COMBINATORIAL EXTENSION OF STABLE BRANCHING RULES FOR CLASSICAL GROUPS

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ABSTRACT. We give new combinatorial formulas for decomposition of the tensor product of integrable highest weight modules over the classical Lie algebras of types B, C, D, and the branching decomposition of an integrable highest weight module with respect to a maximal Levi subalgebra of type A. This formula is based on a combinatorial model of classical crystals called spinor model. We show that our formulas extend in a bijective way various stable branching rules for classical groups to arbitrary highest weights, including the Littlewood restriction rules.

1. INTRODUCTION

1.1. **Stable branching rules.** Let G be a classical group over the complex numbers with a closed subgroup H. For a finite-dimensional irreducible G-module V, there are various branching rules for the multiplicity of an irreducible H-module in V, which are given as a sum of products of Littlewood-Richardson coefficients. These formulas, which originated in the Littlewood restriction rules [32, 33], are often referred to as *stable branching rules* since they hold in a certain range of highest weights of irreducible modules and depend only on the partitions parameterizing highest weights when the ranks of classical groups are sufficiently large. We refer the reader to [12] for a systematic approach to these formulas and detailed exposition on previous works.

The goal of this paper is to give a combinatorial extension of various stable branching rules to arbitrary highest weights, that is, to give a combinatorial formula for the branching multiplicity which holds for arbitrary highest weights and also extends a given stable branching rule in a bijective way.

We recall that there are formulas extending stable branching rules to arbitrary highest weights (for example, [7, 21–24, 36, 41]) where the multiplicities outside a stable range are in general given as an alternating sum of products of Littlewood-Richardson coefficients or determined by certain modification and cancellation of irreducible factors in the decomposition inside a stable range. But compared to these results which are obtained in an algebraic way, relatively not much is known about combinatorial or subtraction-free extension. On the other hand, the theory of crystals [18, 19] has made it possible to provide formulas for the branching multiplicities, which are available for arbitrary highest weights and described explicitly

in terms of various combinatorial objects (for example, [2, 17, 20, 31]). However, most of the known results do not seem to imply the stable branching rules immediately. So one may ask which combinatorial model for classical crystals explains and extends the stable branching formulas more naturally.

1.2. **Combinatorial extensions.** Let us explain our results in detail. Our approach is based on the theory of crystal base and the theory of reductive dual pairs [11].

Let $\mathfrak{g}_\infty$ be the affine Kac-Moody Lie algebra of type B_∞, C_∞, and D_∞ [16], and let $\mathfrak{l}_\infty$ be its maximal Levi subalgebra of type A. We first give a new formula for decomposition of the tensor product of integrable highest weight $\mathfrak{g}_\infty$-modules and the branching decomposition of an integrable highest weight $\mathfrak{g}_\infty$-module with respect to $\mathfrak{l}_\infty$.

Our formula is given in terms of a combinatorial model for the $\mathfrak{g}_\infty$-crystal associated to an integrable highest weight $\mathfrak{g}_\infty$-module, which was recently introduced by the author [26, 27]. A main advantage of this model, which we call *spinor model*, is that it is compatible with the combinatorics of usual Young and Littlewood-Richardson tableaux, and hence it turns out to fit into the stability phenomenon of decomposition numbers very nicely.

The notion of spinor model is motivated by the duality on the fermionic Fock space $\mathcal{F}^{\frac{n}{2}}$, where $(\mathfrak{g}_\infty, G_n)$ forms a reductive dual pair on $\mathcal{F}^{\frac{n}{2}}$ for some complex classical reductive algebraic group G_n [11, 40]. Let $\mathcal{P}(G_n)$ be the set of partitions parameterizing the highest weights for the finite-dimensional irreducible G_n-modules appearing in $\mathcal{F}^{\frac{n}{2}}$. By dual correspondence, we may parameterize the dominant integral weights for $\mathfrak{g}_\infty$ by $\mathcal{P}(G_n)$ for all n. Let $\mathbf{T}^{\mathfrak{g}}(\lambda, n)$ denote the spinor model of the crystal associated to an integrable highest weight $\mathfrak{g}_\infty$-module with highest weight corresponding to $\lambda \in \mathcal{P}(G_n)$. Here $\mathfrak{g} = \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ is understood as a symbol representing the type of $\mathfrak{g}_\infty$.

Now, let $\lambda \in \mathcal{P}(G_{m+n})$, $\mu \in \mathcal{P}(G_m)$, and $\nu \in \mathcal{P}(G_n)$ be given. The multiplicity of $\mathbf{T}^{\mathfrak{g}}(\lambda, m+n)$ in the tensor product $\mathbf{T}^{\mathfrak{g}}(\mu, m) \otimes \mathbf{T}^{\mathfrak{g}}(\nu, n)$ is given by the cardinality of a subset $\mathrm{LR}_{\mu\nu}^\lambda(\mathfrak{g})$ of $\mathbf{T}^{\mathfrak{g}}(\nu, n)$ satisfying certain combinatorial conditions determined by the general theory of crystal base (Proposition 4.4). Hence we may regard elements in $\mathrm{LR}_{\mu\nu}^\lambda(\mathfrak{g})$ as a generalization of Littlewood-Richardson tableaux. Then we show as our first main result that $\mathrm{LR}_{\mu\nu}^\lambda(\mathfrak{g})$ has a nice decomposition into a set of pairs of Littlewood-Richardson tableaux in a stable range. More precisely, we construct an explicit bijection (Theorem 4.5)

$$(1.1) \qquad \mathrm{LR}_{\mu\nu}^\lambda(\mathfrak{g}) \longrightarrow \bigsqcup_{\gamma \in \mathscr{P}} \bigsqcup_{\delta \in \mathscr{P}_\mathfrak{g}} \mathrm{LR}_{\mu'\delta}^\gamma \times \mathrm{LR}_{\gamma\nu'}^{\lambda'} \qquad \text{when } \ell(\lambda) \leq \frac{1}{2}\min\{m, n\}.$$

Here $\mathscr{P}$ is the set of partitions, $\mathscr{P}_\mathfrak{g}$ is a subset of $\mathscr{P}$ given in (4.1), $\mathrm{LR}_{\beta\gamma}^\alpha$ is the set of Littlewood-Richardson tableaux of shape α/β and content γ for $\alpha, \beta, \gamma \in \mathscr{P}$, λ' is the conjugate of λ, and $\ell(\lambda)$ is the length of λ.

Next, we consider the branching decomposition with respect to $\mathfrak{l}_\infty$. Let $\mu \in \mathcal{P}(G_n)$ be given. For $\lambda \in \mathscr{P}$, we denote by $\mathrm{LR}_\lambda^\mu(\mathfrak{g})$ the subset of $\mathbf{T}^{\mathfrak{g}}(\mu, n)$, which consists of the highest weight elements of the $\mathfrak{l}_\infty$-subcrystals in $\mathbf{T}^{\mathfrak{g}}(\mu, n)$ isomorphic to the crystal of Young tableaux of shape λ'. The cardinality of $\mathrm{LR}_\lambda^\mu(\mathfrak{g})$ is

the associated branching multiplicity. We show as our second main result that $\mathrm{LR}^\mu_\lambda(\mathfrak{g})$ decomposes into a set of Littlewood-Richardson tableaux in a stable range by constructing an explicit bijection

$$(1.2) \qquad \mathrm{LR}^\mu_\lambda(\mathfrak{g}) \longrightarrow \bigsqcup_{\delta \in \mathscr{P}_\mathfrak{g}} \mathrm{LR}^{\lambda'}_{\delta\mu'} \qquad \text{when } \ell(\lambda) \leq \frac{n}{2}.$$

Finally, by the reciprocity laws associated to suitable see-saw pairs on $\mathcal{F}^{\frac{n}{2}}$, the bijections in (1.1) and (1.2) recover well-known stable branching rule for the pairs

(I) $\mathrm{G}_{m+n} \supset \mathrm{G}_m \times \mathrm{G}_n$,
(II) $\mathrm{GL}_n \supset \mathrm{G}_n$ (the Littlewood restriction rule),

with $\mathrm{G}_\ell = \mathrm{Sp}_\ell$, O_ℓ (Theorems 5.1 and 5.3), and therefore the sets $\mathrm{LR}^\lambda_{\mu\nu}(\mathfrak{g})$ and $\mathrm{LR}^\mu_\lambda(\mathfrak{g})$ extend these stable branching rules to arbitrary highest weights in a bijective way. On the other hand, considering a subset of tableaux in $\mathbf{T}^\mathfrak{g}(\lambda)$ with entries from $\{1, \ldots, k\}$, we can also describe the tensor product decomposition and branching decomposition for crystals of type B_k, C_k, and D_k in terms of $\mathrm{LR}^\lambda_{\mu\nu}(\mathfrak{g})$ and $\mathrm{LR}^\mu_\lambda(\mathfrak{g})$. That is, we have branching rules for the pairs

(III) $\mathrm{G}_\ell \times \mathrm{G}_\ell \supset \mathrm{G}_\ell$,
(IV) $\mathrm{G}_\ell \supset \mathrm{GL}_{[\ell/2]}$,

with $\mathrm{G}_\ell = \mathrm{Sp}_\ell$, Spin_ℓ, $\mathrm{Osp}_{1|\ell}$, and $k = [\ell/2]$, together with the stable limits (1.1) and (1.2), respectively (see Remark 4.10). In this case, the formula (1.1) for (III) looks different from the known result, and the formulas in (III) and (IV) for Spin_ℓ-modules with half-integer weights seem to be new.

An extension of stable branching formula similar to (1.1) and (1.2) has been obtained in [25] by using a different model of $\mathfrak{g}_\infty$-crystals for $\mathfrak{g} = \mathfrak{b}$, $\mathfrak{c}$ but not for $\mathfrak{g} = \mathfrak{d}$, which is a technically more difficult case. For other combinatorial extensions, we refer the reader to [39] for (II) with $\mathrm{G}_n = \mathrm{Sp}_n$ and [1, Proposition 2.6] for (IV) with $\mathrm{G}_\ell = \mathrm{Sp}_\ell$, SO_ℓ.

Recently, there have been several works studying stability phenomenon from completely different viewpoints from ours. In [6, 37, 38] (see also references therein), the branching multiplicities in a stable range are studied in the context of a non-semisimple tensor category of modules over classical Lie algebras of infinite rank. Also in [41], the extension of stable branching multiplicities in (II) has been studied using q-versions of Brauer's centralizer algebras.

We expect that our combinatorial approach would give a new insight into the structure of representations of classical Lie algebras and groups which are related to the stability phenomenon.

1.3. Application to holomorphic discrete series.
We give an interesting application of spinor model to another stability phenomenon.

In [12] a systematic approach of stable branching rule is given by using Howe's theory of reductive dual pairs [11], which also partly motivated the work in the present paper. Instead of integrable highest weight $\mathfrak{g}_\infty$-modules, a family of infinite-dimensional unitarizable representations of classical Lie algebras of finite rank, which appears in the study of dual pairs acting on bosonic spaces, is considered in [12], and then the irreducibility of their associated generalized Verma modules in a

certain stable range plays a crucial role in giving a unified proof of 10 families of stable branching formulas classified there.

We show that the spinor model $\mathbf{T}^{\mathfrak{g}}(\lambda, n)$ admits a variation to give a combinatorial character formula for these unitarizable representations (Theorem 6.1) thanks to super duality [5]. Hence we obtain the tensor product multiplicity and the branching multiplicity with respect to a maximal Levi subalgebra type A for these irreducible modules. In particular, the branching multiplicity formula implies the irreducibility of associated generalized Verma modules in a stable range called *holomorphic discrete series* (Corollary 6.6).
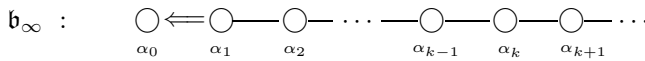
1.4. **Organization.** The paper is organized as follows. In Section 2, we briefly recall necessary background. In Section 3, we review the spinor model of crystals of types BCD. In Section 4, we establish the bijections (1.1) and (1.2). In Section 5, we recover well-known stable branching rules by using the dualities on Fock spaces. In Section 6, we discuss a combinatorial character formula for infinite-dimensional unitarizable representations.

2. PRELIMINARIES

2.1. **Lie algebras of types B, C, and D.** We assume that our base field is $\mathbb{C}$ and $\mathbb{Z}_+$ (resp. $\mathbb{N}$) denotes the set of non-negative (resp. positive) integers. Let $\mathfrak{b}_\infty$, $\mathfrak{c}_\infty$, and $\mathfrak{d}_\infty$ be the Kac-Moody Lie algebras of type B_∞, C_∞, and D_∞, respectively (see [16]). Following the conventions of $\mathfrak{g}_\infty$ ($\mathfrak{g} = \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$) in [3, Section 2.2], we let
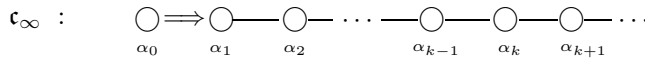
· $I = \mathbb{Z}_+$: the index set for simple roots,
· $\Pi = \{\, \alpha_i \,|\, i \in I \,\}$: the set of simple roots,
· $\Pi^\vee = \{\, \alpha_i^\vee \,|\, i \in I \,\}$: the set of simple coroots,
· $\Lambda_i^{\mathfrak{g}}$: the fundamental weight with respect to α_i,
· $P = \mathbb{Z}\Lambda_0^{\mathfrak{g}} \oplus \bigoplus_{i \geq 1} \mathbb{Z}\epsilon_i$: the weight lattice,

where the Dynkin diagrams of the Cartan matrix ($\langle \alpha_j, \alpha_i^\vee \rangle$), Π, and $\Lambda_i^{\mathfrak{g}}$ are given by

$$\mathfrak{b}_\infty \; : \qquad \underset{\alpha_0}{\bigcirc} \Longleftarrow \underset{\alpha_1}{\bigcirc} \!\!-\!\! \underset{\alpha_2}{\bigcirc} \!-\cdots-\! \underset{\alpha_{k-1}}{\bigcirc} \!\!-\!\! \underset{\alpha_k}{\bigcirc} \!\!-\!\! \underset{\alpha_{k+1}}{\bigcirc} \!-\cdots$$

$$\Pi = \{\, \alpha_0 = -\epsilon_1,\; \alpha_i = \epsilon_i - \epsilon_{i+1} \; (i \geq 1) \,\},$$
$$\Lambda_i^{\mathfrak{b}} = 2\Lambda_0^{\mathfrak{b}} + \epsilon_1 + \cdots + \epsilon_i \quad (i \geq 1),$$

$$\mathfrak{c}_\infty \; : \qquad \underset{\alpha_0}{\bigcirc} \Longrightarrow \underset{\alpha_1}{\bigcirc} \!\!-\!\! \underset{\alpha_2}{\bigcirc} \!-\cdots-\! \underset{\alpha_{k-1}}{\bigcirc} \!\!-\!\! \underset{\alpha_k}{\bigcirc} \!\!-\!\! \underset{\alpha_{k+1}}{\bigcirc} \!-\cdots$$

$$\Pi = \{\, \alpha_0 = -2\epsilon_1,\; \alpha_i = \epsilon_i - \epsilon_{i+1} \; (i \geq 1) \,\},$$
$$\Lambda_i^{\mathfrak{c}} = \Lambda_0^{\mathfrak{c}} + \epsilon_1 + \cdots + \epsilon_i \quad (i \geq 1),$$

$$\Pi = \{\, \alpha_0 = -\epsilon_1 - \epsilon_2,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\ (i \geq 1)\,\},$$

$$\Lambda_i^{\mathfrak{d}} = \begin{cases} \Lambda_0^{\mathfrak{d}} + \epsilon_1, & \text{if } i = 1, \\ 2\Lambda_0^{\mathfrak{d}} + \epsilon_1 + \cdots + \epsilon_i, & \text{if } i > 1. \end{cases}$$

We also let $\mathfrak{l}_\infty$ be the subalgebra of $\mathfrak{g}_\infty$ associated to $\{\, \alpha_i \,|\, i \in I \setminus \{0\}\,\}$, which is of type $A_{+\infty}$ (cf. [16]). We assume that its weight lattice is $\bigoplus_{i \geq 1} \mathbb{Z}\epsilon_i \subset P$.

We will also consider the following Kac-Moody Lie superalgebra $\mathfrak{b}_\infty^\bullet$ of infinite rank, whose Dynkin diagram, Π, and the fundamental weights $\Lambda_i^{\mathfrak{b}^\bullet}$ are given by

$$\Pi = \{\, \alpha_0 = -\epsilon_1,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\ (i \geq 1)\,\},$$

$$\Lambda_i^{\mathfrak{b}^\bullet} = 2\Lambda_0^{\mathfrak{b}} + \epsilon_1 + \cdots + \epsilon_i \quad (i \geq 1),$$

where $\bullet$ denotes a non-isotropic odd simple root (cf. [15]). We use the same notation for the associated data as in the case for $\mathfrak{g}_\infty$ ($\mathfrak{g} = \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$).

For $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}, \mathfrak{d}$, and $k \in \mathbb{N}$, let $\mathfrak{g}_k$ be the subalgebra of $\mathfrak{g}_\infty$ whose Dynkin diagram corresponds to the simple roots $\alpha_0, \ldots, \alpha_{k-1}$. We assume that $k \geq 2$ for $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}$, and $k \geq 4$ for $\mathfrak{g} = \mathfrak{d}$. Let $\mathfrak{l}_k = \mathfrak{l}_\infty \cap \mathfrak{g}_k$ be the corresponding subalgebra of type A_{k-1}.

Let V and W be modules over a Lie algebra $\mathfrak{g}$, where W is irreducible. We define $[V : W] = \dim \mathrm{Hom}_{\mathfrak{g}}(W, V)$, the multiplicity of W in V.

2.2. Dual pairs.
Throughout the paper, G_n denotes one of the following complex reductive algebraic groups: Sp_n, O_n, Spin_n, and Pin_n for $n \geq 2$, where n is even for Sp_n and Pin_n, and n is odd for Spin_n. Following [3, Section 2.2], let $V_{\mathrm{G}_n}^\lambda$ denote the finite-dimensional irreducible representation of G_n corresponding to $\lambda \in \mathcal{P}(\mathrm{G}_n)$ (see also [8, 11]), where

$$\mathcal{P}(\mathrm{Sp}_n) = \{\, \lambda = (\lambda_1, \ldots, \lambda_{\frac{n}{2}}) \,|\, \lambda_i \in \mathbb{Z}_+,\ \lambda_1 \geq \cdots \geq \lambda_{\frac{n}{2}} \,\},$$
$$\mathcal{P}(\mathrm{Pin}_n) = \{\, \lambda = (\lambda_1, \ldots, \lambda_{\frac{n}{2}}) \,|\, \lambda_i \in \mathbb{Z}_+,\ \lambda_1 \geq \cdots \geq \lambda_{\frac{n}{2}} \,\},$$
$$\mathcal{P}(\mathrm{Spin}_n) = \{\, \lambda = (\lambda_1, \ldots, \lambda_{\frac{n-1}{2}}) \,|\, \lambda_i \in \mathbb{Z}_+,\ \lambda_1 \geq \cdots \geq \lambda_{\frac{n-1}{2}} \,\},$$
$$\mathcal{P}(\mathrm{O}_n) = \{\, \lambda = (\lambda_1, \ldots, \lambda_n) \,|\, \lambda_i \in \mathbb{Z}_+,\ \lambda_1 \geq \cdots \geq \lambda_n,\ \lambda_1' + \lambda_2' \leq n \,\}.$$

Here $\lambda' = (\lambda_1', \lambda_2', \ldots)$ is the conjugate partition of $\lambda = (\lambda_1, \lambda_2, \ldots)$. From now on, we mean by $(\mathfrak{g}_\infty, \mathrm{G}_n)$ one of the pairs

$$(2.1) \qquad (\mathfrak{c}_\infty, \mathrm{Sp}_n),\ (\mathfrak{b}_\infty, \mathrm{Pin}_n),\ (\mathfrak{b}_\infty, \mathrm{Spin}_n),\ (\mathfrak{d}_\infty, \mathrm{O}_n),$$

unless otherwise specified. For $\lambda \in \mathcal{P}(G_n)$, we define a dominant integral weight $\Lambda^{\mathfrak{g}}(\lambda)$ for $\mathfrak{g}_\infty$ by

$$(2.2) \qquad \Lambda^{\mathfrak{g}}(\lambda) = \frac{n}{\epsilon}\Lambda_0^{\mathfrak{g}} + \lambda_1'\epsilon_1 + \lambda_2'\epsilon_2 + \cdots,$$

where $\epsilon = 2$ if $\mathfrak{g} = \mathfrak{c}$, and $\epsilon = 1$ otherwise. Note that $\bigsqcup_n \mathcal{P}(G_n)$ parameterizes the set of all dominant integral weights for $\mathfrak{g}_\infty$. We denote by $L(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\lambda))$ the irreducible highest weight $\mathfrak{g}_\infty$-module with highest weight $\Lambda^{\mathfrak{g}}(\lambda)$.

For a positive integer $\ell \geq 1$, let

$$\psi^{+,i}(z) = \sum_{r \in \frac{1}{2}+\mathbb{Z}} \psi_r^{+,i} z^{-r-\frac{1}{2}}, \quad \psi^{-,i}(z) = \sum_{s \in \frac{1}{2}+\mathbb{Z}} \psi_s^{-,i} z^{-s-\frac{1}{2}} \quad (1 \leq i \leq \ell)$$

be ℓ pairs of free fermions with non-trivial commutation relations $[\psi_r^{+,i}, \psi_s^{-,j}] = \delta_{ij}\delta_{r+s,0}$. Let $\mathcal{F}^\ell$ denote the corresponding Fock space generated by the vacuum vector $|0\rangle$, which is annihilated by $\psi_r^{+,i}, \psi_s^{-,i}$ for $r, s > 0$. We introduce a neutral fermionic field $\phi(z) = \sum_{r \in \frac{1}{2}+\mathbb{Z}} \phi_r z^{-r-\frac{1}{2}}$ with non-trivial commutation relations $[\phi_r, \phi_s] = \delta_{r+s,0}$. Denote by $\mathcal{F}^{\frac{1}{2}}$ the Fock space of $\phi(z)$ generated by a vacuum vector that is annihilated by ϕ_r for $r > 0$. We denote by $\mathcal{F}^{\ell+\frac{1}{2}}$ the tensor product of $\mathcal{F}^\ell$ and $\mathcal{F}^{\frac{1}{2}}$. Then we have the following $(\mathfrak{g}_\infty, G_n)$-duality on $\mathcal{F}^{\frac{n}{2}}$.

Proposition 2.1 ([40]). *There exists an action of $\mathfrak{g}_\infty \times G_n$ on $\mathcal{F}^{\frac{n}{2}}$. Furthermore, under this joint action, we have*

$$\mathcal{F}^{\frac{n}{2}} \cong \bigoplus_{\lambda \in \mathcal{P}(G_n)} L(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\lambda)) \otimes V_{G_n}^\lambda.$$

2.3. **Crystals.** Let us give a brief review on crystals (see [9, 19] for more details). Let $\mathfrak{g}$ be a Kac-Moody algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ indexed by a set I. Let $P^\vee$ be the dual weight lattice, $P = \text{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$ the weight lattice, $\Pi^\vee = \{\alpha_i^\vee \,|\, i \in I\}$ the set of simple coroots, and $\Pi = \{\alpha_i \,|\, i \in I\}$ the set of simple roots of $\mathfrak{g}$ such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for $i, j \in I$. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$.

A $\mathfrak{g}$-*crystal* (or *crystal* for short) is a set B together with the maps $\text{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\widetilde{e}_i, \widetilde{f}_i : B \to B \cup \{\mathbf{0}\}$ $(i \in I)$ satisfying certain axioms. For a dominant integral weight Λ for $\mathfrak{g}$, we denote by $\mathbf{B}(\mathfrak{g}, \Lambda)$ the $\mathfrak{g}$-crystal associated to an irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight Λ.

Let B_1 and B_2 be crystals. A tensor product $B_1 \otimes B_2$ is defined to be $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, where

$$\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),$$

$$\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \text{wt}(b_1), \alpha_i^\vee \rangle\},$$

$$\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \langle \text{wt}(b_2), \alpha_i^\vee \rangle, \varphi_i(b_2)\},$$

$$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$

for $i \in I$. Here we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$. Then $B_1 \otimes B_2$ is a crystal.

For $b_i \in B_i$ $(i = 1, 2)$, we write $b_1 \equiv b_2$ if there is an isomorphism of crystals $C(b_1) \to C(b_2)$ mapping b_1 to b_2, where $C(b_i)$ denotes the connected component of b_i in B_i.

2.4. Crystals of Young tableaux and Littlewood-Richardson coefficients.

Let $\mathscr{P}$ be the set of partitions. For $\lambda = (\lambda_i)_{i \geq 1} \in \mathscr{P}$, we denote by $\ell(\lambda)$ the length of λ. We identify λ with a Young diagram as usual. For a skew Young diagram λ/μ, let $SST(\lambda/\mu)$ be the set of semistandard tableaux of shape λ/μ with entries in $\mathbb{N}$. For $T \in SST(\lambda/\mu)$, let $w(T)$ be the column word of T, that is, the word given by reading the entries of T column by column from right to left and from top to bottom in each column. For $T \in SST(\lambda)$ and $a \in \mathbb{N}$, we denote by $a \to T$ the tableau obtained by the column insertion of a into T (cf. [13]). For a semistandard tableau S, we define $(S \to T) = (w(S) \to T)$.

Considering $\mathbb{N}$ as an $\mathfrak{l}_\infty$-crystal (of type $A_{+\infty}$) associated to the natural representation of $\mathfrak{l}_\infty$, we may regard $SST(\lambda/\mu)$ as an $\mathfrak{l}_\infty$-crystal [20]. In particular, $SST(\lambda)$ is isomorphic to $\mathbf{B}(\mathfrak{l}_\infty, \lambda)$ for $\lambda \in \mathscr{P}$, where we identify $\lambda = (\lambda_i)_{i \geq 1}$ with $\sum_{i \geq 1} \lambda_i \epsilon_i$. We denote by H_λ the highest weight element in $SST(\lambda)$ with weight $\sum_{i \geq 1} \lambda_i \epsilon_i$, where each ith row is filled with i. For $T \in SST(\lambda)$ and a semistandard tableau S, we have $(S \to T) \equiv T \otimes S$.

Let $\mathrm{LR}^\lambda_{\mu\nu}$ be the set of Littlewood-Richardson tableaux corresponding to $\lambda, \mu, \nu \in \mathscr{P}$, which is the set of tableaux $T \in SST(\lambda/\mu)$ with weight ν such that $w(T)$ is a lattice word (cf. [13]). Let $c^\lambda_{\mu\nu} = |\mathrm{LR}^\lambda_{\mu\nu}|$ be the Littlewood-Richardson coefficient corresponding to $\lambda, \mu, \nu \in \mathscr{P}$. Recall that $c^\lambda_{\mu\nu}$ is the number of connected components in $SST(\mu) \otimes SST(\nu)$, which is isomorphic to $SST(\lambda)$ as an $\mathfrak{l}_\infty$-crystal. By tensor product rule of crystal, we may regard $\mathrm{LR}^\lambda_{\mu\nu}$ as the set of $T \in SST(\nu)$ such that $\mathrm{wt}(H_\mu) + \mathrm{wt}(T) = \mathrm{wt}(H_\lambda)$ and $\varepsilon_i(T) \leq \varphi_i(H_\mu) = \mu_i - \mu_{i+1}$ for all $i \geq 1$ [35].

3. Spinor model for crystals of types B, C, D

In this section, we briefly review the combinatorial model for $\mathbf{B}(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\lambda))$ $(\lambda \in \mathscr{P}(\mathrm{G}_n))$, which was introduced by the author in [26, 27] (see also [29, Section 2] for more details on the existence of the crystal $\mathbf{B}(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\lambda))$ for $U_q(\mathfrak{g}_\infty)$).

3.1. Crystals of fundamental representations of $\mathfrak{g}_\infty$.

In this subsection, we describe a combinatorial model for $\mathbf{B}(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}_a)$ $(a \in \mathbb{Z}_+)$. It is given by translating the q-deformed Fock space models for classical Lie algebras due to Hayashi [10] in terms of semistandard tableaux of skew shapes with at most two columns.

For a single-columned tableau U, let $\mathrm{ht}(U)$ denote the height of U and $U(i)$ (resp. $U[i]$) the ith entry of U from the bottom (resp. the top) for $i \geq 1$.

For $a, b, c \in \mathbb{Z}_+$, let $\lambda(a, b, c) = (2^{b+c}, 1^a)/(1^b)$, a skew Young diagram with heights $a + c$ and $b + c$ from the left, where a and b denote the heights of lower and upper single columns in the diagram, respectively. For example,

$$\lambda(2, 1, 3) \ = \ \begin{array}{c}\cdot\\\cdot\end{array}$$

Let T be a tableau of shape $\lambda(a,b,c)$, each of whose column is a semistandard tableau. We denote by $T^{\mathtt{L}}$ and $T^{\mathtt{R}}$ the left and right columns of T, respectively. We also denote by $T^{\mathtt{tail}}$ the subtableau of T corresponding to the tail of $\lambda(a,b,c)$, a lower single column of height a, and denote by $T^{\mathtt{body}}$ the subtableau of T above $T^{\mathtt{tail}}$:

$$
T = \begin{array}{|c|c|}
\hline
 & 3 \\
\hline
2 & 4 \\
\hline
4 & 6 \\
\hline
5 & 8 \\
\hline
6 & \\
\cline{1-1}
7 & \\
\cline{1-1}
\end{array}
\qquad
T^{\mathtt{body}} = \begin{array}{|c|c|}
\hline
 & 3 \\
\hline
2 & 4 \\
\hline
4 & 6 \\
\hline
5 & 8 \\
\hline
\end{array}
\qquad
T^{\mathtt{tail}} = \begin{array}{|c|}
\hline
6 \\
\hline
7 \\
\hline
\end{array}
$$

Suppose that T is semistandard, that is, $T \in SST(\lambda(a,b,c))$. One may slide down $T^{\mathtt{R}}$ by k positions for $0 \le k \le \min\{a,b\}$ to have a (not necessarily semistandard) tableau T' of shape $\lambda(a-k, b-k, c+k)$. We define the *residue of T* to be the maximal k such that T' is semistandard and denote it by $\mathfrak{r}_T$. For example, $\mathfrak{r}_S = 1$ and $\mathfrak{r}_T = 2$ when

$$
S = \begin{array}{|c|c|}
\hline
 & 3 \\
\cline{2-2}
 & 4 \\
\hline
4 & 6 \\
\hline
5 & 8 \\
\hline
6 & \\
\cline{1-1}
7 & \\
\cline{1-1}
\end{array}
\qquad
T = \begin{array}{|c|c|}
\hline
 & 4 \\
\cline{2-2}
 & 6 \\
\hline
4 & 7 \\
\hline
5 & 8 \\
\hline
6 & \\
\cline{1-1}
7 & \\
\cline{1-1}
\end{array}
$$

For $a \in \mathbb{Z}_+$, let

$$
\mathbf{T}^{\mathfrak{g}}(a) = \{\, T \mid T \in SST(\lambda(a,b,c)),\ (b,c) \in \mathcal{H}^{\mathfrak{g}},\ \mathfrak{r}_T \le r^{\mathfrak{g}} \,\},
$$

where

$$
\mathcal{H}^{\mathfrak{g}} = \begin{cases}
\{0\} \times \mathbb{Z}_+, & \text{if } \mathfrak{g} = \mathfrak{c}, \\
\mathbb{Z}_+ \times \mathbb{Z}_+, & \text{if } \mathfrak{g} = \mathfrak{b}, \\
2\mathbb{Z}_+ \times 2\mathbb{Z}_+, & \text{if } \mathfrak{g} = \mathfrak{d},
\end{cases}
\qquad
r^{\mathfrak{g}} = \begin{cases}
0, & \text{if } \mathfrak{g} = \mathfrak{c}, \mathfrak{b}, \\
1, & \text{if } \mathfrak{g} = \mathfrak{d}.
\end{cases}
$$

We also put

$$
\mathbf{T}^{\mathrm{sp}} = \bigsqcup_{a\, \in \mathbb{Z}_+} SST((1^a)),
$$

$$
\mathbf{T}^{\mathrm{sp}\,+} = \{\, T \in \mathbf{T}^{\mathrm{sp}} \mid \mathfrak{r}_T = 0 \,\}, \ \ \mathbf{T}^{\mathrm{sp}\,-} = \{\, T \in \mathbf{T}^{\mathrm{sp}} \mid \mathfrak{r}_T = 1 \,\},
$$

$$
\overline{\mathbf{T}}^{\mathfrak{d}}(0) = \bigsqcup_{(b,c) \in \mathcal{H}^{\mathfrak{d}}} SST(\lambda(0,b,c+1)).
$$

Here we define the residue $\mathfrak{r}_T$ of $T \in \mathbf{T}^{\mathrm{sp}}$ to be the residue of $\mathrm{ht}(T)$ modulo 2. (Note that if $T \in SST(\lambda(0,b,0)) \subset T^{\mathfrak{g}}(0)$, then the shape of T is a single column, but $\mathfrak{r}_T = 0$.)

Now, let $\mathbf{B}$ be one of $\mathbf{T}^{\mathfrak{g}}(a)$ ($a \in \mathbb{Z}_+$), $\mathbf{T}^{\mathrm{sp}}$, and $\overline{\mathbf{T}}^{\mathfrak{d}}(0)$, and let $T \in \mathbf{B}$ be given. First, we define $\widetilde{e}_i T$ and $\widetilde{f}_i T$ for $i \in I \setminus \{0\}$ regarding $\mathbf{B}$ as a subset of an $\mathfrak{l}_\infty$-crystal $\bigsqcup_{\lambda \in \mathscr{P}} SST(\lambda)$. Next, we define $\widetilde{e}_0 T$ and $\widetilde{f}_0 T$ as follows:

(1) Suppose that $\mathfrak{g} = \mathfrak{c}$ and $\mathbf{B} = \mathbf{T}^{\mathfrak{c}}(a)$. We define $\widetilde{e}_0 T$ to be the tableau obtained from T by removing a domino $\boxed{1\,1}$ if T has $\boxed{1\,1}$ on its top, and $\mathbf{0}$ otherwise. We define $\widetilde{f}_0 T$ in a similar way by adding $\boxed{1\,1}$ on top of T.

(2) Suppose that $\mathfrak{g} = \mathfrak{b}$. When $\mathbf{B} = \mathbf{T}^{\mathrm{sp}}$, we define $\widetilde{e}_0 T$ to be the tableau obtained from T by removing $\boxed{1}$ if T has $\boxed{1}$ on its top, and $\mathbf{0}$ otherwise. We define $\widetilde{f}_0 T$ in a similar way by adding $\boxed{1}$ on top of T. When $\mathbf{B} = \mathbf{T}^{\mathfrak{b}}(a)$, we define $\tilde{x}_0 T$ $(x = e, f)$ to be $\tilde{x}_0(T^{\mathbb{R}} \otimes T^{\mathbb{L}})$ regarding $\mathbf{T}^{\mathrm{sp}}$ as a regular $\mathfrak{sl}_2$-crystal with respect to $\widetilde{e}_0$ and $\widetilde{f}_0$ and applying the tensor product rule of crystals to $\mathbf{B} \subset (\mathbf{T}^{\mathrm{sp}})^{\otimes 2}$.

(3) Suppose that $\mathfrak{g} = \mathfrak{d}$. When $\mathbf{B} = \mathbf{T}^{\mathrm{sp}}$, we define $\widetilde{e}_0 T$ to be the tableau obtained from T by removing a domino $\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}$ if T has $\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}$ on its top, and $\mathbf{0}$ otherwise. We define $\widetilde{f}_0 T$ in a similar way by adding $\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}$ on top of T. When $\mathbf{B} = \mathbf{T}^{\mathfrak{d}}(a)$ or $\overline{\mathbf{T}}^{\mathfrak{d}}(0)$, we define $\tilde{x}_0 T$ $(x = e, f)$ to be $\tilde{x}_0(T^{\mathbb{R}} \otimes T^{\mathbb{L}})$ as in (2).

We put

$$\mathrm{wt}(T) = \begin{cases} \frac{2}{\epsilon}\Lambda_0^{\mathfrak{g}} + \sum_{i \geq 1} m_i \epsilon_i, & \text{if } \mathbf{B} = \mathbf{T}^{\mathfrak{g}}(a) \text{ or } \overline{\mathbf{T}}^{\mathfrak{d}}(0), \\ \Lambda_0^{\mathfrak{g}} + \sum_{i \geq 1} m_i \epsilon_i, & \text{if } \mathbf{B} = \mathbf{T}^{\mathrm{sp}}, \end{cases}$$

where m_i is the number of occurrences of i in T (cf. (2.2)), and

$$\varepsilon_i(T) = \max\{\, k \,|\, \widetilde{e}_i^k T \neq \mathbf{0} \,\}, \quad \varphi_i(T) = \max\{\, k \,|\, \widetilde{f}_i^k T \neq \mathbf{0} \,\} \quad (i \in I).$$

Then $\mathbf{B}$ is a $\mathfrak{g}_\infty$-crystal with respect to wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in I)$. By [26, Theorem 7.1] for $\mathfrak{g} = \mathfrak{b}, \mathfrak{c}$ and [27, Proposition 4.2] for $\mathfrak{g} = \mathfrak{d}$, we have

Proposition 3.1.

(1) *For* $\mathfrak{g} = \mathfrak{c}$,
$$\mathbf{T}^{\mathfrak{c}}(a) \cong \mathbf{B}(\mathfrak{c}_\infty, \Lambda_a^{\mathfrak{c}}) \quad (a \in \mathbb{Z}_+).$$

(2) *For* $\mathfrak{g} = \mathfrak{b}$,
$$\begin{cases} \mathbf{T}^{\mathfrak{b}}(a) \cong \mathbf{B}(\mathfrak{b}_\infty, \Lambda_a^{\mathfrak{b}}) \quad (a \geq 1), \\ \mathbf{T}^{\mathfrak{b}}(0) \cong \mathbf{B}(\mathfrak{b}_\infty, 2\Lambda_0^{\mathfrak{b}}), \end{cases} \qquad \mathbf{T}^{\mathrm{sp}} \cong \mathbf{B}(\mathfrak{b}_\infty, \Lambda_0^{\mathfrak{b}}).$$

(3) *For* $\mathfrak{g} = \mathfrak{d}$,
$$\begin{cases} \mathbf{T}^{\mathfrak{d}}(a) \cong \mathbf{B}(\mathfrak{d}_\infty, \Lambda_a^{\mathfrak{d}}) \quad (a \geq 2), \\ \mathbf{T}^{\mathfrak{d}}(0) \cong \mathbf{B}(\mathfrak{d}_\infty, 2\Lambda_0^{\mathfrak{d}}), \\ \overline{\mathbf{T}}^{\mathfrak{d}}(0) \cong \mathbf{B}(\mathfrak{d}_\infty, 2\Lambda_1^{\mathfrak{d}}), \end{cases} \qquad \begin{cases} \mathbf{T}^{\mathrm{sp}+} \cong \mathbf{B}(\mathfrak{d}_\infty, \Lambda_0^{\mathfrak{d}}), \\ \mathbf{T}^{\mathrm{sp}-} \cong \mathbf{B}(\mathfrak{d}_\infty, \Lambda_1^{\mathfrak{d}}). \end{cases}$$

Note that the highest weight element in $\mathbf{B}$ is given by $H_{(1^a)}$, $\emptyset$, $H_{(1)}$, and $H_{(2)}$, when $\mathbf{B} = \mathbf{T}^{\mathfrak{g}}(a)$, $\mathbf{T}^{\mathrm{sp},+}$, $\mathbf{T}^{\mathrm{sp},-}$, and $\overline{\mathbf{T}}^{\mathfrak{d}}(0)$, respectively.

3.2. Admissibility. To describe a $\mathfrak{g}_\infty$-crystal $\mathbf{B}(\mathfrak{g}_\infty, \Lambda)$ associated to arbitrary integral dominant weight $\Lambda = \Lambda_{a_1}^{\mathfrak{g}} + \cdots + \Lambda_{a_r}^{\mathfrak{g}}$, we embed $\mathbf{B}(\mathfrak{g}_\infty, \Lambda)$ into the tensor product $\mathbf{B}(\mathfrak{g}_\infty, \Lambda_{a_1}^{\mathfrak{g}}) \otimes \cdots \otimes \mathbf{B}(\mathfrak{g}_\infty, \Lambda_{a_r}^{\mathfrak{g}})$ and then describe a connected component of the highest weight element with weight Λ. For this, we need to characterize an explicit condition for $T_i \otimes T_j \in \mathbf{B}(\mathfrak{g}_\infty, \Lambda_{a_i}^{\mathfrak{g}}) \otimes \mathbf{B}(\mathfrak{g}_\infty, \Lambda_{a_j}^{\mathfrak{g}})$ to be in the same connected component including the highest weight element of weight $\Lambda_{a_i}^{\mathfrak{g}} + \Lambda_{a_j}^{\mathfrak{g}}$. This condition is called *admissibility*, which is an analogue of semistandardness between two adjacent columns in type A.

For $a \in \mathbb{Z}_+$, let $T \in \mathbf{T}^{\mathfrak{g}}(a)$ be given. Associated to T, we introduce the following two pairs of single-columned tableaux $({}^{\mathbb{L}}T, {}^{\mathbb{R}}T)$ and $(T^{\mathbb{L}^*}, T^{\mathbb{R}^*})$, which will play a crucial role in describing admissibility.

First, we define $({}^{\mathsf{L}}T, {}^{\mathsf{R}}T)$ to be the pair determined by the following algorithm:

(1) Let $y_i = T^{\mathsf{R}}(i)$ for $1 \leq i \leq \mathrm{ht}(T^{\mathsf{R}})$. First, slide down the box $\boxed{y_1}$ in T^{R} as far as the entry of T^{L} in the same row is no greater than y_1. If no entry of T^{L} is greater than y, we place $\boxed{y_1}$ next to the bottom of T^{L}.

(2) Next, slide down $\boxed{y_2}$ until it is above $\boxed{y_1}$ and the entry of T^{L} in the same row is no greater than y_2. Repeat the same process with the other boxes $\boxed{y_3}, \boxed{y_4}, \ldots$ until there is no moving down.

(3) Slide each box $\boxed{x}$ in T^{L} to the right if its right position is empty (indeed the number of such boxes is $a - \mathfrak{r}_T$).

(4) Define ${}^{\mathsf{R}}T$ to be the tableau determined by the boxes $\boxed{y_i}$ in T^{R} together with boxes $\boxed{x}$ which have moved from T^{L} by (3), and define ${}^{\mathsf{L}}T$ to be the tableau with the remaining boxes on the left.

For example, we have
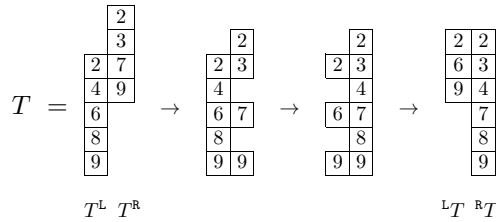
where the pairs $(T^{\mathsf{L}}, {}^{\mathsf{R}}T)$ and $({}^{\mathsf{L}}T, T^{\mathsf{R}})$ are arranged to share the same bottom lines, respectively.

Next, when $\mathfrak{r}_T = 1$, we define $(T^{\mathsf{L}^*}, T^{\mathsf{R}^*})$ to be the pair determined by the following algorithm:

(1) Let $x_i = T^{\mathsf{L}}[i]$ for $1 \leq i \leq \mathrm{ht}(T^{\mathsf{L}})$. First, slide upward $\boxed{x_1}$ until the entry of T^{R} in the same row is no smaller than x_1. If no entry of T^{L} is smaller than x_1, we place $\boxed{x_1}$ next to the top of T^{R}.

(2) Next, slide upward $\boxed{x_2}$ until it is below $\boxed{x_1}$ and the entry of T^{R} in the same row is no smaller than x_2. Repeat the same process with the other boxes $\boxed{x_3}, \boxed{x_4}, \ldots$ until there is no moving up.

(3) Choose the lowest box $\boxed{y}$ in T^{R} whose left position is empty, and then slide it to the left (there exists at least one such $\boxed{y}$ since $\mathfrak{r}_T = 1$).

(4) Define T^{L^*} to be the tableau determined by the boxes $\boxed{x_i}$ in T^{L} together with $\boxed{y}$, and define T^{R^*} to be the tableau given by the remaining boxes on the right.

For example, we have

$$
T \;=\; \begin{array}{c}\boxed{\begin{smallmatrix}2\\3\\2\ 7\\4\ 9\\6\\8\\9\end{smallmatrix}}\end{array} \quad\to\quad \begin{array}{c}\boxed{\begin{smallmatrix}2\ 2\\3\\4\ 7\\6\ 9\\8\\9\end{smallmatrix}}\end{array} \quad\to\quad \begin{array}{c}\boxed{\begin{smallmatrix}2\ 2\\3\\4\ 7\\6\ 9\\8\\9\end{smallmatrix}}\end{array} \quad\to\quad \begin{array}{c}\boxed{\begin{smallmatrix}2\ 2\\3\ 7\\4\ 9\\6\\8\\9\end{smallmatrix}}\end{array}
$$
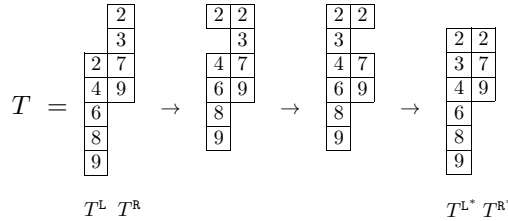
$$T^{\mathrm{L}}\ T^{\mathrm{R}} \qquad\qquad\qquad\qquad T^{\mathrm{L}^*}\ T^{\mathrm{R}^*}$$

Note that the pairs $(T^{\mathrm{L}}, T^{\mathrm{L}^*})$ and $(T^{\mathrm{R}}, T^{\mathrm{R}^*})$ are arranged to share the same bottom lines, respectively.

We refer the reader to [26, Section 6] and [27, Section 3] for more details on the well-definedness of $({}^{\mathrm{L}}T, {}^{\mathrm{R}}T)$ and $(T^{\mathrm{L}^*}, T^{\mathrm{R}^*})$.

Definition 3.2. Let $a, a' \in \mathbb{Z}_+$ be given with $a \geq a'$. We say that a pair (T, S) is admissible and write $T \prec S$ if it is one of the following cases:
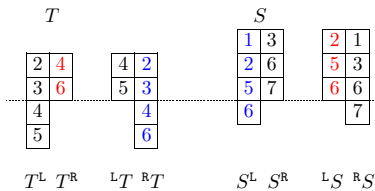
(1) $(T, S) \in \mathbf{T}^{\mathfrak{g}}(a) \times \mathbf{T}^{\mathfrak{g}}(a')$ or $\mathbf{T}^{\mathfrak{g}}(a) \times \mathbf{T}^{\mathrm{sp}}$ with

$$\mathrm{ht}(T^{\mathrm{R}}) \leq \mathrm{ht}(S^{\mathrm{L}}) - a' + 2\mathfrak{r}_T \mathfrak{r}_S,$$

$$\begin{cases} T^{\mathrm{R}}(i) \leq {}^{\mathrm{L}}S(i), & \text{if } \mathfrak{r}_T = 0 \text{ or } \mathfrak{r}_S = 0, \\ T^{\mathrm{R}^*}(i) \leq {}^{\mathrm{L}}S(i), & \text{if } \mathfrak{r}_T = \mathfrak{r}_S = 1, \end{cases}$$

$$\begin{cases} {}^{\mathrm{R}}T(i + a - a') \leq S^{\mathrm{L}}(i), & \text{if } \mathfrak{r}_T = 0 \text{ or } \mathfrak{r}_S = 0, \\ {}^{\mathrm{R}}T(i + a - a' + \varepsilon) \leq S^{\mathrm{L}^*}(i), & \text{if } \mathfrak{r}_T = \mathfrak{r}_S = 1, \end{cases}$$

for $i \geq 1$ (here $\varepsilon = 1$ if $S \in \mathbf{T}^{\mathrm{sp}-}$ and 0 otherwise, and we assume that $a' = \mathfrak{r}_S$, $S = S^{\mathrm{L}} = {}^{\mathrm{L}}S = S^{\mathrm{L}^*}$ when $S \in \mathbf{T}^{\mathrm{sp}}$);

(2) $(T, S) \in \mathbf{T}^{\mathfrak{d}}(a) \times \overline{\mathbf{T}}^{\mathfrak{d}}(0)$ with $T \prec S^{\mathrm{L}}$ in the sense of (1), regarding $S^{\mathrm{L}} \in \mathbf{T}^{\mathrm{sp}-}$;

(3) $(T, S) \in \overline{\mathbf{T}}^{\mathfrak{d}}(0) \times \overline{\mathbf{T}}^{\mathfrak{d}}(0)$ or $\overline{\mathbf{T}}^{\mathfrak{d}}(0) \times \mathbf{T}^{\mathrm{sp}-}$ with $(T^{\mathrm{R}}, S^{\mathrm{L}}) \in \overline{\mathbf{T}}^{\mathfrak{d}}(0)$.

Example 3.3. For $T \in \mathbf{T}^{\mathfrak{c}}(2)$ and $S \in \mathbf{T}^{\mathfrak{c}}(1)$ below, we have

$$T \qquad\qquad\qquad\qquad\qquad S$$

$$T^{\mathrm{L}}\ T^{\mathrm{R}} \qquad {}^{\mathrm{L}}T\ {}^{\mathrm{R}}T \qquad\quad S^{\mathrm{L}}\ S^{\mathrm{R}} \qquad {}^{\mathrm{L}}S\ {}^{\mathrm{R}}S$$

where the dashed line denotes the line separating the body and tail of T and S. Then $(^{\text{R}}T, S^{\text{L}})$ (in blue) and $(T^{\text{R}}, {}^{\text{L}}S)$ (in red) form semistandard tableaux

which implies that $T \prec S$.

The following lemma will be used in the next section.

Lemma 3.4. *Let* $(T, S) \in \mathbf{T}^{\mathfrak{g}}(a) \times \mathbf{T}^{\mathfrak{g}}(a')$ $(a \geq a')$ *such that* $T \prec S$. *Then*

(1) $T^{\text{L}}(i + a - a') \leq S^{\text{L}}(i)$ *for* $i \geq 1$, *if* $\mathfrak{r}_T = 0$ *or* $\mathfrak{r}_S = 0$,
(2) $T^{\text{R}}(1) \leq S^{\text{R}}(1)$.

Proof. (1) By definition of $^{\text{R}}T$, we have $T^{\text{L}}(i + a - a') \leq {}^{\text{R}}T(i + a - a')$ for $i \geq 1$. Since $\mathfrak{r}_T = 0$ or $\mathfrak{r}_S = 0$, we have $T^{\text{L}}(i+a-a') \leq S^{\text{L}}(i)$ for $i \geq 1$ by Definition 3.2(1).

(2) Suppose first that $\mathfrak{g} = \mathfrak{b}$ or $\mathfrak{c}$. In this case, we always have $\mathfrak{r}_T = \mathfrak{r}_S = 0$. We have $^{\text{L}}S(1) \leq S^{\text{R}}(1)$ by definition of $^{\text{L}}S$, and hence $T^{\text{R}}(1) \leq {}^{\text{L}}S(1) \leq S^{\text{R}}(1)$ by Definition 3.2(1).

Now, we suppose that $\mathfrak{g} = \mathfrak{d}$. If $\mathfrak{r}_T = 0$ or $\mathfrak{r}_S = 0$, then we also have $T^{\text{R}}(1) \leq S^{\text{R}}(1)$ by the same argument as in $\mathfrak{g} = \mathfrak{b}$ or $\mathfrak{c}$. So we may assume that $\mathfrak{r}_T = \mathfrak{r}_S = 1$. Let $x = T^{\text{R}}(1)$. Suppose that $\boxed{x}$ moves to the right when we construct T^{L^*}. This implies that $T^{\text{L}}(a) < x < T^{\text{L}}(a - 1)$, and hence $^{\text{R}}T(a) = x$. On the other hand, we have $S^{\text{L}^*}(a') = S^{\text{L}}(a') \leq S^{\text{R}}(1)$ or $S^{\text{L}^*}(a') = S^{\text{R}}(1)$ since $\mathfrak{r}_S = 1$ depending on whether $\boxed{y}$ with $y = S^{\text{R}}(1)$ moves to the left or not when we construct S^{L^*}. By Definition 3.2(1), we have $x = {}^{\text{R}}T(a) \leq S^{\text{L}^*}(a') \leq S^{\text{R}}(1)$. Next, suppose that $\boxed{x}$ does not move to the right when we construct T^{L^*}. This implies that $T^{\text{R}^*}(1) = x$. By Definition 3.2(1), we have $x \leq {}^{\text{L}}S(1) \leq S^{\text{R}}(1)$. This completes the proof. $\square$

3.3. **Highest weight crystals.** Let $(\mathfrak{g}_\infty, \mathrm{G}_n)$ be one of the dual pairs in (2.1) and $\lambda \in \mathcal{P}(\mathrm{G}_n)$.

Suppose that $(\mathfrak{g}_\infty, \mathrm{G}_n) \neq (\mathfrak{d}_\infty, \mathrm{O}_n)$. We put

$$\widehat{\mathbf{T}}^{\mathfrak{g}}(\lambda, n) = \begin{cases} \mathbf{T}^{\mathfrak{g}}(\lambda_1) \times \cdots \times \mathbf{T}^{\mathfrak{g}}(\lambda_{\frac{n}{2}}), & \text{for } (\mathfrak{c}_\infty, \mathrm{Sp}_n) \text{ or } (\mathfrak{b}_\infty, \mathrm{Pin}_n), \\ \mathbf{T}^{\mathfrak{g}}(\lambda_1) \times \cdots \times \mathbf{T}^{\mathfrak{g}}(\lambda_{\frac{n-1}{2}}) \times \mathbf{T}^{\mathrm{sp}}, & \text{for } (\mathfrak{b}_\infty, \mathrm{Spin}_n). \end{cases}$$

Suppose that $(\mathfrak{g}_\infty, \mathrm{G}_n) = (\mathfrak{d}_\infty, \mathrm{O}_n)$. Let $q_\pm$ and $r_\pm$ be non-negative integers such that

$$\begin{cases} n - 2\lambda_1' = 2q_+ + r_+, & \text{if } n - 2\lambda_1' \geq 0, \\ 2\lambda_1' - n = 2q_- + r_-, & \text{if } n - 2\lambda_1' \leq 0, \end{cases}$$

where $r_\pm = 0, 1$. Let $\overline{\lambda} = (\overline{\lambda}_i)_{i \geq 1} \in \mathscr{P}$ be such that $\overline{\lambda}_1' = n - \lambda_1'$ and $\overline{\lambda}_i' = \lambda_i'$ for $i \geq 2$. Let $M_+ = \lambda_1'$, $M_- = \overline{\lambda}_1' = n - \lambda_1'$, and $L = M_\pm + q_\pm$. Note that $2L + r_\pm = n$. Then we put

$$\widehat{\mathbf{T}}^{\mathfrak{d}}(\lambda, n) = \begin{cases} \mathbf{T}^{\mathfrak{d}}(\lambda_1) \times \cdots \times \mathbf{T}^{\mathfrak{d}}(\lambda_{M_+}) \times \mathbf{T}^{\mathfrak{d}}(0)^{\times q_+} \times (\mathbf{T}^{\mathrm{sp}+})^{r_+}, & \text{if } n - 2\lambda_1' \geq 0, \\ \mathbf{T}^{\mathfrak{d}}(\overline{\lambda}_1) \times \cdots \times \mathbf{T}^{\mathfrak{d}}(\overline{\lambda}_{M_-}) \times \overline{\mathbf{T}}^{\mathfrak{d}}(0)^{\times q_-} \times (\mathbf{T}^{\mathrm{sp}-})^{r_-}, & \text{if } n - 2\lambda_1' \leq 0. \end{cases}$$

Definition 3.5. For $\lambda \in \mathcal{P}(G_n)$, we define

$$\mathbf{T}^{\mathfrak{g}}(\lambda, n) = \{\, \mathbf{T} = (T_i)_{i \geq 1} \in \widehat{\mathbf{T}}^{\mathfrak{g}}(\lambda, n) \mid T_i \prec T_{i+1} \text{ for } i \geq 1 \,\}.$$

We regard $\widehat{\mathbf{T}}^{\mathfrak{g}}(\lambda, n)$ as a $\mathfrak{g}_{\infty}$-crystal by identifying $\mathbf{T} = (T_1, T_2, \ldots) \in \widehat{\mathbf{T}}^{\mathfrak{g}}(\lambda, n)$ with $\cdots \otimes T_2 \otimes T_1$ and regard $\mathbf{T}^{\mathfrak{g}}(\lambda, n)$ as its subcrystal. Let $\mathbf{H}_\lambda$ be the element $(T_1, T_2, \ldots)$ in $\mathbf{T}^{\mathfrak{g}}(\lambda, n)$ such that each T_k is a highest weight element for all $k \geq 1$, or $\widetilde{e}_i T_k = \mathbf{0}$ for all $i \in I$. Then we have the following by [26, Theorem 7.4] for $\mathfrak{g} = \mathfrak{b}, \mathfrak{c}$ and [27, Theorem 4.4] for $\mathfrak{g} = \mathfrak{d}$.

Theorem 3.6. *For $\lambda \in \mathcal{P}(G_n)$, we have*

$$\mathbf{T}^{\mathfrak{g}}(\lambda, n) \cong \mathbf{B}(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\lambda)),$$

with highest weight element $\mathbf{H}_\lambda$ of weight $\Lambda^{\mathfrak{g}}(\lambda)$.

Remark 3.7. Following [14], one can see that $\mathbf{T}^{\mathfrak{b}}(\lambda, n)$ for $\lambda \in \mathcal{P}(G_n)$ (G = Pin or Spin) coincides with the crystal graph associated to an integrable highest weight module over $\mathfrak{b}_\infty^\bullet$ with highest weight $\Lambda^{\mathfrak{b}}(\lambda)$. So the tensor product and branching multiplicities for $\mathfrak{b}_\infty$ are equal to those for $\mathfrak{b}_\infty^\bullet$. Recall that $\mathfrak{b}_k^\bullet \cong \mathfrak{osp}_{1|2k}$ for $k \in \mathbb{N}$.

Remark 3.8. For $k \geq 2$ and $\lambda \in \mathcal{P}(G_n)$ with $\lambda_1 \leq k$, let $\mathbf{T}_k^{\mathfrak{g}}(\lambda, n)$ be the subset of $\mathbf{T}^{\mathfrak{g}}(\lambda, n)$ with entries from $\{1, \ldots, k\}$. Then $\mathbf{T}_k^{\mathfrak{g}}(\lambda, n)$ is a $\mathfrak{g}_k$-crystal associated to a finite-dimensional irreducible $\mathfrak{g}_k$-module with highest weight $\Lambda^{\mathfrak{g}}(\lambda)$ for $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}, \mathfrak{d}$ with respect to $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in \{0, \ldots, k-1\}$.

4. Stability in tensor product and branching decomposition

4.1. **Separation lemma.** We put

$$
\begin{aligned}
\mathscr{P}_{\mathfrak{b}} &= \mathscr{P}, \\
\mathscr{P}_{\mathfrak{c}} &= \{\, 2\lambda = (2\lambda_i)_{i \geq 1} \mid \lambda \in \mathscr{P} \,\}, \\
\mathscr{P}_{\mathfrak{d}} &= \{\, (2\lambda)' \mid \lambda \in \mathscr{P} \,\}.
\end{aligned}
\tag{4.1}
$$

Consider a $\mathfrak{g}_\infty$-crystal $\mathbf{T}^{\mathfrak{g}}(\lambda, n)$ for $\lambda \in \mathcal{P}(G_n)$. Let $\mathbf{T} = (T_1, \ldots, T_r) \in \mathbf{T}^{\mathfrak{g}}(\lambda, n)$ be given. We put $w(\mathbf{T}) = w(T_r) \ldots w(T_1)$ and define $L(\mathbf{T})$ to be the maximal length of a weakly decreasing subword of $w(\mathbf{T})$.

Lemma 4.1. *For $\mathbf{T} \in \mathbf{T}^{\mathfrak{g}}(\lambda, n)$, we have $L(\mathbf{T}) \geq L(\mathbf{H}_\lambda) = \ell(\lambda)$.*

Proof. It is clear that $L(\mathbf{H}_\lambda) = \ell(\lambda)$. For arbitrary $\mathbf{T}$, it follows directly from Lemma 3.4(1) that $L(\mathbf{T}) \geq \ell(\lambda)$. $\square$

We put

$$\mathbf{T}^{\mathtt{tail}} = (T_1^{\mathtt{tail}}, \ldots, T_r^{\mathtt{tail}}), \quad \mathbf{T}^{\mathtt{body}} = (T_1^{\mathtt{body}}, \ldots, T_r^{\mathtt{body}}). \tag{4.2}$$

For $\mathfrak{g} = \mathfrak{d}$ and $T_i \in \overline{\mathbf{T}}^{\mathfrak{d}}(0)$ or $\mathbf{T}^{\mathtt{sp}-}$, we assume that $T_i^{\mathtt{tail}}$ is the subtableau of T_i consisting of the boxes at the bottom and $T_i^{\mathtt{body}}$ is the subtableau consisting of the other part of T_i.

We assume that $T_i^{\mathtt{body}}$ and $T_i^{\mathtt{tail}}$ are separated by a common horizontal line. Then we may regard $\mathbf{T}^{\mathtt{tail}}$ as a tableau of shape λ', where $T_i^{\mathtt{tail}}$ is the ith column from the left for $1 \leq i \leq r$, and regard $\mathbf{T}^{\mathtt{body}}$ as a tableau of shape μ^π for some $\mu \in \mathscr{P}$ when $\mathfrak{r}_{T_i} = 0$ for $1 \leq i \leq r$. Here μ^π denotes a skew Young diagram obtained by $180°$-rotation of μ.

Example 4.2. Let $\mathbf{T} = (T_1, T_2) \in \mathbf{T}^{\flat}(2, 1)$ be given by

$$T_1 = \begin{array}{cc} & \boxed{4} \\ \boxed{5} & \boxed{8} \\ \boxed{6} & \\ \boxed{7} & \end{array} \qquad T_2 = \begin{array}{cc} \boxed{3} & \boxed{5} \\ \boxed{4} & \boxed{8} \\ \boxed{7} & \boxed{9} \\ \boxed{8} & \end{array}$$

where the entries in $T_i^{\mathtt{body}}$ and $T_i^{\mathtt{tail}}$ $(i = 1, 2)$ are in red and blue, respectively. Then we have

$$\mathbf{T}^{\mathtt{tail}} = \begin{array}{cc} \boxed{6} & \boxed{8} \\ \boxed{7} & \end{array} \qquad \mathbf{T}^{\mathtt{body}} = \begin{array}{cccc} & & \boxed{3} & \boxed{5} \\ & \boxed{4} & \boxed{4} & \boxed{8} \\ \boxed{5} & \boxed{8} & \boxed{7} & \boxed{9} \end{array}$$

Note that $\mathbf{T}^{\mathtt{body}}$ is not semistandard.

The following lemma will play a crucial role in this paper.

Lemma 4.3. *If* $L(\mathbf{T}) \le \frac{n}{2}$*, then we have*

(1) $\mathbf{T}^{\mathtt{tail}} \in SST(\lambda')$ *and* $\mathbf{T}^{\mathtt{body}} \in SST(\mu^{\pi})$ *for some* $\mu \in \mathscr{P}_{\mathfrak{g}}$*,*

(2) $\mathbf{T} \equiv_A \mathbf{T}^{\mathtt{body}} \otimes \mathbf{T}^{\mathtt{tail}}$*, where* $\equiv_A$ *denotes the equivalence as elements of* $\mathfrak{l}_{\infty}$*-crystals.*

Proof.

Case 1. Suppose that $\mathfrak{g} = \mathfrak{c}$. Let $\mathbf{T} = (T_1, \ldots, T_{n/2}) \in \mathbf{T}^{\mathfrak{c}}(\lambda, n)$ with $L(\mathbf{T}) \le n/2$. We may regard $\mathbf{T}^{\mathtt{tail}}$ as a tableau of shape λ' and $\mathbf{T}^{\mathtt{body}}$ as a tableau of shape μ^{π} for some $\mu \in \mathscr{P}_{\mathfrak{c}}$. We first have by Lemma 3.4(1) that $\mathbf{T}^{\mathtt{tail}}$ is semistandard and hence $\mathbf{T}^{\mathtt{tail}} \in SST(\lambda')$.

Next, we claim that $\mathbf{T}^{\mathtt{body}}$ is semistandard. Since $T_i^{\mathtt{body}}$ is a semistandard tableau of shape $\lambda(0, 0, c_i)$ for some $c_i \in \mathbb{Z}_+$ with $(T_i^{\mathtt{body}})^{\mathtt{R}} = T_i^{\mathtt{R}}$ and $(T_i^{\mathtt{body}})^{\mathtt{L}}(k) = T_i^{\mathtt{L}}(\lambda_i + k)$ for $1 \le i \le n/2$ and $k \ge 1$, it is enough to show that $(T_i^{\mathtt{body}})^{\mathtt{R}}(k) \le (T_{i+1}^{\mathtt{body}})^{\mathtt{L}}(k)$ for all i and k.

Suppose that $(T_i^{\mathtt{body}})^{\mathtt{R}}(k_0) > (T_{i+1}^{\mathtt{body}})^{\mathtt{L}}(k_0)$ for some i and k_0. Since $(T_i^{\mathtt{body}})^{\mathtt{R}}(k_0) > (T_{i+1}^{\mathtt{body}})^{\mathtt{L}}(k_0)$ and $T_i \prec T_{i+1}$, at least one of the boxes in $(T_{i+1}^{\mathtt{body}})^{\mathtt{L}}$ moves to the right to form an entry in ${}^{\mathtt{R}}T_{i+1}$. This implies that $T_{i+1}^{\mathtt{tail}}$ is not empty, and $T_{i+1}^{\mathtt{tail}}[k_1] \le (T_{i+1}^{\mathtt{body}})^{\mathtt{R}}(1)$ for some $k_1 \ge 1$. By Lemma 3.4 we have a weakly increasing sequence

$$T_1^{\mathtt{tail}}[k_1] \le T_2^{\mathtt{tail}}[k_1] \le \cdots \le T_i^{\mathtt{tail}}[k_1] \le T_{i+1}^{\mathtt{tail}}[k_1]$$
$$\le T_{i+1}^{\mathtt{R}}(1) \le T_{i+2}^{\mathtt{R}}(1) \le \cdots \le T_{n/2}^{\mathtt{R}}(1),$$

which yields a weakly decreasing subword of $w(\mathbf{T})$ of length $n/2 + 1$. This is a contradiction, which proves our claim. Therefore, $\mathbf{T}^{\mathtt{body}} \in SST(\mu^{\pi})$.

Finally, we claim that $\mathbf{T} \equiv_A \mathbf{T}^{\mathtt{body}} \otimes \mathbf{T}^{\mathtt{tail}}$. By the semistandardness of $\mathbf{T}^{\mathtt{body}}$ and Lemma 3.4(2), we have for $2 \le i \le n/2$,

(4.3) $$\cdots \le (T_{i-1}^{\mathtt{body}})^{\mathtt{L}}(1) \le (T_{i-1}^{\mathtt{body}})^{\mathtt{R}}(1) \le (T_i^{\mathtt{body}})^{\mathtt{L}}(1) < T_i^{\mathtt{tail}}[1].$$

Since $\equiv_A$ coincides with the usual Knuth equivalence, one can check by (4.3) and considering the column insertion of tableaux that

$$
\begin{aligned}
(4.4) \quad & T_i^{\texttt{body}} \otimes T_i^{\texttt{tail}} \otimes \left(T_{i-1}^{\texttt{body}} \otimes \cdots \otimes T_1^{\texttt{body}} \right) \\
& \equiv_A T_i^{\texttt{body}} \otimes \left(T_{i-1}^{\texttt{body}} \otimes \cdots \otimes T_1^{\texttt{body}} \right) \otimes T_i^{\texttt{tail}}.
\end{aligned}
$$

Applying (4.4) successively, we have

$$
\begin{aligned}
\mathbf{T} &= T_{n/2} \otimes \cdots \otimes T_1 \\
&\equiv_A \left(T_{n/2}^{\texttt{body}} \otimes T_{n/2}^{\texttt{tail}} \right) \otimes \cdots \otimes \left(T_1^{\texttt{body}} \otimes T_1^{\texttt{tail}} \right) \\
&\equiv_A \left(T_{n/2}^{\texttt{body}} \otimes \ldots \otimes T_1^{\texttt{body}} \right) \otimes \left(T_{n/2}^{\texttt{tail}} \otimes \cdots \otimes T_1^{\texttt{tail}} \right) \equiv_A \mathbf{T}^{\texttt{body}} \otimes \mathbf{T}^{\texttt{tail}}.
\end{aligned}
$$

Case 2. Suppose that $\mathfrak{g} = \mathfrak{b}$. The proof is almost identical to Case 1. We leave the details to the reader.

Case 3. Suppose that $\mathfrak{g} = \mathfrak{d}$. Let $\mathbf{T} = (T_1, \ldots, T_r) \in \mathbf{T}^{\mathfrak{d}}(\lambda, n)$ with $L(\mathbf{T}) \leq n/2$. By Lemma 4.1, we have $\ell(\lambda) \leq n/2$, and hence $n - 2\lambda_1' \geq 0$.

We first claim that $\mathfrak{r}_{T_i} = 0$ for all $1 \leq i \leq r$. Suppose that $\mathfrak{r}_{T_i} = 1$ for some $1 \leq i \leq \lambda_1'$, and let j be the smallest one such that $\mathfrak{r}_{T_j} = 1$. By Lemma 3.4, we have a sequence

$$
T_1^{\texttt{tail}}[1] \leq \cdots \leq T_j^{\texttt{tail}}[1] \leq T_j^{\texttt{R}}(1) \leq T_{j+1}^{\texttt{R}}(1) \leq \cdots \leq T_r^{\texttt{R}}(1).
$$

Note that $T_{j+1}^{\texttt{R}}(1), \ldots, T_r^{\texttt{R}}(1)$ are non-empty since $T_i \prec T_{i+1}$ for $j \leq i \leq r - 1$. Reading from right to left, we get a subword of $w(\mathbf{T})$ with length $r + 1$. Since $2\lambda_1' + 2q_+ + r_+ = n$ and $r = \lambda_1' + q_+ + r_+$, we have $r + 1 > n/2$. This is a contradiction, which proves our claim.

Now by Lemma 3.4(1), we have $\mathbf{T}^{\texttt{tail}} \in SST(\lambda')$. By the same arguments as in Case 1 we conclude that $\mathbf{T}^{\texttt{body}} \in SST(\mu^\pi)$ and $\mathbf{T} \equiv_A \mathbf{T}^{\texttt{body}} \otimes \mathbf{T}^{\texttt{tail}}$. $\qquad\square$

4.2. **Stable tensor product and branching rules.** Let $\mu \in \mathcal{P}(\mathrm{G}_m)$ and $\nu \in \mathcal{P}(\mathrm{G}_n)$ be given. For $\lambda \in \mathcal{P}(\mathrm{G}_{m+n})$, let

$$
(4.5) \quad
\begin{aligned}
&\mathsf{LR}_{\mu\nu}^\lambda(\mathfrak{g}) = \left\{ \mathbf{T} \in \mathbf{T}^{\mathfrak{g}}(\nu, n) \,\middle|\,
\begin{array}{l}
(1) \ \mathrm{wt}(\mathbf{T}) + \mathrm{wt}(\mathbf{H}_\mu) = \mathrm{wt}(\mathbf{H}_\lambda) \\
(2) \ \varepsilon_i(\mathbf{T}) \leq \langle \Lambda^{\mathfrak{g}}(\mu), \alpha_i^\vee \rangle \ \text{for } i \in I
\end{array}
\right\}, \\
&c_{\mu\nu}^\lambda(\mathfrak{g}) = \left| \mathsf{LR}_{\mu\nu}^\lambda(\mathfrak{g}) \right|.
\end{aligned}
$$

Then we have the following formula of tensor product multiplicity for integrable highest weight $\mathfrak{g}_\infty$-modules.

Proposition 4.4. *For $\mu \in \mathcal{P}(\mathrm{G}_m)$, $\nu \in \mathcal{P}(\mathrm{G}_n)$, and $\lambda \in \mathcal{P}(\mathrm{G}_{m+n})$, we have*

$$
c_{\mu\nu}^\lambda(\mathfrak{g}) = [L(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\mu)) \otimes L(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\nu)) : L(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\lambda))].
$$

Proof. By tensor product rule of crystals, we have $\mathsf{LR}_{\mu\nu}^\lambda(\mathfrak{g}) = \{ \mathbf{T} \in \mathbf{T}^{\mathfrak{g}}(\nu, n) \,|\, \mathbf{H}_\mu \otimes \mathbf{T} \equiv \mathbf{H}_\lambda \}$, and hence $c_{\mu\nu}^\lambda(\mathfrak{g})$ is the multiplicity of $\mathbf{B}(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\lambda))$ in $\mathbf{B}(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\mu)) \otimes \mathbf{B}(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\nu))$. $\qquad\square$

Now we have the following stable tensor product rule, which is the first main result in this paper.

Theorem 4.5. *Let* $\mu \in \mathcal{P}(G_m)$, $\nu \in \mathcal{P}(G_n)$, *and* $\lambda \in \mathcal{P}(G_{m+n})$ *be given. If* $\ell(\lambda) \leq \frac{1}{2}\min\{m,n\}$, *then the map* $\mathbf{T} \mapsto (\mathbf{T}^{\mathtt{body}}, \mathbf{T}^{\mathtt{tail}})$ *gives a bijection*

$$\mathrm{LR}^{\lambda}_{\mu\nu}(\mathfrak{g}) \longrightarrow \bigsqcup_{\gamma \in \mathscr{P}} \bigsqcup_{\delta \in \mathscr{P}_{\mathfrak{g}}} \mathrm{LR}^{\gamma}_{\mu'\delta} \times \mathrm{LR}^{\lambda'}_{\gamma\nu'}.$$

Proof. Let $\mathbf{T} \in \mathrm{LR}^{\lambda}_{\mu\nu}(\mathfrak{g})$ be given. By definition, $\mathbf{H}_{\mu} \otimes \mathbf{T} \equiv \mathbf{H}_{\lambda}$ as a $\mathfrak{g}_{\infty}$-crystal element, and hence as an $\mathfrak{l}_{\infty}$-crystal element. Recall that $\equiv_A$ denotes the equivalence of $\mathfrak{l}_{\infty}$-crystal elements. Note that $\mathbf{H}_{\lambda} \equiv_A H_{\lambda'}$. Since $\ell(\lambda) \leq \frac{1}{2}\min\{m,n\}$, we have $L(\mathbf{T}) \leq L(\mathbf{H}_{\mu} \otimes \mathbf{T}) = \ell(\lambda) \leq \frac{1}{2}\min\{m,n\} \leq n/2$. Here we understand $L(\mathbf{H}_{\mu} \otimes \mathbf{T})$ as the maximal length of a weakly decreasing subword of $w(\mathbf{H}_{\mu})w(\mathbf{T})$. By Lemma 4.3, we have

  · $\mathbf{T}^{\mathtt{body}} \in SST(\delta^{\pi})$ for some $\delta \in \mathscr{P}_{\mathfrak{g}}$,
  · $\mathbf{T}^{\mathtt{tail}} \in SST(\nu')$,
  · $\mathbf{T} \equiv_A \mathbf{T}^{\mathtt{body}} \otimes \mathbf{T}^{\mathtt{tail}}$.

It is clear that the correspondence $\mathbf{T} \mapsto (\mathbf{T}^{\mathtt{body}}, \mathbf{T}^{\mathtt{tail}})$ is injective.

Since $\mathbf{H}_{\mu} \otimes \mathbf{T}^{\mathtt{body}} \equiv_A H_{\mu'} \otimes \mathbf{T}^{\mathtt{body}} \equiv_A H_{\gamma}$ for some $\gamma \in \mathscr{P}$ and there exists a unique tableau $U \in SST(\delta)$ such that $U \equiv_A \mathbf{T}^{\mathtt{body}}$, we may regard $\mathbf{T}^{\mathtt{body}}$ as an element in $\mathrm{LR}^{\gamma}_{\mu'\delta}$ (see Section 2.4). Then we have $H_{\gamma} \otimes \mathbf{T}^{\mathtt{tail}} \equiv_A \mathbf{H}_{\lambda} \equiv_A H_{\lambda'}$, which implies that $\mathbf{T}^{\mathtt{tail}} \in \mathrm{LR}^{\lambda'}_{\gamma\nu'}$.

Conversely, suppose that a pair $(U,V) \in \mathrm{LR}^{\gamma}_{\mu'\delta} \times \mathrm{LR}^{\lambda'}_{\gamma\nu'}$ is given for $\gamma \in \mathscr{P}$ and $\delta \in \mathscr{P}_{\mathfrak{g}}$. Here we assume that $U \in SST(\delta^{\pi})$ such that $H_{\mu'} \otimes U \equiv_A H_{\gamma}$, and $V \in SST(\nu')$ such that $H_{\gamma} \otimes U \equiv_A H_{\lambda'}$. We also assume that $\mathfrak{g} = \mathfrak{c}$ since the arguments for $\mathfrak{g} = \mathfrak{b}, \mathfrak{d}$ are very similar.

Note that $\ell(\delta'), \ell(\nu) \leq \ell(\lambda) \leq \frac{1}{2}\min\{m,n\}$. Let $U^{(n-i+1)}$ be the ith column of U from the right for $1 \leq i \leq n$, where $U^{(i)}$ is empty for $1 \leq i \leq n/2$. Let $V^{(i)}$ be the ith column of V from the left for $i \geq 1$. We claim that for all $1 \leq i \leq n/2$

$$(4.6) \qquad\qquad V^{(i)}[1] > U^{(2i)}(1).$$

Otherwise, we have a weakly decreasing subword of $w(U)w(V)$ given by

$$U^{(n)}(1) \geq U^{(n-1)}(1) \geq \cdots \geq U^{(2i)}(1) \geq V^{(i)}[1] \geq V^{(i-1)}[1] \geq \cdots \geq V^{(1)}[1],$$

for some i, which is of length $(n-2i+1)+i = n-i+1 > n/2$. Since $H_{\mu'} \otimes U \otimes V \equiv_A H_{\lambda'}$, this implies that $\ell(\lambda) > n/2$, which is a contradiction.

For $1 \leq i \leq n/2$, let T_i be the tableau of shape $\lambda(\nu_i, 0, c)$ for some $c \in \mathbb{Z}_{+}$ such that $(T_i^{\mathtt{body}})^{\mathtt{L}} = U^{(2i-1)}$, $(T_i^{\mathtt{body}})^{\mathtt{R}} = U^{(2i)}$, and $T_i^{\mathtt{tail}} = V^{(i)}$. By (4.6), it is straightforward to see that $T_i \in \mathbf{T}^{\mathfrak{g}}(\nu_i)$ and $T_i \prec T_{i+1}$. Therefore, $\mathbf{T} = (T_1, \ldots, T_{n/2}) \in \mathbf{T}^{\mathfrak{g}}(\nu, n)$ with $\mathbf{T}^{\mathtt{body}} = U$ and $\mathbf{T}^{\mathtt{tail}} = V$. By Lemma 4.3, we have $\mathbf{H}_{\mu} \otimes \mathbf{T} \equiv_A H_{\mu'} \otimes U \otimes V \equiv_A H_{\lambda'}$, and hence $\widetilde{e}_i(\mathbf{H}_{\mu} \otimes \mathbf{T}) = 0$ for $i \in I \setminus \{0\}$.

We see from the definition of $\widetilde{e}_0$ and $\widetilde{f}_0$ that $\varphi_0(\mathbf{H}_{\mu}) = \frac{m}{2} - \ell(\mu)$ and $\varepsilon_0(\mathbf{T}) \leq z/2$, where z is the number of 1's in U. Since $H_{\mu'} \otimes U \otimes V \equiv_A H_{\lambda'}$, we have $z + \ell(\mu) \leq \ell(\lambda)$, and then

$$\varepsilon_0(\mathbf{T}) + \ell(\mu) \leq z + \ell(\mu) \leq \ell(\lambda) \leq \frac{m}{2},$$

which implies that $\varepsilon_0(\mathbf{T}) \leq \frac{m}{2} - \ell(\mu) = \varphi_0(\mathbf{H}_{\mu})$ and $\widetilde{e}_0(\mathbf{H}_{\mu} \otimes \mathbf{T}) = 0$. Hence $\mathbf{T} \in \mathrm{LR}^{\lambda}_{\mu\nu}(\mathfrak{g})$.

Therefore, it follows that the map $\mathbf{T} \mapsto (\mathbf{T}^{\mathtt{body}}, \mathbf{T}^{\mathtt{tail}})$ gives a bijection. $\square$

Corollary 4.6. *Let* $\mu \in \mathcal{P}(G_m)$, $\nu \in \mathcal{P}(G_n)$, *and* $\lambda \in \mathcal{P}(G_{m+n})$ *be given. If* $\ell(\lambda) \leq \frac{1}{2}\min\{m, n\}$, *then*

$$c_{\mu\nu}^{\lambda}(\mathfrak{g}) = \sum_{\gamma \in \mathscr{P}} \sum_{\delta \in \mathscr{P}_{\mathfrak{g}}} c_{\mu\delta'}^{\gamma} c_{\gamma\nu}^{\lambda}.$$

Proof. It follows directly from Theorem 4.5 and the invariance of Littlewood-Richardson coefficients under conjugate of partitions. $\square$

Let $\mu \in \mathcal{P}(G_n)$ be given. For $\lambda \in \mathscr{P}$, let

$$(4.7) \qquad \mathrm{LR}_{\lambda}^{\mu}(\mathfrak{g}) = \left\{ \mathbf{T} \in \mathbf{T}^{\mathfrak{g}}(\mu, n) \;\middle|\; \begin{array}{l} (1)\ \mathrm{wt}(\mathbf{T}) = \frac{n}{\epsilon}\Lambda_0^{\mathfrak{g}} + \sum_{i \geq 1} \lambda_i' \epsilon_i \\ (2)\ \varepsilon_i(\mathbf{T}) = 0 \ \text{ for } i \in I \setminus \{0\} \end{array} \right\},$$

$$c_{\lambda}^{\mu}(\mathfrak{g}) = \left| \mathrm{LR}_{\lambda}^{\mu}(\mathfrak{g}) \right|.$$

Proposition 4.7. *For* $\mu \in \mathcal{P}(G_n)$ *and* $\lambda \in \mathscr{P}$, *we have*

$$c_{\lambda}^{\mu}(\mathfrak{g}) = \left[L(\mathfrak{g}_{\infty}, \Lambda^{\mathfrak{g}}(\mu)) : L(\mathfrak{l}_{\infty}, \lambda') \right],$$

where $L(\mathfrak{l}_{\infty}, \lambda')$ *is the irreducible* $\mathfrak{l}_{\infty}$-*module with highest weight* $\sum_{i \geq 1} \lambda_i' \epsilon_i$.

Proof. By definition, the connected component of $\mathbf{T} \in \mathrm{LR}_{\lambda}^{\mu}(\mathfrak{g})$ in $\mathbf{T}^{\mathfrak{g}}(\mu, n)$ with respect to $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I \setminus \{0\}$ is isomorphic to $SST(\lambda')$ as an $\mathfrak{l}_{\infty}$-crystal. Hence $c_{\lambda}^{\mu}(\mathfrak{g})$ is the multiplicities of $SST(\lambda')$ or $\mathbf{B}(\mathfrak{l}_{\infty}, \lambda')$ in $\mathbf{B}(\mathfrak{g}_{\infty}, \Lambda^{\mathfrak{g}}(\lambda))$. $\square$

Then we have the following stable branching rule, which is the second main result in this paper.

Theorem 4.8. *Let* $\mu \in \mathcal{P}(G_n)$ *and* $\lambda \in \mathscr{P}$ *be given. If* $\ell(\lambda) \leq \frac{n}{2}$, *then the map* $\mathbf{T} \mapsto \mathbf{T}^{\mathtt{tail}}$ *gives a bijection*

$$\mathrm{LR}_{\lambda}^{\mu}(\mathfrak{g}) \longrightarrow \bigsqcup_{\delta \in \mathscr{P}_{\mathfrak{g}}} \mathrm{LR}_{\delta\mu'}^{\lambda'}.$$

Proof. Let $\mathbf{T} \in \mathrm{LR}_{\lambda}^{\mu}(\mathfrak{g})$ be given. By definition, $\mathbf{T} \equiv_A H_{\lambda'}$. Since $L(\mathbf{T}) = \ell(\lambda) \leq \frac{n}{2}$, we have by Lemma 4.3:

- $\mathbf{T}^{\mathtt{body}} \in SST(\delta^{\pi})$ for some $\delta \in \mathscr{P}_{\mathfrak{g}}$,
- $\mathbf{T}^{\mathtt{tail}} \in SST(\mu')$,
- $\mathbf{T} \equiv_A \mathbf{T}^{\mathtt{body}} \otimes \mathbf{T}^{\mathtt{tail}}$.

Moreover, since $\mathbf{T} \equiv_A \mathbf{T}^{\mathtt{body}} \otimes \mathbf{T}^{\mathtt{tail}} \equiv_A H_{\lambda'}$, we have $\mathbf{T}^{\mathtt{body}} \equiv_A H_{\delta}$ and we may regard $\mathbf{T}^{\mathtt{tail}}$ as an element in $\mathrm{LR}_{\delta\mu'}^{\lambda'}$. Hence $\mathbf{T} \mapsto \mathbf{T}^{\mathtt{tail}}$ is an injective map from $\mathrm{LR}_{\lambda}^{\mu}(\mathfrak{g})$ to $\bigsqcup_{\delta \in \mathscr{P}_{\mathfrak{g}}} \mathrm{LR}_{\delta\mu'}^{\lambda'}$.

Conversely, suppose that $V \in \mathrm{LR}_{\delta\mu'}^{\lambda'}$ is given for $\delta \in \mathscr{P}_{\mathfrak{g}}$. Let us assume that $\mathfrak{g} = \mathfrak{c}$ since the arguments for $\mathfrak{g} = \mathfrak{b}, \mathfrak{d}$ are similar. Let U be the unique tableau in $SST(\delta^{\pi})$ such that $U \equiv_A H_{\delta}$. We assume that $V \in SST(\mu')$ such that $U \otimes V \equiv_A H_{\lambda'}$. Let $U^{(n-i+1)}$ be the ith column of U from the right, and let $V^{(i)}$ be the ith column of V from the left for $1 \leq i \leq n$. Then $U^{(i)}$ is empty for $1 \leq i \leq n/2$, and $V^{(i)}$ is empty for $n/2 < i \leq n$ since $\ell(\delta'), \ell(\mu) \leq \ell(\lambda) \leq n/2$. By the same argument as in Theorem 4.5, we can show that $V^{(i)}[1] > U^{(2i)}(1)$ for all $1 \leq i \leq n/2$, and hence there exists a unique $\mathbf{T} \in \mathbf{T}^{\mathfrak{g}}(\mu, n)$ such that $\mathbf{T}^{\mathtt{body}} = U$ and $\mathbf{T}^{\mathtt{tail}} = V$. Since $\mathbf{T} \equiv_A U \otimes V \equiv_A H_{\lambda'}$, we have $\mathbf{T} \in \mathrm{LR}_{\lambda}^{\mu}(\mathfrak{g})$.

Therefore $\mathbf{T} \mapsto \mathbf{T}^{\mathtt{tail}}$ gives a bijection. $\square$

Corollary 4.9. *Let $\mu \in \mathcal{P}(G_n)$ and $\lambda \in \mathscr{P}$. If $\ell(\lambda) \leq \frac{n}{2}$, then*

$$c_\lambda^\mu(\mathfrak{g}) = \sum_{\delta \in \mathscr{P}_\mathfrak{g}} c_{\delta'\mu}^\lambda.$$

Remark 4.10. For $k \geq 2$ and $\lambda \in \mathcal{P}(G_n)$ with $\lambda_1 \leq k$, let $\mathbf{T}_k^\mathfrak{g}(\lambda, n)$ be as in Remark 3.8. Since $\mathbf{T}_k^\mathfrak{g}(\lambda, n)$ for $\lambda \in \bigsqcup_n \mathcal{P}(G_n)$ with $\lambda_1 \leq k$ give crystals for all finite-dimensional irreducible $\mathfrak{g}_k$-modules, we have combinatorial extensions of the stable branching rules for the pairs $G_\ell \times G_\ell \supset G_\ell$ and $G_\ell \supset GL_{[\ell/2]}$, where $G_\ell = Sp_\ell$, $Spin_\ell$, $Osp_{1|\ell}$ with $k = [\ell/2]$ by Theorems 4.5 and 4.8 (see also Remark 3.7).

5. Branching rules for classical groups

5.1. **Branching rule for $G_{m+n} \supset G_m \times G_n$.** Suppose that $G = Sp$ or O, that is, $\mathfrak{g} = \mathfrak{c}$ or $\mathfrak{d}$. Note that for $\lambda \in \mathcal{P}(G_{m+n})$, $V_{G_{m+n}}^\lambda$ can be viewed as a $G_m \times G_n$-module since there is a natural embedding of $G_m \times G_n$ into G_{m+n}. For $\mu \in \mathcal{P}(G_m)$ and $\nu \in \mathcal{P}(G_n)$, let $[V_{G_{m+n}}^\lambda : V_{G_m}^\mu \otimes V_{G_n}^\nu]$ denote the multiplicity of $V_{G_m}^\mu \otimes V_{G_n}^\nu$ in $V_{G_{m+n}}^\lambda$.

Theorem 5.1. *For $\lambda \in \mathcal{P}(G_{m+n})$, $\mu \in \mathcal{P}(G_m)$, and $\nu \in \mathcal{P}(G_n)$, we have*

$$\left[V_{G_{m+n}}^\lambda : V_{G_m}^\mu \otimes V_{G_n}^\nu\right] = c_{\mu\nu}^\lambda(\mathfrak{g}),$$

where $c_{\mu\nu}^\lambda(\mathfrak{g})$ is given in (4.5). Furthermore, if $\ell(\lambda) \leq \frac{1}{2}\min\{m, n\}$, then

$$\left[V_{G_{m+n}}^\lambda : V_{G_m}^\mu \otimes V_{G_n}^\nu\right] = \sum_{\gamma \in \mathscr{P}} \sum_{\delta \in \mathscr{P}_\mathfrak{g}} c_{\mu\delta'}^\gamma c_{\gamma\nu}^\lambda.$$

Proof. By Propositions 2.1 and 4.4, we have

$$\mathcal{F}^{\frac{m}{2}} \otimes \mathcal{F}^{\frac{n}{2}} \cong \left(\bigoplus_{\mu \in \mathcal{P}(G_m)} L(\mathfrak{g}_\infty, \Lambda^\mathfrak{g}(\mu)) \otimes V_{G_m}^\mu\right) \otimes \left(\bigoplus_{\nu \in \mathcal{P}(G_n)} L(\mathfrak{g}_\infty, \Lambda^\mathfrak{g}(\nu)) \otimes V_{G_n}^\nu\right)$$

$$\cong \bigoplus_{\mu \in \mathcal{P}(G_m)} \bigoplus_{\nu \in \mathcal{P}(G_n)} (L(\mathfrak{g}_\infty, \Lambda^\mathfrak{g}(\mu)) \otimes L(\mathfrak{g}_\infty, \Lambda^\mathfrak{g}(\nu))) \otimes \left(V_{G_m}^\mu \otimes V_{G_n}^\nu\right)$$

$$\cong \bigoplus_{\mu \in \mathcal{P}(G_m)} \bigoplus_{\nu \in \mathcal{P}(G_n)} \left(\bigoplus_{\lambda \in \mathcal{P}(G_{m+n})} L(\mathfrak{g}_\infty, \Lambda^\mathfrak{g}(\lambda))^{\oplus c_{\mu\nu}^\lambda(\mathfrak{g})}\right) \otimes \left(V_{G_m}^\mu \otimes V_{G_n}^\nu\right)$$

$$\cong \bigoplus_{\lambda \in \mathcal{P}(G_{m+n})} L(\mathfrak{g}_\infty, \Lambda^\mathfrak{g}(\lambda)) \otimes \left(\bigoplus_{\mu \in \mathcal{P}(G_m)} \bigoplus_{\nu \in \mathcal{P}(G_n)} \left(V_{G_m}^\mu \otimes V_{G_n}^\nu\right)^{\oplus c_{\mu\nu}^\lambda(\mathfrak{g})}\right).$$

Since $\mathcal{F}^{\frac{m}{2}} \otimes \mathcal{F}^{\frac{n}{2}} \cong \mathcal{F}^{\frac{m+n}{2}}$, we have by Proposition 2.1

$$\left[V_{G_{m+n}}^\lambda : V_{G_m}^\mu \otimes V_{G_n}^\nu\right] = c_{\mu\nu}^\lambda(\mathfrak{g}).$$

The second equality follows from Corollary 4.6. $\qquad\square$

Remark 5.2. By the same argument as in Theorem 5.1, we also have a branching rule for the closed subgroup $Spin_m \times Spin_n \subset Pin_{m+n}$ for m, n odd in the case when $\mathfrak{g} = \mathfrak{b}$.

5.2. **Branching rule for** $\mathrm{GL}_n \supset \mathrm{G}_n$. Suppose that $\mathrm{G} = \mathrm{Sp}$ or O. Note that G_n is a subgroup of GL_n. For $\lambda \in \mathscr{P}$ with $\ell(\lambda) \leq n$, let $V_{\mathrm{GL}_n}^\lambda$ be the finite-dimensional irreducible GL_n-module corresponding to λ, and for $\mu \in \mathcal{P}(\mathrm{G}_n)$, let $\left[V_{\mathrm{GL}_n}^\lambda : V_{\mathrm{G}_n}^\mu\right]$ denote the multiplicity of $V_{\mathrm{G}_n}^\mu$ in $V_{\mathrm{GL}_n}^\lambda$ as a G_n-module. Then we have the following, which extends the Littlewood restriction rule [32, 33].

Theorem 5.3. *For $\lambda \in \mathscr{P}$ with $\ell(\lambda) \leq n$, and $\mu \in \mathcal{P}(\mathrm{G}_n)$, we have*

$$\left[V_{\mathrm{GL}_n}^\lambda : V_{\mathrm{G}_n}^\mu\right] = c_\lambda^\mu(\mathfrak{g}),$$

where $c_\lambda^\mu(\mathfrak{g})$ is given in (4.7). Furthermore, if $\ell(\lambda) \leq \frac{n}{2}$, then

$$\left[V_{\mathrm{GL}_n}^\lambda : V_{\mathrm{G}_n}^\mu\right] = \sum_{\delta \in \mathscr{P}_\mathfrak{g}} c_{\delta'\mu}^\lambda.$$

Proof. We first consider an action of $\mathfrak{gl}_n = \bigoplus_{1 \leq i,j \leq n} \mathbb{C}e_{ij}$ as operators on $\mathcal{F}^{\frac{n}{2}}$ by left multiplication as follows:

For $n = 2\ell$ and $\mathrm{G}_n = \mathrm{O}_{2\ell}$, put

$$(5.1) \qquad e_{ij} = \begin{cases} \sum_{r \in \frac{1}{2}+\mathbb{Z}_+} \psi_{-r}^{+,i}\psi_r^{-,j}, & \text{if } 1 \leq i,j \leq \ell, \\ \sum_{r \in \frac{1}{2}+\mathbb{Z}_+} \psi_{-r}^{+,i}\psi_r^{+,j-\ell}, & \text{if } 1 \leq i \leq \ell \text{ and } \ell < j \leq n, \\ \sum_{r \in \frac{1}{2}+\mathbb{Z}_+} \psi_{-r}^{-,i-\ell}\psi_r^{-,j}, & \text{if } \ell < i \leq n \text{ and } 1 \leq j \leq \ell, \\ \sum_{r \in \frac{1}{2}+\mathbb{Z}_+} \psi_{-r}^{-,i-\ell}\psi_r^{+,j-\ell}, & \text{if } \ell < i,j \leq n. \end{cases}$$

For $n = 2\ell$ and $\mathrm{G}_n = \mathrm{Sp}_{2\ell}$, put

$$(5.2) \qquad e_{ij} = \begin{cases} \sum_{r \in \frac{1}{2}+\mathbb{Z}_+} \psi_{-r}^{+,i}\psi_r^{-,j}, & \text{if } 1 \leq i,j \leq \ell, \\ \sum_{r \in \frac{1}{2}+\mathbb{Z}_+} (-1)^{r+\frac{1}{2}}\psi_{-r}^{+,i}\psi_r^{+,j-\ell}, & \text{if } 1 \leq i \leq \ell \text{ and } \ell < j \leq n, \\ \sum_{r \in \frac{1}{2}+\mathbb{Z}_+} (-1)^{r+\frac{1}{2}}\psi_{-r}^{-,i-\ell}\psi_r^{-,j}, & \text{if } \ell < i \leq n \text{ and } 1 \leq j \leq \ell, \\ \sum_{r \in \frac{1}{2}+\mathbb{Z}_+} \psi_{-r}^{-,i-\ell}\psi_r^{+,j-\ell}, & \text{if } \ell < i,j \leq n. \end{cases}$$

For $n = 2\ell + 1$ and $\mathrm{G}_n = \mathrm{O}_{2\ell+1}$, we put e_{ij} to be the same as in (5.1) except we assume that $\psi_r^{\pm,n} = \phi_r^\pm$. Then one can check that (5.1) and (5.2) define an action of GL_n on $\mathcal{F}^{\frac{n}{2}}$ such that its restriction to G_n coincides with the action of G_n in Proposition 2.1. Recall that there exists an action of $\mathfrak{l}_\infty \times \mathrm{GL}_n$ on $\mathcal{F}^{\frac{n}{2}}$ such that

$$(5.3) \qquad \mathcal{F}^{\frac{n}{2}} \cong \bigoplus_{\lambda \in \mathscr{P}} L(\mathfrak{l}_\infty, \lambda') \otimes V_{\mathrm{GL}_n}^\lambda.$$

On the other hand, by Proposition 2.1 we have

$$\mathcal{F}^{\frac{n}{2}} \cong \bigoplus_{\mu \in \mathcal{P}(\mathrm{G}_n)} L(\mathfrak{g}_\infty, \Lambda^\mathfrak{g}(\mu)) \otimes V_{\mathrm{G}_n}^\mu$$

$$\cong \bigoplus_{\mu \in \mathcal{P}(\mathrm{G}_n)} \left(\bigoplus_{\lambda \in \mathscr{P}} L(\mathfrak{l}_\infty, \lambda')^{\oplus c_\lambda^\mu(\mathfrak{g})}\right) \otimes V_{\mathrm{G}_n}^\mu$$

$$\cong \bigoplus_{\lambda \in \mathscr{P}} L(\mathfrak{l}_\infty, \lambda') \otimes \left(\bigoplus_{\mu \in \mathcal{P}(\mathrm{G}_n)} (V_{\mathrm{G}_n}^\mu)^{\oplus c_\lambda^\mu(\mathfrak{g})}\right).$$

Combining with (5.3), we have

$$\left[V_{\mathrm{GL}_n}^\lambda : V_{\mathrm{G}_n}^\mu\right] = c_\lambda^\mu(\mathfrak{g}).$$

The second equality follows from Corollary 4.9. $\square$

Example 5.4. Let $\lambda = (2,2) \in \mathscr{P}$. The decomposition number $[V_{\mathrm{GL}_n}^{\lambda} : V_{\mathrm{O}_n}^{\mu}]$ for each $\mu \in \mathcal{P}(\mathrm{O}_n)$ is equal to the number of $\mathbf{T} \in \mathbf{T}^{\mathfrak{o}}(\mu, n)$ such that $\mathbf{T} \equiv_A H_{\lambda'}$. Using this fact, it is straightforward to check that

$$[V_{\mathrm{GL}_n}^{\lambda} : V_{\mathrm{O}_n}^{\mu}] \neq 0 \iff \mu \in \begin{cases} \{ (2,2,0,\ldots,0), (2,0,\ldots,0), (0,\ldots,0) \}, & \text{if } n \geq 4, \\ \{ (2,0,0), (0,0,0) \}, & \text{if } n = 3, \\ \{ (0,0) \}, & \text{if } n = 2. \end{cases}$$

For $n = 4$, we have $\mathbf{T}^{\mathfrak{o}}(\mu, 4) \subset \widehat{\mathbf{T}}^{\mathfrak{o}}(\mu, 4) = \mathbf{T}^{\mathfrak{o}}(\mu_1) \times \mathbf{T}^{\mathfrak{o}}(\mu_2)$ for $\mu \in \mathcal{P}(\mathrm{O}_4)$ with $[V_{\mathrm{GL}_4}^{\lambda} : V_{\mathrm{O}_4}^{\mu}] \neq 0$ and the tableaux $\mathbf{T} = (T_1, T_2) \in \mathbf{T}^{\mathfrak{o}}(\mu, 4)$ such that $\mathbf{T} \equiv_A H_{\lambda'}$ are as follows:

$$\left(\begin{array}{c} \boxed{\begin{array}{c} 1 \\ 2 \end{array}} \end{array} , \begin{array}{c} \boxed{\begin{array}{c} 1 \\ 2 \end{array}} \end{array} \right) \in \mathbf{T}^{\mathfrak{o}}((2,2,0,0), 4) \qquad \left(\begin{array}{c} \boxed{\begin{array}{c} 1 \\ 2 \end{array}} \end{array} , \begin{array}{c} \boxed{\begin{array}{c} 1 \\ 2 \end{array}} \end{array} \right) \in \mathbf{T}^{\mathfrak{o}}((2,0,0,0), 4)$$

$$\left(\quad , \begin{array}{c} \boxed{\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array}} \end{array} \right) \in \mathbf{T}^{\mathfrak{o}}((0,0,0,0), 4)$$

where the dashed line denotes the line separating the body and the tail of T_i's for $i = 1, 2$. Hence we have

$$V_{\mathrm{GL}_4}^{(2,2)} = V_{\mathrm{O}_4}^{(2,2,0,0)} \oplus V_{\mathrm{O}_4}^{(2,0,0,0)} \oplus V_{\mathrm{O}_4}^{(0,0,0,0)}.$$

It is not difficult to see that the same decomposition of $V_{\mathrm{GL}_n}^{(2,2)}$ as above holds for $n \geq 4$, which is equal to the stable formula in Theorem 5.3.

Let us consider the decomposition of $V_{\mathrm{GL}_n}^{\lambda}$ $(n = 2, 3)$, where λ is outside the stable range. For $n = 3$, we have $\mathbf{T}^{\mathfrak{o}}(\mu, 3) \subset \widehat{\mathbf{T}}^{\mathfrak{o}}(\mu, 3) = \mathbf{T}^{\mathfrak{o}}(\mu_1) \times \mathbf{T}^{\mathrm{sp}+}$ for $\mu \in \mathcal{P}(\mathrm{O}_3)$ with $[V_{\mathrm{GL}_3}^{\lambda} : V_{\mathrm{O}_3}^{\mu}] \neq 0$, and the tableaux $\mathbf{T} = (T_1, T_2) \in \mathbf{T}^{\mathfrak{o}}(\mu, 3)$ such that $\mathbf{T} \equiv_A H_{\lambda'}$ are

$$\left(\begin{array}{c} \boxed{\begin{array}{c} 1 \\ 2 \end{array}} \end{array} , \begin{array}{c} \boxed{\begin{array}{c} 1 \\ 2 \end{array}} \end{array} \right) \in \mathbf{T}^{\mathfrak{o}}((2,0,0), 3) \qquad \left(\begin{array}{c} \boxed{\begin{array}{c} 1 \\ 2 \end{array}} \end{array} , \begin{array}{c} \boxed{\begin{array}{c} 1 \\ 2 \end{array}} \end{array} \right) \in \mathbf{T}^{\mathfrak{o}}((0,0,0), 3).$$

This implies that

$$V_{\mathrm{GL}_3}^{(2,2)} = V_{\mathrm{O}_3}^{(2,0,0)} \oplus V_{\mathrm{O}_3}^{(0,0,0)}.$$

For $n = 2$,

$$\boxed{\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array}} \in \mathbf{T}^{\mathfrak{o}}((0,0), 2)$$

is the unique tableau in $\mathbf{T}^{\mathfrak{o}}((0,0), 2)$ such that $\mathbf{T} \equiv_A H_{\lambda'}$, and hence

$$V_{\mathrm{GL}_2}^{(2,2)} = V_{\mathrm{O}_2}^{(0,0)}.$$

Example 5.5. Let $\lambda = (3,1) \in \mathscr{P}$. As in Example 5.4, we can check that

$$[V_{\mathrm{GL}_n}^{\lambda} : V_{\mathrm{O}_n}^{\mu}] \neq 0 \iff \mu \in \begin{cases} \{ (3,1,0,\ldots,0), (2,0,\ldots,0), (1,1,0,\ldots,0) \}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \geq 3, \\ \{ (2,0), (1,1) \}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n = 2. \end{cases}$$

For $n = 4$, we have $\mathbf{T}^{\mathbf{0}}(\mu, 4) \subset \widehat{\mathbf{T}}^{\mathbf{0}}(\mu, 4) = \mathbf{T}^{\mathbf{0}}(\mu_1) \times \mathbf{T}^{\mathbf{0}}(\mu_2)$ for $\mu \in \mathcal{P}(\mathrm{O}_4)$ with $[V_{\mathrm{GL}_4}^{\lambda} : V_{\mathrm{O}_4}^{\mu}] \neq 0$, and the tableaux $\mathbf{T} = (T_1, T_2) \in \mathbf{T}^{\mathbf{0}}(\mu, 4)$ such that $\mathbf{T} \equiv_A H_{\lambda'}$ are as follows:

Hence we have

$$V_{\mathrm{GL}_4}^{(3,1)} = V_{\mathrm{O}_4}^{(3,1,0,0)} \oplus V_{\mathrm{O}_4}^{(2,0,0,0)} \oplus V_{\mathrm{O}_4}^{(1,1,0,0)}.$$

The above decomposition also holds for $n \geq 4$.

For $n = 3$, we have $\mathbf{T}^{\mathbf{0}}(\mu, 3) \subset \widehat{\mathbf{T}}^{\mathbf{0}}(\mu, 3) = \mathbf{T}^{\mathbf{0}}(\mu_1) \times \mathbf{T}^{\mathrm{sp}\pm}$ for $\mu \in \mathcal{P}(\mathrm{O}_3)$ with $[V_{\mathrm{GL}_3}^{\lambda} : V_{\mathrm{O}_3}^{\mu}] \neq 0$, and the tableaux $\mathbf{T} = (T_1, T_2) \in \mathbf{T}^{\mathbf{0}}(\mu, 3)$ such that $\mathbf{T} \equiv_A H_{\lambda'}$ are as follows:

Hence we have

$$V_{\mathrm{GL}_3}^{(3,1)} = V_{\mathrm{O}_3}^{(3,1,0)} \oplus V_{\mathrm{O}_3}^{(2,0,0)} \oplus V_{\mathrm{O}_3}^{(1,1,0)}.$$

Finally, for $n = 2$, the tableaux $T \in \mathbf{T}^{\mathbf{0}}(\mu, 2)$ such that $T \equiv_A H_{\lambda'}$ are given by

and hence

$$V_{\mathrm{GL}_2}^{(3,1)} = V_{\mathrm{O}_2}^{(2,0)} \oplus V_{\mathrm{O}_2}^{(1,1)}.$$

Remark 5.6. Let $\lambda^{(1)}, \ldots, \lambda^{(r)} \in \mathscr{P}$ with $\ell(\lambda^{(i)}) \leq n$ for $1 \leq i \leq r$. For $\lambda \in \mathscr{P}$ with $\ell(\lambda) \leq n$, let

$$c_{\lambda^{(1)} \cdots \lambda^{(r)}}^{\lambda} = \left[V_{\mathrm{GL}_n}^{\lambda^{(1)}} \otimes \cdots \otimes V_{\mathrm{GL}_n}^{\lambda^{(r)}} : V_{\mathrm{GL}_n}^{\lambda} \right].$$

Then for $\mu \in \mathcal{P}(\mathrm{G}_n)$, we have

$$(5.4) \qquad \left[V_{\mathrm{GL}_n}^{\lambda^{(1)}} \otimes \cdots \otimes V_{\mathrm{GL}_n}^{\lambda^{(r)}} : V_{\mathrm{G}_n}^\mu \right] = \sum_{\lambda \in \mathscr{P}} \left[V_{\mathrm{GL}_n}^\lambda : V_{\mathrm{G}_n}^\mu \right] c_{\lambda^{(1)} \cdots \lambda^{(r)}}^\lambda.$$

In particular, if $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| \leq \frac{n}{2}$, where $|\lambda^{(i)}|$ denotes the sum of non-zero parts of $\lambda^{(i)}$, then $\ell(\lambda) \leq \frac{n}{2}$, and hence (5.4) becomes

$$(5.5) \qquad \left[V_{\mathrm{GL}_n}^{\lambda^{(1)}} \otimes \cdots \otimes V_{\mathrm{GL}_n}^{\lambda^{(r)}} : V_{\mathrm{G}_n}^\mu \right] = \sum_{\lambda \in \mathscr{P}} \sum_{\delta \in \mathscr{P}_{\mathfrak{g}}} c_{\delta' \mu}^\lambda c_{\lambda^{(1)} \cdots \lambda^{(r)}}^\lambda.$$

By using Theorem 5.3, we obtain a new combinatorial formula for (5.4) or a combinatorial extension of (5.5) to arbitrary $\lambda^{(i)}$'s. We should remark that a q-analogue of (5.4) is introduced in [28], and its stable limit is closely related to the energy function on a tensor product of finite affine crystals (see for example [30]). It would be interesting to find a combinatorial interpretation of the q-analogue of (5.4) or (5.5) in terms of spinor model.

6. Character of holomorphic discrete series

6.1. **Superization of $\mathbf{T}^{\mathfrak{g}}(\lambda, n)$.** Let $\mathcal{A}$ be a linearly ordered countable set with a $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$. For a skew Young diagram λ/μ, we define the notion of $\mathcal{A}$-semistandard tableaux of shape λ/μ as usual, that is, (1) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (2) the entries in $\mathcal{A}_0$ (resp. $\mathcal{A}_1$) are strictly increasing in each column (resp. row). We denote by $SST_{\mathcal{A}}(\lambda/\mu)$ the set of $\mathcal{A}$-semistandard tableaux of shape λ/μ. Let $x_{\mathcal{A}} = \{ x_a \mid a \in \mathcal{A} \}$ be the set of formal commuting variables indexed by $\mathcal{A}$. For $\lambda \in \mathscr{P}$, let $s_\lambda(x_{\mathcal{A}}) = \sum_T x_{\mathcal{A}}^T$ be the super Schur function corresponding to λ, where the sum is over $T \in SST_{\mathcal{A}}(\lambda)$ and $x_{\mathcal{A}}^T = \prod_a x_a^{m_a}$ with m_a the number of occurrences of a in T.

For $\lambda \in \mathcal{P}(\mathrm{G}_n)$, one can define $\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, n)$ by replacing ($\mathbb{N}$-)semistandard tableaux with $\mathcal{A}$-semistandard tableaux in Definition 3.5. (See [26, Section 6] and [27, Section 3] for more details, where $\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, n)$ is denoted by $\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, n/\epsilon)$.) When $\mathcal{A}$ is (a subset of) $\mathbb{N}$, we assume that $\mathcal{A}_0 = \mathcal{A}$ and it is equipped with the usual linear ordering on $\mathbb{N}$. In particular, we have $\mathbf{T}_{\mathbb{N}}^{\mathfrak{g}}(\lambda, n) = \mathbf{T}^{\mathfrak{g}}(\lambda, n)$.

We define the character of $\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, n)$ by

$$\mathrm{ch}\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, n) = z^{n/\epsilon} \sum_{\mathbf{T}} \prod_{i \geq 1} x_{\mathcal{A}}^{T_i},$$

where z is another formal variable and the sum is over $\mathbf{T} = (T_1, T_2, \ldots) \in \mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, n)$. It is shown that under certain choices of $\mathcal{A}$ (more precisely, for $\mathcal{A}$ with $0 < |\mathcal{A}_0| < \infty$), $\{ \mathrm{ch}\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, n) \mid \lambda \in \bigsqcup_n \mathcal{P}(\mathrm{G}_n) \}$ gives the characters of a family of irreducible representations of an ortho-symplectic Lie superalgebra in a semisimple tensor category [26, Theorem 7.6] and [27, Theorem 4.6].

6.2. **Unitarizable representations.** From now on, we assume that $2 \leq k \leq \infty$ and $(\mathfrak{g}_k, \mathrm{G}_n)$ denotes one of the following pairs:

$$(6.1) \qquad (\mathfrak{d}_k, \mathrm{Sp}_n), \quad (\mathfrak{b}_k^\bullet, \mathrm{Pin}_n), \quad (\mathfrak{b}_k^\bullet, \mathrm{Spin}_n), \quad (\mathfrak{c}_k, \mathrm{O}_n).$$

Put

$$\mathcal{P}(\mathrm{G}_n)_k = \{ \lambda \mid \lambda \in \mathcal{P}(\mathrm{G}_n), \ \ell(\lambda) \leq k \}.$$

For $\lambda \in \mathscr{P}(G_n)_k$, we define a weight $\Lambda_-^{\mathfrak{g}}(\lambda)$ for $\mathfrak{g}_k$ by

$$\Lambda_-^{\mathfrak{g}}(\lambda) = -\frac{n}{\epsilon}\Lambda_0^{\mathfrak{g}} + \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots.$$

Note that $\Lambda_0^{\mathfrak{g}} = -(\epsilon/2)(\epsilon_1 + \cdots + \epsilon_k)$ for $k < \infty$. We denote by $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ the irreducible highest weight $\mathfrak{g}_k$-module with highest weight $\Lambda_-^{\mathfrak{g}}(\lambda)$. Let $\mathfrak{h}_k$ be the Cartan subalgebra of $\mathfrak{g}_k$. We define the character of $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ by $\mathrm{ch}L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda)) = \sum_\mu \dim L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))_\mu e^\mu$, where $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))_\mu$ is the μ-weight space and e^μ is the basis element of the group algebra $\mathbb{Q}[\mathfrak{h}_k^*]$.

It is well-known that $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ for $\lambda \in \mathscr{P}(G_n)_k$ form a family of infinite-dimensional unitarizable representations which appear in the decomposition of the Segal-Shale representation of the metaplectic group with respect to the reductive dual pair $(\mathfrak{g}_k, G_n)$ (see [12] and many related references therein for $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$ and $k < \infty$, and also [4, 40] for $\mathfrak{g} = \mathfrak{b}^\bullet$ or $k = \infty$). For $\lambda \in \mathscr{P}(G_n)_k$, $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ is not an integrable $\mathfrak{g}_k$-module, but it decomposes into a sum of integrable highest weight $\mathfrak{l}_k$-modules. For $\mu \in \mathscr{P}(G_m)_k$ and $\nu \in \mathscr{P}(G_n)_k$, the tensor product $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\mu)) \otimes L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\nu))$ is completely reducible and is a direct sum of $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$'s for $\lambda \in \mathscr{P}(G_{m+n})_k$.

Let us assume that $(\mathfrak{g}_k^\vee, \mathfrak{g}_k)$ denotes one of the pairs

$$(6.2) \qquad\qquad (\mathfrak{c}_k, \mathfrak{d}_k), \ (\mathfrak{b}_k, \mathfrak{b}_k^\bullet), \ (\mathfrak{d}_k, \mathfrak{c}_k)$$

such that $(\mathfrak{g}_\infty^\vee, G_n)$ and $(\mathfrak{g}_\infty, G_n)$ are the dual pairs with the same G_n in (2.1) and (6.1), respectively. It was recently observed from the theory of super duality [5] that there is an equivalence between two parabolic BGG categories of modules over $\mathfrak{g}_\infty^\vee$ and $\mathfrak{g}_\infty$ with respect to $\mathfrak{l}_\infty$, which sends $L(\mathfrak{g}_\infty^\vee, \Lambda^{\mathfrak{g}^\vee}(\lambda))$ to $L(\mathfrak{g}_\infty, \Lambda_-^{\mathfrak{g}}(\lambda))$ for $\lambda \in \bigsqcup_n \mathscr{P}(G_n)$.

Let $\mathbb{N}' = \{p' | p \in \mathbb{N}\}$ be the set of primed positive integers of odd degree with a linear ordering $<$ where $p' < q'$ if and only if $p < q$. For $\lambda \in \mathscr{P}(G_n)_k$, we put

$$\mathbb{T}_k^{\mathfrak{g}}(\lambda, n) = \begin{cases} \mathbf{T}_{\mathbb{N}'}^{\mathfrak{g}^\vee}(\lambda, n), & \text{if } k = \infty, \\ \mathbf{T}_{[k]'}^{\mathfrak{g}^\vee}(\lambda, n), & \text{if } k < \infty, \end{cases}$$

where $[k]' = \{1', 2', \ldots, k'\} \subset \mathbb{N}'$. Now, we are in a position to give a combinatorial character formula for $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$.

Theorem 6.1. *For $\lambda \in \mathscr{P}(G_n)_k$, we have*

$$\mathrm{ch}L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda)) = \mathrm{ch}\mathbb{T}_k^{\mathfrak{g}}(\lambda, n),$$

where $z = e^{-\Lambda_0^{\mathfrak{g}}}$ and $x_{i'} = e^{\epsilon_i}$ for $i \geq 1$.

Proof. The proof is almost the same as in [26, Theorem 7.6] and [27, Theorem 4.6]. First consider the case when $k = \infty$. Let $(\mathfrak{g}_k^\vee, \mathfrak{g}_k)$ be as in (6.2). It follows from [5, Theorem 4.6] (see also [5, Section 2.3]) that if

$$(6.3) \qquad\qquad \mathrm{ch}L(\mathfrak{g}_\infty^\vee, \Lambda^{\mathfrak{g}^\vee}(\lambda)) = z^{-n/\epsilon}\sum_{\mu \in \mathscr{P}} K_{\lambda\mu}s_\mu(x_{\mathbb{N}})$$

for some non-negative integers $K_{\lambda\mu}$, then

$$(6.4) \qquad \mathrm{ch}L(\mathfrak{g}_\infty, \Lambda_-^{\mathfrak{g}}(\lambda)) = z^{n/\epsilon}\sum_{\mu \in \mathscr{P}} K_{\lambda\mu}s_{\mu'}(x_{\mathbb{N}}) = z^{n/\epsilon}\sum_{\mu \in \mathscr{P}} K_{\lambda\mu}s_\mu(x_{\mathbb{N}'}).$$

Hence by [26, Theorem 6.12, Corollary 6.13] and [27, Theorem 3.9], we have

$$\mathrm{ch}L(\mathfrak{g}_\infty, \Lambda_-^{\mathfrak{g}}(\lambda)) = \mathrm{ch}\mathbf{T}_{\mathbb{N}'}^{\mathfrak{g}^\vee}(\lambda, n) = \mathrm{ch}\mathbb{T}_\infty^{\mathfrak{g}}(\lambda, n).$$

Since $\mathrm{ch}L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ for $k \in \mathbb{N}$ is obtained by specializing $x_{i'} = 0$ for $i > k$ in $\mathrm{ch}L(\mathfrak{g}_\infty, \Lambda_-^{\mathfrak{g}}(\lambda))$ [5, Lemma 3.2], we have $\mathrm{ch}L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda)) = \mathrm{ch}\mathbb{T}_k^{\mathfrak{g}}(\lambda, n)$. □

As an immediate corollary, we have the following tensor product and branching multiplicity formula.

Corollary 6.2.

(1) *For* $\mu \in \mathscr{P}(\mathrm{G}_m)_k$, $\nu \in \mathscr{P}(\mathrm{G}_n)_k$, *and* $\lambda \in \mathscr{P}(\mathrm{G}_{m+n})_k$, *we have*

$$c_{\mu\nu}^{\lambda}(\mathfrak{g}^\vee) = \left[L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\mu)) \otimes L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\nu)) : L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda)) \right].$$

(2) *For* $\mu \in \mathscr{P}(\mathrm{G}_n)_k$ *and* $\lambda \in \mathscr{P}$ *with* $\ell(\lambda) \leq k$,

$$c_{\lambda}^{\mu}(\mathfrak{g}^\vee) = [L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\mu)) : L(\mathfrak{l}_k, \lambda)].$$

Proof. Let ω denote the involution on the ring of symmetric functions sending $s_\lambda(x_\mathbb{N})$ to $s_{\lambda'}(x_\mathbb{N})$ for $\lambda \in \mathscr{P}$. We see from (6.3) and (6.4) that $z^{-n/\epsilon}\mathrm{ch}L(\mathfrak{g}_\infty, \Lambda_-^{\mathfrak{g}}(\lambda))$ $= z^{n/\epsilon}\omega\left(\mathrm{ch}L(\mathfrak{g}_\infty^\vee, \Lambda_-^{\mathfrak{g}^\vee}(\lambda))\right)$. Hence (1) and (2) for $k = \infty$ follow from Theorem 6.1 and Propositions 4.4 and 4.7. The proof for the case when $k < \infty$ is obtained by specializing $x_{i'} = 0$ for $i > k$ in the characters. □

6.3. **Characters in a stable range.** Fix $k < \infty$. Let $\mathfrak{g}_k^+$ be the Borel subalgebra of $\mathfrak{g}_k$ with respect to the simple roots in Section 2.1, and let $\mathfrak{p}_k = \mathfrak{l}_k + \mathfrak{g}_k^+$ be the parabolic subalgebra associated to $\mathfrak{l}_k$. For $\lambda \in \mathscr{P}(\mathrm{G}_n)_k$, let $V(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda)) = \mathrm{Ind}_{\mathfrak{p}_k}^{\mathfrak{g}_k} L(\mathfrak{l}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ be the generalized Verma module with highest weight $\Lambda_-^{\mathfrak{g}}(\lambda)$ whose maximal irreducible quotient is $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$.

In this subsection, we prove that $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ is equal to $V(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$, and hence its character has a nice product form when λ is in a certain stable range (these modules are referred to as *holomorphic discrete series*). It is an already well-known result when $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$, and plays a crucial role in establishing various families of stable branching rules for classical groups in a unified way [12]. Nevertheless the proof here is given in a completely different and purely combinatorial way by decomposing $\mathbb{T}_k^{\mathfrak{g}}(\lambda, n)$ into $\mathfrak{l}_k$-crystals. Moreover the result for $\mathfrak{g} = \mathfrak{b}^\bullet$ obtained here seems to be new.

We put $[k] = \{1, \ldots, k\} \subset \mathbb{N}$ as a $\mathbb{Z}_2$-graded linearly ordered set, which can be viewed as the crystal of the natural representation of $\mathfrak{l}_k$. As in Section 2.4, one can regard $SST_{[k]}(\lambda/\mu)$ as an $\mathfrak{l}_k$-crystal for a skew Young diagram λ/μ. For $T \in SST_\mathcal{A}(\lambda/\mu)$, we define T' to be the tableau of shape λ'/μ' obtained by flipping T with respect to the main diagonal and replacing i (resp. i') with i' (resp. i) when $\mathcal{A}$ is $[k]$ (resp. $[k]'$).

Let $\lambda \in \mathscr{P}(\mathrm{G}_n)_k$ be given. We regard $\mathbb{T}_k^{\mathfrak{g}}(\lambda, n)$ as an $\mathfrak{l}_k$-crystal by identifying $\mathbb{T} = (T_1, \ldots, T_r) \in \mathbb{T}_k^{\mathfrak{g}}(\lambda, n)$ with $(T_1)' \otimes \cdots \otimes (T_r)'$. For $\mathbb{T} = (T_1, \ldots, T_r) \in \mathbb{T}_k^{\mathfrak{g}}(\lambda, n)$, we put $w(\mathbb{T}) = w(T_r) \ldots w(T_1)$, and define $L(\mathbb{T})$ to be the maximal length of a strictly decreasing subword of $w(\mathbb{T})$. It is not difficult to see that if $\mathbb{T} \equiv_A S$ for some $S \in SST_{[k]}(\mu)$, then $L(\mathbb{T}) = \ell(\mu)$.

Lemma 6.3. *We have* $k \geq L(\mathbb{T}) \geq \ell(\lambda)$.

Proof. Note that $\mathbb{T}_k^{\mathfrak{g}}(\lambda, n)$ can be defined in the same way as in $\mathbf{T}^{\mathfrak{g}}(\lambda, n)$ except that we replace all $\leq$ with $<$ in Definition 3.2 since the elements in $[k]'$ are of odd degree. Then Lemma 3.4 also holds for $(T, S) \in \mathbb{T}_k^{\mathfrak{g}}(a) \times \mathbb{T}_k^{\mathfrak{g}}(a')$ $(a \geq a')$ such that $T \prec S$, where $\leq$ is also replaced with $<$. This implies that there exists a strictly decreasing subword $T_{\ell(\lambda)}^{\texttt{tail}}(1) \cdots T_1^{\texttt{tail}}(1)$, and hence $L(\mathbb{T}) \geq \ell(\lambda)$. On the other hand, we have $k \geq L(\mathbb{T})$ since the entries of $\mathbb{T}$ are from $[k]'$. $\qquad\square$

Let us define $\mathbb{T}^{\texttt{tail}}$ and $\mathbb{T}^{\texttt{body}}$ in the same way as in (4.2).

Lemma 6.4. *If $L(\mathbb{T}) \leq \frac{n}{2}$, then we have*
  (1) $\mathbb{T}^{\texttt{tail}} \in SST_{[k]'}(\lambda')$ *and* $\mathbb{T}^{\texttt{body}} \in SST_{[k]'}(\mu^{\pi})$ *for some* $\mu \in \mathscr{P}_{\mathfrak{g}^{\vee}}$,
  (2) $\mathbb{T} \equiv_A (\mathbb{T}^{\texttt{tail}})' \otimes (\mathbb{T}^{\texttt{body}})'$.

Proof. The proof is almost parallel to that of Lemma 4.3. $\qquad\square$

Now, we have the following characterization of $\mathbb{T}_k^{\mathfrak{g}}(\lambda, n)$ as an $\mathfrak{l}_k$-crystal in a stable range, which is the main result in this section.

Theorem 6.5. *Let $\lambda \in \mathcal{P}(G_n)_k$ be given. If $n \geq 2k$, then we have an isomorphism of $\mathfrak{l}_k$-crystals*

$$\mathbb{T}_k^{\mathfrak{g}}(\lambda, n) \longrightarrow \bigsqcup_{\substack{\mu \in \mathscr{P}_{\mathfrak{g}} \\ \ell(\mu) \leq k}} SST_{[k]}(\lambda) \otimes SST_{[k]}(\mu^{\pi}),$$

which maps $\mathbb{T}$ to $(\mathbb{T}^{\texttt{tail}})' \otimes (\mathbb{T}^{\texttt{body}})'$.

Proof. Let $\mathbb{B}$ denote the right-hand side of the above isomorphism. Since $2k \leq n$, we have $L(\mathbb{T}) \leq k \leq \frac{n}{2}$ for all $\mathbb{T} \in \mathbb{T}_k^{\mathfrak{g}}(\lambda, n)$. By Lemma 6.4, the map $\mathbb{T} \mapsto (\mathbb{T}^{\texttt{tail}})' \otimes (\mathbb{T}^{\texttt{body}})'$ is a well-defined morphism of $\mathfrak{l}_k$-crystals from $\mathbb{T}_k^{\mathfrak{g}}(\lambda, n)$ to $\mathbb{B}$. The bijectiveness of the map can be proved by similar arguments as in Theorem 4.5. $\quad\square$

We recover the following stability of $\mathrm{ch}L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ in a purely combinatorial way (cf. [12, Theorem 3.2] and references therein).

Corollary 6.6. *Let $\lambda \in \mathcal{P}(G_n)_k$ be given. If $n \geq 2k$, then we have*

$$\mathrm{ch}L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda)) = z^{n/\epsilon} \Delta^{\mathfrak{g}}(x_{[k]}) s_\lambda(x_{[k]}),$$

where

$$\Delta^{\mathfrak{g}}(x_{[k]}) = \begin{cases} \prod_{1 \leq i \leq k}(1 - x_i)^{-1} \prod_{1 \leq i < j \leq k}(1 - x_i x_j)^{-1}, & \text{if } \mathfrak{g} = \mathfrak{b}^{\bullet}, \\ \prod_{1 \leq i \leq k}(1 - x_i^2)^{-1} \prod_{1 \leq i < j \leq k}(1 - x_i x_j)^{-1}, & \text{if } \mathfrak{g} = \mathfrak{c}, \\ \prod_{1 \leq i < j \leq k}(1 - x_i x_j)^{-1}, & \text{if } \mathfrak{g} = \mathfrak{d}. \end{cases}$$

In particular, we have $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda)) = V(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$.

Proof. It follows from Theorem 6.5 and the well-known identity (cf. [34])

$$\Delta^{\mathfrak{g}}(x_{[k]}) = \sum_{\substack{\mu \in \mathscr{P}_{\mathfrak{g}} \\ \ell(\mu) \leq k}} s_\mu(x_{[k]})$$

that $\mathrm{ch}L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda)) = z^{n/\epsilon} \Delta^{\mathfrak{g}}(x_{[k]}) s_\lambda(x_{[k]})$. Let $\mathfrak{u}_k^-$ be the opposite nilradical of $\mathfrak{p}_k$, and let $U(\mathfrak{u}_k^-)$ be its universal enveloping algebra. We have $\mathrm{ch}U(\mathfrak{u}_k^-) = \Delta^{\mathfrak{g}}(x_{[k]})$, and hence $\mathrm{ch}V(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda)) = z^{n/\epsilon} \Delta^{\mathfrak{g}}(x_{[k]}) s_\lambda(x_{[k]})$. Since $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ is a maximal quotient of $V(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$, we have $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda)) = V(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$. $\qquad\square$

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Remark 6.7. One can deduce Corollary 6.6 from Theorem 4.8, Corollary 6.2 (2), and the Littlewood identity (cf. [34]). Recall that the stability of $\mathrm{ch}L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ in Corollary 6.6 plays a crucial role in obtaining the stable tensor product multiplicities for $L(\mathfrak{g}_k, \Lambda_-^{\mathfrak{g}}(\lambda))$ in [12]. We can also recover this stable tensor product multiplicity directly from Corollaries 4.6 and 6.2(1).

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