

QUOTIENTS OF DEL PEZZO SURFACES OF HIGH DEGREE

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ABSTRACT. In this paper we study quotients of del Pezzo surfaces of degree four and more over arbitrary field \mathbb{k} of characteristic zero by finite groups of automorphisms. We show that if a del Pezzo surface X contains a point defined over the ground field and the degree of X is at least five, then the quotient is always \mathbb{k} -rational. If the degree of X is equal to four, then the quotient can be non- \mathbb{k} -rational only if the order of the group is 1, 2, or 4. For these groups we construct examples of non- \mathbb{k} -rational quotients.

1. INTRODUCTION

In this paper we study a question about rationality of quotients of del Pezzo surfaces over arbitrary field \mathbb{k} of characteristic zero by finite groups of automorphisms. We say that a surface S is \mathbb{k} -rational if there exists a birational map $S \dashrightarrow \mathbb{P}_{\mathbb{k}}^2$ defined over \mathbb{k} . If for the algebraic closure $\bar{\mathbb{k}}$ of \mathbb{k} such a map defined over $\bar{\mathbb{k}}$ exists for a surface $\bar{S} = S \otimes_{\mathbb{k}} \bar{\mathbb{k}}$ and $\mathbb{P}_{\bar{\mathbb{k}}}^2$, we say that S is rational. Note that in many other papers, for these notions the authors use terms *rational surface* and *geometrically rational surface* respectively.

Let \mathbb{k} be any field of characteristic zero. We want to know when quotients of \mathbb{k} -rational surfaces by finite groups are \mathbb{k} -rational. From results of the G -equivariant minimal model program we know that any quotient of a \mathbb{k} -rational surface is birationally equivalent to a quotient of a conic bundle or a del Pezzo surface by the same group (see [Isk79, Theorem 1]).

In [Tr16] it was shown that non- \mathbb{k} -rational quotients of \mathbb{k} -rational surfaces form a birationally unbounded family. But all examples considered in [Tr16] are quotients of conic bundles.

In [Tr14] it was shown that any quotient of the projective plane (which is a del Pezzo surface of degree 9) is \mathbb{k} -rational. In this paper we consider quotients of del Pezzo surfaces of degree no less than 4. We show that if the set of \mathbb{k} -points on the surface is non-empty, then its quotient is \mathbb{k} -rational except for a small number of cases. The main result of the paper is the following.

Theorem 1.1. *Let \mathbb{k} be a field of characteristic zero, let X be a del Pezzo surface over \mathbb{k} such that $X(\mathbb{k}) \neq \emptyset$, and let G be a finite subgroup of automorphisms of X . If $K_X^2 \geq 5$, then the quotient variety X/G is \mathbb{k} -rational. If $K_X^2 = 4$, the order of G is equal to 1, 2, or 4, and all non-trivial elements of G have only isolated fixed*

Received by the editors June 2, 2015 and, in revised form, October 12, 2016.

2010 *Mathematics Subject Classification.* Primary 14E08, 14M20; Secondary 14E07.

The author was supported by the Russian Academic Excellence Project '5-100', Young Russian Mathematics award, and the grant RFFI 15-01-02164-a.

points, then X/G can not be \mathbb{k} -rational. In all other possibilities of G if $K_X^2 = 4$, then X/G is \mathbb{k} -rational.

For a surface X admitting a structure of conic bundle such that $K_X^2 \geq 5$ and $X(\mathbb{k}) \neq \emptyset$ the quotient X/G is \mathbb{k} -rational for any finite subgroup $G \subset \text{Aut}(X)$ by [Tr16, Proposition 1.6]. Therefore we have the following corollary.

Corollary 1.2. *Let \mathbb{k} be a field of characteristic zero, let X be a smooth rational surface over \mathbb{k} such that $X(\mathbb{k}) \neq \emptyset$, and let G be a finite subgroup of automorphisms of X . If $K_X^2 \geq 5$, then the quotient variety X/G is \mathbb{k} -rational.*

Note that a minimal del Pezzo surface X of degree 4 such that $X(\mathbb{k}) \neq \emptyset$ is not \mathbb{k} -rational by the Iskovskikh rationality criterion (see [Isk96, Chapter 4]). This gives us an example of a del Pezzo surface of degree 4 such that its quotient by the trivial group is not \mathbb{k} -rational. For groups G of order 2 and 4 we explicitly construct examples such that X is G -minimal and \mathbb{k} -rational and X/G is not \mathbb{k} -rational. Also for these groups we construct examples of non- \mathbb{k} -rational quotients of non- \mathbb{k} -rational del Pezzo surfaces of degree 4, \mathbb{k} -rational quotients of non- \mathbb{k} -rational del Pezzo surfaces of degree 4, and \mathbb{k} -rational quotients of G -minimal \mathbb{k} -rational del Pezzo surfaces of degree 4.

To prove Theorem 1.1 we consider case-by-case del Pezzo surfaces of degrees from 9 to 4 and study their quotients by finite groups. The cases of degrees 9 and 6 are considered in [Tr14] (see Theorems 2.21 and 2.22 below). The cases of degrees 8, 5, and 4 are considered in Propositions 3.1, 4.1 and 5.1 respectively. If the degree is 7, then a del Pezzo surface is never G -minimal, and its quotient is birationally equivalent to a quotient of a del Pezzo surface of degree 8 or 9. Therefore we do not consider this case.

The plan of this paper is as follows. In Section 2 we review some notions and facts about minimal rational surfaces, groups, singularities, and quotients. In Section 3 we study quotients of del Pezzo surfaces of degree 8 and show that they are all \mathbb{k} -rational. In Section 4 we study quotients of del Pezzo surfaces of degree 5 and show that they are all \mathbb{k} -rational. In Section 5 we show that quotients of del Pezzo surfaces of degree 4 are \mathbb{k} -rational for all non-trivial groups except three cases. In Section 6 we show that for the remaining three cases the quotient of a del Pezzo surface of degree 4 can be non- \mathbb{k} -rational and give explicit examples of non- \mathbb{k} -rational quotients of \mathbb{k} -rational surfaces.

Notation. Throughout this paper \mathbb{k} is any field of characteristic zero, and $\bar{\mathbb{k}}$ is its algebraic closure. For a surface X we denote $X \otimes \bar{\mathbb{k}}$ by \bar{X} . For a surface X we denote the Picard group (resp. G -invariant Picard group) by $\text{Pic}(X)$ (resp. $\text{Pic}(X)^G$). The number $\rho(X) = \text{rk Pic}(X)$ (resp. $\rho(X)^G = \text{rk Pic}(X)^G$) is the Picard number (resp. the G -invariant Picard number) of X . If two surfaces X and Y are \mathbb{k} -birationally equivalent, then we write $X \approx Y$. If two divisors A and B are linearly equivalent, then we write $A \sim B$. The rational ruled (Hirzebruch) surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$ is denoted by \mathbb{F}_n .

2. PRELIMINARIES

2.1. G -minimal rational surfaces. In this subsection we review main notions and results of the G -equivariant minimal model program following the papers [Man67], [Isk79], and [DI09]. Throughout this subsection G is a finite group.

Definition 2.1. A *rational variety* X is a variety over \mathbb{k} such that $\overline{X} = X \otimes \overline{\mathbb{k}}$ is birationally equivalent to $\mathbb{P}_{\mathbb{k}}^n$.

A \mathbb{k} -*rational variety* X is a variety over \mathbb{k} such that X is birationally equivalent to $\mathbb{P}_{\mathbb{k}}^n$.

A variety X over \mathbb{k} is a \mathbb{k} -*unirational variety* if there exists a \mathbb{k} -rational variety Y and a dominant rational map $\varphi : Y \dashrightarrow X$.

Definition 2.2. A G -*surface* is a pair (X, G) where X is a projective surface over \mathbb{k} and G is a finite subgroup of $\text{Aut}_{\mathbb{k}}(X)$. A morphism of G -surfaces $f : X \rightarrow X'$ is called a G -*morphism* if for each $g \in G$ one has $fg = gf$.

A smooth G -surface (X, G) is called G -*minimal* if any birational morphism of smooth G -surfaces $(X, G) \rightarrow (X', G)$ is an isomorphism.

Let (X, G) be a smooth G -surface. A G -minimal surface (Y, G) is called a *minimal model* of (X, G) or G -*minimal model* of X if there exists a birational G -morphism $X \rightarrow Y$.

The following theorem is a classical result about the G -equivariant minimal model program.

Theorem 2.3. Any birational G -morphism $f : X \rightarrow Y$ of smooth G -surfaces can be factorized in the following way:

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n = Y,$$

where each f_i is a contraction of a set Σ_i of disjoint (-1) -curves on X_i , such that Σ_i is defined over \mathbb{k} and is G -invariant. In particular,

$$K_Y^2 - K_X^2 \geq \rho(X)^G - \rho(Y)^G.$$

The classification of G -minimal rational surfaces is well-known due to V. Iskovskikh and Yu. Manin (see [Isk79] and [Man67]). We introduce some important notions before surveying it.

Definition 2.4. A smooth rational G -surface (X, G) admits a *conic bundle* structure if there exists a G -equivariant map $\varphi : X \rightarrow B$ such that any scheme fibre is isomorphic to a reduced conic in $\mathbb{P}_{\mathbb{k}}^2$ and B is a smooth curve.

Definition 2.5. A *del Pezzo surface* is a smooth projective surface X such that the anticanonical class $-K_X$ is ample.

A *singular del Pezzo surface* is a normal projective surface X such that the anticanonical class $-K_X$ is ample and all singularities of X are Du Val singularities.

The number $d = K_X^2$ is called the *degree* of a (singular) del Pezzo surface X .

A del Pezzo surface \overline{X} over $\overline{\mathbb{k}}$ is isomorphic to $\mathbb{P}_{\overline{\mathbb{k}}}^2$, $\mathbb{P}_{\overline{\mathbb{k}}}^1 \times \mathbb{P}_{\overline{\mathbb{k}}}^1$ or a blowup of $\mathbb{P}_{\overline{\mathbb{k}}}^2$ at up to 8 points in general position (see [Man74, Theorem 2.5]). The configuration of (-1) -curves on a del Pezzo surface plays an important role in studying its geometry. Throughout this paper we will use the notation from the following remark.

Remark 2.6. Let \overline{X} be a del Pezzo surface of degree d , $4 \leq d \leq 7$. Then \overline{X} can be realized as a blowup $f : \overline{X} \rightarrow \mathbb{P}_{\overline{\mathbb{k}}}^2$ at $n = 9 - d$ points p_1, \dots, p_n in general position. Put $E_i = f^{-1}(p_i)$ and $L = f^*(l)$, where l is the class of a line on $\mathbb{P}_{\overline{\mathbb{k}}}^2$. One has

$$-K_{\overline{X}} \sim 3L - \sum_{i=1}^n E_i.$$

The (-1) -curves on \overline{X} are E_i , the proper transforms $L_{ij} \sim L - E_i - E_j$ of the lines passing through a pair of points p_i and p_j , and for $d = 4$, the proper transform

$$Q \sim 2L - \sum_{i=1}^5 E_i$$

of the conic passing through the five points p_1, p_2, p_3, p_4, p_5 .

In this notation one has

$$E_i \cdot E_j = 0, \quad E_i \cdot L_{ij} = 1, \quad E_i \cdot L_{jk} = 0,$$

$$L_{ij} \cdot L_{ik} = 0, \quad L_{ij} \cdot L_{kl} = 1,$$

$$E_i \cdot Q = 1, \quad L_{ij} \cdot Q = 0,$$

where i, j, k , and l are different numbers from the set $\{1, 2, 3, 4, 5\}$.

Theorem 2.7 ([Isk79, Theorem 1]). *Let X be a G -minimal rational G -surface. Then either X admits a G -equivariant conic bundle structure with $\text{Pic}(X)^G \cong \mathbb{Z}^2$ or X is a del Pezzo surface with $\text{Pic}(X)^G \cong \mathbb{Z}$.*

Theorem 2.8 (cf. [Isk79, Theorem 4], [Isk79, Theorem 5]). *Let X admit a G -equivariant conic bundle structure, $G \subset \text{Aut}(X)$. Then:*

- (i) *If $K_X^2 = 3, 5, 6, 7$ or $X \cong \mathbb{F}_1$, then X is not G -minimal.*
- (ii) *If $K_X^2 = 8$, then X is isomorphic to \mathbb{F}_n , and X is G -minimal if $n \neq 1$.*
- (iii) *If $K_X^2 \neq 3, 5, 6, 7, 8$ and $\rho(X)^G = 2$, then X is G -minimal.*

The following theorem is an important criterion of \mathbb{k} -rationality over an arbitrary perfect field \mathbb{k} .

Theorem 2.9 ([Isk96, Chapter 4]). *A minimal rational surface X over a perfect field \mathbb{k} is \mathbb{k} -rational if and only if the following two conditions are satisfied:*

- (i) $X(\mathbb{k}) \neq \emptyset$;
- (ii) $K_X^2 \geq 5$.

An important class of rational surfaces is the class of toric surfaces.

Definition 2.10. *A toric variety is a normal variety over \mathbb{k} containing an algebraic torus as a Zariski dense subset, such that the action of the torus on itself by left multiplication extends to the whole variety.*

A variety X is called a \mathbb{k} -form of a toric variety if \overline{X} is toric.

Obviously, a \mathbb{k} -form of a toric variety is rational.

The following lemma is well-known (see, for example, [Tr14, Lemma 2.9]).

Lemma 2.11. *Let X be a G -minimal rational surface such that $X(\mathbb{k}) \neq \emptyset$. The following are equivalent:*

- (i) *X is a \mathbb{k} -form of a toric surface;*
- (ii) $K_X^2 \geq 6$;
- (iii) *X is isomorphic to $\mathbb{P}_{\mathbb{k}}^2$, a smooth quadric $Q \subset \mathbb{P}_{\mathbb{k}}^3$, a del Pezzo surface of degree 6, or a minimal rational ruled surface \mathbb{F}_n ($n \geq 2$).*

Corollary 2.12. *Let X be a smooth rational G -surface such that $X(\mathbb{k}) \neq \emptyset$ and $\rho(X)^G + K_X^2 \geq 7$. Then there exists a G -minimal model Y of X such that Y is a \mathbb{k} -form of a toric surface. In particular, X is \mathbb{k} -rational.*

Proof. By Theorem 2.7 there exists a birational G -morphism $f : X \rightarrow Z$ such that $\rho(Z)^G \leq 2$. By Theorem 2.3 one has

$$K_Z^2 \geq K_X^2 + \rho(X)^G - \rho(Z)^G \geq 7 - \rho(Z)^G.$$

If $\rho(Z)^G = 1$, then $K_Z^2 \geq 6$ and Z is a \mathbb{k} -form of a toric surface by Lemma 2.11. In this case we put $Y = Z$.

If $\rho(Z)^G = 2$ and $K_Z^2 = 5$, then Z is not G -minimal by Theorem 2.8(i). Therefore there exists a minimal model Y of Z such that $K_Y^2 \geq 6$ and Y is a \mathbb{k} -form of a toric surface by Lemma 2.11.

The set $X(\mathbb{k})$ is not empty. Thus $Y(\mathbb{k}) \neq \emptyset$ and $X \approx Y$ is \mathbb{k} -rational by Theorem 2.9. □

2.2. Groups. In this subsection we collect some results and notation concerning groups used in this paper.

We use the following notation:

- \mathfrak{C}_n denotes a cyclic group of order n ;
- \mathfrak{D}_{2n} denotes a dihedral group of order $2n$;
- \mathfrak{S}_n denotes a symmetric group of degree n ;
- \mathfrak{A}_n denotes an alternating group of degree n ;
- $(i_1 i_2 \dots i_j)$ denotes a cyclic permutation of i_1, \dots, i_j ;
- \mathfrak{V}_4 denotes a Klein group isomorphic to \mathfrak{C}_2^2 ;
- $\langle g_1, \dots, g_n \rangle$ denotes a group generated by g_1, \dots, g_n ;
- $A \bullet B$ is an extension of B by A ; i.e., if $G \cong A \bullet B$, then there exists an exact sequence

$$1 \rightarrow A \rightarrow G \rightarrow B \rightarrow 1;$$

- for surjective homomorphisms $\alpha : A \rightarrow D$ and $\beta : B \rightarrow D$ we denote by $A \triangle_D B$ the diagonal product of A and B over their common homomorphic image D that is the subgroup of $A \times B$ of pairs $(a; b)$ such that $\alpha(a) = \beta(b)$;
- $\text{diag}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$;
- $i = \sqrt{-1}$;
- $\xi_n = e^{\frac{2\pi i}{n}}$;
- $\omega = \xi_3 = e^{\frac{2\pi i}{3}}$.

To find fixed points of groups acting on a del Pezzo surface of degree 8 we apply the following well-known lemma. For the proof see, for example, [Tr16, Lemma 3.4].

Lemma 2.13. *Elements $g_1, g_2 \in \text{PGL}_2(\overline{\mathbb{k}})$ such that the group $H = \langle g_1, g_2 \rangle$ is finite have the same pair of fixed points on $\mathbb{P}_{\overline{\mathbb{k}}}^1$ if and only if the group H is cyclic. Otherwise the elements g_1 and g_2 do not have a common fixed point.*

The group \mathfrak{S}_5 often appears as a group of automorphisms of a rational surface. Therefore it is important to know its subgroups and normal subgroups of these subgroups. The following lemma is an easy exercise.

Lemma 2.14. *Any non-trivial subgroup $G \subset \mathfrak{S}_5$ contains a normal subgroup N conjugate in \mathfrak{S}_5 to one of the following groups:*

- $\mathfrak{C}_2 = \langle (12) \rangle$,
- $\mathfrak{C}_2 = \langle (12)(34) \rangle$,
- $\mathfrak{C}_3 = \langle (123) \rangle$,
- $\mathfrak{A}_4 = \langle (12)(34), (13)(24) \rangle$,
- $\mathfrak{C}_5 = \langle (12345) \rangle$,
- \mathfrak{A}_5 .

The following lemma is well-known.

Lemma 2.15. *Let a group \mathfrak{C}_3 act on $\mathbb{P}_{\mathbb{k}}^2$ and not have curves of fixed points. Then the group \mathfrak{C}_3 has three isolated fixed points and acts on the tangent space of $\mathbb{P}_{\mathbb{k}}^2$ at each fixed point as $\text{diag}(\omega, \omega^2)$.*

Proof. The action of any element of finite order n on $\mathbb{P}_{\mathbb{k}}^2$ can be diagonalized in a way such that the entries of the diagonal matrix corresponding to this element are roots of unity of the n -th degree. Therefore one can choose coordinates on $\mathbb{P}_{\mathbb{k}}^2$ in which the action of \mathfrak{C}_3 on $\mathbb{P}_{\mathbb{k}}^2$ has form $\text{diag}(\omega^a, \omega^b, \omega^c)$, where $a, b, c \in \{0, 1, 2\}$. If $a = b = c$, then the action is trivial, and if two of these numbers are equal, then the group \mathfrak{C}_3 has a curve of fixed points. Thus the numbers a, b , and c are distinct, the group \mathfrak{C}_3 has three isolated fixed points

$$(1 : 0 : 0), \quad (0 : 1 : 0), \quad (0 : 0 : 1),$$

and it acts on the tangent space of $\mathbb{P}_{\mathbb{k}}^2$ at each fixed point as $\text{diag}(\omega, \omega^2)$. \square

Remark 2.16. One can check that if we blow up a \mathfrak{C}_3 -fixed point p such that the group acts on the tangent space at p as $\text{diag}(\omega, \omega^2)$, then on the exceptional divisor there are two fixed points of the group \mathfrak{C}_3 and the group acts on the tangent spaces at these points as $\text{diag}(\omega, \omega)$. So starting from $\mathbb{P}_{\mathbb{k}}^2$ we can study actions of \mathfrak{C}_3 on del Pezzo surfaces.

2.3. Singularities. In this subsection we review some results about quotient singularities and their resolutions.

All singularities appearing in this paper are toric singularities. These singularities are locally isomorphic to the quotient of \mathbb{A}^2 by the cyclic group generated by $\text{diag}(\xi_m, \xi_m^q)$. Such a singularity is denoted by $\frac{1}{m}(1, q)$. If $\text{gcd}(m, q) > 1$, then the group

$$\mathfrak{C}_m \cong \langle \text{diag}(\xi_m, \xi_m^q) \rangle$$

contains a reflection and the quotient singularity is isomorphic to a quotient singularity with smaller m .

A toric singularity can be resolved by some weighted blowups. Therefore it is easy to describe numerical properties of a quotient singularity. We list here these properties for singularities appearing in our paper.

Remark 2.17. Let the group \mathfrak{C}_m act on a smooth surface X and $f : X \rightarrow S$ be the quotient map. Let p be a singular point on S of type $\frac{1}{m}(1, q)$. Let C and D be curves passing through p such that $f^{-1}(C)$ and $f^{-1}(D)$ are \mathfrak{C}_m -invariant and tangent vectors of these curves at the point $f^{-1}(p)$ are eigenvectors of the natural action of \mathfrak{C}_m on $T_{f^{-1}(p)}X$ (the curve C corresponds to the eigenvalue ξ_m , and the curve D corresponds to the eigenvalue ξ_m^q).

TABLE 1

| m | q | $K_{\tilde{S}}^2 - K_S^2$ | $\pi_*^{-1}(C)^2 - C^2$ | $\pi_*^{-1}(D)^2 - D^2$ | E_i^2 |
|-----|-----|---------------------------|-------------------------|-------------------------|----------------|
| 2 | 1 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | -2 |
| 3 | 1 | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | -3 |
| 3 | 2 | 0 | $-\frac{2}{3}$ | $-\frac{2}{3}$ | -2, -2 |
| 4 | 1 | -1 | $-\frac{1}{4}$ | $-\frac{1}{4}$ | -4 |
| 4 | 3 | 0 | $-\frac{3}{4}$ | $-\frac{3}{4}$ | -2, -2, -2 |
| 5 | 1 | $-\frac{9}{5}$ | $-\frac{1}{5}$ | $-\frac{1}{5}$ | -5 |
| 5 | 2 | $-\frac{2}{5}$ | $-\frac{2}{5}$ | $-\frac{3}{5}$ | -3, -2 |
| 5 | 3 | $-\frac{2}{5}$ | $-\frac{3}{5}$ | $-\frac{2}{5}$ | -2, -3 |
| 5 | 4 | 0 | $-\frac{4}{5}$ | $-\frac{4}{5}$ | -2, -2, -2, -2 |

Let $\pi : \tilde{S} \rightarrow S$ be the minimal resolution of the singular point p . Table 1 presents some numerical properties of \tilde{S} and S for the singularities with $m \leq 5$.

The exceptional divisor of π is a chain of transversally intersecting exceptional curves E_i whose self-intersection numbers are listed in the last column of Table 1. The curves $\pi_*^{-1}(C)$ and $\pi_*^{-1}(D)$ transversally intersect at a point only the first and the last of these curves respectively and do not intersect other components of exceptional divisor of π .

2.4. Quotients. In this subsection we collect some results about quotients of rational surfaces.

The following lemma is well-known; see, e.g., [Tr14, Lemma 4.2].

Lemma 2.18. *Let \overline{X} be an n -dimensional toric variety over a field $\overline{\mathbb{k}}$ and let G be a finite subgroup in $\text{Aut}(\overline{X})$ conjugate to a subgroup of n -dimensional torus $\overline{\mathbb{T}}^n \subset \overline{X}$ acting on \overline{X} . Then the quotient \overline{X}/G is a toric variety.*

In particular, if G is a finite cyclic subgroup of the connected component of the identity $\text{Aut}^0(\overline{X}) \subset \text{Aut}(\overline{X})$, then the quotient \overline{X}/G is a toric variety.

We use the following definition for convenience.

Definition 2.19. Let X be a G -surface, let $\tilde{X} \rightarrow X$ be its (G -equivariant) minimal resolution of singularities, and let Y be a G -equivariant minimal model of \tilde{X} . We call the surface Y a G -MMP-reduction of X .

Del Pezzo surfaces of degree 8 considered in this paper are toric surfaces. Thus the following proposition is very useful.

Proposition 2.20 ([Tr14, Proposition 4.5]). *Let a group G contain a normal subgroup \mathfrak{C}_p , where p is prime. If X is a G -minimal \mathbb{k} -unirational \mathbb{k} -form of a toric surface, then there exists a G/\mathfrak{C}_p -MMP-reduction Y of X/\mathfrak{C}_p such that Y is a \mathbb{k} -form of a toric surface. In particular, X/\mathfrak{C}_p is \mathbb{k} -rational.*

The quotients of del Pezzo surfaces of degrees 9 and 6 and conic bundles with $K_X^2 \geq 5$ were considered in the author’s papers [Tr14] and [Tr16].

Theorem 2.21 ([Tr14, Theorem 1.3]). *Let $G \subset \mathrm{PGL}_3(\mathbb{k})$ be a finite subgroup. Then $\mathbb{P}_{\mathbb{k}}^2/G$ is \mathbb{k} -rational.*

Theorem 2.22 ([Tr14, Corollary 1.4]). *Let X be a del Pezzo surface of degree 6 over \mathbb{k} such that $X(\mathbb{k}) \neq \emptyset$ and let G be a finite subgroup of automorphisms of X . Then the quotient variety X/G is \mathbb{k} -rational.*

Theorem 2.23 ([Tr16, Proposition 1.6]). *Let X be a conic bundle such that $K_X^2 \geq 5$ and $X(\mathbb{k}) \neq \emptyset$ and let G be a finite subgroup of $\mathrm{Aut}_{\mathbb{k}}(X)$. Then X/G is \mathbb{k} -rational.*

2.5. Singular del Pezzo surfaces. In this subsection we explicitly construct G -MMP-reductions for some singular del Pezzo surfaces.

Lemma 2.24. *Let a finite group G act on a singular del Pezzo surface V of degree 2 with six A_1 singularities. Then there exists a G -MMP-reduction Y of V such that Y is a \mathbb{k} -form of a toric surface.*

Proof. For any del Pezzo surface V of degree 2 with at worst Du Val singularities, the linear system $| -K_V |$ is base point free and defines a double cover

$$f : V \rightarrow \mathbb{P}_{\mathbb{k}}^2$$

branched over a reduced quartic $B \subset \mathbb{P}_{\mathbb{k}}^2$. The singularities of V correspond to the singularities of B . In our case from the local equations one can obtain that B has six nodes. We are going to show that B is a union of four lines.

Consider a conic D on $\mathbb{P}_{\mathbb{k}}^2$ passing through 5 of these nodes. Since

$$D \cdot B = 8 < 10,$$

the curves D and B have a common irreducible component. If this component is an irreducible conic, then B consists either of two irreducible conics or of an irreducible conic and two lines. In both cases the number of nodes is less than six. So B consists of a line and a cubic. This cubic has 3 nodes; thus it consists of three lines, and B consists of four lines l_1, l_2, l_3 , and l_4 , no three passing through a point.

The preimage $f^{-1}(l_i)$ is a rational curve passing through three singular points. From the Hurwitz formula one has

$$f^{-1}(l_i) \cdot f^{-1}(l_j) = \frac{1}{2}.$$

Moreover,

$$K_V \cdot f^{-1}(l_i) = f^* \left(K_{\mathbb{P}_{\mathbb{k}}^2} + \frac{B}{2} \right) \cdot \frac{f^*(l_i)}{2} = \left(K_{\mathbb{P}_{\mathbb{k}}^2} + \frac{B}{2} \right) \cdot l_i = -1.$$

Let $\pi : \tilde{V} \rightarrow V$ be the minimal G -equivariant resolution of singularities. Then the proper transform $\pi_*^{-1} f^{-1}(l_1 + l_2 + l_3 + l_4)$ consists of four disjoint (-1) -curves,

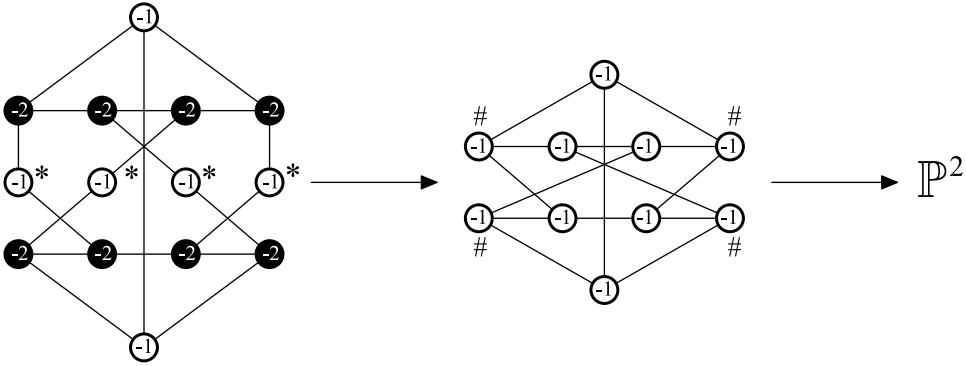


FIGURE 1

and this quadruple is defined over \mathbb{k} . We can G -equivariantly contract these four curves and get a surface Y with

$$K_Y^2 = K_V^2 + 4 = K_V^2 + 4 = 6.$$

So Y is a \mathbb{k} -form of a toric surface by Lemma 2.11. □

Lemma 2.25. *Let a finite group G act on a singular del Pezzo surface V of degree 1 with two A_4 singularities. Then there exists a G -MMP-reduction Y of V such that $\overline{Y} \cong \mathbb{P}_{\mathbb{k}}^2$.*

Proof. Let $\pi : \tilde{V} \rightarrow V$ be the minimal resolution of singularities. The dual graph of curves with negative self-intersection on \tilde{V} is well-known (see [AN06, Table 3, 10e]).

Let us equivariantly contract the four disjoint (-1) -curves marked by $*$, then equivariantly contract the four (-1) -curves marked by $\#$ and get a surface Y . One has

$$K_Y^2 = K_V^2 + 8 = K_V^2 + 8 = 9.$$

Therefore \overline{Y} is isomorphic to $\mathbb{P}_{\mathbb{k}}^2$ by Theorem 2.7. □

3. DEL PEZZO SURFACE OF DEGREE 8

In this section we prove the following proposition.

Proposition 3.1. *Let X be a del Pezzo surface of degree 8 such that $X(\mathbb{k}) \neq \emptyset$ and let G be a finite subgroup of $\text{Aut}_{\mathbb{k}}(X)$. Then X/G is \mathbb{k} -rational.*

We start with several auxiliary assertions.

Lemma 3.2. *Let X be a del Pezzo surface of degree 8 such that $X(\mathbb{k}) \neq \emptyset$ and let G be a finite subgroup of $\text{Aut}_{\mathbb{k}}(X)$. Suppose that $\rho(X)^G = 1$. Then X is isomorphic to a smooth quadric $Q \subset \mathbb{P}_{\mathbb{k}}^3$ and the group G is isomorphic to $A\Delta_D A$ or $(A\Delta_D A) \rtimes \mathfrak{C}_2$, where A is one of the following groups: \mathfrak{C}_n , \mathfrak{D}_{2n} , \mathfrak{A}_4 , \mathfrak{S}_4 , or \mathfrak{A}_5 , and D is a subgroup of A .*

Proof. If \overline{X} is isomorphic to the blowup of $\mathbb{P}_{\mathbb{k}}^2$ at a point, then X is not minimal. Therefore \overline{X} is isomorphic to $\mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1$ and

$$\text{Aut}(\overline{X}) \cong (\text{PGL}_2(\overline{\mathbb{k}}) \times \text{PGL}_2(\overline{\mathbb{k}})) \rtimes \mathfrak{C}_2.$$

Let $\pi_1 : \overline{X} \rightarrow \mathbb{P}_{\mathbb{k}}^1$ and $\pi_2 : \overline{X} \rightarrow \mathbb{P}_{\mathbb{k}}^1$ be the projections on the first and the second factors of $\mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1$ respectively. The group $\text{Pic}(\overline{X}) \cong \mathbb{Z}^2$ is generated by classes of $a = \pi_1^{-1}(p)$ and $b = \pi_2^{-1}(q)$, where p and q are points on the first and the second factors respectively.

The group

$$G_0 = G \cap (\text{PGL}_2(\overline{\mathbb{k}}) \times \text{PGL}_2(\overline{\mathbb{k}}))$$

preserves the bundles π_1 and π_2 . Thus G_0 naturally acts on the factors of $\mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1$. Let $A \subset \text{PGL}_2(\overline{\mathbb{k}})$ and $B \subset \text{PGL}_2(\overline{\mathbb{k}})$ be the images of G_0 under the projections of $\text{PGL}_2(\overline{\mathbb{k}}) \times \text{PGL}_2(\overline{\mathbb{k}})$ onto its factors. Then the group G_0 is a group $A\Delta_D B$ for some D . If the groups A and B are not isomorphic, then $G = G_0$, any element $g \in \text{Gal}(\overline{\mathbb{k}}/\mathbb{k}) \times G$ preserves the factors of $\mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1$, and one has $ga \sim a$ and $gb \sim b$, so that $\rho(X)^G = 2$. Thus $A \cong B$, and the group G_0 is $A\Delta_D A$ for some D . Therefore the group G is $A\Delta_D A$ or $(A\Delta_D A) \bullet \mathfrak{C}_2$, where A is a finite subgroup of $\text{PGL}_2(\overline{\mathbb{k}})$, i.e., $\mathfrak{C}_n, \mathfrak{D}_{2n}, \mathfrak{A}_4, \mathfrak{S}_4$, or \mathfrak{A}_5 . \square

Throughout the rest of this section we use the notation introduced in Lemma 3.2.

Lemma 3.3. *Let a group $G \cong A \times B$ act on a smooth quadric $X \subset \mathbb{P}_{\mathbb{k}}^3$ such that $X(\mathbb{k}) \neq \emptyset$, let the group A act trivially on $\pi_2(\overline{X})$, and let the group B act trivially on $\pi_1(\overline{X})$. Then X/G is isomorphic to a smooth quadric in $\mathbb{P}_{\mathbb{k}}^3$.*

Proof. One has

$$\overline{X}/G \cong (\mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1) / (A \times B) = (\mathbb{P}_{\mathbb{k}}^1/A) \times (\mathbb{P}_{\mathbb{k}}^1/B) \cong \mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1.$$

Thus X/G is isomorphic to a smooth quadric in $\mathbb{P}_{\mathbb{k}}^3$. \square

Lemma 3.4. *If a group $G \cong \mathfrak{C}_n \Delta_D \mathfrak{C}_m$ acts on a smooth quadric $X \subset \mathbb{P}_{\mathbb{k}}^3$, then X/G is a \mathbb{k} -form of a toric surface.*

Proof. The groups \mathfrak{C}_n and \mathfrak{C}_m are subgroups of tori $\mathbb{T}_1 \subset \text{Aut}(\pi_1(\overline{X}))$ and $\mathbb{T}_2 \subset \text{Aut}(\pi_2(\overline{X}))$ respectively. One has $\mathfrak{C}_n \times \mathfrak{C}_m \subset \mathbb{T}_1 \times \mathbb{T}_2$. Thus the group $G \cong \mathfrak{C}_n \Delta_D \mathfrak{C}_m$ is a subgroup of the torus $\mathbb{T}_1 \times \mathbb{T}_2 \subset \text{Aut}(\overline{X})$. Therefore X/G is a \mathbb{k} -form of a toric surface by Lemma 2.18. \square

Remark 3.5. Let a finite group $H \cong H \Delta_H H$ act on a smooth quadric $X \subset \mathbb{P}_{\mathbb{k}}^3$ and faithfully act on both factors of $\overline{X} \cong \mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1$. Then by Lemma 2.13 each cyclic subgroup C of H has four isolated fixed points that are the intersection of C -invariant fibres of π_1 and π_2 . In the neighbourhood of these points the group C acts as $\langle \text{diag}(\xi_n, \xi_n^m) \rangle$, $\langle \text{diag}(\xi_n^{n-1}, \xi_n^m) \rangle$, $\langle \text{diag}(\xi_n, \xi_n^{n-m}) \rangle$, and $\langle \text{diag}(\xi_n^{n-1}, \xi_n^{n-m}) \rangle$, where $n = \text{ord } H$. If two elements $g_1, g_2 \in H$ do not lie in a common cyclic subgroup of H , then they do not have common fixed points by Lemma 2.13.

Lemma 3.6. *Let a finite group G act on a smooth quadric $X \subset \mathbb{P}_{\mathbb{k}}^3$ and let*

$$N \cong \mathfrak{V}_4 \cong \mathfrak{V}_4 \Delta_{\mathfrak{V}_4} \mathfrak{V}_4$$

be a normal subgroup in G acting faithfully on both factors of $\overline{X} \cong \mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1$. Then there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface.

Proof. By Remark 3.5 there are 12 points on \overline{X} each of which is fixed by a non-trivial element of N , and no curves of fixed points for these elements. Therefore

$$K_{X/N}^2 = \frac{K_X^2}{4} = 2,$$

and the surface \overline{X}/N is a singular del Pezzo surface of degree 2 with six A_1 singularities.

By Lemma 2.24 there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface. □

The group \mathfrak{A}_5 has two different representations in $\mathrm{PGL}_2(\overline{\mathbb{k}})$ up to conjugation. Let us consider a group $\mathfrak{A}_5 \cong \mathfrak{A}_5 \triangleleft_{\mathfrak{A}_5} \mathfrak{A}_5$ acting faithfully on both factors of $\mathbb{P}_{\overline{\mathbb{k}}}^1 \times \mathbb{P}_{\overline{\mathbb{k}}}^1$. If the actions of \mathfrak{A}_5 on both factors are conjugate, then we call such an action on $\mathbb{P}_{\overline{\mathbb{k}}}^1 \times \mathbb{P}_{\overline{\mathbb{k}}}^1$ *diagonal*. If the actions of \mathfrak{A}_5 on both factors are not conjugate, then we call such an action on $\mathbb{P}_{\overline{\mathbb{k}}}^1 \times \mathbb{P}_{\overline{\mathbb{k}}}^1$ *twisted diagonal*.

Lemma 3.7. *Let a finite group G act on a smooth quadric $X \subset \mathbb{P}_{\mathbb{k}}^3$ and let*

$$N \cong \mathfrak{A}_5 \cong \mathfrak{A}_5 \triangleleft_{\mathfrak{A}_5} \mathfrak{A}_5$$

be a normal subgroup in G acting faithfully on both factors of $\overline{X} \cong \mathbb{P}_{\overline{\mathbb{k}}}^1 \times \mathbb{P}_{\overline{\mathbb{k}}}^1$ such that the action is diagonal and $\rho(X)^G = 1$. Then there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface.

Proof. Each non-trivial cyclic subgroup in \mathfrak{A}_5 is conjugate to $\langle(12)(34)\rangle$, $\langle(123)\rangle$, or $\langle(12345)\rangle$.

There are 15 subgroups conjugate to $\langle(12)(34)\rangle$. By Remark 3.5 each of these groups has four fixed points and acts on the neighbourhood of these points as $\langle\mathrm{diag}(-1, -1)\rangle$. The stabilizer subgroup of each of these points has order 2. Therefore there exist two A_1 singular points on \overline{X}/N .

There are 10 subgroups conjugate to $\langle(123)\rangle$. By Remark 3.5 each of these groups has four fixed points and acts on the neighbourhood of these points as $\langle\mathrm{diag}(\omega, \omega)\rangle$ or $\langle\mathrm{diag}(\omega, \omega^2)\rangle$. The stabilizer subgroup of each of these points has order 3. Therefore there exist one A_2 singular point and one $\frac{1}{3}(1, 1)$ singular point on \overline{X}/N .

There are 6 subgroups conjugate to $\langle(12345)\rangle$. By Remark 3.5 each of these groups has four fixed points and acts on the neighbourhood of these points as $\langle\mathrm{diag}(\xi_5, \xi_5)\rangle$ or $\langle\mathrm{diag}(\xi_5, \xi_5^4)\rangle$ since the action is diagonal. The stabilizer subgroup of each of these points has order 5. Therefore there exist one A_4 singular point and one $\frac{1}{5}(1, 1)$ singular point on \overline{X}/N .

Hence the set of singular points of \overline{X}/N is the following: two A_1 points, one A_2 point, one $\frac{1}{3}(1, 1)$ point, one A_4 point, and one $\frac{1}{5}(1, 1)$ point. Non-trivial elements of the group N have only isolated fixed points. Thus

$$K_{X/N}^2 = \frac{K_X^2}{60} = \frac{2}{15}, \quad \rho(X/N)^{G/N} = \rho(X)^G = 1.$$

Let $f : X \rightarrow X/N$ be the quotient morphism, let $r : \widetilde{X}/N \rightarrow X/N$ be the minimal resolution of the singularities, and let F_1 and F_2 be \mathfrak{C}_5 -invariant fibres of the projections π_1 and π_2 respectively. Note that there exists an element g of the group $G \times \mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ such that $gF_1 = F_2$ since $\rho(X)^G = 1$.

One has

$$K^2_{\widetilde{X/N}} = K^2_{X/N} - \frac{1}{3} - \frac{9}{5} = -2, \quad \rho(\widetilde{X/N})^{G/N} \geq \rho(X/N)^{G/N} + 6 = 7.$$

Moreover the curves $r_*^{-1}f(F_1)$ and $r_*^{-1}f(F_2)$ are two disjoint curves on $\widetilde{X/N}$ with self-intersection numbers equal to -1 (see Table 1). One can G/N -equivariantly contract this pair of curves and then G/N -equivariantly contract the transforms of two (-2) -curves that are the ends of the chain of rational curves in the preimage of the A_4 singular point. We obtain a surface Z such that $K^2_Z = 2$ and $\rho(Z)^{G/N} \geq 5$. By Corollary 2.12 there exists a G/N -minimal model Y of Z such that Y is a \mathbb{k} -form of a toric surface. \square

Lemma 3.8. *Let a finite group G act on a smooth quadric $X \subset \mathbb{P}^3_{\mathbb{k}}$ and let*

$$N \cong \mathfrak{A}_5 \cong \mathfrak{A}_5 \triangleleft_{\mathfrak{A}_5} \mathfrak{A}_5$$

be a normal subgroup in G acting faithfully on both factors of $\overline{X} \cong \mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$ such that the action is twisted diagonal and $\rho(X)^G = 1$. Then there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface.

Proof. As in the proof of Lemma 3.7 we can apply Remark 3.5 and find the list of singularities of \overline{X}/N : two A_1 points, one A_2 point, one $\frac{1}{3}(1, 1)$ point, and two $\frac{1}{5}(1, 2)$ points.

One has

$$K^2_{\widetilde{X/N}} = K^2_{X/N} - \frac{1}{3} - 2 \cdot \frac{2}{5} = -1, \quad \rho(\widetilde{X/N})^{G/N} \geq \rho(X/N)^{G/N} + 5 = 6.$$

Moreover, for f, r, F_1, F_2 defined as in the proof of Lemma 3.7, the curves $r_*^{-1}f(F_1)$ and $r_*^{-1}f(F_2)$ are two disjoint curves on \widetilde{X}/N with self-intersection numbers equal to -1 (see Table 1). One can G/N -equivariantly contract this pair of curves and then G/N -equivariantly contract the transforms of two (-2) -curves which are irreducible components of the preimages of the $\frac{1}{5}(1, 2)$ singular points. We obtain a surface Z such that $K^2_Z = 3$ and $\rho(Z)^{G/N} \geq 4$. By Corollary 2.12 there exists a G/N -minimal model Y of Z such that Y is a \mathbb{k} -form of a toric surface. \square

Now we prove Proposition 3.1.

Proof of Proposition 3.1. We can assume that $\overline{X} \cong \mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$ and $\rho(X)^G = 1$ since otherwise $\rho(X)^G = 2$, so that X admits a G -equivariant conic bundle structure by Theorem 2.7 and X/G is \mathbb{k} -rational by Theorem 2.23.

Let $f_1 : G \rightarrow \text{Aut}\left(\mathbb{P}^1_{\mathbb{k}}\right)$ and $f_2 : G \rightarrow \text{Aut}\left(\mathbb{P}^1_{\mathbb{k}}\right)$ be homomorphisms to the groups of automorphisms of the first and second factor of $\mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$ respectively. Then the group

$$K = \text{Ker } f_1 \times \text{Ker } f_2$$

is a normal subgroup of G . Then by Lemma 3.3 the surface X/K is a del Pezzo surface of degree 8 and

$$(X/K) / (G/K) = X/G.$$

So we can replace X by X/K and assume that K is trivial.

Since K is trivial then by Lemma 3.2 the group G is isomorphic to $A \triangleleft_A A$ or $(A \triangleleft_A A) \bullet \mathfrak{C}_2$, where A is one of the following groups: $\mathfrak{C}_n, \mathfrak{D}_{2n}, \mathfrak{A}_4, \mathfrak{S}_4$, or \mathfrak{A}_5 . For

each of these groups we find a normal subgroup $N \triangleleft G$ such that there exists a G/N -MMP-reduction Y of X/N that is a \mathbb{k} -form of a toric surface.

- If G is \mathfrak{C}_2 , then there exists an MMP-reduction Y of X/G such that Y is a \mathbb{k} -form of a toric surface by Proposition 2.20.
- If G is $\mathfrak{C}_n \triangle \mathfrak{C}_n \mathfrak{C}_n$, $(\mathfrak{C}_n \triangle \mathfrak{C}_n \mathfrak{C}_n) \bullet \mathfrak{C}_2$, $\mathfrak{D}_{2n} \triangle \mathfrak{D}_{2n} \mathfrak{D}_{2n}$, or $(\mathfrak{D}_{2n} \triangle \mathfrak{D}_{2n} \mathfrak{D}_{2n}) \bullet \mathfrak{C}_2$, then N is $\mathfrak{C}_n \triangle \mathfrak{C}_n \mathfrak{C}_n$. Any G/N -MMP-reduction of X/N is a \mathbb{k} -form of a toric surface by Lemma 3.4.
- If G is $\mathfrak{A}_4 \triangle \mathfrak{A}_4 \mathfrak{A}_4$, $(\mathfrak{A}_4 \triangle \mathfrak{A}_4 \mathfrak{A}_4) \bullet \mathfrak{C}_2$, $\mathfrak{S}_4 \triangle \mathfrak{S}_4 \mathfrak{S}_4$, or $(\mathfrak{S}_4 \triangle \mathfrak{S}_4 \mathfrak{S}_4) \bullet \mathfrak{C}_2$, then N is $\mathfrak{A}_4 \triangle \mathfrak{A}_4 \mathfrak{A}_4$. There exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface by Lemma 3.6.
- If G is $\mathfrak{A}_5 \triangle \mathfrak{A}_5 \mathfrak{A}_5$ or $(\mathfrak{A}_5 \triangle \mathfrak{A}_5 \mathfrak{A}_5) \bullet \mathfrak{C}_2$, then N is $\mathfrak{A}_5 \triangle \mathfrak{A}_5 \mathfrak{A}_5$. There exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface by Lemmas 3.7 and 3.8.

Therefore Y is a \mathbb{k} -form of a toric surface. Moreover, $Y(\mathbb{k}) \neq \emptyset$ since $X(\mathbb{k}) \neq \emptyset$.

If the surface Y is $\mathbb{P}_{\mathbb{k}}^2$, \mathbb{F}_n , or a del Pezzo surface of degree 6, then

$$Y/(G/N) \approx X/G$$

is \mathbb{k} -rational by Theorems 2.21, 2.23, and 2.22 respectively. If the surface \overline{Y} is $\mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1$ we apply the procedure above with the smaller group G/N . As a result we obtain that X/G is \mathbb{k} -rational. □

4. DEL PEZZO SURFACE OF DEGREE 5

Let X be a del Pezzo surface of degree 5. The group $\text{Aut}(\overline{X})$ is isomorphic to

$$W(A_4) \cong \mathfrak{S}_5$$

(see e.g. [DI09, Subsection 6.3] or [Dol12, Theorem 8.5.8]). This group is generated by a subgroup \mathfrak{S}_4 and the element (12345). In the notation of Remark 2.6 for any $\sigma \in \mathfrak{S}_4$ one has $\sigma(E_i) = E_{\sigma(i)}$ and $\sigma(L_{ij}) = L_{\sigma(i)\sigma(j)}$.

In this section we prove the following proposition.

Proposition 4.1. *Let X be a del Pezzo surface of degree 5 such that $X(\mathbb{k}) \neq \emptyset$ and let G be a subgroup of $\text{Aut}_{\mathbb{k}}(X)$. Then X/G is \mathbb{k} -rational.*

To prove Proposition 4.1 we show that in all other cases either the surface X is not G -minimal or there is a normal subgroup $N \triangleleft G$ such that there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface. So the proof of Proposition 4.1 is reduced to Theorems 2.21, 2.23, 2.22, and Proposition 3.1.

Lemma 4.2. *Let a finite group G act on a del Pezzo surface X of degree 5 and let N be a non-trivial normal subgroup in G . If the group N is isomorphic to \mathfrak{C}_2 , \mathfrak{C}_3 , or \mathfrak{A}_4 , then X is not G -minimal.*

Proof. If $N \cong \mathfrak{C}_2$, then it is conjugate to $\langle(12)\rangle$ or $\langle(12)(34)\rangle$. In the first case there are exactly four N -invariant (-1) -curves on X : E_3, E_4, L_{12} , and L_{34} . But only the curve L_{34} intersects every other N -invariant (-1) -curve. Thus L_{34} is G -invariant and defined over \mathbb{k} so it can be contracted. In the second case there are exactly two orbits consisting of disjoint (-1) -curves on X : E_1 and E_2, E_3 and E_4 . Thus this quadruple is G -invariant and defined over \mathbb{k} so it can be contracted.

If $N \cong \mathfrak{C}_3$, then it is conjugate to $\langle(123)\rangle$. There is exactly one N -invariant (-1) -curve E_4 on X . Thus this curve is G -invariant and defined over \mathbb{k} so it can be contracted.

If $N \cong \mathfrak{A}_4$, then it is conjugate to $\langle(12)(34), (13)(24)\rangle$. There is exactly one N -orbit consisting of four disjoint (-1) -curves on X : E_1, E_2, E_3 , and E_4 . Thus this quadruple is G -invariant and defined over \mathbb{k} so it can be contracted. \square

Lemma 4.3. *Let the group \mathfrak{C}_5 act on a del Pezzo surface of degree 5. Then the group \mathfrak{C}_5 has two fixed points on \overline{X} and acts on the tangent spaces of \overline{X} at these points as $\langle\text{diag}(\xi_5, \xi_5^4)\rangle$.*

Proof. Let us consider the following Cremona transformation of $\mathbb{P}_{\mathbb{k}}^2$:

$$g : (x : y : z) \mapsto ((y - x)z : (z - x)y : yz).$$

The order of g is 5. This transformation is regular on a del Pezzo surface of degree 5 which is the blowup of $\mathbb{P}_{\mathbb{k}}^2$ at four points: $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, and $(1 : 1 : 1)$. All del Pezzo surfaces of degree 5 are isomorphic; thus any element of order 5 is conjugate to g in $\text{Aut}(\overline{X})$.

The element g has two fixed points: $(\sqrt{5} - 1 : 2 : \sqrt{5} + 1)$ and $(\sqrt{5} + 1 : -2 : \sqrt{5} - 1)$ on $\mathbb{P}_{\mathbb{k}}^2$. One can easily check that the element g acts on the tangent spaces of $\mathbb{P}_{\mathbb{k}}^2$ at the fixed points as $\text{diag}(\xi_5, \xi_5^4)$. \square

Lemma 4.4. *Let a finite group G act on a del Pezzo surface X of degree 5 and let $N \cong \mathfrak{C}_5$ be a normal subgroup in G . Then there exists a G/N -MMP-reduction Y of X/N such that $\overline{Y} \cong \mathbb{P}_{\mathbb{k}}^2$.*

Proof. By Lemma 4.3 on the quotient \overline{X}/N there are two A_4 singularities, $-K_{\overline{X}/N}$ is ample, and

$$K_{X/N}^2 = \frac{K_X^2}{5} = 1.$$

This means that X/N is a singular del Pezzo surface of degree 1.

By Lemma 2.25 there exists a G/N -MMP-reduction Y of X/N such that $\overline{Y} \cong \mathbb{P}_{\mathbb{k}}^2$. \square

Lemma 4.5. *Let a finite group G act on a del Pezzo surface X of degree 5 and let $N \cong \mathfrak{A}_5$ be a normal subgroup in G . Then there exists a G/N -MMP-reduction Y of X/N that is isomorphic to \mathbb{F}_3 .*

Proof. Let us consider fixed points of elements of N . The stabilizer of such a point is a subgroup of \mathfrak{A}_5 having a faithful representation in $\text{GL}_2(\overline{\mathbb{k}})$ (see e.g. [Pop14, Lemma 4]). Any subgroup of \mathfrak{A}_5 is isomorphic to $\mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{A}_4, \mathfrak{S}_3, \mathfrak{A}_4, \mathfrak{C}_5, \mathfrak{D}_{10}$, or \mathfrak{A}_5 . The groups \mathfrak{A}_4 and \mathfrak{A}_5 do not have faithful two-dimensional representations. For the groups $\mathfrak{A}_4, \mathfrak{S}_3$, and \mathfrak{D}_{10} faithful two-dimensional representations are generated by reflections; thus images of points with such stabilizers are smooth points on the quotient surface. All other groups are cyclic groups of prime order.

An element of order 5 has exactly two fixed points on X , and this element acts on the tangent spaces of \overline{X} at the fixed points as $\text{diag}(\xi_5, \xi_5^4)$ by Lemma 4.3.

Each element of order 3 in \mathfrak{A}_5 is conjugate to $\langle(123)\rangle$. The unique (-1) -curve on X invariant with respect to the group $\langle(123)\rangle$ is E_4 . Let us $\langle(123)\rangle$ -equivariantly contract the four (-1) -curves E_i and get $\mathbb{P}_{\mathbb{k}}^2$. The group $\langle(123)\rangle$ acts on $\mathbb{P}_{\mathbb{k}}^2$ and has no curves of fixed points since the line passing through p_1 and p_2 (see the notation of

Remark 2.6) does not contain $\langle(123)\rangle$ -fixed points. Therefore the action of $\langle(123)\rangle$ on $\mathbb{P}_{\mathbb{k}}^2$ is conjugate to $\langle\text{diag}(1, \omega, \omega^2)\rangle$ and it has 3 fixed points, one of which is p_4 by Lemma 2.15. In the tangent space of \overline{X} at the other two fixed points the group $\langle(123)\rangle$ acts as $\langle\text{diag}(\omega, \omega^2)\rangle$. On the (-1) -curve E_4 the group $\langle(123)\rangle$ has two fixed points. This group acts on the tangent space of \overline{X} at these points as $\langle\text{diag}(\omega, \omega)\rangle$ by Remark 2.16. There are 20 elements of order 3 in \mathfrak{A}_5 and they have 20 fixed points on (-1) -curves. The stabilizer of such a point is \mathfrak{C}_3 , so all these points are permuted by the group \mathfrak{A}_5 .

Consider a group

$$\mathfrak{V}_4 = \langle(12)(34), (13)(24)\rangle.$$

One can \mathfrak{V}_4 -equivariantly contract the four (-1) -curves E_i and get $\mathbb{P}_{\mathbb{k}}^2$. The group \mathfrak{V}_4 acts on $\mathbb{P}_{\mathbb{k}}^2$, and each non-trivial element has a line of fixed points not passing through the points p_i . Thus each of these elements has a curve of fixed points in \overline{X} whose class in $\text{Pic}(\overline{X})$ is L .

The images of L in $\text{Pic}(\overline{X})$ under the action of \mathfrak{C}_5 are

$$2L - E_1 - E_2 - E_3, 2L - E_1 - E_2 - E_4, 2L - E_1 - E_3 - E_4, \text{ and } 2L - E_2 - E_3 - E_4.$$

Thus the ramification divisor of the quotient morphism $f : X \rightarrow X/\mathfrak{A}_5$ is a member of the linear system $| -9K_X |$. By the Hurwitz formula

$$K_{X/\mathfrak{A}_5}^2 = \frac{1}{60}(K_X + 9K_X)^2 = \frac{25}{3}.$$

Moreover, there is one $\frac{1}{3}(1, 1)$ singularity and maybe some Du Val singularities on $\overline{X}/\mathfrak{A}_5$. Let $\pi : Y \rightarrow X/\mathfrak{A}_5$ be the minimal resolution of singularities. One has

$$K_Y^2 = K_{X/\mathfrak{A}_5}^2 - \frac{1}{3} = 8, \quad \rho(\overline{Y}) = 10 - K_{\overline{Y}}^2 = 2.$$

Therefore the only singularity on $\overline{X}/\mathfrak{A}_5$ is $\frac{1}{3}(1, 1)$, and Y is isomorphic to \mathbb{F}_3 since $K_Y^2 = 8$ and Y contains a (-3) -curve. □

Now we prove Proposition 4.1.

Proof of Proposition 4.1. By Lemma 2.14 each group $G \subset \mathfrak{S}_5$ has a normal subgroup N isomorphic to $\mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{V}_4, \mathfrak{C}_5$, or \mathfrak{A}_5 .

If N is isomorphic to $\mathfrak{C}_2, \mathfrak{C}_3$, or \mathfrak{V}_4 , then X is not G -minimal by Lemma 4.2, and X/G is \mathbb{k} -rational by Theorems 2.21, 2.23, 2.22 and Proposition 3.1.

If N is isomorphic to \mathfrak{C}_5 , then there exists a G/N -MMP-reduction Y of X/N such that \overline{Y} is isomorphic to $\mathbb{P}_{\mathbb{k}}^2$ by Lemma 4.4.

If N is isomorphic to \mathfrak{A}_5 , then any G/N -MMP-reduction Y of X/N is isomorphic to \mathbb{F}_3 by Lemma 4.5.

In the last two cases one has $Y(\mathbb{k}) \neq \emptyset$ since $X(\mathbb{k}) \neq \emptyset$. Thus

$$Y/(G/N) \approx X/G$$

is \mathbb{k} -rational by Theorems 2.21 and 2.23 respectively. □

Now we can prove Corollary 1.2.

Proof of Corollary 1.2. Let Y be a G -minimal model of X . Then $K_Y^2 \geq K_X^2 \geq 5$ by Theorem 2.3, $Y(\mathbb{k}) \neq \emptyset$ since $X(\mathbb{k}) \neq \emptyset$, and Y is either a del Pezzo surface or admits a conic bundle structure by Theorem 2.7.

Therefore $X/G \approx Y/G$ is \mathbb{k} -rational by Theorems 2.21, 2.23, 2.22, and Propositions 3.1 and 4.1. \square

5. DEL PEZZO SURFACE OF DEGREE 4

Let X be a del Pezzo surface of degree 4. The group $\text{Aut}(\overline{X})$ is a subgroup of the group

$$W(D_5) \cong \mathfrak{C}_2^4 \rtimes \mathfrak{S}_5$$

(see e.g. [DI09, Subsection 6.4] or [Dol12, Proposition 8.6.7]). The group $\mathfrak{C}_2^4 \rtimes \mathfrak{S}_5$ is generated by subgroups \mathfrak{S}_5 and \mathfrak{C}_2^4 . In the notation of Remark 2.6 for any $\sigma \in \mathfrak{S}_5$ one has $\sigma(E_i) = E_{\sigma(i)}$, $\sigma(L_{ij}) = L_{\sigma(i)\sigma(j)}$, and $\sigma(Q) = Q$.

The surface \overline{X} is isomorphic to a surface of degree 4 in $\mathbb{P}_{\mathbb{k}}^4$ given by equations

$$\sum_{i=1}^5 x_i^2 = 0, \quad \sum_{i=1}^5 a_i x_i^2 = 0.$$

The group \mathfrak{C}_2^4 acts on $\mathbb{P}_{\mathbb{k}}^4$ and \overline{X} as a diagonal subgroup of $\text{PGL}_5(\overline{\mathbb{k}})$. There are involutions of two kinds in such diagonal groups: ι_{ijkl} and ι_{ij} . These involutions switch signs of coordinates x_i, x_j, x_k, x_l and x_i, x_j , respectively.

In this section we prove the following proposition.

Proposition 5.1. *Let X be a del Pezzo surface of degree 4 such that $X(\mathbb{k}) \neq \emptyset$ and let G be a subgroup of $\text{Aut}_{\mathbb{k}}(X)$. Then X/G is \mathbb{k} -rational if G is not conjugate to any of the groups $\langle id \rangle$, $\mathfrak{C}_2 = \langle \iota_{12} \rangle$, $\mathfrak{A}_4 = \langle \iota_{12}, \iota_{13} \rangle$, or $\mathfrak{C}_4 = \langle (12)(34)\iota_{15} \rangle$.*

In Section 6 we will show that in the latter three cases the quotient can be non- \mathbb{k} -rational. Now we show that in all other cases the quotient of X is \mathbb{k} -rational.

To prove Proposition 5.1 we show that in any of the remaining cases either the surface X is not G -minimal or there is a normal subgroup $N \triangleleft G$ such that there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface. So the proof of Proposition 5.1 is reduced to Theorems 2.21, 2.23, 2.22, and Propositions 3.1 and 4.1.

Now we are going to prove some auxillary lemmas.

The following lemma immediately follows from the results of [DI09, Subsection 6.4]. We give the proof for the convenience of the reader.

Lemma 5.2. *Let a finite group G act on a del Pezzo surface X of degree 4 and let*

$$h : \text{Aut}(X) \rightarrow \mathfrak{S}_5$$

be the natural homomorphism. Then the group $h(G)$ does not contain subgroups conjugate to $\mathfrak{C}_2 = \langle (12) \rangle$ and $\mathfrak{A}_4 = \langle (12)(34), (13)(24) \rangle$.

Proof. The group \mathfrak{C}_2^4 acts on \overline{X} . The group $\text{Aut}(\overline{X}) \subset \mathfrak{C}_2^4 \rtimes \mathfrak{S}_5$ contains a subgroup \mathfrak{C}_2^4 . Therefore if the group G contains an element hg , where $g \in \mathfrak{S}_5$, $h \in \mathfrak{C}_2^4$, then the group $\text{Aut}(\overline{X})$ contains the element g . Thus it is sufficient to prove that there are no subgroups in $\text{Aut}(\overline{X})$ conjugate to $\mathfrak{C}_2 = \langle (12) \rangle$ and $\mathfrak{A}_4 = \langle (12)(34), (13)(24) \rangle$.

Suppose that the group $\mathfrak{C}_2 = \langle (12) \rangle$ acts on \overline{X} . One can \mathfrak{C}_2 -equivariantly contract five (-1) -curves E_1, E_2, E_3, E_4 , and E_5 and get $\mathbb{P}_{\mathbb{k}}^2$ with the action of \mathfrak{C}_2 . The group \mathfrak{C}_2 has a unique isolated fixed point on $\mathbb{P}_{\mathbb{k}}^2$ and a unique line of fixed points on $\mathbb{P}_{\mathbb{k}}^2$. The points p_3, p_4 , and p_5 (see the notation of Remark 2.6) are fixed by

the group \mathfrak{C}_2 . These three points do not lie on a line, so one of these points is the isolated fixed point. The points p_1 and p_2 are permuted by the group \mathfrak{C}_2 . Therefore the group \mathfrak{C}_2 acts faithfully on the line passing through these two points. Thus this line contains the isolated fixed point of \mathfrak{C}_2 . The proper transform of this line on \overline{X} is a (-2) -curve, but there are no (-2) -curves on a del Pezzo surface of degree 4. Therefore the group $\mathfrak{C}_2 = \langle(12)\rangle$ cannot act on \overline{X} .

Suppose that the group $\mathfrak{V}_4 = \langle(12)(34), (13)(24)\rangle$ acts on \overline{X} . One can \mathfrak{V}_4 -equivariantly contract five (-1) -curves E_1, E_2, E_3, E_4 , and E_5 and get $\mathbb{P}_{\mathbb{k}}^2$ with the action of \mathfrak{V}_4 . The point p_5 is fixed by the group \mathfrak{V}_4 . Thus this point is the unique isolated fixed point on $\mathbb{P}_{\mathbb{k}}^2$ of an element of \mathfrak{V}_4 . Therefore as above in the case of the group $\mathfrak{C}_2 = \langle(12)\rangle$ three of the points p_1, p_2, p_3, p_4 , and p_5 lie on a line, and the group \mathfrak{V}_4 cannot act on \overline{X} . \square

Lemma 5.3. *Let a finite group G act on a del Pezzo surface X of degree 4 and let*

$$N \cong \mathfrak{C}_2 = \langle(12)(34)\rangle$$

be a normal subgroup in G . Then there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface.

Proof. Let us N -equivariantly contract five (-1) -curves E_1, E_2, E_3, E_4 , and E_5 on \overline{X} and get a $\mathbb{P}_{\mathbb{k}}^2$ with the action of N . The group N has a unique isolated fixed point on $\mathbb{P}_{\mathbb{k}}^2$ and a unique line of fixed points. As in the proof of Lemma 5.2 the point p_5 lies on this line. Thus on the surface \overline{X} the group N has two isolated fixed points $L_{12} \cap L_{34}, Q \cap E_5$, and a curve of fixed points whose class in $\text{Pic}(\overline{X})$ is $L - E_5$.

Let $f : X \rightarrow X/N$ be the quotient morphism and let

$$\pi : \widetilde{X/N} \rightarrow X/N$$

be the minimal resolution of singularities. By the Hurwitz formula

$$K_{X/N}^2 = \frac{1}{2} (K_X - L + E_5)^2 = 4.$$

There are exactly two A_1 singularities on X/N . The proper transforms $\pi_*^{-1}f(L_{12}), \pi_*^{-1}f(L_{34}), \pi_*^{-1}f(Q)$, and $\pi_*^{-1}f(E_5)$ are four disjoint G/N -invariant (-1) -curves defined over \mathbb{k} (see Table 1). One can G/N -equivariantly contract this quadruple and get a surface Y such that $K_Y^2 = 8$. Thus there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface by Lemma 2.11. \square

Remark 5.4. Note that any involution in $W(D_5)$ is conjugate to $(12), (12)(34), 12\iota_{34}, \iota_{12}$, or ι_{1234} . By Lemma 5.2 an element conjugate to (12) or $(12)\iota_{34}$ cannot act on a del Pezzo surface of degree 4. From the proof of Lemma 5.3 one can see that an element conjugate to $(12)(34)$ has a curve of fixed points, and for an element conjugate to ι_{1234} there exists a hyperplane section consisting of fixed points. Thus if an involution acting on a del Pezzo surface of degree 4 does not have curves of fixed points, then it is conjugate to ι_{12} . For the groups of order 4 there are only two cases for which all elements of order 2 are conjugate to ι_{12} : either $\langle\iota_{12}, \iota_{13}\rangle$ or $\langle(12)(34)\iota_{15}\rangle$. Therefore the conditions on the group G in Proposition 5.1 are that $\text{ord } G$ is 1, 2, or 4, and non-trivial elements of G have only isolated fixed points.

Lemma 5.5. *Let a finite group G act on a del Pezzo surface X of degree 4 and let*

$$N \cong \mathfrak{C}_3 = \langle(123)\rangle$$

be a normal subgroup in G . Then there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface.

Proof. Let $\sigma : \overline{X} \rightarrow \mathbb{P}_{\mathbb{k}}^2$ be the N -equivariant contraction of the five (-1) -curves E_1, E_2, E_3, E_4 , and E_5 . The group N has three isolated fixed points on $\mathbb{P}_{\mathbb{k}}^2$, two of which are p_4 and p_5 (see the notation of Remark 2.6). Denote the third fixed point by p . The group N acts on the tangent space of $\mathbb{P}_{\mathbb{k}}^2$ at these fixed points as $\langle \text{diag}(\omega, \omega^2) \rangle$ by Lemma 2.15. Therefore there are five fixed points on \overline{X} : $E_4 \cap L_{45}, E_5 \cap L_{45}, E_5 \cap Q, E_4 \cap Q$, and $\sigma^{-1}(p)$. The group N acts on the tangent space of \overline{X} at the point $\sigma^{-1}(p)$ as $\langle \text{diag}(\omega, \omega^2) \rangle$ and on the tangent spaces of \overline{X} at the other fixed points as $\langle \text{diag}(\omega, \omega) \rangle$ by Remark 2.16.

Let C_1, C_2, C_3 , and C_4 be N -invariant curves on X with classes

$$2L - E_1 - E_2 - E_3 - E_4, \quad 2L - E_1 - E_2 - E_3 - E_5, \quad L - E_5, \quad \text{and} \quad L - E_4$$

passing through $\sigma^{-1}p$ and another fixed point (these curves are proper transforms of lines passing through p and p_4 or p and p_5 and conics passing through p, p_1, p_2, p_3, p_4 or p, p_1, p_2, p_3, p_5).

Assume that there is another N -invariant irreducible rational curve C with self-intersection number 0 on X whose class in $\text{Pic}(\overline{X})$ is

$$aL - b(E_1 + E_2 + E_3) - cE_4 - dE_5.$$

The curve C is irreducible; thus the numbers a, b, c, d are non-negative. One has $C^2 = 0$ and $C(K_X + C) = -2$. That means that the following system of equations holds:

$$\begin{cases} a^2 - 3b^2 - c^2 - d^2 = 0, \\ 3a - 3b - c - d = 2. \end{cases}$$

One can check that all possibilities for the class of C are

$$2L - E_1 - E_2 - E_3 - E_4, \quad 2L - E_1 - E_2 - E_3 - E_5, \quad L - E_5, \quad \text{and} \quad L - E_4.$$

Therefore there are no rational curves on X with self-intersection number 0 which differ from C_1, C_2, C_3 , and C_4 .

Let $f : X \rightarrow X/N$ be the quotient morphism and let

$$\pi : \widetilde{X/N} \rightarrow X/N$$

be the minimal resolution of singularities. Then there are four $\frac{1}{3}(1, 1)$ singularities $f(E_4 \cap L_{45}), f(E_5 \cap L_{45}), f(E_5 \cap Q), f(E_4 \cap Q)$ and one A_2 singularity $f(\sigma^{-1}(p))$ on \overline{X}/N . Consider eight curves $f(C_1), f(C_2), f(C_3), f(C_4), f(E_4), f(L_{45}), f(E_5)$, and $f(Q)$. This eighttuple is G/N -equivariant and defined over \mathbb{k} . The intersection number of any two of these curves is 0, $\frac{1}{3}$, or $\frac{2}{3}$. Thus their proper transforms on $\widetilde{X/N}$ are eight disjoint (-1) -curves (see Table 1). One can G/N -equivariantly contract these curves and get a surface Y . Then

$$K_Y^2 = K_{\widetilde{X/N}}^2 + 8 = K_{X/N}^2 - \frac{4}{3} + 8 = \frac{1}{3}K_X^2 + \frac{28}{3} = 8.$$

Thus there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface by Lemma 2.11. □

Lemma 5.6. *Let a finite group G act on a del Pezzo surface X of degree 4 and let*

$$N \cong \mathfrak{C}_5 = \langle (12345) \rangle$$

be a normal subgroup in G . Then X is not G -minimal.

Proof. The group N has a unique invariant (-1) -curve Q on X . Thus this curve is G -invariant and defined over \mathbb{k} so it can be contracted. \square

Lemma 5.7. *Let a finite group G act on a del Pezzo surface X of degree 4 and contain an element conjugate to ι_{1234} . Then there exists a normal subgroup $N \triangleleft G$ such that a G/N -MMP-reduction Y of X/N is a \mathbb{k} -form of a toric surface.*

Proof. Note that the set of fixed points of an element conjugate to ι_{1234} is a hyperplane section of X in $\mathbb{P}_{\mathbb{k}}^4$.

Let N be a subgroup of G generated by elements conjugate to ι_{1234} . Note that each element conjugate to ι_{1234} has a curve of fixed points on X that is a member of the linear system $| -K_X |$. The group N is normal, and one of the following possibilities holds:

- If N is generated by one element conjugate to ι_{1234} , then $N \cong \mathfrak{C}_2$ and by the Hurwitz formula

$$K_{X/N}^2 = \frac{1}{2} (2K_X)^2 = 8.$$

- If N is generated by two elements conjugate to ι_{1234} , then $N \cong \mathfrak{C}_2^2$ and by the Hurwitz formula

$$K_{X/N}^2 = \frac{1}{4} (3K_X)^2 = 9.$$

- If N is generated by three elements conjugate to ι_{1234} , then $N \cong \mathfrak{C}_2^3$ and by the Hurwitz formula

$$K_{X/N}^2 = \frac{1}{8} (4K_X)^2 = 8.$$

- If N is generated by four elements conjugate to ι_{1234} , then N contains the fifth element conjugate to ι_{1234} . One has $N \cong \mathfrak{C}_2^4$ and by the Hurwitz formula

$$K_{X/N}^2 = \frac{1}{16} (6K_X)^2 = 9.$$

The surface X/N has at worst Du Val singularities. Hence for any G/N -MMP-reduction Y of X/N one has $K_Y^2 \geq K_{X/N}^2 \geq 8$. Thus Y is a \mathbb{k} -form of a toric surface by Lemma 2.11. \square

Lemma 5.8. *Let a finite group G act on a del Pezzo surface X of degree 4 and let*

$$N \cong \mathfrak{A}_4 = \langle \iota_{12}, \iota_{13} \rangle$$

be a normal subgroup in G . Then the surface X/N is G/N -birationally equivalent to a del Pezzo surface Y of degree 4. If $\rho(X)^G = 1$ and for each non-trivial element of N all its fixed points are in one orbit of the group $G \times \text{Gal}(\bar{\mathbb{k}}/\mathbb{k})$, then Y is G/N -minimal.

Proof. Each non-trivial element of N has 4 fixed points on X . The hyperplane sections $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ cut out from X elliptic curves C_1 , C_2 , and C_3 defined over \mathbb{k} . Each of these curves contains 8 points each of which is fixed by a non-trivial element of N .

Let $f : X \rightarrow X/N$ be the quotient map and let

$$\pi : \widetilde{X/N} \rightarrow X/N$$

be the minimal resolution of singularities. Then $f(C_i)$ is a rational curve containing four A_1 singularities. One has

$$f(C_i) \cdot f(C_j) = \frac{1}{4}C_i \cdot C_j = \frac{1}{4}K_X^2 = 1.$$

Thus $\pi_*^{-1}f(C_1)$, $\pi_*^{-1}f(C_2)$, and $\pi_*^{-1}f(C_3)$ are three disjoint (-1) -curves defined over \mathbb{k} (see Table 1). We can contract these three curves and get a surface Y such that

$$K_Y^2 = K_{\widetilde{X/N}}^2 + 3 = K_{X/N}^2 + 3 = \frac{1}{4}K_X^2 + 3 = 4.$$

Note that the curves $\pi_*^{-1}f(C_1)$, $\pi_*^{-1}f(C_2)$, and $\pi_*^{-1}f(C_3)$ intersect all (-2) -curves on $\widetilde{X/N}$, since the curves $f(C_1)$, $f(C_2)$, and $f(C_3)$ pass through all A_1 singular points on X/N . Therefore Y does not contain curves with self-intersection less than -1 . Thus Y is a del Pezzo surface of degree 4.

Suppose that $\rho(X)^G = 1$ and for each non-trivial element of N all its fixed points are in one orbit of the group $G \times \text{Gal}(\overline{\mathbb{k}}/\mathbb{k})$. Let k be the number of conjugacy classes in G containing non-trivial elements of N . Then the fixed points of non-trivial elements of N lie in k orbits of the group $G \times \text{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ and curves C_1 , C_2 , and C_3 form k orbits of the group $G \times \text{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ (so that in particular $k \leq 3$). Therefore

$$\rho(Y)^{G/N} = \rho(\widetilde{X/N})^{G/N} - k = \rho(X/N)^{G/N} = \rho(X)^G = 1.$$

Thus Y is G/N -minimal. □

Now we prove Proposition 5.1.

Proof of Proposition 5.1. If the group G contains an element conjugate to ι_{1234} , then by Lemma 5.7 there exists a normal subgroup $N \triangleleft G$ such that a G/N -MMP-reduction Y of X/N is a \mathbb{k} -form of a toric surface.

If the group $G \cap \mathfrak{C}_2^4$ is conjugate to the group $\mathfrak{A}_4 = \langle \iota_{12}, \iota_{13} \rangle$, then the group G is conjugate to a subgroup of

$$\mathfrak{A}_4 \rtimes (\mathfrak{S}_3 \times \mathfrak{C}_2) = \langle \iota_{12}, \iota_{13}, (123), (12), (45) \rangle.$$

Such a group G cannot contain a subgroup conjugate to $\mathfrak{C}_2 = \langle (12) \rangle$ by Lemma 5.2. If the group G does not contain an element of order 3, then either $G = \mathfrak{A}_4$ or G contains a normal subgroup conjugate to $N = \langle (12)(45) \rangle$ and there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface by Lemma 5.3. Otherwise by Lemma 5.8 the quotient X/N is G/N -birationally equivalent to a del Pezzo surface Z of degree 4, and the group G/N contains an element of order 3. So we can replace X by Z , G by G/N , and start the proof from the beginning with a smaller group.

If the group $G \cap \mathfrak{C}_2^4$ is conjugate to the group $\mathfrak{C}_2 = \langle \iota_{12} \rangle$, then the group G is conjugate to a subgroup of

$$\mathfrak{C}_2 \times (\mathfrak{C}_2 \times \mathfrak{S}_3) = \langle \iota_{12}, (12), (345), (34) \rangle.$$

Such a group G cannot contain a subgroup conjugate to $\mathfrak{C}_2 = \langle (12) \rangle$ by Lemma 5.2. If such a group G contains a subgroup conjugate to $N = \langle (345) \rangle$, then this

group is normal and there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface by Lemma 5.5. Otherwise either the group G is conjugate to $\langle \iota_{12} \rangle$ or $\langle \iota_{15}(12)(34) \rangle$ or the group G contains a normal subgroup conjugate to $N = \langle (12)(34) \rangle$, and there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface by Lemma 5.3.

If the group $G \cap \mathfrak{C}_2^4$ is trivial, then G is isomorphic to a subgroup of \mathfrak{S}_5 . By Lemma 5.2 the group G cannot contain subgroups conjugate to $\mathfrak{C}_2 = \langle (12) \rangle$, \mathfrak{A}_4 , and \mathfrak{A}_5 . Thus by Lemma 2.14 the group G contains a normal subgroup N conjugate to $\mathfrak{C}_2 = \langle (12)(34) \rangle$, $\mathfrak{C}_3 = \langle (123) \rangle$, or $\mathfrak{C}_5 = \langle (12345) \rangle$. In the last case the surface X is not G -minimal by Lemma 5.6, and the quotient X/G is \mathbb{k} -rational by Theorems 2.21, 2.23, 2.22, and Propositions 3.1 and 4.1. In the other two cases there exists a G/N -MMP-reduction Y of X/N such that Y is a \mathbb{k} -form of a toric surface by Lemmas 5.3 and 5.5.

In all cases $Y(\mathbb{k}) \neq \emptyset$ since $X(\mathbb{k}) \neq \emptyset$. Thus

$$Y/(G/N) \approx X/G$$

is \mathbb{k} -rational by Theorems 2.21, 2.23, 2.22, and Proposition 3.1. □

6. EXAMPLES OF NON-RATIONAL QUOTIENTS

In this section we show that for the groups $\mathfrak{C}_2 = \langle \iota_{12} \rangle$, $\mathfrak{A}_4 = \langle \iota_{12}, \iota_{13} \rangle$ and $\mathfrak{C}_4 = \langle (12)(34)\iota_{15} \rangle$ acting on a del Pezzo surface of degree 4 the quotient may be non- \mathbb{k} -rational. We use the notation of Section 5.

We start with the quotients of a del Pezzo surface of degree 4 by the group $\mathfrak{C}_2 = \langle \iota_{12} \rangle$.

Lemma 6.1. *Let a finite group G act on a del Pezzo surface X of degree 4 and let*

$$N \cong \mathfrak{C}_2 = \langle \iota_{12} \rangle$$

be a normal subgroup in G . Then the surface X/N is G/N -birationally equivalent to a conic bundle Y with $K_Y^2 = 2$. If $\rho(X)^G = 1$ and all fixed points of ι_{12} are in one orbit of the group $G \times \text{Gal}(\bar{\mathbb{k}}/\mathbb{k})$, then Y is G/N -minimal.

Proof. The element ι_{12} has four fixed points on X cut out by the plane $x_1 = x_2 = 0$. Let \mathcal{C} be a one-dimensional linear subsystem of $|-K_X|$ spanned by the curves $x_1 = 0$ and $x_2 = 0$. A general member of \mathcal{C} is an N -invariant elliptic curve passing through all fixed points of N .

Let $f : X \rightarrow X/N$ be the quotient morphism and let $\pi : Y \rightarrow X/N$ be the minimal resolution of singularities. A general member of the linear system $f_*\mathcal{C}$ is a smooth conic passing through four A_1 singularities of X/N . Therefore the linear system $\pi_*^{-1}f_*\mathcal{C}$ gives a conic bundle structure

$$\varphi_{\pi_*^{-1}f_*\mathcal{C}} : Y \rightarrow \mathbb{P}_{\mathbb{k}}^1.$$

One has

$$K_Y^2 = K_{X/N}^2 = \frac{1}{2}K_X^2 = 2.$$

If all fixed points of ι_{12} are in one orbit of the group $G \times \text{Gal}(\bar{\mathbb{k}}/\mathbb{k})$, then

$$\rho(Y)^{G/N} = \rho(X/N)^{G/N} + 1 = \rho(X)^G + 1.$$

Thus if $\rho(X)^G = 1$, then $\rho(Y)^{G/N} = 2$ and Y is G/N -minimal by Theorem 2.8(iii). □

Remark 6.2. Note that in Lemma 6.1 if one of the ι_{12} -fixed points $p \in \overline{X}$ is defined over \mathbb{k} and G -fixed, then $\pi^{-1}(f(p))$ is a section E of the conic bundle $Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ defined over \mathbb{k} such that $E^2 = -2$. The conic bundle $Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ has 6 singular fibres since $K_Y^2 = 2$. Therefore one can G/N -equivariantly contract 6 components of singular fibres meeting E and get a surface of degree 8 which is \mathbb{k} -rational by Theorem 2.9.

Now we construct an explicit example satisfying the conditions of Lemma 6.1.

Example 6.3. Consider a surface X in $\mathbb{P}_{\mathbb{k}}^4$ given by the equations

$$(6.4) \quad x_1^2 + x_3^2 - x_4^2 - x_5^2 = 0, \quad -x_2^2 + 2x_3^2 - x_4^2 - 4x_5^2 = 0.$$

Note that $X(\mathbb{k}) \neq \emptyset$ since the \mathbb{k} -point $(0 : 1 : 1 : 1 : 0)$ lies on X .

The normal subgroup \mathfrak{C}_2^4 of $W(D_5)$ acts on X by switching signs of coordinates: elements ι_{ij} and ι_{ijkl} switch signs of coordinates x_i, x_j and x_i, x_j, x_k, x_l , respectively. In particular, the group $G = \langle \iota_{12} \rangle$ acts on $\mathbb{P}_{\mathbb{k}}^4$ and switches signs of coordinates x_1 and x_2 . The group G has four fixed points $(0 : 0 : \pm\sqrt{3} : \pm\sqrt{2} : 1)$ on \overline{X} .

One can check that sixteen (-1) -curves on the surface \overline{X} are given by the following parametrization:

$$\left(\pm(\alpha x + y) : \pm \left(i \left(1 + \sqrt{6} \right) x + i \left(2\sqrt{2} + \sqrt{3} \right) y \right) : \pm(x - \alpha y) : \pm(\alpha x - y) : \pm(x + \alpha y) \right)$$

where $\alpha = \sqrt{2} + \sqrt{3}$. These curves are defined over any field containing $i, \sqrt{2}$, and $\sqrt{3}$.

Example 6.5. Suppose that in Example 6.3 the field \mathbb{k} does not contain $i, \sqrt{2}$, and $\sqrt{3}$, and

$$\text{Gal} \left(\mathbb{k} \left(i, \sqrt{2}, \sqrt{3} \right) / \mathbb{k} \right) \cong \mathfrak{C}_2^3.$$

For instance, this holds for $\mathbb{k} = \mathbb{Q}$. In this case the image of the group $\text{Gal} \left(\mathbb{k} \left(i, \sqrt{2}, \sqrt{3} \right) / \mathbb{k} \right)$ in $W(D_5)$ is $\langle \iota_{1345}, \iota_{15}, \iota_{45} \rangle$. The surface X admits a structure of a minimal conic bundle by Theorem 2.8(iii), since $\rho(X) = 2$. Thus the surface X is not \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$, and the four fixed points of G on X are permuted by the Galois group. Thus the surface Y admits a structure of a minimal conic bundle with $K_Y^2 = 2$ by Lemma 6.1 and Y is not \mathbb{k} -rational by Theorem 2.9. This gives us an example of a non- \mathbb{k} -rational quotient of a $(G$ -minimal) non- \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{C}_2 .

Now assume that in Example 6.3 the field \mathbb{k} contains $i\sqrt{2}$ but does not contain $\sqrt{2}, \sqrt{3}$, and $\sqrt{6}$. In this case the image of the Galois group

$$\text{Gal} \left(\mathbb{k} \left(\sqrt{2}, \sqrt{3} \right) / \mathbb{k} \right) \cong \mathfrak{C}_2^2$$

in $W(D_5)$ is $\langle \iota_{34}, \iota_{45} \rangle$. The quadruple of (-1) -curves E_1, E_5, L_{23}, E_4 is defined over \mathbb{k} . Thus one can Galois-equivariantly contract this quadruple and get a del Pezzo surface of degree 8. Therefore X is \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$, and the four fixed points of G on X are permuted by the Galois group. Also the surface Y admits a structure of a minimal conic bundle with $K_Y^2 = 2$ by Lemma 6.1 and Y is not \mathbb{k} -rational by Theorem 2.9. This gives us an example of

a non- \mathbb{k} -rational quotient of a (G -minimal) \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{C}_2 .

If the field \mathbb{k} contains $\sqrt{2}$ and $\sqrt{3}$ but does not contain i , then the image of the Galois group

$$\text{Gal}(\mathbb{k}(i)/\mathbb{k}) \cong \mathfrak{C}_2$$

in $W(D_5)$ is $\langle \iota_{1345} \rangle$. The surface X admits a structure of a minimal conic bundle by Theorem 2.8(iii), since $\rho(X) = 2$. Thus the surface X is not \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$. But on the conic bundle $Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ all sections with self-intersection -2 are defined over \mathbb{k} . Therefore we can Galois-equivariantly contract six components of the singular fibres intersecting one section of the conic bundle $Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ with self-intersection -2 and get a del Pezzo surface of degree 8 which is \mathbb{k} -rational by Theorem 2.9. This gives us an example of a \mathbb{k} -rational quotient of a G -minimal non- \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{C}_2 .

Now assume that in Example 6.3 the field \mathbb{k} contains $i\sqrt{2}$ and $\sqrt{3}$ but does not contain $\sqrt{2}$. In this case the image of the Galois group

$$\text{Gal}(\mathbb{k}(\sqrt{2})/\mathbb{k}) \cong \mathfrak{C}_2$$

in $W(D_5)$ is $\langle \iota_{34} \rangle$. The pair of (-1) -curves E_1 and L_{25} is defined over \mathbb{k} . Thus one can Galois-equivariantly contract this pair and get a del Pezzo surface of degree 6. Therefore X is \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 2$; therefore X admits a structure of a G -minimal conic bundle by Theorem 2.8(iii). Note that singular fibres of the conic bundle $Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ correspond to reducible members of the linear system spanned by the curves $x_1 = 0$ and $x_2 = 0$. One has $K_Y^2 = 2$; therefore the conic bundle $Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ has six singular fibres. The proper transforms of these six fibres are cut out from $\bar{X} \subset \mathbb{P}_{\mathbb{k}}^4$ by the following six hyperplanes:

$$x_2 = \pm ix_1, \quad x_2 = \pm i\sqrt{2}x_1, \quad x_2 = \pm 2ix_1.$$

On the conic bundle $Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ we can Galois-equivariantly contract four components of the singular fibres corresponding to the hyperplane sections $x_2 = \pm ix_1$, $x_2 = \pm 2ix_1$ and get a del Pezzo surface of degree 6 which is \mathbb{k} -rational by Theorem 2.9. This gives us an example of a \mathbb{k} -rational quotient of a G -minimal \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{C}_2 .

Now we show that the quotient of a del Pezzo surface of degree 4 by a group \mathfrak{A}_4 can be non- \mathbb{k} -rational.

Example 6.6. Let us consider the quotient of the surface X given by equations (6.4) by the group $G = \langle \iota_{12}, \iota_{13} \rangle$. The element ι_{12} has four fixed points $(0 : 0 : \pm\sqrt{3} : \pm\sqrt{2} : 1)$, the element ι_{13} has four fixed points $(0 : \pm i\sqrt{3} : 0 : \pm i : 1)$, and the element ι_{23} has four fixed points $(\pm i\sqrt{3} : 0 : 0 : \pm 2i : 1)$.

Suppose that in Example 6.3 the field \mathbb{k} does not contain i , $\sqrt{2}$, and $\sqrt{3}$, and

$$\text{Gal}(\mathbb{k}(i, \sqrt{2}, \sqrt{3})/\mathbb{k}) \cong \mathfrak{C}_2^3.$$

As in Example 6.3 the sixteen (-1) -curves on X are defined over the field $\mathbb{k}(i, \sqrt{2}, \sqrt{3})$. The image of the group $\text{Gal}(\mathbb{k}(i, \sqrt{2}, \sqrt{3})/\mathbb{k})$ in $W(D_5)$ is $\langle \iota_{1345}, \iota_{15}, \iota_{45} \rangle$. In this case $\rho(X) = 2$, and X admits a structure of a minimal conic bundle by Theorem 2.8(iii). Thus the surface X is not \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$, and for each non-trivial element in G its four fixed points are permuted by the Galois group. Thus the quotient surface X/G is birationally

equivalent to a minimal del Pezzo surface Y of degree 4 by Lemma 5.8. The surface Y is not \mathbb{k} -rational by Theorem 2.9. This gives us an example of a non- \mathbb{k} -rational quotient of a (G -minimal) non- \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{A}_4 .

Now assume that in Example 6.3 the field \mathbb{k} contains $i\sqrt{2}$ but does not contain $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$. In this case the image of the Galois group

$$\text{Gal}\left(\mathbb{k}\left(\sqrt{2}, \sqrt{3}\right)/\mathbb{k}\right) \cong \mathfrak{C}_2^2$$

in $W(D_5)$ is $\langle \iota_{34}, \iota_{45} \rangle$. The quadruple of (-1) -curves E_1, E_5, L_{23}, E_4 is defined over \mathbb{k} . Thus one can Galois-equivariantly contract this quadruple and get a del Pezzo surface of degree 8. Therefore X is \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$, and for each non-trivial element in G its four fixed points are permuted by the group $G \times \text{Gal}(\bar{\mathbb{k}}/\mathbb{k})$. Thus by Lemma 5.8 the quotient surface X/G is birationally equivalent to a minimal del Pezzo surface Y of degree 4 and Y is not \mathbb{k} -rational by Theorem 2.9. This gives us an example of a non- \mathbb{k} -rational quotient of a G -minimal \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{A}_4 .

Now assume that in Example 6.3 the field \mathbb{k} contains i and $\sqrt{2}$ but does not contain $\sqrt{3}$. In this case the image of the Galois group

$$\text{Gal}\left(\mathbb{k}\left(\sqrt{3}\right)/\mathbb{k}\right) \cong \mathfrak{C}_2$$

in $W(D_5)$ is $\langle \iota_{45} \rangle$. The pair of (-1) -curves E_1 and L_{23} is defined over \mathbb{k} . Thus one can Galois-equivariantly contract this pair and get a del Pezzo surface of degree 6. Therefore X is \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$, but for each non-trivial element in G its four fixed points are not permuted transitively by the Galois group $\text{Gal}\left(\mathbb{k}\left(\sqrt{3}\right)/\mathbb{k}\right)$. Thus by Lemma 5.8 the quotient surface X/G is birationally equivalent to a del Pezzo surface Y of degree 4 with $\rho(Y) = 4$, and Y is \mathbb{k} -rational by Corollary 2.12. This gives us an example of a \mathbb{k} -rational quotient of a G -minimal \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{A}_4 .

Example 6.7. In Example 6.3 consider the quotient of X by the group $G = \langle \iota_{12}, \iota_{14} \rangle$. The element ι_{12} has four fixed points $(0 : 0 : \pm\sqrt{3} : \pm\sqrt{2} : 1)$, the element ι_{14} has four fixed points $(0 : \pm i\sqrt{2} : \pm 1 : 0 : 1)$, and the element ι_{24} has four fixed points $(\pm i : 0 : \pm\sqrt{2} : 0 : 1)$.

Suppose that the field \mathbb{k} contains $\sqrt{2}$ and $\sqrt{3}$ and does not contain i . Then the image of the Galois group

$$\text{Gal}\left(\mathbb{k}(i)/\mathbb{k}\right) \cong \mathfrak{C}_2$$

in $W(D_5)$ is $\langle \iota_{1345} \rangle$. The surface X admits a structure of a minimal conic bundle by Theorem 2.8(iii), since $\rho(X) = 2$. Thus the surface X is not \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$. But for each non-trivial element in G the four fixed points are not permuted transitively by the Galois group $\text{Gal}\left(\mathbb{k}(i)/\mathbb{k}\right)$. Thus by Lemma 5.8 the quotient surface X/G is birationally equivalent to a del Pezzo surface Y of degree 4 with $\rho(Y) = 4$, and Y is \mathbb{k} -rational by Corollary 2.12. This gives us an example of a \mathbb{k} -rational quotient of a (G -minimal) non- \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{A}_4 .

Now we show that the quotient of a del Pezzo surface of degree 4 by a group \mathfrak{C}_4 can be non- \mathbb{k} -rational.

Lemma 6.8. *Let a finite group G act on a del Pezzo surface X of degree 4 and let*

$$N \cong \mathfrak{C}_4 = \langle (12)(34)\iota_{15} \rangle$$

be a normal subgroup in G . Then the surface X/N is G/N -birationally equivalent to a conic bundle Y with $K_Y^2 = 4$.

Proof. After a suitable change of coordinates the group N is generated by an element

$$g : (x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (-x_2 : x_1 : x_4 : x_3 : -x_5).$$

The element g is of order 4, and the element g^2 has four fixed points $p_1, p_2, p_3,$ and p_4 on X cut out by the plane $x_1 = x_2 = 0$. These points are the intersection points of two N -invariant conics C_1 and C_2 in the plane $x_1 = x_2 = 0$. These conics cannot be pointwisely fixed by the group N , since N has only four fixed points. Therefore N faithfully acts on C_1 and C_2 , has two fixed points on C_1 , and has two fixed points on C_2 . Thus after relabelling the points p_i if necessary, we have $gp_1 = p_1, gp_2 = p_2, gp_3 = p_4,$ and $gp_4 = p_3$.

Consider a linear system spanned by the curves $x_3 + x_4 = 0$ and $x_5 = 0$. In this linear system there is exactly one member C passing through the points p_3 and p_4 . There are two g^2 -fixed points on C . An element of an automorphism group of order 2 cannot have two fixed points on an elliptic curve. Therefore C is a singular curve; moreover C has more than one singular point since the element g does not have fixed points on C . Thus C is reducible and consists of two smooth conics meeting each other at points p_3 and p_4 .

Let $f : X \rightarrow X/N$ be the quotient map and let

$$\pi : \widetilde{X/N} \rightarrow X/N$$

be the minimal resolution of singularities. One can show that $\widetilde{X/N}$ admits a conic bundle structure as in the proof of Lemma 6.1.

The points $f(p_1)$ and $f(p_2)$ are A_3 singularities. Exceptional divisors of their resolutions are chains consisting of three (-2) -curves each. The point $f(p_3) = f(p_4)$ is an A_1 singularity. The curves $\pi_*^{-1}f(C_1), \pi_*^{-1}f(C_2),$ and $\pi_*^{-1}f(C)$ are three disjoint (-1) -curves (see Table 1).

Let $\sigma : X \rightarrow Y$ be the contraction of the curves $\pi_*^{-1}f(C_1), \pi_*^{-1}f(C_2),$ and $\pi_*^{-1}f(C)$. Then

$$K_Y^2 = K_{\widetilde{X/N}}^2 + 3 = K_{X/N}^2 + 3 = \frac{1}{4}K_X^2 + 3 = 4,$$

and Y is a conic bundle. □

Remark 6.9. The reducible curves $\pi^{-1}f(p_i)$ are chains of three (-2) -curves. The conic bundle Y obtained in Lemma 6.8 has two sections with self-intersection -2 which are transforms of the central (-2) -curves in these chains. There exists an elliptic curve E such that the surface \overline{Y} is birationally equivalent to a quotient of $\mathbb{P}_{\overline{\mathbb{k}}}^1 \times E$ by an involution (see [DI09, Subsection 5.2] for details). Such a surface is called an *Iskovskikh surface*.

Remark 6.10. In the notation of Lemma 6.8 one can check that if $\rho(X)^G = 1$ and the points p_1 and p_2 are permuted by an element of $G \times \text{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ which does not permute the curves C_1 and C_2 then $\rho(Y)^{G/N} = 2$ and Y is G/N -minimal by Theorem 2.8(iii).

Now we construct an explicit example satisfying the conditions of Lemma 6.8.

Example 6.11. Suppose that the field \mathbb{k} contains i and does not contain $\sqrt{2}$ and $\sqrt{6}$. Consider a surface X in $\mathbb{P}_{\mathbb{k}}^4$ given by the equations

$$4x_1^2 - 4x_2^2 - x_3^2 + x_4^2 = 0, \quad 2x_1^2 + 2x_2^2 - x_3^2 - x_4^2 + 12x_5^2 = 0.$$

Note that $X(\mathbb{k}) \neq \emptyset$ since the \mathbb{k} -point $(1 : i : 2 : 2i : 0)$ lies on X .

The normal subgroup \mathfrak{C}_2^4 of $W(D_5)$ acts on X by switching signs of coordinates: elements ι_{ij} and ι_{ijkl} switch signs of coordinates x_i, x_j and x_i, x_j, x_k, x_l , respectively. The group $G \cong \mathfrak{C}_4$ generated by an element

$$g : (x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (-x_2 : x_1 : x_4 : x_3 : -x_5)$$

acts on X . The element g^2 has four fixed points on \overline{X} :

$$\begin{aligned} p_1 &= (0 : 0 : \sqrt{6} : -\sqrt{6} : 1), & p_2 &= (0 : 0 : -\sqrt{6} : \sqrt{6} : 1), \\ p_3 &= (0 : 0 : \sqrt{6} : \sqrt{6} : 1), & p_4 &= (0 : 0 : -\sqrt{6} : -\sqrt{6} : 1). \end{aligned}$$

One can check that sixteen (-1) -curves on the surface \overline{X} are given by the following parametrization:

$$\left(\pm(x+y) : \pm(x-y) : \pm\sqrt{2}(x+2y) : \pm\sqrt{2}(x-2y) : \pm y \right).$$

These curves are defined over any field containing $\sqrt{2}$, and the image of the group

$$\text{Gal}\left(\mathbb{k}\left(\sqrt{2}\right)/\mathbb{k}\right) \cong \mathfrak{C}_2$$

in $W(D_5)$ is $\langle \iota_{34} \rangle$. The pair of (-1) -curves E_3 and E_4 is defined over \mathbb{k} . Thus one can Galois-equivariantly contract this pair and get a del Pezzo surface of degree 6. Therefore X is \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$, and the points p_1 and p_2 are permuted by an element of $G \times \text{Gal}\left(\overline{\mathbb{k}}/\mathbb{k}\right)$ which does not permute the curves C_1 and C_2 , given by $x_1 = \pm ix_2$. Therefore by Remark 6.10 the quotient surface X/G is birationally equivalent to a minimal conic bundle Y with $K_Y^2 = 4$ and Y is not \mathbb{k} -rational by Theorem 2.9. This gives us an example of a non- \mathbb{k} -rational quotient of a G -minimal \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{C}_4 .

Now assume that the field \mathbb{k} contains $\sqrt{6}$ but does not contain $\sqrt{2}$. In this case the image of the Galois group

$$\text{Gal}\left(\mathbb{k}\left(\sqrt{2}\right)/\mathbb{k}\right) \cong \mathfrak{C}_2$$

in $W(D_5)$ is $\langle \iota_{34} \rangle$. The pair of (-1) -curves E_3 and E_4 is defined over \mathbb{k} . Thus one can Galois-equivariantly contract this pair and get a del Pezzo surface of degree 6. Therefore X is \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$, but the points p_1 and p_2 are not permuted by the Galois group $\text{Gal}\left(\mathbb{k}\left(\sqrt{2}\right)/\mathbb{k}\right)$. Thus by Lemma 6.8 the quotient surface X/G is birationally equivalent to a minimal conic bundle Y with $K_Y^2 = 4$ with $\rho(Y) \geq 3$, and Y is \mathbb{k} -rational by Corollary 2.12. This gives us an example of a \mathbb{k} -rational quotient of a G -minimal \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{C}_4 .

Example 6.12. Suppose that the field \mathbb{k} contains i and does not contain $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$. Consider a surface X in $\mathbb{P}_{\mathbb{k}}^4$ given by the equations

$$4x_1^2 - 4x_2^2 - 3x_3^2 + 3x_4^2 = 0, \quad 2x_1^2 + 2x_2^2 - 3x_3^2 - 3x_4^2 + 36x_5^2 = 0.$$

Note that $X(\mathbb{k}) \neq \emptyset$ since the \mathbb{k} -point $(3 : 3 : 0 : 0 : i)$ lies on X .

The normal subgroup \mathfrak{C}_2^4 of $W(D_5)$ acts on X by switching signs of coordinates: elements ι_{ij} and ι_{ijkl} switch signs of coordinates x_i, x_j and x_i, x_j, x_k, x_l , respectively. The group $G \cong \mathfrak{C}_4$ generated by an element

$$g : (x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (-x_2 : x_1 : x_4 : x_3 : -x_5)$$

acts on X . The element g^2 has four fixed points on \bar{X} :

$$p_1 = (0 : 0 : \sqrt{6} : -\sqrt{6} : 1), \quad p_2 = (0 : 0 : -\sqrt{6} : \sqrt{6} : 1),$$

$$p_3 = (0 : 0 : \sqrt{6} : \sqrt{6} : 1), \quad p_4 = (0 : 0 : -\sqrt{6} : -\sqrt{6} : 1).$$

One can check that sixteen (-1) -curves on the surface \bar{X} are given by the following parametrization:

$$\left(\pm\sqrt{3}(x+y) : \pm\sqrt{3}(x-y) : \pm\sqrt{2}(x+2y) : \pm\sqrt{2}(x-2y) : \pm y \right).$$

These curves are defined over any field containing $\sqrt{2}$ and $\sqrt{3}$, and the image of the group

$$\text{Gal} \left(\mathbb{k} \left(\sqrt{2}, \sqrt{3} \right) / \mathbb{k} \right) \cong \mathfrak{C}_2^2$$

in $W(D_5)$ is $\langle \iota_{12}, \iota_{34} \rangle$. In this case $\rho(X) = 2$, and X admits a structure of a minimal conic bundle by Theorem 2.8(iii). Thus the surface X is not \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$, and the points p_1 and p_2 are permuted by an element of $G \times \text{Gal}(\bar{\mathbb{k}}/\mathbb{k})$ which does not permute the curves C_1 and C_2 , given by $x_1 = \pm ix_2$. Therefore by Remark 6.10 the quotient surface X/G is birationally equivalent to a minimal conic bundle Y with $K_Y^2 = 4$ and Y is not \mathbb{k} -rational by Theorem 2.9. This gives us an example of a non- \mathbb{k} -rational quotient of a (G -minimal) non- \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{C}_4 .

Now assume that the field \mathbb{k} contains $\sqrt{6}$ but does not contain $\sqrt{2}$ and $\sqrt{3}$. In this case the image of the Galois group

$$\text{Gal} \left(\mathbb{k} \left(\sqrt{2} \right) / \mathbb{k} \right) \cong \mathfrak{C}_2$$

in $W(D_5)$ is $\langle \iota_{1234} \rangle$. In this case $\rho(X) = 2$, and X admits a structure of a minimal conic bundle by Theorem 2.8(iii). Thus the surface X is not \mathbb{k} -rational by Theorem 2.9. One has $\rho(X)^G = 1$, but the points p_1 and p_2 are not permuted by the Galois group $\text{Gal}(\mathbb{k}(\sqrt{2})/\mathbb{k})$. Thus by Lemma 6.8 the quotient surface X/G is birationally equivalent to a minimal conic bundle Y with $K_Y^2 = 4$ with $\rho(Y) \geq 3$ and Y is \mathbb{k} -rational by Corollary 2.12. This gives us an example of a \mathbb{k} -rational quotient of a (G -minimal) non- \mathbb{k} -rational del Pezzo surface of degree 4 by the group \mathfrak{C}_4 .

ACKNOWLEDGMENTS

The author is grateful to his adviser Yu. G. Prokhorov and to C. A. Shramov for useful discussions and to I. V. Netay for his help in finding explicit equations of (-1) -curves in Examples 6.5 and 6.11.

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