TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 9, September 2018, Pages 6651–6683
http://dx.doi.org/10.1090/tran/7200
Article electronically published on September 15, 2017

STRONG HYPERCONTRACTIVITY AND LOGARITHMIC SOBOLEV INEQUALITIES ON STRATIFIED COMPLEX LIE GROUPS

NATHANIEL ELDREDGE, LEONARD GROSS, AND LAURENT SALOFF-COSTE

Abstract. We show that for a hypoelliptic Dirichlet form operator A on a stratified complex Lie group, if the logarithmic Sobolev inequality holds, then a holomorphic projection of A is strongly hypercontractive in the sense of Janson. This extends previous results of Gross to a setting in which the operator A is not holomorphic.

1. Introduction

In [10–13], subsets of the current authors, together with Bruce K. Driver, studied properties of elliptic and hypoelliptic heat kernels on complex Lie groups and homogeneous spaces, particularly the Taylor map for L^2 holomorphic functions. Generally, it was shown that hypoelliptic heat kernels and their sub-Laplacians often behave similarly to their elliptic counterparts, such as the Gaussian heat kernel and standard Laplacian on $\mathbb{C}^n$. In this paper we turn our attention to the phenomenon of strong hypercontractivity in the particular case of stratified complex Lie groups.

To motivate this study, let us first consider Euclidean space $\mathbb{R}^n$ equipped with standard Gaussian measure ν. Let $Q(f,g)$ be the Dirichlet form with core $C_c^\infty(\mathbb{R}^n)$ defined by $Q(f,g) = \int_{\mathbb{R}^n} \nabla f \cdot \nabla \bar{g}\, d\nu$, whose generator is the Ornstein–Uhlenbeck operator $Af(x) = -\Delta f(x) + x \cdot \nabla f(x)$. In [38], E. Nelson discovered that the semigroup e^{-tA} enjoys the following property known as **hypercontractivity**.

Theorem 1.1. *For $1 < q \le p < \infty$, let $t_N(p,q) = \frac{1}{2}\log\left(\frac{p-1}{q-1}\right)$. Then for any $t \ge t_N$, e^{-tA} is a contraction from $L^q(\nu)$ to $L^p(\nu)$.*

So the semigroup e^{-tA} improves local integrability of functions with respect to ν; as soon as t exceeds "Nelson's time" $t_N(p,q)$, e^{-tA} maps L^q into L^p. Moreover, Nelson's time is sharp: for $t < t_N(p,q)$, e^{-tA} is unbounded from L^q to L^p. For a short history of this theorem, see the survey [28].

Now replace ν with any smooth measure μ on $\mathbb{R}^n$ and redefine Q and A accordingly. In [24], the second author introduced the notion of a **logarithmic Sobolev**

Received by the editors January 12, 2016, and, in revised form, January 12, 2017.

2010 *Mathematics Subject Classification.* Primary 35R03, 35H20; Secondary 43A15, 32W30.

Key words and phrases. Stratified complex Lie group, hypoelliptic heat kernel, strong hypercontractivity, logarithmic Sobolev inequality, holomorphic L^p space.

The first author was supported in part by a grant from the Simons Foundation (#355659, Nathaniel Eldredge).

The third author was supported in part by National Science Foundation grant DMS 1404435.

inequality, which (in its simplest version) is said to be satisfied by μ if

$$\int_{\mathbb{R}^n} |f|^2 \log |f| \, d\mu \le Q(f) + \|f\|_{L^2(\mu)}^2 \log \|f\|_{L^2(\mu)} \tag{1.1}$$

for all f in the domain of Q.

(Actually, in this paper, we shall study a more general version of (1.1) in which the coefficient of $Q(f)$ is a constant c other than 1, and in which a term of the form $\beta \|f\|_{L^2}^2$ can be added to the right side. See (7.1). The general version can also be used in the theorems in this introduction, making appropriate changes to the constants, but for simplicity we omit the details here.)

It was shown in [24] that in this case the logarithmic Sobolev inequality (1.1) is essentially equivalent to hypercontractivity.

Theorem 1.2. *A smooth measure μ on $\mathbb{R}^n$ satisfies the logarithmic Sobolev inequality* (1.1) *if and only if the corresponding semigroup e^{-tA} is hypercontractive* (*with Nelson's time t_N*).

The early history of these two types of inequalities devolves from two different backgrounds. In 1959 A. J. Stam [40], motivated by problems in information theory, proved an inequality, based on Lebesgue measure rather than on Gauss measure, easily transformable into the Gaussian special case of (1.1). In 1966 E. Nelson [37], motivated by the problem of semiboundedness of Hamiltonian operators in quantum field theory, proved the first version of the hypercontractivity inequality of Theorem 1.1 with dimension dependent bounds. In order to encompass a larger class of Hamiltonians, J. Glimm [21] sharpened Nelson's inequality in 1968 and removed the dimension dependence, thereby enabling the inequality to work in infinite dimensions. Subsequently Nelson [38], in 1973, found the best hypercontractivity constants, which are those presented in Theorem 1.1. Pursuing a different track to semiboundedness of quantum field Hamiltonians, P. Federbush [17] showed in 1969 that semiboundedness would follow from a logarithmic Sobolev inequality much more easily than from hypercontractivity. His semiboundedness theorem essentially asserts that a logarithmic Sobolev inequality implies semiboundedness. In this paper he also gave a derivation of a Gaussian logarithmic Sobolev inequality using delicate Hermite function expansions in infinitely many variables. Although his version of a logarithmic Sobolev inequality is not written in this paper, it follows easily from the identity [17, Equ. (14)] and inequality [17, Equ. (21)]. He thereby recovered semiboundedness for the class of Hamiltonians originally addressed by Nelson, though not the class encompassed by Glimm's improvement. Ironically, using the semiboundedness theorem of Federbush, the sharp logarithmic Sobolev inequality of Stam would have yielded semiboundedness of the large class addressed by Glimm's improvement. But Stam's results were not known among this group of mathematical physicists till 1991, when Eric Carlen [9], discovered Stam's paper and made the connection with the Gaussian logarithmic Sobolev inequalities of the mathematical physics literature. In the meanwhile, the second author [24] proved in 1975 that a family of hypercontractivity bounds, such as those in Theorem 1.1, is equivalent to a logarithmic Sobolev inequality. Best constants are preserved in this equivalence. Theorem 1.2 is a typical case. He also proved the sharp form (1.1) of the Gaussian logarithmic Sobolev inequality, which Carlen later showed to be equivalent to the Euclidean form of Stam. With the help of the equivalence theorem, one can understand better the relation between Stam's and Federbush's versions of the logarithmic Sobolev inequality: the former is equivalent to the strong form of Glimm, while the latter is equivalent to the original form of Nelson.

Generalizations of the equivalence Theorem 1.2 are now known to hold in a wide variety of settings; see [2, 25, 28] for surveys and the recent exposition and historical background in [39].

Let us turn now to the complex setting; replace $\mathbb{R}^n$ by $\mathbb{C}^n$ and suppose that μ is a standard Gaussian measure on $\mathbb{C}^n$. S. Janson discovered in [31] that if one restricts the Ornstein–Uhlenbeck semigroup e^{-tA} to the holomorphic functions $\mathcal{H}$, then one obtains the property of **strong hypercontractivity**, in which the improvement in integrability happens at earlier times:

Theorem 1.3. *For $0 < q \le p < \infty$, let $t_J(p,q) = \frac{1}{2}\log\left(\frac{p}{q}\right)$. Then, for any $t \ge t_J$, e^{-tA} is a contraction from $\mathcal{H} \cap L^q(\mu)$ to $\mathcal{H} \cap L^p(\mu)$.*

Several other proofs of this theorem followed [8, 32, 46]. Note that "Janson's time" $t_J(p,q)$ is less than Nelson's time $t_N(p,q)$ whenever $1 < q < p < \infty$. Moreover Janson's strong hypercontractivity also has content for $0 < q \le p \le 1$. Very roughly, the reason for this is that holomorphic functions are harmonic, and so the second-order differential operator A, when restricted to $\mathcal{H}$, reduces to the first-order operator $Af(z) = z\cdot\nabla f(z)$. Thus it is not surprising that its behavior should be improved in this case. We note for later reference that in this case A is the holomorphic vector field which generates the flow of the dilations $\varphi_t(z) = tz$, meaning that the semigroup e^{-tA} is simply $e^{-tA}f(z) = f(e^{-t}z)$.

In the paper [26], the second author studied such Dirichlet form operators over a complex Riemannian manifold (M,g) equipped with a smooth measure μ, seeking to relate the logarithmic Sobolev inequality to strong hypercontractivity in a general holomorphic context. The result was that the former implies the latter, under fairly mild assumptions. In this result, the spaces $\mathcal{H} \cap L^p(\mu)$ must be replaced with possibly smaller spaces denoted $\mathcal{H}L^p(\mu)$; see Remark 4.6 below for the definitions used in [26], and see [26] for a complete discussion of the issues involved. As in the Euclidean case, the Dirichlet form operator A is given by the Laplacian over M plus a complex vector field Z, so that on holomorphic functions one has $Af = Zf$. If Z is a holomorphic vector field or, equivalently, if the operator A maps $\mathcal{H}$ into $\mathcal{H}$, we will say that A is **holomorphic**. Let $Y = i(Z - \bar{Z})$ be the imaginary part of Z.

Theorem 1.4 ([26, Theorem 2.19])**.** *Suppose that the operator A is holomorphic and that the real vector field Y is Killing. If the logarithmic Sobolev inequality* (1.1) *holds, then for any $t \ge t_J(p,q)$, e^{-tA} is a contraction from $\mathcal{H}L^q(\mu)$ to $\mathcal{H}L^p(\mu)$.*

A second proof was given in [27], which also allows for certain other types of boundary conditions in the case that (M,g) is incomplete.

The present paper is an extension of the results of [26, 27]. As noted, a key assumption of those papers was that A should be holomorphic. This assumption is in some sense natural, since it allows one to work entirely within the holomorphic category, and it is satisfied by many interesting examples. But there are also many apparently innocuous settings in which A is not holomorphic. See [26, 27, 29] and references therein for examples, counterexamples, and necessary and sufficient conditions; the same condition is studied, in other contexts, in [7, 19].

To the best of our knowledge, until now, there have been no strong hypercontractivity results akin to Theorem 1.4 that apply in the case where A is not holomorphic. As such, our goal here is to begin attacking this case by studying a particular class of examples in which A is not holomorphic, yet a strong hypercontractivity theorem can still be proved.

One possible way to approach the case where A is not holomorphic is, as suggested in [27, Section 7], to replace A by $B = P_{\mathcal{H}}A$, its L^2 orthogonal projection onto the holomorphic functions $\mathcal{H}$. Unfortunately, this does not always work, and [27] gives an example of a complex manifold (a cylinder) for which e^{-tB} is not strongly hypercontractive and is not even contractive on $L^p(\mu)$ for small $p < 1$.

In the present paper, we examine a class of spaces in which the operator A is not holomorphic, and yet we are able to show that e^{-tB} is strongly hypercontractive, where B is (at least on a large class of functions) the holomorphic projection of A. We work in the setting of complex stratified Lie groups, where we replace the Laplacian Δ by the hypoelliptic sub-Laplacian and take as our measure the corresponding hypoelliptic heat kernel. A key observation is that stratified Lie groups have a canonical dilation structure, and it turns out that, as in the case of the Gaussian measure on $\mathbb{C}^n$, the operator B is essentially the holomorphic vector field generated by dilations.

The paper is structured as follows.

- In Section 2 we introduce notation and review important properties of stratified complex Lie groups G, their sub-Riemannian geometry, and the hypoelliptic heat kernel ρ_a. We also begin a discussion of holomorphic polynomials on G.
- Section 3 defines the Dirichlet form Q and the operators A, B.
- In Section 4, we study the density properties of holomorphic polynomials, including an orthogonal decomposition of holomorphic functions in $L^2(\rho_a)$ into homogeneous polynomials, and obtain some additional properties of A, B and their domains. Section 4 also defines the function spaces $\mathcal{H}L^p(\rho_a)$ on which we work and discusses related subtleties.
- In Section 5, we show that the operator B is (up to scaling and domain issues) identical to the holomorphic vector field generated by dilations; we take advantage of this to show that (except in trivial cases) the operator A is not holomorphic.
- We then proceed to show in Section 6 that the semigroup e^{-tB} is a contraction on $L^p(\rho_a)$ for $0 < p < \infty$; this is the special case of strong hypercontractivity with $q = p$.
- Section 7 contains the proof of our main theorem, showing that if the logarithmic Sobolev inequality holds, then the semigroup e^{-tB} is strongly hypercontractive.
- In Section 8 we specifically consider the complex Heisenberg group for which the logarithmic Sobolev inequality does indeed hold.

2. Stratified complex groups

2.1. **Definitions.** In this section, we recall the definition of a stratified complex Lie group (respectively, algebra) and its basic properties. A comprehensive reference on stratified Lie groups is [6].

Definition 2.1. Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. We say $\mathfrak{g}$ is **stratified** of step m if it admits a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{j=1}^{m} V_j \tag{2.1}$$

for which

$$[V_1, V_j] = V_{j+1}, \qquad [V_1, V_m] = 0$$

and $V_m \neq 0$. A complex Lie group G is **stratified** if it is connected and simply connected and its Lie algebra $\mathfrak{g}$ is stratified.

Using the Jacobi identity, it is easy to show that in a stratified Lie algebra, we have $[V_k, V_j] \subset V_{j+k}$, where we take $V_{j+k} = 0$ for $j + k > m$. (Proceed by induction on k.) In particular, $\mathfrak{g}$ is nilpotent of step m. As such, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism, so we may as well take $G = \mathfrak{g}$ as sets and let the exponential map be the identity. The group operation on G can then be written explicitly using the Baker–Campbell–Hausdorff formula. We note that in G, the identity element e is 0, and the group inverse is given by $g^{-1} = -g$. We shall use $L_x : G \to G$ to denote the left translation map $L_x(y) = x \cdot y$. We identify $\mathfrak{g}$ with the tangent space T_eG, and for $\xi \in \mathfrak{g}$, $\widetilde{\xi}$ is the left-invariant vector field on G with $\widetilde{\xi}(e) = \xi$.

Since $\mathfrak{g}$ is a finite-dimensional vector space, it carries a translation-invariant Lebesgue measure, which is unique up to scaling. We fix one such measure and denote it by m; integrals with respect to dx, dy, etc., will also be understood to refer to this measure. Then m is also a measure on G. It is easy to verify that m is bi-invariant under the group operation on G, so m is (again up to scaling) the Haar measure on G.

Notation 2.2. We define **convolution** on G by

$$(f * g)(x) = \int_G f(xy^{-1})g(y)dy = \int_G f(z)g(z^{-1}x)dz \tag{2.2}$$

when the Lebesgue integral exists.

Our motivating examples are the complex Heisenberg and Heisenberg–Weyl groups.

Example 2.3. The **complex Heisenberg Lie algebra** is the complex Lie algebra $\mathfrak{h}_3^{\mathbb{C}}$ given by $\mathbb{C}^3$ with the bracket defined by

$$[(z_1, z_2, z_3), (z_1', z_2', z_3')] = (0, 0, z_1 z_2' - z_1' z_2). \tag{2.3}$$

Taking $V_1 = \{(z_1, z_2, 0) : z_1, z_2 \in \mathbb{C}\}$ and $V_2 = \{(0, 0, z_3) : z_3 \in \mathbb{C}\}$, it is clear that $\mathfrak{h}_3^{\mathbb{C}}$ is stratified of step 2. The **complex Heisenberg group** $\mathbb{H}_3^{\mathbb{C}}$ is then $\mathbb{C}^3$ with the group operation $g \cdot h = g + h + \frac{1}{2}[g, h]$, which we may write in coordinates as

$$(z_1, z_2, z_3) \cdot (z_1', z_2', z_3') = (z_1 + z_1', z_2 + z_2', z_3 + z_3' + \frac{1}{2}(z_1 z_2' - z_2 z_1')).$$

Some readers may be used to seeing the Heisenberg group as the group of upper triangular matrices with 1s on the diagonal. Let us note that by mapping the element $(z_1, z_2, z_3) \in \mathbb{H}_3^{\mathbb{C}}$ to the matrix

$$\begin{pmatrix} 1 & z_1 & z_3 + \frac{1}{2}z_1 z_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}$$

we have an embedding of the Lie group $\mathbb{H}_3^{\mathbb{C}}$ into the Lie group $GL(\mathbb{C}, 3)$ of invertible 3×3 complex matrices, whose image is precisely the upper triangular matrices with 1s on the diagonal. So this realization of the complex Heisenberg group is isomorphic to ours. (Note that the slightly strange-looking upper right entry of the matrix above is chosen so that this map is a group homomorphism.)

Example 2.4. Generalizing the previous example, the **complex Heisenberg–Weyl Lie algebra** of dimension $2n+1$ is the complex Lie algebra $\mathfrak{h}_{2n+1}^{\mathbb{C}}$ given by $\mathbb{C}^{2n+1}$ with the bracket defined by

$$[(z_1, \dots, z_{2n+1}), (z_1', \dots, z_{2n+1}')] = \left(0, \dots, 0, \sum_{k=1}^{n} z_{2k-1} z_{2k}' - z_{2k-1}' z_{2k}\right). \tag{2.4}$$

This again is stratified of step 2, taking $V_1 = \{(z_1, \dots, z_{2n}, 0) : z_1, \dots, z_{2n} \in \mathbb{C}\}$ and $V_2 = \{(0, \dots, 0, z_{2n+1}) : z_{2n+1} \in \mathbb{C}\}$. The **complex Heisenberg–Weyl group** $\mathbb{H}_{2n+1}^{\mathbb{C}}$ is again $\mathbb{C}^{2n+1}$ with the group operation $g \cdot h = g + h + \frac{1}{2}[g, h]$.

2.2. **The dilation semigroup.**

Definition 2.5. For $\lambda \in \mathbb{C}$, the **dilation map** on $\mathfrak{g}$ or G is defined by

$$\delta_\lambda(v_1 + \cdots + v_m) = \sum_{k=1}^{m} \lambda^k v_k, \qquad v_j \in V_j, \quad j = 1, \dots, m. \tag{2.5}$$

It is straightforward to verify that for $\lambda \neq 0$, δ_λ is an algebra automorphism of $\mathfrak{g}$ and a group automorphism of G and that

$$\delta_{\lambda\mu} = \delta_\lambda \circ \delta_\mu, \qquad \lambda, \mu \in \mathbb{C}. \tag{2.6}$$

Moreover, δ_λ is linear, so the derivative at the identity of $\delta_\lambda : G \to G$ is $(\delta_\lambda)_* = \delta_\lambda : \mathfrak{g} \to \mathfrak{g}$.

We note that δ_λ scales the Lebesgue measure m by

$$m(\delta_\lambda(A)) = |\lambda|^{2D} m(A), \tag{2.7}$$

where $D := \sum_{j=1}^{m} j \dim_{\mathbb{C}} V_j$ is the homogeneous dimension of G. Thus for an integrable function f, we have

$$\int_G f \circ \delta_\lambda \, dm = |\lambda|^{-2D} \int_G f \, dm. \tag{2.8}$$

We can then consider the vector fields generating this semigroup.

Definition 2.6. We define the real vector fields X, Y on G as

$$(Xf)(z) = \frac{d}{ds}\Big|_{s=0} f(\delta_{e^s} z), \qquad f \in C^\infty(G), \tag{2.9}$$

$$(Yf)(z) = \frac{d}{d\theta}\Big|_{\theta=0} f(\delta_{e^{i\theta}} z), \qquad f \in C^\infty(G) \tag{2.10}$$

and the complex vector field Z by

$$Z = \frac{1}{2}(X - iY). \tag{2.11}$$

Remark 2.7. To remind the reader of standard conventions, we note that the i appearing in (2.11) does not denote the complex structure on $\mathfrak{g}$, but rather ordinary scalar multiplication for complex vector fields. Formally, Z is a smooth section of the complexified tangent bundle $TG \otimes_{\mathbb{R}} \mathbb{C}$, which has a natural complex vector space structure with scalar multiplication $\zeta \cdot (v_x \otimes \eta) = v_x \otimes (\zeta\eta)$, and in which TG embeds naturally via $v_x \mapsto v_x \otimes 1$.

Lemma 2.8. *Z is a holomorphic vector field of type* $(1,0)$.

Proof. Let $z_1, \ldots, z_N$ be complex coordinates on $G \equiv \mathfrak{g}$ relative to a basis of $\mathfrak{g}$ adapted to the decomposition in (2.1). Then $\delta_\lambda z = (\ldots, \lambda^{c_j} z_j, \ldots)$ for positive integers $c_1, \ldots, c_N$. Hence for any function $f \in C^\infty(G)$ we have

$$(Xf)(z) = \sum_{j=1}^{N} \left\{ c_j z_j \frac{\partial f}{\partial z_j} + c_j \bar{z}_j \frac{\partial f}{\partial \bar{z}_j} \right\}$$

and

$$(Yf)(z) = \sum_{j=1}^{N} \left\{ i c_j z_j \frac{\partial f}{\partial z_j} - i c_j \bar{z}_j \frac{\partial f}{\partial \bar{z}_j} \right\}.$$

Thus

$$(Zf)(z) = \sum_{j=1}^{N} c_j z_j \frac{\partial f}{\partial z_j}. \tag{2.12}$$

□

2.3. **Holomorphic polynomials and Taylor series.**

Notation 2.9. $\mathcal{H}$ denotes the vector space of holomorphic functions on G.

The dilations δ_λ on G lead naturally to a notion of homogeneous functions and polynomials on G. These functions were used extensively in [18] in the context of real homogeneous groups. For us, they will be used as a convenient class of holomorphic test functions. In this section, we define these functions and verify a few key properties that will be important in this paper.

Definition 2.10. Let k be a nonnegative integer. A function $f : G \to \mathbb{C}$ is **homogeneous** of degree k if

$$f(\delta_\lambda z) = \lambda^k f(z) \text{ for all } z \in G \text{ and } 0 \neq \lambda \in \mathbb{C}. \tag{2.13}$$

Example 2.11. If G is the complex Heisenberg group with complex coordinates z_1, z_2, z_3, then $z_1^2, z_1 z_2, z_2^2, z_3$ are all homogeneous of degree 2.

Note that if f is homogeneous of degree k, then (2.13) and (2.9), (2.10), (2.11) give

$$Xf(z) = kf(z), \tag{2.14}$$

$$(Yf)(z) = ikf(z) \tag{2.15}$$

and

$$(Zf)(z) = kf(z). \tag{2.16}$$

Notation 2.12. For $k = 0, 1, 2, \ldots$ we will denote by $\mathcal{P}_k$ the set of all holomorphic functions on G which are homogeneous of degree k.

Lemma 2.13. *Every holomorphic function $f \in \mathcal{H}$ has a unique decomposition of the form*

$$f(z) = \sum_{k=0}^{\infty} f_k, \qquad f_k \in \mathcal{P}_k, \tag{2.17}$$

in the sense of pointwise convergence.

Proof. Notice first that the function $G \times \mathbb{C} \ni (z, \lambda) \mapsto \delta_\lambda z \in G$ is holomorphic in the sense that each coordinate of $\delta_\lambda z$, in the basis used in Lemma 2.8, is holomorphic.

Suppose $f : G \to \mathbb{C}$ is holomorphic, so that $(z, \lambda) \mapsto f(\delta_\lambda z)$ is holomorphic on $G \times \mathbb{C}$. Fix an arbitrary $z \in G$. Then the function $u(\lambda) := f(\delta_\lambda z)$ is an entire function on $\mathbb{C}$, and its Taylor expansion

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n a_n(z) \tag{2.18}$$

determines functions $a_n(z)$ which are holomorphic functions on G because

$$a_n(z) = \frac{1}{n!} \frac{d^n}{d\lambda^n}\Big|_{\lambda=0} f(\delta_\lambda z).$$

Now if $\mu \in \mathbb{C}$, then

$$\sum_{n=0}^{\infty} \lambda^n a_n(\delta_\mu z) = f(\delta_\lambda \delta_\mu z) = f(\delta_{\lambda\mu} z) = \sum_{n=0}^{\infty} (\lambda\mu)^n a_n(z) \text{ for all } \lambda \in \mathbb{C}.$$

Hence

$$a_n(\delta_\mu z) = \mu^n a_n(z) \text{ for all } z \in G.$$

Therefore $a_n \in \mathcal{P}_n$. This proves the existence of the functions f_k satisfying (2.17). If $\{g_k\}$ is another set satisfying (2.17), then

$$\sum_{k=0}^{\infty} \lambda^k g_k(z) = \sum_{k=0}^{\infty} g_k(\delta_\lambda z) = f(\delta_\lambda z) = \sum_{k=0}^{\infty} f_k(\delta_\lambda z) = \sum_{k=0}^{\infty} \lambda^k f_k(z)$$

for all $\lambda \in \mathbb{C}$. Hence $g_k(z) = f_k(z)$ for all k and z. □

Notation 2.14. Let $\mathcal{P}$ denote the linear span of $\{\mathcal{P}_k : k \geq 0\}$, i.e., the set of all *finite* sums of homogeneous functions (of possibly different degrees).

Lemma 2.15. *$\mathcal{P}$ is the set of holomorphic polynomials.*

Proof. In the adapted coordinates $z_1, \ldots, z_N$, a monomial $\prod_{j=1}^N z_j^{k_j}$ is homogeneous of degree $\sum_{j=1}^N k_j c_j$. Therefore any holomorphic polynomial lies in $\mathcal{P}$. Conversely, we need to show that a function $f \in \mathcal{P}_k$ is actually a polynomial. If its power series expansion is given by

$$f(z) = \sum_{k_1, \ldots, k_N \geq 0} a_{k_1, \ldots, k_N} z^{k_1} \cdots z^{k_N}, \tag{2.19}$$

then, for all complex $\lambda \neq 0$, we have

$$\lambda^k f(z) = f(\delta_\lambda z) = \sum_{k_1, \ldots, k_N} a_{k_1, \ldots, k_N} z^{k_1} \cdots z^{k_N} \lambda^{\sum_{j=1}^N k_j c_j}. \tag{2.20}$$

Since the coefficient of λ^r on the right must be zero for all z if $r \neq k$ we actually have

$$f(z) = \sum_{\sum_{j=1}^N k_j c_j = k} a_{k_1,\dots,k_N} z^{k_1} \cdots z^{k_N}. \tag{2.21}$$

The subscripts in the sum form a finite set, showing that f is a polynomial. □

Corollary 2.16. *$\mathcal{P}_k$ is finite dimensional.*

Lemma 2.17. *If f is holomorphic and is given by* (2.17)*, then*

$$(Zf)(z) = \sum_{k=0}^{\infty} k f_k(z). \tag{2.22}$$

Proof. Since $f(\delta_\lambda z) = \sum_{k=0}^{\infty} \lambda^k f_k(z)$ for all $\lambda \in \mathbb{C}$ we have

$$(Zf)(z) = (Xf)(z) = \frac{d}{ds}\Big|_{s=0} \sum_{k=0}^{\infty} e^{ks} f_k(z) = \sum_{k=0}^{\infty} k f_k(z).$$

The interchange of derivative and sum is justified since $\sum_{k=0}^{\infty} e^{ks} f_k(z)$ is the Taylor series of the holomorphic function $u(e^s)$, where $u(\lambda) := f(\delta_\lambda z)$ as in the proof of Lemma 2.13, and this can be differentiated termwise. □

We remark for future reference that by (2.14) and (2.16), we have

$$Zf = Xf, \qquad f \in \mathcal{P}. \tag{2.23}$$

Lemma 2.18. *Let $\xi \in V_j$ and $f \in \mathcal{P}_k$. Then $\widetilde{\xi} f \in \mathcal{P}_{k-j}$ if $k \geq j$, and $\widetilde{\xi} f = 0$ if $k < j$.*

Proof. First, since f is holomorphic and $\widetilde{\xi}$ is left-invariant, $\widetilde{\xi} f$ is holomorphic. Next, since δ_λ is a group homomorphism, for any $z \in G$ we have $L_{\delta_\lambda(z)} = \delta_\lambda \circ L_z \circ \delta_{\lambda^{-1}}$. By left-invariance of $\widetilde{\xi}$ we have

$$\begin{aligned}
(\widetilde{\xi} f)(\delta_\lambda z) &= ((L_{\delta_\lambda(z)})_* \xi) f \\
&= (\delta_\lambda (L_z)_* \delta_{\lambda^{-1}} \xi) f \\
&= \lambda^{-j} (\delta_\lambda (L_z)_* \xi) f && \text{since } \xi \in V_j \\
&= \lambda^{-j} ((L_z)_* \xi)(f \circ \delta_\lambda) \\
&= \lambda^{k-j} ((L_z)_* \xi) f && \text{since } f \in \mathcal{P}_k \\
&= \lambda^{k-j} \widetilde{\xi} f(z).
\end{aligned}$$

Thus $f \in \mathcal{P}_{k-j}$. If $k - j < 0$, then the fact that $\widetilde{\xi} f$ is continuous at the identity leads to the conclusion that $f \equiv 0$. □

2.4. Sub-Riemannian geometry on G.

As before, let $\mathfrak{g}$ be a stratified complex Lie algebra with its connected, simply connected complex Lie group G. For this section, we will use J to denote the complex structure on $\mathfrak{g}$. In this section, we collect a number of facts about the sub-Riemannian geometry of G and its hypoelliptic Laplacian. Although much of this development is standard, we shall be rather explicit with our definitions to fix notation and avoid any possible ambiguity.

View $\mathfrak{g}$ as a real vector space, and let $\mathfrak{g}^*$ be its dual space. Let $h : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathbb{R}$ be a symmetric, positive semidefinite, real bilinear form on $\mathfrak{g}^*$. We shall think of

h as a "dual metric" on the dual $\mathfrak{g}^*$, despite the fact that it is degenerate, i.e., only positive semidefinite instead of positive definite. Suppose further that h is Hermitian, i.e., $h(J^*\alpha, J^*\beta) = h(\alpha, \beta)$, where J^* is the adjoint of J. (This ensures that h, in some sense, respects the complex structure of $\mathfrak{g}$.)

Let $K := \{\alpha \in \mathfrak{g}^* : h(\alpha, \alpha) = 0\}$ be the null space of h and let $H = K^0 = \bigcap_{\alpha \in K} \ker \alpha \subset \mathfrak{g}$ be the backward annihilator of K; H is called the **horizontal subspace** of $\mathfrak{g}$. Note that H is invariant under J.

Henceforth we assume the following nondegeneracy condition:

Assumption 2.19. $H = V_1$.

In particular, Hörmander's condition is satisfied: H generates $\mathfrak{g}$. In fact, Hörmander's condition is satisfied if and only if $V_1 \subset H$; we need the opposite inclusion to ensure that h interacts nicely with the dilation structure on G.

Now h induces a natural real-linear map $\Phi : \mathfrak{g}^* \to \mathfrak{g}$ defined by $\alpha(\Phi\beta) = h(\alpha, \beta)$ with kernel K and image H. (Note that $\Phi = J\Phi J^*$.) We may then define a bilinear form $g : H \times H \to \mathbb{R}$ on H by $g(\Phi\alpha, \Phi\beta) = h(\alpha, \beta)$, which is easily seen to be well-defined, Hermitian (i.e., $g(v, w) = g(Jv, Jw)$), and positive definite.

By left translation, we can extend h to a (degenerate) left-invariant dual metric (still denoted by h) on T^*G, defined by $h_x(\alpha_x, \beta_x) = h(L_x^*\alpha_x, L_x^*\beta_x)$ for $\alpha_x, \beta_x \in T_x^*G$. Then H extends to a left-invariant sub-bundle of TG, namely, $v_x \in H_x \subset T_xG$ iff $(L_{x^{-1}})_* v_x \in H$, which happens iff $\alpha_x(v_x) = 0$ for every $\alpha_x \in T_x^*G$ satisfying $h_x(\alpha_x, \alpha_x) = 0$. H_x is the **horizontal subspace** of T_xG, and vectors $v_x \in H_x$ are said to be **horizontal**. The bundle H itself is sometimes called the **horizontal distribution**. We can also extend g to a left-invariant positive definite inner product on H, defined by $g_x(v_x, w_x) = g((L_{x^{-1}})_* v_x, (L_{x^{-1}})_* w_x)$ for $v_x, w_x \in H_x$. g is called a **sub-Riemannian metric**. If we define $\Phi_x : T_x^*G \to T_xG$ by $\Phi_x = (L_x)_* \Phi L_x^*$, then the image of Φ_x is H_x, and we have $g_x(\Phi_x\alpha_x, \Phi_x\beta_x) = h_x(\alpha_x, \beta_x)$. Given a smooth function $f : G \to \mathbb{R}$, we can define its left-invariant **sub-gradient** $\nabla f \in H$ by $\nabla f(x) = \Phi_x(df(x))$.

We wish to consider complex functions, one-forms, vector fields, etc., on G, so we shall now complexify everything in sight. At each $x \in G$, we form the complexified tangent space $T_xG \otimes \mathbb{C}$, which, as mentioned in Remark 2.7, is a complex vector space with the complex scalar multiplication $\zeta \cdot (v_x \otimes \eta) = v_x \otimes (\zeta\eta)$. When taking this tensor product, we view T_xG as a *real* vector space, forgetting that it already has the natural complex structure $J_x = (L_x)_* J (L_{x^{-1}})_*$. This means that $T_xG \otimes \mathbb{C}$ now has *two* distinct complex structures: multiplication by i (i.e., $v_x \otimes \eta \mapsto v_x \otimes i\eta$) and J_x (which we extend to $T_xG \otimes \mathbb{C}$ by complex linearity: $J_x i v_x = i J_x v_x$). A complex vector field can thus be viewed as a smooth section of the complexified tangent bundle $TG \otimes \mathbb{C}$. The complexified horizontal bundle $H \otimes \mathbb{C}$ is naturally contained in $TG \otimes \mathbb{C}$. We likewise form the complexified cotangent space $T_x^*G \otimes \mathbb{C}$ and note that it can be viewed as the complex dual space of $T_xG \otimes \mathbb{C}$. If $f : G \to \mathbb{C}$ is a complex function, written as $f = u + iv$, then its differential df is a complex one-form, a smooth section of $T^*G \otimes \mathbb{C}$ given by $df = du + idv$. $T^*G \otimes \mathbb{C}$ also has two complex structures: multiplication by i and $J_x^* = L_{x^{-1}}^* J^* L_x^*$ (extended by complex linearity). In particular, if f is holomorphic, then we have the Cauchy–Riemann equation $J^* df = i df$; that is, df is a complex one-form of type $(1, 0)$.

Now we extend h to $T^*G \otimes \mathbb{C}$ in such a way as to make it *complex bilinear* with respect to multiplication by i; that is, $h_x(i\alpha_x, \beta_x) = h_x(\alpha_x, i\beta_x) = i h_x(\alpha_x, \beta_x)$. So now h_x is complex bilinear with respect to i and Hermitian with respect to J_x^*. We likewise extend Φ_x to a complex linear map $\Phi_x : T_x^*G \otimes \mathbb{C} \to H_x \otimes \mathbb{C}$, and then defining g_x analogously as before makes it a complex bilinear form on $H_x \otimes \mathbb{C}$. Note that g_x remains Hermitian with respect to J_x. By an abuse of terminology, we shall continue to call g and h the sub-Riemannian metric and dual metric, respectively. We now also have the sub-gradient $\nabla f(x) = \Phi_x(df(x)) \in T_xG \otimes \mathbb{C}$ defined for complex functions.

We can describe this geometry more explicitly by choosing a set of left-invariant real vector fields $X_1, Y_1, \ldots, X_k, Y_k$ which span H, are g-orthonormal, and have $Y_j = JX_j$. Then the sub-gradient is given by

$$\nabla f(x) = \sum_j (X_j f)(x) X_j(x) + (Y_j f)(x) Y_j(x),$$

and for smooth $f_1, f_2 : G \to \mathbb{C}$ we have

$$g(\nabla f_1, \nabla \bar{f}_2) = h(df_1, d\bar{f}_2) = \sum_j \{X_j f_1 X_j \bar{f}_2 + Y_j f_1 Y_j \bar{f}_2\}. \tag{2.24}$$

We shall use $|\nabla f|^2$ as shorthand for $g(\nabla f, \nabla \bar{f})$.

Alternatively, letting

$$\begin{aligned} Z_j &= \frac{1}{2}(X_j - iY_j), \\ \bar{Z}_j &= \frac{1}{2}(X_j + iY_j) \end{aligned} \tag{2.25}$$

so that Z_j and $\bar{Z}_j$ are complex vector fields of type $(1,0)$ and $(0,1)$ respectively, we get

$$\nabla f(x) = 2 \sum_j \Big((Z_j f)(x) \bar{Z}_j(x) + (\bar{Z}_j f)(x) Z_j(x) \Big), \tag{2.26}$$

$$g(\nabla f_1, \nabla \bar{f}_2) = h(df_1, d\bar{f}_2) = 2 \sum_j \Big(Z_j f_1 \bar{Z}_j \bar{f}_2 + \bar{Z}_j f_1 Z_j \bar{f}_2 \Big). \tag{2.27}$$

We remark in passing that X_j and Y_j commute (since, using the fact that $\mathfrak{g}$ is a complex Lie algebra, $[X_j, Y_j] = [X_j, JX_j] = J[X_j, X_j] = 0$), and thus Z_j and $\bar{Z}_j$ commute.

Note that when f is real, we have

$$|\nabla f|^2 := g(\nabla f, \nabla f) = h(df, df) = 4 \sum_j |Z_j f|^2, \tag{2.28}$$

and when f is holomorphic,

$$|\nabla f|^2 = 2 \sum_j |Z_j f|^2. \tag{2.29}$$

Example 2.20. Returning to the example of the complex Heisenberg group begun in Example 2.3, consider $\mathbb{H}_3^{\mathbb{C}} = \mathbb{C}^3$ with its Euclidean coordinates (z_1, z_2, z_3). Let h be the left-invariant dual metric given at the identity $e = 0$ by

$$\begin{aligned} h_e(dz_1, d\bar{z}_1) = h_e(dz_2, d\bar{z}_2) &= 2, \\ h_e(dz_3, d\bar{z}_3) &= 0, \\ h_e(dz_j, d\bar{z}_k) &= 0, \qquad j \neq k. \end{aligned}$$

This makes h Hermitian with respect to the complex structure of $\mathbb{H}_3^{\mathbb{C}}$, so that $h_e(dz_j, dz_k) = h_e(d\bar{z}_j, d\bar{z}_k) = 0$ for all j, k. (The 2 appearing in the first line ensures that the cotangent vectors dx_i, dy_j are orthonormal under h_e.)

From now on, any occurrence of $\mathbb{H}_3^{\mathbb{C}}$ will be understood to carry this dual metric h and the corresponding metric g.

We can choose the left-invariant complex vector fields Z_j discussed in (2.25) to be those which equal $\frac{\partial}{\partial z_j}$ at the identity. They are given by

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial z_1} - \frac{1}{2} z_2 \frac{\partial}{\partial z_3}, \\ Z_2 &= \frac{\partial}{\partial z_2} + \frac{1}{2} z_1 \frac{\partial}{\partial z_3}, \\ Z_3 &= \frac{\partial}{\partial z_3}. \end{aligned}$$

Example 2.21. For the Heisenberg–Weyl group $\mathbb{H}_{2n+1}^{\mathbb{C}}$ of Example 2.4, we may similarly define a left-invariant dual metric h by

$$\begin{aligned} h_e(dz_j, d\bar{z}_j) &= 2, & 1 \le j \le 2n, \\ h_e(dz_{2n+1}, d\bar{z}_{2n+1}) &= 0, & \\ h_e(dz_j, d\bar{z}_k) &= 0, & j \ne k. \end{aligned}$$

Let us see how the dilations interact with the left-invariant real vector fields X_j, Y_j. If $y \in G$ and $\lambda = \alpha + i\beta \in \mathbb{C}$, we have

$$\begin{aligned} (\delta_\lambda)_* X_j(y) &= (\delta_\lambda L_y)_* X_j(e) \\ &= (L_{\delta_\lambda(y)} \delta_\lambda)_* X_j(e) \\ &= (L_{\delta_\lambda(y)})_* (\alpha X_j(e) + \beta J X_j(e)) \\ &= \alpha X_j(\delta_\lambda(y)) + \beta J X_j(\delta_\lambda(y)). \end{aligned} \tag{2.30}$$

The same holds for Y_j. Thus we get

$$\begin{aligned} (\delta_\lambda)_* Z_j(y) &= \lambda Z_j(\delta_\lambda(y)), \\ (\delta_\lambda)_* \bar{Z}_j(y) &= \bar{\lambda} \bar{Z}_j(\delta_\lambda(y)). \end{aligned} \tag{2.31}$$

The **sub-Laplacian** Δ is defined by

$$\Delta = \sum_j X_j^2 + Y_j^2 = 4 \sum_j Z_j \bar{Z}_j. \tag{2.32}$$

It is shown in [44] that Δ, with domain $C_c^\infty(G)$, is a hypoelliptic operator and is essentially self-adjoint on $L^2(m)$. As a consequence of (2.31), we have

$$\Delta(f \circ \delta_\lambda) = |\lambda|^2 (\Delta f) \circ \delta_\lambda. \tag{2.33}$$

Likewise, if $e^{s\Delta/4}$ is the heat semigroup for Δ, we have

$$e^{s\Delta/4}(f \circ \delta_\lambda) = (e^{s|\lambda|^2 \Delta/4} f) \circ \delta_\lambda. \tag{2.34}$$

Finally, we recall the definition of the Carnot–Carathéodory distance on G and some of its basic properties. Suppose $\gamma : [0,1] \to G$ is a smooth path. If $\dot{\gamma}(t) \in H_{\gamma(t)}$ for each t, we say γ is **horizontal**, and we define its **length** by

$$\ell(\gamma) = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt. \tag{2.35}$$

Then for $x, y \in G$, we define the **Carnot–Carathéodory distance** d by

$$d(x, y) = \inf\{\ell(\gamma) : \gamma \text{ horizontal}, \gamma(0) = x, \gamma(1) = y\}.$$

Since Hörmander's condition is satisfied, the Chow–Rashevskii and ball-box theorems [35, 36] imply that $d(x, y) < \infty$ and that d is a metric which induces the manifold topology on G (which indeed is just the Euclidean topology on the finite-dimensional vector space $G = \mathfrak{g}$).

Since we are denoting the complex structure on $\mathfrak{g}$ by J, for $v \in V_1 \subset \mathfrak{g} = T_e G$ we have $(\delta_{\alpha+i\beta})_* v = \delta_{\alpha+i\beta}(v) = \alpha v + \beta J v$. Thus, for $v, w \in V_1$ we have $g((\delta_\lambda)_* v, (\delta_\lambda)_* w) = |\lambda|^2 g(v, w)$. Since δ_λ is a group homomorphism and g is left invariant, it follows that the same holds for $v, w \in H_x$. In particular, $\ell(\delta_\lambda(\gamma)) = |\lambda|\ell(\gamma)$, and so $d(e, \delta_\lambda(x)) = |\lambda| d(e, x)$.

By fixing a basis for $\mathfrak{g}$, we may linearly identify it (noncanonically) with Euclidean space $\mathbb{R}^{\dim_{\mathbb{R}} \mathfrak{g}}$; let $|\cdot|$ denote the pullback of the Euclidean norm onto $\mathfrak{g}$. For $v \in \mathfrak{g}$, write $v = v_1 + \cdots + v_m$ with $v_k \in V_k$ and let

$$|v|_1 = \sum_{k=1}^{m} |v_k|^{1/k}. \tag{2.36}$$

Note that $|\delta_\lambda v|_1 = |\lambda| |v|_1$. Since we have identified G with $\mathfrak{g}$ as a set, $|\cdot|_1$ also makes sense on G. It is shown in [6, Proposition 5.1.4] that there is a constant c such that for every $x \in G$ we have

$$\frac{1}{c}|x|_1 \le d(e, x) \le c|x|_1. \tag{2.37}$$

The proof is simple: since $d(e, \cdot)$ and $|\cdot|_1$ have the same scaling with δ_λ, it suffices to consider x with $|x|_1 = 1$. The set of such x is compact, so $d(e, \cdot)$ attains a finite maximum and a nonzero minimum on this set.

2.5. **Properties of the heat kernel.** It is shown in [44] that the Markovian heat semigroup $e^{s\Delta/4}$ admits a right convolution kernel ρ_s, i.e., $e^{s\Delta/4} f = f * \rho_s$, which we shall call the **heat kernel**; it is also shown that ρ_s is C^∞ and strictly positive. Since $e^{s\Delta/4}$ is Markovian, the heat kernel measure $\rho_s \, dm$ is a probability measure.

Notation 2.22. For $s > 0$ and $0 < p < \infty$, we write $L^p(\rho_s)$ as short for $L^p(G, \rho_s \, dm)$. As usual, for $0 < p < 1$, the vector space $L^p(\rho_s)$ is equipped with the topology induced by the complete translation-invariant metric $d(f, g) = \int |f - g|^p \rho_s \, dm$. Nonetheless $\|f\|_{L^p(\rho_s)}$ will still mean $\left(\int |f|^p \rho_s \, dm\right)^{1/p}$, even for the case $0 < p < 1$ in which it does not define a norm.

Since ρ_s is bounded, and bounded below on compact sets, any sequence converging in $L^p(\rho_s)$ also converges in $L^p_{\mathrm{loc}}(m)$. As such, if f_n are holomorphic functions and $f_n \to f$ in $L^p(\rho_s)$, then we also have $f_n \to f$ uniformly on compact sets, and so f is holomorphic. Thus $L^p(\rho_s) \cap \mathcal{H}$ is closed in $L^p(\rho_s)$.

We record here some estimates for the heat kernel.

Theorem 2.23. *For each $0 < \epsilon < 1$ there are constants C, C' such that for every $x \in G$ and $s > 0$,*

$$\frac{C}{m(B(e, \sqrt{s}))} e^{-d(e,x)^2/(1-\epsilon)s} \le \rho_s(x) \le \frac{C'}{m(B(e, \sqrt{s}))} e^{-d(e,x)^2/(1+\epsilon)s} \tag{2.38}$$

where $m(B(e, \sqrt{s}))$ is the Lebesgue (Haar) measure of the d-ball centered at the origin (or any other point) of radius $\sqrt{s}$.

Proof. The upper bound is Theorem IV.4.2 of [44]. The lower bound is Theorem 1 of [43]. Note that our choice to consider the semigroup $e^{s\Delta/4}$ rather than $e^{s\Delta}$ accounts for a missing factor of 4 in the exponents compared to the results stated in [43, 44]. ☐

Theorem 2.24. *Suppose $\xi_1, \ldots, \xi_k \in \mathfrak{g}$. Let m be a nonnegative integer, $r \geq 0$, and $0 < s < t < \infty$. There is a constant C such that for all $y \in G$,*

$$\sup_{d(x,e)<r} \left| \left(\frac{d^m}{ds^m} \widetilde{\xi}_1 \cdots \widetilde{\xi}_k \rho_s \right) (y \cdot x) \right| \leq C \rho_t(y). \tag{2.39}$$

Proof. This is a special case of Theorem IV.3.1 of [44]. To reduce their statement to ours, note first that it suffices to assume the ξ_i are all in V_1 (since, assuming Hörmander's condition, any other left-invariant vector field may be written as a linear combination of commutators of vector fields from V_1). We can also assume without loss of generality that the ξ_i are orthonormal. Then, in their notation, take $R = 1$, $\alpha = s$, $\beta = t$, and $\delta = r$. ☐

Lemma 2.25. *Let $s > 0$.*

(a) *For every $t > s$ there exists $p > 1$ such that $\rho_t/\rho_s \in L^p(\rho_s)$.*
(b) *For every $p \geq 1$ there exists $t > s$ such that $\rho_t/\rho_s \in L^p(\rho_s)$.*

Proof. Let $\epsilon > 0$. By Theorem 2.23, for any $0 < s < t$, any $p > 1$, and any $\epsilon > 0$ we may find a constant $C(s, t, \epsilon)$ such that

$$\begin{aligned} \left| \frac{\rho_t(x)}{\rho_s(x)} \right|^p \rho_s(x) &= \frac{\rho_t(x)^p}{\rho_s(x)^{p-1}} \\ &\leq C(s,t,\epsilon) \exp\left(- \left(\frac{p}{(1+\epsilon)t} - \frac{p-1}{(1-\epsilon)s} \right) d(e,x)^2 \right) \end{aligned}$$

where the $m(B(e, \sqrt{\cdot}))$ factors have been absorbed into $C(s, t, \epsilon)$. Let $A = A(p, s, t, \epsilon) = \left(\frac{p}{(1+\epsilon)t} - \frac{p-1}{(1-\epsilon)s} \right)$ be the bracketed quantity in the exponent. If $A > 0$, then by (2.37) the right side will be integrable with respect to m, implying the desired conclusion.

For (a), suppose $t > s$ is given. Fix any $\epsilon \in (0, 1)$. As $p \downarrow 1$ we have $A \to \frac{1}{(1+\epsilon)t} > 0$, so for any p sufficiently close to 1 we get $A > 0$ and hence $\rho_t/\rho_s \in L^p(\rho_s)$.

For (b), suppose $s > 0$ and $p \geq 1$ are given. Without loss of generality we can assume $p > 1$ (since $L^1(\rho_s) \supset L^p(\rho_s)$ for any $p > 1$). Choose t with $s < t < \frac{p}{p-1}s$. Then as $\epsilon \downarrow 0$ we have $A \to \frac{p}{t} - \frac{p-1}{s} > 0$, so for any sufficiently small ϵ we get $A > 0$. ☐

Lemma 2.26. *For any $\xi \in \mathfrak{g}$ and any $s > 0$ we have $\widetilde{\xi} \log \rho_s \in \bigcap_{p \geq 1} L^p(\rho_s)$.*

Proof. Fix $p \geq 1$. By Lemma 2.25(b) we can choose $t > s$ such that $\rho_t/\rho_s \in L^p(\rho_s)$. Then by Theorem 2.24, taking any $r > 0$ and $x = e$, there is a constant C such that $\widetilde{\xi} \rho_s \leq C \rho_t$. As such, by the chain rule we have

$$\widetilde{\xi} \log \rho_s = \frac{\widetilde{\xi} \rho_s}{\rho_s} \leq \frac{\rho_t}{\rho_s} \in L^p(\rho_s).$$

☐

Lemma 2.27. *The heat kernel ρ_s obeys the scaling relation*

$$\rho_s(\delta_\lambda(y)) = |\lambda|^{-2D} \rho_{s|\lambda|^{-2}}(y). \tag{2.40}$$

Proof. This follows from the corresponding scaling properties of the semigroup $e^{s\Delta/4}$ (2.34) and of the Haar measure m (2.7). □

3. Dirichlet forms and operators

For the rest of the paper, fix some $a > 0$. Henceforward L^p by itself will, unless otherwise specified, refer to $L^p(\rho_a)$.

Notation 3.1. Let Q_0 be the positive quadratic form on $L^2(\rho_a)$ defined on the domain $C_c^\infty(G)$ by

$$Q_0(f_1, f_2) = \int_G h(df_1, d\bar{f}_2)\, \rho_a\, dz = \int_G g(\nabla f_1, \nabla \bar{f}_2)\, \rho_a\, dz \tag{3.1}$$

and let Q be its closure, with domain $\mathcal{D}(Q)$, so that $(Q, \mathcal{D}(Q))$ is a Dirichlet form on $L^2(\rho_a)$. Note that $\mathcal{D}(Q)$ is a Hilbert space under the **energy norm** $(f,g)_Q = (f,g)_{L^2(\rho_a)} + Q(f,g)$. Let $(A, \mathcal{D}(A))$ be the generator of Q; i.e., A is the unique self-adjoint operator on $L^2(\rho_a)$ having domain $\mathcal{D}(A) \subset \mathcal{D}(Q)$ and satisfying $\int_G (Af_1)\bar{f}_2\, \rho_a\, dz = Q(f_1, f_2)$ for all $f_1 \in \mathcal{D}(A)$, $f_2 \in \mathcal{D}(Q)$.

On smooth functions $f \in \mathcal{D}(A) \cap C^\infty(G)$, integration by parts gives

$$Af = d^* df = -\Delta f - g(\nabla f, \nabla \log \rho_a) = -\Delta f - h(df, d \log \rho_a). \tag{3.2}$$

The operator $A = d^*d$ can be seen as an analogue of the Ornstein–Uhlenbeck operator in this noncommutative Lie group setting. Such operators have attracted substantial interest in the literature, including the study of functional inequalities such as Poincaré inequalities. Papers which study these operators (in the setting of real Lie groups) include [5, 33, 34].

Remark 3.2. When $\mathfrak{g}$ is abelian (i.e., the Lie bracket is 0) then G is Euclidean space $\mathbb{C}^n$ (with its usual additive group structure). If we take h to be the usual positive definite Euclidean inner product, then everything reduces to the Euclidean case: ∇ and Δ are the usual gradient and Laplacian, d is Euclidean distance, ρ_s is the Gaussian heat kernel $\rho_s(z) = (\pi s)^{-n} e^{-|z|^2/s}$, and A is the Ornstein–Uhlenbeck operator.

Definition 3.3. We will say that A is a **holomorphic** operator if it maps holomorphic functions to holomorphic functions, i.e., $A(\mathcal{D}(A) \cap \mathcal{H}) \subset \mathcal{H}$.

In our setting, the operator A is *not* holomorphic (except in the abelian case $G = \mathbb{C}^n$); see Theorem 5.10 below. So our setting stands in contrast to that of [26], in which most of the main results were proved under the hypothesis that the operator A should be holomorphic.

Since the phenomenon of strong hypercontractivity is quite specific to the holomorphic category, it is not reasonable to expect it to hold for an operator that does not preserve holomorphicity. As such, our main object of study will not be A itself, but rather the operator B defined as follows.

Notation 3.4. The restriction $Q|_{\mathcal{H}}$ of Q to the domain $\mathcal{D}(Q) \cap \mathcal{H}$ is a positive closed quadratic form on the Hilbert space $\mathcal{H} \cap L^2(\rho_a)$. Let $(B, \mathcal{D}(B))$ be its generator, so that B is a self-adjoint operator on $\mathcal{H} \cap L^2(\rho_a)$.

We intend to think of B as the "holomorphic projection" of the operator A. In Section 4, we shall discuss the precise sense in which this is true. For now, let us observe that

$$\mathcal{D}(A) \cap \mathcal{H} \subset \mathcal{D}(B). \tag{3.3}$$

To see this, note that for $f \in \mathcal{D}(A) \cap \mathcal{H} \subset \mathcal{D}(Q) \cap \mathcal{H}$ and $g \in \mathcal{D}(Q) \cap \mathcal{H}$, we have $|Q(f,g)| = |(Af,g)_{L^2}| \le \|Af\|_{L^2}\|g\|_{L^2}$, and so f is in the domain of the generator of $Q|_{\mathcal{H}}$, namely B.

4. Density properties of holomorphic polynomials

Notation 4.1. $\mathcal{H}$ will denote the set of holomorphic functions on G.

Theorem 4.2.

(a) *$\mathcal{P}$ is dense in $\mathcal{H} \cap L^p(\rho_a)$ for $1 \le p < \infty$.*
(b) *$\mathcal{P} \subset \mathcal{D}(Q)$ and is a core for $Q|_{\mathcal{H}}$. In particular, from* (a)*, $Q|_{\mathcal{H}}$ is densely defined in $\mathcal{H} \cap L^2(\rho_a)$.*
(c) *If $j \ne k$, then $\mathcal{P}_j \perp \mathcal{P}_k$ in both $L^2(\rho_a)$ and in energy norm.*
(d) *$\mathcal{H} \cap L^2(\rho_a) = \bigoplus_{k=0}^{\infty} \mathcal{P}_k$.*
(e) *$\mathcal{H} \cap \mathcal{D}(Q) = \bigoplus_{k=0}^{\infty} \mathcal{P}_k$ (convergence in energy norm).*
(f) *$\mathcal{P} \subset \mathcal{D}(B)$ and is a core for B.*

Remark 4.3. It is interesting to contrast Theorem 4.2 with [33, Proposition 8] (credited to W. Hebisch), in which it is shown that the result is typically false if we drop the word "holomorphic". Specifically, when G is a (real) stratified Lie group, the (not necessarily holomorphic) polynomials are dense in $L^2(\rho_a)$ if and only if G has step at most 4.

Proof. The proofs are slight variants of the proof of [26, Lemma 5.4].

For (a), to begin, it follows from the upper bound in Theorem 2.23, using polar coordinates and the homogeneity of d, that $\mathcal{P} \subset L^p(\rho_a)$.

Let

$$\begin{aligned} F_n(\theta) &= \frac{1}{2\pi n} \sum_{k=0}^{n-1} \sum_{j=-k}^{k} e^{ij\theta} \\ &= \frac{1}{2\pi n} \frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)} \end{aligned} \tag{4.1}$$

denote Fejer's kernel [42, §13.31]. We observe that

$$\int_{-\pi}^{\pi} F_n(\theta)\, d\theta = 1, \tag{4.2}$$

$$\int_{-\pi}^{\pi} F_n(\theta) e^{i\ell\theta}\, d\theta = 0, \qquad \ell \ge n, \tag{4.3}$$

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} F_n(\theta)\varphi(\theta)\, d\theta = \varphi(0), \qquad \varphi \in C([-\pi,\pi]). \tag{4.4}$$

Define

$$V_\theta f := f \circ \delta_{e^{i\theta}}$$

for any function f on G. If $f \in \mathcal{H}$ and is written $f = \sum_{k=0}^{\infty} f_k$ as in (2.17), with $f_k \in \mathcal{P}_k$, then

$$(V_\theta f)(z) = \sum_{k=0}^{\infty} e^{ik\theta} f_k(z). \tag{4.5}$$

The convergence is uniform on $\theta \in [-\pi, \pi]$ for each $z \in G$ because the function $\theta \mapsto f(\delta_{e^{i\theta}} z)$ is smooth and periodic with period 2π. Now let

$$g_n(z) := \int_{-\pi}^{\pi} F_n(\theta)(V_\theta f)(z)\, d\theta. \tag{4.6}$$

Using (4.5), Fubini's theorem, and (4.3), we see that g_n is a linear combination of $f_0, f_1, \ldots, f_{n-1}$ and is therefore in $\mathcal{P}$. (We can justify the application of Fubini's theorem using the fact that $\sum_{k=0}^{\infty} f_k(z)$ is the Taylor series for $u(\lambda)$, as defined in (2.18), at $\lambda = 1$, and therefore converges absolutely.) Since the map $\delta_{e^{i\theta}} : G \to G$ preserves the measure $\rho_a(x)dx$ (see (2.7), (2.40)), the operators V_θ are isometries in $L^p(G, \rho_a(x)dx)$ for $0 < p < \infty$. Moreover, the map $\theta \mapsto V_\theta$ is strongly continuous in $L^p(\rho_a)$ for $1 \le p < \infty$: for bounded continuous $f : G \to \mathbb{R}$, dominated convergence gives $V_\theta f \to f$ in $L^p(\rho_a)$ as $\theta \to 0$, and the case of general $f \in L^p(\rho_a)$ follows by density.

Thus if $1 \le p < \infty$ and $f \in \mathcal{H} \cap L^p(\rho_a)$, then we have

$$\begin{aligned} \|f - g_n\|_{L^p} &= \left\| \int_{-\pi}^{\pi} F_n(\theta)(f - V_\theta f)\, d\theta \right\|_{L^p} \\ &\le \int_{-\pi}^{\pi} F_n(\theta) \|f - V_\theta f\|_{L^p}\, d\theta \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned} \tag{4.7}$$

by Minkowski's inequality for integrals. This proves part (a).

To prove part (b), recall that by Lemma 2.18, if $f \in \mathcal{P}_k$ and $\xi \in V_1$, then $\widetilde{\xi} f \in \mathcal{P}_{k-1} \subset L^2(\rho_a)$. Hence $|\nabla f|^2$ is in $L^1(\rho_a)$. Moreover, multiplying f by a sequence φ_n of cutoff functions in $C_c^\infty(G)$ which converge to 1 boundedly and such that $\widetilde{\xi}\varphi_n \to 0$ boundedly, one sees that $f \in \mathcal{D}(Q)$. So $\mathcal{P} \subset \mathcal{D}(Q)$. By (2.27) and (2.31), for any smooth f we have

$$|\nabla(f \circ \delta_{e^{i\theta}})|^2(z) = |\nabla f|^2(\delta_{e^{i\theta}} z). \tag{4.8}$$

Since $\rho_a(x)dx$ is preserved by the map $\delta_{e^{i\theta}}$ it follows that

$$Q(V_\theta f) = Q(f) \quad \text{for all} \quad f \in \mathcal{D}(Q)$$

and in particular for all $f \in \mathcal{H} \cap \mathcal{D}(Q)$. So V_θ is unitary on $\mathcal{H} \cap \mathcal{D}(Q)$ in the energy norm, $[\|f\|_{L^2}^2 + Q(f)]^{1/2}$. Now if $f \in \mathcal{H} \cap \mathcal{D}(Q)$ and we define the polynomials g_n as in (4.6), we can differentiate under the integral sign to see that

$$\widetilde{\xi} g_n(z) = \int_{-\pi}^{\pi} F_n(\theta)(\widetilde{\xi} V_\theta f)(z)\, d\theta = \int_{-\pi}^{\pi} F_n(\theta) e^{i\theta} (V_\theta \widetilde{\xi} f)(z)\, d\theta. \tag{4.9}$$

Then, similarly to (4.7), we have

$$\begin{aligned}\left\|\widetilde{\xi} f-\widetilde{\xi} g_n\right\|_{L^2} &=\left\|\int_{-\pi}^{\pi} F_n(\theta)(\widetilde{\xi} f-e^{i\theta} V_\theta \widetilde{\xi} f)\, d\theta\right\|_{L^p} \\ &\leq \int_{-\pi}^{\pi} F_n(\theta)\left\|\widetilde{\xi} f-e^{i\theta} V_\theta \widetilde{\xi} f\right\|_{L^p} d\theta \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{4.10}$$

It follows that $g_n \to f$ in energy norm. Hence $\mathcal{P}$ is a core for $Q \mid \mathcal{H}$.

Now if $f \in \mathcal{P}_n$ and $g \in \mathcal{P}_k$, then $(V_\theta f)(z) = e^{in\theta} f(z)$ and $(V_\theta g)(z) = e^{ik\theta} g(z)$ by (2.13). Hence $(f,g)_{L^2} = (V_\theta f, V_\theta g)_{L^2} = e^{i(n-k)\theta}(f,g)_{L^2}$ for all real θ. So if $n \neq k$, then $(f,g)_{L^2} = 0$. Moreover, $\widetilde{\xi} f \in \mathcal{P}_{n-1}$ and $\widetilde{\xi} g \in \mathcal{P}_{k-1}$ if $\xi \in V_1$. So if $n \neq k$, then $Q(f,g) = 0$. This proves part (c). Parts (d) and (e) now follow from parts (a), (b), and (c).

To prove part (f), assume first that $g \in \mathcal{P}_n$. Let $f \in \mathcal{H} \cap \mathcal{D}(Q)$. By part (e) we may write $f = \sum_{k=0}^{\infty} f_k$ with $f_k \in \mathcal{P}_k$, by part (e), which also yields

$$|Q(g,f)| = |Q(g,f_n)| \leq Q(g)^{1/2} Q(f_n)^{1/2}.$$

Since $\mathcal{P}_n$ is finite dimensional (Corollary 2.16) there is a constant C_n such that $Q(f_n) \leq C_n^2 \|f_n\|_{L^2}^2$. Since the functions f_k are orthogonal in the L^2 inner product we have $\|f_n\|_{L^2}^2 \leq \|f\|_{L^2}^2$. Thus $|Q(g,f)| \leq Q(g)^{1/2} C_n \|f\|_{L^2}$. Hence $g \in \mathcal{D}(B)$ and we have shown $\mathcal{P} \subset \mathcal{D}(B)$.

Now suppose that $h \in \mathcal{D}(B)$. Define $h_n(z) = \int_{-\pi}^{\pi} F_n(\theta)(V_\theta h)(z)\, d\theta$. As we have seen, $h_n \in \mathcal{P}$. We will show that $h_n \to h$ in the graph norm of B, using the fact that V_θ is unitary in both of the Hilbert spaces $\mathcal{H}L^2$ and $\mathcal{H} \cap \mathcal{D}(Q)$. If $g \in \mathcal{H} \cap \mathcal{D}(Q)$, then

$$(V_\theta B h, g) = (Bh, V_{-\theta} g) = Q(h, V_{-\theta} g) = Q(V_\theta h, g). \tag{4.11}$$

Since the left side is continuous in g in the L^2 norm so is $Q(V_\theta h, g)$. Hence $V_\theta h \in \mathcal{D}(B)$ and

$$V_\theta B h = B V_\theta h, \qquad h \in \mathcal{D}(B). \tag{4.12}$$

Although this equality is of interest in itself we will actually use (4.11) a little differently. Multiply equation (4.11) by $F_n(\theta)$ and integrate over $[-\pi, \pi]$. The integral can be taken inside both the L^2 and energy inner products because V_θ is strongly continuous in both spaces. We obtain

$$\left(\int_{-\pi}^{\pi} F_n(\theta) V_\theta B h\, d\theta, g\right) = Q(h_n, g) \quad \forall\, g \in \mathcal{H} \cap \mathcal{D}(Q).$$

So

$$\int_{-\pi}^{\pi} F_n(\theta) V_\theta B h\, d\theta = B h_n.$$

As $n \to \infty$ the left side converges to Bh in L^2 norm. Thus $h_n \to h$ and $Bh_n \to Bh$. Hence $\mathcal{P}$ is a core for B. □

Let us remark on the requirement that $p \geq 1$ in Theorem 4.2(a). Our proof fails for $0 < p < 1$ because the inequality in (4.7) would go the wrong way.

However, in the Euclidean case $G = \mathbb{C}^n$ (see Remark 3.2), where ρ_a is the Gaussian heat kernel, it is known that in fact $\mathcal{P}$ is dense in $L^p(\rho_a)$ for $0 < p < 1$. This is a consequence of a theorem of Wallstén [45, Theorem 3.1], from which it follows that the set $\mathcal{E}$ of holomorphic functions of the form $f(z) = \sum_{j=1}^{m} a_j e^{z_j \cdot \bar{w}_j}$, with $a_j \in \mathbb{C}$ and $w_j \in \mathbb{C}^n$, is dense in $L^p(\rho_a)$. Since $\mathcal{E} \subset L^1$, we have that L^1 is dense in L^p. But since $\mathcal{P}$ is dense in L^1 and the inclusion $L^1 \subset L^p$ is continuous, we have $\mathcal{P}$ dense in L^p as well. Unfortunately for us, Wallstén's argument relies heavily on the simple structure of the Gaussian, and it is not clear whether it can be adapted to a general complex Lie group with a Hörmander metric h.

Question 4.4. *For general (G, h), is $\mathcal{P}$ dense in $L^p(\rho_a)$ for $0 < p < 1$?*

In light of this issue, we adopt the following function spaces on which to prove our main results.

Notation 4.5. For $1 \le p < \infty$, let $\mathcal{H}L^p(\rho_a) = \mathcal{H} \cap L^p(\rho_a)$. For $0 < p < 1$, let $\mathcal{H}L^p(\rho_a)$ be the L^p-closure of $\mathcal{H} \cap L^2(\rho_a)$, which may or may not equal $\mathcal{H} \cap L^p(\rho_a)$.

In particular, by this definition, $\mathcal{P}$ is dense in $\mathcal{H}L^p(\rho_a)$ for every $0 < p < \infty$. Also, for $0 < p < q < \infty$, $\mathcal{H}L^q$ is dense in $\mathcal{H}L^p$.

Remark 4.6. Our spaces $\mathcal{H}L^p$ are defined differently from the spaces $\mathcal{H}^p$ used in [26], but in our current setting they are equal.

- For $p = 2$, [26] defines $\mathcal{H}^2$ as the L^2-closure of $\mathcal{H} \cap \mathcal{D}(Q)$; for us, Theorem 4.2(a,b) shows this equals $\mathcal{H} \cap L^2$.
- For $p > 2$, [26] defines $\mathcal{H}^p$ as $\mathcal{H}^2 \cap L^p$; for us this equals $\mathcal{H} \cap L^2 \cap L^p = \mathcal{H} \cap L^p$.
- For $0 < p < 2$, [26] defines $\mathcal{H}^p$ as the L^p closure of $\mathcal{H}^2$. For $0 < p < 1$ this is precisely our definition; for $1 \le p < 2$, this equals $\mathcal{H} \cap L^p$ since $\mathcal{H}L^2$ is dense in $\mathcal{H}L^p$.

In the cases considered by [26], it was possible that $\mathcal{H}^p$ was very different from $\mathcal{H} \cap L^p$; see the counterexamples in [26, Section 5].

We now return to the question of in what sense B is a "holomorphic projection" of A. Let $P_{\mathcal{H}}$ be an orthogonal projection from L^2 onto the closed subspace $\mathcal{H}L^2$.

Proposition 4.7. *For $f \in \mathcal{D}(A) \cap \mathcal{D}(B)$, we have $Bf = P_{\mathcal{H}} A f$.*

Proof. For any $g \in \mathcal{H} \cap \mathcal{D}(Q)$, we have

$$(Bf, g)_{L^2} = Q(f, g) = (Af, g)_{L^2} = (P_{\mathcal{H}} A f, g)_{L^2}.$$

Since $\mathcal{H} \cap \mathcal{D}(Q)$ is dense in $\mathcal{H} \cap L^2$ we must have $Bf = P_{\mathcal{H}} A f$. □

To make the previous proposition more interesting, we should show that $\mathcal{D}(A) \cap \mathcal{D}(B)$ is reasonably large.

Proposition 4.8. *$\mathcal{P} \subset \mathcal{D}(A)$.*

Proof. Let $f \in \mathcal{P}$, and let $\varphi = -\Delta f - h(df, d \log \rho_a)$ be the function which, as in (3.2), ought to equal Af. Integration by parts shows that for any $\psi \in C_c^\infty(G)$ we have $Q(f, \psi) = \int_G \varphi \bar{\psi} \rho_a \, dm$, so if we can show $\varphi \in L^2(\rho_a)$, we will have $|Q(f, \psi)| \le \|\varphi\|_{L^2} \|\psi\|_{L^2}$, implying that $f \in \mathcal{D}(A)$ and moreover $Af = \varphi$.

Since f is holomorphic, $\Delta f = 0$ so we have

$$\varphi = -h(df, d \log \rho_a) = -\sum_j Z_j f \bar{Z}_j \log \rho_a \tag{4.13}$$

using (2.27) and $\bar{Z}_j f = 0$. By Lemma 2.18, $Z_j f \in \mathcal{P} \subset \bigcap_{q \ge 1} L^q(\rho_a)$, and by Lemma 2.26, $\bar{Z}_j \log \rho_a \in \bigcap_{p \ge 1} L^p(\rho_a)$, so by Hölder's inequality, $\varphi \in L^2(\rho_a)$ as desired. □

(A similar argument would show that any L^2 holomorphic function with its first derivatives in $L^{2+\epsilon}$ is also in $\mathcal{D}(A)$.)

In particular we have $\mathcal{P} \subset \mathcal{D}(A) \cap \mathcal{D}(B)$, so $Bf = P_{\mathcal{H}} Af$ for all polynomials.

In the case that A is holomorphic, we actually have that B is simply the restriction of A to $\mathcal{D}(A) \cap \mathcal{H}$. We already showed in (3.3) that $\mathcal{D}(A) \cap \mathcal{H} \subset \mathcal{D}(B)$. For the other direction, let $f \in \mathcal{D}(B)$; by Theorem 4.2(f) we can find a sequence $p_n \in \mathcal{P}$ with $p_n \to f$ and $Bp_n \to Bf$ in L^2. But $Bp_n = P_{\mathcal{H}} Ap_n = Ap_n$ if A is holomorphic, so Ap_n converges, and since A is closed we have $f \in \mathcal{D}(A)$ and $Af = Bf$.

It is conceivable that even when A is not holomorphic, we might get $\mathcal{D}(B) = \mathcal{D}(A) \cap \mathcal{H}$, in which case B is simply the restriction of $P_{\mathcal{H}} A$ to $\mathcal{D}(A) \cap H$, i.e., the literal holomorphic projection of A. However, we do not have a proof of this.

Question 4.9. *Under what conditions does $\mathcal{D}(B) = \mathcal{D}(A) \cap \mathcal{H}$?*

5. Dilations and the operator B

In this subsection, we show that in fact the operator B is just a constant multiple of the vector field Z introduced in (2.11): $B = \frac{2}{a} Z$. Along the way, we establish some lemmas that will also be useful in future computations.

Remark 5.1. To see that $B = \frac{2}{a} Z$ is a plausible statement, consider the Euclidean case $G = \mathbb{C}^n$ as in Remark 3.2. Here A is the Ornstein–Uhlenbeck operator $Af = -\Delta f + \frac{1}{a} z \cdot \nabla f$; since this is a holomorphic operator, B is simply the restriction of A to holomorphic functions. For holomorphic f we have $\Delta f = 0$ and $z \cdot \nabla f = 2 \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}$. On the other hand, as in (2.12), in this case we have $Zf = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}$ (note that all the c_j are 1).

Notation 5.2. Let us introduce a class of convenient functions with which to work. We will say a function $f : G \to \mathbb{C}$ has **polynomial growth** if there are constants C, N such that $|f(z)| \le C(1 + d(e,z))^N$ for all z. Then we let $C_p^2(G)$ denote the class of all $f \in C^2(G)$ such that $f, \xi_j f, \xi_j \xi_k f, Xf, Yf$ all have polynomial growth.

It is immediate that $\mathcal{P} \subset C_p^2(G)$, and if f, g are in $C_p^2(g)$, then so are $f \circ \delta_\lambda$, $\bar{f}$, $f + g$, and fg. Moreover, if $u : \mathbb{C} \to \mathbb{C}$ is a C^2 function with bounded first and second derivatives, then $u(f)$ is also in C_p^2. This is certainly not the broadest class of functions for which the results below will hold, but it is sufficient for our purposes and simplifies several of the arguments.

Lemma 5.3. *If $f \in C_p^2(G)$, then $s \mapsto \int_G f \rho_s \, dm$ is differentiable and*

$$\frac{d}{ds} \int_G f \rho_s \, dm = \frac{1}{4} \int_G \Delta f \rho_s \, dm = \frac{1}{2s} \int_G Xf \rho_s \, dm. \tag{5.1}$$

Proof. Suppose first that $f \in C_c^\infty(G)$. Let $a(s) = \int_G f \rho_s \, dm$. For the first equality, differentiating under the integral sign and then integrating by parts gives

$$a'(s) = \int_G f \frac{d}{ds} \rho_s \, dm = \frac{1}{4} \int_G f \Delta \rho_s \, dm = \frac{1}{4} \int_G \Delta f \rho_s \, dm.$$

For the second equality, we use (2.34) to observe

$$\begin{aligned}\int_G (f \circ \delta_{e^r})\, \rho_s \, dm &= e^{s\Delta/4}(f \circ \delta_{e^r})(e) \\ &= (e^{se^{2r}\Delta/4} f)(\delta_{e^r}(e)) \\ &= (e^{se^{2r}\Delta/4} f)(e) \\ &= \int_G f\, \rho_{se^{2r}}\, dm \\ &= a(se^{2r}).\end{aligned}$$

Now differentiating under the integral sign with respect to r and then setting $r = 0$, we get

$$\int_G Xf\, \rho_s\, dm = \frac{d}{dr}\Big|_{r=0} a(se^{2r}) = 2s\, a'(s),$$

which establishes the second equality of (5.1).

For the case of general $f \in C_p^2(G)$, let $\psi \in C_c^\infty(G)$ be a cutoff function which equals 1 on a neighborhood of $e \in G$, and set $\psi_n(x) = \psi(\delta_{1/n}(x))$. Then $\psi_n \to 1$ boundedly. It follows from (2.30) that $\widetilde{\xi}_j \psi_n \to 0$ and $\widetilde{\xi}_j \widetilde{\xi}_k \psi_n \to 0$ boundedly, at least for $\xi \in V_1$, and the same for general $\xi \in \mathfrak{g}$ by taking commutators. Then since X, Y commute with $\delta_{1/n}$, we also have $X\psi_n \to 0$, $Y\psi_n \to 0$ boundedly. Hence setting $f_n = \psi_n f$, we have constructed $f_n \in C_c^2(G)$ such that, pointwise,

$$f_n \to f, \qquad \Delta f_n \to \Delta f, \qquad X f_n \to Xf,$$

and moreover such that f_n and its derivatives are controlled by f and its derivatives. In particular, there exist C, N such that for all n, x we have

$$|f_n(x)| + |\Delta f_n(x)| + |X f_n(x)| \le C(1 + d(e,x))^N.$$

Now by integrating (5.1), we have

$$\int_G f_n\, (\rho_t - \rho_s)\, dm = \frac{1}{4} \int_s^t \int_G \Delta f_n\, \rho_\sigma\, dm\, d\sigma = \int_s^t \frac{1}{2\sigma} \int_G X f_n\, \rho_\sigma\, dm\, d\sigma. \tag{5.2}$$

By the Gaussian heat kernel upper bounds of Theorem 2.23, we have

$$\int_G C(1 + d(e,x))^N \sup_{\sigma \in [s,t]} \rho_\sigma(x)\, m(dx) < \infty,$$

and so by Fubini's theorem and dominated convergence, we can pass to the limit in (5.2) as $n \to \infty$ to get

$$\int_G f\, (\rho_t - \rho_s)\, dm = \frac{1}{4} \int_s^t \int_G \Delta f\, \rho_\sigma\, dm\, d\sigma = \int_s^t \frac{1}{2\sigma} \int_G Xf\, \rho_\sigma\, dm\, d\sigma. \tag{5.3}$$

Since the two integrals over G are each continuous functions of σ, then by the fundamental theorem of calculus, this is equivalent to the desired result. □

Lemma 5.4. *For $f \in C_p^2(G)$, we have $\int_G Yf\, \rho_s\, dm = 0$.*

Proof. This is similar to the previous proof. By (2.34) we have

$$\begin{aligned}\int_G (f \circ \delta_{e^{i\theta}})\, \rho_s \, dm &= e^{s\Delta/4}(f \circ \delta_{e^{i\theta}})(e) \\ &= (e^{s|e^{i\theta}|^2\Delta/4} f)(\delta_{e^{i\theta}}(e)) \\ &= (e^{s\Delta/4} f)(e) \\ &= \int_G f \, \rho_s \, dm.\end{aligned}$$

If $f \in C_c^2(G)$ we can differentiate under the integral sign with respect to θ and set $\theta = 0$ to get $\int_G Y f \, \rho_s \, dm = 0$. For $f \in C_p^2(G)$, use cutoff functions. □

Corollary 5.5. *Suppose that* $f, g \in \mathcal{P}$. *Then*

$$(Zf, g)_{L^2(\rho_a)} = (f, Zg)_{L^2(\rho_a)}. \tag{5.4}$$

Proof. $-iY(f\bar{g}) = (Z - \bar{Z})(f\bar{g}) = (Zf)\bar{g} - f\overline{Zg}$. Since $f\bar{g} \in C_p^2(G)$, by Lemma 5.4 the integral with respect to $\rho_a \, dm$ is zero. □

Theorem 5.6. *Let* $a > 0$. *We have*

$$\mathcal{D}(B) = \{f \in \mathcal{H}L^2(\rho_a) : Zf \in L^2(\rho_a)\} \tag{5.5}$$

and

$$Bf = \frac{2}{a} Zf \quad \text{for all} \quad f \in \mathcal{D}(B). \tag{5.6}$$

Proof. We begin by showing that (5.6) holds for $f \in \mathcal{P}$. Suppose that f and g are in $\mathcal{P}$, and let Z_j be as defined in (2.25). First observe that

$$Z_j \bar{Z}_j (f\bar{g}) = Z_j \bar{Z}_j f \cdot \bar{g} + \bar{Z}_j f \cdot Z_j \bar{g} + Z_j f \cdot \bar{Z}_j \bar{g} + f \cdot Z_j \bar{Z}_j \bar{g} = Z_j f \cdot \overline{Z_j g}.$$

The first, second and fourth terms of the middle expression vanish because $\bar{Z}_j f = 0$ and $Z_j \bar{Z}_j \bar{g} = \bar{Z}_j Z_j \bar{g} = 0$ (since Z_j is of type (1,0) and commutes with $\bar{Z}_j$). So by (2.27) and (2.32) we have

$$h(df, d\bar{g}) = \frac{1}{2}\Delta(f\bar{g}).$$

Note that $f\bar{g} \in C_p^2(G)$. Thus multiplying by ρ_a and integrating, we have

$$\begin{aligned}(Bf, g)_{L^2(\rho_a)} &= Q(f, g) \\ &= \frac{1}{2}\int_G \Delta(f\overline{g})\rho_a \, dm \\ &= \frac{1}{a}\int_G X(f\overline{g})\rho_a \, dm && \text{by Lemma 5.3} \\ &= \frac{1}{a}\int_g \{(Xf)\overline{g} + f\overline{Xg}\}\rho_a \, dx \\ &= \frac{1}{a}\int_G \{(Zf)\overline{g} + f\overline{Zg}\}\rho_a \, dx && \text{see (2.23)} \\ &= \frac{1}{a}(Zf, g)_{L^2} + (f, Zg)_{L^2} \\ &= \frac{2}{a}(Zf, g)_{L^2} && \text{by Corollary 5.5.}\end{aligned}$$

Since Bf, Zf are both holomorphic and $\mathcal{P}$ is dense in $\mathcal{H}L^2(\rho_a)$, we conclude that $Bf = \frac{2}{a}Zf$.

Now let $f \in \mathcal{D}(B)$ be arbitrary. Since $\mathcal{P}$ is a core for B, we may find $f_n \in \mathcal{P}$ with $f_n \to f$ and $Bf_n \to Bf$ in L^2, and also uniformly on compact sets. In particular, Zf_n converges uniformly on compact sets, so its limit must be Zf. We conclude that $Bf = \frac{2}{a}Zf$ and have also shown the $\subset$ inclusion of (5.5).

For the other inclusion, suppose $f, Zf \in \mathcal{H}L^2$, and as in (4.6) set

$$g_n(z) = \int_{-\pi}^{\pi} F_n(\theta) f(\delta_{e^{i\theta}}(z))\, d\theta.$$

We showed in Theorem 4.2(a) that $g_n \in \mathcal{P}$ and $g_n \to f$ in L^2. Since the integral is over a compact set and f is smooth, we can differentiate under the integral sign to obtain

$$Zg_n(z) = \int_{-\pi}^{\pi} F_n(\theta)(Zf)(\delta_{e^{i\theta}}(z))\, d\theta.$$

Then as before, we have $Zg_n \to Zf$ in L^2. Hence $Bg_n \to \frac{2}{a}Zf$ in L^2. Since B is a closed operator, we have $f \in \mathcal{D}(B)$. ☐

Corollary 5.7. *We have*

$$e^{-tB}f = f \circ \delta_{e^{-2t/a}} \tag{5.7}$$

for $f \in \mathcal{H} \cap L^2(\rho_a)$ *and* $t \geq 0$.

Proof. For $f \in \mathcal{P}_k \subset \mathcal{D}(B)$, by Theorem 5.6 and (2.16), both sides of (5.7) are equal to $e^{-2tk/a}f$. Hence (5.7) holds for all $f \in \mathcal{P}$. Now if $f \in \mathcal{H} \cap L^2(\rho_a)$, by Theorem 4.2(a) we may choose $f_n \in \mathcal{P}$ with $f_n \to f$ in $L^2(\rho_a)$. Since e^{-tB} is a contraction on L^2, we have $e^{-tB}f_n \to e^{-tB}f$ in L^2, and also $f_n \circ \delta_{e^{-2t/a}} \to f \circ \delta_{e^{-2t/a}}$ pointwise. ☐

Remark 5.8. In light of Theorem 5.6, our goal of understanding strong hypercontractivity for the holomorphic projection of the semigroup e^{-tA} has essentially reduced to the problem of understanding it for the dilation semigroup on G. A related study was undertaken in the papers [22, 23], in which the authors consider the dilation semigroup on real Euclidean space. In these papers, the holomorphic functions are replaced with the class of log-subharmonic functions, and the authors examine the relationship between an appropriate version of strong hypercontractivity and a so-called strong logarithmic Sobolev inequality for such functions. In recent work by the first author [16], these results are extended to real stratified Lie groups.

Remark 5.9. The dilation semigroup also arises from the Ornstein–Uhlenbeck semigroup e^{-tA} in another way. In [33], the author introduces a "Mehler semigroup" e^{-tN} on a stratified Lie group, defined as follows (after adjusting notation and time scaling):

$$(e^{-tN}f)(x) = \int_G f\left(\delta_{e^{-\beta t}}(x) \cdot \delta_{\sqrt{1-e^{-2\beta t}}}(y)\right) \rho_a(y)\, m(dy) \tag{5.8}$$

where we take $\beta = 2/a$ to make our time scaling come out right. The name "Mehler semigroup" is explained by the fact that when $G = \mathbb{R}^n$ (i.e., a stratified Lie group of step 1), then (5.8) is precisely Mehler's formula for the Ornstein–Uhlenbeck semigroup, so in this special case, $e^{-tN} = e^{-tA}$. For a nonabelian group

G, e^{-tN} and e^{-tA} differ, and e^{-tN} is a nonsymmetric semigroup on $L^2(\rho_a)$. A simple computation shows that, formally, the generator of e^{-tN} is $N = -\Delta + \beta X = -\Delta + \frac{2}{a}X$. In particular, when f is holomorphic, we have (still formally)

$$Nf = \frac{2}{a}Xf = \frac{2}{a}Zf = Bf. \tag{5.9}$$

Thus our main Theorem 7.2 below could be restated as giving the strong hypercontractivity of the Mehler semigroup e^{-tN}, still conditionally on the logarithmic Sobolev inequality (7.1).

As a consequence of Theorem 5.6, we can show:

Theorem 5.10. *Except in the abelian case $G = \mathbb{C}^n$, A is not holomorphic.*

Proof. Consider the decomposition $\mathfrak{g} = \bigoplus_{j=1}^m V_j$ as in (2.1), where $V_m \neq 0$ is the center of $\mathfrak{g}$. Excluding the abelian case $G = \mathbb{C}^n$, we have $m > 1$.

Fix a nonzero $\eta \in V_m$ and let $\ell : \mathfrak{g} \to \mathbb{C}$ be a complex linear functional with $\ell(\eta) = 1$ and $\ell = 0$ on $V_1 \oplus \cdots \oplus V_{m-1}$. The exponential map $\exp : \mathfrak{g} \to G$ is a holomorphic diffeomorphism, so we can define a holomorphic function $f : G \to \mathbb{C}$ by $f(\exp(\xi)) = \ell(\xi)$. (Previously we took $G = \mathfrak{g}$ as sets and exp to be the identity, but for now we shall write exp explicitly.) In fact, f is homogeneous of degree m, so $f \in \mathcal{P}_m$. We thus have $f \in \mathcal{D}(A) \cap \mathcal{D}(B)$ by Theorem 4.2(f) and Proposition 4.8. If Af were holomorphic, by Proposition 4.7 we would have $Af = Bf$. We show this is not the case.

Let $g = \exp(\eta) \in G$, so that $f(g) = 1$. By Theorem 5.6 and (2.16), we have $Bf = \frac{2}{a}Zf = \frac{2m}{a}f$, so $Bf(g) = \frac{2m}{a}$.

On the other hand, suppose $\xi \in V_1$. For any $t \in \mathbb{R}$, we have $g \cdot \exp(t\xi) = \exp(\eta)\exp(t\xi) = \exp(\eta + t\xi)$, since $\eta \in V_m$ commutes with ξ. Thus $f(g \cdot \exp(t\xi)) = \ell(\eta + t\xi) = 1$ since $\xi \in V_1$ implies $\ell(\xi) = 0$. Differentiating with respect to t at $t = 0$, we have $\widetilde{\xi} f(g) = 0$. Hence $\nabla f(g) = 0$ and so by (3.2) and (2.32), $Af(g) = 0 \neq Bf(g)$. ∎

As an explicit example, in the complex Heisenberg group $\mathbb{H}_3^{\mathbb{C}}$ with coordinates (z_1, z_2, z_3), one could take $f(z) = z_3$ and verify by direct computation that $Zf(0,0,1) = 2$ while $Af(0,0,1) = 0$.

In the case of stratified Lie groups of step 2, explicit integral formulas for the heat kernel ρ_a are known [20, 41]. So in those cases, to show A is not holomorphic, in light of (3.2) one could compute $\bar{Z}_j \log \rho_a$ and check that it is not holomorphic.

6. Contractivity of e^{-tB}

Theorem 6.1. *Let $0 < p < \infty$. For every $f \in \mathcal{H}L^p(\rho_a)$ and every $t \geq 0$ we have*

$$\|f \circ \delta_{e^{-t}}\|_{L^p(\rho_a)} \leq \|f\|_{L^p(\rho_a)}. \tag{6.1}$$

In particular, e^{-tB} extends continuously to $\mathcal{H}L^p(\rho_a)$ for $0 < p < 2$ and is a contraction on $\mathcal{H}L^p(\rho_a)$ for $0 < p < \infty$.

Proof. First, let us note that for any $g \in L^1(\rho_a)$, the scaling relation (2.34) implies

$$\int_G (g \circ \delta_{e^{-t}})\, \rho_a \, dm = \int_G g\, \rho_{ae^{-2t}}\, dm. \tag{6.2}$$

So if $g \in C_p^2(G)$ with $\Delta g \geq 0$, then Lemma 5.3 implies that this quantity decreases with respect to t; that is,

$$\int_G (g \circ \delta_{e^{-t}})\, \rho_a \, dm \leq \int_G g\, \rho_a \, dm, \quad g \in C_p^2(G), \quad \Delta g \geq 0. \tag{6.3}$$

We would now like to replace g with some approximation of $|f|^p$. To achieve this, let us first suppose that $f \in \mathcal{P}$; the general case will then follow from a density argument. Following [27, Lemma 4.3] we shall introduce a sequence of "subharmonizing" functions.

Let $v \in C_c^\infty((0,\infty))$ be nonnegative, and set

$$u(t) = \int_0^t \frac{1}{s} \int_0^s v(\sigma)\, d\sigma\, ds.$$

Then it is easy to verify that:

- $u \in C^\infty([0,\infty))$;
- $u \geq 0$;
- u', u'' are bounded;
- $tu''(t) + u'(t) = v(t) \geq 0$ for all $t \geq 0$.

As such, if $f \in \mathcal{P}$, then $g := u(|f|^2) \in C_p^2(G)$. Now using the chain rule and the fact that f is holomorphic (so that $\bar{Z}_j f = 0$), we have

$$\begin{aligned}
\frac{1}{4}\Delta g &= \sum_{j=1}^m Z_j \bar{Z}_j u(|f|^2) \\
&= \sum_{j=1}^m Z_j \left[u'(|f|^2) f \overline{Z_j f} \right] \\
&= \sum_{j=1}^m \left\{ u''(|f|^2) \bar{f} Z_j f \cdot f \overline{Z_j f} + u'(|f|^2) |Z_j f|^2 \right\} \\
&= \sum_{j=1}^m \left(|f|^2 u''(|f|^2) + u'(|f|^2) \right) |Z_j f|^2.
\end{aligned}$$

Since $tu''(t) + u'(t) \geq 0$, we have $\Delta g \geq 0$ and so (6.3) holds with $g = u(|f|^2)$.

Now let $v_n \in C_c^\infty((0,\infty))$ be a sequence of nonnegative smooth functions with $v_n(\sigma) \uparrow \left(\frac{p}{2}\right)^2 \sigma^{(p/2)-1}$ for $\sigma > 0$, and as before set $u_n(t) = \int_0^t \frac{1}{s} \int_0^s v_n(\sigma)\, d\sigma\, ds$ and $g_n = u_n(|f|^2)$. As before, g_n satisfies (6.3). By monotone convergence,

$$u_n(t) \uparrow \int_0^t \frac{1}{s} \int_0^s \left(\frac{p}{2}\right)^2 \sigma^{(p/2)-1}\, d\sigma\, ds = t^{p/2}$$

and hence $g_n \uparrow |f|^p$. Hence using (6.3) and monotone convergence, we have

$$\int_G |f \circ \delta_{e^{-t}}|^p \, \rho_a \, dm \leq \int_G |f|^p \rho_a \, dm \tag{6.4}$$

so that (6.1) holds for $f \in \mathcal{P}$.

Now let $f \in \mathcal{H}L^p(\rho_a)$ be arbitrary. As mentioned following Notation 4.5, $\mathcal{P}$ is dense in $\mathcal{H}L^p(\rho_a)$, so we may find a sequence $f_n \in \mathcal{P}$ with $f_n \to f$ in L^p and also pointwise, so that in particular $f_n \circ \delta_{e^{-t}} \to f \circ \delta_{e^{-t}}$ pointwise. Now since (6.1) holds for f_n, we see that $f_n \circ \delta_{e^{-t}}$ is Cauchy in L^p, hence converges in L^p, and the limit must equal the pointwise limit $f \circ \delta_{e^{-t}}$. (In particular, $f \circ \delta_{e^{-t}} \in \mathcal{H}L^p(\rho_a)$.)

Since the p-norm is continuous on L^p, we can pass to the limit in (6.1) to see that it holds for f. ☐

Corollary 6.2. *e^{-tB} is a strongly continuous contraction semigroup on $\mathcal{H}L^p(\rho_a)$ for $0 < p < \infty$.*

Proof. As we noted, $e^{-tB}f = f \circ \delta_{e^{-t}}$. Hence the semigroup property is given by (2.6), and the previous theorem showed the contractivity. To verify strong continuity, we note that for $f \in \mathcal{P}_k$ we have $f \circ \delta_{e^{-t}} \to f$ pointwise, and $|f \circ \delta_{e^{-t}}| = e^{-tk}|f| \le |f|$. So by dominated convergence, $e^{-tB}f = f \circ \delta_{e^{-t}} \to f$ in L^p as $t \to 0$. By linearity, the same holds for any $f \in \mathcal{P}$. For general $f \in \mathcal{H}L^p(\rho_a)$, we use a familiar triangle inequality argument. Since $\mathcal{P}$ is dense in $\mathcal{H}L^p$, for any ϵ we can choose $g \in \mathcal{P}$ with $\|f - g\|_{L^p} < \epsilon$. For $p \ge 1$, Minkowski's triangle inequality gives

$$\begin{aligned}\|e^{-tB}f - f\|_{L^p} &\le \|e^{-tB}(f-g)\|_{L^p} + \|e^{-tB}g - g\|_{L^p} + \|g - f\|_{L^p} \\ &\le 2\epsilon + \|e^{-tB}g - g\|_{L^p}\end{aligned}$$

using the contractivity of e^{-tB} on the first term. Since $g \in \mathcal{P}$, we know that $\|e^{-tB}g - g\|_{L^p} \to 0$ and hence $\limsup_{t\to 0} \|e^{-tB}f - f\|_{L^p} \le 2\epsilon$, implying the desired result since ϵ is arbitrary. For $0 < p < 1$, $\|\cdot\|_{L^p}$ is not a norm, but we get the same result by replacing $\|\cdot\|_{L^p}$ with $\|\cdot\|_{L^p}^p$, which does satisfy the triangle inequality. ☐

7. Strong hypercontractivity for the dilation semigroup

We now state and prove our main theorem.

We say that the heat kernel ρ_a satisfies a **logarithmic Sobolev inequality** if there exist $c > 0$ and $\beta \ge 0$ such that

$$\int_G |f|^2 \log|f| \rho_a \, dm \le cQ(f) + \beta\|f\|^2_{L^2(\rho_a)} + \|f\|^2_{L^2(\rho_a)} \log\|f\|_{L^2(\rho_a)} \tag{7.1}$$

for all f such that $Q(f) < \infty$. (In the case $\beta > 0$, (7.1) is sometimes called a **defective** logarithmic Sobolev inequality.)

Remark 7.1. To the best of our knowledge, it is currently an open problem to determine whether the logarithmic Sobolev inequality (7.1) is satisfied in all complex stratified Lie groups G. As such, our main Theorem 7.2 is necessarily conditional in nature, taking (7.1) as a hypothesis. However, in Section 8 below, we discuss the particular case of the complex Heisenberg and Heisenberg–Weyl groups, for which (7.1) is known to hold [14, 30] and which therefore serve as a concrete example to which our theorem applies. It would be of great interest to have additional examples of groups satisfying (7.1).

For $0 < q \le p < \infty$, let

$$t_J(p, q) := \frac{c}{2} \log\left(\frac{p}{q}\right) \tag{7.2}$$

and

$$M(p, q) := \exp\left(2\beta\left(\frac{1}{q} - \frac{1}{p}\right)\right). \tag{7.3}$$

Theorem 7.2. *Suppose that the logarithmic Sobolev inequality* (7.1) *holds and that $0 < q \le p < \infty$. Then for every $f \in \mathcal{H}L^q(\rho_a)$ and every $t \ge t_J(p, q)$,*

$$\|e^{-tB}f\|_{L^p(\rho_a)} \le M(p, q)\|f\|_{L^q(\rho_a)}. \tag{7.4}$$

Proof. Fix $0 < q \le p < \infty$. We shall concentrate first on the case when $f \in \mathcal{P}$; let us say f has degree D, so $f \in \bigoplus_{k=0}^{D} \mathcal{P}_k$. The general case will then follow by a density argument as in the proof of Theorem 6.1. We also note that it is sufficient to prove that (7.4) holds for $t = t_J(p,q)$, since if this can be shown, then using Theorem 6.1 we conclude that for any $t \ge t_J$,

$$\|e^{-tB} f\|_{L^p} = \|e^{-t_J B}(e^{-(t-t_J)B} f)\|_{L^p} \le M(p,q)\|e^{-(t-t_J)B} f\|_{L^q} \le M(p,q)\|f\|_{L^q}.$$

We adopt similar notation as in [26, Section 4], which we generally follow. Let

$$g_t := e^{-tB} f.$$

Since $\mathcal{P}_k$ is invariant under B (Corollary 5.7 and Lemma 2.17), g_t is a smooth curve in the finite-dimensional space $\bigoplus_{k=0}^{D} \mathcal{P}_k$. Indeed, if $f = \sum_{k=0}^{D} f_k$ with $f_k \in \mathcal{P}_k$, we have $g_t = \sum_{k=0}^{D} e^{-2tk/a} f_k$.

Fix $\epsilon > 0$ and let

$$\begin{aligned} \gamma_t &:= \left(|g_t|^2 + \epsilon\right)^{1/2}, \\ r(t) &:= q e^{2t/c}, \\ v(t) &:= \int \gamma_t(x)^{r(t)} \rho_a(x)\, m(dx), \\ \alpha(t) &:= \|\gamma_t\|_{L^{r(t)}(\rho_a)} = v(t)^{1/r(t)}. \end{aligned}$$

Notice that $\gamma_t \in C_p^2(G)$ (see Notation 5.2) and in particular $v(t)$, $\alpha(t)$ are finite for all t. Also notice that $r(t_J) = p$. Our goal will be to show $\alpha(t_J) \le M(p,q)\alpha(0)$, which when taking $\epsilon \to 0$ turns into (7.4) with $t = t_J$. We will do this by deriving an appropriate differential inequality for α.

Simple calculus shows

$$\alpha'(t) = \alpha(t) v(t)^{-1} \left(r(t)^{-1} v'(t) - \frac{2}{c} v(t) \log \alpha(t) \right). \tag{7.5}$$

To attack this, we differentiate under the integral sign to show

$$\begin{aligned} v'(t) &= \int_G \gamma_t^{r(t)} \left(r'(t) \log \gamma_t + \frac{r(t)}{\gamma_t} \gamma_t' \right) \rho_a \, dm && (7.6) \\ &= \frac{2r(t)}{c} \int_G \gamma_t^{r(t)} \log \gamma_t \, \rho_a \, dm + r(t) \int_G \gamma_t^{r(t)-1} \gamma_t' \, \rho_a \, dm && (7.7) \\ &= \frac{2r(t)}{c} \int_G \gamma_t^{r(t)} \log \gamma_t \, \rho_a \, dm - r(t) \operatorname{Re} \int_G \gamma_t^{r(t)-2} B g_t \cdot \overline{g_t} \, \rho_a \, dm. && (7.8) \end{aligned}$$

To check that differentiation under the integral sign is justified, fix a bounded interval $[t_1, t_2]$ containing t, and note that since $s \mapsto g_s$ is a continuous curve in the holomorphic polynomials of degree D, there is a constant C so that $|g_s(x)| + |g_s'(x)| \le C(1 + d(e,x))^D$ for all $s \in [t_1, t_2]$. Since γ_t is bounded below and r, r' are bounded on $[t_1, t_2]$ by some constant R, it follows that for $t \in [t_1, t_2]$ the integrand on the right side of (7.6) is dominated by some constant times $(C(1 + d(e,x))^D)^{R+1} \rho_a(x)$, which is integrable.

Let $I := r \operatorname{Re} \int_G \gamma^{r-2} Bg \cdot \bar{g} \, \rho_a \, dx$ be the second term in (7.8). (For notational hygiene, we suppressed the explicit dependence on t and will continue to do so when convenient.) We wish to estimate I from below using the logarithmic Sobolev inequality, so we need to convert it into an expression involving Q.

Since g is a polynomial, by Theorem 5.6 and (2.23), we have $Bg = \frac{2}{a} Zg = \frac{2}{a} Xg$, so that

$$I = \frac{2r}{a} \operatorname{Re} \int_G \gamma^{r-2} Xg \cdot \bar{g}\, \rho_a\, dm.$$

But X is a real vector field, so an easy computation shows $X[|g|^2] = 2 \operatorname{Re}[Xg \cdot \bar{g}]$ and hence $X[\gamma^r] = r\gamma^{r-2} \operatorname{Re}[Xg \cdot \bar{g}]$. Since $\gamma^r \in C_p^2(G)$, by Lemma 5.3 we have

$$I = \frac{2}{a} \int_G X[\gamma^r] \rho_a\, dm = \int_G \Delta[\gamma^r] \rho_a\, dm.$$

Now using elementary calculus, we may show:

$$\Delta[\gamma^r] = 4|\nabla \gamma^{r/2}|^2 + r\epsilon \gamma^{r-4} |\nabla g|^2. \tag{7.9}$$

To see this, let Z_j be the vector fields defined in (2.25), which are of type $(1,0)$, so that $\Delta = 4 \sum_j Z_j \bar{Z}_j$. We have

$$\begin{aligned} 4 Z_j \bar{Z}_j [\gamma^r] &= 4 Z_j \left[\frac{r}{2} \gamma^{r-2} \cdot \left(\cancel{\bar{Z}_j g} \cdot \bar{g} + g \cdot \bar{Z}_j \bar{g} \right) \right] \\ &= 2r \cdot \frac{r-2}{2} \gamma^{r-4} \cdot (Z_j g \cdot \bar{g} + g \cdot \cancel{Z_j \bar{g}})\, (g \cdot \bar{Z}_j) \\ &\quad + 2r\gamma^{r-2} \left(Z_j g \cdot \bar{Z}_j \bar{g} + g \cdot \cancel{Z_j \bar{Z}_j \bar{g}} \right) \end{aligned}$$

since $Z_j \bar{Z}_j \bar{g} = \bar{Z}_j Z_j \bar{g} = 0$. Now rearranging,

$$\begin{aligned} 4 Z_j \bar{Z}_j [\gamma^r] &= r(r-2) \gamma^{r-4} |Z_j g|^2 |g|^2 + 2r\gamma^{r-2} |Z_j g|^2 \\ &= r^2 \gamma^{r-4} |Z_j g|^2 |g|^2 + 2r\gamma^{r-4} |Z_j g|^2 (\gamma^2 - |g|^2) \\ &= r^2 \gamma^{r-4} |Z_j g|^2 |g|^2 + 2r\epsilon \gamma^{r-4} |Z_j g|^2 \end{aligned}$$

since $\gamma^2 - |g|^2 = \epsilon$. On the other hand,

$$Z_j[\gamma^{r/2}] = \frac{r}{4} \gamma^{\frac{r-4}{2}} Z_j g \cdot \bar{g}$$

so that

$$4 Z_j \bar{Z}_j [\gamma^r] = 16 |Z_j[\gamma^{r/2}]|^2 + 2r\epsilon \gamma^{r-4} |Z_j g|^2.$$

Summing over j and referring to (2.28–2.29), we obtain (7.9).

In particular, since the second term of (7.9) is nonnegative,

$$\Delta[\gamma^r] \geq 4 |\nabla[\gamma^{r/2}]|^2.$$

So integrating gives

$$I \geq 4Q(\gamma^{r/2}).$$

Now, applying the logarithmic Sobolev inequality (7.1) and noting that $\left\| \gamma_t^{r(t)/2} \right\|_{L^2(\rho_a)}^2 = v(t)$, it follows that

$$I \geq \frac{2r(t)}{c} \int_G \gamma_t^{r(t)} \log \gamma_t\, \rho_a\, dm - \frac{4\beta}{c} v(t) - \frac{2}{c} v(t) \log v(t).$$

Referring back to (7.8), this shows

$$v'(t) \leq \frac{4\beta}{c} v(t) + \frac{2}{c} v(t) \log v(t) = \frac{4\beta}{c} v(t) + \frac{2r(t)}{c} v(t) \log \alpha(t), \tag{7.10}$$

and thus from (7.5)

$$\alpha'(t) \leq \frac{4\beta \alpha(t)}{c r(t)}. \tag{7.11}$$

In other words,

$$\frac{d}{dt}\log\alpha(t) \le \frac{4\beta}{cr(t)} = \frac{4\beta}{cq}e^{-2t/c}, \tag{7.12}$$

so, integrating,

$$\alpha(t) \le \alpha(0)\exp\left(\frac{2\beta}{q}(1-e^{-2t/c})\right) = \alpha(0)\exp\left(2\beta\left(\frac{1}{q}-\frac{1}{r(t)}\right)\right). \tag{7.13}$$

Now let $\epsilon \downarrow 0$, so that $\gamma_t \downarrow |g_t|$, and by dominated convergence, $\alpha(t) \downarrow \|g_t\|_{L^{r(t)}(\rho_a)} = \|e^{-tB}f\|_{L^{r(t)}(\rho_a)}$. Taking $t = t_J$ and recalling that $r(t_J) = p$, (7.13) becomes

$$\|e^{-t_J B}f\|_{L^p(\rho_a)} \le M(p,q)\|f\|_{L^q(\rho_a)}, \tag{7.14}$$

which is precisely (7.4) with $t = t_J$. This completes the proof for $f \in \mathcal{P}$.

For general $f \in \mathcal{H}L^q(\rho_a)$, proceed as in the last paragraph of the proof of Theorem 6.1. Choose a sequence $f_n \in \mathcal{P}$ with $f_n \to f$ in L^q-norm. Then (7.4) holds for f_n. As $n \to \infty$, the right side of (7.4) converges to $M(p,q)\|f\|_{L^q(\rho_a)}$. Since e^{-tB} is a contraction on $\mathcal{H}L^p$ by Theorem 6.1, $e^{-tB}f_n$ is Cauchy in L^p norm, so converges in L^p to some function which can only be $e^{-tB}f$. Hence the left side of (7.4) converges to $\|e^{-tB}f\|_{L^p(\rho_a)}$ as desired. □

8. Application to the complex Heisenberg group

In order for Theorem 7.2 to have content, we need examples of stratified complex groups for which the logarithmic Sobolev inequality (7.1) is satisfied. In this section, we verify that the complex Heisenberg group $\mathbb{H}_3^{\mathbb{C}}$ of Examples 2.3 and 2.20 enjoys that property, as do the complex Heisenberg–Weyl groups $\mathbb{H}_{2n+1}^{\mathbb{C}}$ of Examples 2.4 and 2.21. So for these groups, the hypotheses of our Theorem 7.2 are satisfied. On the other hand, since as shown in Theorem 5.10, the operator A is not holomorphic in this setting, the results of [26] do not apply, so we have proved something new.

Indeed, the papers [14] and [30] showed independently that so-called H-type Lie groups satisfy a gradient estimate which is known to imply the logarithmic Sobolev inequality (7.1). We shall state that result, check that the complex Heisenberg group $\mathbb{H}_3^{\mathbb{C}}$ is an H-type Lie group, and sketch in the steps leading to (7.1). The same argument, *mutatis mutandis*, also applies to the Heisenberg–Weyl groups $\mathbb{H}_{2n+1}^{\mathbb{C}}$. We omit the details because they add notation but no further insight.

Definition 8.1. Suppose $\mathfrak{g}$ is a *real* Lie algebra equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$. For $u, v \in \mathfrak{g}$, define $J_u v$ via

$$\langle J_u v, w\rangle = \langle u, [v,w]\rangle.$$

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, and $\mathfrak{v} = \mathfrak{z}^\perp$. We say $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is **H-type** if:

(1) $[\mathfrak{v},\mathfrak{v}] = \mathfrak{z}$; and
(2) for each $u \in \mathfrak{z}$ with $\|u\| = 1$, J_u maps $\mathfrak{v}$ isometrically onto itself.

An **H-type Lie group** is a connected, simply connected real Lie group G equipped with an inner product $\langle\cdot,\cdot\rangle$ on its Lie algebra $\mathfrak{g}$ such that $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is H-type in the above sense.

Suppose then that $(G, \langle\cdot,\cdot\rangle)$ is an H-type Lie group. By item 1 of Definition 8.1, G is nilpotent, so we may fix a bi-invariant Haar measure m which is simply (a scalar multiple of) Lebesgue measure. Let $\xi_1, \dots, \xi_n$ be an orthonormal basis for $\mathfrak{v} \subset \mathfrak{g}$, let $\widetilde{\xi_1}, \dots, \widetilde{\xi_n}$ be the corresponding left-invariant vector fields, and define the sub-Laplacian by $\Delta = \widetilde{\xi_1}^2 + \cdots + \widetilde{\xi_n}^2$. Also, for sufficiently smooth f let $|\nabla f|^2 := |\widetilde{\xi_1} f|^2 + \cdots + |\widetilde{\xi_n} f|^2$. The main theorem of [14] and [30] is:

Theorem 8.2. *If $(G, \langle\cdot,\cdot\rangle)$ is H-type, then following the above notation, there is a constant K such that for all $t \geq 0$ and $f \in C_c^1(G)$ we have*

$$|\nabla e^{t\Delta/4} f| \leq K e^{t\Delta/4} |\nabla f|. \tag{8.1}$$

Lemma 8.3. *Consider $\mathbb{H}_3^{\mathbb{C}}$ as a 6-dimensional real Lie group. As a set, $\mathfrak{h}_3^{\mathbb{C}} = \mathbb{C}^3 = \mathbb{R}^6$, so equip it with the Euclidean inner product $\langle\cdot,\cdot\rangle$. Then $(\mathbb{H}_3^{\mathbb{C}}, \langle\cdot,\cdot\rangle)$ is H-type.*

Proof. Let $\{e_j, ie_j : j = 1, 2, 3\}$ be the standard basis of $\mathfrak{h}_3^{\mathbb{C}} = \mathbb{C}^3 = \mathbb{R}^6$, which is orthonormal with respect to the (real) Euclidean inner product $\langle\cdot,\cdot\rangle$. Then the center $\mathfrak{z}$ of $\mathfrak{h}_3^{\mathbb{C}}$ is spanned (over $\mathbb{R}$) by $\{e_3, ie_3\}$, so $\mathfrak{v} = \mathfrak{z}^\perp$ is spanned by $\{e_1, ie_1, e_2, ie_2\}$. By inspection of the Lie bracket defined in (2.3), we see that $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$.

Next, we note that for $u, v, w \in \mathfrak{h}_3^{\mathbb{C}}$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\langle J_{\alpha u}(\beta v), w\rangle = \langle \alpha u, [\beta v, w]\rangle = \langle u, [v, \bar\alpha \beta w]\rangle = \langle J_u v, \bar\alpha\beta w\rangle = \langle \alpha\bar\beta J_u v, w\rangle \tag{8.2}$$

so that $J_u v$ is complex-linear in u and conjugate-linear in v. Together with the relations $J_{e_3} e_1 = e_2$, $J_{e_3} e_2 = -e_1$, we easily see that for any $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, we have that $J_{\alpha e_3}$ is an isometry of $\mathfrak{v}$ into itself. ☐

Now we note that when the dual metric h is defined on $(\mathfrak{h}_3^{\mathbb{C}})^*$ as in Example 2.20, the backward annihilator H is precisely $\mathfrak{v}$, and the metric g is just the restriction of $\langle\cdot,\cdot\rangle$ to H. Hence the sub-Laplacian Δ used in Theorem 8.2 is the same as that defined in (2.32), and for smooth real f, the squared gradient $|\nabla f|$ of Theorem 8.2 is equal to $h(df, df)$ in the notation of Section 2.4.

Theorem 8.4. *It follows from Theorem 8.2 that the logarithmic Sobolev inequality (7.1) holds for $\mathbb{H}_3^{\mathbb{C}}$, with $c = 2K^2 a$ and $\beta = 0$, where K is the constant from Theorem 8.2.*

Proof. This can be proved by an elementary, though clever, argument in the style of Γ_2-calculus, which can be found in [4, Theorem 6.1]. The essence of this argument, which is an equivalence between gradient bounds and the logarithmic Sobolev inequality, goes back to [3]. ☐

Corollary 8.5. *Theorem 7.2 holds for the complex Heisenberg and Heisenberg–Weyl groups $\mathbb{H}_{2n+1}^{\mathbb{C}}$, with $t_J(p, q) = K^2 a \log\left(\frac{p}{q}\right)$ and $M(p, q) = 1$, where K is the constant from Theorem 8.2.*

Remark 8.6. The foregoing argument would apply to any complex stratified Lie group which is H-type. Since the complex stratified groups and the H-type groups are each rather large classes, one might think there would be many more such examples. However, there are actually no more: the first author has shown in [15] that the complex Heisenberg–Weyl Lie algebras are the only complex Lie algebras which are H-type under a Hermitian inner product.

Acknowledgments

The authors would like to thank Bruce K. Driver for helpful discussions regarding this paper. They would also like to thank the anonymous referees for several very helpful suggestions, including the relevance of the paper [33].

References

[1] D. Bakry, *On Sobolev and logarithmic Sobolev inequalities for Markov semigroups*, New trends in stochastic analysis (Charingworth, 1994), World Sci. Publ., River Edge, NJ, 1997, pp. 43–75. MR1654503

[2] Dominique Bakry, *L'hypercontractivité et son utilisation en théorie des semigroupes* (French), Lectures on probability theory (Saint-Flour, 1992), Lecture Notes in Math., vol. 1581, Springer, Berlin, 1994, pp. 1–114, DOI 10.1007/BFb0073872. MR1307413

[3] Dominique Bakry and Michel Émery, *Hypercontractivité de semi-groupes de diffusion* (French, with English summary), C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 15, 775–778. MR772092

[4] Dominique Bakry, Fabrice Baudoin, Michel Bonnefont, and Djalil Chafaï, *On gradient bounds for the heat kernel on the Heisenberg group*, J. Funct. Anal. **255** (2008), no. 8, 1905–1938, DOI 10.1016/j.jfa.2008.09.002. MR2462581

[5] Fabrice Baudoin, Martin Hairer, and Josef Teichmann, *Ornstein-Uhlenbeck processes on Lie groups*, J. Funct. Anal. **255** (2008), no. 4, 877–890, DOI 10.1016/j.jfa.2008.05.004. MR2433956

[6] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007. MR2363343

[7] Eugenio Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259–290. MR645743

[8] Eric A. Carlen, *Some integral identities and inequalities for entire functions and their application to the coherent state transform*, J. Funct. Anal. **97** (1991), no. 1, 231–249, DOI 10.1016/0022-1236(91)90022-W. MR1105661

[9] Eric A. Carlen, *Superadditivity of Fisher's information and logarithmic Sobolev inequalities*, J. Funct. Anal. **101** (1991), no. 1, 194–211, DOI 10.1016/0022-1236(91)90155-X. MR1132315

[10] Bruce K. Driver and Leonard Gross, *Hilbert spaces of holomorphic functions on complex Lie groups*, New trends in stochastic analysis (Charingworth, 1994), World Sci. Publ., River Edge, NJ, 1997, pp. 76–106. MR1654507

[11] Bruce K. Driver, Leonard Gross, and Laurent Saloff-Coste, *Holomorphic functions and subelliptic heat kernels over Lie groups*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 5, 941–978, DOI 10.4171/JEMS/171. MR2538496

[12] Bruce K. Driver, Leonard Gross, and Laurent Saloff-Coste, *Surjectivity of the Taylor map for complex nilpotent Lie groups*, Math. Proc. Cambridge Philos. Soc. **146** (2009), no. 1, 177–195, DOI 10.1017/S0305004108001692. MR2461876

[13] Bruce K. Driver, Leonard Gross, and Laurent Saloff-Coste, *Growth of Taylor coefficients over complex homogeneous spaces*, Tohoku Math. J. (2) **62** (2010), no. 3, 427–474, DOI 10.2748/tmj/1287148621. MR2742018

[14] Nathaniel Eldredge, *Gradient estimates for the subelliptic heat kernel on H-type groups*, J. Funct. Anal. **258** (2010), no. 2, 504–533, DOI 10.1016/j.jfa.2009.08.012. MR2557945

[15] Nathaniel Eldredge, *On complex H-type Lie algebras*, preprint, arXiv:1406.2396, 2014.

[16] Nathaniel Eldredge, *Strong hypercontractivity and strong logarithmic Sobolev inequalities for log-subharmonic functions on stratified Lie groups*, preprint, arXiv:1706.07517, 2017.

[17] Paul Federbush, *Partially alternate derivation of a result of Nelson*, J. Math. Physics **10** (1969), no. 1, 50–52.

[18] G. B. Folland and Elias M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982. MR657581

[19] Akito Futaki, *Kähler-Einstein metrics and integral invariants*, Lecture Notes in Mathematics, vol. 1314, Springer-Verlag, Berlin, 1988. MR947341

[20] Bernard Gaveau, *Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents*, Acta Math. **139** (1977), no. 1-2, 95–153, DOI 10.1007/BF02392235. MR0461589

[21] James Glimm, *Boson fields with nonlinear self-interaction in two dimensions*, Comm. Math. Phys. **8** (1968), 12–25.

[22] Piotr Graczyk, Todd Kemp, and Jean-Jacques Loeb, *Hypercontractivity for log-subharmonic functions*, J. Funct. Anal. **258** (2010), no. 6, 1785–1805, DOI 10.1016/j.jfa.2009.08.014. MR2578455

[23] Piotr Graczyk, Todd Kemp, and Jean-Jacques Loeb, *Strong logarithmic Sobolev inequalities for log-subharmonic functions*, Canad. J. Math. **67** (2015), no. 6, 1384–1410, DOI 10.4153/CJM-2015-015-8. MR3415657

[24] Leonard Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061–1083, DOI 10.2307/2373688. MR0420249

[25] Leonard Gross, *Logarithmic Sobolev inequalities and contractivity properties of semigroups*, Dirichlet forms (Varenna, 1992), Lecture Notes in Math., vol. 1563, Springer, Berlin, 1993, pp. 54–88, DOI 10.1007/BFb0074091. MR1292277

[26] Leonard Gross, *Hypercontractivity over complex manifolds*, Acta Math. **182** (1999), no. 2, 159–206, DOI 10.1007/BF02392573. MR1710181

[27] Leonard Gross, *Strong hypercontractivity and relative subharmonicity*, J. Funct. Anal. **190** (2002), no. 1, 38–92, DOI 10.1006/jfan.2001.3883. MR1895529

[28] Leonard Gross, *Hypercontractivity, logarithmic Sobolev inequalities, and applications: a survey of surveys*, Diffusion, quantum theory, and radically elementary mathematics, Math. Notes, vol. 47, Princeton Univ. Press, Princeton, NJ, 2006, pp. 45–73. MR2325763

[29] Leonard Gross and Zhongmin Qian, *Holomorphic Dirichlet forms on complex manifolds*, Math. Z. **246** (2004), no. 3, 521–561, DOI 10.1007/s00209-003-0588-x. MR2073455

[30] Jun-Qi Hu and Hong-Quan Li, *Gradient estimates for the heat semigroup on H-type groups*, Potential Anal. **33** (2010), no. 4, 355–386, DOI 10.1007/s11118-010-9173-1. MR2726903

[31] Svante Janson, *On hypercontractivity for multipliers on orthogonal polynomials*, Ark. Mat. **21** (1983), no. 1, 97–110, DOI 10.1007/BF02384302. MR706641

[32] Svante Janson, *On complex hypercontractivity*, J. Funct. Anal. **151** (1997), no. 1, 270–280, DOI 10.1006/jfan.1997.3144. MR1487778

[33] Françoise Lust-Piquard, *Ornstein-Uhlenbeck semi-groups on stratified groups*, J. Funct. Anal. **258** (2010), no. 6, 1883–1908, DOI 10.1016/j.jfa.2009.11.012. MR2578458

[34] Tai Melcher, *Hypoelliptic heat kernel inequalities on Lie groups*, Stochastic Process. Appl. **118** (2008), no. 3, 368–388, DOI 10.1016/j.spa.2007.04.012. MR2389050

[35] Richard Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, vol. 91, American Mathematical Society, Providence, RI, 2002. MR1867362

[36] Alexander Nagel, Elias M. Stein, and Stephen Wainger, *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math. **155** (1985), no. 1-2, 103–147, DOI 10.1007/BF02392539. MR793239

[37] Edward Nelson, *A quartic interaction in two dimensions*, Mathematical Theory of Elementary Particles (Proc. Conf., Dedham, Mass., 1965), M.I.T. Press, Cambridge, Mass., 1966, pp. 69–73. MR0210416

[38] Edward Nelson, *The free Markoff field*, J. Functional Analysis **12** (1973), 211–227. MR0343816

[39] Barry Simon, *Harmonic analysis*, A Comprehensive Course in Analysis, Part 3, American Mathematical Society, Providence, RI, 2015. MR3410783

[40] A. J. Stam, *Some inequalities satisfied by the quantities of information of Fisher and Shannon*, Information and Control **2** (1959), 101–112. MR0109101

[41] Thomas Taylor, *A parametrix for step-two hypoelliptic diffusion equations*, Trans. Amer. Math. Soc. **296** (1986), no. 1, 191–215, DOI 10.2307/2000569. MR837807

[42] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, Oxford, 1958. Reprint of the second (1939) edition. MR3155290

[43] N. Th. Varopoulos, *Small time Gaussian estimates of heat diffusion kernels. II. The theory of large deviations*, J. Funct. Anal. **93** (1990), no. 1, 1–33, DOI 10.1016/0022-1236(90)90136-9. MR1070036

[44] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, Cambridge, 1992. MR1218884

[45] Robert Wallstén, *The S^p-criterion for Hankel forms on the Fock space,* $0 < p < 1$, Math. Scand. **64** (1989), no. 1, 123–132, DOI 10.7146/math.scand.a-12251. MR1036432

[46] Zheng-Fang Zhou, *The contractivity of the free Hamiltonian semigroup in the L_p space of entire functions*, J. Funct. Anal. **96** (1991), no. 2, 407–425, DOI 10.1016/0022-1236(91)90067-F. MR1101263

School of Mathematical Sciences, University of Northern Colorado, 501 20th Street, Box 122, Greeley, Colorado 80639
E-mail address: neldredge@unco.edu

Department of Mathematics, Cornell University, 301 Malott Hall, Ithaca, New York 14853
E-mail address: gross@math.cornell.edu

Department of Mathematics, Cornell University, 301 Malott Hall, Ithaca, New York 14853
E-mail address: lsc@math.cornell.edu