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MAJORIZATION IN C*-ALGEBRAS

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Abstract. We investigate the closed convex hull of unitary orbits of selfadjoint elements in arbitrary unital C*-algebras. Using a notion of majorization against unbounded traces, a characterization of these closed convex hulls is obtained. Furthermore, for C*-algebras satisfying Blackadar's strict comparison of positive elements by traces or for collections of C*-algebras with a uniform bound on their nuclear dimension, an upper bound for the number of unitary conjugates in a convex combination required to approximate an element in the closed convex hull within a given error is shown to exist. This property, however, fails for certain "badly behaved" simple nuclear C*-algebras.

1. Introduction

The relation of majorization between selfadjoint matrices is an important and well-studied relation (see [And94] and the references therein). It is thus natural to pursue its study in the more general realm of operator algebras. This has been done for von Neumann algebra factors ([Kam83, HN91]) and for various classes of simple C*-algebras ([Sko16, NS16]). A basic result on matrix majorization due to Uhlmann gives two equivalent ways of defining the majorization relation: Given selfadjoint matrices a and b, the following conditions on a and b are equivalent:

(1) a belongs to the convex hull of the unitary conjugates of b,
(2) $\mathrm{Tr}(a) = \mathrm{Tr}(b)$ and $\mathrm{Tr}((a-t)_+) \leqslant \mathrm{Tr}((b-t)_+)$ for all $t \in \mathbb{R}$. Here $(a-t)_+$ is the element obtained from a by functional calculus with the function $x \mapsto (x-t)_+ := \max(x-t, 0)$ and Tr is the trace.

When either of these conditions holds a is said to be majorized by b. We show in this paper that the equivalence above has a natural generalization to arbitrary C*-algebras. In order to formulate a suitable version of (2) we must now look at possibly unbounded traces. Let A be a C*-algebra. We call a map $\tau\colon A_+ \to [0,\infty]$ a trace if it is linear (additive, $\mathbb{R}_+$-homogeneous, and maps 0 to 0) and satisfies that $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$. We will always assume that traces are lower semicontinuous, i.e., such that $\tau(a) \leqslant \liminf_n \tau(a_n)$ if $a_n \to a$. We do not assume, however, that traces are densely finite. We denote the cone of all lower semicontinuous traces by $\mathrm{T}(A)$. We prove below the following theorem.

Theorem 1.1. *Let A be a unital C*-algebra. Let $a, b \in A$ be selfadjoint elements. The following are equivalent:*

(i) $a \in \overline{\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}}$,
(ii) $\tau((a-t)_+) \leqslant \tau((b-t)_+)$ *and* $\tau((-a-t)_+) \leqslant \tau((-b-t)_+)$ *for all* $\tau \in \mathrm{T}(A)$ *and all* $t \in \mathbb{R}$.

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In this theorem $\mathrm{co}(\cdot)$ denotes the convex hull of a set and $\mathrm{U}(A)$ the unitary group of A. If (i) holds we say that a is majorized by b. If A is a simple C*-algebra with at least one bounded trace, then condition (ii) of Theorem 1.1 takes the following form, which is closer to the matrix case: $\tau((a-t)_+) \leqslant \tau((b-t)_+)$ and $\tau(a) = \tau(b)$ for all bounded traces τ and all $t \in \mathbb{R}$ (Corollary 4.6(i)). However, since we allow for traces that are not densely finite, Theorem 1.1 covers the simple purely infinite C*-algebras as well (Corollary 4.6(ii)); indeed, it covers all C*-algebras. A related theorem, also valid for all C*-algebras, is [Rob09, Theorem 1.1], which shows that the agreement of two positive elements on all traces in $\mathrm{T}(A)$ is equivalent to the Cuntz–Pedersen relation.

A few words on the proof of Theorem 1.1: We use a well-known Hahn–Banach argument going back to Day ([Day57]) to reduce the proof to the von Neumann algebra A^{**}. In the von Neumann algebra setting, we deal first with finite von Neumann algebras using arguments inspired by the II_1 factor case and then extend the proof to the general case. In the process we obtain a formula for the distance from a to $\overline{\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}}$ in terms of tracial inequalities (the zero distance case of this formula is Theorem 1.1).

In the context of majorization of matrices one encounters the following phenomenon: For any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $a, b \in M_n(\mathbb{C})$ are selfadjoint matrices of norm at most 1 and a is majorized by b, then there exists a convex combination of at most N unitary conjugates of b which is within a distance of ε from a. Here the number N does not depend on a or b, as long as they are contractions, or on the matrix size n (see [Sko16, Theorem 6.1] for an explicit formula). We refer to this property as uniform majorization. (In the language of continuous logic of C*-algebras, the fact that N depends solely on ε implies that the relation of majorization is uniformly definable within the class of matrix C*-algebras; see [FHL$^+$16].) Uniform majorization does not hold for general C*-algebras and may fail even in a single C*-algebra. We show below that the C*-algebra constructed in [Rob15, Theorem 1.4] does not have uniform majorization (Example 5.7). This C*-algebra, which is simple and nuclear, fails to have various regularity properties of great significance in the classification of simple nuclear C*-algebras; to wit, it has neither strict comparison of positive elements by traces nor finite nuclear dimension. We prove below that these very same regularity properties serve to ensure uniform majorization.

Theorem 1.2. *For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if A is a unital C*-algebra with strict comparison of positive elements by traces and $a, b \in A$ are selfadjoint contractions such that $a \in \overline{\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}}$, then*

$$\Big\| a - \frac{1}{N}\sum_{i=1}^{N} u_i b u_i^* \Big\| < \varepsilon$$

for some $u_1, \ldots, u_N \in \mathrm{U}(A)$.

A version of Theorem 1.2 for C*-algebras with finite nuclear dimension is also valid (Theorem 5.6). We obtain the following interesting application of uniform majorization: Let A be a unital C*-algebra with either strict comparison by traces or finite nuclear dimension. Let $B \subseteq A_\infty$ be a separable C*-subalgebra of the sequence algebra $A_\infty := \prod_{i=1}^\infty A / \bigoplus_{i=1}^\infty A$. Then for every selfadjoint $a \in A_\infty$ the set $\overline{\mathrm{co}(\{uau^* \mid u \in \mathrm{U}(A_\infty)\})}$ has nonempty intersection with $B' \cap A_\infty$.

This paper is organized as follows: In Section 2 we define the majorization and submajorization relations and prove some of their general properties which will be needed later on. In Section 3 we prove Theorem 1.1 when A is a von Neumann algebra (at this point we assume that a and b are positive contractions as a matter of convenience). In Section 4 we prove Theorem 1.1 together with a more general distance formula and we derive some corollaries of these theorems. In Section 5 we investigate the property of uniform majorization described above. The proof of Theorem 1.2, unlike the more hands-on methods used in [Sko16], does not yield and explicit formula for the number N in terms of ε.

2. Preliminaries on majorization and submajorization

Let A be a C*-algebra. Let us denote by A_+ and A_{sa} the sets of positive and selfadjoint elements of A, respectively. If A is unital, we let $\mathrm{U}(A)$ denote the unitary group of A. If $a \in A_{\mathrm{sa}}$ and $t \in \mathbb{R}$ we denote by $(a-t)_+$ the element obtained from a by functional calculus with the function $x \mapsto (x-t)_+ := \max(x-t, 0)$.

Given $a, b \in A_{\mathrm{sa}}$ let us say that a is submajorized by b, and denote it by $a \prec_{\mathrm{c}} b$, if

$$a \in \overline{\mathrm{co}(\{dbd^* \mid \|d\| \leqslant 1\})}.$$

Suppose that A is unital. Let us say that a is majorized by b, and denote it by $a \prec_{\mathrm{u}} b$, if

$$a \in \overline{\mathrm{co}(\{uau^* \mid u \in \mathrm{U}(A)\})}.$$

It is possible to extend the relation of majorization to nonunital C*-algebras simply by passing to the unitization. However, we will always assume that A is unital when discussing majorization. Both submajorization and majorization are preorder relations.

We use the following lemma quite frequently and without reference.

Lemma 2.1. *Let $a_1, a_2, b_1, b_2 \in A_{\mathrm{sa}}$ be such that $a_1 \prec_{\mathrm{c}} b_1$, $a_2 \prec_{\mathrm{c}} b_2$, $a_1 a_2 = 0$ and $b_1 b_2 = 0$. Then $a_1 + a_2 \prec_{\mathrm{c}} b_1 + b_2$.*

Proof. Let $\varepsilon > 0$. Suppose that

$$\|a_1 - \frac{1}{N}\sum_{i=1}^{N} d_{i,1} b_1 d_{i,1}^*\| < \varepsilon \text{ and } \|a_2 - \frac{1}{N}\sum_{i=1}^{N} d_{i,2} b_2 d_{i,2}^*\| < \varepsilon,$$

for some contractions $d_{i,1}, d_{i,2} \in A$. Multiplying by an approximate unit of $\overline{|a_1|A|a_1|}$ on the left and on the right of the first equation and replacing b_1 by $|b_1|^{\frac{1}{n}} b_1 |b_1|^{\frac{1}{n}}$ for large enough n we can assume that $d_{i,1} \in \overline{|a_1|A|b_1|}$ for all i. Similarly, we can assume that $d_{i,2} \in \overline{|a_2|A|b_2|}$ for all i. Define $d_i = d_{i,1} + d_{i,2}$ for all i. A straightforward calculation exploiting that $a_1 a_2 = b_1 b_2 = 0$ shows that

$$\|(a_1 + a_2) - \frac{1}{N}\sum_{i=1}^{N} d_i (b_1 + b_2) d_i^*\| < 2\varepsilon.$$

This proves the lemma. □

Lemma 2.2. *Let $a, b \in A_{\mathrm{sa}}$.*

(i) *If $a \leqslant b$, then $a_+ \prec_{\mathrm{c}} b_+$.*
(ii) *If $\|a - b\| \leqslant r$, then $(a-r)_+ \prec_{\mathrm{c}} b_+$.*
(iii) *If $a \prec_{\mathrm{c}} b$, then $(a-t)_+ \prec_{\mathrm{c}} (b-t)_+$ for all $t \in [0, \infty)$.*

Proof.

(i) Assume first that $a \geqslant 0$ (so $b \geqslant 0$). Since $a \leqslant b$, a is in the hereditary C*-subalgebra generated by b. Hence, $b^{\frac{1}{n}} a b^{\frac{1}{n}} \to a$, which shows that $a \prec_{\mathrm{c}} b$, as desired. Suppose now that $a \in A_{\mathrm{sa}}$. Let $\varepsilon > 0$. Let $c \in C^*(a)$ be a positive contraction such that $ca = (a-\varepsilon)_+$. Multiplying by $c^{1/2}$ on the left and on the right of $a \leqslant b$ we get

$$(a-\varepsilon)_+ \leqslant c^{\frac{1}{2}} b c^{\frac{1}{2}} \leqslant c^{\frac{1}{2}} b_+ c^{\frac{1}{2}} \prec_{\mathrm{c}} b_+.$$

Since submajorization is transitive and we have already shown that the order on positive elements is stronger than the submajorization relation, $(a-\varepsilon)_+ \prec_{\mathrm{c}} b_+$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$ we are done.

(ii) We have that $a - r \leqslant b$. So we can apply (i) to get that $(a-r)_+ \prec_{\mathrm{c}} b_+$.

(iii) Choose $b' = \frac{1}{N}\sum_{i=1}^N d_i b d_i^*$, with $\|d_i\| \leqslant 1$ for all i, such that $\|a - b'\| < \varepsilon$. From $a - t - \varepsilon \leqslant b' - t$ we get, by (i), that $(a-t-\varepsilon)_+ \prec_{\mathrm{c}} (b'-t)_+$. Also,

$$b' - t \leqslant \frac{1}{N}\sum_{i=1}^N d_i(b-t)d_i^* \leqslant \frac{1}{N}\sum_{i=1}^N d_i(b-t)_+ d_i^*.$$

Hence, by (i), $(b'-t)_+$ is submajorized by $\frac{1}{N}\sum_{i=1}^N d_i(b-t)_+ d_i^*$, which in turn is submajorized by $(b-t)_+$. By the transitivity of submajorization, $(a-t-\varepsilon)_+ \prec_{\mathrm{c}} (b-t)_+$ for all $\varepsilon > 0$, from which the desired result follows. □

Proposition 2.3. *Let $a, b \in A_{\mathrm{sa}}$. Then $a \prec_{\mathrm{c}} b$ if and only if $a_+ \prec_{\mathrm{c}} b_+$ and $a_- \prec_{\mathrm{c}} b_-$.*

Proof. Suppose first that $a \prec_{\mathrm{c}} b$. Let $a_n \in A_{\mathrm{sa}}$ be elements such that $a_n \to a$ and each a_n is a finite convex combination of elements of the form dbd^*. Since $(a_n)_+ \to a_+$ it suffices to show that $(a_n)_+ \prec_{\mathrm{c}} b_+$ for all n. Put differently, it suffices to assume that $a = \frac{1}{N}\sum_{i=1}^N d_i b d_i^*$ for some $\|d_i\| \leqslant 1$. In this case we have that

$$a \leqslant \frac{1}{N}\sum_{i=1}^N d_i b_+ d_i^*.$$

By Lemma 2.2,

$$a_+ \prec_{\mathrm{c}} \Big(\frac{1}{N}\sum_{i=1}^N d_i b_+ d_i^*\Big)_+ = \frac{1}{N}\sum_{i=1}^N d_i b_+ d_i^*.$$

The rightmost side is clearly submajorized by b_+. Thus, $a_+ \prec_{\mathrm{c}} b_+$. Since $-a \prec_{\mathrm{c}} -b$ we also have that $a_- = (-a)_+ \prec_{\mathrm{c}} (b_+) = b_-$. This proves one implication.

Suppose now that $a_+ \prec_{\mathrm{c}} b_+$ and $a_- \prec_{\mathrm{c}} b_-$. By Lemma 2.1 we have that $a_+ - a_- \prec_{\mathrm{c}} b_+ - b_-$, i.e., $a \prec_{\mathrm{c}} b$, as desired. □

In light of the previous proposition we will largely focus on the study of the submajorization relation among positive elements. It will be easy enough to extend our main results to selfadjoint elements relying on this proposition.

We call trace on A a map $\tau\colon A_+ \to [0,\infty]$ that is $\mathbb{R}^+$-linear, maps 0 to 0, and satisfies that $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$. Notice that ∞ is in the range of τ and that we do not assume that τ is densely finite. We denote by $\mathrm{T}(A)$ the cone of all lower semicontinuous traces on A. The reader is referred to [ERS11] for basic facts on $\mathrm{T}(A)$. Observe that for each closed two-sided ideal $I \subseteq A$ the map $\tau_I\colon A_+ \to [0,\infty]$ defined as $\tau_I(a) = 0$ if $a \in I_+$ and $\tau_I(a) = \infty$ otherwise is a lower semicontinuous trace. In particular, if we choose $I = \{0\}$ we get a trace that is ∞ everywhere except at 0.

Let $a, b \in A_+$. We say that a is tracially submajorized by b if

$$\tau((a-t)_+) \leqslant \tau((b-t)_+) \text{ for all } \tau \in \mathrm{T}(A) \text{ and all } t \in [0, \infty).$$

We denote this relation by $a \prec_{\mathrm{T}} b$.

The following proposition clarifies the meaning of tracial submajorization in C*-algebras with "very few" traces.

Proposition 2.4. *Suppose that the C*-algebra A has no l.s.c. traces other than the traces τ_I associated to its closed two-sided ideals (e.g., A is purely infinite). Let $a, b \in A_+$. Then $a \prec_{\mathrm{T}} b$ if and only if $\|\pi_I(a)\| \leqslant \|\pi_I(b)\|$ for all quotient maps $\pi_I : A \to A/I$.*

Proof. Let I be a closed two-sided ideal of A. Denote by $\pi_I : A \to A/I$ the quotient map. Let $t \in [0, \infty)$. Consider the inequality $\tau_I((a-t)_+) \leqslant \tau_I((b-t)_+)$. The right side is ∞ for all $t < \|\pi_I(b)\|$. So in this case the inequality is trivially valid. On the other hand, if $t \geqslant \|\pi_I(b)\|$, then the inequality is valid if and only if the left side is 0, i.e., if $(a - \|\pi(b)\|)_+ \in I$. This is equivalent to $\|\pi_I(a)\| \leqslant \|\pi_I(b)\|$, as desired. ☐

We will show below that in any C*-algebra tracial submajorization is equivalent to submajorization (for positive elements), but this will entail first elucidating independently some of the properties of both relations.

Lemma 2.5. *Let $B \subseteq A$ be a hereditary C*-subalgebra. Let $a, b \in B_+$.*

(i) *If $a \prec_{\mathrm{c}} b$ in A, then $a \prec_{\mathrm{c}} b$ in B.*

(ii) *If $a \prec_{\mathrm{T}} b$ in A, then $a \prec_{\mathrm{T}} b$ in B.*

Proof.

(i) Let $(e_\lambda)_\lambda$ be an approximate unit of B consisting of contractions. Let $\varepsilon > 0$. Say $d_1, \ldots, d_N \in A$ are contractions such that

$$\Big\| a - \frac{1}{N} \sum_{i=1}^{N} d_i b d_i^* \Big\| < \varepsilon.$$

Call the left side of the above inequality ε' and choose $\varepsilon' < \varepsilon'' < \varepsilon$. We have

$$\begin{aligned} \Big\| a - \frac{1}{N} \sum_{i=1}^{N} e_\lambda d_i b d_i^* e_\lambda \Big\| &\leqslant \|a - e_\lambda a e_\lambda\| + \Big\| e_\lambda \Big(a - \frac{1}{N} \sum_{i=1}^{N} d_i b d_i^* \Big) e_\lambda \Big\| \\ &\leqslant \|a - e_\lambda a e_\lambda\| + \varepsilon'. \end{aligned}$$

Since $e_\lambda a e_\lambda \to a$, there exists λ_0 such that the left side is less than ε'' for all $\lambda \geqslant \lambda_0$. Moreover, since $e_\lambda b e_\lambda \to b$, there exists λ_1 such that

$$\|a - \frac{1}{N} \sum_{i=1}^{N} e_\lambda d_i (e_\lambda b e_\lambda) d_i^* e_\lambda\| < \varepsilon$$

for all $\lambda \geqslant \lambda_1$. Notice that $e_\lambda d_i e_\lambda \in B$ for all i. Thus, $a \prec_{\mathrm{c}} b$ in B.

(ii) It suffices to show that every l.s.c. trace on B extends to A. Let us sketch the proof of this known fact: Given positive elements $e, f \in A_+$, let us write $e \precsim_{\mathrm{CP}} f$ if $e = \sum_{i=1}^{\infty} x_i^* x_i$ and $\sum_{i=1}^{\infty} x_i x_i^* \leqslant f$ for some $x_i \in A$, where the series are convergent in norm. This transitive relation is studied in [CP79] and [Rob09]. To define an extension of a trace τ on B to A we set

$$\tilde{\tau}(x) = \sup\{\tau(y) \mid y \in B_+,\, y \precsim_{\mathrm{CP}} x\},$$

for all $x \in A_+$. Then $\tilde{\tau}$ is an l.s.c. trace on A extending τ. The proof of this claim may be found in the proof of [CP79, Lemma 4.6]. □

Let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable, infinite-dimensional, Hilbert space. We regard A embedded in $A \otimes \mathcal{K}$ in the usual manner, i.e., by placing the elements of A in the upper-left corner of an infinite matrix whose entries are 0 everywhere else.

Proposition 2.6. *Let $a, b \in A_+$. Then $a \prec_{\mathrm{c}} b$ in A if and only if $a \prec_{\mathrm{u}} b$ in $(A \otimes \mathcal{K})^\sim$ (i.e., in the unitization of the stabilization of A).*

Proof. Suppose that $a \prec_{\mathrm{u}} b$ in $(A \otimes \mathcal{K})^\sim$. Since A is a hereditary C*-subalgebra of $(A \otimes \mathcal{K})^\sim$, we have $a \prec_{\mathrm{c}} b$ in A by Lemma 2.5(i).

Let us prove the opposite implication. We consider first the case when $a \leqslant b$. Let $n \in \mathbb{N}$. We have

$$b = \begin{pmatrix} a^{1/2} & \cdots & (b-a)^{1/2} \end{pmatrix} \begin{pmatrix} a^{1/2} \\ \vdots \\ (b-a)^{1/2} \end{pmatrix}$$

and

$$\begin{pmatrix} a^{1/2} \\ \vdots \\ (b-a)^{1/2} \end{pmatrix} \begin{pmatrix} a^{1/2} & \cdots & (b-a)^{1/2} \end{pmatrix} = \begin{pmatrix} a & \cdots & a^{1/2}(b-a)^{1/2} \\ \vdots & & \vdots \\ (b-a)^{1/2}a^{1/2} & \cdots & b-a \end{pmatrix} \in M_n(A),$$

where the omitted entries are all zeros. By changing n and averaging we find that for any $\varepsilon > 0$ we can choose $x_1, \dots, x_N \in A \otimes \mathcal{K}$ such that

$$\|a - \frac{1}{N}\sum_{i=1}^{N} x_i x_i^*\| < \varepsilon \text{ and } b = x_i^* x_i \text{ for all } i.$$

But for all $x \in A \otimes \mathcal{K}$ the elements x^*x and xx^* are approximately unitarily equivalent in $(A \otimes \mathcal{K})^\sim$ ([BRT+12, Lemma 4.3.3]). This shows that $a \prec_{\mathrm{u}} b$ in $(A \otimes \mathcal{K})^\sim$, as desired.

Suppose now that $a = dbd^*$, with $\|d\| \leqslant 1$. Let $x = db^{1/2}$. Then $a = xx^*$ and $x^*x \leqslant b$. We have already shown that $x^*x \prec_{\mathrm{u}} b$ in $(A \otimes \mathcal{K})^\sim$. But, as remarked above, x^*x and xx^* are approximately unitarily equivalent in $(A \otimes \mathcal{K})^\sim$. So $a = x^*x \prec_{\mathrm{u}} xx^* \prec_{\mathrm{u}} b$.

Consider the general case. Suppose that $a \prec_{\mathrm{c}} b$. Then a is a limit of convex combinations of elements of the form dbd^*, with $\|d\| \leqslant 1$. We have already shown that each of these elements is majorized by b in $(A \otimes \mathcal{K})^\sim$. It follows that $a \prec_{\mathrm{u}} b$ in $(A \otimes \mathcal{K})^\sim$, as desired. □

Proposition 2.7. *Let $E\colon A \to A$ be a positive contractive map that is also trace decreasing, i.e., $\tau(E(a)) \leqslant \tau(a)$ for all $\tau \in \mathrm{T}(A)$ and all $a \in A_+$. Then $E(a) \prec_{\mathrm{T}} a$ for all $a \in A_+$.*

Proof. Let $t \in [0, \infty)$ and let $\tau \in \mathrm{T}(A)$. Let $\varepsilon > 0$. Since E is positive and contractive we have that

$$E(a) - t \cdot 1 \leqslant E(a - t \cdot 1) \leqslant E((a-t)_+).$$

Let $c \in C^*(E(a))$ be a positive contraction such that $(E(a)-t)c = (E(a)-t-\varepsilon)_+$. Then

$$(E(a) - t - \varepsilon)_+ \leqslant c^{\frac{1}{2}} E((a-t)_+) c^{\frac{1}{2}}.$$

Evaluating both sides on τ and using the fact that E is trace decreasing we get that

$$\tau((E(a) - t - \varepsilon)_+) \leqslant \tau(E((a-t)_+)) \leqslant \tau((a-t)_+).$$

Letting $\varepsilon \to 0$ and using that τ is lower semicontinuous we get the desired inequality. □

Proposition 2.8. *Let $a, b \in A_+$. If $a \prec_{\mathrm{c}} b$, then $a \prec_{\mathrm{T}} b$.*

Proof. First suppose that a is exactly a convex combination of elements of the form dbd^*, with $\|d\| \leqslant 1$. Say $a = \sum_{i=1}^n t_i d_i b d_i^*$, where $\|d_i\| \leqslant 1$ for all i, $0 \leqslant t_i \leqslant 1$ for all i, and $\sum_{i=1}^n t_i = 1$. Let $E\colon A \to A$ be defined as $E(y) = \sum_{i=1}^n t_i d_i y d_i^*$ for all $y \in A$. Then E is positive, contractive, and trace decreasing. By Proposition 2.7, $a = E(b) \prec_{\mathrm{T}} b$, as desired.

Suppose now that $a, b \in A_+$ are arbitrary elements such that $a \prec_{\mathrm{c}} b$. Let $a_n \to a$, where a_n is a finite convex combination of elements of the form dbd^*, with $\|d\| \leqslant 1$. Then $a_n \prec_{\mathrm{T}} b$ for all n by the previous case. Let $\tau \in \mathrm{T}(A)$ and $t \in [0, \infty)$. Then $\tau((a_n - t)_+) \leqslant \tau((b-t)_+)$ for all n and $(a_n - t)_+ \to (a-t)_+$. By the lower semicontinuity τ,

$$\tau((a-t)_+) \leqslant \liminf_n \tau((a_n - t)_+) \leqslant \tau((b-t)_+),$$

as desired. (What we have shown is that the set of elements tracially submajorized by b is closed.) □

3. von Neumann algebra case

In this section we work exclusively in the setting of von Neumann algebras. The main results of this section, Propositions 3.13 and 3.15, characterize submajorization and majorization in a von Neumann algebra in terms of tracial submajorization. They are stepping stones towards proving the same results for all C*-algebras. (We take up this task in the next section.)

Throughout this section M denotes a von Neumann algebra. We also fix the following notation: The center of M is denoted by Z. Elements of Z are often regarded as continuous functions on $\widehat{Z}$ (the spectrum of Z). Given $a \in M$ we denote by $c_a \in Z$ the central carrier or central support projection of a.

Lemma 3.1. *Let $a, b \in M_+$ be such that $a \prec_{\mathrm{T}} b$. Let $\lambda \in Z_+$. The following are true:*

(i) $\lambda a \prec_{\mathrm{T}} \lambda b$.
(ii) $(a - \lambda)_+ \prec_{\mathrm{T}} (b - \lambda)_+$.
(iii) $a + \lambda \prec_{\mathrm{T}} b + \lambda$.

Proof.

(i) Let $\tau \in \mathrm{T}(M)$ and $t \in [0,\infty)$. We must show that $\tau((\lambda a-t)_+) \leqslant \tau((\lambda b-t)_+)$. Suppose first that $\lambda = e$ is a central projection. Then $\tau((ea-t)_+) = \tau(e(a-t)_+)$, and since $x \mapsto \tau(e\cdot x)$ is a trace in $\mathrm{T}(M)$, $\tau(e(a-t)_+) \leqslant \tau(e(b-t)_+)$. So $\tau((ea-t)_+) \leqslant \tau((eb-t)_+)$, as desired.

Suppose now that λ has finite spectrum. Then $\lambda = \sum_{i=1}^n \alpha_i e_i$, where $e_1,\dots,e_n$ are pairwise orthogonal central projections and where $\alpha_1,\dots,\alpha_n > 0$ are positive scalars. We have

$$\begin{aligned}\tau((\lambda a-t)_+) &= \tau\Big(\Big(\sum_{i=1}^n \alpha_i e_i a - t\Big)_+\Big)\\ &= \tau\Big(\sum_{i=1}^n \alpha_i\Big(e_i a - \frac{t}{\alpha_i}\Big)_+\Big) \leqslant \tau\Big(\sum_{i=1}^n \alpha_i\Big(e_i b - \frac{t}{\alpha_i}\Big)_+\Big) = \tau((\lambda b-t)_+).\end{aligned}$$

Finally, suppose that λ is an arbitrary positive central element. Making use of the Borel functional calculus on λ, construct an increasing sequence of positive central elements $(\lambda_n)_n$ each with finite spectrum and such that $\lambda_n \nearrow \lambda$ in norm. We have already proven that $\tau((\lambda_n a-t)_+) \leqslant \tau((\lambda_n b-t)_+)$ for all n. Observe that $(\lambda_n a-t)_+ \nearrow (\lambda a-t)_+$ and $(\lambda_n b-t)_+ \nearrow (\lambda b-t)_+$. So passing to the limit as $n\to\infty$ and using the fact that τ is l.s.c. we get that $\tau((\lambda a-t)_+) \leqslant \tau((\lambda b-t)_+)$, as desired.

(ii) It suffices to show that $\tau((a-\lambda)_+) \leqslant \tau((b-\lambda)_+)$ for all $\tau \in \mathrm{T}(M)$. Making use of the Borel functional calculus on λ, construct a decreasing sequence of positive central elements $(\lambda_n)_n$ with finite spectrum and such that $\lambda_n \searrow \lambda$ in norm. Then $(a-\lambda_n)_+ \nearrow (a-\lambda)_+$ and $(b-\lambda_n)_+ \nearrow (b-\lambda)_+$. So, arguing as in (i), the proof is reduced to the case of λ with finite spectrum.

Say $\lambda = \sum_{i=1}^n \alpha_i e_i$, where $e_1,\dots,e_n$ are pairwise orthogonal central projections adding up to 1 and $\alpha_i \geqslant 0$ are scalars. Then

$$\tau(e_i(a-\lambda)_+) = \tau(e_i(a-\alpha_i)_+) \leqslant \tau(e_i(b-\alpha_i)_+) = \tau(e_i(b-\lambda)_+),$$

for all i. Adding over all i we get the result.

(iii) We can reduce the proof to the case of a λ with finite spectrum by choosing an increasing sequence $(\lambda_n)_n$ such that $\lambda_n \nearrow \lambda$ in norm and arguing as in (i). Passing to central cut-downs e_iM, where $e_1,\dots,e_n$ are central projections adding up to 1, we are further reduced to the case that λ is a nonnegative scalar. So assume that this is the case. Then $(a+\lambda-t)_+ = a+(\lambda-t)$ if $t\leqslant\lambda$ and $(a+\lambda-t)_+ = (a-(t-\lambda))_+$ otherwise. This calculation shows that $a \prec_{\mathrm{T}} b$ implies that $a+\lambda \prec_{\mathrm{T}} b+\lambda$. □

Proposition 3.2. *Let $a,b\in M$ be positive elements with finite spectrum. Then*

$$a = \sum_{i=1}^n \alpha_i P_i, \quad b = \sum_{i=1}^n \beta_i Q_i \tag{3.1}$$

for some $(P_i)_{i=1}^n$ *and* $(Q_i)_{i=1}^n$*, pairwise orthogonal projections in* M *adding up to* 1 *such that* $P_i \sim Q_i$ *for all* i*, and some decreasing sequences of positive central elements* $(\alpha_i)_{i=1}^n$ *and* $(\beta_i)_{i=1}^n$ *such that* $\|\alpha_i\| \leqslant \|a\|$ *and* $\|\beta_i\| \leqslant \|b\|$ *for all* i*.*

Proof. Since a and b have finite spectrum, we have decompositions

$$a = \sum_{i=1}^{l} \mu_i E_i, \quad b = \sum_{j=1}^{m} \nu_j F_j, \tag{3.2}$$

where $(E_i)_{i=1}^l$ and $(F_j)_{j=1}^m$ are pairwise orthogonal projections adding up to 1, and $(\alpha_i)_{i=1}^m$ and $(\beta_j)_{j=1}^m$ are nonnegative scalars. We further assume that both sequences have been arranged in decreasing order.

We will prove the representation for a and b in (3.1) by induction on $l+m$. The base case is $l+m=2$, i.e., $l=m=1$. In this case both a and b are scalar multiples of the identity. The desired representation has already been achieved.

Suppose that the desired representation is true for all pairs a and b as in (3.2) such that $l+m$ is less than a given number. Now suppose that $l+m$ is that given number. Observe that if $(e_k)_{k=1}^N$ are central projections adding up to 1 and the desired representation has been obtained for $e_k a$ and $e_k b$ in $e_k M$ for all k, then adding up these representations—adding zero terms if necessary so that they have the same number of terms—we get the desired representation for a and b. Now recall that there is a central projection e such that $eE_1 \precsim eF_1$ and $(1-e)F_1 \precsim (1-e)E_1$ ([KR97, Theorem 6.2.7]). Hence, reducing the proof to eM and $(1-e)M$, we can assume that E_1 and F_1 are Murray–von Neumann comparable. By symmetry, it suffices to assume that $E_1 \precsim F_1$. Recall also that for any projection $P \in M$ there exists a central projection e such that eP is a finite projection and $(1-e)P$ is properly infinite ([KR97, Proposition 6.3.7]). Applying this to E_1 and reducing the proof to each central cut-down, we can assume that E_1 is either finite or properly infinite.

Case 1: E_1 *is finite.* Let us find $F_1' \leqslant F_1$ such that $E_1 \sim F_1'$. Since E_1 is finite, there exists a unitary u such that $uE_1u^* = F_1'$ ([KR97, Exercise 6.9.7]). Since it is sufficient to obtain the desired representation for uau^* and b, let us rename uau^* as a and assume that $E_1 = F_1'$. Let

$$a' = \sum_{i=2}^{l} \mu_i E_i, \quad b' = \nu_1(F_1 - F_1') + \sum_{j=2}^{m} \nu_j F_j.$$

Notice that the total number of projections supporting a' and b' is now $l+m-1$. We can thus apply the induction hypothesis in the von Neumann algebra $(1-F_1')M(1-F_1')$ to get

$$a' = \sum_{i=1}^{n} \alpha_i P_i, \quad b' = \sum_{i=1}^{n} \beta_i Q_i.$$

We also have by induction that $\alpha_1 \leqslant \|a'\|$ and $\beta_1 \leqslant \|b'\|$. The map $x \mapsto (1-F_1')x$ is a surjective homomorphism from Z to the center of $(1-F_1')M(1-F_1')$ (by [SS08, Theorem 5.4.1]). Thus, the decreasing central elements $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ in $(1-F_1')M(1-F_1')$ can be lifted to central elements in Z. Moreover, as is clear for any surjective map between abelian von Neumann algebras, the decreasing order and the inequalities $\alpha_1 \leqslant \|a'\|$ and $\beta_1 \leqslant \|b'\|$ can be maintained after this lifting. Let us continue to denote these central liftings by α_i and β_i. Notice that $\mu_1 \geqslant \mu_2 = \|a'\| \geqslant \alpha_i$ for all i and $\nu_1 \geqslant \|b'\| \geqslant \beta_i$ for all i. So

$$a = \mu_1 E_1 + \sum_{i=1}^{n} \alpha_i P_i, \quad b = \nu_1 F_1' + \sum_{i=1}^{n} \beta_i Q_i$$

are the desired representations for a and b.

Case 2: *E_1 is properly infinite.* We can find a central projection e such that $eE_1 \precsim e(1-E_1)$ and $(1-e)(1-E_1) \precsim (1-e)E_1$. By passing to the corresponding central cut-down, we arrive at two cases.

Case 2(a): $E_1 \precsim 1 - E_1$. Let us again find $F_1' \leqslant F_1$ such that $E_1 \sim F_1'$. Let us moreover choose F_1' such that $F_1' \precsim 1 - F_1'$. We can easily achieve this by exploiting that E_1 is properly infinite. We claim that $1 - E_1 \sim 1 \sim 1 - F_1'$. Indeed, say $E_1' \leqslant 1 - E_1$ is such that $E_1 \sim E_1'$. Since we have assumed that E_1 is properly infinite, $E_1 + E_1' \sim E_1$. Hence

$$1 = (1 - E_1 - E_1') + E_1' + E_1 \sim (1 - E_1 - E_1') + E_1' = 1 - E_1.$$

We prove similarly that $1 \sim 1 - F_1'$, thereby establishing our claim. From $E_1 \sim F_1'$ and $1 - E_1 \sim 1 - F_1'$ we again deduce—as in the case where E_1 is finite—that there exists a unitary u such that $uE_1u^* = F_1'$. We can now continue arguing as in the case where E_1 is finite to complete the induction step.

Case 2(b): $1 - E_1 \precsim E_1$. Since E_1 is properly infinite, $E_1 \sim 1$. (Proof: We have $E_1 \leqslant 1$. So, by Cantor–Bernstein, it suffices to show that $1 \precsim E_1$. Indeed,

$$1 = (1 - E_1) + E_1 \precsim E_1 \oplus E_1 \sim E_1.)$$

Moreover, since $E_1 \precsim F_1$, we have $F_1 \sim 1$ as well. We can thus decompose E_1 and F_1 as follows: $E_1 = E_1' + E_1''$ and $F_1 = F_1' + F_1''$, where E_1', E_1'', F_1', F_1'' are projections such that $E_1' \sim F_1' \sim 1$, $E_1'' \sim 1 - F_1$, and $F_1'' \sim 1 - E_1$. Notice that $E_1' \sim F_1'$ and that

$$1 - E_1' = (1 - E_1) + E_1'' \sim F_1'' + (1 - F_1) = 1 - F_1'.$$

So there exists a unitary u such that $uE_1'u^* = F_1'$. It suffices to find the desired representations for uau^* and b. Let us relabel uau^* as a and assume that $E_1' = F_1'$. We have that

$$a = \mu_1 F_1' + \mu_1 E_1'' + \sum_{i=2}^{l} \mu_i E_i,$$

while b has the form

$$b = \nu_1 F_1' + \nu_1 F_1'' + \sum_{j=2}^{m} \nu_j F_j.$$

It is thus clear that it suffices to find the desired representations for

$$a' = \mu_1 E_1'' + \sum_{i=2}^{l} \mu_i E_i, \quad b' = \nu_1 F_1'' + \sum_{j=2}^{m} \nu_j F_j$$

in the von Neumann algebra $(1 - F_1')M(1 - F_1')$ and then lift the central coefficients to M (as in Case 1 above). Notice that the number of projections supporting a' and b' is still $l + m$. However, repeating the arguments used above we will find ourselves in either Case 1 or Case 2(a). More specifically, working in the von Neumann algebra $(1-F_1')M(1-F_1')$, we can find central projections e_1, e_2, e_3, e_4 adding up to the unit $1-F_1'$ and such that

(1) either $e_iE_1'' \precsim e_iF_1''$ or $e_iF_1'' \precsim e_iE_1''$ for all $i = 1, 2, 3, 4$,
(2) e_iE_1'' is either finite or properly infinite for all $i = 1, 2, 3, 4$.

Passing to the algebra $e_i(1-F_1')M(1-F_1')$, let us assume first that $E_1'' \precsim F_1''$. Then

$$E_1'' \precsim F_1'' \sim 1 - E_1 = (1-E_1') - E_1'' = (1-F_1') - E_1''.$$

So we can continue arguing as in Cases 1 and 2(a). Similarly, if $F_1'' \precsim E_1''$, then $F_1'' \precsim (1-F_1') - F_1''$, so again we can continue arguing as in Cases 1 and 2(a). This completes the induction.

□

Lemma 3.3. *Let $a, b \in M_+$ be positive elements with finite spectrum represented as in* (3.1) *of Proposition* 3.2*. If $a \prec_{\mathrm{T}} b$, then*

$$\sum_{i=1}^{k} \alpha_i P_i \prec_{\mathrm{T}} \sum_{i=1}^{k} \beta_i Q_i \tag{3.3}$$

for $k = 1, \dots, n$.

Proof. Since $P_i \sim Q_i$ for all i and both sets of projections add up to 1, there exists a unitary u such that $uQ_iu^* = P_i$ for all i. Let us relabel ubu^* as b and assume that $P_i = Q_i$ for all i.

We prove the lemma by induction on k. Let us first prove that $\alpha_1 c_{P_1} \leqslant \beta_1 c_{P_1}$, which clearly implies the case $k = 1$. Passing to the central cut-down $c_{P_1}M$ if necessary, we may assume that $c_{P_1} = 1$ (since $a \prec_{\mathrm{T}} b$ implies that $ac_{P_1} \prec_{\mathrm{T}} bc_{P_1}$ in $c_{P_1}M$). Suppose for the sake of contradiction that $\alpha_1 \nleqslant \beta_1$. Then there exists a projection $e \in Z$ and a scalar $\varepsilon > 0$ such that $\alpha_1 e \geqslant \beta_1 e + \varepsilon e$. Since the central coefficients $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ are decreasing, we deduce that $\|ea\| > \|eb\|$. But this contradicts that $ea \prec_{\mathrm{T}} eb$. Therefore, $\alpha_1 \leqslant \beta_1$.

Suppose that the lemma is true for $k-1$. To prove (3.3) it suffices to do it on each central cut-down e_iM of a partition of unity by central projections $e_1, \dots, e_N$. Since Z is an abelian von Neumann algebra, given any two positive elements $\alpha, \beta \in Z$ it is possible to find a projection $e \in Z$ such that $e\alpha \geqslant e\beta$ and $(1-e)\alpha \leqslant (1-e)\beta$. Thus, we can reduce the proof to two cases: $\alpha_k \geqslant \beta_k$ or $\alpha_k \leqslant \beta_k$. The second case follows at once from the induction hypothesis. Let us assume that $\alpha_k \geqslant \beta_k$. We have that $(a - \beta_{k+1})_+ \prec_{\mathrm{T}} (b - \beta_{k+1})_+$, by Lemma 3.1(ii). Hence

$$\begin{aligned}\sum_{i=1}^{k}(\alpha_i - \beta_{k+1})P_i &\leqslant \sum_{i=1}^{n}(\alpha_i - \beta_{k+1})_+ P_i \\ &= (a-\beta_{k+1})_+ \prec_{\mathrm{T}} (b-\beta_{k+1})_+ = \sum_{i=1}^{k}(\beta_i - \beta_{k+1})P_i.\end{aligned}$$

The above tracial submajorization holds in the hereditary subalgebra $(P_1 + \cdots + P_k)M(P_1 + \cdots + P_k)$ (by Lemma 2.5(ii)). Since $\beta_{k+1}(P_1 + \cdots + P_k)$ is a central element of this von Neumann algebra, we can add it on both sides by Lemma 3.1(iii). This yields (3.3). □

Lemma 3.4. *Let $a, b, c \in M_+$ be such that $a \prec_{\mathrm{u}} c$ and $b \prec_{\mathrm{u}} c$. Then for any central element $0 \leqslant \lambda \leqslant 1$ we have that $\lambda a + (1-\lambda)b \prec_{\mathrm{u}} c$.*

Proof. By a simple limiting argument it suffices to consider the case that λ has finite spectrum (see the proof of Lemma 3.1). Say $\lambda = \sum_{i=1}^n \alpha_i e_i$, where $e_1, \ldots, e_n$ are pairwise orthogonal central projections adding up to 1 and $\alpha_i \in [0,1]$ for all i. In order to show that $\lambda a+(1-\lambda)b \prec_{\mathrm{u}} c$ it suffices to show that $e_i(\alpha_i a+(1-\alpha_i)b) \prec_{\mathrm{u}} e_i c$ in $e_i M$ for all i. But $e_i a, e_i b \prec_{\mathrm{u}} e_i c$ for all i and $e_i\alpha_i a + e_i(1-\alpha_i)b$ is a scalar convex combination of $e_i a$ and $e_i b$. The lemma is thus proved. □

Lemma 3.5. *Let $P, Q \in M$ be orthogonal projections and let $\mu, \nu \in Z_+$. There exists $\rho \in Z_+$ such that $\min(\mu,\nu) \leqslant \rho \leqslant \max(\mu,\nu)$ and such that for any central element $0 \leqslant \lambda \leqslant 1$ we have*

$$\mu' P + \nu' Q \prec_{\mathrm{u}} \mu P + \nu Q,$$

where

$$\mu' = \mu\lambda + (1-\lambda)\rho,$$
$$\nu' = \nu\lambda + (1-\lambda)\rho.$$

Proof. By Dixmier's approximation theorem ([KR97, Theorem 8.3.5]) applied in the von Neumann algebra $(P+Q)M(P+Q)$ we have that

$$\rho P + \rho Q \prec_{\mathrm{u}} \mu P + \nu Q,$$

for some $\min(\mu,\nu) \leqslant \rho \leqslant \max(\mu,\nu)$ in the center of $(P+Q)M(P+Q)$. We can lift ρ to an element in the center of M satisfying the same inequalities. Let $\lambda \in Z$ be such that $0 \leqslant \lambda \leqslant 1$. Then, by the previous lemma,

$$(\mu\lambda + \rho(1-\lambda))P + (\nu\lambda + \rho(1-\lambda))Q \prec_{\mathrm{u}} \mu P + \nu Q,$$

as desired. □

Remark 3.6. In the case that M is finite one can show that $\rho = \frac{\mu E(P)+\nu E(Q)}{E(P+Q)}$, where $E\colon M \to Z$ is the center-valued trace.

In the following proposition we assume that M is a finite von Neumann algebra. We denote by $E\colon M \to Z$ the center-valued trace of M.

Proposition 3.7. *Suppose that M is a finite von Neumann algebra. Let $a, b \in M_+$ be positive elements of the form* (3.1) *in Proposition* 3.2. *If*

$$\sum_{i=1}^k \alpha_i E(P_i) \leqslant \sum_{i=1}^k \beta_i E(Q_i) \tag{3.4}$$

for all $k = 1, \ldots, n$, then $a \prec_{\mathrm{c}} b$.

Proof. Conjugating b by a unitary we may assume that $P_i = Q_i$ for all i. Passing to central cut-downs $e_j M$ for suitable projections $e_1, \ldots, e_N \in Z$ that partition the unit, we may assume that $c_{P_i} = 1$ for all i. Assuming these simplifications, we prove the proposition by induction on n. More specifically, we will show by induction on n that if $P_1, \ldots, P_n$ are pairwise orthogonal projections in a finite von Neumann algebra such that $c_{P_i} = 1$ for all i, and $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ are decreasing positive central elements such that

$$\sum_{i=1}^k \alpha_i E(P_i) \leqslant \sum_{i=1}^k \beta_i E(P_i) \text{ for all } k = 1, \ldots, n, \tag{3.5}$$

then $a = \sum_{i=1}^n \alpha_i P_i \prec_c \sum_{i=1}^n \beta_i P_i = b$. We do not assume (mostly as a matter of convenience) that the projections P_i add up to 1.

Consider the case $n = 1$. From the inequality (3.5) we get that $\alpha_1 E(P_1) \leqslant \beta_1 E(P_1)$. Since $c_{P_1} = 1$ this implies that $\alpha_1 \leq \beta_1$, which in turn implies that $a \leqslant b$. By Lemma 2.2(i), $a \prec_c b$, as desired.

Suppose now, by induction, that the desired result is valid whenever the number of projections is less than n. Consider the case of n projections. Let us apply Lemma 3.5 to $\beta_1 P_1 + \beta_2 P_2$ with a suitable $0 \leqslant \lambda \leqslant 1$ (to be specified soon) so as to obtain $\rho \in Z$ and $\beta_1' P_1 + \beta_2' P_2$ majorized by $\beta_1 P_1 + \beta_2 P_2$. Since $\beta_1 \geqslant \beta_2$ we have $\beta_1 \geqslant \rho \geqslant \beta_2$ and $\beta_1' \geqslant \beta_2'$. Let us choose λ such that the $\widehat{Z}$ (the spectrum of Z) partitions into two clopen sets satisfying that

(C1) $\beta_1' = \alpha_1$ on the first set,
(C2) $\beta_1' = \beta_2' \geqslant \alpha_1$ on the second set.

To see that this is possible, notice that the inequality $\beta_1' \geqslant \alpha_1$, put in terms of λ, has the form

$$\kappa\lambda \geqslant \gamma,$$

for some $\kappa \in Z_+$ and some $\gamma \in Z_{\text{sa}}$ such that $\kappa \geqslant \gamma$ (in fact, $\kappa = \beta_1 - \rho \in Z_+$ and $\gamma = \alpha_1 - \rho \in Z_{\text{sa}}$). Let us choose $\lambda = \gamma_+/\kappa$, where the fraction is defined to be zero outside the set $\overline{\{x \in \widehat{Z} \mid \kappa(x) > 0\}}$. (Recall that we regard elements of Z as continuous functions on its spectrum $\widehat{Z}$.) The quotient γ_+/κ is well defined in Z since $\kappa \geqslant \gamma_+$ and Z is an abelian von Neumann algebra. Observe that $0 \leqslant \lambda \leqslant 1$. Let us partition $\widehat{Z}$ into the sets $\overline{\{x \in \widehat{Z} \mid \lambda(x) > 0\}}$ and its complement. These sets are clopen since $\widehat{Z}$ is extremally disconnected. On the first set we have that $\kappa\lambda = \gamma$, which, put back in terms of β_1', implies that $\beta_1' = \alpha_1$. Thus, we are in case (C1) above. On the second set we have that $\lambda = 0$. This implies that $\beta_1' = \beta_2' \geqslant \alpha_1$, i.e., we are in case (C2). Thus, λ is as desired.

Let

$$b' = \beta_1' P_1 + \beta_2' P_2 + \sum_{i>2} \beta_i P_i.$$

Then (3.5) continues to hold for a and b'. Indeed, for $k = 1$ because $\beta_1' \geqslant \alpha_1$, and for $k > 1$ because

$$\beta_1' E(P_1) + \beta_2' E(P_2) = \beta_1 E(P_1) + \beta_2 E(P_2).$$

Since $b' \prec_u b$, in order to prove the proposition it suffices to show that $a \prec_c b'$. So let us rename b' as b, β_1' as β_1, and β_2' as β_2.

We can restrict to the two clopen sets described above and prove the proposition in each case. (In other words, if $e_1, e_2 \in Z$ are the central projections corresponding to these sets, then $e_1 a$ and $e_1 b$ continue to satisfy (3.5) in $e_1 M$ (keep in mind that the center valued trace of $e_1 M$ is $e_1 E(\cdot)$) and similarly for $e_2 a$ and $e_2 b$ in $e_2 M$. Moreover, it suffices to show that $e_i a \prec_c e_i b$ in $e_i M$ for $i = 1, 2$.) We claim that after restricting to the first set we are done by induction. Indeed, from (3.5), and keeping in mind that $\beta_1 = \alpha_1$ on this set, we obtain that the elements

$$a' = \sum_{i=2}^n \alpha_i P_i \text{ and } b'' = \sum_{i=2}^n \beta_i P_i$$

satisfy the induction hypothesis. So $a' \prec_c b''$. By Lemma 2.5 this relation holds in the hereditary subalgebra $(P_2 + \cdots + P_n)M(P_2 + \cdots + P_n)$. Therefore,

$$a = \alpha_1 P_1 + a' \prec_c \beta_1 P_1 + b'' = b,$$

as desired.

Let us restrict to the second set where $\beta_1 = \beta_2 \geqslant \alpha_1$. Suppose more generally that for some $1 < k \leqslant n$ we have that $\beta_1 = \cdots = \beta_k \geqslant \alpha_1$. Assume first that $k < n$. Let us apply Lemma 3.5 to $\beta_1(P_1 + \cdots + P_k) + \beta_{k+1}P_{k+1}$, yielding the element $\beta_1'(P_1 + \cdots + P_k) + \beta_{k+1}' P_{k+1}$ majorized by $\beta_1(P_1 + \cdots + P_k) + \beta_{k+1}P_{k+1}$. We choose $0 \leqslant \lambda \leqslant 1$ such that there exist two clopen sets such that

(C1') $\beta_1' = \alpha_1$ on the first set,
(C2') $\beta_1' = \beta_{k+1}' \geqslant \alpha_1$, on the second set.

Such a choice is possible by the discussion above. Observe that the conditions in (3.5) continue to hold for a and

$$b' = \beta_1'(P_1 + \cdots + P_k) + \beta_{k+1}' P_{k+1} + \sum_{i=k+2}^{n} \beta_i P_i.$$

They hold for $l \leqslant k$ because $\beta_1' \geqslant \alpha_1$ and for $l \geqslant k+1$ because

$$\beta_1' E(P_1 + \cdots + P_k) + \beta_{k+1}' E(P_{k+1}) = \beta_1 E(P_1 + \cdots + P_k) + \beta_{k+1} E(P_{k+1}).$$

We have already shown how to deal with the set where $\beta_1' = \alpha_1$ by using the induction hypothesis. It remains to consider the case when $k = n$, i.e., $\beta_1 = \cdots = \beta_n \geqslant \alpha_1$. But in this case we clearly have that $a \leqslant b$. So, by Lemma 2.2(i), $a \prec_c b$. ☐

Proposition 3.8. *Suppose that M is a finite von Neumann algebra. Let $a, b \in M_+$. If $a \prec_T b$, then $a \prec_c b$.*

Proof. We can reduce the proof to the case that a and b have finite spectrum. For suppose $\|a - a'\| < \varepsilon$ and $\|b - b'\| < \varepsilon$ for some $\varepsilon > 0$ and some $a', b' \in M_+$ of finite spectrum (whose existence is guaranteed by the Borel functional calculus on a and b). Then, relying on Lemma 2.2, we deduce that $(a' - 2\varepsilon)_+ \prec_c (a - \varepsilon)_+$ and $(b - \varepsilon)_+ \prec_c b'$. Hence $(a' - 2\varepsilon)_+ \prec_T b'$. Suppose we have shown that $(a' - 2\varepsilon)_+ \prec_c b'$. Then, again using Lemma 2.2, we obtain that $(a - 4\varepsilon)_+ \prec_c (a' - 3\varepsilon)_+ \prec_c (b' - \varepsilon)_+ \prec_c b$. Since $\varepsilon > 0$ can be arbitrarily small, we arrive at $a \prec_c b$, as desired. So let us assume that a and b have finite spectrum.

Express a and b as in (3.1) of Proposition 3.2:

$$a = \sum_{i=1}^{n} \alpha_i P_i, \quad b = \sum_{i=1}^{n} \beta_i P_i.$$

(We have conjugated b by a unitary so that the projections in a and b are the same.) We can take central cut-downs and reduce to the case that $c_{P_i} = 1$ for all $i = 1, \ldots, n$. From $a \prec_T b$ we deduce from Lemma 3.3 that

$$\tau\Big(\sum_{i=1}^{k} \alpha_i P_i\Big) \leqslant \tau\Big(\sum_{i=1}^{k} \beta_i P_i\Big)$$

for all $\tau \in \mathrm{T}(M)$ and all $k = 1, \ldots, n$. Letting τ range through traces of the form $\delta_x \circ E$, where δ_x is a point evaluation on the center, we deduce that (3.4) from Proposition 3.7 holds. The desired result now follows from Proposition 3.7. ☐

Recall that $E\colon M \to Z$ denotes the center-valued trace of M (whenever M is assumed to be a finite von Neumann algebra).

Proposition 3.9. *Suppose that M is a finite von Neumann algebra. Let $a, b \in M_+$ be positive contractions of the form* (3.1) *in Proposition* 3.2. *Let $r \geqslant 0$. If*

$$\text{(a)}\quad \sum_{i=1}^{k}(\alpha_i - r)_+ E(P_i) \leqslant \sum_{i=1}^{k} \beta_i E(Q_i) \text{ for all } k = 1, \dots, n$$

and

$$\text{(a')}\quad \sum_{i=k}^{n}(1-\alpha_i - r)_+ E(P_i) \leqslant \sum_{i=k}^{n}(1-\beta_i) E(Q_i) \text{ for all } k = 1, \dots, n,$$

then there exists $b' \in M_+$ such that $b' \prec_{\mathrm{u}} b$ and $\|a - b'\| \leqslant r$.

Proof. Conjugating b by a unitary we may assume that $P_i = Q_i$ for all i. Passing to central cut-downs $e_j M$, for suitable projections $e_1, \dots, e_N \in Z$ that partition the unit, we may also assume that $c_{P_i} = 1$ for all i. We will proceed by induction on n under the additional assumptions that $P_i = Q_i$ and $c_{P_i} = 1$ for all i.

If $n = 1$, then $a = \alpha_1 \cdot 1$ and $b = \beta_1 \cdot 1$ are multiples of the identity. From condition (a) we deduce that $(\alpha_1 - r)_+ \leqslant \beta_1$, whereas from (a') we deduce that $(1 - \alpha_1 - r)_+ \leqslant 1 - \beta_1$. Together they imply that $\|\alpha_1 - \beta_1\| \leqslant r$.

Let us assume now by induction that the proposition is true when the number of projections P_i is less than a given n. Let a and b be as in the statement of the lemma. From condition (a) with $k = 1$ and from $c_{P_1} = 1$ we deduce that $\beta_1 \geqslant (\alpha_1 - r)_+$. Just as we did before in the proof of Proposition 3.7, let us apply Lemma 3.5 in $\beta_1 P_1 + \beta_2 P_2$ with a suitable central element $0 \leqslant \lambda \leqslant 1$ (to be specified soon) so as to obtain $\rho \in Z_+$ and an element $\beta_1' P_1 + \beta_2' P_2$ majorized by $\beta_1 P_1 + \beta_2 P_2$. We have that

$$\beta_1 \geqslant \beta_1' \geqslant \rho \geqslant \beta_2' \geqslant \beta_2$$

and that

$$\beta_1 E(P_1) + \beta_2 E(P_2) = \beta_1' E(P_1) + \beta_2' E(P_2).$$

Let

$$b' = \beta_1' P_1 + \beta_2' P_2 + \sum_{i=3}^{n} \beta_i P_i.$$

Then for any $\lambda \in Z$ such that $0 \leqslant \lambda \leqslant 1$ the inequalities in (a), applied now to a and b', hold except possibly for $k = 1$. The inequalities in (a') also hold for a and b', except possibly for $k = 2$. Let us choose λ such that each point of the spectrum of Z is either $\lambda = 0$ or one of these two inequalities, $k = 1$ in (a) or $k = 2$ in (a'), becomes an equality while the other one remains valid. More specifically, we choose a central element $0 \leqslant \lambda \leqslant 1$ such that the center is partitioned into three clopen sets satisfying the following conditions:

(C1) $\beta_1' = (\alpha_1 - r)_+$, $\beta_1' \geqslant \beta_2'$, and

$$\sum_{i=2}^{n}(1-\alpha_i - r)_+ E(P_i) \leqslant (1-\beta_2')E(P_2) + \sum_{i=3}^{n}(1-\beta_i)E(P_i)$$

on the first set,

(C2) $\beta_1' \geqslant (\alpha_1 - r)_+$, $\beta_1' \geqslant \beta_2'$, and

$$\sum_{i=2}^{n}(1-\alpha_i - r)_+ E(P_i) = (1-\beta_2')E(P_2) + \sum_{i=3}^{n}(1-\beta_i)E(P_i) \tag{3.6}$$

on the second set,

(C3) $\beta_1' \geqslant (\alpha_1 - r)_+$, $\beta_1' = \beta_2'$, and

$$\sum_{i=2}^{n}(1-\alpha_i - r)_+ E(P_i) \leqslant (1-\beta_2')E(P_2) + \sum_{i=3}^{n}(1-\beta_i)E(P_i)$$

on the third set.

To see that such a choice of λ is possible, notice first that the inequalities

$$\beta_1' \geqslant (\alpha_1 - r)_+$$

and

$$\sum_{i=2}^{n}(1-\alpha_i - r)_+ E(P_i) \leqslant (1-\beta_2')E(P_2) + \sum_{i=3}^{n}(1-\beta_i)E(P_i),$$

when put in terms of λ, take the general form

$$\kappa_1\lambda \geqslant \gamma_1 \text{ and } \kappa_2\lambda \geqslant \gamma_2$$

for some $\kappa_1, \kappa_2 \in Z_+$ and $\gamma_1, \gamma_2 \in Z_{sa}$ such that $\kappa_1 \geqslant \gamma_1$ and $\kappa_2 \geqslant \gamma_2$ (i.e., the inequalities are valid for $\lambda = 1$). (In fact, $\kappa_1 = \beta_1 - \rho$, $\gamma_1 = (\alpha_1 - r)_+ - \rho$, $\kappa_2 = (\rho - \beta_2)E(P_2)$, and

$$\gamma_2 = \sum_{i\geqslant 2}(1-\alpha_i - r)_+ E(P_i) - \sum_{i\geqslant 2}(1-\beta_i)E(P_i) - (1-\rho)E(P_2).)$$

Let us choose

$$\lambda = \max((\gamma_1)_+/\kappa_1, (\gamma_2)_+/\kappa_2).$$

These fractions are well defined in Z because Z is an abelian von Neumann algebra and $\kappa_1 \geqslant (\gamma_1)_+$ and $\kappa_2 \geqslant (\gamma_2)_+$. Let us show that λ is as desired. It is clear that $0 \leqslant \lambda \leqslant 1$. Exploiting that $\widehat{Z}$ is extremally disconnected, let us partition $\widehat{Z}$ into four clopen sets X_1, X_2, X_3, X_4 such that $\gamma_1 \leqslant 0$ and $\gamma_2 \leqslant 0$ on X_1, $\gamma_1 \geqslant 0$ and $\gamma_2 \leqslant 0$ on X_2, $\gamma_1 \leqslant 0$ and $\gamma_2 \geqslant 0$ on X_3, and $\gamma_1 \geqslant 0$ and $\gamma_2 \geqslant 0$ on X_4. It is straightforward to check that $\lambda = 0$ on X_1. Thus, on this set we find ourselves in case (C3) above. It can also be checked that $\kappa_2\lambda = \gamma_2$ on X_2 and $\kappa_1\lambda = \gamma_1$ on X_3. This value of λ yields cases (C1) and (C2) above, respectively. Finally, partition X_4 into two clopen sets such that $\gamma_1\kappa_2 \geqslant \gamma_2\kappa_1$ on one set and $\gamma_1\kappa_2 \leqslant \gamma_2\kappa_1$ on the second. On the first of these sets we have that $\kappa_1\lambda = \gamma_1$ and on the other that $\kappa_2\lambda = \gamma_2$ (again yielding cases (C1) and (C2) above).

Since $b' \prec_{\mathrm{u}} b$, it suffices to prove that $a \prec_{\mathrm{u}} b'$. Equivalently, it suffices to prove the proposition with b' in place of b. So let us rename β_1' and β_2' as β_1 and β_2 and now assume that the conditions (C1)–(C3) for the three clopen sets described above hold for β_1 and β_2.

Let us show that on the clopen sets satisfying (C1) and (C2) we can argue by induction. Indeed, restricting to the first set (while retaining the same names for our variables) we have that

$$\sum_{i=2}^{k}(\alpha_i - r)_+E(P_i) \leqslant \sum_{i=2}^{k}\beta_iE(Q_i) \text{ for all } k = 2,\dots,n,$$
$$\sum_{i=k}^{n}(1-\alpha_i - r)_+E(P_i) \leqslant \sum_{i=k}^{n}(1-\beta_i)E(Q_i) \text{ for all } k = 2,\dots,n.$$

Thus

$$a'' = \sum_{i=2}^{n}\alpha_iP_i \text{ and } b'' = \sum_{i=2}^{n}\beta_1P_i$$

satisfy conditions (a) and (a') in the algebra PMP, where $P = P_2 + \cdots + P_n$. (To see this we use that the center-valued trace $E_P\colon PMP \to PZ$ can be computed to be $E_P(x) = \frac{E(x)}{E(P)}P$.) Hence, by the induction hypothesis applied in PMP, there exists $b''' \in PMP$ majorized by b'' and within r distance of a''. The element $\beta_1P_1 + b''$ is within r of a and $\beta_1P_1 + b'' \prec_{\mathrm{u}} b$. This proves the induction step.

Suppose now that we are in the second set. Consider the elements

$$a'' = \sum_{i=2}^{n}(1-(1-\alpha_i - r)_+)P_i, \quad b'' = \sum_{i=2}^{n}\beta_iP_i$$

in PMP, where $P = P_2 + \cdots + P_n$. From condition (a') applied to a and b we get that a'' and b'' satisfy condition (a') with $r = 0$. Moreover, from (3.6) we deduce that the center-valued traces of these two elements agree, i.e., $E_P(a'') = E_P(b'')$ (recall that we have relabeled β_2' as β_2, so (3.6) is now valid with β_2 in place of β_2'). This in turn implies that a'' and b'' satisfy condition (a) with $r = 0$ as well. By the induction hypothesis with $r = 0$ applied in the von Neumann algebra PMP we get that $a'' \prec_{\mathrm{u}} b''$ in PMP. Notice that

$$1-(1-\alpha_i - r)_+ = \min(1, \alpha_i + r).$$

From this we easily deduce that a'' is within a distance r of $\sum_{i=2}^{n}\alpha_iP_i$. Now, from condition (a') applied to a and b with $k = 1$, and keeping equality (3.6) in mind, we deduce that $(\alpha_1 + r)E(P_1) \geqslant \beta_1E(P_1)$. This implies that $\alpha_1 + r \geqslant \beta_1$ (since $c_{P_1} = 1$, which implies that the subset of $\widehat{Z}$, where $E(P_1)$ is strictly positive is dense in $\widehat{Z}$). Similarly, from condition (a) with $k = 1$ we deduce that $\beta_1 \geqslant \alpha_1 - r$. So $\|\alpha_1 - \beta_1\| \leqslant r$. Therefore, $\beta_1P_1 + a''$ is within a distance r of a and $\beta_1P_1 + a'' \prec_{\mathrm{u}} b$. This again proves the induction step in this case.

Let us now examine the third set, where $\beta_1 = \beta_2$ while conditions (a) and (a') remain valid. Suppose more generally that for some $k = 2,\dots,n$ we have that $\beta_1 = \cdots = \beta_k$ while the conditions (a) and (a') are valid. Suppose first that $k < n$. Let us apply Lemma 3.5 to the element $\beta_1(P_1 + \cdots + P_k) + \beta_{k+1}P_{k+1}$ with a suitable central element $0 \leqslant \lambda \leqslant 1$ (to be specified soon). Call $\beta_1'(P_1 + \cdots + P_k) + \beta_{k+1}'P_{k+1}$ the resulting element. As before, we can choose λ such that conditions (a) and (a') remain valid for a and

$$b' = \beta_1'(P_1 + \cdots + P_k) + \beta_{k+1}'P_{k+1} + \sum_{i>k+1}\beta_iP_i$$

and such that either one of the following three cases occurs after restricting to suitable clopen sets that partition $\widehat{Z}$:

(C1') $\beta_1' = (\alpha_1 - r)_+$,

(C2')

$$\sum_{i=k+1}^{n} (1-\alpha_i - r)_+ E(P_i) = (1-\beta_{k+1}')E(P_{k+1}) + \sum_{i=k+2}^{n} (1-\beta_i)E(P_i), \tag{3.7}$$

(C3') $\beta_1' = \beta_{k+1}'$.

Let us rename β_1' and β_{k+1}' as β_1 and β_{k+1}, respectively. We have already dealt with the first of these three cases. The second is dealt with similarly as before: The elements

$$a'' = \sum_{i=k+1}^{n} (1-(1-\alpha_i - r)_+)P_i \text{ and } b'' = \sum_{i=k+1}^{n} \beta_i P_i$$

satisfy the induction hypotheses with $r = 0$ in the von Neumann algebra PMP, where $P = P_{k+1} + \cdots + P_n$. On the other hand, keeping in mind equality (3.7), we deduce that

$$a''' = \sum_{i=1}^{k} \alpha_i P_i, \quad b''' = \sum_{i=1}^{k} \beta_i P_i$$

satisfy conditions (a) and (a') with the same r in the von Neumann algebra $(1-P)M(1-P)$. We can thus apply the induction hypothesis in both cases to get the desired result.

The remaining case to be considered is when $k = n$, i.e., $\beta_1 = \cdots = \beta_n$, and conditions (a) and (a') are valid. From condition (a) with $k = 1$ we deduce that $\beta_1 + r \geqslant \alpha_1$, while from condition (a') with $k = n$ we deduce that $\alpha_n + r \geqslant \beta_n$ (here we use that $c_{P_i} = 1$ for all i). This clearly implies that $\|\alpha_i - \beta_i\| \leqslant r$ for all i, implying that $\|a' - b'\| \leqslant r$, as desired. ☐

Proposition 3.10. *Suppose that M is a finite von Neumann algebra. Let $r \geqslant 0$. If $a, b \in M_+$ are contractions such that $(a-r)_+ \prec_{\mathrm{T}} b$ and $(1-a-r)_+ \prec_{\mathrm{T}} 1-b$, then a is within a distance r of $\overline{\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(M)\}}$.*

Proof. Let $\varepsilon > 0$. Let a' and b' be positive contractions of finite spectrum such that $\|a - a'\| < \varepsilon/2$ and $\|b - b'\| < \varepsilon/2$ (whose existence is guaranteed by the Borel functional calculus on a and b). Then, using Lemma 2.2, we find that $(a'-r-\varepsilon)_+ \prec_{\mathrm{T}} b'$ and $(1-a'-r-\varepsilon) \prec_{\mathrm{T}} 1-b'$ (see the proof of Proposition 3.8). Let us express a' and b' in the form of (3.1) from Proposition 3.2:

$$a' = \sum_{i=1}^{n} \alpha_i P_i, \quad b' = \sum_{i=1}^{n} \beta_i Q_i.$$

Conjugating b' by a unitary assume that $Q_i = P_i$ for all i. Cutting down the center by central projections we assume that $c_{P_i} = 1$ for all i. By Lemma 3.3, from $(a'-r-\varepsilon)_+ \prec_{\mathrm{T}} b'$ we deduce that

(a) $\displaystyle\sum_{i=1}^{k} (\alpha_i - r - \varepsilon)_+ E(P_i) \leqslant \sum_{i=1}^{k} \beta_i E(P_i)$ for all $k = 1, \ldots, n$, and from $(1-r-\varepsilon-a')_+ \prec_{\mathrm{T}} (1-b')$ that

(a') $\displaystyle\sum_{i=k}^{n}(1-\alpha_i-r-\varepsilon)_+E(P_i) \leqslant \sum_{i=k}^{n}(1-\beta_i)E(P_i)$ for all $k=1,\dots,n$.

By Proposition 3.9, there exists b'' majorized by b' and within $r+\varepsilon$ distance of a'. Since ε can be arbitrarily small, this proves the proposition. □

We now proceed to extend Propositions 3.8 and 3.10 to arbitrary von Neumann algebras. This is accomplished in Propositions 3.13 and 3.15 below.

Lemma 3.11. *Let $a, b \in M_+$ be as follows:*

$$a=\sum_{i=1}^{n}\alpha_iP_i, \quad b=\sum_{i=1}^{n}\beta_iP_i,$$

where $(P_i)_{i=1}^n$ are orthogonal projections adding up to 1 and such that $c_{P_i}=1$ for all i and where $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ are decreasing nonnegative scalar coefficients. Suppose that $a \prec_{\mathrm{T}} b$. Then:

(a) *For all traces $\tau \in \mathrm{T}(M)$ and all $k=1,\dots,n$ we have*

$$\tau\Big(\sum_{i=1}^{k}\alpha_iP_i\Big) \leqslant \tau\Big(\sum_{i=1}^{k}\beta_iP_i\Big).$$

(b) $\alpha_1 \leqslant \beta_1$.

(c) *For each $k=2,\dots,n$, if $\alpha_k>\beta_k$, then $P_k \propto \sum_{i<k}P_i$. (Here $P \propto Q$ means that $P \prec_{\mathrm{c}} Q^{\oplus N}$ for some N.)*

Proof. Conditions (a) and (b) follow at once from Lemma 3.3. Suppose now that $\alpha_k>\beta_k>0$ for some k. By Lemma 3.3, $\sum_{i=1}^{k}\alpha_iP_i \prec_{\mathrm{T}} \sum_{i=1}^{k}\beta_iP_i$. Hence,

$$\tau\Big(\sum_{i=1}^{k}(\alpha_i-\beta_k)P_j\Big) \leqslant \tau\Big(\sum_{i=1}^{k-1}(\beta_i-\beta_k)P_i\Big)$$

for all $\tau \in \mathrm{T}(M)$. Since we have assumed that $\alpha_k-\beta_k>0$, this implies that

$$\tau(P_k) \leqslant N\tau(P_1+\cdots+P_{k-1})$$

for all $\tau \in \mathrm{T}(M)$ and some suitable positive integer N (e.g., $N \geqslant \frac{\beta_i-\beta_k}{\alpha_k-\beta_k}$ for all i). By [KR97, Theorem 8.4.3 (vii)], this implies that $P_k \propto P_1+\cdots+P_{k-1}$. □

We start with the submajorization result. First, a lemma.

Lemma 3.12. *Let $P_1,\dots,P_n$ be pairwise orthogonal projections such that P_1 is properly infinite and $P_i \precsim P_1$ for all i. Let $\alpha_1,\dots,\alpha_n$ be central positive elements such that $\alpha_i \leqslant \alpha_1$ for all i. Then*

$$\sum_{i=1}^{n}\alpha_iP_i \prec_{\mathrm{c}} \alpha_1P_1.$$

Proof. Let us write $P_1=P_1'+Q_2+\cdots+Q_n$, where $P_1',Q_2,\dots,Q_n$ are pairwise orthogonal projections such that $P_1' \sim P_1$ and $Q_i \sim P_i$ for all $i \geqslant 2$. Let $v \in M$ be a partial isometry such that $vP_1'v^*=P_1$ and $vQ_iv^*=P_i$ for $i \geqslant 2$. Then

$$v(\alpha_1P_1)v^*=\alpha_1vP_1'v^*+\sum_{i=2}^{n}\alpha_1vQ_iv^*=\sum_{i=1}^{n}\alpha_1P_i \geqslant \sum_{i=1}^{n}\alpha_iP_i.$$

The result now follows from Lemma 2.2(i). ◻

Proposition 3.13. *If $a, b \in M_+$ are such that $a \prec_{\mathrm{T}} b$, then $a \prec_{\mathrm{c}} b$.*

Proof. Arguing as in the proof of Proposition 3.8 we can reduce the proof to the case that a and b have finite spectra. We then put them in the form (3.1) from Proposition 3.2 assuming further that $P_i = Q_i$ for all i (conjugating b by a unitary if necessary):

$$a = \sum_{i=1}^{n} \alpha_i P_i, \quad b = \sum_{i=1}^{n} \beta_i P_i.$$

Again arguments as in the proof of Proposition 3.8 allow us to assume that the coefficients $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ have finite spectrum.

Notice that if e is a central projection, then the hypothesis of the theorem hold for ea and eb in eM (by Lemma 3.1). On the other hand, if central projections $(e_j)_{j=1}^N$ partition the unit and we have proven the theorem for $e_j a$ and $e_j b$ in $e_j M$ for all j, then we conclude the same for a and b. This allows us to make the following reductions:

(1) each P_i is either finite or properly infinite for all i,
(2) the projections P_i are pairwise orthogonal, pairwise Murray–von Neumann comparable, and add up to 1,
(3) $c_{P_i} = 1$ for all i.

Recall that we have assumed that the central coefficients $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ have finite spectra. By passing to cut-downs of M by central projections we can assume that these coefficients are scalars. Observe that the decreasing ordering of $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ is maintained by doing this and that properties (1)–(3) above are not destroyed in the process. Thus, we further assume that

(4) the coefficients $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ are decreasing scalars.

We proceed by induction on the number of projections. If $n = 1$, then Lemma 3.11(b) implies that $\alpha_1 \leqslant \beta_1$. Hence $a \leqslant b$.

Let us now consider the general case. The case when all the projections P_i are finite has already been dealt with in Proposition 3.8. So let us assume that one of the projections is properly infinite. Let P_k be a projection larger than the rest in the Murray–von Neumann sense. By assumption, P_k is properly infinite.

Case $k < n$. From $a \prec_{\mathrm{T}} b$ we know, by Lemma 3.3, that $\sum_{i=1}^k \alpha_i P_i \prec_{\mathrm{T}} \sum_{i=1}^k \beta_i P_i$. This relation also holds in the hereditary subalgebra $(P_1 + \cdots + P_k)M(P_1 + \cdots + P_k)$ (by Lemma 2.5). By induction, $\sum_{i=1}^k \alpha_i P_i \prec_{\mathrm{c}} \sum_{i=1}^k \beta_i P_i$. Hence

$$\sum_{i=1}^{k} \alpha_i P_i + \sum_{i>k} \beta_i P_i \prec_{\mathrm{c}} b.$$

But $\sum_{i=k}^n \alpha_i P_i \prec_{\mathrm{c}} \alpha_k P_k + \sum_{i>k} \beta_i P_i$, by Lemma 3.12. Hence,

$$a = \sum_{i=1}^{n} \alpha_i P_i \prec_{\mathrm{c}} \sum_{i=1}^{k-1} \alpha_i P_i + \alpha_k P_k + \sum_{i>k} \beta_i P_i \prec_{\mathrm{c}} b.$$

Case $k = n$. Suppose that P_n is the largest projection in the Murray–von Neumann sense (and it is properly infinite). If $\alpha_n > \beta_n$, then from condition (c) of Lemma 3.11 we get that $P_n \propto \bigoplus_{i<n} P_i$. But we have assumed that the projections P_i are

pairwise Murray–von Neumann comparable. So $P_n \propto P_{k'}$ for some $k' < n$. The projection $P_{k'}$ is also properly infinite (since it cannot be finite). Hence, $P_n \precsim P_{k'}$. We are then in a case previously dealt with, since $P_{k'}$ is properly infinite and larger than the other projections. So let us assume that $\alpha_n \leqslant \beta_n$.

By Lemma 3.3 we have $\sum_{i=1}^{n-1} \alpha_i P_i \prec_{\mathrm{c}} \sum_{i=1}^{n-1} \beta_i P_i$, which also holds in the hereditary subalgebra $(P_1 + \cdots + P_{n-1})M(P_1 + \cdots + P_{n-1})$ (by Lemma 2.5). Hence, by induction,

$$\sum_{i=1}^{n-1} \alpha_i P_i \prec_{\mathrm{c}} \sum_{i=1}^{n-1} \beta_i P_i.$$

Since $\alpha_n \leqslant \beta_n$ we get that $a \prec_{\mathrm{c}} b$, as desired. □

Lemma 3.14. *Let P be a properly infinite projection such that $P \sim 1$. Let $a, b \in (1-P)M(1-P)$ be positive contractions.*

(i) *If $(1-P) - a \prec_{\mathrm{c}} (1-P) - b$, then $\beta P + a \prec_{\mathrm{u}} \beta P + b$ for any scalar β such that $a, b \leqslant \beta \leqslant 1$.*
(ii) *If $a \prec_{\mathrm{c}} b$, then $a + \alpha P \prec_{\mathrm{u}} b + \alpha P$ for any scalar $\alpha \geqslant 0$ such that $a, b \geqslant \alpha(1-P)$.*

Proof. (i) By Lemma 2.2,

$$((1-P) - a - t)_+ \prec_{\mathrm{c}} ((1-P) - t - b)_+$$

for any $t \in [0, \infty)$. Choosing $t = 1 - \beta$ we obtain that $\beta(1-P) - a \prec_{\mathrm{c}} \beta(1-P) - b$ in $(1-P)M(1-P)$. Since $1 - P \precsim P$ and P is properly infinite, we can find countably many orthogonal copies of $1 - P$ in PMP. So $(1-P)M(1-P) \otimes \mathcal{K}$ embeds in M mapping $(1-P)M(1-P)$ to itself. By Proposition 2.6, submajorization in a C*-algebra is equivalent to majorization in the unitization of the stabilization of that C*-algebra. Hence $\beta(1-P) - a \prec_{\mathrm{u}} \beta(1-P) - b$ in M. So,

$$\beta - (\beta(1-P) - a) = \beta P + a$$

is majorized by $\beta P + b$ in M, as desired.

(ii) By (i) applied to $a' = (1-P) - a$ and $b' = (1-P) - b$ with $\beta = 1 - \alpha$, we get that

$$(1-\alpha)P + (1-P) - a \prec_{\mathrm{u}} (1-\alpha)P + (1-P) - b.$$

Hence

$$1 - ((1-\alpha)P + (1-P) - a) = a + \alpha P$$

is majorized by $b + \alpha P$, as desired. □

Proposition 3.15. *Let $r \in [0, \infty)$. Let $a, b \in M_+$ be contractions such that $(a - r)_+ \prec_{\mathrm{T}} b$ and $(1 - a - r)_+ \prec_{\mathrm{T}} 1 - b$. Then a is within a distance r of $\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(M)\}$.*

Proof. Let $a, b \in M_+$ be as in the statement of the theorem. Let $\varepsilon > 0$. Using the Borel functional calculus on a and b, let us find contractions with finite spectrum $a', b' \in M_+$ such that $\|a - a'\| < \varepsilon/2$ and $\|b - b'\| < \varepsilon/2$. Express them in the form

$$a' = \sum_{i=1}^{n} \alpha_i P_i, \quad b' = \sum_{i=1}^{n} \beta_i Q_i$$

as in (3.1) of Proposition 3.2. From $(a - r)_+ \prec_{\mathrm{T}} b$ and $(1 - a - r)_+ \prec_{\mathrm{T}} 1 - b$ we deduce that $(a' - r - \varepsilon)_+ \prec_{\mathrm{T}} b'$ and $(1 - a' - r - \varepsilon)_+ \prec_{\mathrm{T}} 1 - b'$. Having proven the theorem for a' and b' it is clear that, by letting $\varepsilon \to 0$, we deduce the theorem for a and b. So let us instead assume that a and b, as in the statement of the theorem, have finite spectra. Conjugating b by a unitary, we may also assume that $P_i = Q_i$ for all i. We assume further that the central coefficients $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ have finite spectra, which can be attained by a small enough approximation when moving from a, b to a', b', respectively.

Notice that if e is a central projection, then the hypothesis of the theorem holds for ea and eb in eM (by Lemma 3.1). On the other hand, if central projections $(e_j)_{j=1}^N$ partition the unit and we have proven the theorem for $e_j a$ and $e_j b$ in $e_j M$ for all j, then we conclude the same for a and b. This allows us to make the following reductions:

(1) each projection P_i is either finite or properly infinite for all i,
(2) the projections P_i are pairwise orthogonal, pairwise Murray–von Neumann comparable, and add up to 1,
(3) $c_{P_i} = 1$ for all i.

In the case that all the projections $P_1, \dots, P_n$, are finite, the unit 1 is finite, and so the desired conclusion follows from Proposition 3.10. Thus, we can make the following additional assumption:

(4) at least one of the projections P_i is properly infinite.

Recall that we have assumed that the central coefficients $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ have finite spectra. Passing to cut-downs of M by central projections that partition the unit, we can assume that these coefficients are scalars. Observe that the decreasing ordering of the coefficients $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ is maintained by doing this and that properties (1)–(4) above are not destroyed in the process. Thus, we further assume that

(5) the coefficients $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ are decreasing scalars.

By Lemma 3.11, $(a-r)_+ \prec_{\mathrm{T}} b$ implies the following conditions:

(a) $\tau(\sum_{i=1}^k (\alpha_i - r)_+ P_i) \leqslant \tau(\sum_{i=1}^k \beta_i P_i)$ for all $\tau \in \mathrm{T}(M)$ and $k = 1, \dots, n$.
(b) $\beta_1 \geqslant \alpha_1 - r$.
(c) If for some $k \geqslant 2$ we have that $\alpha_k - r > \beta_k$, then $P_k \propto P_{k'}$ for some $k' < k$. That is, P_k is Murray–von Neumann smaller than finitely many copies of some $P_{k'}$ with $k' < k$. (Indeed, by Lemma 3.11, $P_k \propto \bigoplus_{i<k} P_i$. But we have assumed that the projections P_i are pairwise Murray–von Neumann comparable. So $\bigoplus_{i<k} P_i \propto P_{k'}$ for some $k' < k$.)

Let us call the conditions stated above left-to-right conditions. One derives similar conditions from $(1-a-r)_+ \prec_{\mathrm{T}} 1-b$. They take the form

(a') $\tau(\sum_{i=k}^n (1-\alpha_i - r)_+ P_i) \leqslant \tau(\sum_{i=k}^n (1-\beta_i) P_i)$ for all $\tau \in \mathrm{T}(M)$ and $k = 1, \dots, n$.
(b') $\beta_n \leqslant \alpha_n + r$.
(c') If for some $k \leqslant n-1$ we have that $\alpha_k + r < \beta_k$, then P_k is Murray–von Neumann smaller than finitely many copies of some $P_{k'}$ with $k' > k$.

We'll call the above right-to-left conditions.

Let $k = 1, \ldots, n$ be the least index such that P_k is larger (in the Murray–von Neumann sense) than the other projections. By assumption P_k is also properly infinite. Notice that by conditions (b) and (c) we cannot have that $\alpha_k > \beta_k + r$. So either $|\alpha_k - \beta_k| \leqslant r$ or $\beta_k > \alpha_k + r$. We consider these two cases next:

Case $|\alpha_k - \beta_k| \leqslant r$. Let us write $P_k = P_k' + P_k''$, where $P_k \sim P_k' \sim P_k''$. Consider the pair of elements

$$a' = \sum_{i=1}^{k-1} \max(\alpha_i - r, \beta_k) P_i + \beta_k P_k', \quad b' = \sum_{i=1}^{k-1} \beta_i P_i + \beta_k P_k'$$

and the pair

$$a'' = \beta_k P_k'' + \sum_{i=k+1}^{n} \min(\alpha_i + r, \beta_k) P_i, \quad b'' = \beta_k P_k'' + \sum_{i=k+1}^{n} \beta_i P_i.$$

We claim that $a' \prec_{\mathrm{u}} b'$ in PMP, where $P = P_1 + \cdots + P_{k-1} + P_k'$, and that $a'' \prec_{\mathrm{u}} b''$ in $(1-P)M(1-P)$. Since $b = b' + b''$ and $\|a - (a' + a'')\| \leqslant r$, the desired result will follow from this claim.

Let us prove that $a' \prec_{\mathrm{u}} b'$ in PMP. If $k = 1$, this holds trivially, so assume that $k > 1$. Let $1 \leqslant i_0 \leqslant k-1$ be the largest index such that $\alpha_i - r \geqslant \beta_k$ and if there is no such index set $i_0 = 0$. From $(a-r)_+ \prec_{\mathrm{T}} b$ and Lemma 3.3 we get that

$$\sum_{i=1}^{i_0} (\alpha_i - r) P_i \prec_{\mathrm{T}} \sum_{i=1}^{i_0} \beta_i P_i.$$

Furthermore, by Proposition 3.13, the above relation is in fact a submajorization. On the other hand,

$$\sum_{i=i_0+1}^{k-1} \beta_k P_i \leqslant \sum_{i=i_0+1}^{k-1} \beta_i P_i.$$

Hence,

$$\sum_{i=1}^{k-1} \max(\alpha_i - r, \beta_k) P_i \prec_{\mathrm{c}} \sum_{i=1}^{k-1} \beta_i P_i.$$

That $a' \prec_{\mathrm{u}} b'$ now follows from Lemma 3.14(ii).

The proof that $a'' \prec_{\mathrm{u}} b''$ in $(1-P)M(1-P)$ is entirely analogous (recall that we have written $P = P_1 + \cdots + P_{k-1} + P_k'$): By Lemma 3.14(i), it suffices to check that

$$(1 - P - P_k'') - \sum_{i=k+1}^{n} \min(\alpha_i + r, \beta_k) P_i$$

is submajorized by

$$(1 - P - P_k'') - \sum_{i=k+1}^{n} \beta_i P_i.$$

To check this, let $i_0 \geqslant k+1$ be the largest index such that $\alpha_i + r \geqslant \beta_k$. Then

$$\sum_{i=k+1}^{i_0} (1 - \beta_k) P_i \leqslant \sum_{i=k+1}^{i_0} (1 - \beta_i) P_i.$$

On the other hand, from $(1-a-r)_+ \prec_{\mathrm{T}} 1-b$ and Lemma 3.3 we get that

$$\sum_{i=i_0+1}^{n} (1-\alpha_i - r)P_i \prec_{\mathrm{T}} \sum_{i=i_0+1}^{n} (1-\beta_i)P_i.$$

Moreover, by Proposition 3.13, this relation is of submajorization. Hence

$$\sum_{i=k+1}^{n} (1-\min(\alpha_i + r, \beta_k))P_i \prec_{\mathrm{c}} \sum_{i=k+1}^{n} (1-\beta_i)P_i,$$

as desired.

Case $\beta_k > \alpha_k + r$. By condition (c'), there must exist an index $k' > k$ such that $P_{k'}$ is also properly infinite and larger than every other projection. Let k' be the largest such index. Notice that we cannot have that $\beta_{k'} > \alpha_{k'} + r$ by conditions (b') and (c') from the right-to-left conditions. So we must have that either $|\alpha_{k'} - \beta_{k'}| \leqslant r$ or that $\alpha_{k'} > \beta_{k'} + r$. The first of these two cases has already been dealt with. So let us assume that $\alpha_{k'} > \beta_{k'} + r$.

We claim that b majorizes

$$a' = \sum_{i=1}^{k-1} (\alpha_i - r)_+ P_i + \sum_{i=k}^{k'} \alpha_i P_i + \sum_{i=k'+1}^{n} (\alpha_i + r)P_i.$$

Since a' is within a distance r of a, this is sufficient to complete the proof of this case. Let us prove our claim. Notice first that, as argued in the previous paragraphs, from Lemma 3.14(i) we obtain the majorization

$$\beta_k P_k + \sum_{i=k'+1}^{n} (\alpha_i + r)P_i \prec_{\mathrm{u}} \beta_k P_k + \sum_{i=k'+1}^{n} \beta_i P_i \tag{3.8}$$

in $(P_k + P_{k'+1} + \cdots + P_n)M(P_k + P_{k'+1} + \cdots + P_n)$. Similarly, from Lemma 3.14(ii) we obtain that

$$\sum_{i=1}^{k-1} (\alpha_i - r)P_i + \beta_{k'} P_{k'} \prec_{\mathrm{u}} \sum_{i=1}^{k-1} \beta_i P_i + \beta_{k'} P_{k'} \tag{3.9}$$

in $(P_1 + \cdots + P_{k-1} + P_{k'})M(P_1 + \cdots + P_{k-1} + P_{k'})$. We will be done once we have shown that

$$\sum_{i=k}^{k'} \alpha_i P_i \prec_{\mathrm{u}} \sum_{i=k}^{k'} \beta_i P_i$$

in $(P_k + \cdots + P_{k'})M(P_k + \cdots + P_{k'})$. Let us show this. We have

$$\sum_{i=k}^{k'-1} \alpha_i P_i \prec_{\mathrm{c}} \sum_{i=k}^{k'-1} \beta_i P_i$$

by Lemma 3.12. So

$$\sum_{i=k}^{k'-1} \alpha_i P_i + \beta_{k'} P_{k'} \prec_{\mathrm{u}} \sum_{i=k}^{k'} \beta_i P_i$$

by Lemma 3.14(ii). Repeating the same argument, symmetrically,

$$\sum_{i=k+1}^{k'} (1-\alpha_i)P_i \prec_{\mathrm{c}} \sum_{i=k+1}^{k'-1} (1-\alpha_i)P_i + (1-\beta_{k'})P_{k'}$$

by Lemma 3.12. So

$$\sum_{i=k}^{k'} \alpha_i P_i \prec_{\mathrm{u}} \sum_{i=k}^{k'-1} \alpha_i P_i + \beta_{k'} P_{k'}$$

by Lemma 3.14(i).

□

4. Majorization and submajorization in C*-algebras

Proposition 4.1. *Let A be a C*-algebra. Let $a, b \in A$.*

(i) *The distance from a to* $\mathrm{co}\{dbd^* \mid d \in A^{**},\ \|d\| \leqslant 1\}$ *is equal to the distance from a to* $\mathrm{co}\{dbd^* \mid d \in A,\ \|d\| \leqslant 1\}$.

(ii) *Suppose A is unital. Then the distance from a to* $\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A^{**})\}$ *is equal to the distance from a to* $\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}$.

Proof.

(i) It is clear that the distance from a to $\mathrm{co}\{dbd^* \mid d \in A,\ \|d\| \leqslant 1\}$ is greater than or equal to the distance from a to $\mathrm{co}\{dbd^* \mid d \in A^{**},\ \|d\| \leqslant 1\}$. Denote the latter distance by r. Let $\varepsilon > 0$. Suppose that

$$\Big\| a - \frac{1}{n}\sum_{i=1}^{n} d_i b d_i^* \Big\| < r + \varepsilon$$

for some contractions $d_1, \ldots, d_n \in A^{**}$. For each $i = 1, \ldots, n$ let us find a net of contractions $(d_{i,\lambda})_\lambda$ in A such that $d_{i,\lambda} \to d_i$ in the ultrastrong* topology. Such a net exists by Kaplansky's density theorem. Then the ultrastrong* closure of the set

$$\Big\{ a - \frac{1}{n}\sum_{i=1}^{n} d_{i,\lambda} b d_{i,\lambda}^* \mid \lambda \Big\}$$

intersects the ball $B_{r+\varepsilon}(0)$. By the Hahn–Banach theorem, the convex hull of this set also intersects that ball. A convex combination of elements of this set again has the form $a - a'$ with a' a convex combination of elements of the form dbd^* with $d \in A$ a contraction.

(ii) It is clear that the distance from a to $\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}$ is greater than or equal to the distance from a to $\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A^{**})\}$. Denote the latter distance by r. Let $\varepsilon > 0$. Suppose that

$$\Big\| a - \frac{1}{n}\sum_{i=1}^{n} u_i b u_i^* \Big\| < r + \varepsilon$$

for some unitaries $u_i \in A^{**}$. By Kaplansky's density theorem for unitaries, there exist nets of unitaries $(u_{i,\lambda})_\lambda$ in A converging to u_i in the ultrastrong* topology. Then the ultrastrong* closure of the set

$$\Big\{ a - \frac{1}{n}\sum_{i=1}^{n} u_{i,\lambda} b u_{i,\lambda}^* \mid \lambda \Big\}$$

intersects the ball $B_{r+\varepsilon}(0)$. This implies that the convex hull of this set also intersects that ball. But a convex combination of elements of this set again has the form $a - a'$ with a' a convex combination of elements of the form ubu^* with $u \in \mathrm{U}(A)$. □

Theorem 4.2. *Let A be a C*-algebra. Let $a, b \in A_{\mathrm{sa}}$. The distance from a to the set* $\mathrm{co}\{dbd^* \mid d \in A,\ \|d\| \leqslant 1\}$ *is equal to the infimum $r \in [0,\infty)$ such that*

$$\tau((a-r-t)_+) \leqslant \tau((b-t)_+) \text{ for all } t \in [0,\infty) \text{ and all } \tau \in \mathrm{T}(A), \tag{4.1}$$

$$\tau((-a-r-t)_+) \leqslant \tau((-b-t)_+) \text{ for all } t \in [0,\infty) \text{ and all } \tau \in \mathrm{T}(A). \tag{4.2}$$

Moreover, if r is such infimum, then $(a-r)_+ - (a+r)_- \prec_{\mathrm{c}} b$.

Proof. Let $\tilde{r} \in (0,\infty)$ be such that $\|a-b'\| < \tilde{r}$ for some $b' \prec_{\mathrm{c}} b$. Then $(a-\tilde{r})_+ \prec_{\mathrm{c}} b'_+$ by Lemma 2.2(ii). Also, $b' \prec_{\mathrm{c}} b$ implies that $(b')_+ \prec_{\mathrm{c}} b_+$ by Proposition 2.3. Hence $(a-\tilde{r})_+ \prec_{\mathrm{c}} b_+$. Starting from $\|(-a)-(-b')\| < \tilde{r}$ and following the same line of reasoning we obtain that $(a+\tilde{r})_- \prec_{\mathrm{c}} b_-$. Since submajorization implies tracial submajorization (Proposition 2.8), $(a-\tilde{r})_+ \prec_{\mathrm{T}} b_+$ and $(a+\tilde{r})_- \prec_{\mathrm{T}} b_-$. These relations translate at once into (4.1) and (4.2) (for the number $\tilde{r}$).

Assume now that (4.1)–(4.2) hold for some $r \in [0,\infty)$. Let us show that $(a-r)_+ - (a+r)_- \prec_{\mathrm{c}} b$. Since the distance from a to $(a-r)_+ - (a+r)_-$ is r, this will complete the proof of the theorem. As remarked above, (4.1)–(4.2) can be restated as saying that $(a-r)_+ \prec_{\mathrm{T}} b_+$ and $(a+r)_- \prec_{\mathrm{T}} b_-$. In view of Proposition 2.3, it remains to show that $(a-r)_+ \prec_{\mathrm{c}} b_+$ and $(a+r)_- \prec_{\mathrm{c}} b_-$. This boils down to showing that if $c, d \in A_+$ are such that $c \prec_{\mathrm{T}} d$, then $c \prec_{\mathrm{c}} d$. Let us prove this. It is clear from $c \prec_{\mathrm{T}} d$ in A that $c \prec_{\mathrm{T}} d$ in the von Neumann algebra A^{**} (indeed, in any C*-algebra containing A), since traces in $\mathrm{T}(A^{**})$ restrict to traces in $\mathrm{T}(A)$. Then, by Proposition 3.13, $c \prec_{\mathrm{c}} d$ in A^{**}. Finally, by Proposition 4.1(i), $c \prec_{\mathrm{c}} d$ in A, as desired. ☐

Theorem 4.3. *Let A be a unital C*-algebra. Let $a, b \in A$ be selfadjoint elements. Then the distance from a to* $\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}$ *is equal to the infimum $r \in [0,\infty)$ such that*

$$\tau((a-r-t)_+) \leqslant \tau((b-t)_+) \text{ for all } t \in \mathbb{R} \text{ and all } \tau \in \mathrm{T}(A), \tag{4.3}$$

$$\tau((-a-r-t)_+ \leqslant \tau((-b-t)_+) \text{ for all } t \in \mathbb{R} \text{ and all } \tau \in \mathrm{T}(A). \tag{4.4}$$

Proof. If we replace a by $a + s\cdot 1$ and b by $b + s \cdot 1$ for some $s \in \mathbb{R}$, then neither the infimum r satisfying (4.3)–(4.4) nor the distance from a to $\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}$ is changed. Thus, by choosing a sufficiently large s we may assume that a and b are positive. A simple calculation also shows that if we replace a by a/s' and b by b/s' for some $s' \in (0,\infty)$, then both the infimum r satisfying (4.3)–(4.4) and the distance from a to $\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}$ get multiplied by a factor of $1/s'$. Thus, by choosing a sufficiently large s' we may assume that a and b are positive contractions. We do so henceforth.

Let $r \in (0,\infty)$ be any number satisfying (4.3)–(4.4). From (4.3) we deduce that $(a-r)_+ \prec_{\mathrm{T}} b$, while from (4.4) we deduce that $(1-a-r)_+ \prec_{\mathrm{T}} 1-b$. Thus, by Proposition 3.15, a is within a distance r of $\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A^{**})\}$. Then, by Proposition 4.1(ii), a is within a distance r of $\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}$. This proves one inequality.

Let $\tilde{r} \in (0,\infty)$ be any number such that $\|a-b'\| < \tilde{r}$ for some $b' \in \mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}$. By Lemma 2.2(ii), $(a-\tilde{r})_+ \prec_{\mathrm{c}} b' \prec_{\mathrm{c}} b$. Since submajorization implies tracial submajorization (for positive elements) we have that $(a-\tilde{r})_+ \prec_{\mathrm{T}} b$. That is,

$$\tau((a-\tilde{r}-t)_+) \leqslant \tau((b-t)_+)$$

for all $t \in [0,\infty)$ and all $\tau \in \mathrm{T}(A)$. Using that b is positive we can extend this inequality to all $t < 0$. Indeed, if $t < 0$, then $(a - \tilde{r} - t)_+ \leqslant (a - \tilde{r})_+ - t$, and so

$$\begin{aligned}\tau((a-\tilde{r}-t)_+) &\leqslant \tau((a-\tilde{r})_+) + \tau(-t)\\ &\leqslant \tau(b) + \tau(-t)\\ &= \tau((b-t)_+),\end{aligned}$$

for all $\tau \in \mathrm{T}(A)$. Thus, (4.3) holds for $\tilde{r}$. Applying the same arguments starting from $\|(1-a)-(1-b')\| < \tilde{r}$ we deduce that $\tau((1-a-\tilde{r}-t)_+ \leqslant \tau((1-b-t)_+)$ for all $\tau \in \mathrm{T}(A)$ and all $t \in \mathbb{R}$. This is equivalent to (4.4). ☐

Proof of Theorem 1.1. This is the case $r = 0$ of Theorem 4.3. ☐

Remark 4.4. The following observation, whose verification is left to the reader, will be useful below: if $a, b \in A$ are positive contractions the tracial inequalities in Theorem 1.1(ii) are equivalent to the tracial submajorizations $a \prec_{\mathrm{T}} b$ and $1-a \prec_{\mathrm{T}} 1-b$.

Let us explore some consequences of Theorem 1.1.

The following simple properties of the relation $\prec_{\mathrm{u}}$ follow from Theorem 1.1 (although they do not seem to follow directly from the definition of $\prec_{\mathrm{u}}$).

Corollary 4.5. *Let $a, b, c, d \in A_{\mathrm{sa}}$. Let $k \in \mathbb{N}$.*

(i) *If $a \otimes 1_k \prec_{\mathrm{u}} b \otimes 1_k$ in $M_k(A)$ then $a \prec_{\mathrm{u}} b$ in A.*
(ii) *If $a \leqslant b \leqslant c$, $a \prec_{\mathrm{u}} d$ and $c \prec_{\mathrm{u}} d$, then $b \prec_{\mathrm{u}} d$.*
(iii) *If $b \leqslant c \leqslant d$, $a \prec_{\mathrm{u}} b$ and $a \prec_{\mathrm{u}} d$, then $a \prec_{\mathrm{u}} c$.*

Proof.

(i) Let $t \in \mathbb{R}$ and $\tau \in \mathrm{T}(A)$. Then τ extends to a lower semicontinuous trace on $M_k(A)$ by setting $\tau((a_{i,j})) = \sum_{i=1}^k \tau(a_{i,i})$. We have

$$\tau((a-t)_+) = \frac{1}{k}\tau((a\otimes 1_k - t)_+) \leqslant \frac{1}{k}\tau((b\otimes 1_k - t)_+) = \tau((b-t)_+).$$

We deduce similarly that $\tau((-a-t)_+) \leqslant \tau((-b-t)_+)$. By Theorem 1.1, $a \prec_{\mathrm{u}} b$.

(ii) Let $t \in \mathbb{R}$ and $\tau \in \mathrm{T}(A)$. We have $(b+t)_+ \prec_{\mathrm{c}} (c+t)_+$ by Lemma 2.2(i). Hence, $\tau((b+t)_+) \leqslant \tau((c+t)_+)$, and since $c \prec_{\mathrm{u}} d$, $\tau((b+t)_+) \leqslant \tau((d+t)_+)$. Exploiting that $a \leqslant b$ and $a \prec_{\mathrm{u}} d$ we deduce similarly that $\tau((-b+t)_+) \leqslant \tau((-d+t)_+)$.

(iii) The proof is very similar to (ii). ☐

Corollary 4.6. *Let A be a simple unital C*-algebra. Let $a, b \in A_{\mathrm{sa}}$.*

(i) *If A has at least one non-zero bounded trace, then $a \prec_{\mathrm{u}} b$ if and only if $\tau(a) = \tau(b)$ and $\tau((a-t)_+) \leqslant \tau((b-t)_+)$ for all $t \in \mathbb{R}$ and all bounded traces τ on A.*
(ii) *If A has no bounded traces, then $a \prec_{\mathrm{u}} b$ if and only if* $\mathrm{sp}(a) \subseteq \mathrm{co}(\mathrm{sp}(b))$.

Proof. The implications starting with $a \prec_{\mathrm{u}} b$ in both (i) and (ii) are straightforward from Theorem 1.1. To prove the converse we will show in both cases that the tracial inequalities from Theorem 1.1 hold.

(i) Let us suppose that A has at least one nonzero bounded trace. Since A is simple and unital, $\mathrm{T}(A)$ consists of the bounded traces on A and the trace $\tau_\infty(a) := \infty$ for all $a \neq 0$ and $\tau_\infty(0) := 0$. We have assumed that $\tau((a-t)_+) \leqslant \tau((b-t)_+)$ for all bounded traces and all $t \in \mathbb{R}$. Let us show that

$$\tau_\infty((a-t)_+) \leqslant \tau_\infty((b-t)_+) \tag{4.5}$$

for all $t \in \mathbb{R}$. It suffices to show that the left side is zero for all $t \geqslant \|b_+\|$. We have that

$$\tau((a-\|b_+\|)_+) \leqslant \tau((b-\|b_+\|)_+) = 0$$

for all bounded traces τ. Since A has at least one nonzero bounded trace—which is necessarily faithful because A is simple—we get that $(a-\|b_+\|)_+ = 0$. This implies (4.5).

Let us prove that $\tau((-a-t)_+) \leqslant \tau((-b-t)_+)$ for all $\tau \in \mathrm{T}(A)$ and all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$. Let τ be a bounded trace (which we assume defined on all A). Observe that

$$(-c-t)_+ = (c+t)_+ - (c+t)$$

for any selfadjoint element c. Thus, as $\tau(a) = \tau(b)$,

$$\begin{aligned}\tau((-a-t)_+) &= \tau((a+t)_+) - \tau(a+t)\\ &\leqslant \tau((b+t)_+) - \tau(b+t) = \tau((-b-t)_+).\end{aligned}$$

To get that $\tau_\infty((-a-t)_+) \leqslant \tau_\infty((-b-t)_+)$ we proceed as in the previous paragraph. Exploiting the existence of a nonzero (faithful) bounded trace we deduce that $(a+\|b_-\|)_- = 0$ (since $(b+\|b_-\|)_- = 0$), from which the desired inequality readily follows.

(ii) Suppose that A has no nonzero bounded traces. Then $\mathrm{T}(A)$ consists only of τ_∞ and the zero trace. Since $\mathrm{sp}(a) \subseteq \mathrm{co}(\mathrm{sb}(b))$, we have that $\|a_+\| \leqslant \|b_+\|$ and $\|a_-\| \leqslant \|b_-\|$. It is readily verified from this that $\tau_\infty((a-t)_+) \leqslant \tau_\infty((b-t)_+)$ and $\tau_\infty((-a-t)_+) \leqslant \tau_\infty((-b-t)_+)$ for all $t \in \mathbb{R}$, as desired. □

Theorem 4.7. *Let A be a unital C*-algebra. Let a be a selfadjoint element in A. Then $0 \in \overline{\mathrm{co}\{uau^* \mid u \in \mathrm{U}(A)\}}$ if and only if*

- (a) *$\tau(a) = 0$ for all bounded traces τ on A and*
- (b) *in no nonzero quotient of A can the image of a be either invertible and positive or invertible and negative.*

Proof. The necessity of the conditions is relatively straightforward. Since all the elements in the set $\overline{\mathrm{co}\{uau^* \mid u \in \mathrm{U}(A)\}}$ agree on bounded traces, we have (a). If $a \geqslant \alpha\cdot 1$ for some $\alpha \in (0,\infty)$, then the same holds for all elements in $\overline{\mathrm{co}\{uau^* \mid u \in \mathrm{U}(A)\}}$, which prevents 0 from belonging to this set. Similarly, we cannot have that $a \leqslant -\alpha 1$. Moreover, if 0 is in the closure of the convex hull of the unitary conjugates of a, the same holds for the image of a on any quotient. So we have (b).

Suppose now that (a) and (b) hold. To prove the theorem we use Theorem 1.1. We must check that $\tau((0-t)_+) \leqslant \tau((a-t)_+)$ for all $t \in \mathbb{R}$ and all $\tau \in \mathrm{T}(A)$. This boils down to showing that $\tau(t) \leqslant \tau((a+t)_+)$ for all $t > 0$ and all $\tau \in \mathrm{T}(A)$. Let $t > 0$. Suppose first that τ is a bounded trace (so assume that it is defined on all A). Evaluating τ on $(a+t)_+ \geqslant a+t$ we get $\tau((a+t)_+) \geqslant \tau(t)$, as desired. Suppose now that τ is unbounded. Since $\tau(t) = \infty$ we must show that $\tau((a+t)_+) = \infty$.

Equivalently, we must show that $(a+t)_+$ is full, i.e., it generates A as a closed two-sided ideal. But if this were not the case, then in the quotient by the closed two-sided ideal generated by $(a+t)_+$ we would have that $\overline{a}+t \leqslant 0$ (where $\overline{a}$ denotes the image of a in this quotient). This contradicts (2). Thus, $(a+t)_+$ is full, as desired. Since $-a$ satisfies (1) and (2) too, we also arrive at $\tau((-0-t)_+) \leqslant \tau((-a-t)_+)$ for all $t \in \mathbb{R}$ and all $\tau \in \mathrm{T}(A)$. By Theorem 1.1, $0 \prec_{\mathrm{u}} a$, as desired. ☐

5. Uniform majorization

In this section we discuss the majorization relation in the context of regularity properties of C*-algebras. We show that one has a uniform version of majorization holding across all C*-algebras of either one of the following classes:

(1) C*-algebras satisfying Blackadar's strict comparison of positive elements by traces,
(2) C*-algebras having a uniform bound on their nuclear dimension.

In both cases we derive the uniform majorization from the preservation of the relation of tracial submajorization under products of C*-algebras in the given class (Propositions 5.1 and 5.5).

Let us recall some definitions. Let A be a C*-algebra. Let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable infinite-dimensional Hilbert space. Let $\tau \in \mathrm{T}(A)$. We can extend τ to a trace on $(A \otimes \mathcal{K})_+$ by setting

$$\tau((a_{i,j})) = \sum_{i=1}^{\infty} \tau(a_{i,i})$$

for all $(a_{i,j})_{i,j} \in (A \otimes \mathcal{K})_+$. From τ we obtain a "dimension function" $d_\tau\colon (A \otimes \mathcal{K})_+ \to [0, \infty]$ defined as

$$d_\tau(a) = \lim_n \tau(a^{\frac{1}{n}})$$

for all $a \in (A \otimes \mathcal{K})_+$. (Alternatively, $d_\tau(a)$ is the norm of the restriction of τ to $\overline{a(A \otimes \mathcal{K})a}$.)

Next, let us recall the definition of the Cuntz comparison relation among positive elements: Given positive elements $a, b \in A \otimes \mathcal{K}$, a is said to be Cuntz subequivalent to b if there exist $e_1, e_2, \ldots \in A \otimes \mathcal{K}$ such that $e_n b e_n^* \to a$. We denote this relation by $a \precsim_{\mathrm{Cu}} b$.

The C*-algebra A is said to have the property of strict comparison of positive elements by traces if for all $a, b \in (A \otimes \mathcal{K})_+$ and $\varepsilon > 0$ we have that

$$d_\tau(a) \leqslant (1-\varepsilon) d_\tau(b) \text{ for all } \tau \in \mathrm{T}(A) \text{ implies that } a \precsim_{\mathrm{Cu}} b. \tag{5.1}$$

(Note: A number of different variations on "strict comparison" exist in the literature; e.g., one may restrict τ to be a bounded trace or allow it to be a 2-quasitrace; one may restrict a, b to be in A, etc.)

We will make use of the topology on $\mathrm{T}(A)$ introduced in [ERS11]. Let us recall it here: a net $(\tau_\lambda)_\lambda$ in $\mathrm{T}(A)$ converges to τ if for all $a \in A_+$ and $\varepsilon > 0$ we have

$$\limsup_\lambda \tau_\lambda((a-\varepsilon)_+) \leqslant \tau(a) \leqslant \liminf_\lambda \tau_\lambda(a).$$

It is shown in [ERS11, Theorem 3.7] that $\mathrm{T}(A)$ is compact and Hausdorff under this topology.

The following variation on the strict comparison property has been introduced in [NR16]: Let $K \subseteq \mathrm{T}(A)$ be a compact set. Then A is said to have strict comparison of positive elements by traces in K if for all $a, b \in (A \otimes \mathcal{K})_+$ and $\varepsilon > 0$ it suffices to let τ range through K in (5.1) for this implication to hold.

The following proposition is essentially obtained in [NR16]:

Proposition 5.1. *Let $A_1, A_2, \dots$ be C*-algebras with strict comparison of positive elements by traces. Let $a = (a_n)_{n=1}^\infty$ and $b = (b_n)_{n=1}^\infty$ be positive elements in $\prod_{n=1}^\infty A_n$ such that $a_n \prec_{\mathrm{T}} b_n$ for all n. Then $a \prec_{\mathrm{T}} b$ in $\prod_{n=1}^\infty A_n$.*

Proof. Let us regard $\mathrm{T}(A_n)$ embedded in $\mathrm{T}(\prod_{n=1}^\infty A_n)$ via the map induced by the projection from $\prod_{n=1}^\infty A_n$ onto A_n. Let $K = \overline{\bigcup_{n=1}^\infty \mathrm{T}(A_n)} \subseteq \mathrm{T}(\prod_{n=1}^\infty A_n)$. In the course of the proof of [NR16, Theorem 4.1] it is shown that the C*-algebra $\prod_{n=1}^\infty A_n$ has strict comparison of positive elements by traces in K. The elements a and b from the statement of the theorem satisfy that $\tau((a-t)_+) \leqslant \tau((b-t)_+)$ for all $\tau \in \bigcup_{n=1}^\infty \mathrm{T}(A_n)$ and $t \geqslant 0$ (this holds by assumption). Let us show that these inequalities extend to all traces in K. Let $\tau \in K$ and choose a net $\tau_\lambda \to \tau$ with $\tau_\lambda \in \bigcup_{n=1}^\infty \mathrm{T}(A_n)$. From the definition of the topology in $\mathrm{T}(A)$ we get that

$$\begin{aligned}\tau((a-t-\varepsilon)_+) &\leqslant \liminf \tau_\lambda((a-t-\varepsilon)_+)\\ &\leqslant \liminf \tau_\lambda((b-t-\varepsilon)_+)\\ &\leqslant \tau((b-t)_+)\end{aligned}$$

for all $t \geqslant 0$ and $\varepsilon > 0$. Thus, $\tau((a-t-\varepsilon)_+) \leqslant \tau((b-t)_+)$. Letting $\varepsilon \to 0$ and using the lower semicontinuity of τ we get that $\tau((a-t)_+) \leqslant \tau((b-t)_+)$ for all $\tau \in K$ and all $t \geqslant 0$. Now, [NR16, Lemma 3.4] asserts that if a C*-algebra A has strict comparison by traces in a compact set K, then for any given $c, d \in A_+$, if $\tau(c) \leqslant \tau(d)$ for all $\tau \in K$, then $\tau(c) \leqslant \tau(d)$ for all $\tau \in \mathrm{T}(A)$. Applied in $A = \prod_{n=1}^\infty A_n$ with K as above, this lemma implies that $a \prec_{\mathrm{T}} b$, as desired. □

Proof of Theorem 1.2. Let $\varepsilon > 0$. Suppose for the sake of contradiction that no N as in the statement of the theorem exists. Then there exist unital C*-algebras $A_1, A_2, \dots$ with strict comparison by traces and selfadjoint contractions $a_n, b_n \in A_n$ such that $a_n \prec_{\mathrm{u}} b_n$ for all n but

$$\|a_n - \frac{1}{n}\sum_{i=1}^n u_i b_n u_i^*\| \geqslant \varepsilon$$

for all n-tuples of unitaries $u_1, \dots, u_n \in A_n$. Let $a_n' = \frac{a_n+1}{2}$ and $b_n' = \frac{b_n+1}{2}$. Observe that these are positive contractions such that $a_n' \prec_{\mathrm{u}} b_n'$ for all n and

$$\|a_n' - \frac{1}{n}\sum_{i=1}^n u_i b_n' u_i^*\| \geqslant \frac{\varepsilon}{2}$$

for all n-tuples of unitaries $u_1, \dots, u_n \in A_n$. Consider the positive elements $a = (a_n')_{n=1}^\infty$ and $b = (b_n')_{n=1}^\infty$ in $\prod_{n=1}^\infty A_n$. Since $a_n' \prec_{\mathrm{u}} b_n'$ for all n we have, by Proposition 5.1, that $a \prec_{\mathrm{T}} b$. Also, $1 - a_n' \prec_{\mathrm{T}} 1 - b_n'$ for all n, and so $1 - a \prec_{\mathrm{T}} 1 - b$. By Theorem 1.1 (keeping Remark 4.4 in mind), we have that $a \prec_{\mathrm{u}} b$. Hence, there exists $N \in \mathbb{N}$ and unitaries $w_1, w_2, \dots, w_N \in \prod_{n=1}^\infty A_n$ such that

$$\|a - \frac{1}{N}\sum_{i=1}^N w_i b w_i^*\| < \frac{\varepsilon}{2}.$$

Projecting onto A_N we arrive at a contradiction. □

Theorem 5.2. *For each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if A is a C*-algebra with strict comparison of positive elements by traces and $a, b \in A_{\mathrm{sa}}$ are contractions such that $a \prec_{\mathrm{c}} b$, then*

$$\|a - \frac{1}{N}\sum_{i=1}^{N} d_i b d_i^*\| < \varepsilon$$

for some contractions $d_1, \ldots, d_N \in A$.

Proof. It is easy to argue, using Proposition 2.3, that it suffices to prove the theorem letting a and b range through all positive contractions. One can then proceed as in the proof of Theorem 1.2, arguing by contradiction and relying on Proposition 5.1. The details are left to the reader. □

Next we prove the same uniform majorization among C*-algebras with a uniform bound in their nuclear dimension. We start by recalling the definition of nuclear dimension and some background facts.

A completely positive contractive (c.p.c.) map $\phi\colon A \to B$ is called of order zero if it preserves orthogonality, i.e., $ab = 0$ implies $\phi(a)\phi(b) = 0$ for all $a, b \in A$. By [WZ09, Theorem 2.3], such a map has the form $\phi(a) = h\pi_\phi(a)$, where $\pi_\phi\colon A \to \mathrm{M}(C^*(\phi(A))$ is a homomorphism and where $h \in \mathrm{M}(C^*(\phi(A)))$ is a positive element commuting with $\pi_\phi(A)$. (Here $\mathrm{M}(C^*(\phi(A)))$ denotes the multiplier algebra of the C*-algebra generated by $\phi(A)$.) With the aid of this theorem one can easily deduce the preservation of various relations under c.p.c. order zero maps. For example, if $a = x^*x$ and $b = xx^*$ for some $x \in A$, then $\phi(a) = y^*y$ and $\phi(b) = yy^*$ for some $y \in B$ (we can choose $y = h^{1/2}\pi_\phi(x)$). The submajorization relation is also preserved under c.p.c. order zero maps. For if $a, b \in A_{\mathrm{sa}}$ are such that $a = \frac{1}{N}\sum_{i=1}^{N} d_i b d_i^*$ for some contractions $d_i \in A$, then $\phi(a) = \frac{1}{N}\sum_{i=1}^{N}\pi_\phi(d_i)\phi(b)\pi_\phi(d_i)^*$. Although the contractions $\pi_\phi(d_i)$ belong to $M(C^*(\phi(A)))$ rather than B, by Lemma 2.5(i) we still have that $\phi(a) \prec_{\mathrm{c}} \phi(b)$ in B. If, more generally, $a \prec_{\mathrm{c}} b$ in A, then an argument passing to limits readily proves that $\phi(a) \prec_{\mathrm{c}} \phi(b)$ in B.

Let $m \in \mathbb{N}$. Following Winter and Zacharias [WZ10] we say that a C*-algebra A has nuclear dimension at most m if for each finite set $F \subset A$ and $\varepsilon > 0$ there exist c.p.c. maps $A \xrightarrow{\psi_k} C_k \xrightarrow{\phi_k} A$ with $k = 0, 1, \ldots, m$ such that C_k is a finite-dimensional C*-algebra for all k, ϕ_k is an order zero map for all k, and

$$\|a - \sum_{k=0}^{m}\phi_k\psi_k(a)\| < \varepsilon \quad \text{for all } a \in F.$$

In [WZ10, Proposition 3.2], Winter and Zacharias show that it is possible to arrange for the maps ψ_k to be asymptotically of order zero. In this way one obtains c.p.c. order zero maps

$$A \xrightarrow{\overline{\psi}_k} N_k \xrightarrow{\overline{\phi}_k} A_\infty$$

for $k = 0, \ldots, m$ such that

$$\iota = \sum_{k=0}^{m}\overline{\phi}_k\overline{\psi}_k.$$

Here $A_\infty = (\prod_\lambda A_\lambda)/(\bigoplus_\lambda A_\lambda)$ is a sequence algebra over some upward directed set Λ, $\iota\colon A \to A_\infty$ denotes the canonical embedding of A in A_∞ as "constant sequences", and $N_k = (\prod_\lambda C_{k,\lambda})/(\bigoplus_\lambda C_{k,\lambda})$, where $C_{k,\lambda}$ is a finite-dimensional C*-algebra for all $\lambda \in \Lambda$ and all $k = 0, \ldots, m$.

Lemma 5.3. *Each C*-algebra N_k as defined above has the property of strict comparison of positive elements by traces.*

Proof. This is a consequence of N_k being the quotient of a product of finite-dimensional C*-algebras. More specifically, as remarked in the proof of Proposition 5.1, a product of C*-algebras with strict comparison by traces again has strict comparison by traces (in fact, by traces ranging in a suitable compact set K). Since each $C_{k,\lambda}$ is finite dimensional it has strict comparison by traces. Thus, the same holds for $\prod_\lambda C_{k,\lambda}$. Also, the property of strict comparison by traces also passes to quotients. Indeed, by [NR16, Proposition 3.6 (i)], strict comparison by traces is equivalent to "strict comparison by 2-quasitraces and 2-quasitraces are traces". It is clear that if all the lower semicontinuous 2-quasitraces of a C*-algebra are traces, the same holds for its quotients. Strict comparison by 2-quasitraces also passes to quotients since, by [ERS11, Proposition 6.2], it is equivalent to almost unperforation in the Cuntz semigroup and the latter passes to quotients by [RT17, Proposition 2.2] (it is called 0-comparison in this reference). □

Lemma 5.4. *For each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if A is a C*-algebra of nuclear dimension at most m and $a, b \in A_+$ are such that $a \prec_{\mathrm{T}} b$, then*

$$(a-\varepsilon)_+ = \sum_{i=1}^{N(m+1)} x_i^* x_i \text{ and } \sum_{i=1}^{N(m+1)} x_i x_i^* \leqslant b$$

for some $x_1, \dots, x_{N(m+1)} \in A$.

Proof. Let $A \xrightarrow{\psi_k} N_k \xrightarrow{\phi_k} A_\infty$, for $k = 0, \dots, m$, be c.p.c. order zero maps as in the discussion above. Fix $k = 0, \dots, m$. We have remarked above that c.p.c. order zero maps preserve the submajorization relation (which, by Theorem 4.2, agrees with $\prec_{\mathrm{T}}$). Hence, $a \prec_{\mathrm{T}} b$ implies that $\psi_k(a) \prec_{\mathrm{T}} \psi_k(b)$ in N_k. By Lemma 5.3 the C*-algebra algebra N_k has the property of strict comparison of positive elements. Hence, by Theorem 5.2, there exists a number $N \in \mathbb{N}$ and elements $d_{k,1}, \dots, d_{k,N} \in N_k$ such that

$$\Big\|\psi_k(a) - \sum_{i=1}^{N} d_{k,i}^* \psi_k(b) d_{k,i}\Big\| < \frac{\varepsilon}{m+1} \text{ and } \sum_{i=1}^{N} d_{k,i} d_{k,i}^* \leqslant 1.$$

The number N depends only on ε and m. If we set $y_{k,i} = (\psi_k(b))^{\frac{1}{2}} d_{k,i}$ then we can rewrite these inequalities as

$$\Big\|\psi_k(a) - \sum_{i=1}^{N} y_{k,i}^* y_{k,i}\Big\| < \frac{\varepsilon}{m+1} \text{ and } \sum_{i=1}^{N} y_{k,i} y_{k,i}^* \leqslant \psi_k(b).$$

Applying ϕ_k on both inequalities and using that it is an order zero map we deduce that

$$\Big\|\phi_k\psi_k(a) - \sum_{i=1}^{N} \tilde{y}_{k,i}^* \tilde{y}_{k,i}\Big\| < \frac{\varepsilon}{m+1} \text{ and } \sum_{i=1}^{N} \tilde{y}_{k,i} \tilde{y}_{k,i}^* \leqslant \phi_k\psi_k(b)$$

for some $\tilde{y}_{k,i} \in A_\infty$. Adding over all k we get that

$$\Big\|a - \sum_{k=0}^{m}\sum_{i=1}^{N} \tilde{y}_{k,i}^* \tilde{y}_{k,i}\Big\| < \varepsilon \text{ and } \sum_{k=0}^{m}\sum_{i=1}^{N} \tilde{y}_{k,i} \tilde{y}_{k,i}^* \leqslant b.$$

We can lift the elements $\tilde{y}_{k,i}$ to $\prod_\lambda A$ so that these inequalities are preserved. Then from those lifts we find elements $x_{k,i} \in A$ such that the same inequalities hold in A; namely,

$$\Big\|a - \sum_{k=0}^{m}\sum_{i=1}^{N} x_{k,i}^* x_{k,i}\Big\| < \varepsilon \text{ and } \sum_{k=0}^{m}\sum_{i=1}^{N} x_{k,i}x_{k,i}^* \leqslant b.$$

By a well-known lemma of Kirchberg and Rørdam, if $\|a - a'\| < \varepsilon$, then $(a-\varepsilon)_+ = da'd^*$ for some contraction $d \in A$ ([KR02, Lemma 2.2]). Applying this lemma with $a' = \sum_{k=0}^{m}\sum_{i=1}^{N} x_{k,i}^* x_{k,i}$ we can turn the inequalities above into the relations claimed by the lemma. □

Proposition 5.5. *Let $A_1, A_2, \dots$ be a sequence of C*-algebras with uniformly bounded nuclear dimensions. Let $a = (a_n)_{n=1}^\infty$ and $b = (b_n)_{n=1}^\infty$ be positive elements in $\prod_{n=1}^\infty A_n$ such that $a_n \prec_{\mathrm{T}} b_n$ for all n. Then $a \prec_{\mathrm{T}} b$ in $\prod_{n=1}^\infty A_n$*

Proof. It suffices to show that $\tau(a) \leqslant \tau(b)$ for all $\tau \in \mathrm{T}(\prod_{n=1}^\infty A_n)$, for then the same argument applied to $(a-t)_+$ and $(b-t)_+$ in place of a and b gives us that $\tau((a-t)_+) \leqslant \tau((b-t)_+)$ for all τ. Let $\varepsilon > 0$. From the previous lemma we deduce that for each n there exist $x_{1,n}, \dots, x_{N(m+1),n} \in A_n$ such that

$$(a_n - \varepsilon)_+ = \sum_{i=1}^{N(m+1)} x_{i,n}^* x_{i,n} \text{ and } \sum_{i=1}^{N(m+1)} x_{i,n}x_{i,n}^* \leqslant b_n.$$

The sequences $(x_{i,n})_n$ are necessarily bounded. So if we set $x_i = (x_{i,n})_n \in \prod_{n=1}^\infty A_n$, then

$$(a-\varepsilon)_+ = \sum_{i=1}^{N(m+1)} x_i^* x_i \text{ and } \sum_{i=1}^{N(m+1)} x_i x_i^* \leqslant b.$$

This implies that $\tau((a-\varepsilon)_+) \leqslant \tau(b)$ for all lower semicontinuous traces τ on $\prod_{n=1}^\infty A_n$. Since $\varepsilon > 0$ is arbitrary, we get that $\tau(a) \leqslant \tau(b)$ for all τ, as desired. □

Theorem 5.6. *Let $m \in \mathbb{N}$. For every $\varepsilon > 0$ there exists N such that if A is a unital C*-algebra with nuclear dimension at most m and $a, b \in A$ are selfadjoint contractions such that $a \in \overline{\mathrm{co}\{ubu^* \mid u \in \mathrm{U}(A)\}}$, then*

$$\Big\|a - \frac{1}{N}\sum_{i=1}^{N} u_i a u_i^*\Big\| < \varepsilon$$

for some $u_1, \dots, u_N \in \mathrm{U}(A)$.

Proof. The same proof of Theorem 1.2 applies here relying on Proposition 5.5 rather than on Proposition 5.1. □

Example 5.7. In [Rob15, Theorem 1.4] an example is given of a simple unital C*-algebra A with a unique tracial state τ such that for each $n \in \mathbb{N}$ there exists a selfadjoint element $a_n \in A$ of norm 1 such that $\tau(a_n) = 0$ and the distance from a_n to the set $\{\sum_{i=1}^n [b_i^*, b_i] \mid b_i \in A\}$ is 1. In this C*-algebra the property of uniform majorization cannot hold. Indeed, by Haagerup and Zsidó's theorem from [HZ84], we have $0 \prec_{\mathrm{u}} a_n$ for all n. We claim, however, that no convex combination of at most n unitary conjugates of a_n can have norm less than 1. For suppose that there were unitaries $u_1, \dots, u_n \in A$ such that

$$\Big\| \sum_{i=1}^{n} t_i u_i a_n u_i^* \Big\| < 1$$

for some $t_i \in [0,1]$ such that $\sum_{i=1}^n t_i = 1$. Then

$$\Big\| a_n - \sum_{i=1}^{n} [b_i^*, b_i] \Big\| < 1,$$

where $b_i = t_i^{\frac{1}{2}} u_i (1 + a_n)^{\frac{1}{2}}$ for $i = 1, \dots, n$. This contradicts the property of a_n. Thus, no such unitaries exist.

Theorem 5.8. *Let $A_1, A_2, \dots$ be unital C*-algebras with strict comparison of positive elements by traces or with a uniform bound on their nuclear dimensions. Let $A = \prod_{i=1}^\infty A_i / \bigoplus_{i=1}^\infty A_i$ and let $B \subseteq A$ be a separable C*-subalgebra. Then for each $a \in A_{\mathrm{sa}}$ we have that*

$$\overline{\mathrm{co}(\{uau^* \mid u \in \mathrm{U}(A)\})} \cap (B' \cap A) \neq \varnothing.$$

Proof. Let $(a_n)_n \in \prod_{n=1}^\infty A_n$ be a lift of a with $a_n \in (A_n)_{\mathrm{sa}}$ and $\|a_n\| \leqslant \|a\|$ for all n. Let $(b_n^{(1)})_n, (b_n^{(2)})_n, \dots \in \prod_{n=1}^\infty A_n$ be lifts of a sequence $b^{(1)}, b^{(2)}, \dots \in B$ dense in B. [KR14, Lemma 6.4] asserts that given an element and a finite set in a C*-algebra we can find a convex combination of unitary conjugates of the given element that almost commutes with the given finite set. (This is derived from Dixmier's approximation property in A^{**}.) Applying this lemma, we can find for each $a_n \in A_n$ a selfadjoint element $a_n' \prec_{\mathrm{u}} a_n$ such that $\|[a_n', b_i^{(j)}]\| \leqslant \frac{1}{n}\|a\|\|b^{(j)}\|$ for all $1 \leqslant i, j \leqslant n$. Let a' denote the image of $(a_n')_n$ in A. Then a' commutes with $b^{(j)}$ for all j, and so $a' \in B' \cap A$. On the other hand, from the fact that $a_n' \prec_{\mathrm{u}} a_n$ for all n we get that $(a_n')_n \prec_{\mathrm{u}} (a_n)_n$ in $\prod_{n=1}^\infty A_n$. In the case that all the C*-algebras have strict comparison by traces, this follows from Proposition 5.1. If their nuclear dimensions are uniformly bounded, this follows from Proposition 5.5. Passing to the quotient we get that $a' \prec_{\mathrm{u}} a$ in A. That is, $a' \in \overline{\mathrm{co}(\{uau^* \mid u \in \mathrm{U}(A)\})}$. ☐

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