)^2 + \varepsilon}\, d\gamma \leq cd\|\varphi'\|_\infty \|\partial_e f\|_2^{-1/(d-1)} \varepsilon^{-1+1/(2d-2)} \int_0^\infty (s+1)^{-2} s^{1/(2d-2)}\, ds.$$

Thus,

$$\int \varphi'(f)\, d\gamma \leq c_1(d)\|\partial_e f\|_2^{-1/(d-1)} \|\varphi'\|_\infty \varepsilon^{1/(2d-2)} + c_2(d)\varepsilon^{-1/2}.$$

Taking $\varepsilon = \|\varphi'\|_\infty^{-2+2/d}$, we obtain

$$\int \varphi'(f)\, d\gamma \leq (c_1(d)\|\partial_e f\|_2^{-1/(d-1)} + c_2(d))\|\varphi'\|_\infty^{1-1/d}.$$

Since this estimate is valid for every vector $e \in \mathbb{R}^n$ of unit length, we have

$$\int \varphi'(f)\, d\gamma \leq \Big(c_1(d)\|\nabla f\|_*^{-1/(d-1)} + c_2(d) \Big) \|\varphi'\|_\infty^{1-1/d}.$$

Applying the last estimate to the polynomial $f\|\nabla f\|_*^{-1}$, we find that

$$\int \varphi'(f\|\nabla f\|_*^{-1})\, d\gamma \leq (c_1(d) + c_2(d))\|\varphi'\|_\infty^{1-1/d}.$$

Let $\psi(t) = \varphi(t\|\nabla f\|_*^{-1})$, $C(d) = c_1(d) + c_2(d)$. Then

$$\int \psi'(f)\, d\gamma = \|\nabla f\|_*^{-1} \int \varphi'(f\|\nabla f\|_*^{-1})\, d\gamma$$

$$\leq C(d)\|\nabla f\|_*^{-1} \|\varphi'\|_\infty^{1-1/d} = C(d)\|\nabla f\|_*^{-1/d} \|\psi'\|_\infty^{1-1/d},$$

which proves the theorem. $\qquad\square$

Corollary 5.8. *Let γ be a centered Radon Gaussian measure on a locally convex space. Then, for every $d \in \mathbb{N}$, there is a number $C(d)$ that depends only on d such that for every γ-measurable polynomial f of degree d on X and every function $\varphi \in C_b^\infty(\mathbb{R})$, we have*

$$\int_X \varphi'(f)\,d\gamma \leq C(d)\|\nabla f\|_*^{-1/d}\|\varphi\|_\infty^{1/d}\|\varphi'\|_\infty^{1-1/d}.$$

Therefore, $\gamma \circ f^{-1}$ belongs to the Nikolskii–Besov class $B^{1/d}(\mathbb{R})$ provided that f is not a constant a.e.

From the previous theorem one derives the following assertion, which is an analog of Theorem 5.2 in this case.

Theorem 5.9. *Let $d \in \mathbb{N}$, $a > 0$. Then there is a number $C = C(d, a)$ such that, whenever f and g are real polynomials on $\mathbb{R}^n$ of degree d with $\|\nabla f\|_* \geq a$ and $\|\nabla g\|_* \geq a$, one has*

$$d_{\mathrm{TV}}(\gamma_n \circ f^{-1}, \gamma_n \circ g^{-1}) \leq C d_{\mathrm{K}}(\gamma_n \circ f^{-1}, \gamma_n \circ g^{-1})^\theta, \quad \theta = \frac{1}{d+1}.$$

We observe that in some cases the quantity $\|\nabla f_n\|_*$ tends to zero while the distance $d_{\mathrm{K}}(\gamma \circ f_n^{-1}, \gamma \circ g^{-1})$ tends to zero as well (in this relation we mention the papers [24] and [30]). In these cases the above estimate provides no useful information. As an example we can take $f_n(x) = n^{-1/2}\sum_{i=1}^n(x_i^2 - 1)$ and $g = x_1$ (i.e., the distribution of g is Gaussian). In this case $\|\nabla f_n\|_* \to 0$, so the previous theorem is not applicable (but Corollaries 5.4 and 5.5 are applicable). On the other hand, in some model examples the quantity $\|\nabla f_n\|_*$ is separated from zero. For example, we can consider $f_n(x) = (2n^2 - 2n)^{-1/2}\sum_{1 \leq i,j \leq n; i \neq j} x_i x_j$. For this f_n we have $\|\nabla f_n\|_* = \sqrt{(n-1)/2n} > 1/2$. Hence the above theorem provides a rate of convergence in total variation to a Gaussian law in terms of weak convergence.

We also note that a sharper estimate with a constant dependent only on the variances of f and g and the same exponent as above has been established in [22].

It can happen that the optimal power is $1/d$; the following simple example shows that one cannot get any better exponent (and that the order $1/d$ of the Nikolskii–Besov class above is optimal).

Example 5.10. Let us consider the monomial x^d with even d on the real line with the standard Gaussian measure γ. Let ϱ be its distribution density. It is obvious that $\varrho(t) = 0$ if $t < 0$ and that ϱ is strictly monotonically decreasing on $(0, +\infty)$. Let us also consider $x^d - h$, $h > 0$. The Kantorovich distance between the laws of x^d and $x^d - h$ equals h, and the variation distance is given by

$$\int_{-\infty}^{+\infty} |\varrho(t-h) - \varrho(t)|\,dt = \int_0^h \varrho(t)\,dt + \int_h^{+\infty}(\varrho(t-h) - \varrho(t))\,dt = 2\gamma(|x| \leq h^{1/d}).$$

It is readily verified that the latter expression for small h behaves like $h^{1/d}$.

Remark 5.11. It is still unknown whether the set of distributions of polynomials of a fixed degree d is closed in the weak topology (equivalently, in the metrics d_{K} and d_{FM}). The answer is positive for $d = 1$ (which is trivial) and for $d = 2$ (which was proved in [3] and [34]). Some asymptotic properties of polynomial distributions are discussed in [2] and [10].

6. BOUNDS VIA L^2-NORMS

In this section, γ is the standard Gaussian measure on $\mathbb{R}^n$ (in this case we also use the symbol γ_n) or on $\mathbb{R}^\infty$. The following result was announced in [16] (we present it in our terms; in [16] multiple stochastic integrals of order d are used).

Theorem A. Let $g \in \mathcal{H}_d$ and $g \neq 0$. Then there is a constant $C(d, g)$ depending only on d and g such that for every $f \in \mathcal{H}_d$ one has

$$\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{\mathrm{TV}} \leq C(d, g)\|f - g\|_2^{1/d}.$$

The announcement does not contain details of proof, and also the form of dependence of $C(d, g)$ on g is not indicated. In relation to this estimate Nourdin and Poly [29] proved the following result (also presented here in our terms).

Theorem B. Let $d \in \mathbb{N}$, $a > 0$, $b > 0$. Then there exists a number $C(d, a, b) > 0$ such that for every pair of polynomials f, g of degree d with $\sigma_f \in [a, b]$ one has

$$\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{\mathrm{TV}} \leq C(d, a, b)\|f - g\|_2^{1/(2d)}.$$

While the power of the L^2-norm in this theorem is twice smaller (which makes the estimate worse) than in Theorem A, Nourdin and Poly managed to clarify dependence of $C(d, g)$ on g: this constant depends only on the bounds for the variance. In this section, we first prove an intermediate result between Theorem A and Theorem B and then give its multidimensional extension. The next theorem gives an analog of the Davydov–Martynova estimate with a constant worse than in the Nourdin–Poly estimate, but with a better dependence on the L^2-norm (which differs from the announcement in [16] by only a logarithmic factor). We recall that $\|\cdot\|_*$ is defined by (5.3).

Theorem 6.1. *There is a constant $c(d)$ depending only on d such that for every pair of polynomials f, g of degree $d > 1$ one has*

$$\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{\mathrm{TV}} \leq c(d)\big(\|\nabla g\|_*^{-1/(d-1)} + \sigma_g + 1\big)\|f - g\|_2^{1/d}\Big(\big|\ln\|f - g\|_2\big|^{d/2} + 1\Big).$$

Proof. If $\|f - g\|_2 \geq 1/e$, then

$$\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{\mathrm{TV}} \leq 1 \leq e^{1/d}\|f - g\|_2^{1/d}\Big(\big|\ln\|f - g\|_2\big|^{d/2} + 1\Big).$$

We now consider the case $\|f - g\|_2 \leq 1/e$. Fix a function $\varphi \in C_0^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$, a vector $e \in \mathbb{R}^n$ of unit length, and a number $\varepsilon \in (0, 1/e)$. Consider the function

$$\Phi(t) = \int_{-\infty}^t \varphi(\tau)d\tau.$$

Note that

$$\partial_e(\Phi(f) - \Phi(g)) = \partial_e f \varphi(f) - \partial_e g \varphi(g) = (\varphi(f) - \varphi(g))\partial_e g + \varphi(f)(\partial_e f - \partial_e g).$$

Thus, we have (omitting indication of limits of integration in case of $\mathbb{R}^n$)

$$\int \varphi(f) - \varphi(g)d\gamma = \int (\varphi(f) - \varphi(g))\frac{(\partial_e g)^2}{(\partial_e g)^2 + \varepsilon}d\gamma$$
$$+ \varepsilon \int (\varphi(f) - \varphi(g))((\partial_e g)^2 + \varepsilon)^{-1}d\gamma$$
$$= \int \frac{\partial_e g \partial_e (\Phi(f) - \Phi(g))}{(\partial_e g)^2 + \varepsilon}d\gamma - \int \frac{\varphi(f)(\partial_e f - \partial_e g)\partial_e g}{(\partial_e g)^2 + \varepsilon}d\gamma$$
$$+ \varepsilon \int (\varphi(f) - \varphi(g))((\partial_e g)^2 + \varepsilon)^{-1}d\gamma.$$

Let us estimate each term separately. First, let us consider the last term. Using the Carbery–Wright inequality in the same manner as in the derivation of (4.4) we obtain

$$\varepsilon \int (\varphi(f) - \varphi(g))((\partial_e g)^2 + \varepsilon)^{-1}d\gamma \leq 2\varepsilon \int ((\partial_e g)^2 + \varepsilon)^{-1}d\gamma$$
$$\leq 2dc_1 \|\partial_e g\|_2^{-1/(d-1)}\varepsilon^{1/(2d-2)} \int_0^\infty (s+1)^{-2}s^{1/(2d-2)}ds$$
$$= c_1(d)\|\partial_e g\|_2^{-1/(d-1)}\varepsilon^{1/(2d-2)}.$$

Now we estimate the second term:

$$-\int \frac{\varphi(f)(\partial_e f - \partial_e g)\partial_e g}{(\partial_e g)^2 + \varepsilon}d\gamma \leq \int \frac{|\partial_e f - \partial_e g| \, |\partial_e g|}{(\partial_e g)^2 + \varepsilon}d\gamma$$
$$\leq 2^{-1}\varepsilon^{-1/2} \int |\partial_e f - \partial_e g|d\gamma \leq c_2(d)\varepsilon^{-1/2}\|f - g\|_2.$$

Finally, let us estimate the first term. Integrating by parts we obtain

$$\int \frac{\partial_e g \partial_e (\Phi(f) - \Phi(g))}{(\partial_e g)^2 + \varepsilon}d\gamma$$
$$= -\int (\Phi(f) - \Phi(g))\left[\frac{\partial_e^2 g - \langle x, e\rangle \partial_e g}{(\partial_e g)^2 + \varepsilon} - 2\frac{(\partial_e g)^2 \partial_e^2 g}{((\partial_e g)^2 + \varepsilon)^2}\right]d\gamma$$
$$\leq 3\int |f - g|\frac{|\partial_e^2 g|}{(\partial_e g)^2 + \varepsilon}d\gamma + 2^{-1}\varepsilon^{-1/2}\int |f - g| \, |\langle x, e\rangle|d\gamma$$
$$\leq 3\int_{\{|f-g|\geq\|f-g\|_2 t\}} |f - g|\frac{|\partial_e^2 g|}{(\partial_e g)^2 + \varepsilon}d\gamma + 3t\|f - g\|_2 \int \frac{|\partial_e^2 g|}{(\partial_e g)^2 + \varepsilon}d\gamma$$
$$+ 2^{-1}\varepsilon^{-1/2}\|f - g\|_2 \leq c_3(d)\varepsilon^{-1}\|f - g\|_2\sigma_g\big(\gamma(|f - g| \geq \|f - g\|_2 t)\big)^{1/3}$$
$$+ 3t\|f - g\|_2 \int \frac{|\partial_e^2 g|}{(\partial_e g)^2 + \varepsilon}d\gamma + 2^{-1}\varepsilon^{-1/2}\|f - g\|_2.$$

Note that writing γ as the product of γ_1 and γ_{n-1}, we have

$$
\int \frac{|\partial_e^2 g|}{(\partial_e g)^2 + \varepsilon} d\gamma = \int_{\langle e \rangle^\perp} \int_{\langle e \rangle} \frac{|\partial_e^2 g|}{(\partial_e g)^2 + \varepsilon} d\gamma_1 \, d\gamma_{n-1}
$$

$$
= \varepsilon^{-1/2} \int_{\langle e \rangle^\perp} \int_{\langle e \rangle} \frac{|\partial_e^2 g \varepsilon^{-1/2}|}{(\partial_e g \varepsilon^{-1/2})^2 + 1} d\gamma_1 \, d\gamma_{n-1} \le d(2\pi \varepsilon)^{-1/2} \int_{\langle e \rangle^\perp} \int \frac{d\tau}{\tau^2 + 1} d\gamma_{n-1}
$$

$$
= d\varepsilon^{-1/2} (2\pi)^{-1/2} \int \frac{1}{\tau^2 + 1} d\tau = 3^{-1} c_4(d) \varepsilon^{-1/2}.
$$

Recall (see [8, Corollary 5.5.7]) that

$$
\gamma\big(x \colon |f(x)| \ge t\|f\|_2\big) \le c_r \exp(-rt^{2/d}), \quad r < \frac{d}{2e},
$$

where c_r depends only on r. Thus, for $t \ge 1$ and some $c \in (0, 1/2)$ we obtain

$$
\int [\varphi(f) - \varphi(g)] d\gamma \le c_5(d) \Big(\|\partial_e g\|_2^{-1/(d-1)} \varepsilon^{1/(2d-2)}
$$

$$
+ \varepsilon^{-1} \|f - g\|_2 \sigma_g \exp\big(-ct^{2/d}\big) + t\|f - g\|_2 \varepsilon^{-1/2} \Big).
$$

Setting $t = (2c)^{-d/2} (\ln \varepsilon^{-1})^{d/2}$, $\varepsilon = \|f - g\|_2^{2(d-1)/d}$ (recall that $\|f - g\|_2 < 1/e$, hence $t \ge 1$), we obtain that the right-hand side is estimated by

$$
c(d) \Big(\|\partial_e g\|_2^{-1/(d-1)} \|f - g\|_2^{1/d} + \sigma_g \|f - g\|_2^{1/d} + \big|\ln \|f - g\|_2 \big|^{d/2} \|f - g\|_2^{1/d} \Big)
$$

$$
\le c(d) \big(\|\partial_e g\|_2^{-1/(d-1)} + \sigma_g + 1 \big) \big|\ln \|f - g\|_2 \big|^{d/2} \|f - g\|_2^{1/d}.
$$

Now taking inf over e and sup over φ we obtain the desired estimate. $\qquad \square$

Our next theorem is a multidimensional analog of Theorem A. We need a lemma.

Lemma 6.2. *Let A and B be a pair of square $k \times k$-matrices. Then*

$$
|\det A - \det B| \le \|A - B\|_{HS} \big(\|A\|_{HS}^2 + \|B\|_{HS}^2 \big)^{(k-1)/2}.
$$

Proof. Let a_i and b_i, $i = 1, \ldots, k$, be the columns of the matrices A and B, respectively. The determinant of the matrix A is a multilinear function in $a_1, \ldots, a_k$. We denote this function by $\Delta(a_1, \ldots, a_k)$. We have

$$
|\det A - \det B| = |\Delta(a_1, \ldots, a_k) - \Delta(b_1, \ldots, b_k)|
$$

$$
\le \sum_{i=1}^k |\Delta(b_1, \ldots, b_{i-1}, a_i, \ldots, a_k) - \Delta(b_1, \ldots, b_i, a_{i+1}, \ldots a_k)|
$$

$$
= \sum_{i=1}^k |\Delta(b_1, \ldots, b_{i-1}, a_i - b_i, a_{i+1}, \ldots, a_k)| \le \sum_{i=1}^k |b_1| \ldots |b_{i-1}| |a_i - b_i| |a_{i+1}| \ldots |a_k|
$$

$$
\le \Big(\sum_{i=1}^k |a_i - b_i|^2 \Big)^{1/2} \Big(\sum_{i=1}^k (|a_i|^2 + |b_i|^2) \Big)^{(k-1)/2}
$$

$$
= \|A - B\|_{HS} (\|A\|_{HS}^2 + \|B\|_{HS}^2)^{(k-1)/2}.
$$

The lemma is proved. $\qquad \square$

Theorem 6.3. *Let $k, d \in \mathbb{N}$, $a > 0$, $b > 0$, $\tau > 0$. Then there exists a number $C(d, k, a, b, \tau) > 0$ such that, for every pair of mappings $f = (f_1, \ldots, f_k)$ and $g = (g_1, \ldots, g_k) \colon \mathbb{R}^n \to \mathbb{R}^k$, where all f_i, g_i are polynomials of degree d and*

$$\int_{\mathbb{R}^n} \Delta_f \, d\gamma \geq a, \quad \max_{i \leq k} \sigma_{f_i} \leq b,$$

one has

$$(6.1) \qquad \|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{\mathrm{TV}} \leq C(d, k, a, b, \tau)\|f - g\|_2^\theta, \ \theta = \frac{1}{4k(d-1) + \tau}.$$

Proof. Fix $\varphi \in C_0^\infty(\mathbb{R}^k)$ with $\|\varphi\|_\infty \leq 1$. Let $f^i = (g_1, \ldots, g_i, f_{i+1}, \ldots, f_k)$, $f^0 = f$, $f^k = g$. Consider the function

$$\Phi_i(y_1, \ldots, y_k) = \int_{-\infty}^{y_i} \varphi(y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_k) dt.$$

Note that for each i we have

$$\nabla(\Phi_i(f^{i-1})) - \nabla(\Phi_i(f^i)) = \sum_{j=1}^k \left(\partial_{y_j} \Phi_i(f^{i-1}) \nabla f_j^{i-1} - \partial_{y_j} \Phi_i(f^i) \nabla f_j^i\right),$$

which can be written as

$$\sum_{j=1}^k \left(\partial_{y_j} \Phi_i(f^{i-1}) - \partial_{y_j} \Phi_i(f^i)\right) \nabla f_j^{i-1} + \partial_{y_i} \Phi_i(f^i)(\nabla f_i^{i-1} - \nabla f_i^i)$$

$$= \sum_{j=1}^k \left(\partial_{y_j} \Phi_i(f^{i-1}) - \partial_{y_j} \Phi_i(f^i)\right) \nabla f_j^{i-1} + \varphi(f^i)(\nabla f_i - \nabla g_i).$$

Thus,

$$\left(\langle \nabla \Phi_i(f^{i-1}) - \nabla \Phi_i(f^i), \nabla f_m^{i-1} \rangle\right)_{m=1}^k$$
$$= M_{f^{i-1}} \left(\partial_{y_j} \Phi_i(f^{i-1}) - \partial_{y_j} \Phi_i(f^i)\right)_{j=1}^k + \varphi(f^i) \left(\langle \nabla f_i - \nabla g_i, \nabla f_m^{i-1} \rangle\right)_{m=1}^k.$$

Recall that $\Delta_f \cdot M_f^{-1} = A_f$ (see (4.1)). Hence, denoting the elements of the matrix A_{f^i} by $a_{f^i}^{s,r}$, we obtain

$$(6.2) \quad \Delta_{f^{i-1}}(\varphi(f^{i-1}) - \varphi(f^i)) = \Delta_{f^{i-1}}(\partial_{y_i} \Phi_i(f^{i-1}) - \partial_{y_i} \Phi_i(f^i))$$

$$= \sum_{j=1}^k \langle \nabla \Phi_i(f^{i-1}) - \nabla \Phi_i(f^i), \nabla f_j^{i-1} \rangle a_{f^{i-1}}^{j,i} - \varphi(f^i) \sum_{j=1}^k \langle \nabla f_i - \nabla g_i, \nabla f_j^{i-1} \rangle a_{f^{i-1}}^{j,i}.$$

Next we observe that (all integrals below are taken over $\mathbb{R}^n$)

$$(6.3) \quad \int [\varphi(f) - \varphi(g)] \, d\gamma$$

$$= \sum_{i=1}^k \int \frac{\Delta_{f^{i-1}}(\varphi(f^{i-1}) - \varphi(f^i))}{\Delta_f + \varepsilon} \, d\gamma + \sum_{i=1}^k \int \frac{(\Delta_{f^{i-1}} - \Delta_{f^i})\varphi(f^i)}{\Delta_f + \varepsilon} \, d\gamma$$

$$+ \int \frac{(\Delta_g - \Delta_f)\varphi(g)}{\Delta_f + \varepsilon} d\gamma + \int \varepsilon(\varphi(f) - \varphi(g))(\Delta_f + \varepsilon)^{-1} \, d\gamma.$$

Let us estimate each term separately. Let

$$\beta = (2k(d-1))^{-1}.$$

Recall (see (4.4)) that

$$\int (\Delta_f + \varepsilon)^{-p}\, d\gamma \le c(p,d)^p \varepsilon^{-p+\beta}\left(\int \Delta_f\, d\gamma\right)^{-\beta}.$$

Using this inequality, we estimate the last term in the right-hand side of (6.3):

$$\int \varepsilon(\varphi(f) - \varphi(g))(\Delta_f + \varepsilon)^{-1} d\gamma \le 2c(1,d)\varepsilon^\beta \left(\int \Delta_f\, d\gamma\right)^{-\beta}.$$

The second and the third terms in (6.3) can be estimated as follows. By Lemma 6.2 we have

$$|\Delta_g - \Delta_f| \le \left(\|M_f\|_{HS}^2 + \|M_g\|_{HS}^2\right)^{(k-1)/2}\|M_f - M_g\|_{HS}$$

$$\le \sqrt{2}\Big(\sum_{i=1}^k \big(|\nabla f_i|^2 + |\nabla g_i|^2\big)\Big)^{k-1/2}\Big(\sum_{i=1}^k |\nabla f_i - \nabla g_i|^2\Big)^{1/2},$$

where we used the estimates $\|M_f\|_{HS}^2 = \sum_{i,j}\langle \nabla f_i, \nabla f_j\rangle^2 \le \left(\sum_i |\nabla f_i|^2\right)^2$ and

$$\|M_f - M_g\|_{HS}^2 = \sum_{i,j}\big(\langle \nabla f_i, \nabla f_j\rangle - \langle \nabla g_i, \nabla g_j\rangle\big)^2$$

$$\le 2\sum_{i,j}\Big[\big(\langle \nabla f_i, \nabla f_j\rangle - \langle \nabla f_i, \nabla g_j\rangle\big)^2 + \big(\langle \nabla f_i, \nabla g_j\rangle - \langle \nabla g_i, \nabla g_j\rangle\big)^2\Big]$$

$$\le 2\sum_{i,j}\big[|\nabla f_i|^2|\nabla f_j - \nabla g_j|^2 + |\nabla f_i - \nabla g_i|^2|\nabla g_j|^2\big]$$

$$= 2\sum_i |\nabla f_i - \nabla g_i|^2 \sum_i \big(|\nabla f_i|^2 + |\nabla g_i|^2\big).$$

Similarly,

$$|\Delta_{f^{i-1}} - \Delta_{f^i}| \le 2^k\Big(\sum_{i=1}^k (|\nabla f_i|^2 + |\nabla g_i|^2)\Big)^{k-1/2}\Big(\sum_{i=1}^k |\nabla f_i - \nabla g_i|^2\Big)^{1/2}.$$

Using these estimates we obtain

$$\int \frac{(\Delta_{f^{i-1}} - \Delta_{f^i})\varphi(f^i)}{\Delta_f + \varepsilon} d\gamma \le \int \frac{|\Delta_{f^{i-1}} - \Delta_{f^i}|}{\Delta_f + \varepsilon}\, d\gamma$$

$$\le 2^k \int \Big(\sum_{i=1}^k (|\nabla f_i|^2 + |\nabla g_i|^2)\Big)^{k-1/2}\Big(\sum_{i=1}^k |\nabla f_i - \nabla g_i|^2\Big)^{1/2}(\Delta_f + \varepsilon)^{-1}\, d\gamma$$

$$\le C(k,d)\Big(\sum_{i=1}^k (\sigma_{f_i}^2 + \sigma_{g_i}^2)\Big)^{k-1/2}\|f - g\|_2 \varepsilon^{-1}.$$

Similarly,

$$\int \frac{(\Delta_g - \Delta_f)\varphi(g)}{\Delta_f + \varepsilon}\, d\gamma \le C(k,d)\Big(\sum_{i=1}^k (\sigma_{f_i}^2 + \sigma_{g_i}^2)\Big)^{k-1/2}\|f - g\|_2 \varepsilon^{-1}.$$

Let us now consider the first term in the right-hand side of (6.3). By (6.2) we have

$$(6.4) \quad \int \frac{\Delta_{f^{i-1}}(\varphi(f^{i-1}) - \varphi(f^i))}{\Delta_f + \varepsilon} \, d\gamma$$

$$= \int (\Delta_f + \varepsilon)^{-1} \sum_{j=1}^{k} \langle \nabla \Phi_i(f^{i-1}) - \nabla \Phi_i(f^i), \nabla f_j^{i-1} \rangle a_{f^{i-1}}^{j,i} \, d\gamma$$

$$- \int \varphi(f^i)(\Delta_f + \varepsilon)^{-1} \sum_{j=1}^{k} \langle \nabla f_i - \nabla g_i, \nabla f_j^{i-1} \rangle a_{f^{i-1}}^{j,i} \, d\gamma.$$

The second term in (6.4) can be estimated in the following way:

$$\int \varphi(f^i)(\Delta_f + \varepsilon)^{-1} \langle \nabla f_i - \nabla g_i, \nabla f_j^{i-1} \rangle a_{f^{i-1}}^{j,i} \, d\gamma$$

$$\leq \int (\Delta_f + \varepsilon)^{-1} |\nabla f_i - \nabla g_i| \, |\nabla f_j^{i-1}| \, |a_{f^{i-1}}^{j,i}| \, d\gamma$$

$$\leq \varepsilon^{-1}(k-1)^2! \int \Big(\sum_{i=1}^{k} (|\nabla f_i|^2 + |\nabla g_i|^2) \Big)^{k-1/2} \Big(\sum_{i=1}^{k} |\nabla f_i - \nabla g_i|^2 \Big)^{1/2} \, d\gamma$$

$$\leq C(k,d)\varepsilon^{-1} \Big(\sum_{i=1}^{k} (\sigma_{f_i}^2 + \sigma_{g_i}^2) \Big)^{k-1/2} \|f - g\|_2.$$

Finally, let us consider the first term in (6.4). Fix $p > 1$. Integrating by parts we have

$$\int (\Delta_f + \varepsilon)^{-1} \langle \nabla \Phi_i(f^{i-1}) - \nabla \Phi_i(f^i), \nabla f_j^{i-1} \rangle a_{f^{i-1}}^{j,i} \, d\gamma = - \int (\Phi_i(f^{i-1}) - \Phi_i(f^i))$$

$$\times \Big(\frac{a_{f^{i-1}}^{j,i} L f_j^{i-1}}{\Delta_f + \varepsilon} - \frac{a_{f^{i-1}}^{j,i} \langle \nabla f_j^{i-1}, \nabla \Delta_f \rangle}{(\Delta_f + \varepsilon)^2} + \frac{\langle \nabla f_j^{i-1}, \nabla a_{f^{i-1}}^{j,i} \rangle}{\Delta_f + \varepsilon} \Big) \, d\gamma$$

$$\leq \int |f_i - g_i| \Big(\frac{|a_{f^{i-1}}^{j,i} L f_j^{i-1}|}{\Delta_f + \varepsilon} + \frac{|a_{f^{i-1}}^{j,i} \langle \nabla f_j^{i-1}, \nabla \Delta_f \rangle|}{(\Delta_f + \varepsilon)^2} + \frac{|\langle \nabla f_j^{i-1}, \nabla a_{f^{i-1}}^{j,i} \rangle|}{\Delta_f + \varepsilon} \Big) \, d\gamma,$$

which is estimated by

$$\varepsilon^{-1} \|f - g\|_2 \Big(\|a_{f^{i-1}}^{j,i} L f_j^{i-1}\|_2 + \|\langle \nabla f_j^{i-1}, \nabla a_{f^{i-1}}^{j,i} \rangle\|_2 \Big)$$

$$+ C(p,k,d)\|f - g\|_2 \|a_{f^{i-1}}^{j,i} \langle \nabla f_j^{i-1}, \nabla \Delta_f \rangle\|_2 \|(\Delta_f + \varepsilon)^{-2}\|_p$$

$$\leq C(k,d)\varepsilon^{-1} \Big(\sum_{i=1}^{k} (\sigma_{f_i}^2 + \sigma_{g_i}^2) \Big)^{k-1/2} \|f - g\|_2$$

$$+ C_1(p,k,d)\|f - g\|_2 \Big(\sum_{i=1}^{k} (\sigma_{f_i}^2 + \sigma_{g_i}^2) \Big)^{2k-1/2} \varepsilon^{-2+\beta/p} \Big(\int \Delta_f \, d\gamma \Big)^{-\beta/p}.$$

Now the left-hand side of (6.3) can be estimated by

$$C_2(p,k,d)\left(\varepsilon^\beta\left(\int \Delta_f\,d\gamma\right)^{-\beta} + \left(\sum_{i=1}^k(\sigma_{f_i}^2 + \sigma_{g_i}^2)\right)^{k-1/2}\|f-g\|_2\varepsilon^{-1}\right.$$
$$\left. + \|f-g\|_2\left(\sum_{i=1}^k(\sigma_{f_i}^2 + \sigma_{g_i}^2)\right)^{2k-1/2}\varepsilon^{-2+\beta/p}\left(\int \Delta_f\,d\gamma\right)^{-\beta/p}\right).$$

If $\|f-g\|_2 \geq 1$, the desired estimate (6.1) is trivial. Assume that $\|f-g\|_2 \leq 1$. Whenever $\varepsilon \leq 1$ we have $\varepsilon^{-1} \leq \varepsilon^{-2+\beta/p}$. Let $\tau = (p-1)/p$. Setting $\varepsilon = \|f-g\|^\alpha$ with $\alpha = (2+\beta\tau)^{-1}$, we have

$$\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{\mathrm{TV}} \leq C_2(p,k,d)R(f,g)\|f-g\|^\theta,\ \theta = \frac{1}{4k(d-1)+\tau},$$

where

$$R(f,g) = \left(\int \Delta_f\,d\gamma\right)^{-\beta} + \left(\sum_{i=1}^k(\sigma_{f_i}^2 + \sigma_{g_i}^2)\right)^{k-1/2}$$
$$+ \left(\sum_{i=1}^k(\sigma_{f_i}^2 + \sigma_{g_i}^2)\right)^{2k-1/2}\left(\int \Delta_f\,d\gamma\right)^{-\beta/p}.$$

Since $|\sigma_{f_i} - \sigma_{g_i}| \leq 2\|f-g\|_2 \leq 2$, the desired estimate is proved. $\square$

Remark 6.4. Theorem 4.2 yields an analog of estimate (6.1) with the power of the L^2-norm equal to $1/(4k(d-1)+1+\tau)$. Hence Theorem 6.3 provides a better rate of convergence.

Finally, we draw the reader's attention to the recent surveys [11] and [28] on distributions of polynomials.

Acknowledgments

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