THE APOLLONIAN STRUCTURE OF BIANCHI GROUPS

KATHERINE E. STANGE

ABSTRACT. We study the orbit of $\widehat{\mathbb{R}}$ under the Möbius action of the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_K)$ on $\widehat{\mathbb{C}}$, where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field K. The orbit $\mathcal{S}_K$, called a Schmidt arrangement, is a geometric realisation, as an intricate circle packing, of the arithmetic of K. We give a simple geometric characterisation of certain subsets of $\mathcal{S}_K$ generalizing Apollonian circle packings, and show that $\mathcal{S}_K$, considered with orientations, is a disjoint union of all primitive integral such K-Apollonian packings. These packings are described by a new class of thin groups of arithmetic interest called K-Apollonian groups. We make a conjecture on the curvatures of these packings, generalizing the local-to-global conjecture for Apollonian circle packings.

1. INTRODUCTION

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. The Bianchi group $\mathrm{PSL}_2(\mathcal{O}_K)$ acts on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ via Möbius transformations. Its action permutes the collection of circles of $\widehat{\mathbb{C}}$ (where straight lines are considered circles through ∞). The orbit of the extended real line $\widehat{\mathbb{R}}$ under $\mathrm{PSL}_2(\mathcal{O}_K)$ forms a delicately intertwined collection of circles, some examples of which are shown in Figures 1, 4, and 7. This is the *Schmidt arrangement* of K, denoted $\mathcal{S}_K$, named, in [42], in honour of the work of Asmus Schmidt generalizing continued fractions to the complex setting [33–37]. The individual images of $\widehat{\mathbb{R}}$ are called K-*Bianchi circles*.

The Schmidt arrangement naturally exhibits various aspects of the number theory of $\mathcal{O}_K$, and this was the topic of study of an earlier paper of the author [42]. Most vividly, $\mathcal{S}_K$ is connected if and only if $\mathcal{O}_K$ is Euclidean.

The author's interest arose from $\mathcal{S}_{\mathbb{Q}(i)}$ (Figure 1), which made its appearance in [14] as an *Apollonian superpacking* in the study of Apollonian circle packings (apparently the authors of [14] were unaware of the relation to Schmidt's work; the picture is the same but the definition is different). This paper gives a new description of the relationship between $\mathcal{S}_{\mathbb{Q}(i)}$ and Apollonian circle packings, and introduces a family of new circle packings arising from the Schmidt arrangements of imaginary quadratic fields in an analogous way. These packings, which share many of the remarkable arithmetic properties of Apollonian circle packings, are isolated
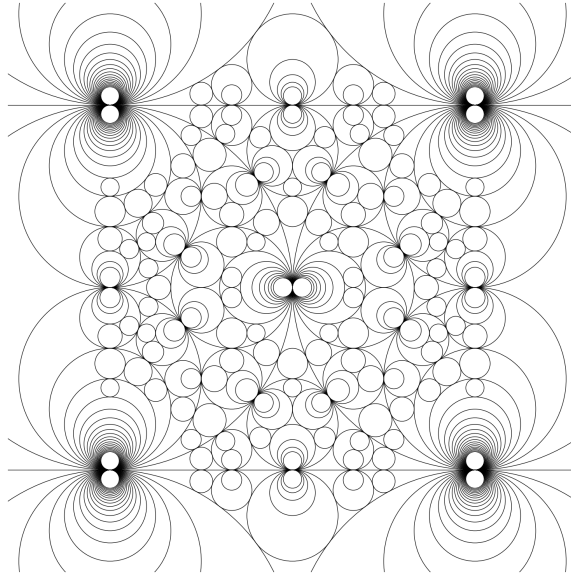
FIGURE 1. The Schmidt arrangement of $\mathcal{S}_{\mathbb{Q}(i)}$. The image includes those circles of curvature ≤ 20 intersecting the closure of the fundamental parallelogram of the ring of integers.
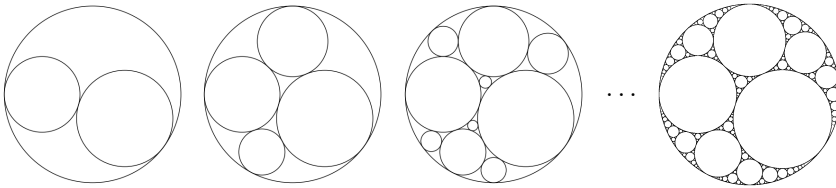

FIGURE 2. The iteration process generating an Apollonian circle packing.

by a simple geometric characterisation, and give rise to associated thin subgroups of $\mathrm{PSL}_2(\mathcal{O}_K)$; we refer to this as the Apollonian structure of $\mathrm{PSL}_2(\mathcal{O}_K)$.

A *Descartes quadruple* of circles in $\widehat{\mathbb{C}}$ is a quadruple such that every pair of circles is tangent. The curvatures (inverse radii) a, b, c, d of such a quadruple satisfy the *Descartes quadratic relation*, famously stated by Descartes in a 1643 letter to Princess Elizabeth of Bohemia:

$$(1) \qquad\qquad (a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2).$$

The study of Descartes quadruples has a lively history and there are several excellent expositions; see for example [10, 32]. If one begins with three mutually tangent circles with curvatures a, b, c, then there are exactly two circles, called *Soddy circles*, which complete the triple to a Descartes quadruple, with curvatures d and d' solving (1); these satisfy

$$d + d' = 2(a + b + c).$$

Adding these two new circles to our original triple, we have a set of five circles. For each mutually tangent triple in the set, we can again find two Soddy circles

which complete it to a quadruple. Any of these which are not already included in our collection are now added, thereby expanding the set of circles. If we continue this process ad infinitum, we obtain an infinite collection of circles which is called an *Apollonian circle packing*. See Figure 2. The remarkable fact for the number theorist is that if the first four circles had integer curvatures, then the generation rule implies that every circle in the packing has an integral curvature. For an example, see Figure 3. The natural question is to determine which integers occur as curvatures.

As it happens, $\mathcal{S}_{\mathbb{Q}(i)}$ is the union of all possible primitive, integral Apollonian circle packings, considered up to suitable symmetries and an appropriate scaling (integral refers to integer curvatures; primitive means they share no common factor). This is a result of [14, Theorem 6.3] (using the definition of the Schmidt arrangement as an Apollonian superpacking), further studied in [41]. In particular, the curvatures of $\mathbb{Q}(i)$-Bianchi circles are all integral.

Apollonian circle packings have generated great interest recently, in large part because of their connection to thin groups. The central conjecture is a local-global principle for the curvatures of a packing.

Conjecture 1.1 (Graham, Lagarias, Mallows, Wilks, and Yan [12], Fuchs and Sanden [11])**.** *Let $\mathcal{P}$ be a primitive integral Apollonian circle packing, and let S be the set of residue classes modulo 24 of the curvatures in $\mathcal{P}$. Then all sufficiently large integers with residues in S occur as curvatures in $\mathcal{P}$.*

Significant progress has been made toward this conjecture, most notably that it holds for a set of integers of density one [3] (positive density was first shown in [2]). For an excellent overview and further references, see [10]; see also the series of papers [12–14] which are central to the field, and the exposition [32]. For the related question of the multi-set of integral curvatures appearing in a packing, a gateway to the literature is the survey [29].

These results depend on an analysis of the Apollonian group, a matrix group which describes the relations between curvatures of tangent circles. The Apollonian group is Zariski dense in O_Q (where O_Q is the orthogonal group for the Descartes form (1)), and yet it is of infinite index in $O_Q(\mathbb{Z})$; in other words, it is a *thin group*. Thin groups are not as accessible as arithmetic groups, but, remarkably, still share some of their properties, most notably a version of strong approximation. The Apollonian group has garnered so much interest in part because of its position as a 'naturally occurring' thin group of arithmetic interest. For an overview of the arithmetic of thin groups, and the discoveries rapidly unfolding in recent years, see [21]. It is one of the principal goals of this paper to place the Apollonian group in a new 'naturally occurring' infinite family of thin groups of arithmetic interest.

We begin by defining K-Apollonian packings for imaginary quadratic fields $K \neq \mathbb{Q}(\sqrt{-3})$.

Definition 1.2. One says that a collection of circles $\mathcal{P}$ *straddles* a circle C if it intersects both the interior and exterior of C nontrivially. We say that a collection of circles $\mathcal{P}$ is *tangency-connected* if the graph whose vertices are circles and whose edges indicate tangencies is a connected graph. We define a K-*Apollonian packing* to be any maximal tangency-connected subset $\mathcal{P}$ of circles in $\mathcal{S}_K$ under the condition that $\mathcal{P}$ does not straddle any circle of $\mathcal{S}_K$.
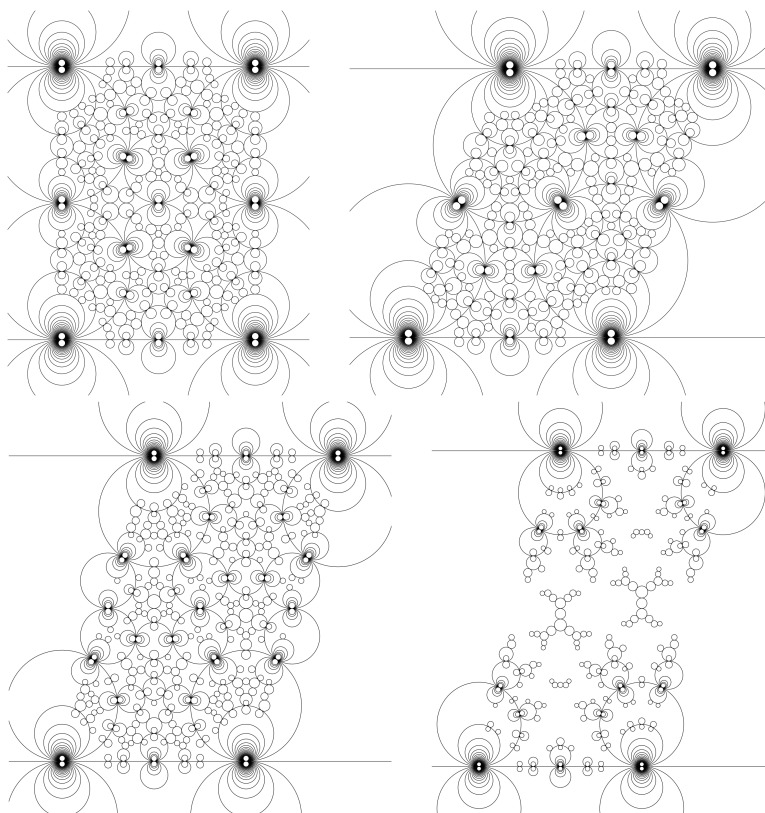
FIGURE 3. A primitive integral Apollonian packing with curvatures shown. The outer circle has curvature -6 (indicating that its interior is outside).

For an example, see Figure 5.

Note that, in particular, this implies that the circles in $\mathcal{P}$ can be oriented to have disjoint interiors. An equivalent definition of a K-Apollonian packing is that it corresponds to a connected component of the *immediate tangency graph*, the graph whose vertices are oriented circles, and whose edges indicate *immediate tangency*, that is, tangency between two circles of disjoint interiors which straddle no other K-Bianchi circle (Proposition 4.10).

Now we can state the fundamental relationship between Schmidt arrangements and K-Apollonian packings.

Theorem 1.3. *For an imaginary quadratic field $K \neq \mathbb{Q}(\sqrt{-3})$, each circle of $\mathcal{S}_K$ participates in exactly two K-Apollonian packings: one disjoint with its interior and one disjoint with its exterior. In particular, $\mathcal{S}_K$ is equal to the union of all K-Apollonian packings.*

FIGURE 4. Schmidt arrangements $\mathcal{S}_K$ of imaginary quadratic fields K. Clockwise from top left: $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-7})$, $\mathbb{Q}(\sqrt{-15})$, $\mathbb{Q}(\sqrt{-11})$. In each case, the image includes those circles of curvature ≤ 20 intersecting the closure of the fundamental parallelogram of $\mathcal{O}_K$.

The Eisenstein case ($K = \mathbb{Q}(\sqrt{-3})$) presents the difficulty that $\mathcal{S}_K$ has circles intersecting other than tangently, as in Figure 7. **For this reason, throughout the paper, we will assume that $K \neq \mathbb{Q}(\sqrt{-3})$.** This special case is being further investigated in [16].

Let $\Delta < 0$ be the discriminant of K. The curvature of a K-Bianchi circle is of the form $n\sqrt{-\Delta}$, for some $n \in \mathbb{Z}$; we may call n the *reduced curvature* of the circle. With this definition, we may extend Conjecture 1.1 for Apollonian circle packings.

Conjecture 1.4. *Let K be an imaginary quadratic field with $\Delta \neq -3$. Let $\mathcal{P}$ be a K-Apollonian packing, and for any $M \in \mathbb{Z}$, let S_M be the set of residue classes modulo M of the reduced curvatures in $\mathcal{P}$. There exists an M dividing 24 such that all sufficiently large integers whose residues modulo M lie in S_M occur as curvatures in $\mathcal{P}$.*
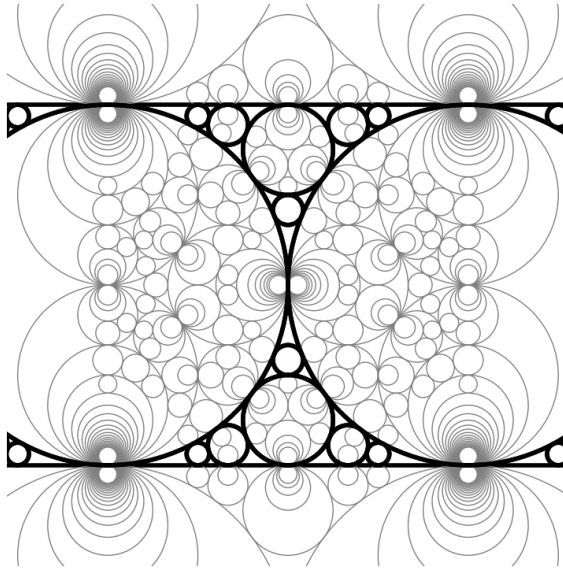
FIGURE 5. An Apollonian packing (shown in darker lines) lying in the Schmidt arrangement of $\mathbb{Q}(i)$ according to Definition 1.2.

Furthermore, fixing K, the minimal M which will suffice for all K-Apollonian packings is determined by the following formulae for its valuations with respect to 2 and 3:

$$v_2(M) = \begin{cases} 3 & \Delta \equiv 28 \pmod{32} \\ 2 & \Delta \equiv 8, 12, 20, 24 \pmod{32} \\ 1 & \Delta \equiv 0, 4, 16 \pmod{32} \\ 0 & \textit{otherwise} \end{cases}, \quad v_3(M) = \begin{cases} 1 & \Delta \equiv 5, 8 \pmod{12} \\ 0 & \textit{otherwise} \end{cases}.$$

Further detailed predictions and supporting evidence is given in Section 16. See Figure 6 for an example of packing with curvatures shown.

The bulk of the paper is devoted to the study of K-Apollonian packings. In particular, we define *K-Apollonian groups* for every $K \neq \mathbb{Q}(\sqrt{-3})$. Let Möb denote the group of Möbius transformations, including $\mathfrak{c}$, complex conjugation. Then a *K-Apollonian group* is one satisfying the conclusions of the following theorem.

Theorem 1.5 (Summary of results of Sections 9–15)**.** *Let $K \neq \mathbb{Q}(\sqrt{-3})$ be an imaginary quadratic field. Then there exists a finitely generated Kleinian group $\mathcal{A} < \langle \mathrm{PSL}_2(\mathcal{O}_K), \mathfrak{c} \rangle < $ Möb with the following properties:*

(1) *The limit set of $\mathcal{A}$ is the closure of the K-Apollonian packing containing $\widehat{\mathbb{R}}$, where $\widehat{\mathbb{R}}$ is oriented so its interior is the lower half plane.*

(2) *Any K-Apollonian packing is the orbit under $\mathcal{A}$ of some finite collection of circles.*

(3) *$\mathcal{A}$ has Zariski closure $A = SO_{3,1}$ or $O_{3,1}$, under the isomorphism Möb $\cong O_{3,1}^+(\mathbb{R})$, and is of infinite index in $A(\mathbb{Z})$. In other words, it is thin.*
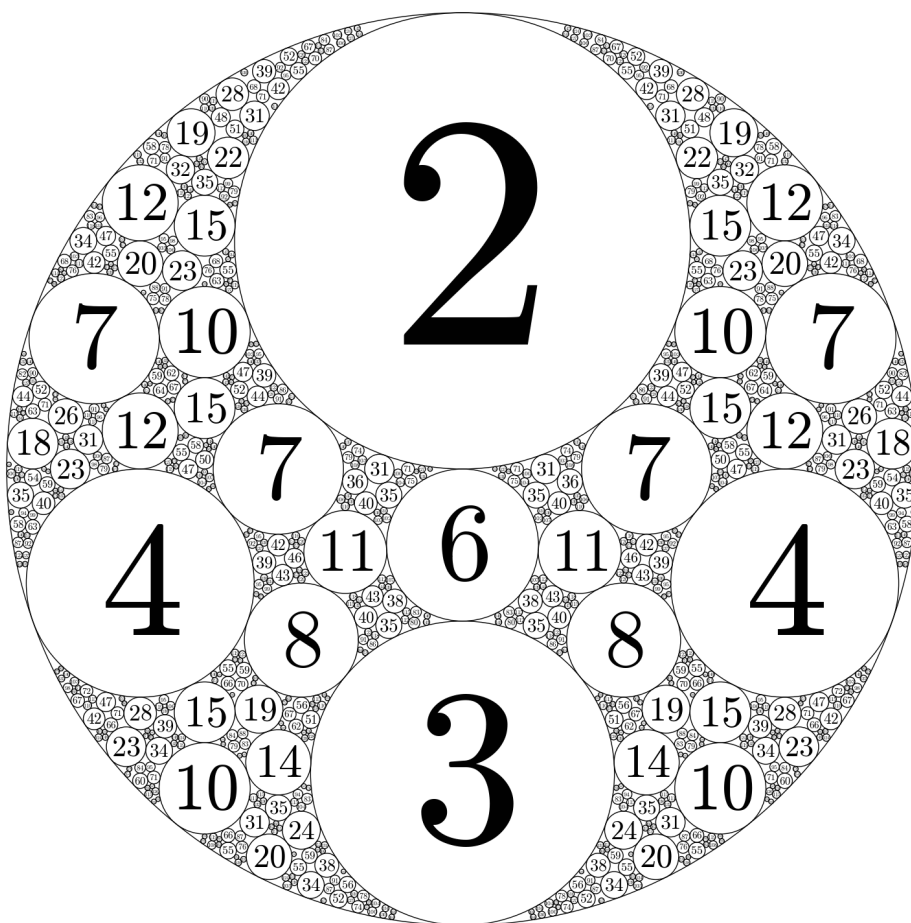
FIGURE 6. An example of $\mathbb{Q}(\sqrt{-2})$-Apollonian packing, shown with reduced curvatures (the outer circle has reduced curvature -1). In this case, it is conjectured that all sufficiently large integers that are not congruent to 1 modulo 4 appear as reduced curvatures.

(4) *It is possible to define* clusters *of circles, being unordered collections of $4 \leq n < \infty$ circles in a certain geometric arrangement, so that the set of clusters in any K-Apollonian packing is a principal homogeneous space*[1] *for* $\mathcal{A}$.

The last property deserves some further explanation. In the study of traditional Apollonian circle packings, the 'clusters' are Descartes quadruples (any four circles which are mutually tangent). We have seen that, given three mutually tangent circles, there are exactly two Descartes quadruples containing these three. Therefore, given one Descartes quadruple and one circle of that quadruple, there is a

---

[1]By a *principal homogeneous space for a group G* we mean a nonempty set on which the group G acts freely and transitively.

swap that replaces that circle with the unique choice that gives a new Descartes quadruple. For each quadruple, there are four such swaps. It turns out that the space of Descartes quadruples is a principal homogeneous space for the traditional Apollonian group, whose four generators encode the four swaps. This group has two manifestations, sometimes called *algebraic* and *geometric*. The geometric manifestation is the one discussed in the theorem above, but both aspects are discussed in this paper.

There are many K-Apollonian groups (even for the traditional Apollonian circle packing), and we give some particularly simple and symmetric examples with pleasing presentations. In particular, in cases where $\mathcal{O}_K$ is Euclidean, we give an example which is a free product of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$. In each case, we give generators explicitly, and describe their interpretation as 'swaps' on 'clusters'.

Various Apollonian-like circle packings have appeared in the literature. Guettler and Mallows studied *generalized Apollonian packings*, which have also come to be called *Apollonian 3-packings* or *octahedral Apollonian packings*, circle packings in which each curvilinear triangle is packed with three (not just one) new circles [15]. They generalize much of the general theory of Apollonian packings, including the analogue of the Apollonian superpacking [15, Figure 4]. Zhang later extended some of the local-to-global theory to Apollonian 3-packings, most notably a density one result on curvatures [45]; his paper contains an Apollonian-type group for the 3-packings. It appears that the Apollonian 3-packings are dual to $\mathbb{Q}(\sqrt{-2})$-Apollonian packings in the following sense: in $\mathcal{S}_{\mathbb{Q}(\sqrt{-2})}$, every loop of four tangent circles has its tangency points along a new circle; this collection of new circles is the super 3-packing suitably rotated and scaled.[2]

Butler, Graham, Guettler and Mallows study circle packings given by certain clusters (which they call *configurations*) and recursive *rules* for filling curvilinear triangles [4]. They are particularly interested in which fields are needed to define the curvatures and centres of such packings. Some of the K-Apollonian packings in the present paper come under their rubric, but not all. For example, some K-Apollonian packings are formed recursively by filling curvilinear 4- or 6-gons.

The study of a three-dimensional analogue of Apollonian circle packings with integer curvatures began with Soddy's *Nature* article of 1937 entitled *The Bowl of Integers and the Hexlet* [40]. The three-dimensional case has the distinction that the associated local-global conjecture is known to hold [20]. Nakamura extends this result to *orthoplicial Apollonian sphere packings* [26], which generalize the Apollonian-3-packing to three dimensions in the same way Soddy's bowl of integers generalizes the classical Apollonian circle packing.

The methods of the paper give rise to a few results that may be of independent interest. The following theorem strengthens the results of [42].

Theorem 1.6 (Theorem 5.1). *Suppose $\mathcal{O}_K$ is non-Euclidean. Then the immediate tangency graph of the Schmidt arrangement for K is an infinite forest of trees of infinite valence. In particular, the tangency graph of a K-Apollonian packing is a tree.*

---

[2]The inspiration for this observation arises from the beautiful *Glowing Limit* poster [24] advertising *Indra's Pearls* [25], which happens to depict this duality, albeit without reference to $\mathbb{Q}(\sqrt{-2})$.
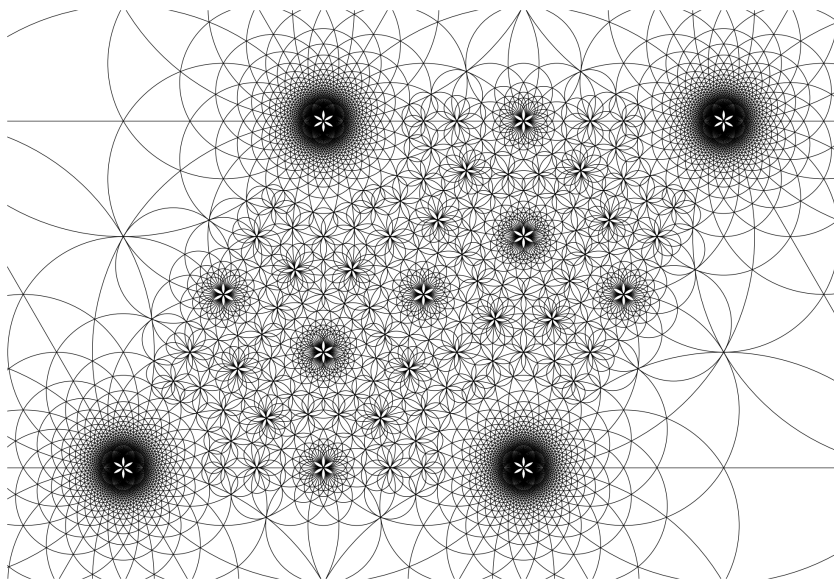
FIGURE 7. Schmidt arrangement $\mathcal{S}_{\mathbb{Q}(\sqrt{-3})}$. The image includes those circles of curvature bounded by 20 intersecting the fundamental parallelogram of the ring of integers or its boundary.

The following is a straightforward strengthening of results of [28], although we provide an independent geometric proof of the infinite index which may be of interest.

Theorem 1.7 (Theorem 9.5). *Whenever $\mathcal{O}_K$ is non-Euclidean, the subgroup of* $\mathrm{PSL}_2(\mathcal{O}_K)$ *generated by elementary matrices is a thin group.*

And finally, the paper contains in Section 10 a discussion of *topographical groups*. These are subgroups of $\mathrm{PGL}_2(\mathbb{Z})$ for which unordered superbases are a principal homogeneous space. There are only two normal such subgroups, which are isomorphic under the outer automorphism of $\mathrm{PGL}_2(\mathbb{Z})$. They have a Cayley graph isomorphic to the *topograph* of Conway and Fung [5].

It is natural to ask about extending the results of the present paper to Schmidt arrangements for congruence subgroups of the Bianchi group, which is work in progress [16], and Bianchi groups for nonmaximal orders, for which Schmidt arrangements have been studied by Sheydvasser [39].

A note on the figures. The figures in this paper were produced with Sage Mathematics Software [43]. Figures 2, 4, 7, and 8 appeared previously in [41, 42].

2. NOTATION

Throughout the paper, K is an imaginary quadratic field with discriminant $-3 \neq \Delta < 0$ and ring of integers $\mathcal{O}_K$. The ring $\mathcal{O}_K$ has an integral basis $1, \tau$, where

$$\tau^2 = \begin{cases} \Delta/4, & \Delta \equiv 0 \pmod 4, \\ \tau + (\Delta - 1)/4, & \Delta \equiv 1 \pmod 4. \end{cases}$$

It is convenient to write

$$\epsilon = Tr(\tau) = \begin{cases} 0, & \Delta \equiv 0 \ (\text{mod } 4), \\ 1, & \Delta \equiv 1 \ (\text{mod } 4). \end{cases}$$

In other words,

$$\tau, \overline{\tau} = \frac{\epsilon}{2} \pm \frac{\sqrt{\Delta}}{2}.$$

Write $N : K \to \mathbb{Q}$ for the norm map $\alpha \mapsto \alpha\overline{\alpha}$. In particular,

$$N(\tau) = -\frac{\Delta - \epsilon}{4}.$$

We follow [13] in writing Möb for the *conformal group*, the group of conformal maps of $\widehat{\mathbb{C}}$, including Möbius transformations and reflections (i.e. allowing complex conjugation); and writing Möb$_+$ for the group of Möbius transformations without reflections, which is isomorphic to $\text{PSL}_2(\mathbb{C})$ via the usual matrix representation of a Möbius transformation. We also write $\text{GM}^* := \text{Möb} \times \{\pm I\}$ for the *extended Möbius group*. The group Möb is isomorphic to the isometry group of three-dimensional hyperbolic space, $\mathbb{H}^3$, via its action on the boundary of the upper-half-space model.

Warning: For the remainder of the paper, $K \neq \mathbb{Q}(\sqrt{-3})$.

3. Preliminaries

We recall some basic facts about Schmidt arrangements from [42], to which the reader is referred for further details. Let K be an imaginary quadratic field. Recall that throughout the paper, $K \neq \mathbb{Q}(\sqrt{-3})$. Among the remaining cases, $K = \mathbb{Q}(i)$ is special in several regards, as we will describe.

Definition 3.1. If $M \in \text{PSL}_2(\mathcal{O}_K)$, then $M(\widehat{\mathbb{R}})$ is called a *K-Bianchi circle*. The collection of all K-Bianchi circles is called the *Schmidt arrangement*, denoted $\mathcal{S}_K$. We will also refer to the union of these circles as $\mathcal{S}_K$, without fear of confusion.

It is convenient to consider oriented circles.

Definition 3.2 ([42, Definitions 3.1, 3.2, 3.3]). An *oriented circle* is a circle together with an *orientation*, which is a direction of travel, specified as either *positive/counterclockwise* or *negative/clockwise*. The *interior* of an oriented circle is the area to your left as you travel along the circle according to its orientation (in other words, the region not containing ∞ for a positively oriented circle). For lines (circles through ∞) besides $\widehat{\mathbb{R}}$, positive orientation indicates travel in the direction of increasing imaginary part. For $\widehat{\mathbb{R}}$, positive orientation indicates travel to the right. An *oriented K-Bianchi circle* is an oriented circle whose underlying circle is a K-Bianchi circle. Write $\widehat{\mathcal{S}}_K$ for the collection of oriented K-Bianchi circles.

The map $\widehat{\mathcal{S}}_K \to \mathcal{S}_K$ which forgets orientation is two-to-one.

Let us also set the convention that $\widehat{\mathbb{R}}$, when considered an oriented circle, denotes the negatively oriented circle (whose interior is the lower half plane). The group $\text{PGL}_2(\mathbb{C})$ has a natural action on oriented circles via Möbius transformation, so that, with this convention, $M(\widehat{\mathbb{R}})$ is naturally endowed with an orientation. The stabilizer of $\widehat{\mathbb{R}}$ is $\text{PSL}_2(\mathbb{R})$. Furthermore, by Proposition 3.4 of [42], this action restricts, in the case $K \neq \mathbb{Q}(i)$, to a transitive action of $\text{PGL}_2(\mathcal{O}_K)$ on oriented K-Bianchi circles, with $\text{Stab}(\widehat{\mathbb{R}}) = \text{PSL}_2(\mathbb{Z})$. In the case of $K = \mathbb{Q}(i)$, $\text{PSL}_2(\mathcal{O}_K)$ is already transitive on oriented K-Bianchi circles with the same stabilizer. If one

considers the orbit of $\widehat{\mathbb{R}}$ under $\mathrm{PGL}_2(\mathbb{Z}[i])$, one obtains a strictly larger collection of circles than the Schmidt arrangement. However, the new circles are not interesting: the set consists of two copies of the Schmidt arrangement at right angles. See [42, Section 3].

Oriented circles are best described as parameterized by four real parameters, as follows.

Proposition 3.3 ([42, Proposition 3.5])**.** *Let C be an oriented circle in $\widehat{\mathbb{C}}$ (including those through ∞). Then the circle C can be given uniquely in the form*

$$\left\{ X/Y \in \widehat{\mathbb{C}} : bX\overline{X} - aY\overline{X} - \overline{a}X\overline{Y} + b'Y\overline{Y} = 0 \right\},$$

where the following hold:

(1) *$b, b' \in \mathbb{R}$, $a \in \mathbb{C}$,*
(2) *we have*

(2) $$b'b = a\overline{a} - 1,$$

(3) *b has sign equal to the orientation of C, and*
(4) *if $b = 0$, in which case C must be a line, then a, as a vector, is a unit vector pointing from exterior to interior orthogonal to C.*

Furthermore, if C' is the image of C under $z \mapsto 1/z$, and C has parameters (b, b', a) according to the requirements above, then C' has parameters $(b', b, \overline{a})$ according to the requirements above. Finally,

(1) *$b = 0$ if and only if $\infty \in C$,*
(2) *$b' = 0$ if and only if $0 \in C$,*
(3) *if $\infty \notin C$, then C has radius $1/|b|$,*
(4) *if $\infty \notin C$, then C has centre a/b.*

Definition 3.4 ([42, Definition 3.6])**.** For any circle C in $\widehat{\mathbb{C}}$, expressing it as in Proposition 3.3, we call b the *curvature* (elsewhere sometimes called a *bend*), b' the *co-curvature*, and a the *curvature-centre*.

Proposition 3.5 ([42, Proposition 3.7])**.** *Consider an oriented circle expressed as the image of $\widehat{\mathbb{R}}$ under a transformation of the form*

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \ |\alpha\delta - \beta\gamma| = 1.$$

The curvature of the circle is given by

$$i(\beta\overline{\delta} - \overline{\beta}\delta),$$

the co-curvature of the circle is given by

$$i(\alpha\overline{\gamma} - \overline{\alpha}\gamma),$$

and the curvature-centre is given by

$$i(\alpha\overline{\delta} - \gamma\overline{\beta}).$$

The curvature and co-curvature of an oriented K-Bianchi circle are integer multiples of $\sqrt{-\Delta}$; the integer alone will be referred to as the *reduced curvature* or *reduced co-curvature*, respectively.

Proposition 3.6 ([42, Propositions 4.1 and 4.4]). *Two K-Bianchi circles may intersect only at K-points, and only tangently.*[3]

It is useful to be very explicit about which circles are tangent. We say that $\alpha, \beta \in \mathcal{O}_K$ are coprime if the ideals they generate are coprime, i.e., $(\alpha) + (\beta) = (1)$.

Proposition 3.7 ([42, Proposition 4.3]). *Let $\alpha/\beta \in K$ be such that α and β are coprime. Suppose $|\mathcal{O}_K^*| = n$. Then the collection of oriented K-Bianchi circles passing through α/β is a union of n generically different $\mathbb{Z}$-families, one for each $u \in \mathcal{O}_K^*$. The family associated to u consists of the images of $\widehat{\mathbb{R}}$ under the transformations*

$$\begin{pmatrix} \alpha & u\gamma + k\tau\alpha \\ \beta & u\delta + k\tau\beta \end{pmatrix}, \quad k \in \mathbb{Z},$$

where γ, δ is a particular solution to $\alpha\delta - \beta\gamma = 1$. Furthermore,

(1) *The curvatures of the circles in one family form an equivalence class modulo $\sqrt{-\Delta}N(\beta)$.*
(2) *The centres of the circles in a given family lie on a single line through α/β.*
(3) *The family given by unit u contains the same circles as the family given by $-u$, but with opposite orientations.*

4. K-Apollonian packings and immediate tangency

In this section we give two equivalent definitions of a K-Apollonian packing. This allows us to prove Theorem 1.3 of the introduction, describing the Schmidt arrangement as a union of its K-Apollonian packings.

Definition 4.1. The *tangency graph* of $\mathcal{P} \subset \mathcal{S}_K$ is the graph whose vertices are the circles of $\mathcal{P}$ and whose edges indicate tangencies. We say that $\mathcal{P}$ is *tangency-connected* if its tangency graph is connected. We say that $\mathcal{P}' \subset \widehat{\mathcal{S}}_K$ is *tangency-connected* if the underlying set of unoriented circles is so. We say $\mathcal{P}$ *straddles* a circle C if it intersects both the interior and exterior of C nontrivially.

This definition of straddling coincides with that of [42, Definition 4.5] in the cases we are considering ($K \neq \mathbb{Q}(\sqrt{-3})$, where circles intersect only tangently), and is simpler to state.

We repeat Definition 1.2 of the introduction here:

Definition 4.2. We define a *K-Apollonian packing of unoriented circles* to be any maximal tangency-connected subset $\mathcal{P}$ of circles of $\mathcal{S}_K$ under the condition that $\mathcal{P}$ does not straddle any circle of $\mathcal{S}_K$.

We will now extend the definition of a K-Apollonian packing to oriented circles.

Definition 4.3. We define a *K-Apollonian packing of oriented circles* to be any maximal tangency-connected subset $\mathcal{P}$ of oriented circles of $\widehat{\mathcal{S}}_K$ with disjoint interiors under the condition that $\mathcal{P}$ does not straddle any circle of $\widehat{\mathcal{S}}_K$.

The following lemma verifies that these two definitions correspond nicely under the orientation-forgetting map.

---

[3]This does not hold for $K = \mathbb{Q}(\sqrt{-3})$. The reader is reminded, one final time, that $K \neq \mathbb{Q}(\sqrt{-3})$ throughout the paper.

Lemma 4.4. *A collection of circles $\mathcal{P} \subset \mathcal{S}_K$ is a K-Apollonian packing of unoriented circles if and only if it can be obtained from a K-Apollonian packing of oriented circles by forgetting orientation.*

Proof. First, we show that any tangency-connected, nonstraddling $\mathcal{P} \subset \mathcal{S}_K$ lifts (under the orientation-forgetting map) to a unique $\mathcal{P}' \subset \widehat{\mathcal{S}}_K$ which is nonstraddling and tangency-connected with disjoint interiors. For, suppose $\mathcal{P} \subset \mathcal{S}_K$ does not straddle any circle of $\mathcal{S}_K$. Then, in particular, it cannot intersect both the interior and exterior of any $C \in \mathcal{P}$. Therefore, $\mathcal{P}$ can be lifted to $\mathcal{P}' \subset \widehat{\mathcal{S}}_K$ by choosing the orientation of each circle $C \in \mathcal{P}$ in such a way that $\mathcal{P}$ is disjoint from the interior of C. This lift is unique if $\mathcal{P}$ contains at least two circles (an assumption we are safe in making; for example, by Proposition 3.7, K-Apollonian packings contain more than one circle). The resulting collection has disjoint interiors and the nonstraddling property. Furthermore, $\mathcal{P}'$ is tangency-connected if and only if $\mathcal{P}$ is.

Conversely, any $\mathcal{P}' \subset \widehat{\mathcal{S}}_K$ which is nonstraddling and tangency-connected with disjoint interiors is surely nonstraddling and tangency-connected when orientation is forgotten, and this operation is inverse to the lifting just described.

What remains then, is to show that such pairs $(\mathcal{P}, \mathcal{P}')$ have the property that $\mathcal{P}$ is maximal with respect to being tangency-connected and nonstraddling if and only if $\mathcal{P}'$ is maximal with respect to being tangency-connected, nonstraddling, and having disjoint interiors.

A K-Apollonian packing in $\widehat{\mathcal{S}}_K$ cannot contain both orientations of the same unoriented circle without violating the disjoint interiors requirement (unless it contained only those two circles, but then it would not be maximal). It follows then, that if there is a way to add another circle to $\mathcal{P}'$ in $\widehat{\mathcal{S}}_K$ under the stipulated conditions it corresponds to a way to add another circle to $\mathcal{P}$ in $\mathcal{S}_K$, and vice versa. $\square$

With this result in place, we may henceforth consider oriented circles. We wish to give a different characterisation of being a K-Apollonian packing of oriented circles, and show that it is equivalent. We will need the following notion.

Definition 4.5. Two oriented K-Bianchi circles $C_1, C_2 \in \widehat{\mathcal{S}}_K$ are *immediately tangent* if they are externally tangent in such a way that the pair straddles no circles of $\widehat{\mathcal{S}}_K$.

In other words, C_1 and C_2 are *immediately tangent* if they are consecutive in an unoriented family of Proposition 3.7, and taken with disjoint interiors.

We recall the result that for any K-Bianchi circle, there is exactly one K-Bianchi circle immediately tangent to it at a fixed K-rational point. More precisely:

Proposition 4.6 ([42, Proposition 6.2]). *Let $C \in \widehat{\mathcal{S}}_K$ be an oriented K-Bianchi circle with K-rational point x. Then there exists*

$$M_C = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O}_K)$$

such that $C = M_C(\widehat{\mathbb{R}})$ and $x = \alpha/\beta$. Furthermore, there exists exactly one oriented K-Bianchi circle $C' \in \widehat{\mathcal{S}}_K$ immediately tangent to C at x, and it is given by $C' = M_{C'}(\widehat{\mathbb{R}})$ where

$$M_{C'} = \begin{pmatrix} \alpha & -\gamma + \tau\alpha \\ \beta & -\delta + \tau\beta \end{pmatrix} = M_C \begin{pmatrix} 1 & \tau \\ 0 & -1 \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O}_K).$$

Definition 4.7. Let $\mathcal{P}$ be a subset of $\widehat{\mathcal{S}}_K$. We say that $\mathcal{P}$ is *closed under immediate tangency* if it has the property that for any circle $C \in \mathcal{P}$, any circle $C' \in \widehat{\mathcal{S}}_K$ that is immediately tangent to C is also contained in $\mathcal{P}$. We say that $\mathcal{P}$ is a *K-Apollonian packing defined by tangency* if it is minimal among nonempty subsets of $\widehat{\mathcal{S}}_K$ which are closed under immediate tangency.

There are infinitely many circles immediately tangent to any one circle, by Proposition 4.6, so K-Apollonian packings defined by tangency are infinite sets of circles.

This definition can be rephrased slightly, in the language of graph theory.

Definition 4.8. The *immediate tangency graph* of $\widehat{\mathcal{S}}_K$ is the graph whose vertices are the circles of $\widehat{\mathcal{S}}_K$ and whose edges are given by the symmetric relation of immediate tangency.

With this definition, the following proposition is immediate.

Proposition 4.9. *A subset $\mathcal{P}$ of $\widehat{\mathcal{S}}_K$ is a K-Apollonian packing defined by tangency if and only if it is a connected component of the immediate tangency graph.*

We now show that Definitions 4.3 and 4.7 for a K-Apollonian packing coincide.

Proposition 4.10. *A subset $\mathcal{P}$ of $\widehat{\mathcal{S}}_K$ is a K-Apollonian packing if and only if it is a K-Apollonian packing defined by tangency.*

Proof. Let $\mathcal{P}$ be a K-Apollonian packing. Let C be a circle of $\mathcal{P}$ and let $z \in C$ be a K-rational point. Let C' be the unique circle immediately tangent to C at z. Our first goal is to show that $\mathcal{P}$ contains C'. This demonstrates that $\mathcal{P}$ is closed under immediate tangency.

If not, since C is in the exterior of C', then all of $\mathcal{P}$ is exterior to C'. Suppose we add C' to $\mathcal{P}$. This creates a larger tangency-connected set with disjoint interiors. We show that this larger set is not straddling any circle of $\widehat{\mathcal{S}}_K$, which is a contradiction to maximality. To see this, suppose it did straddle a circle. Since $\mathcal{P}$ did not, it must be that C' is inside some circle D that $\mathcal{P}$ is exterior to. Since they touch at z, D is tangent to C and C' at z and separates them (one is interior to D, the other exterior). In other words, C and C' straddle D, but this is a contradiction to immediate tangency.

Hence $C' \in \mathcal{P}$, and we have demonstrated that $\mathcal{P}$ is closed under immediate tangency.

To show that $\mathcal{P}$ is minimal as a set closed under immediate tangency, let us imagine removing a collection of circles $\mathcal{S}$ from $\mathcal{P}$, to leave behind a subset $\mathcal{S}' \subset \mathcal{P}$ which is closed under immediate tangency. Then it is impossible that any circle of $\mathcal{S}$ touches any circle of $\mathcal{S}'$ since any two circles of $\mathcal{P}$ that are touching are immediately tangent (otherwise they would violate the nonstraddling property). Hence $\mathcal{P}$ is tangency-disconnected, a contradiction. Therefore $\mathcal{P}$ is a K-Apollonian packing defined by tangency.

Conversely, suppose $\mathcal{P}$ is a K-Apollonian packing defined by tangency. First, we show that $\mathcal{P}$ straddles no circle $D \in \widehat{\mathcal{S}}_K$. By definition, D is not tangent immediately to any of the circles in its interior. No circle of $\mathcal{P}$ exterior to D is immediately tangent to a circle interior to D, as such a tangency would straddle D. Hence $\mathcal{P}$ is not minimal; its circles inside D could be removed without violating closure under immediate tangency. This is a contradiction. Therefore, we have shown that $\mathcal{P}$ straddles no circles.

We have already observed (Proposition 4.9) that $\mathcal{P}$ is tangency-connected. Next we show that $\mathcal{P}$ has disjoint interiors. Suppose not: then some circle C lies inside another, C'. Since $\mathcal{P}$ is tangency-connected, we can assume without loss of generality that C is tangent to C' at some point z. But since $\mathcal{P}$ also includes the circle immediately tangent to C' at z, this gives a collection of three distinct circles tangent at a point. This is disallowed by the nonstraddling property just shown.

Finally, we must show $\mathcal{P}$ is maximal. Suppose there were another circle $C \notin \mathcal{P}$ such that $\mathcal{P} \cup \{C\}$ is tangency-connected. We will show $\mathcal{P} \cup \{C\}$ straddles some circle of $\mathcal{S}_K$. Let $z \in C$ be a point of tangency with a circle $D \in \mathcal{P}$. This is not an immediate tangency, since $C \notin \mathcal{P}$. Then there is some $C' \neq D$, $C' \in \mathcal{P}$, immediately tangent to C. Thus we have three distinct circles of $\mathcal{P} \cup \{C\}$ sharing a single tangency point: as a set they must straddle one of their members. □

The following statement is an immediate consequence of the two equivalent definitions.

Theorem 4.11. *The K-Apollonian packings of $\widehat{\mathcal{S}}_K$ form a collection of disjoint subsets of $\widehat{\mathcal{S}}_K$ whose union is all of $\widehat{\mathcal{S}}_K$.*

Proof. This is evident from Definition 4.7, as the K-Apollonian packings are exactly the connected components of the immediate tangency graph on $\widehat{\mathcal{S}}_K$. □

Proof of Theorem 1.3. Forget orientations in Theorem 4.11. □

Note that the packings are not disjoint as collections of *unoriented* circles, however. A circle generally belongs to two packings, depending on the orientation assigned: one living in its interior and another in its exterior.

It is an open question whether tangency-connectedness corresponds exactly to the topological notions of connectedness or path-connectedness for subsets of Schmidt arrangements (it is clearly stronger for arbitrary unions of circles in $\widehat{\mathbb{C}}$). See [42, Section 7] for more on this distinction. In particular, it is shown in [42, Theorem 7.1] that $\mathcal{S}_K$ is connected if and only if it is tangency-connected if and only if $\mathcal{O}_K$ is Euclidean.

Definition 4.12. The K-Apollonian packing containing $\widehat{\mathbb{R}}$ is called the *fundamental packing*, and is denoted $\mathcal{P}_K$.

See Figure 5 for the fundamental packing of $\mathbb{Q}(i)$. Other fundamental packings are shown in Figures 16, 19 and 22.

5. Loops in the tangency graph

The purpose of this section is to prove the following theorem, from which Theorem 1.6 of the introduction follows. Note that [42, Theorem 7.5] already guarantees that $\mathcal{S}_K$ has infinitely many components when $\Delta \leq -15$.

Theorem 5.1. *Let K be such that $\Delta \leq -15$. Then the immediate tangency graph of $\widehat{\mathcal{S}}_K$ contains no loops.*

The proof for $\Delta < -15$ will rely on a simple graph theory principle.

Lemma 5.2. *Consider a graph G with vertex set V and a function $f : V \to \mathbb{R}$. Direct each edge of G according to the direction of increase of f (whenever it is not constant). Suppose that at any vertex $v \in V$, all but at most one edge is directed outward. Then G contains no loops.*

Proof. Suppose there is a loop. Let v be a vertex of that loop. By assumption, at least one of the two edges of the loop adjacent to v is directed outward. Moving to the next vertex w along this edge, we enter w along an inward directed edge, and, therefore, continuing along the loop, leave w along an outward directed edge. Hence, as we travel around the loop in this direction, the value of f is increasing, so we can never return to v. □

Lemma 5.3. *Let K be such that $\Delta < -15$. Let C be a K-Bianchi circle. Then there is at most one point $x \in C$ at which C is tangent to a K-Bianchi circle of absolute curvature less than or equal to the absolute curvature of C.*

By *absolute curvature* we mean the absolute value of the curvature. Note that, at the point x, the circle may be tangent to several circles of absolute curvature less than its own.

Proof. Suppose $C = M(\widehat{\mathbb{R}})$ for

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix},$$

and let $\Lambda = \beta\mathbb{Z} + \delta\mathbb{Z}$. Let b be the reduced curvature of C. Then the covolume of Λ as a lattice in $\mathbb{C}$ is $|b\sqrt{-\Delta}/2|$. There is at most one element of Λ, up to multiples, whose norm is less than its covolume, so for all but one $x \in \Lambda$, we have

$$(3) \qquad\qquad N(x) \geq |b\sqrt{-\Delta}/2|.$$

On the other hand, if C' is tangent to C at the unique point with denominator x (here we use that $b \neq 0$), and has curvature b' where $|b'| \leq |b|$, then it follows from Proposition 3.7 that

$$|kN(x) - b| = |b'| \leq |b|$$

for some $k \in \mathbb{Z}^{>0}$, whence

$$(4) \qquad\qquad N(x) \leq 2|b|.$$

Comparing the inequalities (3) and (4) gives the desired result. □

The proof of Theorem 5.1 will rely on this lemma for $\Delta < -15$. However, the lemma fails for $\Delta = -15$, where we will require an entirely different method.

Proposition 5.4. *The tangency graph of $K = \mathbb{Q}(\sqrt{-15})$ contains no loops.*

Proof. Let $K = \mathbb{Q}(\sqrt{-15})$. First, we remark that $\mathrm{PSL}_2(\mathcal{O}_K)$ is transitive on pairs (C, z) where $C \in \widehat{\mathcal{S}}_K$ and $z \in C$ is a tangency point between circles of $\widehat{\mathcal{S}}_K$. Thus, we need only show that $\widehat{\mathbb{R}}$ (with interior below) and any $C \in \mathcal{S}_K$ of the form $ki + \widehat{\mathbb{R}}$, $k \in \mathbb{Z}^{>0}$ (with interior above) do not participate as adjacent vertices in any cycle of the graph. Suppose for a contradiction, that they do. We will call this the *postulated cycle*.

Note that $k \geq \frac{\sqrt{15}}{2}$ (equality occurs if C is the circle immediately tangent to $\widehat{\mathbb{R}}$ (oriented negatively) at ∞). Then the finitely many other circles making up the cycle must form a chain of tangent circles reaching from some tangency point x of $\widehat{\mathbb{R}}$ up to some tangency point y of C. In particular, the chain consists of finitely many circles. Each is of curvature at least $\sqrt{15}$ or else equal to 0. Therefore, the portion of the chain that lies below $\widehat{\mathbb{R}} + \frac{\sqrt{-15}}{2}$ is bounded away from ∞.
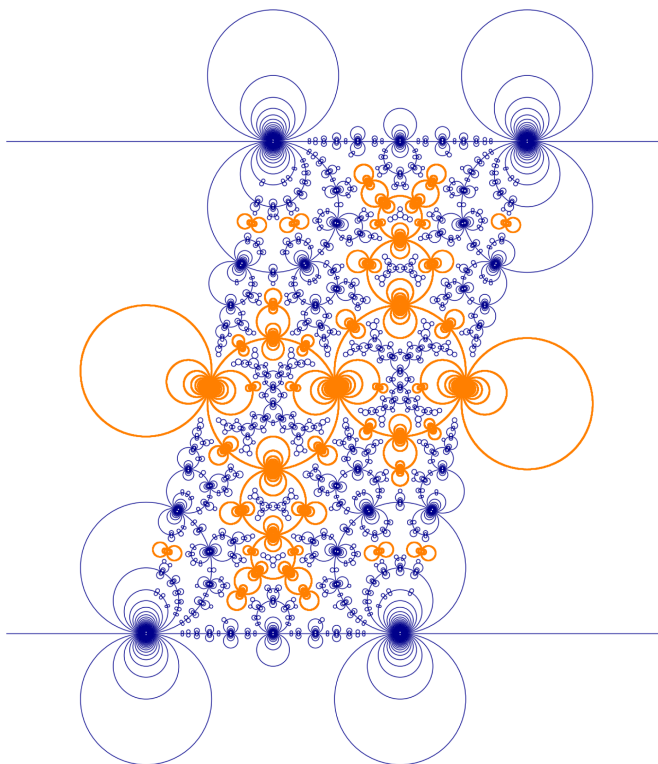
FIGURE 8. The Schmidt arrangement of $\mathbb{Q}(\sqrt{-15})$, in blue, with ghost circles, shown in yellow.

In [42, Section 7], it was shown that $\mathcal{S}_K$ is disconnected by demonstrating the existence of a *ghost circle* which is contained in the complement of $\mathcal{S}_K$. The ghost circle G is the circle of radius $1/\sqrt{15}$ centred at

$$\frac{1}{2} - \frac{7\sqrt{-15}}{30}.$$

Its existence immediately implies the existence of infinitely many other ghost circles formed by $\mathrm{PSL}_2(\mathcal{O}_K)$ images of G. The union of these circles is contained in the complement of $\mathcal{S}_K$. These images include an infinite 'ghost chain' of tangent circles extending horizontally to ∞ in both directions, and separating $\widehat{\mathbb{R}}$ and $\widehat{\mathbb{R}} + \frac{\sqrt{-15}}{2}$. To see this, illustrated in Figure 8, let G' be the reflection of G in $\widehat{\mathbb{R}}$, and let G'' be the translation of G by $\frac{-1+\sqrt{-15}}{2}$. The circles G' and G'' are tangent at $\frac{1+\sqrt{-15}}{4}$. The union of translates of the pair G' and G'' by $\mathbb{Z}$ is the 'ghost chain' of tangent circles which contradicts the existence of the postulated cycle, and the theorem is proved. $\square$

Proof of Theorem 5.1. Suppose first that $\Delta < -15$. Let G be the immediate tangency graph of $\widehat{\mathcal{S}}_K$. Let f be the function of absolute curvatures on the vertices of the graph. The edges of G may be labelled by the points of tangency. If one does so, then at any vertex, there is at most one adjacent edge with any given

label, by Proposition 4.6. Combining this observation with Lemma 5.3 verifies the hypotheses of Lemma 5.2, and we are done.

The case of $\Delta = -15$ is completed by Proposition 5.4. $\square$

In the Euclidean cases, we will see that the immediate tangency graph does contain loops.

6. The space of circles

We introduce an embedding of the space of circles in Minkowski space which is described in [13] and elaborated upon beautifully by Kocik [19]; it is a natural viewpoint from the perspective of the spin homomorphism and hyperbolic space as the space of Hermitian forms; see, for example, [8]. Associating a circle to a Hermitian form goes back to Bianchi [6, Chapter XV, p. 272]. See also [18].

Let $\mathbb{M}$ be the vector space $\mathbb{R}^4$ endowed with an inner product of signature $3, 1$ given by the Gram matrix

$$G_M = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The associated quadratic form will be denoted M, for Minkowski. This space is identified with the collection of Hermitian matrices via

$$\begin{pmatrix} b' \\ b \\ r \\ m \end{pmatrix} \leftrightarrow \begin{pmatrix} b' & r + mi \\ r - mi & b \end{pmatrix}.$$

Under this identification, the self-product of a vector under G_M is equal to the determinant of the corresponding matrix. In what follows, we will use this identification implicitly.

The space $\mathbb{M}$ (considered as Hermitian matrices T) comes with a metric-preserving action of $\mathrm{PGL}_2(\mathbb{C})$ via

$$\gamma \cdot T := \frac{\gamma T \gamma^\dagger}{\mathrm{N}(\det \gamma)}, \quad \gamma \in \mathrm{PGL}_2(\mathbb{C}),$$

where $\dagger$ indicates the conjugate transpose. There is also an action by Möb, where complex conjugation $\mathfrak{c}$ acts by $\mathfrak{c} \cdot T = \overline{T}$. This allows us to define a map

$$\rho : \mathrm{Möb} \to \mathrm{O}_M^+(\mathbb{R})$$

where $\mathrm{O}_M^+(\mathbb{R})$ is the collection of time-preserving isometries of $\mathbb{M}$ (in our description, this is equivalent to preserving the sign of $b + b'$), also known as the orthochronous Lorentz group. This map is an isomorphism, which we call the *exceptional isomorphism*, closely related to the famous spin homomorphism. It can also be extended to GM^*, so that we have the following isomorphisms and subgroup relations:

$$\begin{array}{ccccc} \mathrm{Möb}_+ & \hookrightarrow & \mathrm{Möb} & \hookrightarrow & \mathrm{GM}^* \\ \rho \downarrow \cong & & \rho \downarrow \cong & & \rho \downarrow \cong \\ SO_M^+(\mathbb{R}) & \hookrightarrow & O_M^+(\mathbb{R}) & \hookrightarrow & O_M(\mathbb{R}) \end{array}$$

For more on this, see [13].

Two Hermitian forms $H_i(u,v)$, $i = 1,2$, on a $\mathbb{C}$-vector space V are called *equivalent* or *isometric* over $\mathbb{C}$ if $H_1(\phi(u), \phi(v)) = H_2(u,v)$ for some $\mathbb{C}$-isomorphism $\phi : V \to V$. The equivalence class of a Hermitian form and its determinant form a complete set of invariants for the orbits of the $\mathrm{PGL}_2(\mathbb{C})$-action on Hermitian forms described above. All Hermitian forms of positive determinant are equivalent over $\mathbb{C}$ (see, for example, [23] and [8, Chapter 9]). Therefore, the action of $\mathrm{PGL}_2(\mathbb{C})$ on the locus $M = 1$ is transitive.

Let $\mathcal{C}irc$ be the collection of circles in $\widehat{\mathbb{C}}$. This is a set with a Möb action (where $\mathfrak{c}$ acts by conjugation on $\widehat{\mathbb{C}}$). Denote by $\widehat{\mathcal{C}irc}$ the collection of oriented circles, which is also a set with an action by Möb.

We define a Pedoe map

$$\pi : \widehat{\mathcal{C}irc} \to \mathbb{M}$$

which will endow the space of circles with the structure of a hypersurface in an inner product space (see [19]). The map is defined as follows:

$$\pi(C) = \begin{pmatrix} b' & r + mi \\ r - mi & b \end{pmatrix} \leftrightarrow \begin{pmatrix} b' \\ b \\ r \\ m \end{pmatrix},$$

where b is the curvature, b' the co-curvature, and $r + mi$ the curvature-centre of C. The image of π is a Hermitian matrix of determinant 1 (by Proposition 3.3), hence the image lies on the hyperboloid $M = 1$ in $\mathbb{M}$.

Proposition 6.1. *The map* $\pi : \widehat{\mathcal{C}irc} \longrightarrow \{v \in \mathbb{M} : M(v) = 1\}$ *is a Möb-equivariant bijection via* ρ.

Proof. Bijectivity follows from Proposition 3.3 (note that $M(v) = 1$ is exactly (2)). That π respects the Möb-action is a direct computation. $\square$

In particular, the oriented K-Bianchi circles are in bijection with the orbit of $(0,0,0,1)^t \in \mathbb{M}$ under $\rho(\mathrm{Möb}(\mathcal{O}_K))$.

One may verify that π may also be computed in the following way. Express a circle C as $C = M_C(\mathbb{R})$ for

$$M_C = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C}), \quad \mathrm{N}(\det(M_C)) = 1.$$

Then

$$\pi(C) = i(N - N^\dagger), \text{ where } N = \begin{pmatrix} \alpha\overline{\gamma} & \alpha\overline{\delta} \\ \beta\overline{\gamma} & \beta\overline{\delta} \end{pmatrix}.$$

It is a brief computation that this map agrees with π. Note that, if we write $-C$ for the circle C with opposite orientation, then $\pi(-C) = -\pi(C)$.

The inner product on Minkowski space now gives us an inner product on circles. It carries geometric information about the circles in question. Following Kocik, we will call this the *Pedoe product*.

Proposition 6.2 (Proposition 2.4 of [19])**.** *Let* $v_i = \pi(C_i)$ *for two circles* C_1, C_2 *which are not disjoint. Then* $\langle v_1, v_2 \rangle = \cos\theta$, *where* θ *is the angle between the two circles as in Figure* 9. *In particular,*

(1) $\langle v_1, v_2 \rangle = -1$ *if and only if the circles are tangent externally*
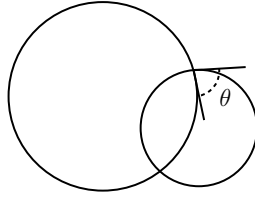
FIGURE 9. The angle of intersection of two circles.

(2) $\langle v_1, v_2 \rangle = 1$ *if and only if the circles are tangent internally*

(3) $\langle v_1, v_2 \rangle = 0$ *if and only if the circles are mutually orthogonal*

This extends via the Law of Cosines, so that the inner product of disjoint circles is

$$(5) \qquad \langle v_1, v_2 \rangle = \frac{1}{2} \left(d^2 b_1 b_2 - b_2/b_1 - b_1/b_2 \right),$$

where b_1 and b_2 are the curvatures and d is the distance between centres.

7. The Gaussian integers and Descartes configurations

The study of Apollonian circle packings depends on the study of the action of the so-called Apollonian group on the space of Descartes configurations. We will develop a similar theory for K-Apollonian packings in other imaginary quadratic fields. Therefore, we will review briefly the case of Apollonian circle packings for comparison. See [13] for details.

We will consider Apollonian circle packings formed from circles in $\widehat{\mathbb{C}}$ (any Apollonian circle packing may be realized this way). Let W_D be a matrix whose columns are $\pi(C_i)$ for any four oriented circles $D : C_1, C_2, C_3, C_4$. By Proposition 6.2, these four circles are in Descartes configuration, or this is the case once all orientations are reversed, if and only if

$$(6) \qquad W_D^t G_M W_D = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} =: R.$$

Let $\mathcal{D}$ denote the space of matrices satisfying this equation, known as the space of Descartes configurations.[4] Since G_M, R, and the elements of $\mathcal{D}$ are all invertible, we obtain

$$W_D R^{-1} W_D^t = G_M^{-1}.$$

This collection of 16 quadratic equations in the curvatures, co-curvatures, and curvature-centres of four circles are the full 'Descartes relations' as in [22]: the circles are in Descartes configuration if and only if these 16 equalities hold. In particular, the upper left corner of this matrix equality is the relation on curvatures (1).

---

[4]Note that these are ordered, oriented configurations in the sense of [14].

Given four circles in Descartes configuration (write $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{M}$), and a chosen subset of three of them, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, the fourth circle $\mathbf{v}_4$ can be swapped out for its alternative $\mathbf{v}_4'$, i.e., the unique other circle which forms a Descartes configuration with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. This takes one configuration in an Apollonian circle packing $\mathcal{P}$ to another in the same packing. The relation that describes this swap in $\mathbb{M}$ is very simple:

$$\mathbf{v}_i + \mathbf{v}_i' = 2 \left(\sum_{j \neq i} \mathbf{v}_j \right).$$

In particular, if the original curvatures are a, b, c, d, then the new curvature is

$$d' = 2(a + b + c) - d.$$

There are four ways to swap out a circle. This is accomplished by right multiplication on $\mathcal{D}$ by the following four matrices of $O_R(\mathbb{R})$ of order two:

$$(7) \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Right multiplication by one of the matrices (7) corresponds to inversion in a circle orthogonal to three of the four circles of the quadruple, as in Figure 10. For example, the four swaps on the so-called *base cluster* shown in Figure 11 are accomplished by the following Möbius transformations (given in the same order as (7)):

$$(8) \qquad z \mapsto \frac{(2i+1)\overline{z} - 2}{2\overline{z} + 2i - 1}, \quad z \mapsto -\overline{z} + 2, \quad z \mapsto \frac{\overline{z}}{2\overline{z} - 1}, \quad z \mapsto -\overline{z}.$$

The group generated by the matrices (7) is commonly referred to as the algebraic Apollonian group, which we will denote $\widehat{\mathcal{A}_{\mathbb{Q}(i)}}$. The group generated by the Möbius transformations (8) is commonly referred to as the geometric Apollonian group, which we will denote $\mathcal{A}_{\mathbb{Q}(i)}$. While the algebraic Apollonian group does not act on individual circles, only ordered quadruples, the geometric Apollonian group is a group of Möbius transformations acting on circles, unordered or ordered quadruples. However, both groups have the property that the orbit of the base quadruple gives the full fundamental Apollonian packing. Furthermore, the two are isomorphic as groups. The following section will elaborate on the isomorphism.

Theorem 7.1 (Graham, Lagarias, Mallows, Wilks, Yan [13, Proof of Theorem 4.3])**.** *There are no relations on the matrices of* (7) *besides the fact that they are of order two. Therefore, the group* $\widehat{\mathcal{A}_{\mathbb{Q}(i)}}$*, and also* $\mathcal{A}_{\mathbb{Q}(i)}$*, is a free product of the four copies of* $\mathbb{Z}/2\mathbb{Z}$*.*

This group is the basic tool in arithmetic results concerning curvatures in Apollonian circle packings, because of the following.
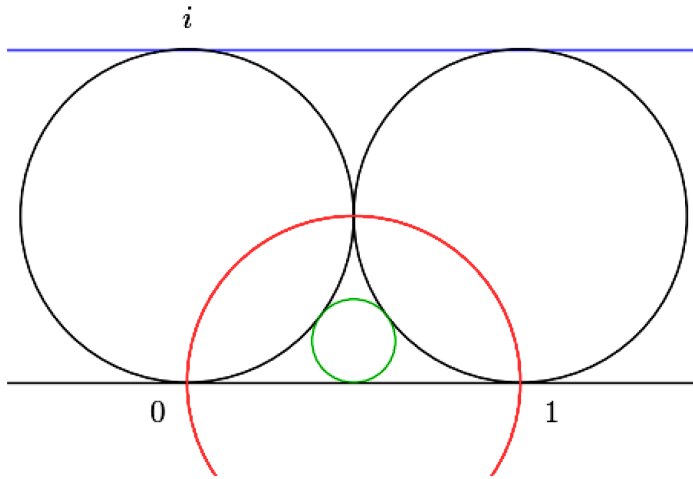
FIGURE 10. A swap for $\mathcal{S}_{\mathbb{Q}(i)}$. The blue and black circles (including two straight lines) form the base quadruple (see Figure 11) of the $\mathbb{Q}(i)$-Apollonian packing containing $\widehat{\mathbb{R}}$. If swapping by preserving the black circles, the blue circle is replaced with the green one, by inversion in the red circle. The new quadruple is formed of the black and green circles.
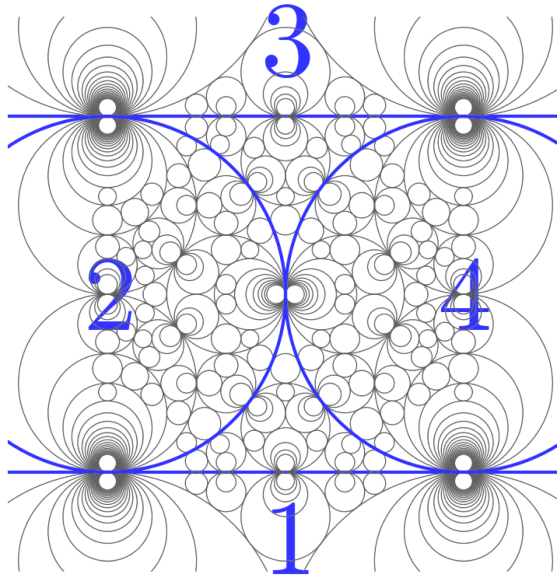

FIGURE 11. Base quadruple for $\mathcal{S}_{\mathbb{Q}(i)}$. The coordinates of the four circles in $\mathbb{M}$ are given, in the labelled order, by the columns of the following matrix: $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ -1 & 1 & 1 & 1 \end{pmatrix}$.

Theorem 7.2 (Graham, Lagarias, Mallows, Wilks, Yan [13, Theorem 4.3]). *Let $\mathcal{P}$ be an Apollonian circle packing.*

(1) *The full set of Descartes configurations contained in $\mathcal{P}$ is a union of 48 orbits of $\widehat{\mathcal{A}_{\mathbb{Q}(i)}}$.*

(2) *Fix a Descartes quadruple $D \in \mathcal{P}$. Then there are 48 matrices $W_D \in \mathcal{D}$ (i.e., satisfying (6)) representing this quadruple. Each of the 48 orbits contains exactly one of these 48 matrices.*

The 48 matrices representing a quadruple are formed by reordering the circles 24 ways, and reversing the orientation of all four simultaneously.

8. Cluster spaces and the algebraic-geometric correspondence

Definition 8.1. A *cluster space* is a set of the form
$$S_R := \{W \in M_{4\times4}(\mathbb{R}) : W^t G_M W = R\},$$
where R is a fixed invertible matrix.

The space $\mathcal{D}$ of Descartes quadruples of the last section is such a space. When the columns of the W lie on $M = 1$ (i.e., the diagonal of R consists of 1's), this can be considered the collection of quadruples of circles, considered in $\mathbb{M}$, which are in a particular configuration (specified by R) with respect to the Pedoe product. In general, we will loosen this requirement, and may interpret columns as sums of circles.

The purpose of the next two results is to show that S_R is a principal homogeneous space under left and right actions on S_R by matrix groups isomorphic to $\mathrm{O}_M(\mathbb{R})$.

Definition 8.2. The left action of $\mathrm{O}_M(\mathbb{R})$ on S_R by matrix multiplication on the left is called the *geometric action*.

Proposition 8.3. *The set S_R is a principal homogeneous space for the geometric action.*

Proof. Since $N \in \mathrm{O}_M(\mathbb{R})$ preserves the form M,
$$(NW)^t G_M NW = W^t G_M W,$$
and so this action preserves S_R. If R is invertible, then $W \in S_R$ are invertible and so the element of $\mathrm{O}_M(\mathbb{R})$ taking any W_1 to W_2, namely $N := W_2 W_1^{-1} \in \mathrm{O}_M(\mathbb{R})$, exists and is unique. That is, $\mathrm{O}_M(\mathbb{R})$ is freely transitive on S_R. $\square$

The reason for the name *geometric* is that, restricting to $\mathrm{O}_M^+(\mathbb{R})$, the left action can be considered a Möbius action on circles via the exceptional isomorphism ρ.

The special case of $R = G_M$ gives $S_R = \mathrm{O}_M(\mathbb{R})$.

Definition 8.4. Write $\mathrm{O}_R(\mathbb{R})$ for the matrices preserving the quadratic form associated to Gram matrix R. The right action of $\mathrm{O}_R(\mathbb{R})$ on S_R by matrix multiplication on the right is called the *algebraic action*.

Proposition 8.5. *The set S_R is a principal homogeneous space for the algebraic action. Furthermore, $\mathrm{O}_R(\mathbb{R}) \cong \mathrm{O}_M(\mathbb{R})$.*

Proof. Let $W_0 \in S_R$. Then $\mathrm{O}_R(\mathbb{R}) = W_0^{-1} \mathrm{O}_M(\mathbb{R}) W_0 \cong \mathrm{O}_M(\mathbb{R})$ preserves the quadratic form given by Gram matrix R, which form is isomorphic to M over $\mathbb{R}$. Therefore right multiplication by this group preserves S_R. The proof is as for the last proposition. $\square$

The algebraic action is sometimes called the Apollonian action. It acts by linear combination on 4-tuples of vectors in $\mathbb{M}$. It cannot be thought of as arising from an action on circles or vectors alone; it only acts on S_R.

The isomorphism between $\mathrm{O}_R(\mathbb{R})$ and $\mathrm{O}_M(\mathbb{R})$ given in Proposition 8.5 is called the *algebraic-geometric correspondence* depending on $W_0 \in S_R$, which we will denote

$$\sigma_{W_0} : \mathrm{O}_M(\mathbb{R}) \to \mathrm{O}_R(\mathbb{R}), \quad M \mapsto W_0^{-1} M W_0.$$

Both the geometric and algebraic actions have a manifestion in the action of Möbius. To discuss this, we have to interpret S_R as *clusters* of circles.

Definition 8.6. Let $n \geq 4$, $n \in \mathbb{Z}$. A *cluster type* is a finite-to-one function

$$f : S_R \to \mathcal{C}irc^n,$$

on a cluster space, with image lying in the collection of n-tuples of oriented circles, and having the form

$$f(W) = \left(\pi^{-1} \left(\sum_{i=1}^{4} a_{i,j} W_i \right) \right)_{j=1}^{n},$$

where $a_{i,j} \in \mathbb{Z}$ and $W_1, \ldots, W_4$ denote the columns of W. In other words, a cluster type determines a collection of circles by linear combination on the columns of W. A cluster type is given by the data of an invertible matrix R and the $a_{i,j} \in \mathbb{Z}$, $1 \leq i \leq 4$, $1 \leq j \leq n$. A collection of circles in the image of a given cluster type is called an *ordered cluster* of the given type. An *unordered cluster* is any collection obtained from an ordered cluster by forgetting ordering.

The previous definition is motivated by the notion of a *Descartes quadruple*, which is the cluster type given by $n = 4$,

$$R = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix},$$

and $a_{i,j} = \delta_{i,j}$.

It is immediate to verify that under these definitions, the Möbius action corresponds to the geometric action, which justifies its name. The algebraic action can also be interpreted as an action of Möb, via the algebraic-geometric correspondence. This is complicated by the need for a *base cluster* W_0 to define the algebraic-geometric correspondence, and by the need to consider ordered clusters as elements of Möb. The precise statement is as follows, and is verified by direct computation.

Proposition 8.7. *Fixing an ordered base cluster B given by $W_B \in S_R$, the left- and right-multiplication action of Möb on Möb corresponds to the algebraic and geometric actions of $\mathrm{O}_R(\mathbb{R})$ and $\mathrm{O}_M(\mathbb{R})$, respectively, on S_R, in the sense that the following diagram commutes (where all arrows between sets are bijections, all*

arrows between groups are isomorphisms, and tightly curved arrows represent group actions):
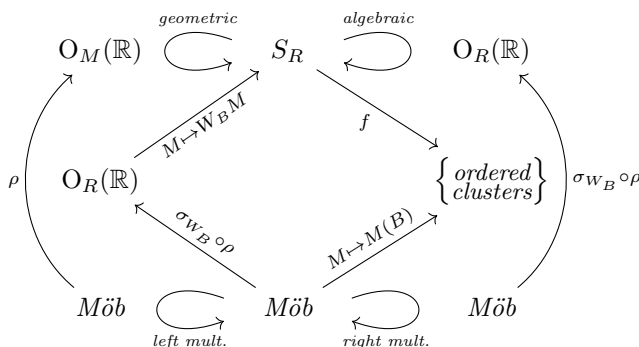
In other words, an element of Möb can be interpreted as its image on the base cluster, and then the right multiplication action of Möb on Möb 'is' the algebraic action on clusters. When the right multiplication of Möb on Möb is interpreted in this way, we will simply refer to it as the algebraic action. A consequence of Proposition 8.7 is the following.

Proposition 8.8. *The geometric and algebraic orbits of the base cluster under any subgroup $G <$ Möb agree.*

9. Apollonian groups, weak and strong, algebraic and geometric

In this section we define the various flavours of Apollonian groups for a general imaginary quadratic field. Recall that $\mathcal{P}_K$ denotes the fundamental packing (Definition 4.12), and let $\overline{\mathcal{P}_K}$ denote its closure in $\widehat{\mathbb{C}}$. The *limit set* (or *residual set*) $\Lambda(G)$ of a subgroup $G \subset$ Möb is the accumulation set of the orbit of the origin.

Definition 9.1. A *weak Apollonian group* for the imaginary quadratic field $K \neq \mathbb{Q}(\sqrt{-3})$, or *weak K-Apollonian group*, is a finitely generated Kleinian group $\mathcal{A}_K < \langle \mathrm{PSL}_2(\mathcal{O}_K), \mathfrak{c} \rangle <$ Möb whose limit set is the closure of the fundamental Apollonian packing for K, i.e., $\Lambda(\mathcal{A}_K) = \overline{\mathcal{P}_K}$.

There is no reason to assume that there is a unique weak K-Apollonian group for a given field K. It is easy to give examples of weak K-Apollonian groups.

Theorem 9.2. *Let K be an imaginary quadratic field. Let $\mathcal{A}$ be the subgroup of Möb generated by $\mathrm{PSL}_2(\mathbb{Z})$ and the matrix $V = \begin{pmatrix} 1 & \tau \\ 0 & -1 \end{pmatrix}$. Then $\mathcal{A}$ is a weak K-Apollonian group.*

Proof. As a subset of the Bianchi group, $\mathcal{A}$ is Kleinian. Since $\mathrm{PSL}_2(\mathbb{Z})$ is finitely generated, so is $\mathcal{A}$. We use Proposition 4.6. Given a circle $M(\widehat{\mathbb{R}})$, the immediately tangent circles are exactly those given by $M'(\widehat{\mathbb{R}})$ where $M' \in M\,\mathrm{PSL}_2(\mathbb{Z})V$. Therefore the orbit of $\widehat{\mathbb{R}}$ includes all of the fundamental packing. Since we have $\mathrm{PSL}_2(\mathbb{Z}) < \mathcal{A}$, all of $\widehat{\mathbb{Q}}$ is in the orbit of 0, so that $\widehat{\mathbb{R}}$ is in the limit set $\Lambda(\mathcal{A})$, and so are all its images, i.e., all of $\mathcal{P}_K$. On the other hand, the orbit of 0 lies within $\mathcal{P}_K$. Therefore $\Lambda(\mathcal{A}) = \overline{\mathcal{P}_K}$. $\qquad\square$

The proof illustrates that the fundamental packing is exactly the orbit of $\widehat{\mathbb{R}}$ under $\mathcal{A}$. All other K-Apollonian packings are orbits of left cosets of $\mathcal{A}$.

Theorem 9.3. *Any K-Apollonian packing is of Hausdorff dimension $\delta_K > 1$.*

Proof. It suffices to prove this for the packing $\mathcal{P}_K$, by the Möbius action. But $\mathcal{P}_K$ is the limit set of a weak K-Apollonian group by Theorem 9.2. For a finitely-generated Kleinian group, the limit set must be one of the following: totally disconnected, a circle, or of Hausdorff dimension > 1 (see, for example, [1, Corollary 1.8] and the citations therein). However, $\mathcal{P}_K$, since it contains $\widehat{\mathbb{R}}$ and other circles, is neither totally disconnected, nor a circle. $\qquad\square$

Let X be a subgroup of $H(\mathbb{Z})$, where H is a semi-simple Lie group, and let $G = \mathrm{Zcl}(X)$ be its Zariski closure in H. Then X is called *thin* if it is of infinite index in $G(\mathbb{Z})$. We are interested in the case $H = \mathrm{O}_M^+ \cong \mathrm{Möb}$. We will consider a weak K-Apollonian group $\mathcal{A}$ to be a subgroup of $\mathrm{O}_M^+(\mathbb{Z}) < \mathrm{O}_M(\mathbb{Z})$.

Theorem 9.4. *Any weak Apollonian group $\mathcal{A}$ for K is thin, and its Zariski closure is either O_M or SO_M.*

In any given situation, to determine which Zariski closure is obtained, it suffices to check whether the generators of $\mathcal{A}$ all lie in SO_M.

Proof. First, we show that $\mathcal{A}$ is of infinite index in $O_M(\mathbb{Z})$. For, its index is equal to the number of K-Apollonian packings in $\mathcal{S}_K$, which is infinite (for example, there is a strip packing in every horizontal strip $k \le (z - \overline{z})i/\sqrt{-\Delta} \le k + 1$ for $k \in \mathbb{Z}$). On the other hand, $\mathcal{A}$ is infinite, as its limit set is infinite.

Let $\mathcal{G}$ be the Zariski closure of $\mathcal{A}$. It is necessarily an algebraic subgroup of O_M defined over $\mathbb{Z}$. Therefore $\mathcal{G}(\mathbb{R})$ must be a Lie subgroup of $O_M(\mathbb{R})$. Our proof imitates that in [9, Lemma 1.6(ii)]. The classification of the a priori possibilities for $\mathcal{G}$ are:

  (1) A finite subgroup.
  (2) A torus or parabolic subgroup.
  (3) A subgroup fixing a form of signature $(1, 2)$.
  (4) O_M.
  (5) SO_M.

We eliminate the first three possibilities in turn. First, $\mathcal{G}(\mathbb{Z})$ is not finite as $\mathcal{A}$ is not finite. The second two possibilities are subgroups of dimension ≤ 2. Any finitely generated Kleinian subgroup in those dimensions is geometrically finite. Therefore, the limit set has Hausdorff dimension at most 1. However, the residual set of $\mathcal{A}$ has Hausdorff dimension > 1, by Theorem 9.3. Therefore, $\mathcal{G} = O_M$ or SO_M. In either case, as a subgroup of infinite index in $O_M(\mathbb{Z})$, $\mathcal{A}$ is of infinite index in $\mathcal{G}(\mathbb{Z})$. Therefore, it is thin. $\qquad\square$

We now observe that the methods above also provide a proof that the subgroup $E_2(\mathcal{O}_K)$ of $\mathrm{PSL}_2(\mathcal{O}_K)$ generated by elementary matrices is thin whenever $\mathcal{O}_K$ is non-Euclidean.

Theorem 9.5. *When $\mathcal{O}_K$ is non-Euclidean, the groups $E_2(\mathcal{O}_K)$ are thin.*

Proof. The group $E_2(\mathcal{O}_K)$ is generated by $\mathrm{PSL}_2(\mathbb{Z})$ and the matrix

$$W = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}.$$

First we show that $\Lambda(E_2(\mathcal{O}_K))$ is the tangency-connected component of $\widehat{\mathbb{R}}$ in $\mathcal{S}_K$. Consider a circle given by $M(\widehat{\mathbb{R}})$. Then the circles $MNW^n(\widehat{\mathbb{R}})$, where $N \in \mathrm{PSL}_2(\mathbb{Z})$ are exactly those tangent to $\widehat{\mathbb{R}}$ in $\mathcal{S}_K$ (see Proposition 3.7). Therefore, the orbit of $\widehat{\mathbb{R}}$ under $E_2(\mathcal{O}_K)$ is the tangency-connected component of $\mathcal{S}_K$ containing $\widehat{\mathbb{R}}$. Since $\mathrm{PSL}_2(\mathbb{Z}) < E_2(\mathcal{O}_K)$, all of $\widehat{\mathbb{R}}$ is in the limit set, so the entire tangency-connected component of $\mathcal{S}_K$ is $\Lambda(E_2(\mathcal{O}_K))$.

By [42, Theorem 7.1], there are infinitely many tangency-connected components in $\mathcal{S}_K$. Hence $E_2(\mathcal{O}_K)$ is of infinite index in $\mathrm{PSL}_2(\mathcal{O}_K)$. Note that this fact was proven in [28] by different methods.

By the same method as Theorem 9.3, this component has Hausdorff dimension exceeding 1. The rest of the proof of thinness is as in Theorem 9.4. $\qquad\square$

Next we define *strong K-Apollonian groups*, also called simply *K-Apollonian groups*. These are weak Apollonian groups with extra structure captured in their relationship to a certain cluster type. We are motivated by the traditional Apollonian group, which is described by its relationship to Descartes quadruples.

Definition 9.6. Let $\mathcal{A} < \mathrm{M\ddot{o}b}$ and fix a cluster space and corresponding cluster type. Suppose that $\mathcal{A}$ is a weak Apollonian group for K for which the set of unordered clusters in $\mathcal{P}_K$ is a principal homogeneous space for $\mathcal{A}$. Then we say that $\mathcal{A}$ is a *strong K-Apollonian group with respect to the cluster type* or simply a *K-Apollonian group* when no confusion will occur. If a base cluster is fixed, then $\mathcal{A}$ corresponds under the algebraic-geometric correspondence to a subgroup $\widehat{\mathcal{A}}$ of $O_R(\mathbb{Z})$. In this case, $\widehat{\mathcal{A}}$ is called an *algebraic K-Apollonian group*, and $\mathcal{A}$ is called a *geometric K-Apollonian group* when distinction is necessary.

The group $\widehat{\mathcal{A}}$ acts on *ordered* clusters but not on unordered clusters or circles.

Theorem 9.7. *The group $\mathcal{A}_{\mathbb{Q}(i)}$ defined in Section 7 is a geometric $\mathbb{Q}(i)$-Apollonian group, and $\widehat{\mathcal{A}_{\mathbb{Q}(i)}}$ is an algebraic $\mathbb{Q}(i)$-Apollonian group.*

Proof. Theorem 7.2, via Proposition 8.8, tells us that the geometric action of $\mathcal{A}_{\mathbb{Q}(i)}$ is transitive on the set of unordered Descartes quadruples. Furthermore, since the algebraic action is induced by the action of $O_R(\mathbb{R})$ on S_R, we also know from Theorem 7.2 that there are no automorphisms of a cluster in $\mathcal{A}_{\mathbb{Q}(i)}$'s algebraic action, hence no elements fixing the base cluster in its geometric action. This proves that the set of unordered quadruples is a principle homogeneous space for $\mathcal{A}_{\mathbb{Q}(i)}$'s geometric action. That its limit set is the strip packing is well known. $\qquad\square$

A general method of proving groups are Apollonian will be developed in Section 11.

10. Topographical Groups

For this section, we concern ourselves with certain special subgroups of $\mathrm{PGL}_2(\mathbb{Z})$.

Definition 10.1. A *superbasis* is a triple (a, b, c) of points of $\widehat{\mathbb{Q}}$ which are pairwise distinct modulo all primes.

The use of the terminology *superbasis* is borrowed from Conway and Fung [5, The First Lecture]: for them, $\mathbf{a} = [a_1, a_2], \mathbf{b} = [b_1, b_2], \mathbf{c} = [c_1, c_2] \in \mathbb{Z}^2$ form a superbasis if they are primitive vectors (i.e., $\gcd(a_1, a_2) = \gcd(b_1, b_2) = \gcd(c_1, c_2) = 1$),

and each pair forms a basis for $\mathbb{Z}^2$. It is evident that the two definitions are naturally in bijection, where the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{P}^1(\mathbb{Z})$ represent the elements $a = a_1/a_2, b = b_1/b_2, c = c_1/c_2 \in \widehat{\mathbb{Q}}$.

Conway shows that the graph whose vertices are unordered superbases, where an edge indicates that two superbases share a basis, is a single tree of valence three, shown in Figure 12. Conway calls this graph the *topograph*.

A matrix of $\mathrm{PGL}_2(\mathbb{Z})$ with columns $\mathbf{a}$ and $\mathbf{b}$ can be interpreted as the superbasis $\mathbf{a}, \mathbf{b}, \mathbf{a}+\mathbf{b}$ as above. This gives a six-to-one map

$$\phi : \mathrm{PGL}_2(\mathbb{Z}) \to \{\text{unordered superbases}\}.$$

The matrices which are interpreted as the superbasis $0, 1, \infty$, in some order, are

$$S := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right\} < \mathrm{PGL}_2(\mathbb{Z}).$$

The set S is a group isomorphic to S_3. Left cosets of S inside $\mathrm{PGL}_2(\mathbb{Z})$ are in bijection with unordered superbases.

Definition 10.2. A *topographical group* is a subgroup of $\mathrm{PGL}_2(\mathbb{Z})$ for which ϕ gives an isomorphism from its Cayley graph under right multiplication (with respect to suitable generators) to Conway's topograph.

By the Cayley graph under right multiplication, it is meant that the edges g and gs (not sg) are joined for each generator s.

There are only finitely many topographical groups. For, suppose H has a Cayley graph isomorphic to Conway's topograph under ϕ. In particular, it is generated by three elements of order two which take $\{0, 1, \infty\}$ to $\{0, -1, \infty\}$, $\{0, 1, 1/2\}$, and $\{1, 2, \infty\}$, respectively. There are only finitely many possibilities mapping $\{0, 1, \infty\}$ to each of these sets, and most possibilities are not of order two. The remaining elements are:

$$\gamma_1 = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Write $\Pi = \mathrm{PGL}_2(\mathbb{Z})$ and $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Write $\Pi(N)$ and $\Gamma(N)$ for their congruence subgroups of level N, respectively.

Theorem 10.3. *The only topographical groups which are normal in Π are $G := \Gamma^3 = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ and $P := \Pi(2) = \langle \rho_1, \rho_2, \rho_3 \rangle$.*

Note that these two groups are normal but not characteristic: the outer automorphism of $\mathrm{PGL}_2(\mathbb{Z})$ maps one to the other (see [7, 44] for more on the outer automorphism). One is a congruence subgroup of $\mathrm{PGL}_2(\mathbb{Z})$ and the other is not; this is a general phenomenon with regard to the outer automorphism [17].

Proof. Suppose a group H is normal and unordered bases form a principal homogeneous space for H under the Möbius action. Then H is normal of index 6 in $\Pi = \mathrm{PGL}_2(\mathbb{Z})$. From the classification of normal subgroups of small index in $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ due to Newman [27], we immediately observe that $H \cap \Gamma$ is one of $\Gamma(2)$, Γ', or Γ^3, where Γ' represents the commutatator subgroup, and $\Gamma(2)$ the congruence subgroup of level 2. The first two of these groups are of index 6 in Γ and the latter is of index 3. All of these groups contain $\Gamma(12)$. But the only normal
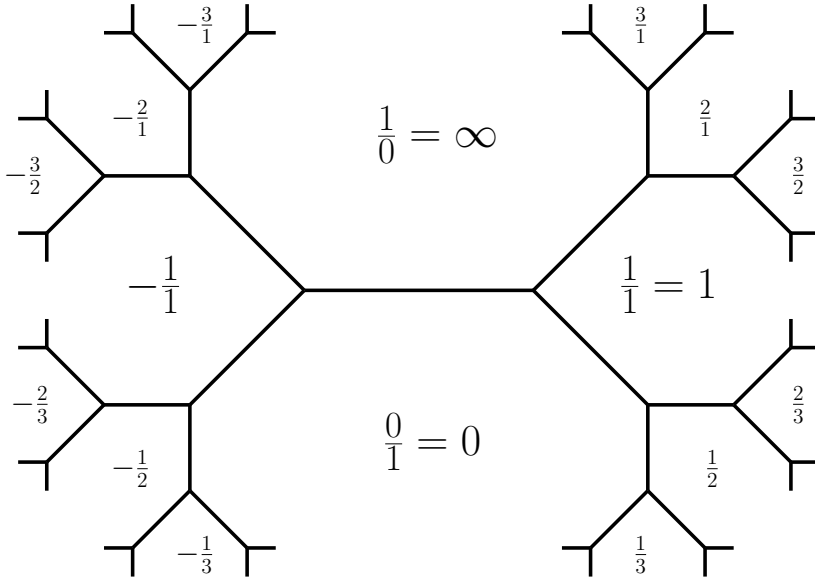
FIGURE 12. Conway's topograph is a valence three tree embedded in the plane such that it breaks up the plane into many infinite regions, each labelled with an element of $\mathbb{P}^1(\mathbb{Z})$. The vertices represent unordered superbases a, b, c, where a, b, c, are given by the three regions adjacent to the vertex. For example, the right-central vertex corresponds to the superbasis $0, 1, \infty$.

subgroups of Π of index 6 containing $\Gamma(12)$ are $G = \Gamma^3$ and $P = \Pi(2)$ (see [31]), whose intersections with Γ are exactly Γ^3 and $\Gamma(2)$, respectively.

Let
$$G_0 = \langle \gamma_1, \gamma_2, \gamma_3 \rangle,$$
where
$$\gamma_1 = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
and
$$P_0 = \langle \rho_1, \rho_2, \rho_3 \rangle,$$
where
$$\rho_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.$$
The generators γ_i and ρ_i are each of order two. Next we verify that $G_0 = G$ and $P_0 = P$. If we set, as usual,
$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Z}),$$
then $\gamma_1 = TST^{-1}$, $\gamma_2 = T^{-1}ST$, and $\gamma_3 = S$. Since $\gamma_1\gamma_2\gamma_3 = T^3$, $G = \Gamma^3 < G_0$. But Γ^3 is normal, so it contains all conjugates of the element S, hence $G_0 < \Gamma^3 = G$. That the ρ_i generate $P = \Pi(2)$ is immediate.

Finally, to verify that G and P are topographical is now a simple computation using the Cayley graph on the generators above. □

We will need the following lemma. Let

$$B = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$

which is the stabilizer of ∞.

Lemma 10.4.

$$G \cap B = \langle \gamma_1 \gamma_2 \gamma_3 \rangle = \left\{ \begin{pmatrix} 1 & 3n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

We will use the notation $G_B := G \cap B$.

Proof. We use the identification of the Cayley graph of G under right multiplication with the topograph, as in the last theorem. One can draw the topograph in the plane so that the infinite 'regions' between branches are labelled with elements of $\widehat{\mathbb{Q}}$, and the 'shoreline' consists of those superbases and bases containing that element, in the sense of Conway (see [5, The First Lecture] for details). The stabilizer of ∞ is exactly those elements of G which map $\infty, 0, -1$ to a superbasis surrounding the region ∞, *and keep ∞ in first position*. Travelling along the 'shore' of the ∞ region by explicit computation, we see that this is exactly the group generated by $\gamma_1 \gamma_2 \gamma_3$. Finally,

$$\gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

$\square$

The structure of $\mathcal{A}_{\mathbb{Q}(i)}$ and the collection of quadruples has much in common with the structure of topographical groups and the collection of superbases. In particular, the collection of (unordered) Descartes quadruples can be considered the vertices of a graph, where edges indicate that two quadruples share a subset of three circles. This graph is a tree of valence four which is the Cayley graph of $\mathcal{A}_{\mathbb{Q}(i)}$ with respect to the four given generators. Each generator swaps out one circle of a quadruple, changing it into an adjacent quadruple. For more on this, see [41]. We will imitate this structure for other fields in later sections.

Finally, it is worth remarking that superbases can be interpreted as the tangency points of a triple of intervals covering $\widehat{\mathbb{R}}$; such intervals can be put in bijection with elements of $\mathbb{R}^3$ lying on a hypersurface; this develops a story analogous to that of Section 6 with the exceptional isomorphism $\mathrm{O}_{2,1}^+(\mathbb{R}) \cong \mathrm{PGL}_2(\mathbb{R})$ in place of $\mathrm{O}_{3,1}^+(\mathbb{R}) \cong \mathrm{PGL}_2(\mathbb{C})$, and quadratic forms in place of Hermitian forms. This has pleasing connections to the rational parametrization of Pythagorean triples; see [30].

11. Sufficient conditions for Apollonian groups

In this section we provide a set of sufficient conditions guaranteeing a group is Apollonian. The main theorem, Theorem 11.3, will be used to verify all the example Apollonian groups in the remainder of the paper.

Let $\mathrm{M\ddot{o}b}_{\widehat{\mathbb{R}}}$ be the set of Möbius transformations fixing $\widehat{\mathbb{R}}$. Define

$$\eta : \mathrm{M\ddot{o}b}_{\widehat{\mathbb{R}}} \to \mathrm{PGL}_2(\mathbb{R})$$

to be the restriction of the action of such a Möbius transformation to $\widehat{\mathbb{R}}$. This map is surjective.

We will need the following technical term for the theorem.

Definition 11.1. The *base prong* for $\mathcal{S}_K$ is the collection of four K-Bianchi circles consisting of $\widehat{\mathbb{R}}$ and three circles immediately tangent to $\widehat{\mathbb{R}}$ at 0, 1, and ∞. A *three-prong* is a collection of four circles that has the same pairwise Pedoe products as the *base prong*. Equivalently, the *three-prongs* are the clusters in the cluster space containing the base prong. A three-prong is said to be *centred on* any of its circles which is tangent to the other three.

In the case of $\mathbb{Q}(i)$, three-prongs are exactly Descartes quadruples and a prong is centred on all of its circles. In other cases, the central circle is unique. Evidently, the base prong exists and is unique for a particular Schmidt arrangement $\mathcal{S}_K$. The matrix R for which the base prongs correspond to the cluster space S_R is given explicitly in the next section, but is not needed at the moment.

Lemma 11.2. *Three-prongs centred on $\widehat{\mathbb{R}}$ are in bijection with superbases.*

Proof. Any K-Bianchi circle tangent to $\widehat{\mathbb{R}}$ at α/β (in lowest terms) is given by

$$\begin{pmatrix} \alpha & -\gamma + \alpha k\tau \\ \beta & -\delta + \beta k\tau \end{pmatrix},$$

where $\alpha\delta - \beta\gamma = 1$, $\alpha, \delta, \beta, \gamma \in \mathbb{Z}$, and where $k = 1$ if and only if the circle is immediately tangent (Proposition 3.7). Therefore, its Pedoe embedding into $\mathbb{M}$ is given by Proposition 3.5:

$$b = k(\overline{\tau} - \tau)\beta^2 i, \quad b' = k(\overline{\tau} - \tau)\alpha^2 i, \quad a = i(-(\alpha\delta - \beta\gamma) + k(\overline{\tau} - \tau)\alpha\beta) = i(k(\overline{\tau} - \tau) - 1).$$

Then the Pedoe product of two K-Bianchi circles tangent to $\widehat{\mathbb{R}}$ at α_1/β_1 and α_2/β_2 (in lowest terms) is

$$1 + \frac{1}{2}k_1 k_2 (\overline{\tau} - \tau)^2 (\alpha_1 \beta_2 - \alpha_2 \beta_1)^2,$$

where $k_1, k_2 \in \mathbb{Z}$. Therefore, we obtain

$$1 + \frac{1}{2}(\overline{\tau} - \tau)^2$$

if and only if α/β and γ/δ satisfy $\alpha\delta - \beta\gamma = \pm 1$ and the circles are immediately tangent. The result follows. $\square$

Theorem 11.3. *Let $\mathcal{A} < \langle \mathrm{PSL}_2(\mathcal{O}_K), \mathfrak{c} \rangle < M\ddot{o}b$ be a finitely generated Kleinian group. Suppose that $\mathcal{A}$ takes circles in $\mathcal{P}_K$ to circles in $\mathcal{P}_K$. Suppose further that there is a cluster type, and a cluster (call it the* base cluster)*, for which the following conditions hold:*

- (i) *The base cluster is contained in $\mathcal{P}_K$.*
- (ii) *The base cluster is the unique cluster containing the base prong.*
- (iii) *In the orbit of the base cluster, there are clusters containing the three base prongs centred on $\widehat{\mathbb{R}}$ with tangencies at $(0, 1/2, 1)$, $(1, 2, \infty)$, and $(-1, 0, \infty)$.*
- (iv) *There are elements of $\mathcal{A}$ taking a cluster with three-prong centred on $\widehat{\mathbb{R}}$ to a cluster with three-prong centred on the circles immediately tangent to $\widehat{\mathbb{R}}$ at 0, 1, and ∞.*
- (v) *No automorphism of the base cluster is contained in $\mathcal{A}$.*

(vi) *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two three-prongs centred on the same circle C, but oth-
erwise sharing no circles, and if $\mathcal{C}_1$ and $\mathcal{C}_2$ are clusters containing $\mathcal{B}_1$ and
$\mathcal{B}_2$, respectively, then $\mathcal{C}_1$ and $\mathcal{C}_2$ have no circles in common besides C.*

(vii) *The base cluster is tangency-connected.*

Then, $\mathcal{A}$ is an Apollonian group for K.

Proof. First, we will show that all clusters are in the orbit of the base cluster. The
orbit of the base cluster under the geometric (left multiplication) or algebraic (right
multiplication) actions is the same (see Proposition 8.8). This allows us to focus
on the algebraic action when needed.

To begin, we will show that any cluster in $\mathcal{P}_K$ which contains a three-prong
centred on $\widehat{\mathbb{R}}$ is in the orbit of the base cluster. It suffices to observe that

(1) there is exactly one cluster in $\mathcal{P}_K$ containing any given three-prong centred
on any fixed circle (this is by assumptions (i) and (ii), and the Möbius
action),

(2) the three-prongs centred on $\widehat{\mathbb{R}}$ are in bijection with superbases (Lemma
11.2), and

(3) the orbit of the base cluster contains a cluster centred on $\widehat{\mathbb{R}}$ at any super-
basis. This follows by induction from (iii). More precisely, the topograph
is connected, and the three superbases specified are the three superbases
adjacent to $(0, 1, \infty)$ in the topograph.

Let C be the circle immediately tangent to $\widehat{\mathbb{R}}$ at 0, 1, or ∞. By assumptions (iv)
and (i), there is, in the orbit of the base cluster, a cluster containing a three-prong
centred on C. But then by the last observation, we see that in the orbit of the base
cluster there is a cluster containing a three-prong centred on any circle immediately
tangent to $\widehat{\mathbb{R}}$.

By induction we see that in fact for any circle C in $\mathcal{P}_K$, since it is obtained from
$\widehat{\mathbb{R}}$ by a finite chain of immediate tangencies, there is in the orbit of the base cluster,
a cluster containing a three-prong centred on C. Every cluster contains some three-
prong centred on some circle, since the base cluster does by assumption, and that
all clusters have the same pairwise Pedoe products. So by observation (1), every
cluster is in the orbit of the base cluster. Hence all clusters are in the orbit of any
cluster, by the algebraic action.

That $\mathcal{A}$ acts freely and transitively on the clusters of $\mathcal{P}_K$ now follows from
assumption (v).

Now we must show that $\Lambda(\mathcal{A}) = \overline{\mathcal{P}_K}$. First, the immediate tangency points of
any fixed circle $C \in \mathcal{P}_K$ with other circles of $\mathcal{P}_K$ are dense in C, by Proposition 4.6.
From this observation, it suffices to show that all tangency points of all clusters lie
in $\Lambda(\mathcal{A})$ to conclude that $\mathcal{P}_K \subset \Lambda(\mathcal{A})$. Since the orbit of 0 is contained in $\mathcal{P}_K$, it
would follow that $\Lambda(\mathcal{A}) = \overline{\mathcal{P}_K}$.

First, 0 is in $\Lambda(\mathcal{A})$, and, therefore, by the action of $\mathcal{A}$, every cluster contains at
least one point of $\Lambda(\mathcal{A})$ among its tangency points.

Next we will show that there is a sequence of clusters whose tangency points all
approach 0. First observe that there is a sequence of disjoint superbases approach-
ing 0. To finish the proof, we will show that the tangency points of the clusters
containing the superbases also approach 0. Consider any two superbases in this
sequence which are disjoint; then the clusters they define may not share any circles
besides $\widehat{\mathbb{R}}$ by (vi). Since there are only finitely many circles of curvature below any

fixed bound, the curvatures in the cluster (besides that of $\widehat{\mathbb{R}}$) must be approaching ∞. For a tangency connected cluster (item (vii)), this implies the tangency points are all approaching 0.

By the observation of the previous paragraph, there is a sequence of clusters whose tangency points all approach 0. By Möbius symmetry, for any tangency point x of $\mathcal{P}_K$, there is a sequence of clusters whose tangency points all approach x. Since every cluster carries a point of $\Lambda(\mathcal{A})$ among its tangency points, this shows that $x \in \Lambda(\mathcal{A})$ and we are done. $\square$

In the following sections we define example Apollonian groups for each imaginary quadratic field. These definitions are not unique, and the author has endeavoured to find pleasing choices.

12. K-CLUSTERS FOR IMAGINARY QUADRATIC FIELDS

In this section, we develop a general theory that yields an Apollonian group for any imaginary quadratic K (save $\mathbb{Q}(\sqrt{-3})$, as usual). In the Euclidean cases, it is possible to replace this group with one which is freely generated by elements of order two, and this is the purpose of later sections. We begin, however, taking K generally.

Definition 12.1. Given a set, D, of four oriented circles $C_1, \ldots, C_4$ corresponding to vectors $\mathbf{v}_1, \ldots, \mathbf{v}_4$ in $\mathbb{M}$, define the matrix W_D formed of the columns

$$(9) \qquad \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4.$$

We say that $C_1, \ldots, C_4$ form a K-*cluster*, or K-*quadruple*, if W_D satisfies the relationship $W_D^t G_M W_D = R$, where the matrix $R = R_0$ is defined to be

$$R_0 := \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1+\frac{1}{2}\Delta & 1+\frac{1}{2}\Delta \\ -1 & 1+\frac{1}{2}\Delta & 1 & 1+\frac{1}{2}\Delta \\ -1 & 1+\frac{1}{2}\Delta & 1+\frac{1}{2}\Delta & 1 \end{pmatrix}.$$

Definition 12.2. In the case that $\Delta \equiv 1 \pmod 4$, it will be convenient to change variables so that W_D is the matrix represented by the columns

$$(10) \qquad \mathbf{v}_1, \frac{1}{2}(-\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4), \frac{1}{2}(\mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4), \frac{1}{2}(\mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4).$$

Then, R becomes

$$R_1 := \begin{pmatrix} 1 & -1/2 & -1/2 & -1/2 \\ -1/2 & \frac{1-\Delta}{4} & \frac{1+\Delta}{4} & \frac{1+\Delta}{4} \\ -1/2 & \frac{1+\Delta}{4} & \frac{1-\Delta}{4} & \frac{1+\Delta}{4} \\ -1/2 & \frac{1+\Delta}{4} & \frac{1+\Delta}{4} & \frac{1-\Delta}{4} \end{pmatrix}.$$

The change of variables is given by $R_0 = S^t R_1 S$, where

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The interpretation of the cluster space as a space of clusters of circles is unchanged. This is done to simplify the generators in the next definition.
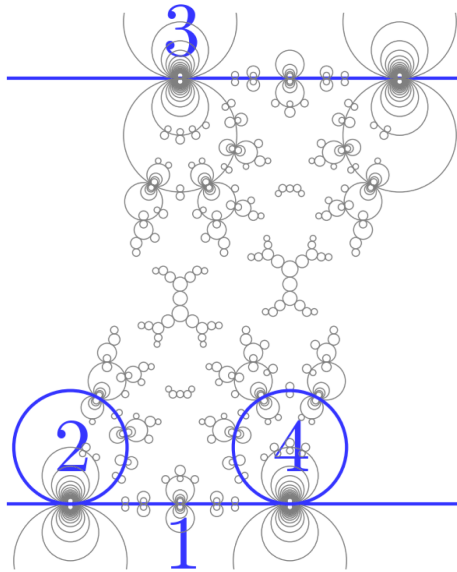
FIGURE 13. The base K-cluster, here shown for $\mathcal{S}_{\mathbb{Q}(\sqrt{-15})}$.

We remark that

$$R_0^{-1} = \frac{1}{\Delta} \begin{pmatrix} \Delta+3 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad R_1^{-1} = \frac{1}{\Delta} \begin{pmatrix} \Delta+3 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{pmatrix}.$$

In particular, that tells us the "Descartes equation" for K-clusters: the curvatures a, b, c, d of four circles in a K-cluster satisfy $2(a^2 + b^2 + c^2 + d^2) - (a+b+c+d)^2 - (\Delta+4)a^2 = 0$.

In particular, a K-cluster always consists of a three-prong as defined in the last section, and S_R is the cluster space of three-prongs. In the case $\Delta \neq -4$, these three circles are disjoint from one another, as in Figure 13. The definition above reduces to the usual Descartes quadruple (where they are all mutually tangent) when $\Delta = -4$.

Definition 12.3. We define the matrix group $\widehat{\mathcal{A}_K^0} \subset O_R(\mathbb{R})$ to be the group generated by the following generators. For the case $\Delta \equiv 0 \pmod 4$:

(11)
$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1+\Delta/4 & 1 & 1+\Delta/4 \\ 0 & 1 & 0 & 0 \\ 1 & -1-\Delta/4 & 0 & -1-\Delta/4 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

while for the case $\Delta \equiv 1 \pmod 4$:

(12)
$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \frac{\Delta+3}{4} & 0 \\ 1 & 1 & -\frac{\Delta-1}{4} & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -\frac{\Delta+3}{4} & 0 \end{pmatrix}.$$

We wish to associate to this a group of Möbius transformations via the algebraic-geometric correspondence. Define the *base cluster* D to be given by the images of $\widehat{\mathbb{R}}$ under

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & \tau \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & \tau-1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1-\tau \\ 1 & -\tau \end{pmatrix}.$$

The associated matrix W_D to this K-cluster is, for $\Delta \equiv 0 \pmod 4$,

$$W_D = W_D^0 := \begin{pmatrix} 0 & 0 & \eta & \eta \\ 0 & \eta & 0 & \eta \\ 0 & 0 & 0 & \eta \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

and for $\Delta \equiv 1 \pmod 4$,

$$W_D = W_D^1 := \begin{pmatrix} 0 & \eta & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & \eta/2 & \eta/2 & -\eta/2 \\ -1 & 1/2 & 1/2 & 1/2 \end{pmatrix},$$

where $\eta = i(\overline{\tau} - \tau) = \sqrt{-\Delta}$. Note that $W_D^0 = W_D^1 S$.

The four generators of $\widehat{\mathcal{A}_K^0}$, under the algebraic-geometric correspondence via the base quadruple, become

(13)
$$\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1-\tau \\ 0 & -1 \end{pmatrix}.$$

The reader will recognise the first three matrices as $\gamma_1, \gamma_2, \gamma_3$, so that $G < \widehat{\mathcal{A}_K^0}$. The last matrix takes $\widehat{\mathbb{R}}$ to the circle tangent to $\widehat{\mathbb{R}}$ at ∞. The group generated by these Möbius transformations will be called $\mathcal{A}_K^0$. Note that $\mathcal{A}_{\mathbb{Q}(i)} \neq \mathcal{A}_{\mathbb{Q}(i)}^0$, where $\mathcal{A}_{\mathbb{Q}(i)}$ is as defined in Section 7.

Theorem 12.4. *Suppose $\Delta \leq -15$. Then $\mathcal{A}_K^0$ (or $\widehat{\mathcal{A}_K^0}$) has the presentation*

$$\langle s_1, s_2, s_3, r : s_1^2 = s_2^2 = s_3^2 = r^2 = 1, rs_1s_2s_3 = s_3s_2s_1r \rangle.$$

In fact, this presentation may be realised by taking s_1, s_2, s_3, r to be the matrices (11) *or* (12), *depending on whether $\Delta \equiv 0$ or $1 \pmod 4$.*

To prove this, we need a result from the Bass-Serre theory of groups acting on trees (in our case, the tree will be a tangency-connected component of the graph of immediate tangencies of $\mathcal{P}$; see Definition 4.8).

Theorem 12.5 (Serre [38, Section 1.4]). *Suppose that a group G acts on a tree X with inversion (i.e., there exists $g \in G$ and e an edge of X such that $g \cdot e$ is again e, but with orientation reversed). Suppose the action is transitive on vertices and transitive on edges. Let v be a vertex of X and let e be an adjacent edge. Let G_v*

be the stabilizer of v, let G_e be the stabilizer of e, and let G' be the stabilizer of v and e (hence preserving the direction of e). Then G' is of index 2 in G_e and

$$G \cong G_v *_{G'} G_e,$$

where $*$ denotes the free product with amalgamation.

We will also need a lemma.

Lemma 12.6. *The stabilizer of $\widehat{\mathbb{R}}$ in $\mathcal{A}_K^0$ is G.*

Proof. Write $\gamma_1, \gamma_2, \gamma_3, \rho$ for the generators (13) of $\mathcal{A}_K^0$. It is clear that G is in the stabilizer, since $G = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$.

To show the opposite containment, let w be a word in the γ_i and ρ which lies in $\mathrm{PGL}_2(\mathbb{R})$. Build up the word w character-by-character from the left, applying the algebraic right action to the base cluster, to produce a series of three-prongs. A character changes the central circle of the three-prong only if it is equal to ρ. When it changes, it changes to an immediately tangent circle. Since the immediate tangency graph is a tree (Theorem 5.1), the sequence of central circles must be of the form

$$\widehat{\mathbb{R}}, C_1, C_2, \ldots, C_{n-1}, C_n, C_{n-1}, \ldots, C_2, C_1, \widehat{\mathbb{R}}.$$

Define the middle substring $s = \rho \gamma_{i_1} \cdots \gamma_{i_j} \rho$ of the word corresponding to the central circles

$$C_{n-1}, C_n, C_{n-1}.$$

Then this word, acting on the right, fixes C_{n-1} and, therefore, is an element of $\mathrm{PGL}_2(\mathbb{R})$.

There are two possible cases: $s \in G$ or $s \notin G$. In the first case, we can replace this part of the word with a string formed of γ_i's. Repeating this process, we either eliminate all ρ's, so that $w \in G$, or we arrive in the second case. Therefore, it suffices to show that whenever $\gamma \in G$ and $\rho \gamma \rho \in \mathrm{PGL}_2(\mathbb{R})$, then $\rho \gamma \rho \in G$.

Let $G' = \rho G \rho$. Then it is a direct computation that $G' \cap \mathrm{PGL}_2(\mathbb{R}) = G_B$, and we are done. $\square$

Proof of Theorem 12.4. The group $\widehat{\mathcal{A}_K^0}$ is isomorphic to the group $\mathcal{A}_K^0$ of Möbius transformations given by generators (13). As Möbius transformations act on circles, preserving tangencies, and $\mathcal{A}_K^0$ takes a K-Apollonian packing $\mathcal{P}$ back to itself, it acts on the (immediate) tangency tree of $\mathcal{P}$ (it is a tree by Theorem 5.1). The fourth generator of (13), call it r, reverses the edge between circle $\widehat{\mathbb{R}}$ and $\widehat{\mathbb{R}} + \tau$. Note that the topographical group G is a subgroup of $\mathcal{A}_K^0$.

The orbit of rG acting on $\widehat{\mathbb{R}}$ is the collection of K-Bianchi circles immediately tangent to $\widehat{\mathbb{R}}$. Thus $\mathcal{A}_K^0$ maps $\widehat{\mathbb{R}}$ to all immediately tangent circles. Therefore the action is transitive on vertices. The orbit of ∞ under G is $\widehat{\mathbb{Q}}$. This implies that the action is transitive on edges (which correspond to tangency points).

The stabilizer of ∞ is $\langle G_B, r \rangle$ (see Lemma 10.4 for the definition of G_B). Since conjugation by r acts as the nontrivial automorphism of $G_B \cong \mathbb{Z}$ (a simple direct computation), we have that this stabilizer is isomorphic to the nontrivial semi-direct product $\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$. The stabilizer of $\widehat{\mathbb{R}}$ is G by Lemma 12.6. The stabilizer of the directed edge $\widehat{\mathbb{R}} \xrightarrow{\infty} (\widehat{\mathbb{R}} + \tau)$ is G_B.

With these data, Theorem 12.5 describes the structure of $\mathcal{A}_K^0$ as $G *_{G_B} \langle G_B, r \rangle$. Recalling that G has presentation $\langle \gamma_1, \gamma_2, \gamma_3 \rangle$ and with reference to Lemma 10.4, the presentation is now a direct computation. $\square$

Theorem 12.7. *Suppose $\Delta \leq -15$. Then the group $\mathcal{A}_K^0$ is a geometric Apollonian group for K, and $\widehat{\mathcal{A}_K^0}$ is an algebraic Apollonian group for K.*

Proof. This proof uses Theorem 11.3. That $\mathcal{A}_K^0$ is a finitely-generated Kleinian group is evident. Since its generators take $\widehat{\mathbb{R}}$ to itself or immediately tangent circles, it takes elements of $\mathcal{P}_K$ to $\mathcal{P}_K$. It now suffices to verify some details about the base cluster and $\mathcal{A}_K^0$. Items (i) and (ii) are immediate.

It was noted above that $\mathcal{A}_K^0$ contains the topographical group G, which implies item (iii).

For item (iv), it suffices to exhibit such elements: the last element of (13), together with its conjugations by γ_1 and γ_3, will suffice.

To verify (v), we must appeal to the tangency tree. In particular, the automorphisms of the base cluster must stabilize $\widehat{\mathbb{R}}$; this stabilizer is exactly G by Lemma 12.6. But G contains no automorphisms of superbases, so there are no automorphisms of the base cluster in $\mathcal{A}_K^0$.

Finally, items (vi) and (vii) are immediate. $\square$

13. Cubes and cubicles in $K = \mathbb{Q}(\sqrt{-2})$

In the case of $K = \mathbb{Q}(\sqrt{-2})$, there is a very pretty Apollonian group (an alternative to the general one in the last section), and we will elaborate on it somewhat here.

In the graph of tangencies, we find cubes, as in Figure 14. A cube is defined to be eight circles in the following arrangement specified by the Pedoe products. Here W_C is a 4×8 matrix which has as columns the eight $\pi(C_i)$:

$$(14) \qquad W_C^t G_M W_C = \begin{pmatrix} 1 & -1 & -3 & -1 & -5 & -3 & -1 & -3 \\ -1 & 1 & -1 & -3 & -3 & -5 & -3 & -1 \\ -3 & -1 & 1 & -1 & -1 & -3 & -5 & -3 \\ -1 & -3 & -1 & 1 & -3 & -1 & -3 & -5 \\ -5 & -3 & -1 & -3 & 1 & -1 & -3 & -1 \\ -3 & -5 & -3 & -1 & -1 & 1 & -1 & -3 \\ -1 & -3 & -5 & -3 & -3 & -1 & 1 & -1 \\ -3 & -1 & -3 & -5 & -1 & -3 & -1 & 1 \end{pmatrix}.$$

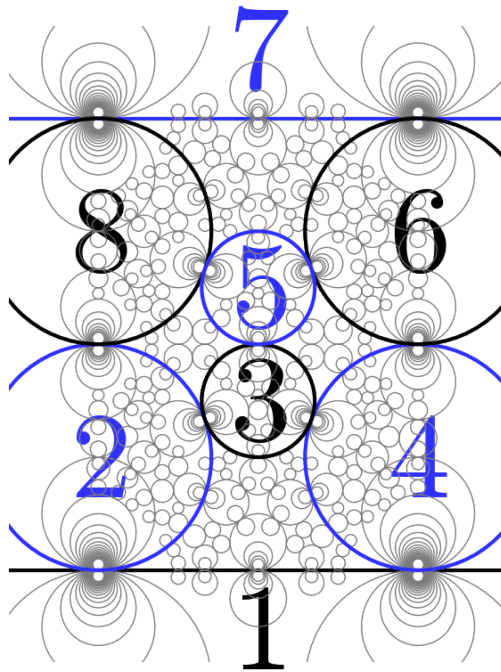This matrix represents the following arrangement of tangencies:
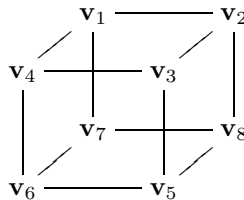
FIGURE 14. Base cube for $\mathcal{S}_{\mathbb{Q}(\sqrt{-2})}$. The coordinates of the eight circles in $\mathbb{M}$ are given, in the labelled order, by the columns of the following matrix:

$$\begin{pmatrix} 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & 4\sqrt{2} & 4\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 4\sqrt{2} & 2\sqrt{2} & 4\sqrt{2} & 2\sqrt{2} & 0 & 2\sqrt{2} \\ 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 \\ -1 & 1 & 3 & 1 & 5 & 3 & 1 & 3 \end{pmatrix}.$$

The two cubicles forming the cube are shown in blue and black, respectively.

It is evident that a cube contains eight $\mathbb{Q}(\sqrt{-2})$-clusters as defined in the previous section, corresponding to the eight vertices of the cube (considered up to reordering the three circles tangent to a central one). Any such $\mathbb{Q}(\sqrt{-2})$-cluster defines a unique cube (this follows from verifying the fact for the base cluster).

We define a cubicle to be a subset of four circles of a given cube so that no two are tangent. There are two cubicles in a cube. A cubicle satisfies this arrangement:[5]

$$(15) \qquad\qquad W_D^t G_M W_D = \begin{pmatrix} 1 & -3 & -3 & -3 \\ -3 & 1 & -3 & -3 \\ -3 & -3 & 1 & -3 \\ -3 & -3 & -3 & 1 \end{pmatrix}.$$

A cubicle is contained in a unique cube. By considering a single cubicle in $\mathbb{Q}(\sqrt{-2})$

---

[5]Incidentally, this gives a relation on the curvatures of a cubicle: $8(a^2 + b^2 + c^2 + d^2) = 3(a + b + c + d)^2$ which we do not need here.
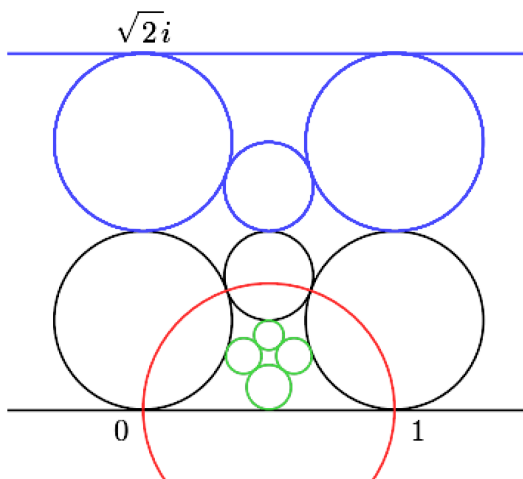
FIGURE 15. The blue and black circles (including two straight lines) form the base cube of the $\mathbb{Q}(\sqrt{-2})$-Apollonian packing containing $\widehat{\mathbb{R}}$. If swapping through the side represented by black circles, the blue circles are replaced with green ones, by inversion in the red circle. The new cube is formed of the black and green circles.

(we use Figure 14), one can obtain equations that determine the cube from the cubicle:[6]

$$2\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_3 - \mathbf{v}_6 + \mathbf{v}_8,$$
$$2\mathbf{v}_4 = \mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_6 - \mathbf{v}_8,$$
$$2\mathbf{v}_5 = -\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_6 + \mathbf{v}_8,$$
$$2\mathbf{v}_7 = \mathbf{v}_1 - \mathbf{v}_3 + \mathbf{v}_6 + \mathbf{v}_8.$$

Any individual face of a cube consists of four $\mathbb{Q}(\sqrt{-2})$-Bianchi circles satisfying the following relations (in particular, they are tangent in a loop):

$$(16) \qquad W_D^t G_M W_D = \begin{pmatrix} 1 & -1 & -3 & -1 \\ -1 & 1 & -1 & -3 \\ -3 & -1 & 1 & -1 \\ -1 & -3 & -1 & 1 \end{pmatrix}.$$

Then there are exactly two ways to complete a face to a cube of $\mathbb{Q}(\sqrt{-2})$-Bianchi circles (again, this need only be verified on the base cube). This implies that given a cube, there are six *swaps* one can perform which fix one side and swap out the remaining four circles. Geometrically, this is accomplished by reflecting in the circle C orthogonal to all circles in the fixed side, as shown in Figure 15.

These six swaps can be described in terms of the cube as follows. Fixing $\mathbf{v}_1, \ldots, \mathbf{v}_4$, the two ways to complete to a cube containing this side are to add

---

[6]It is also nice to observe that all body diagonal sums agree: $\mathbf{v}_1 + \mathbf{v}_5 = \mathbf{v}_2 + \mathbf{v}_6 = \mathbf{v}_3 + \mathbf{v}_7 = \mathbf{v}_4 + \mathbf{v}_8$, and that on each individual face, diagonal sums again agree: $\mathbf{v}_1 + \mathbf{v}_3 = \mathbf{v}_2 + \mathbf{v}_4$, etc.
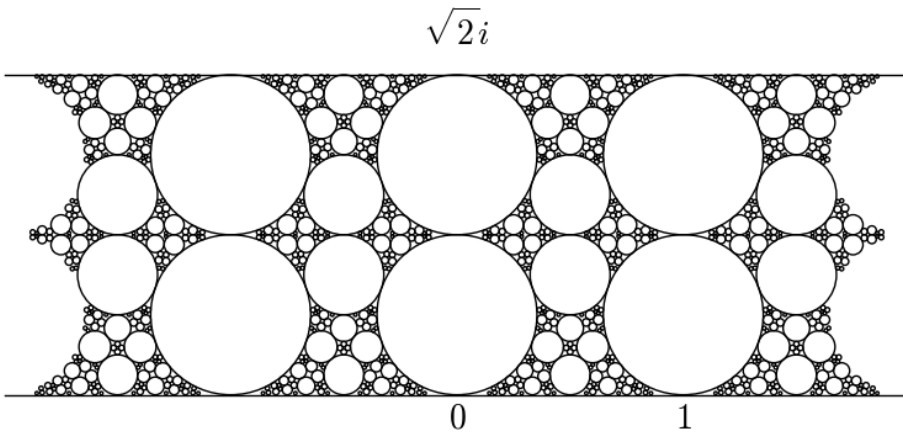
$$\sqrt{2}\,i$$

FIGURE 16. The $\mathbb{Q}(\sqrt{-2})$-Apollonian packing containing $\widehat{\mathbb{R}}$.

$\mathbf{v}_5, \ldots, \mathbf{v}_8$ or $\mathbf{v}'_5, \ldots, \mathbf{v}'_8$ where

$$\mathbf{v}_5 + \mathbf{v}'_5 = -2\mathbf{v}_1 + 4\mathbf{v}_2 + 4\mathbf{v}_4,$$
$$\mathbf{v}_6 + \mathbf{v}'_6 = 4\mathbf{v}_1 - 2\mathbf{v}_2 + 4\mathbf{v}_3,$$
$$\mathbf{v}_7 + \mathbf{v}'_7 = 4\mathbf{v}_2 - 2\mathbf{v}_3 + 4\mathbf{v}_4,$$
$$\mathbf{v}_8 + \mathbf{v}'_8 = 4\mathbf{v}_1 + 4\mathbf{v}_3 - 2\mathbf{v}_4.$$

If we express a cube as a real 4×4 matrix whose columns are the cubicle $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6, \mathbf{v}_8$, then these swaps correspond to right multiplication by

$$(17) \qquad \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 3 & 1 & 0 & 3 \\ 3 & 0 & 1 & 3 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 3 & 1 & 3 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{pmatrix}.$$

Definition 13.1. Call the group generated by these matrices $\widehat{\mathcal{A}_{\mathbb{Q}(\sqrt{-2})}}$.

The Möbius transformations which realise the six swaps on the base cube of Figure 14 are given by (in the same order as the generators (17)),

$$(18) \qquad z \mapsto \frac{\overline{z}}{2\overline{z} - 1}, \quad z \mapsto -\overline{z} + 2, \quad z \mapsto \frac{(3 + 2\sqrt{-2})\overline{z} - 4}{4\overline{z} - 3 + 2\sqrt{-2}},$$

$$z \mapsto -\overline{z}, \quad z \mapsto \frac{(1 + 2\sqrt{-2})\overline{z} - 2}{4\overline{z} - 1 + 2\sqrt{-2}}, \quad z \mapsto \frac{(1 + 2\sqrt{-2})\overline{z} - 4}{2\overline{z} - 1 + 2\sqrt{-2}}.$$

Definition 13.2. Call the group generated by these transformations $\mathcal{A}_{\mathbb{Q}(\sqrt{-2})}$.

Definition 13.3. The *cube graph* is the graph whose vertices are all cubes (considered without regard to the ordering of the constituent circles); and whose edges indicate cubes sharing a side.

Theorem 13.4. *The cube graph is a forest of trees of valence six. Consequently,* $A_{\mathbb{Q}(\sqrt{-2})}$ *is freely generated by the generators* (17). *In particular, it is a free product of six copies of* $\mathbb{Z}/2\mathbb{Z}$.

Proof. Let H be the cube graph. Any one component of the graph of ordered cubes under the six available swaps is the Cayley graph of $\mathcal{A}_{\mathbb{Q}(\sqrt{-2})}$ under right multiplication and generators (17). We wish to demonstrate that any component of H is a tree. If so, then the Cayley graph cannot have a loop, as it would reduce to a loop in the cube graph under forgetting orientation. The theorem follows.

Let G be the tangency graph of a single packing, i.e., the graph whose vertices are circles, and whose edges represent tangencies. This graph is the same for any packing, so assume the packing is bounded away from ∞. This graph can be embedded on the sphere $\widehat{\mathbb{C}}$ by placing each vertex at the centre of the corresponding circle, so that edges pass through tangency points (the exception being the bounding circle, where we can take its centre to be ∞). Therefore, it is planar.

Suppose that H has a loop. This loop is a finite loop of $n \geq 2$ cubes lying in one packing, so its circles create an induced subgraph L of G (for that packing). We will demonstrate that L cannot exist inside the planar graph G.

To do so, we will consider building the graph L cube-by-cube. Begin with one adjacent pair of cubes in L, sharing four vertices in a cycle C. The cycle breaks the plane up into two regions. The two cubes lie in the two different regions (except that they both include the boundary). (By the Möbius action, verification on the base cube suffices.) Continuing to add cubes to form L, at each stage we are adding edges and vertices of G inside an existing face of the construction. In particular, we are either adding a cube inside the cycle C or outside it. Finally, to complete the loop L, we must join one cube inside C to one outside C, using another cycle $C' \neq C$. This is impossible inside the planar graph G. $\qquad\square$

An alternate method of proof is to show that L contains $4n$ vertices and $8n$ edges, but no cycles of length 3, which is impossible for a planar graph. Interestingly, a similar count works for $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-7})$, but fails for $\mathbb{Q}(\sqrt{-11})$. The proof above generalizes to all cases with little modification.

Theorem 13.5. *The group* $\mathcal{A}_{\mathbb{Q}(\sqrt{-2})}$ *is a geometric Apollonian group for* $\mathbb{Q}(\sqrt{-2})$ *and* $\widehat{\mathcal{A}_{\mathbb{Q}(\sqrt{-2})}}$ *is an algebraic Apollonian group for* $\mathbb{Q}(\sqrt{-2})$.

Proof. Our cluster type is that given above (cubes), and the base cube is as in Figure 14. We will verify the hypotheses of Theorem 11.3; many of the verifications are simple and similar to those in the proof of Theorem 12.7. Item (iii) is verified by generators 1, 2, and 4 of (18). Item (iv) is verified by identity transformation, since the base cube contains a three-prong centred on any of its circles.

That no automorphism of the base cluster is contained in $\mathcal{A}_{\mathbb{Q}(\sqrt{-2})}$ (item (v)) is a consequence of Theorem 13.4: such an automorphism would indicate a nontrivial path in the Cayley graph whose endpoints were two different orderings of the same cube. Forgetting orientation, this would give a loop in the cube graph. $\qquad\square$

14. Tents and belts in $K = \mathbb{Q}(\sqrt{-7})$

In this case, and for $\mathbb{Q}(\sqrt{-11})$, we will be somewhat more brief in our description. We define a $\mathbb{Q}(\sqrt{-7})$-Descartes configuration, called a *tent*, to be five circles in the

arrangement specified by the relation

$$W_D^t G_M W_D = \begin{pmatrix} 1 & -1 & -5/2 & -1 & -5/2 \\ -1 & 1 & -1 & -5/2 & -1 \\ -5/2 & -1 & 1 & -1 & -5/2 \\ -1 & -5/2 & -1 & 1 & -1 \\ -5/2 & -1 & -5/2 & -1 & 1 \end{pmatrix},$$

on the matrix W_D whose columns are the five circles. The graph of tangencies looks like this:
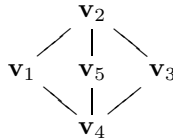

There is one relation among these circles:

$$\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_5 = 2(\mathbf{v}_2 + \mathbf{v}_4).$$

Three of the five circles $(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5)$ are distinguished as being tangent to fewer other circles in the tent (2 instead of 3). Call such a circle a *peak*. Removing a peak leaves four circles in a cycle, called a *belt*. Given a belt, there are exactly two tents containing it. Given a belt $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, the two peaks $\mathbf{v}_5$ and $\mathbf{v}_5'$ that complete it to a tent satisfy

$$\mathbf{v}_5 + \mathbf{v}_5' = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4.$$

The other peak $\mathbf{v}_5'$ is tangent to $\mathbf{v}_1$ and $\mathbf{v}_3$, thus:
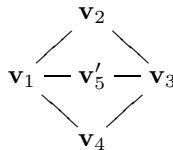
Therefore, $\mathbf{v}_2$ and $\mathbf{v}_4$ become new peaks. It is appropriate to renumber the resulting tent so we have

$$\mathbf{v}_1' = \mathbf{v}_2, \quad \mathbf{v}_2' = \mathbf{v}_1, \quad \mathbf{v}_3' = \mathbf{v}_4, \quad \mathbf{v}_4' = \mathbf{v}_3, \quad \mathbf{v}_5' = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 - \mathbf{v}_5.$$

An example is shown in Figure 18.

We will write a tent as a 4×4 matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$; call this a *tentbase*. Four circles form a tentbase if and only if the matrix of their columns, W_D, satisfies

$$W_D^t G_M W_D = \begin{pmatrix} 1 & -1 & -5/2 & -1 \\ -1 & 1 & -1 & -5/2 \\ -5/2 & -1 & 1 & -1 \\ -1 & -5/2 & -1 & 1 \end{pmatrix} =: R.$$

There are three moves that will replace a tent with another that shares a belt: each of the three peaks can be swapped out. These correspond to multiplying the tentbase on the right by these three matrices of order two:

$$(19) \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 & -1 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 3 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$
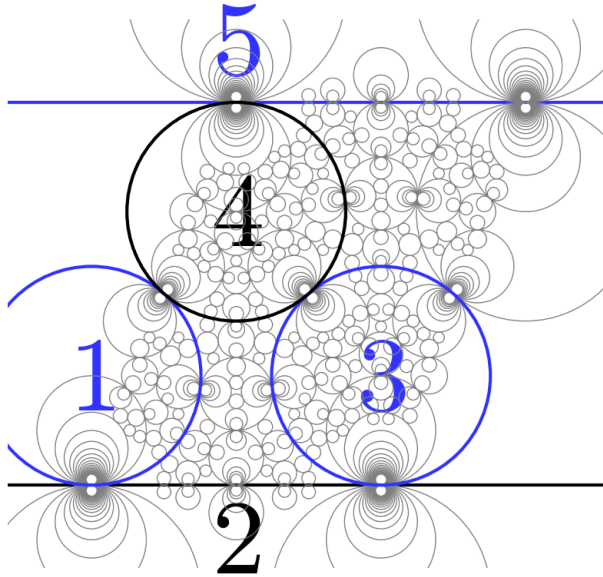
FIGURE 17. Base tent for $\mathcal{S}_{\mathbb{Q}(\sqrt{-7})}$. The coordinates of the five circles in $\mathbb{M}$ are given, in the labelled order, by the columns of the following matrix:

$$\begin{pmatrix} 0 & 0 & \sqrt{7} & \sqrt{7} & \sqrt{7} \\ \sqrt{7} & 0 & \sqrt{7} & \sqrt{7} & 0 \\ 0 & 0 & \sqrt{7} & \sqrt{7}/2 & 0 \\ 1 & -1 & 1 & 5/2 & 1 \end{pmatrix}.$$

Circles tangent to three others in the tent are shown in black, while those tangent to two others are shown in blue.

Definition 14.1. Denote the subgroup of $O_R(\mathbb{R})$ generated by the matrices (19) by $\widehat{\mathcal{A}_{\mathbb{Q}(\sqrt{-7})}}$.

The three swaps (19) are not realised as inversions in circles as in the case of $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-2})$. On the base tent of Figure 17, they correspond, via the algebraic-geometric correspondence, to the following Möbius transformations, respectively:

$$(20) \qquad z \mapsto \frac{\frac{1+\sqrt{-7}}{2}z + \frac{1-\sqrt{-7}}{2}}{\frac{3+\sqrt{-7}}{2}z + \frac{-1-\sqrt{-7}}{2}}, \quad z \mapsto -z - \frac{-3-\sqrt{-7}}{2}, \quad z \mapsto \frac{z}{\frac{1-\sqrt{-7}}{2}z - 1}.$$

Definition 14.2. The 7-*tent graph* is the graph whose vertices are all tents (considered without regard to ordering); and whose edges indicate tents sharing a belt.

Theorem 14.3. *The 7-tent graph is a forest of trees of valence three. Consequently, $A_{\mathbb{Q}(\sqrt{-7})}$ is freely generated by the generators (19). In particular, it is a free product of three copies of $\mathbb{Z}/2\mathbb{Z}$.*

Proof. The proof is exactly as in Theorem 13.4: any two tents lie in the two regions created by their shared belt. $\qquad\square$

Theorem 14.4. *The group $A_{\mathbb{Q}(\sqrt{-7})}$ is a geometric Apollonian group for $\mathbb{Q}(\sqrt{-7})$ and the group $\widehat{A_{\mathbb{Q}(\sqrt{-7})}}$ is an algebraic Apollonian group for $\mathbb{Q}(\sqrt{-7})$.*
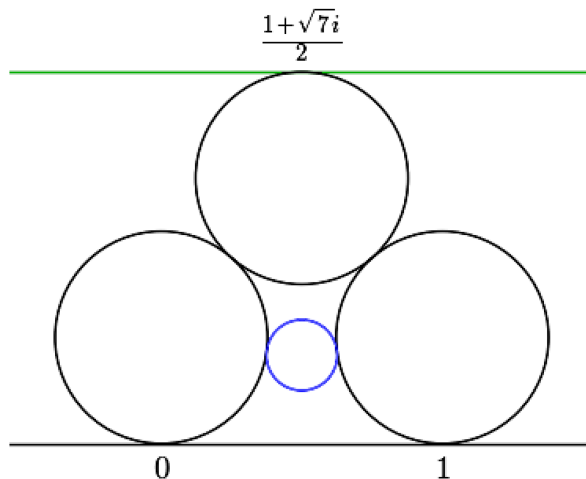
$$\frac{1+\sqrt{7}i}{2}$$

FIGURE 18. The green and black circles (including two straight lines) form the base tent of the $\mathbb{Q}(\sqrt{-7})$-Apollonian packing containing $\widehat{\mathbb{R}}$. If swapping out the peak given by the green circle, the green circle is replaced with the blue one. The new tent is formed of the black and blue circles. Note that the black circles are not individually fixed; they are permuted.
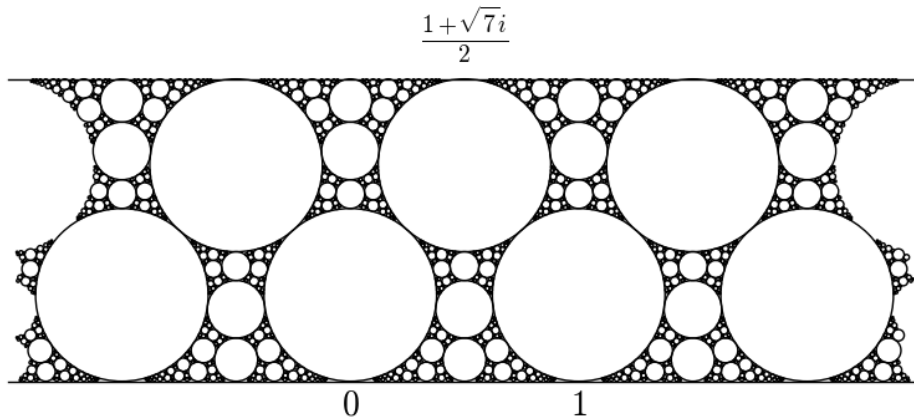
$$\frac{1+\sqrt{7}i}{2}$$

FIGURE 19. The $\mathbb{Q}(\sqrt{-7})$-Apollonian packing containing $\widehat{\mathbb{R}}$.

Proof. Tents are the clusters, and the base tent is as in Figure 17. Most of the verifications of the hypotheses of Theorem 11.3 are simple or similar to those in the proof of Theorems 12.7 and 13.5; we consider only (iii), (iv), and (v).

Composing each pair of generators (20) in each possible order, we obtain six transformations. It is a computation to verify that, applying these six transformations to the base cluster, we obtain among the results clusters containing the three-prongs on $(0, 1/2, 1)$, $(1, 2, \infty)$, and $(-1, 0, \infty)$. This verifies (iii).

Item (iv) is verified by the three swaps (20). Finally, (v) is verified, as in the proof of Theorem 13.5, by Theorem 14.3. □

15. Tents and belts in $K = \mathbb{Q}(\sqrt{-11})$

This case is similar to the case $\mathbb{Q}(\sqrt{-7})$. A *tent* D consists of ten circles, and contains four *belts* (loops of tangent circles) of six circles each, in the following arrangement:
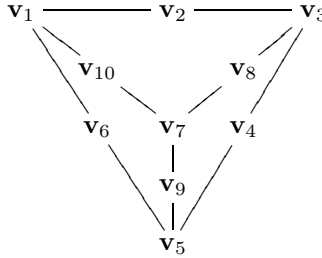
Let W_D denote the matrix whose columns are the vectors in $\mathbb{M}$ corresponding to these circles, say $\mathbf{v}_i$, $i = 1, \ldots, 10$ of $\mathbb{M}$. We have

$$W_D^t G_M W_D$$

$$= \begin{pmatrix}
1 & -1 & -9/2 & -13/2 & -9/2 & -1 & -9/2 & -13/2 & -13/2 & -1 \\
-1 & 1 & -1 & -9/2 & -13/2 & -9/2 & -13/2 & -9/2 & -10 & -9/2 \\
-9/2 & -1 & 1 & -1 & -9/2 & -13/2 & -9/2 & -1 & -13/2 & -13/2 \\
-13/2 & -9/2 & -1 & 1 & -1 & -9/2 & -13/2 & -9/2 & -9/2 & -10 \\
-9/2 & -13/2 & -9/2 & -1 & 1 & -1 & -9/2 & -13/2 & -1 & -13/2 \\
-1 & -9/2 & -13/2 & -9/2 & -1 & 1 & -13/2 & -10 & -9/2 & -9/2 \\
-9/2 & -13/2 & -9/2 & -13/2 & -9/2 & -13/2 & 1 & -1 & -1 & -1 \\
-13/2 & -9/2 & -1 & -9/2 & -13/2 & -10 & -1 & 1 & -9/2 & -9/2 \\
-13/2 & -10 & -13/2 & -9/2 & -1 & -9/2 & -1 & -9/2 & 1 & -9/2 \\
-1 & -9/2 & -13/2 & -10 & -13/2 & -9/2 & -1 & -9/2 & -9/2 & 1
\end{pmatrix}.$$

The ten circles of a tent span a vector space of dimension 4. A presentation of the relations is:

$$5\mathbf{v}_2 = -4\mathbf{v}_1 - 4\mathbf{v}_3 + \mathbf{v}_5 + \mathbf{v}_7,$$
$$5\mathbf{v}_4 = \mathbf{v}_1 - 4\mathbf{v}_3 - 4\mathbf{v}_5 + \mathbf{v}_7,$$
$$5\mathbf{v}_6 = -4\mathbf{v}_1 + \mathbf{v}_3 - 4\mathbf{v}_5 + \mathbf{v}_7,$$
$$5\mathbf{v}_8 = \mathbf{v}_1 - 4\mathbf{v}_3 + \mathbf{v}_5 - 4\mathbf{v}_7,$$
$$5\mathbf{v}_9 = \mathbf{v}_1 + \mathbf{v}_3 - 4\mathbf{v}_5 - 4\mathbf{v}_7,$$
$$5\mathbf{v}_{10} = -4\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_5 - 4\mathbf{v}_7.$$

A tent contains four belts. Within a belt, the sum of opposite circles is invariant, e.g.,

$$\mathbf{v}_1 + \mathbf{v}_4 = \mathbf{v}_2 + \mathbf{v}_5 = \mathbf{v}_3 + \mathbf{v}_6.$$

Since there are four belts, one obtains four vectors; these are independent, and we therefore represent a tent as a matrix with these four columns, say

$$\mathbf{v}_1 + \mathbf{v}_4, \mathbf{v}_1 + \mathbf{v}_8, \mathbf{v}_1 + \mathbf{v}_9, \mathbf{v}_3 + \mathbf{v}_9.$$

There are exactly two tents containing a single belt. Therefore, there are four swaps one can perform. Each swap preserves one belt and changes three belts. An example
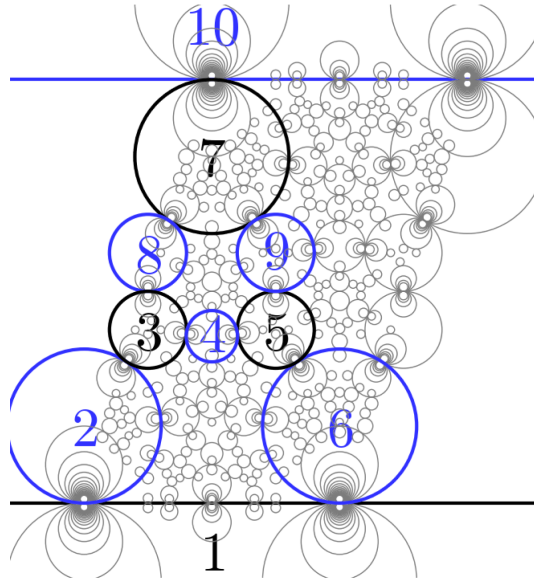
FIGURE 20. Base tent for $\mathcal{S}_{\mathbb{Q}(\sqrt{-11})}$. The coordinates of the ten circles in $\mathbb{M}$ are given, in the labelled order, by the columns of the following matrix:

$$\begin{pmatrix} 0 & 0 & \sqrt{11} & 2\sqrt{11} & 2\sqrt{11} & \sqrt{11} & 2\sqrt{11} & 2\sqrt{11} & 3\sqrt{11} & \sqrt{11} \\ 0 & \sqrt{11} & 2\sqrt{11} & 3\sqrt{11} & 2\sqrt{11} & \sqrt{11} & \sqrt{11} & 2\sqrt{11} & 2\sqrt{11} & 0 \\ 0 & 0 & \sqrt{11}/2 & 3\sqrt{11}/2 & 3\sqrt{11}/2 & \sqrt{11} & \sqrt{11}/2 & \sqrt{11}/2 & 3\sqrt{11}/2 & 0 \\ -1 & 1 & 9/2 & 13/2 & 9/2 & 1 & 9/2 & 13/2 & 13/2 & 1 \end{pmatrix}.$$

Circles tangent to three others in the tent are shown in black, while those tangent to two others are shown in blue.

is shown in Figure 21. The resulting matrices, in terms of the representation above, are

(21)

$$\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 3 & 1 & 3 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 3 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 3 & 3 & 3 & 1 \end{pmatrix}.$$

Note that the circles of the belt are permuted; for example, the first of these has the effect

$$\mathbf{v}_1' = \mathbf{v}_4, \mathbf{v}_2' = \mathbf{v}_5, \mathbf{v}_3' = \mathbf{v}_6, \mathbf{v}_4' = \mathbf{v}_1, \mathbf{v}_5' = \mathbf{v}_2, \mathbf{v}_6' = \mathbf{v}_3.$$

Definition 15.1. Denote the subgroup of $O_R(\mathbb{R})$ generated by these four matrices by $\widehat{\mathcal{A}_{\mathbb{Q}(\sqrt{-11})}}$.
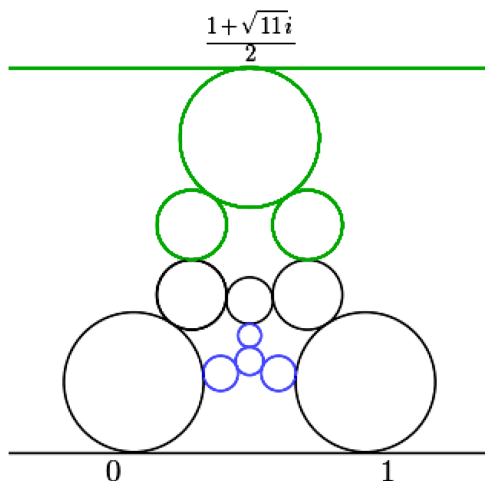
$$\frac{1+\sqrt{11}i}{2}$$

FIGURE 21. The green and black circles (including two straight lines) form the base tent of the $\mathbb{Q}(\sqrt{-11})$-Apollonian packing containing $\widehat{\mathbb{R}}$. If swapping out the peak given by the green circles, the green circles are replaced with the blue ones. The new tent is formed of the black and blue circles. Note that the black circles are not individually fixed; they are cyclically permuted.
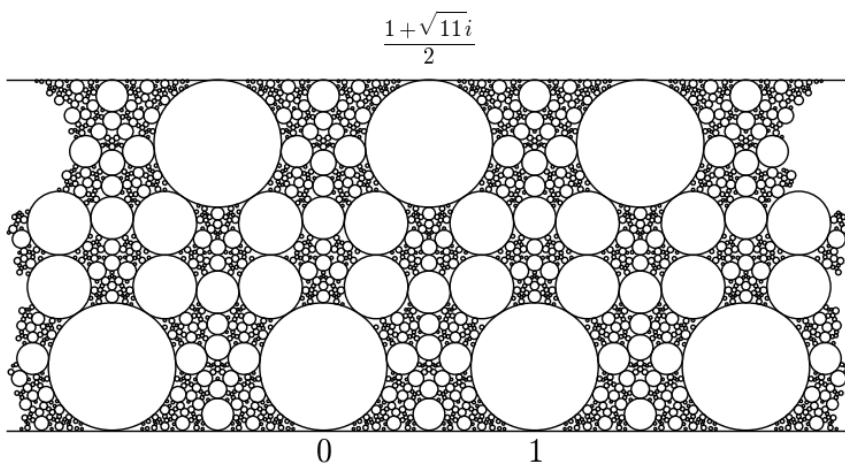
$$\frac{1+\sqrt{11}i}{2}$$

FIGURE 22. The $\mathbb{Q}(\sqrt{-11})$-Apollonian packing containing $\widehat{\mathbb{R}}$.

The four Möbius maps performing these swaps on the base tent of Figure 20 (in the same order as (21)) are:

$$(22) \qquad \begin{aligned} & z \mapsto \frac{\frac{3+\sqrt{-11}}{2}\overline{z} - 2}{3\overline{z} + \frac{-3+\sqrt{-11}}{2}}, \quad z \mapsto \frac{\frac{1+\sqrt{-11}}{2}\overline{z} - 2}{2\overline{z} + \frac{-1+\sqrt{-11}}{2}}, \\[2ex] & z \mapsto \frac{\frac{3+\sqrt{-11}}{2}\overline{z} - 3}{2\overline{z} + \frac{-3+\sqrt{-11}}{2}}, \quad z \mapsto \frac{(2+\sqrt{-11})\overline{z} - 4}{4\overline{z} - 2 + \sqrt{-11}}. \end{aligned}$$

TABLE 1. For each discriminant Δ, the table shows the smallest power of 2, M, that explains the obstructions at the prime 2, and the observed residue sets S_M modulo M.

Δ	M	S_M
$\Delta \equiv 0, 4, 16 \pmod{32}$	2	$S_2 = \{0, 1\}, \{1\}$
$\Delta \equiv 8, 24 \pmod{32}$	4	$S_4 = \{0, 2, 3\}, \{0, 1, 2\}$
$\Delta \equiv 12 \pmod{32}$	4	$S_4 = \{0, 1\}, \{1, 2\}, \{2, 3\}, \{0, 3\}$
$\Delta \equiv 20 \pmod{32}$	4	$S_4 = \{1\}, \{3\}, \{0, 1, 2, 3\}$
$\Delta \equiv 28 \pmod{32}$	8	$S_8 = \{0, 1, 4\}, \{2, 3, 6, 7\}, \{0, 4, 5\}$
otherwise	none	none

TABLE 2. For each discriminant Δ, the obstruction at the prime 3 is explained by the first power, 3. The table shows the observed residue sets S_3 modulo 3.

Δ	S_3
$\Delta \equiv 5, 8 \pmod{12}$	$S_3 = \{0, 1\}, \{0, 2\}$
otherwise	none

Definition 15.2. Denote the subgroup of Möb generated by these four matrices by $\mathcal{A}_{\mathbb{Q}(\sqrt{-11})}$.

Definition 15.3. The 11-*tent graph* is the graph whose vertices are all tents (considered without regard to ordering); and whose edges indicate tents sharing a belt.

Theorem 15.4. *The* 11-*tent graph is a forest of trees of valence four. Consequently, $A_{\mathbb{Q}(\sqrt{-11})}$ is freely generated by the generators* (21). *In particular, it is a free product of four copies of $\mathbb{Z}/2\mathbb{Z}$.*

Proof. The proof is exactly as in Theorem 13.4 and Theorem 14.3. $\square$

Theorem 15.5. *The group $\mathcal{A}_{\mathbb{Q}(\sqrt{-11})}$ is a geometric Apollonian group for $\mathbb{Q}(\sqrt{-11})$, and the group $\widehat{\mathcal{A}_{\mathbb{Q}(\sqrt{-11})}}$ is an algebraic Apollonian group for $\mathbb{Q}(\sqrt{-11})$.*

Proof. The proof is very similar to Theorem 14.4. As before, compositions of generators verify (iii), and the single generators verify (iv). $\square$

16. CURVATURES IN K-APOLLONIAN PACKINGS

This section is devoted to some computational data supporting Conjecture 1.4. We first record a basic result on curvatures.

Theorem 16.1. *Let $\mathcal{P}$ be a K-Apollonian circle packing. Then the reduced curvatures of $\mathcal{P}$ have no common factor.*

Proof. $\mathcal{P}$ is generated by immediate tangency. Theorem 4.7 of [42] states that if C_1 and C_2 are immediately tangent, then

$$\mathrm{curv}(C_1) + \mathrm{curv}(C_2) = N(x),$$

where x is the denominator of the point of tangency. If all the curvatures of the packing are divisible by some prime p, then $p \mid N(x)$. Fixing C_1, the value of x for all immediately tangent C_2 are exactly those x in a rank two $\mathbb{Z}$-lattice $\beta\mathbb{Z} + \delta\mathbb{Z} \subset \mathcal{O}_K$.

Here, β and δ can be taken to be the lower row of some matrix $M \in \mathrm{PSL}_2(\mathcal{O}_K)$ taking $\widehat{\mathbb{R}}$ to C_1 (in particular, $(\beta, \delta) = \mathcal{O}_K$).

Therefore, p divides $N(\beta)$, $N(\delta)$ and $N(\beta + \delta)$. If $p \mid N(x)$, then $x \in \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ above p. Therefore at least two of β, δ and $\beta + \delta$ lie in the same prime ideal; but this is a contradiction to the fact that any pair of them generates $\mathcal{O}_K$. $\qquad\square$

Using Sage Mathematics Software [43], some computer experiments were performed. The author computed the complete set of reduced curvatures in various K-Apollonian packings modulo various moduli n. The results suggest Conjecture 1.4. In particular, each Apollonian packing was observed to omit certain modular equivalence classes. The behaviour of individual primes is independent, so that we can discuss the *obstruction at a prime p* as all equivalence classes modulo powers of p that cannot occur. The obstruction is *explained* by p^k if p^k is the largest power of p needed to describe the obstruction. In experiments, the only obstructions that occurred were at 2 (always explained by 2, 4 or 8) and at 3 (always explained by 3). This explains the number 24 in Conjecture 1.4. Tables 1 and 2 give the observed allowable sets of residues. It is conjectured that these tables are complete.

Acknowledgments

References

[1] Christopher J. Bishop and Peter W. Jones, *Hausdorff dimension and Kleinian groups*, Acta Math. **179** (1997), no. 1, 1–39, DOI 10.1007/BF02392718. MR1484767

[2] Jean Bourgain and Elena Fuchs, *A proof of the positive density conjecture for integer Apollonian circle packings*, J. Amer. Math. Soc. **24** (2011), no. 4, 945–967, DOI 10.1090/S0894-0347-2011-00707-8. MR2813334

[3] Jean Bourgain and Alex Kontorovich, *On the local-global conjecture for integral Apollonian gaskets*, Invent. Math. **196** (2014), no. 3, 589–650, DOI 10.1007/s00222-013-0475-y. With an appendix by Péter P. Varjú. MR3211042

[4] Steve Butler, Ron Graham, Gerhard Guettler, and Colin Mallows, *Irreducible Apollonian configurations and packings*, Discrete Comput. Geom. **44** (2010), no. 3, 487–507, DOI 10.1007/s00454-009-9216-9. MR2679050

[5] John H. Conway, *The sensual (quadratic) form*, Carus Mathematical Monographs, vol. 26, Mathematical Association of America, Washington, DC, 1997. With the assistance of Francis Y. C. Fung. MR1478672

[6] Leonard Eugene Dickson, *History of the theory of numbers. Vol. III: Quadratic and higher forms.*, With a chapter on the class number by G. H. Cresse, Chelsea Publishing Co., New York, 1966. MR0245501

[7] Joan L. Dyer, *Automorphism sequences of integer unimodular groups*, Illinois J. Math. **22** (1978), no. 1, 1–30. MR0460483

[8] J. Elstrodt, F. Grunewald, and J. Mennicke, *Groups acting on hyperbolic space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. Harmonic analysis and number theory. MR1483315

[9] Elena Fuchs, *Strong approximation in the Apollonian group*, J. Number Theory **131** (2011), no. 12, 2282–2302, DOI 10.1016/j.jnt.2011.05.010. MR2832824

[10] Elena Fuchs, *Counting problems in Apollonian packings*, Bull. Amer. Math. Soc. (N.S.) **50** (2013), no. 2, 229–266, DOI 10.1090/S0273-0979-2013-01401-0. MR3020827

[11] Elena Fuchs and Katherine Sanden, *Some experiments with integral Apollonian circle pack-ings*, Exp. Math. **20** (2011), no. 4, 380–399, DOI 10.1080/10586458.2011.565255. MR2859897

[12] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan, *Apollonian circle packings: number theory*, J. Number Theory **100** (2003), no. 1, 1–45, DOI 10.1016/S0022-314X(03)00015-5. MR1971245

[13] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan, *Apollonian circle packings: geometry and group theory. I. The Apollonian group*, Discrete Comput. Geom. **34** (2005), no. 4, 547–585, DOI 10.1007/s00454-005-1196-9. MR2173929

[14] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan, *Apollonian circle packings: geometry and group theory. II. Super-Apollonian group and integral packings*, Discrete Comput. Geom. **35** (2006), no. 1, 1–36, DOI 10.1007/s00454-005-1195-x. MR2183489

[15] Gerhard Guettler and Colin Mallows, *A generalization of Apollonian packing of circles*, J. Comb. **1** (2010), no. 1, [ISSN 1097-959X on cover], 1–27, DOI 10.4310/JOC.2010.v1.n1.a1. MR2675919

[16] Andrew Jensen, Cherry Ng, Evan Oliver, Tyler Schrock, and Katherine E. Stange, *The Schmidt arrangement of the Eisenstein integers and Apollonian-like packings arising from congruence subgroups*, In preparation.

[17] Gareth A. Jones and John S. Thornton, *Automorphisms and congruence subgroups of the extended modular group*, J. London Math. Soc. (2) **34** (1986), no. 1, 26–40, DOI 10.1112/jlms/s2-34.1.26. MR859146

[18] Vladimir V. Kisil, *Starting with the group* $\mathrm{SL}_2(\mathbf{R})$, Notices Amer. Math. Soc. **54** (2007), no. 11, 1458–1465. MR2361159

[19] Jerzy Kocik, *A theorem on circle configuarations*, 2007. `arXiv:0706.0372`.

[20] Alex Kontorovich, *The local-global principle for integral soddy sphere packings*, `arXiv:1208.5441`.

[21] Alex Kontorovich, *From Apollonius to Zaremba: local-global phenomena in thin orbits*, Bull. Amer. Math. Soc. (N.S.) **50** (2013), no. 2, 187–228, DOI 10.1090/S0273-0979-2013-01402-2. MR3020826

[22] Jeffrey C. Lagarias, Colin L. Mallows, and Allan R. Wilks, *Beyond the Descartes circle the-orem*, Amer. Math. Monthly **109** (2002), no. 4, 338–361, DOI 10.2307/2695498. MR1903421

[23] D. W. Lewis, *The isometry classification of Hermitian forms over division algebras*, Linear Algebra Appl. **43** (1982), 245–272, DOI 10.1016/0024-3795(82)90258-0. MR656449

[24] David Mumford, Caroline Series, and David Wright, *Indra's pearls*, Cambridge University Press, Cambridge, 2015. The vision of Felix Klein; With cartoons by Larry Gonick; Paperback edition with corrections; For the 2002 edition see [MR1913879]. MR3558870

[25] David Mumford, Caroline Series, and David Wright, *Indra's pearls*, Cambridge University Press, New York, 2002. The vision of Felix Klein. MR1913879

[26] Kei Nakamura, *The local-global principle for integral bends in orthoplicial apollonian sphere packings*, `arXiv:1401.2980`.

[27] Morris Newman, *Classification of normal subgroups of the modular group*, Trans. Amer. Math. Soc. **126** (1967), 267–277, DOI 10.2307/1994453. MR0204375

[28] Bogdan Nica, *The unreasonable slightness of* E_2 *over imaginary quadratic rings*, Amer. Math. Monthly **118** (2011), no. 5, 455–462, DOI 10.4169/amer.math.monthly.118.05.455. MR2805033

[29] Hee Oh, *Apollonian circle packings: dynamics and number theory*, Jpn. J. Math. **9** (2014), no. 1, 69–97, DOI 10.1007/s11537-014-1384-6. MR3173439

[30] Dan Romik, *The dynamics of Pythagorean triples*, Trans. Amer. Math. Soc. **360** (2008), no. 11, 6045–6064, DOI 10.1090/S0002-9947-08-04467-X. MR2425702

[31] Recep Sahin, Sebahattin Ikikardes, and Özden Koruoğlu, *Some normal subgroups of the extended Hecke groups* $\overline{H}(\lambda_p)$, Rocky Mountain J. Math. **36** (2006), no. 3, 1033–1048, DOI 10.1216/rmjm/1181069444. MR2254377

[32] Peter Sarnak, *Integral Apollonian packings*, Amer. Math. Monthly **118** (2011), no. 4, 291–306, DOI 10.4169/amer.math.monthly.118.04.291. MR2800340

[33] Asmus L. Schmidt, *Farey triangles and Farey quadrangles in the complex plane*, Math. Scand. **21** (1967), 241–295 (1969), DOI 10.7146/math.scand.a-10863. MR0245525

[34] Asmus L. Schmidt, *Diophantine approximation of complex numbers*, Acta Math. **134** (1975), 1–85, DOI 10.1007/BF02392098. MR0422168

[35] Asmus L. Schmidt, *Diophantine approximation in the field* $\mathbf{Q}(i(11^{1/2}))$, J. Number Theory **10** (1978), no. 2, 151–176, DOI 10.1016/0022-314X(78)90033-1. MR0485715

[36] Asmus L. Schmidt, *Diophantine approximation in the Eisensteinian field*, J. Number Theory **16** (1983), no. 2, 169–204, DOI 10.1016/0022-314X(83)90040-9. MR698164

[37] Asmus L. Schmidt, *Diophantine approximation in the field* $\mathbb{Q}(i\sqrt{2})$, J. Number Theory **131** (2011), no. 10, 1983–2012, DOI 10.1016/j.jnt.2011.04.002. MR2811562

[38] Jean-Pierre Serre, *Trees*, Springer-Verlag, Berlin-New York, 1980. Translated from the French by John Stillwell. MR607504

[39] Arseniy Sheydvasser, untitled preprint.

[40] Frederick Soddy, *The bowl of integers and the hexlet*, Nature, 139:77–79, 1937.

[41] Katherine E. Stange, *The sensual Apollonian circle packing*, Expo. Math. **34** (2016), no. 4, 364–395, DOI 10.1016/j.exmath.2016.01.001. MR3578004

[42] Katherine E. Stange, *Visualizing imaginary quadratic fields*, CMS Notes **48** (2016), no. 4, 16–17. MR3559745

[43] W. A. Stein et al., *Sage Mathematics Software (Version 6.4)*. The Sage Development Team, 2015. http://www.sagemath.org.

[44] A. Muhammed Uludağ and Hakan Ayral, *Jimm, a Fundamental Involution*, 2015. arXiv:1501.03787.

[45] Xin Zhang, *On the Local-global Principle for Integral Apollonian-3 Circle Packings*, ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–State University of New York at Stony Brook. MR3347084

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, CAMPUX BOX 395, BOULDER, COLORADO 80309-0395

Email address: kstange@math.colorado.edu