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ON MULTIPLICATIVELY DEPENDENT VECTORS OF ALGEBRAIC NUMBERS

FRANCESCO PAPPALARDI, MIN SHA, IGOR E. SHPARLINSKI,
AND CAMERON L. STEWART

Abstract. In this paper, we give several asymptotic formulas for the number of multiplicatively dependent vectors of algebraic numbers of fixed degree, or within a fixed number field, and bounded height.

1. Introduction

1.1. **Background.** Let n be a positive integer, and let G be a multiplicative group, and let $\nu = (\nu_1, \dots, \nu_n)$ be in G^n. We say that ν is multiplicatively dependent if there is a non-zero vector $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ for which

$$\nu^{\mathbf{k}} = \nu_1^{k_1} \cdots \nu_n^{k_n} = 1. \tag{1.1}$$

We denote by $\mathcal{M}_n(G)$ the set of multiplicatively dependent vectors in G^n.

For instance, the set $\mathcal{M}_n(\mathbb{C}^*)$ of multiplicatively dependent vectors in $(\mathbb{C}^*)^n$ is of Lebesgue measure zero, since it is a countable union of sets of measure zero. In fact, the ongoing project [24] aims at studying in detail the density of these vectors. Further, if we fix an exponent vector $\mathbf{k}$ the subvariety of $(\mathbb{C}^*)^n$ determined by (1.1) is an algebraic subgroup of $(\mathbb{C}^*)^n$.

For multiplicatively dependent vectors of algebraic numbers there are two kinds of questions which have been extensively studied. The first question concerns the exponents in (1.1). Given a multiplicatively dependent vector ν it follows from the work of Loxton and van der Poorten [14, 21], Matveev [18], and Loher and Masser [13, Corollary 3.2] (attributed to K. Yu) that there is a relation of the form (1.1) with a non-zero vector $\mathbf{k}$ with small coordinates. The second question is to find comparison relations among the heights of the coordinates. For example, Stewart [27, Theorem 1] has given an inequality for the heights of the coordinates of such a vector (of low multiplicative rank, in the terminology of Section 1.2), and a lower bound for the sum of the heights of the coordinates is implied in [28].

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In this paper, we obtain several asymptotic formulas for the number of multiplicatively dependent n-tuples whose coordinates are algebraic numbers of fixed degree, or within a fixed number field, and bounded height. Equivalently (see [23]), we count n-tuples of algebraic numbers in a fixed algebraic number field, or of fixed degree, and given height which occur in some proper algebraic subgroup of the algebraic group G_m^n, where G_m is the multiplicative group of an algebraic closure of $\mathbb{Q}$. Aside from the results mentioned above, to the best of our knowledge, this natural question has never been addressed in the literature.

We remark that the above question is interesting in its own right, but is also partially motivated by the works [20,25], where multiplicatively independent vectors play an important role.

1.2. **Rank of multiplicative independence.** The following notion plays a crucial role in our argument and is also of independent interest.

Let $\overline{\mathbb{Q}}$ be an algebraic closure of the rational numbers $\mathbb{Q}$. For each ν in $(\overline{\mathbb{Q}}^*)^n$, we define s, the *multiplicative rank* of ν, in the following way. If ν has a coordinate which is a root of unity, we put $s = 0$; otherwise let s be the largest integer with $1 \le s \le n$ for which any s coordinates of ν form a multiplicatively independent vector. Notice that

$$0 \le s \le n-1 \tag{1.2}$$

whenever ν is multiplicatively dependent.

1.3. **Conventions and notation.** For any algebraic number α, let

$$f(x) = a_d x^d + \cdots + a_1 x + a_0$$

be the minimal polynomial of α over the integers $\mathbb{Z}$ (so with content 1 and positive leading coefficient). Suppose that f is factored as

$$f(x) = a_d (x - \alpha_1) \cdots (x - \alpha_d)$$

over the complex numbers $\mathbb{C}$. The *naive height* $\mathrm{H}_0(\alpha)$ of α is given by

$$\mathrm{H}_0(\alpha) = \max\{|a_d|, \ldots, |a_1|, |a_0|\},$$

and $\mathrm{H}(\alpha)$, the height of α, also known as the *absolute Weil height* of α, is defined by

$$\mathrm{H}(\alpha) = \left(a_d \prod_{i=1}^{d} \max\{1, |\alpha_i|\} \right)^{1/d}.$$

Let K be a number field of degree d (over $\mathbb{Q}$). We use the following standard notation:

- r_1 and r_2 for the number of real and pairs of complex conjugate embeddings of K, respectively, and $r = r_1 + r_2 - 1$;
- D, h, R, and ζ_K for the discriminant, class number, regulator, and Dedekind zeta function of K, respectively;
- w for the number of roots of unity in K.

Note that r is exactly the rank of the unit group of the ring of algebraic integers of K. As usual, let $\zeta(s)$ be the Riemann zeta function.

For any real number x, let $\lceil x \rceil$ denote the smallest integer greater than or equal to x, and let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

We always implicitly assume that H is large enough, in particular so that the logarithmic expressions $\log H$ and $\log\log H$ are well-defined.

In the sequel, we use the Landau symbols O and o and the Vinogradov symbol $\ll$. We recall that the assertions $U = O(V)$ and $U \ll V$ are both equivalent to the inequality $|U| \le cV$ with some positive constant c, while $U = o(V)$ means that $U/V \to 0$. We also use the asymptotic notation $\sim$.

For a finite set S we use $|S|$ to denote its cardinality.

Throughout the paper, the implied constants in the symbols O and $\ll$ only depend on the given number field K, the given degree d, or the dimension n.

1.4. **Counting vectors within a number field.** Let K be a number field of degree d. Denote the set of algebraic integers of K of height at most H by $\mathcal{B}_K(H)$ and the set of algebraic numbers of K of height at most H by $\mathcal{B}^*_K(H)$. Set

$$B_K(H) = |\mathcal{B}_K(H)| \qquad \text{and} \qquad B^*_K(H) = |\mathcal{B}^*_K(H)|\,.$$

Put

$$C_1(K) = \frac{2^{r_1}(2\pi)^{r_2}d^r}{|D|^{1/2}r!}.$$

It follows directly from the work of Widmer [31, Theorem 1.1] (taking $n = e = 1$ there) that

$$B_K(H) = C_1(K)H^d(\log H)^r + O\left(H^d(\log H)^{r-1}\right). \tag{1.3}$$

If $r = 0$, then (1.3) can be improved to (see [2, Theorem 1.1])

$$B_K(H) = C_1(K)H^d + O(H^{d-1}). \tag{1.4}$$

We remark that the estimate in (1.3) is stated in [12, Chapter 3, Theorem 5.2] without the explicit constant $C_1(K)$, and moreover Barroero [3] has obtained similar estimates for the number of algebraic S-integers with fixed degree and bounded height.

Define

$$C_2(K) = \frac{2^{2r_1}(2\pi)^{2r_2}2^r hR}{|D|w\zeta_K(2)}.$$

Schanuel [22, Corollary to Theorem 3] proved in 1979 (see also [17, equation (1.5)]) that

$$B^*_K(H) = C_2(K)H^{2d} + O\left(H^{2d-1}(\log H)^{\sigma(d)}\right), \tag{1.5}$$

where $\sigma(1) = 1$ and $\sigma(d) = 0$ for $d > 1$. Note that the height in [22] is our height to the power d.

For any positive integer n, we denote by $L_{n,K}(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic integers of height at most H, and we denote by $L^*_{n,K}(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic numbers of height at most H.

Put

$$C_3(n,K) = \frac{n(n+1)}{2}wC_1(K)^{n-1}.$$

Theorem 1.1. *Let K be a number field of degree d over $\mathbb{Q}$ and let n be an integer with $n \ge 2$. We have*

$$\begin{aligned} L_{n,K}(H) = {}& C_3(n,K)H^{d(n-1)}(\log H)^{r(n-1)} \\ & + O\left(H^{d(n-1)}(\log H)^{r(n-1)-1}\right). \end{aligned} \tag{1.6}$$

If furthermore $K = \mathbb{Q}$ or is an imaginary quadratic field, we have

$$L_{n,K}(H) = C_3(n,K)H^{d(n-1)} + O\left(H^{d(n-3/2)}\right). \tag{1.7}$$

We remark that when $K = \mathbb{Q}$ a better error term than that given in (1.7) is stated in Theorem 1.4 below; more precisely, see (1.16).

We estimate $L^*_{n,K}(H)$ next. Put

$$C_4(n,K) = n^2 w C_2(K)^{n-1}.$$

Theorem 1.2. *Let K be a number field of degree d, and let n be an integer with $n \geq 2$. Then, we have*

$$L^*_{n,K}(H) = C_4(n,K)H^{2d(n-1)} + O\left(H^{2d(n-1)-1}g(H)\right), \tag{1.8}$$

where

$$g(H) = \begin{cases} \log H & \text{if } d = 1 \text{ and } n = 2, \\ \exp(c \log H/\log\log H) & \text{if } d = 1 \text{ and } n > 2, \\ 1 & \text{if } d > 1 \text{ and } n \geq 2, \end{cases}$$

and c is a positive number depending only on n.

We now outline the strategy of the proofs. Given a number field K, we define $L_{n,K,s}(H)$ and $L^*_{n,K,s}(H)$ to be the number of multiplicatively dependent n-tuples of multiplicative rank s whose coordinates are algebraic integers in $\mathcal{B}_K(H)$ and algebraic numbers in $\mathcal{B}^*_K(H)$ respectively. It follows from (1.2) that

$$\begin{cases} L_{n,K}(H) = L_{n,K,0}(H) + \cdots + L_{n,K,n-1}(H), \\ \\ L^*_{n,K}(H) = L^*_{n,K,0}(H) + \cdots + L^*_{n,K,n-1}(H). \end{cases} \tag{1.9}$$

The main term in (1.6) comes from the contributions of $L_{n,K,0}(H)$ and $L_{n,K,1}(H)$ in (1.9), and the main term in Theorem 1.2 comes from the contributions of $L^*_{n,K,0}(H)$ and $L^*_{n,K,1}(H)$ in (1.9). To prove Theorems 1.1 and 1.2, we make use of (1.9) and the following result.

Proposition 1.3. *Let K be a number field of degree d. Let n and s be integers with $n \geq 2$ and $0 \leq s \leq n-1$. Then, there exist positive numbers c_1 and c_2 which depend on n and K, such that*

$$L_{n,K,s}(H) < H^{d(n-1)-d(\lceil (s+1)/2 \rceil - 1)} \exp(c_1 \log H/\log\log H) \tag{1.10}$$

and

$$L^*_{n,K,s}(H) < H^{2d(n-1)-d(\lceil (s+1)/2 \rceil - 1)} \exp(c_2 \log H/\log\log H). \tag{1.11}$$

In Section 5, we show that when $s = n-1$ (1.10) cannot be improved by much; see Theorems 5.2 and 5.4. In particular, it does not hold with $\exp(c_1 \log H/\log\log H)$ replaced by a quantity which is $o((\log H)^{(k-1)^2})$, where $K = \mathbb{Q}$ and $n = 2k$.

1.5. **Counting vectors of fixed degree.** Let d be a positive integer, and let $\mathcal{A}_d(H)$, respectively $\mathcal{A}^*_d(H)$, be the set of algebraic integers of degree d (over $\mathbb{Q}$), respectively algebraic numbers of degree d, of height at most H. We set

$$A_d(H) = |\mathcal{A}_d(H)| \qquad \text{and} \qquad A^*_d(H) = |\mathcal{A}^*_d(H)|.$$

Put

$$C_5(d) = d2^d \prod_{j=1}^{\lfloor (d-1)/2 \rfloor} \frac{d(2j)^{d-2j-1}}{(2j+1)^{d-2j}}$$

and

$$C_6(d) = \frac{d2^d}{\zeta(d+1)} \prod_{j=1}^{\lfloor (d-1)/2 \rfloor} \frac{(d+1)(2j)^{d-2j}}{(2j+1)^{d-2j+1}}.$$

It follows from the work of Barroero [2, Theorem 1.1] that (see also [2, equation (1.2)] for a previous estimate with a weaker error term which follows from [6, Theorem 6])

$$A_d(H) = C_5(d)H^{d^2} + O\left(H^{d(d-1)}(\log H)^{\rho(d)}\right), \tag{1.12}$$

where $\rho(2) = 1$ and $\rho(d) = 0$ for any $d \neq 2$.

Further, Masser and Vaaler [16, equation (7)] have shown that (see also [17, equation (1.5)])

$$A_d^*(H) = C_6(d)H^{d(d+1)} + O\left(H^{d^2}(\log H)^{\vartheta(d)}\right), \tag{1.13}$$

where $\vartheta(1) = \vartheta(2) = 1$ and $\vartheta(d) = 0$ for any $d \geq 3$.

For any positive integer n, we denote by $M_{n,d}(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic integers in $\mathcal{A}_d(H)$, and we denote by $M_{n,d}^*(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic numbers in $\mathcal{A}_d^*(H)$.

For each positive integer d, we define $w_0(d)$ to be the number of roots of unity of degree d. Let φ denote Euler's totient function. Since $\varphi(k) \gg k/\log\log k$ for any integer $k \geq 3$, it follows that

$$w_0(d) \ll d^2 \log\log d, \tag{1.14}$$

where $d \geq 3$ and the implied constant is absolute. We remark that $w_0(d)$ can be zero, such as for an odd integer $d > 1$.

Given positive integers n and d, we define $C_7(n,d)$ and $C_8(n,d)$ as

$$C_7(n,d) = (nw_0(d) + n(n-1))\, C_5(d)^{n-1}$$

and

$$C_8(n,d) = (nw_0(d) + 2n(n-1))\, C_6(d)^{n-1}.$$

Theorem 1.4. *Let d and n be positive integers with $n \geq 2$. Then, the following hold.*

(i) *We have*

$$M_{n,d}(H) = C_7(n,d)H^{d^2(n-1)} + O\left(H^{d^2(n-1)-d/2}\right). \tag{1.15}$$

Furthermore if $d = 2$ or d is odd, we have

$$\begin{aligned} M_{n,d}(H) = {} & C_7(n,d)H^{d^2(n-1)} \\ & + O\left(H^{d^2(n-1)-d}\exp(c_0 \log H/\log\log H)\right) \end{aligned} \tag{1.16}$$

and

$$M_{2,d}(H) = C_7(2,d)H^{d^2} + O\left(H^{d^2-d}(\log H)^{\rho(d)}\right), \tag{1.17}$$

where c_0 is a positive number which depends only on n and d, and $\rho(d)$ has been defined in (1.12).

(ii) *We have*

$$M^*_{n,d}(H) = C_8(n,d)H^{d(d+1)(n-1)} + O\left(H^{d(d+1)(n-1)-d/2}\log H\right). \tag{1.18}$$

Furthermore if $d = 2$ or d is odd, we have

$$\begin{aligned} M^*_{n,d}(H) = {} & C_8(n,d)H^{d(d+1)(n-1)} \\ & + O\left(H^{d(d+1)(n-1)-d}\exp(c\log H/\log\log H)\right) \end{aligned} \tag{1.19}$$

and

$$M^*_{2,d}(H) = C_8(2,d)H^{d(d+1)} + O\left(H^{d^2}(\log H)^{\vartheta(d)}\right), \tag{1.20}$$

where c is a positive number which depends only on n and d, and $\vartheta(d)$ is defined in (1.13).

We remark that the case when $d = 1$ actually has been included in Theorems 1.1 and 1.2. However, in this case the error term in (1.16) is $H^{n-2+o(1)}$, which is better than that in (1.7) taken with $d = 1$.

The strategy to prove Theorem 1.4 is similar to that in proving Theorems 1.1 and 1.2. For each integer s with $0 \le s \le n-1$, we define $M_{n,d,s}(H)$ and $M^*_{n,d,s}(H)$ to be the number of multiplicatively dependent n-tuples of multiplicative rank s whose coordinates are algebraic integers in $\mathcal{A}_d(H)$ and algebraic numbers in $\mathcal{A}^*_d(H)$ respectively. Just as in (1.9) we have

$$\begin{cases} M_{n,d}(H) = M_{n,d,0}(H) + \cdots + M_{n,d,n-1}(H), \\ M^*_{n,d}(H) = M^*_{n,d,0}(H) + \cdots + M^*_{n,d,n-1}(H). \end{cases} \tag{1.21}$$

For the proof of Theorem 1.4, we make use of (1.21) and the following result.

Proposition 1.5. *Let d, n, and s be integers with $d \ge 1$, $n \ge 2$, and $0 \le s \le n-1$. Then, there exist positive numbers c_1 and c_2, which depend on n and d, such that*

$$M_{n,d,s}(H) < H^{d^2(n-1)-d(\lceil (s+1)/2\rceil -1)}\exp(c_1\log H/\log\log H) \tag{1.22}$$

and

$$\begin{aligned} M^*_{n,d,s}(H) < {} & H^{d(d+1)(n-1)-d(\lceil (s+1)/2\rceil -1)} \\ & \exp(c_2\log H/\log\log H). \end{aligned} \tag{1.23}$$

We remark that the estimate (1.22) yields an improvement on the upper bound of $H^{d^2(n-1)}$ and (1.23) yields an improvement of the upper bound $H^{d(d+1)(n-1)}$ for s at least 2.

2. Preliminaries

2.1. **Weil height.** We first record a well-known result about the absolute Weil height; see [12, Chapter 3]. Let α be a non-zero algebraic number, and let k be an integer. Then

$$\mathrm{H}(\alpha^k) = \mathrm{H}(\alpha)^{|k|}. \tag{2.1}$$

There is also a well-known comparison between the naive height H_0 and the absolute Weil height H; see [15, equation (6)]. Let α be an algebraic number of degree d. Then

$$\mathrm{H}_0(\alpha) \le (2\mathrm{H}(\alpha))^d . \tag{2.2}$$

For the proofs of Theorems 1.1 and 1.2, we need the following result.

Lemma 2.1. *Let α be an algebraic number of degree d, and let a be the leading coefficient of the minimal polynomial of α over the integers. Then*

$$\mathrm{H}(a\alpha) \le 2^{d-1}\mathrm{H}(\alpha)^d.$$

Proof. By definition, we have

$$\mathrm{H}(\alpha) = \left(a \prod_{i=1}^{d} \max\{1, |\alpha_i|\} \right)^{1/d},$$

where $\alpha_1, \ldots, \alpha_d$ are the roots of the minimal polynomial of α. Then, $a\alpha$ is an algebraic integer, and

$$\mathrm{H}(a\alpha) = \left(\prod_{i=1}^{d} \max\{1, |a\alpha_i|\} \right)^{1/d}.$$

Thus

$$\mathrm{H}(a\alpha)^d \le a^d \prod_{i=1}^{d} \max\{1, |\alpha_i|\} = a^{d-1}\mathrm{H}(\alpha)^d,$$

which, together with (2.2), implies that

$$\mathrm{H}(a\alpha)^d \le (2\mathrm{H}(\alpha))^{d(d-1)}\, \mathrm{H}(\alpha)^d = 2^{d(d-1)}\mathrm{H}(\alpha)^{d^2},$$

and so

$$\mathrm{H}(a\alpha) \le 2^{d-1}\mathrm{H}(\alpha)^d$$

as required. □

2.2. **Multiplicative structure of algebraic numbers.** Let K be a number field, and let H be a positive real number. We denote by $U_K(H)$ the number of units in the ring of algebraic integers of K of height at most H.

Lemma 2.2. *Let K be a number field, and let r be the rank of the unit group as defined in Section* 1.3. *Then, there exists a positive number c, depending on K, such that*

$$U_K(H) < c(\log H)^r.$$

Proof. This is [12, part (ii) of Theorem 5.2 of Chapter 3]. □

The next result shows that if algebraic numbers $\alpha_1, \ldots, \alpha_n$ are multiplicatively dependent, then we can find a relation as (1.1), where the exponents are not too large. Such a result has found application in transcendence theory; see for example [1, 18, 21, 26].

Lemma 2.3. *Let $n \geq 2$, and let $\alpha_1, \ldots, \alpha_n$ be multiplicatively dependent non-zero algebraic numbers of degree at most d and height at most H. Then, there is a positive number c, which depends only on n and d, and there are rational integers $k_1, \ldots, k_n$, not all zero, such that*

$$\alpha_1^{k_1} \cdots \alpha_n^{k_n} = 1$$

and

$$\max_{1 \leq i \leq n} |k_i| < c(\log H)^{n-1}.$$

Proof. This follows from [21, Theorem 1]. For an explicit constant c, we refer to [13, Corollary 3.2]. □

Let x and y be positive real numbers with y larger than 2, and let $\psi(x, y)$ denote the number of positive integers not exceeding x which contain no prime factors greater than y. Put

$$Z = \left(\log\left(1 + \frac{y}{\log x}\right)\right) \frac{\log x}{\log y} + \left(\log\left(1 + \frac{\log x}{y}\right)\right) \frac{y}{\log y}$$

and

$$u = (\log x)/(\log y).$$

Lemma 2.4. *For $2 < y \leq x$, we have*

$$\begin{aligned}&\psi(x, y)\\ &\quad = \exp\left(Z\left(1 + O((\log y)^{-1}) + O((\log\log x)^{-1}) + O((u+1)^{-1})\right)\right).\end{aligned}$$

Proof. This is [4, Theorem 1]. □

2.3. **Counting special algebraic numbers.** In this section, we count two special kinds of algebraic numbers.

Lemma 2.5. *Let K be a number field of degree d, and let u and v be non-zero integers with $u > 0$. Then, there is a positive number c, which depends on K, such that the number of elements α in K of height at most H, whose minimal polynomial has leading coefficient u and constant coefficient v, is at most*

$$\exp(c \log H / \log\log H).$$

Proof. Let $c_1, c_2, \ldots$ denote positive numbers depending on K. Let $N_{K/\mathbb{Q}}$ be the norm function from K to $\mathbb{Q}$. Suppose that α is an element of K of height at most H whose minimal polynomial has leading coefficient u and constant coefficient v. Then, we see that $u\alpha$ is an algebraic integer in K and

$$N_{K/\mathbb{Q}}(\alpha) = (-1)^d v/u \qquad \text{and} \qquad N_{K/\mathbb{Q}}(u\alpha) = (-1)^d u^{d-1} v.$$

By Lemma 2.1, we further have $\mathrm{H}(u\alpha) \leq 2^{d-1} H^d$. Note that u is fixed, so the number of such α does not exceed the number of algebraic integers $\beta \in K$ of height at most $2^{d-1} H^d$ and satisfying

$$N_{K/\mathbb{Q}}(\beta) = (-1)^d u^{d-1} v. \tag{2.3}$$

We say that two algebraic integers β_1 and β_2 in K are equivalent if the principal integral ideals $\langle \beta_1 \rangle$ and $\langle \beta_2 \rangle$ are equal. We note that, using [5, Chapter 3, equation (7.8)], the number E of equivalence classes of solutions of (2.3) is at most

$\tau(|u^{d-1}v|)^d$, where, for any positive integer k, $\tau(k)$ denotes the number of positive integers which divide k. By Wigert's Theorem (see [11, Theorem 317]),

$$E < \exp\left(c_1 \log(3|uv|)/\log\log(3|uv|)\right). \tag{2.4}$$

Further by (2.2) u and v are at most $(2H)^d$ in absolute value, hence

$$E < \exp(c_2 \log H/\log\log H). \tag{2.5}$$

Besides, if two solutions β_1 and β_2 of (2.3) are equivalent, then β_1/β_2 is a unit η in the ring of algebraic integers of K. But

$$\mathrm{H}(\eta) \le \mathrm{H}(\beta_1)\mathrm{H}((\beta_2)^{-1}) \le 2^{2(d-1)}H^{2d}.$$

By Lemma 2.2 the number of such units is at most

$$U_K(2^{2(d-1)}H^{2d}) \le c_3(\log H)^r. \tag{2.6}$$

Our result now follows from (2.5) and (2.6). □

We remark that if we set $u = 1$, then Lemma 2.5 gives an upper bound for the number of algebraic integers in K of norm $\pm v$ and of height at most H.

Given integer $d \ge 1$, let $\mathcal{C}_d^*(H)$ be the set of algebraic numbers α of degree d and height at most H such that $\alpha\eta$ is also of degree d for some root of unity $\eta \ne \pm 1$, and let $\mathcal{C}_d(H)$ be the set of algebraic integers contained in $\mathcal{C}_d^*(H)$. Here, we want to estimate the sizes of $\mathcal{C}_d(H)$ and $\mathcal{C}_d^*(H)$.

For this we need some preparation. Given a polynomial $f = a_dX^d + \cdots + a_1X + a_0 \in \mathbb{Q}[X]$ of degree d, we call it *degenerate* if it has two distinct roots whose quotient is a root of unity. Besides, we define its *height* as

$$\mathrm{H}(f) = \max\{|a_d|, \dots, |a_1|, |a_0|\},$$

and we denote by G_f the *Galois group* of the splitting field of f over $\mathbb{Q}$. Let S_d be the full symmetric group on d symbols.

Define

$$\mathcal{E}_d(H) = \{\text{monic } f \in \mathbb{Z}[X] \text{ of degree } d : \mathrm{H}(f) \le H \text{ and } G_f \ne S_d\}$$

and

$$\mathcal{E}_d^*(H) = \{f \in \mathbb{Z}[X] \text{ of degree } d : \mathrm{H}(f) \le H \text{ and } G_f \ne S_d\}.$$

The study of the sizes of $\mathcal{E}_d(H)$ and $\mathcal{E}_d^*(H)$ was initiated by van der Waerden [29]. Here, we recall a recent result due to Dietmann [8, Theorem 1]:

$$|\mathcal{E}_d(H)| \ll H^{d-1/2}. \tag{2.7}$$

Besides, by a result of Cohen [7, Theorem 1] (taking $K = \mathbb{Q}, s = n + 1$, and $r = 1$ there), we directly have

$$|\mathcal{E}_d^*(H)| \ll H^{d+1/2} \log H. \tag{2.8}$$

We also put

$$\mathcal{F}_d(H) = \{\text{monic } f \in \mathbb{Z}[X] \text{ of degree } d : \mathrm{H}(f) \le H,\ f \text{ is degenerate}\}$$

and

$$\mathcal{F}_d^*(H) = \{f \in \mathbb{Z}[X] \text{ of degree } d : \mathrm{H}(f) \le H,\ f \text{ is degenerate}\}.$$

Applying [10, Theorems 1 and 4], we have

$$|\mathcal{F}_d(H)| \ll H^{d-1} \qquad \text{and} \qquad |\mathcal{F}_d^*(H)| \ll H^d. \tag{2.9}$$

We are now ready to prove the following lemma.

Lemma 2.6. *We have*

(i) *for any integer $d \ge 1$,*

$$|\mathcal{C}_d(H)| \ll H^{d(d-1/2)} \quad \textit{and} \quad |\mathcal{C}_d^*(H)| \ll H^{d(d+1/2)} \log H;$$

(ii) *for $d = 2$ or for d odd,*

$$|\mathcal{C}_d(H)| \ll H^{d(d-1)} \qquad \textit{and} \qquad |\mathcal{C}_d^*(H)| \ll H^{d^2}.$$

Proof. Pick an arbitrary element $\alpha \in \mathcal{C}_d(H)$. We let f be its minimal polynomial over $\mathbb{Z}$, and let the d roots of f be $\alpha_1, \ldots, \alpha_d$ with $\alpha_1 = \alpha$. Since α is of height at most H, by (2.2) we have

$$\mathrm{H}(f) \le (2H)^d.$$

By definition, there is a root of unity $\eta \ne \pm 1$ such that $\alpha\eta$ is also of degree d. If $\eta \in \mathbb{Q}(\alpha)$, then under an isomorphism sending α to α_i, η is mapped to one of its conjugates η_i in $\mathbb{Q}(\alpha_i)$, which implies that $\eta \in \mathbb{Q}(\alpha_i)$ for any $1 \le i \le d$. Indeed, the image η_i of η in $\mathbb{Q}(\alpha_i)$ multiplicatively generates the same group as η, and thus η is a power of η_i, so $\eta \in \mathbb{Q}(\alpha_i)$. Hence, $\bigcap_{i=1}^d \mathbb{Q}(\alpha_i) \ne \mathbb{Q}$, so we must have $G_f \ne S_d$, that is,

$$f \in \mathcal{E}_d((2H)^d). \tag{2.10}$$

Furthermore, since f is irreducible, in this case $d \ne 2$. We also note that since η is of even degree $\varphi(k)$, where $k > 2$ is the smallest positive integer with $\eta^k = 1$, this case does not happen when d is odd.

Now, we assume that $\eta \notin \mathbb{Q}(\alpha)$. Let $K = \mathbb{Q}(\eta, \alpha_1, \ldots, \alpha_d)$, and let G be the Galois group $\mathrm{Gal}(K/\mathbb{Q})$, where K is indeed a Galois extension over $\mathbb{Q}$. We construct a disjoint union $G = \bigcup_{i=1}^d G_i$, where

$$G_i = \{\phi \in G \ : \ \phi(\alpha) = \alpha_i\}.$$

So, for each $1 \le i \le d$,

$$G_i\alpha\eta = \{\phi(\alpha\eta) \ : \ \phi \in G_i\} = \{\alpha_i\phi(\eta) \ : \ \phi \in G_i\}.$$

Since $\alpha\eta$ is of degree d, we have

$$\left| \bigcup_{i=1}^d G_i\alpha\eta \right| = d. \tag{2.11}$$

Note that $\alpha_1 = \alpha$; then $G_1 = \mathrm{Gal}(K/\mathbb{Q}(\alpha))$. Since $\eta \notin \mathbb{Q}(\alpha)$, there exist two morphisms $\phi_1, \phi_2 \in G_1$ such that $\phi_1(\eta) \ne \phi_2(\eta)$. That is, $|G_1\alpha\eta| \ge 2$. Trivially, $|G_i\alpha\eta| \ge 1$ for $2 \le i \le d$. We now see from (2.11) that there are two distinct indices i, j such that $G_i\alpha\eta \cap G_j\alpha\eta \ne \emptyset$, which implies that α_i/α_j is a root of unity and thus f is degenerate, that is,

$$f \in \mathcal{F}_d((2H)^d). \tag{2.12}$$

Hence, if $\alpha \in \mathcal{C}_d(H)$, then we have either (2.10) or (2.12). So, combining (2.7) with (2.9), we derive the first inequality in (i). If $d = 2$ or d is odd, by the above discussion we always have (2.12), and thus the first inequality in (ii) follows from (2.9). Similar arguments also apply to estimate $|\mathcal{C}_d^*(H)|$ by using (2.8) and (2.9). $\square$

3. Proofs of Propositions 1.3 and 1.5

3.1. **Proof of Proposition 1.3.** Let $c_3, c_4, \dots$ denote positive numbers depending on n and K. Let $\nu = (\nu_1, \dots, \nu_n)$ be a multiplicatively dependent vector of multiplicative rank s whose coordinates are from K and have height at most H. Set $m = s + 1$. Then, there are m distinct integers $j_1, \dots, j_m$ from $\{1, \dots, n\}$ for which $\nu_{j_1}, \dots, \nu_{j_m}$ are multiplicatively dependent, and there are non-zero integers $k_{j_1}, \dots, k_{j_m}$ for which

$$\nu_{j_1}^{k_{j_1}} \cdots \nu_{j_m}^{k_{j_m}} = 1, \tag{3.1}$$

and further, by Lemma 2.3, we can assume that

$$\max\{|k_{j_1}|, \dots, |k_{j_m}|\} < c_3 (\log H)^{m-1}. \tag{3.2}$$

Let P be the set of indices i for which k_i is positive, and let N be the set of indices i for which k_i is negative. Then

$$\prod_{i \in P} \nu_i^{k_i} = \prod_{i \in N} \nu_i^{-k_i}. \tag{3.3}$$

Plainly, either $|P|$ or $|N|$ is at least $\lceil m/2 \rceil$.

Let $I = \{j_1, \dots, j_m\}$, and let I_0 be the subset of I consisting of the indices i for which k_i is positive if $|P| \geq \lceil m/2 \rceil$, and otherwise let I_0 be the subset of I consisting of the indices i for which k_i is negative. Note that

$$|I_0| \geq \left\lceil \frac{m}{2} \right\rceil. \tag{3.4}$$

It follows from (3.3) that

$$\prod_{i \in I_0} \nu_i^{|k_i|} = \prod_{i \in I \setminus I_0} \nu_i^{|k_i|}. \tag{3.5}$$

For each coordinate ν_i, $i \in I$, let a_i be the leading coefficient of the minimal polynomial of ν_i over the integers. Note that $a_i \nu_i$ is an algebraic integer and that we can rewrite (3.5) as

$$\prod_{i \in I_0} (a_i \nu_i)^{|k_i|} = \prod_{i \in I_0} a_i^{|k_i|} \prod_{i \in I \setminus I_0} \nu_i^{|k_i|}. \tag{3.6}$$

We first establish (1.10). Accordingly, we fix non-zero algebraic integers $\nu_i \in \mathcal{B}_K(H)$ for i from $\{1, \dots, n\} \setminus I_0$ and estimate the number of solutions of (3.5) in algebraic integers ν_i, $i \in I_0$, from $\mathcal{B}_K(H)$. Observe that the number of cases when we consider an equation of the form (3.5) is, by (3.2), at most

$$\binom{n}{m} \left(2c_3 (\log H)^{(m-1)}\right)^m B_K(H)^{n-|I_0|}$$

and, by (1.3) and (3.4), is at most

$$c_4 H^{d(n - \lceil m/2 \rceil)} (\log H)^{c_5}. \tag{3.7}$$

Let $q_1, \dots, q_t$ be the primes which divide

$$\prod_{i \in I \setminus I_0} N_{K/\mathbb{Q}}(\nu_i),$$

where $N_{K/\mathbb{Q}}$ is the norm from K to $\mathbb{Q}$. Since the height of ν_i is at most H, it follows from (2.2) that

$$|N_{K/\mathbb{Q}}(\nu_i)| \le (2H)^d, \quad i = 1, 2, \ldots, n, \tag{3.8}$$

and since $|I \backslash I_0| \le n$, we see that

$$\left| \prod_{i \in I \backslash I_0} N_{K/\mathbb{Q}}(\nu_i) \right| \le (2H)^{dn}. \tag{3.9}$$

Let $p_1, \ldots, p_k$ be the first k primes, where k satisfies

$$p_1 \cdots p_k \le \left| \prod_{i \in I \backslash I_0} N_{K/\mathbb{Q}}(\nu_i) \right| < p_1 \cdots p_{k+1}.$$

Let T denote the number of positive integers up to $(2H)^d$ which are composed only of primes from $\{q_1, \ldots, q_t\}$. We see that T is bounded from above by the number of positive integers up to $(2H)^d$ which are composed of primes from $\{p_1, \ldots, p_k\}$. By (3.9), we obtain

$$\sum_{\text{prime } p \le p_k} \log p \ll \log H,$$

which, combined with the prime number theorem, yields

$$p_k < c_6 \log H.$$

Therefore we have

$$T \le \psi\left((2H)^d, c_6 \log H\right),$$

and thus by Lemma 2.4,

$$T < \exp(c_7 \log H / \log\log H). \tag{3.10}$$

It follows that if $(\nu_i, i \in I_0)$ is a solution of (3.5), then $|N_{K/\mathbb{Q}}(\nu_i)|$ is composed only of primes from $\{q_1, \ldots, q_t\}$, and so $N_{K/\mathbb{Q}}(\nu_i)$ is one of at most $2T$ integers of absolute value at most $(2H)^d$. Let a be one of those integers.

By Lemma 2.5, the number of algebraic integers α from K of height at most H for which

$$N_{K/\mathbb{Q}}(\alpha) = a \tag{3.11}$$

is at most $\exp(c_8 \log H / \log\log H)$. Therefore, by (3.10) and (3.11), the number of $|I_0|$-tuples $(\nu_i, i \in I_0)$ that give a solution of (3.5) is at most $\exp(c_9 \log H / \log\log H)$. Recalling $m = s + 1$, we see that our bound (1.10) now follows from (3.7).

We now establish (1.11). We first remark by (2.2) and Lemma 2.1 that

$$0 < a_i \le (2H)^d \tag{3.12}$$

and

$$\mathrm{H}(a_i \nu_i) \le 2^{d-1} H^d, \tag{3.13}$$

for $i = 1, \ldots, n$. Moreover, without loss of generality we can assume that $I \setminus I_0$ is not empty. Indeed, if $I \setminus I_0$ is empty, then we can replace an arbitrary coordinate $\nu_i, i \in I$, by its inverse ν_i^{-1}.

In view of (3.6), we proceed by fixing a_i for i in I_0 and ν_i for i in $\{1,\dots,n\}\backslash I$. Since $I\backslash I_0$ is non-empty, say that it contains i_1. We further fix ν_i for i in $I\backslash I_0$ with $i \neq i_1$, and then the corresponding leading coefficient a_i is also fixed. Let

$$\beta = \prod_{i\in I_0} a_i^{|k_i|} \prod_{\substack{i\in I\backslash I_0 \\ i\neq i_1}} (a_i\nu_i)^{|k_i|},$$

which is actually a fixed non-zero algebraic integer; then $N_{K/\mathbb{Q}}(\beta)$ is a fixed non-zero integer. Note that the left-hand side of (3.6) is an algebraic integer, so $\beta\nu_{i_1}$ is an algebraic integer, and then $N_{K/\mathbb{Q}}(\beta\nu_{i_1})$ is also an algebraic integer. Thus, the leading coefficient a_{i_1} divides $N_{K/\mathbb{Q}}(\beta)$. It follows that the prime factors of a_{i_1} divide

$$\prod_{i\in I_0} a_i \prod_{\substack{i\in I\backslash I_0 \\ i\neq i_1}} N_{K/\mathbb{Q}}(a_i\nu_i).$$

Since the heights of $\nu_1,\dots,\nu_n$ are at most H, we see, as in the proof of the estimate (3.10), that there are at most $\exp(c_{10}\log H/\log\log H)$ possibilities for the leading coefficient a_{i_1}. Note that by (2.2) there are at most $2(2H)^d$ possibilities for the constant coefficient of the minimal polynomial of ν_{i_1}. Thus, by Lemma 2.5, there are at most

$$H^d \exp(c_{11}\log H/\log\log H) \tag{3.14}$$

possible values of ν_{i_1} that we need to consider. In total we have, by (1.5), (3.12), and (3.14), at most

$$\binom{n}{m}\left(2c_3(\log H)^{(m-1)}\right)^m (2H)^{d|I_0|} H^{2d(n-|I_0|-1)} H^d \exp(c_{11}\log H/\log\log H)$$

equations of the form (3.6). Since $|I_0| \geq \lceil \frac{m}{2} \rceil$, the number of such equations is at most

$$H^{2dn-d(\lceil \frac{m}{2}\rceil+1)} \exp(c_{12}\log H/\log\log H). \tag{3.15}$$

Let us put

$$\gamma_0 = \prod_{i\in I_0} a_i^{|k_i|} \prod_{i\in I\backslash I_0} (a_i\nu_i)^{|k_i|} \tag{3.16}$$

and

$$\gamma_1 = \prod_{i\in I\backslash I_0} a_i^{|k_i|}.$$

Notice that once ν_i is fixed for i in $I\backslash I_0$, so is a_i, and thus γ_1 is fixed. Then, (3.6) can be rewritten as

$$\gamma_1 \prod_{i\in I_0} (a_i\nu_i)^{|k_i|} = \gamma_0, \tag{3.17}$$

and we seek an estimate for the number of solutions of (3.17) in algebraic numbers ν_i from $\mathcal{B}_K^*(H)$ with leading coefficient a_i for $i \in I_0$.

Note that γ_0 is an algebraic integer and γ_1 is an integer. Let $q_1,\dots,q_t$ be the prime factors of

$$\prod_{i\in I_0} a_i \prod_{i\in I\backslash I_0} N_{K/\mathbb{Q}}(a_i\nu_i).$$

Then, by (3.16) and (3.17), for each index $i \in I_0$ the prime factors of $N_{K/\mathbb{Q}}(a_i\nu_i)$ are from $\{q_1, \ldots, q_t\}$. It follows from (3.12), (3.13), and (2.2) that

$$\left| \prod_{i \in I_0} a_i \prod_{i \in I \setminus I_0} N_{K/\mathbb{Q}}(a_i \nu_i) \right| \le (2H)^{d|I_0|} (2^d H^d)^{d|I \setminus I_0|} \le (2H)^{d^2 n}.$$

We can now argue as in our proof of (1.10) that the number of solutions of (3.17) in algebraic integers $a_i\nu_i$, $i \in I_0$, from K of height at most $2^{d-1}H^d$ is at most $\exp(c_{13} \log H / \log\log H)$. The result (1.11) now follows from (3.15).

3.2. **Proof of Proposition 1.5.** Let $c_3, c_4, \ldots$ denote positive numbers depending on n and d. Let $\nu = (\nu_1, \ldots, \nu_n)$ be a multiplicatively dependent vector of multiplicative rank s whose coordinates are from $\mathcal{A}_d^*(H)$. Set $m = s + 1$. Then, there are m distinct integers $j_1, \ldots, j_m$ from $\{1, \ldots, n\}$ for which $\nu_{j_1}, \ldots, \nu_{j_m}$ are multiplicatively dependent, and there are non-zero integers $k_{j_1}, \ldots, k_{j_m}$ for which (3.1) holds, and by Lemma 2.3, we can suppose that (3.2) holds. Let $I = \{j_1, \ldots, j_m\}$ and I_0 be defined as in the proof of Proposition 1.3, so that (3.4) and (3.5) hold.

We first establish (1.22). Fixing non-zero algebraic integers $\nu_i \in \mathcal{A}_d(H)$ for $i \in \{1, \ldots, n\} \setminus I_0$, we want to estimate the number of solutions of (3.5) in algebraic integers $\nu_i \in \mathcal{A}_d(H)$ for $i \in I_0$. The number of cases when we consider an equation of the form (3.5) is, by (3.2), at most

$$\binom{n}{m} \left(2c_3 (\log H)^{m-1}\right)^m A_d(H)^{n-|I_0|},$$

which, by (1.12), is at most

$$c_4 H^{d^2(n-|I_0|)} (\log H)^{m(m-1)}. \tag{3.18}$$

For each $i \in I_0$, by (3.5) the prime factors of $N_{\mathbb{Q}(\nu_i)/\mathbb{Q}}(\nu_i)$ divide

$$\prod_{j \in I \setminus I_0} N_{\mathbb{Q}(\nu_j)/\mathbb{Q}}(\nu_j).$$

Exactly as in the proof of Proposition 1.3, we can apply (2.2) and Lemma 2.4 to conclude that, for $i \in I_0$, $N_{\mathbb{Q}(\nu_i)/\mathbb{Q}}(\nu_i)$ is one of at most T integers, where, as in (3.10),

$$T < \exp(c_5 \log H / \log\log H).$$

Then, estimating the number of possible choices of the minimal polynomial of ν_i over the integers by using (2.2), we see that there are at most

$$d\left(2(2H)^d + 1\right)^{d-1} \exp(c_5 \log H / \log\log H) \tag{3.19}$$

possible values of each ν_i for $i \in I_0$. We now fix $|I_0| - 1$ of the terms ν_i with i in I_0. Let $i_0 \in I_0$ denote the index of the term which is not fixed. Then, ν_{i_0} is a solution of

$$x^{|k_{i_0}|} = \eta_0, \tag{3.20}$$

where

$$\eta_0 = \prod_{\substack{i \in I_0 \\ i \ne i_0}} \nu_i^{-|k_i|} \prod_{i \in I \setminus I_0} \nu_i^{|k_i|}.$$

If ν_{i_0} and μ_{i_0} are two solutions of (3.20) from $\mathcal{A}_d(H)$, then ν_{i_0}/μ_{i_0} is a $|k_{i_0}|$-th root of unity. But the degree of ν_{i_0}/μ_{i_0} is at most d^2, and so there are at most c_6 possibilities for ν_{i_0}/μ_{i_0} when d is fixed. It follows from (3.19) that each equation (3.5) has at most

$$H^{d(d-1)(|I_0|-1)}\exp(c_7\log H/\log\log H) \tag{3.21}$$

solutions. Thus by (3.18) and (3.21), we have

$$M_{n,d,s}(H) < H^{d^2(n-|I_0|)+d(d-1)(|I_0|-1)}\exp(c_8\log H/\log\log H). \tag{3.22}$$

Further, by (3.4),

$$d^2(n-|I_0|)+d(d-1)(|I_0|-1) \le d^2(n-1)-d\left(\left\lceil\frac{m}{2}\right\rceil-1\right). \tag{3.23}$$

Now, (1.22) follows from (3.22) and (3.23).

We next establish (1.23). For each $i\in I$, let a_i denote the leading coefficient of the minimal polynomial of ν_i over the integers. Without loss of generality, we can assume that $I\setminus I_0$ is not empty. Indeed, if $I\setminus I_0$ is empty, then we can replace an arbitrary coordinate $\nu_i, i\in I$, by its inverse ν_i^{-1}.

In view of (3.6), we proceed by first fixing positive integers a_i for $i\in I_0$. Since $I\backslash I_0$ is non-empty, say that it contains i_1. We next fix ν_i for i in $i\in\{1,\dots,n\}\backslash I_0$ with $i\ne i_1$, and then the corresponding a_i is also fixed. Let

$$\beta=\prod_{i\in I_0}a_i^{|k_i|}\prod_{\substack{i\in I\backslash I_0\\ i\ne i_1}}(a_i\nu_i)^{|k_i|},$$

which is a fixed non-zero algebraic integer. Notice that the left-hand side of (3.6) is an algebraic integer, so $\beta\nu_{i_1}$ is also an algebraic integer, and thus as in the proof of (1.11) the prime factors of the leading coefficient a_{i_1} divide

$$\prod_{i\in I_0}a_i\prod_{\substack{i\in I\backslash I_0\\ i\ne i_1}}N_{\mathbb{Q}(\nu_i)/\mathbb{Q}}(a_i\nu_i).$$

Since the heights of $\nu_1,\dots,\nu_n$ are at most H and their degrees are all equal to d, we see, as in the proof of (3.10), that there are at most $\exp(c_9\log H/\log\log H)$ possibilities for the leading coefficient a_{i_1}. Then, combining this result with (2.2), we know that the number of the possibilities for the minimal polynomial of ν_{i_1} is at most

$$H^{d^2}\exp(c_{10}\log H/\log\log H).$$

Thus, there are at most

$$H^{d^2}\exp(c_{11}\log H/\log\log H) \tag{3.24}$$

possible values of ν_{i_1} that we need to consider.

Hence, the number of cases of the equation (3.6) to be considered is, by (3.2), (3.12), and (3.24), at most

$$\binom{n}{m}\left(2c_3(\log H)^{m-1}\right)^m(2H)^{d|I_0|}A_d^*(H)^{n-|I_0|-1}H^{d^2}$$
$$\exp(c_{11}\log H/\log\log H),$$

which, by (1.13), is at most

$$H^{d(d+1)(n-|I_0|-1)+d|I_0|+d^2}\exp(c_{12}\log H/\log\log H). \tag{3.25}$$

We now estimate the number of solutions of (3.6) in algebraic numbers $\nu_i \in \mathcal{A}_d^*(H)$ for $i \in I_0$ with minimal polynomial having leading coefficient a_i. It follows from (3.6) that for each $i \in I_0$ the prime factors of $N_{\mathbb{Q}(\nu_i)/\mathbb{Q}}(a_i\nu_i)$ divide

$$\prod_{j \in I_0} a_j \prod_{j \in I \setminus I_0} N_{\mathbb{Q}(\nu_j)/\mathbb{Q}}(a_j\nu_j).$$

Thus, by (2.2), Lemma 2.1, and Lemma 2.4, as in the proof of (3.10), there is a set of at most T integers, where

$$T < \exp(c_{13} \log H / \log\log H),$$

and $N_{\mathbb{Q}(\nu_i)/\mathbb{Q}}(a_i\nu_i)$ belongs to that set. Since a_i is fixed, the norm $N_{\mathbb{Q}(\nu_i)/\mathbb{Q}}(\nu_i)$ also belongs to a set of cardinality at most T for $i \in I_0$. Notice that for the minimal polynomial of $\nu_i, i \in I_0$, if $N_{\mathbb{Q}(\nu_i)/\mathbb{Q}}(\nu_i)$ is fixed, then the constant coefficient is also fixed, because the leading coefficient a_i has already been fixed. Hence, counting possible choices of the minimal polynomial of ν_i by using (2.2), we see that there are at most

$$H^{d(d-1)} \exp(c_{14} \log H / \log\log H) \tag{3.26}$$

possible values of ν_i for $i \in I_0$. We now fix $|I_0| - 1$ of the coordinates ν_i with $i \in I_0$ and argue as before to conclude from (3.26) that each equation (3.6) has at most

$$H^{d(d-1)(|I_0|-1)} \exp(c_{15} \log H / \log\log H) \tag{3.27}$$

solutions. Thus, by (3.25) and (3.27), we obtain

$$\begin{aligned} M_{n,d,s}^*(H) < H^{d(d+1)(n-|I_0|-1)+d|I_0|+d^2+d(d-1)(|I_0|-1)} \\ \exp(c_{16} \log H / \log\log H). \end{aligned} \tag{3.28}$$

Observing that

$$\begin{aligned} d(d+1)(n - |I_0| - 1) + d\,|I_0| + d^2 + d(d-1)(|I_0| - 1) \\ = d(d+1)(n-1) - d(|I_0| - 1), \end{aligned}$$

our result (1.23) now follows from (3.4) and (3.28).

4. Proof of main results

4.1. **Proof of Theorem 1.1.** By (1.9) and (1.10), there is a positive number c which depends on n and K such that

$$\begin{aligned} L_{n,K}(H) = L_{n,K,0}(H) + L_{n,K,1}(H) \\ + O(H^{d(n-1)-d} \exp(c \log H / \log\log H)). \end{aligned} \tag{4.1}$$

Each such vector ν of multiplicative rank 0 has an index i_0 for which ν_{i_0} is a root of unity. Accordingly, we have

$$nw(B_K(H) - w - 1)^{n-1} \le L_{n,K,0}(H) \le nwB_K(H)^{n-1},$$

and thus by (1.3)

$$\begin{aligned} L_{n,K,0}(H) = nwC_1(K)^{n-1} H^{d(n-1)} (\log H)^{r(n-1)} \\ + O\left(H^{d(n-1)} (\log H)^{r(n-1)-1}\right). \end{aligned} \tag{4.2}$$

We next estimate $L_{n,K,1}(H)$. Each such vector ν of rank 1 has a pair of indices (i_0, i_1), two coordinates ν_{i_0} and ν_{i_1} from $\mathcal{B}_K(H)$, and non-zero integers k_{i_0} and k_{i_1}

such that $\nu_{i_0}^{k_{i_0}}\nu_{i_1}^{k_{i_1}} = 1$. There are $n(n-1)/2$ pairs (i_0, i_1). By Lemma 2.3, the number of such vectors associated with two distinct such pairs (i_0, i_1) and (i_2, i_3) is

$$O\left(B_K(H)^{n-2}(\log H)^4\right). \tag{4.3}$$

We now estimate the number of n-tuples ν whose coordinates are from $\mathcal{B}_K(H)$ for which

$$\nu_{i_0}^{k_{i_0}}\nu_{i_1}^{k_{i_1}} = 1$$

with (k_{i_0}, k_{i_1}) equal to (t,t) or $(t,-t)$ for some non-zero integer t. We have $(B_K(H) - w - 1)^{n-2}$ choices for the coordinates of ν associated with indices different from i_0 and i_1, because they are non-zero and not roots of unity. Also there are $B_K(H) - w - 1$ choices for the i_0-th coordinate, and once it is determined, say ν_{i_0}, then the i_1-th coordinate is of the form $\eta\nu_{i_0}$ or $\eta\nu_{i_0}^{-1}$, where η is a root of unity from K. Note that

$$\mathrm{H}(\eta\nu_{i_0}) = \mathrm{H}(\nu_{i_0}) = \mathrm{H}(\eta\nu_{i_0}^{-1}),$$

and that $\eta\nu_{i_0}^{-1}$ is only counted when ν_{i_0} is a unit in the ring of algebraic integers of K. Thus, we have

$$(B_K(H) - w - 1)^{n-2}\left((B_K(H) - w - 1)\, w + (U_K(H) - w)\, w\right) \tag{4.4}$$

such vectors of rank 1 associated with (i_0, i_1). So, by (1.3), (4.3), (4.4), and Lemma 2.2, the number of such vectors of rank 1 associated with an exponent vector $\mathbf{k}$ with $k_{i_0} = t$, $k_{i_1} = \pm t$ for t a non-zero integer is

$$\begin{aligned}\frac{n(n-1)}{2} wC_1(K)^{n-1}H^{d(n-1)}(\log H)^{r(n-1)}& \\ + O\left(H^{d(n-1)}(\log H)^{r(n-1)-1}\right)&.\end{aligned} \tag{4.5}$$

It remains to estimate the number of such vectors of multiplicative rank 1 associated with an exponent vector $\mathbf{k}$ with $k_{i_0} = t_1$ and $k_{i_1} = t_2$ with $t_1 \neq \pm t_2$ and t_1 and t_2 non-zero integers. Let $\nu_1, \nu_2 \in \mathcal{B}_K(H)$ be associated with $t_1, -t_2$ respectively. In this case

$$\nu_1^{t_1} = \nu_2^{t_2}.$$

We first consider the case when t_1 and t_2 are of opposite signs. Then, ν_1 and ν_2 are units in the ring of algebraic integers of K, and so by Lemma 2.2 the number of such vectors is

$$O\left((\log H)^{2r}B_K(H)^{n-2}\right). \tag{4.6}$$

It remains to consider the case when t_1 and t_2 are both positive. Without loss of generality, we assume that $0 < t_1 < t_2$, and also $t_2 \ll \log H$ by Lemma 2.3.

If $t_2 = 2t_1$, then ν_1 is determined by ν_2^2 up to a root of unity contained in K, and also we have $\mathrm{H}(\nu_2) \le H^{1/2}$. So, the number of such pairs (ν_1, ν_2) is $O(H^{d/2}(\log H)^r)$ by using (1.3), and thus the number of such vectors of rank 1 is

$$O\left(H^{d/2}(\log H)^r B_K(H)^{n-2}\right). \tag{4.7}$$

If t_1 divides t_2 and $t_2/t_1 \ge 3$, then we have $\mathrm{H}(\nu_2) \le H^{1/3}$, and so as in the above the number of such vectors of rank 1 is

$$O\left(H^{d/3}(\log H)^{r+1}B_K(H)^{n-2}\right). \tag{4.8}$$

Now, we assume that t_1 does not divide t_2. Let t be the greatest common divisor of t_1 and t_2. Note that $t_1/t \geq 2$ and $t_2/t \geq 3$. Put

$$\gamma = \nu_1^{t_1} = \nu_2^{t_2}, \tag{4.9}$$

and let β be a root of $x^{t_1t_2} - \gamma$. Observe that

$$\beta^{t_1} = \eta_1\nu_2 \quad \text{and} \quad \beta^{t_2} = \eta_2\nu_1$$

for some t_1t_2-th roots of unity η_1 and η_2. There exist integers u and v with $ut_1 + vt_2 = t$, and so

$$\beta^t = \beta^{t_1u}\beta^{t_2v} = \eta_1^u\nu_2^u\eta_2^v\nu_1^v = \eta\alpha$$

for η a t_1t_2-th root of unity and α an algebraic integer of K. Therefore

$$(\eta\alpha)^{t_2/t} = \beta^{t_2} = \eta_2\nu_1, \tag{4.10}$$

and so

$$\mathrm{H}(\alpha)^{t_2/t} = \mathrm{H}(\nu_1). \tag{4.11}$$

Since $\mathrm{H}(\nu_1) \leq H$, we see, from (4.10) and (4.11), that ν_1 is determined up to a t_1t_2-th root of unity, by an algebraic integer of K of height at most $H^{t/t_2} \leq H^{1/3}$. Thus, by (1.3) and Lemma 2.3, the number of such pairs (ν_1, ν_2) is

$$O(H^{d/3}(\log H)^{r+3});$$

hence the number of such vectors of rank 1 is

$$O\left(H^{d/3}(\log H)^{r+3}B_K(H)^{n-2}\right). \tag{4.12}$$

Thus, by (1.3), (4.5), (4.6), (4.7), (4.8), and (4.12), we get

$$\begin{aligned} L_{n,K,1}(H) = \frac{n(n-1)}{2}&wC_1(K)^{n-1}H^{d(n-1)}(\log H)^{r(n-1)} \\ &+ O\left(H^{d(n-1)}(\log H)^{r(n-1)-1}\right). \end{aligned} \tag{4.13}$$

The estimate (1.6) now follows from (4.1), (4.2), and (4.13).

Finally, assume that K is the rational number field $\mathbb{Q}$ or an imaginary quadratic field. Then, $r = 0$, and so $B_K(H) = C_1(K)H^d + O(H^{d-1})$ by (1.4). Repeating the above process, we obtain

$$L_{n,K,0}(H) = nwC_1(K)^{n-1}H^{d(n-1)} + O(H^{d(n-1)-1})$$

and

$$L_{n,K,1}(H) = \frac{n(n-1)}{2}wC_1(K)^{n-1}H^{d(n-1)} + O\left(H^{d(n-3/2)}\right),$$

where the second error term comes from (4.7) (and also (4.4) when $d = 2$). Hence, noticing (4.1) and $d = 1$ or 2, we obtain (1.7).

4.2. **Proof of Theorem 1.2.** By (1.9) and (1.11), we have

$$\begin{aligned} L_{n,K}^*(H) = L_{n,K,0}^*(H) &+ L_{n,K,1}^*(H) \\ &+ O\left(H^{2d(n-1)-d}\exp(c_2\log H/\log\log H)\right). \end{aligned} \tag{4.14}$$

As in the proof of Theorem 1.1, we obtain, by using (1.5) in place of (1.3),

$$L_{n,K,0}^*(H) = nwC_2(K)^{n-1}H^{2d(n-1)} + O\left(H^{2d(n-1)-1}(\log H)^{\sigma(d)}\right), \tag{4.15}$$

where $\sigma(1) = 1$ and $\sigma(d) = 0$ for $d > 1$.

Similarly, we find that

$$L^*_{n,K,1}(H) = n(n-1)wC_2(K)^{n-1}H^{2d(n-1)} + O\left(H^{2d(n-1)-1}(\log H)^{\sigma(d)}\right), \tag{4.16}$$

where the main difference from the proof of (4.13) is that the contribution from the exponent vectors (k_{i_0}, k_{i_1}) equal to (t,t) is the same as when (k_{i_0}, k_{i_1}) is equal to $(t,-t)$.

The desired result now follows from (4.14), (4.15), and (4.16) by noticing that

$$L^*_{2,K}(H) = L^*_{2,K,0}(H) + L^*_{2,K,1}(H).$$

4.3. **Proof of Theorem 1.4.** We first establish (1.15). By (1.21) and (1.22), we have

$$M_{n,d}(H) = M_{n,d,0}(H) + M_{n,d,1}(H) + O\left(H^{d^2(n-1)-d}\exp(c_1 \log H/\log\log H)\right). \tag{4.17}$$

Note that each such vector ν of multiplicative rank 0 has a coordinate which is a root of unity of degree d. So, in view of the definition of $w_0(d)$ in (1.14) we have

$$nw_0(d)\left(A_d(H) - w_0(d)\right)^{n-1} \le M_{n,d,0}(H) \le nw_0(d)A_d(H)^{n-1},$$

and thus by (1.12) and (1.14),

$$M_{n,d,0}(H) = nw_0(d)C_5(d)^{n-1}H^{d^2(n-1)} + O\left(H^{d^2(n-1)-d}(\log H)^{\rho(d)}\right). \tag{4.18}$$

We remark that $M_{n,d,0}(H) = 0$ if $w_0(d) = 0$.

Moreover, arguing as in the proof of Theorem 1.1, we find that the main contribution to $M_{n,d,1}(H)$ comes from vectors associated with an exponent vector $\mathbf{k}$ which has two non-zero components, one of which is t and the other of which is $\pm t$ with t a non-zero integer. Notice that the number $U_d(H)$ of algebraic integers which are units of degree d and height at most H satisfies (by using (2.2))

$$U_d(H) = O\left(H^{d(d-1)}\right). \tag{4.19}$$

We then deduce from (1.12), (1.14), (4.19), and Lemma 2.6 that

$$M_{n,d,1}(H) = n(n-1)C_5(d)^{n-1}H^{d^2(n-1)} + O\left(H^{d^2(n-1)-d/2}\right). \tag{4.20}$$

If furthermore $d=2$ or d is odd, then

$$M_{n,d,1}(H) = n(n-1)C_5(d)^{n-1}H^{d^2(n-1)} + O\left(H^{d^2(n-1)-d}\log H\right). \tag{4.21}$$

Here, we need to note that for an algebraic integer α of degree d and a root of unity $\eta \ne \pm 1$, $\alpha\eta$ might not be of degree d.

The desired asymptotic formula (1.15) now follows from (4.17), (4.18), and (4.20). In order to show (1.16), we use (4.21) instead of (4.20). Besides, (1.17) follows from (4.18) and (4.21) by noticing that

$$M_{2,d}(H) = M_{2,d,0}(H) + M_{2,d,1}(H).$$

Finally, we prove (1.18), (1.19), and (1.20). By (1.21) and (1.23), we have

$$\begin{aligned} M^*_{n,d}(H) =& M^*_{n,d,0}(H) + M^*_{n,d,1}(H) \\ & + O\left(H^{d(d+1)(n-1)-d}\exp(c_2 \log H/\log\log H)\right). \end{aligned} \tag{4.22}$$

As before, we have, by using (1.13),

$$\begin{aligned} M^*_{n,d,0}(H) = & \, nw_0(d)C_6(d)^{n-1}H^{d(d+1)(n-1)} \\ & + O\left(H^{d(d+1)(n-1)-d}(\log H)^{\vartheta(d)}\right). \end{aligned} \tag{4.23}$$

As in (4.20) and (4.21), we find that

$$\begin{aligned} M^*_{n,d,1}(H) = & \, 2n(n-1)C_6(d)^{n-1}H^{d(d+1)(n-1)} \\ & + O\left(H^{d(d+1)(n-1)-d/2}\log H\right). \end{aligned} \tag{4.24}$$

If furthermore $d = 2$ or d is odd, we have

$$\begin{aligned} M^*_{n,d,1}(H) = & \, 2n(n-1)C_6(d)^{n-1}H^{d(d+1)(n-1)} \\ & + O\left(H^{d(d+1)(n-1)-d}(\log H)^{\vartheta(d)}\right). \end{aligned} \tag{4.25}$$

So, (1.18) follows from (4.22), (4.23), and (4.24); then using (4.25) instead of (4.24) gives (1.19). In order to deduce (1.20), we apply (4.23) and (4.25) and notice that

$$M^*_{2,d}(H) = M^*_{2,d,0}(H) + M^*_{2,d,1}(H).$$

5. Lower bound

In this section, we prove that (1.10) is sharp, apart from a factor $H^{o(1)}$, when $n = s + 1$ is even.

In order to estabish the case $K = \mathbb{Q}$, we need the following slight extension of [19, Lemma 2.3].

Lemma 5.1. *Let k and q be integers with $k \ge 2$ and $q \ge 2$. Let $\gamma = (\gamma_1, \ldots, \gamma_k)$ with $\gamma_1, \ldots, \gamma_k$ positive real numbers. Then, there exists a positive number $\Gamma(q,\gamma)$ such that for $T \to \infty$, we have*

$$\sum_{\substack{a_1\cdots a_k = b_1 \cdots b_k \\ \gcd(a_i b_i, q) = 1 \\ 1 \le a_i, b_i \le T^{\gamma_i} \\ i = 1, \ldots, k}} \cdots \sum 1 \quad \sim \quad \Gamma(q,\gamma) T^{\gamma} (\log T)^{(k-1)^2},$$

where $\gamma = \gamma_1 + \cdots + \gamma_k$.

Proof. The proof proceeds along the same lines as in the proof of [19, Lemma 2.3]. The only difference is that the primes p which divide q are now excluded from the Euler products that appear in [19]. ☐

We show that apart from the factor $\exp(c_1 \log H/\log\log H)$ the estimate (1.10) in Proposition 1.3 is sharp when n is even, $s = n - 1$, and $K = \mathbb{Q}$.

Theorem 5.2. *Let $n = 2k$, where k is an integer with $k > 1$. Then, for sufficiently large H, there exists a positive number c depending only on n such that*

$$L_{n,\mathbb{Q},n-1}(H) \ge cH^k (\log H)^{(k-1)^2}.$$

Proof. Fix $n-2$ distinct odd primes p_i, q_i, $i=2,\dots,k$. Given positive integers $a_1,\dots,a_k,b_1,\dots,b_k$, we first set

$$\nu_1 = 2p_2\cdots p_k a_1 \qquad \text{and} \qquad \nu_{k+1} = 2q_2\cdots q_k b_1.$$

After this we set

$$\nu_i = q_i a_i \qquad \text{and} \qquad \nu_{k+i} = p_i b_i, \quad i=2,\dots,k.$$

Clearly, if $a_1\cdots a_k = b_1\cdots b_k$ with $\gcd(a_ib_i, 2p_2q_2\cdots p_kq_k)=1$ for any $2\le i\le k$, then the integer vector $\nu=(\nu_1,\dots,\nu_n)$ is multiplicatively dependent of rank $n-1$ by noticing that $\nu_1\cdots\nu_k = \nu_{k+1}\cdots\nu_n$ and that there is no non-empty subset $\{i_1,\dots,i_m\}$ of $\{1,\dots,n\}$ of size less than n for which

$$\nu_{i_1}^{j_{i_1}}\cdots\nu_{i_m}^{j_{i_m}} = 1,$$

with $j_{i_1},\dots,j_{i_m}$ non-zero integers.

For sufficiently large H, we choose such integers $a_i,b_i\le c_1H$ for some positive number c_1 depending only on the above fixed primes such that we have $|\nu_i|\le H$ for each $1\le i\le n$. Then, each such vector ν contributes to $L_{n,\mathbb{Q},n-1}(H)$. Now applying Lemma 5.1 to count such vectors (taking $T=c_1H$ and $\gamma_i=1$ for each $i=1,\dots,k$), we derive

$$L_{n,\mathbb{Q},n-1}(H)\ge cH^k(\log H)^{(k-1)^2},$$

where c is a positive number depending on n. ☐

To get a more general result, we need the following result, which might be of independent interest.

Lemma 5.3. *Let K be a number field of degree d, and let m be a positive integer. Assume that m has t distinct prime factors and each prime factor of m is greater than dt. Then, for sufficiently large H, there exists a positive number c depending only on m and K such that*

$$|\{\alpha\in\mathcal{B}_K(H)\ :\ \gcd(\alpha,m)=1\}|\ge cH^d(\log H)^r,$$

where r is the rank of the unit group of K.

Proof. Applying (1.3), it suffices to show that for each $\alpha\in\mathcal{B}_K(H)$ with $\gcd(\alpha,m)\ne 1$, there is a uniform way to construct an element $\beta\in\mathcal{B}_K(cH)$ with $\gcd(\beta,m)=1$, where the constant c depends only on m and K.

Now, given $\alpha\in\mathcal{B}_K(H)$ with $\gcd(\alpha,m)\ne1$, let $\alpha_i=\alpha+i$ for $i=0,1,\dots,dt$. Assume that for each $0\le i\le dt$, we have $\gcd(\alpha_i,m)\ne1$. Note that in the prime decomposition of the ideal $\langle m\rangle$ in K there are at most dt distinct prime ideals, but the number of such α_i is $dt+1$. So, there exist $0\le i<j\le dt$ such that the two ideals $\langle\alpha_i\rangle$ and $\langle\alpha_j\rangle$ have a common prime factor, say $\mathfrak{p}$, which corresponds to a prime factor of m, say p. Then, $\alpha_i,\alpha_j\in\mathfrak{p}$, and then $\alpha_j-\alpha_i=j-i\in\mathfrak{p}$, and thus $p\mid j-i$, which contradicts the assumption $p>dt$.

Therefore, there must exist $0\le j\le dt$ such that $\gcd(\alpha+j,m)=1$. This in fact completes the proof. ☐

Using Lemma 5.3 instead of Lemma 5.1, we can get a slightly weaker but more general result.

Theorem 5.4. *Let $n = 2k$, where k is an integer with $k > 1$, and let K be a number field of degree d. Then, for sufficiently large H, there exists a positive number c depending on n and K such that*

$$L_{n,K,n-1}(H) \geq cH^{dk}(\log H)^{rk}.$$

Proof. Following the strategy in the proof of Theorem 5.2 and letting $a_i = b_i \in \mathcal{B}_K(H)$ for each $1 \leq i \leq k$, one can directly get the desired result by choosing sufficiently large primes p_i, q_i and using Lemma 5.3. □

Similarly, to understand the tightness of (1.22), we need the following simple statement:

Lemma 5.5. *Let d and m be two positive integers. Assume that the prime factors of m are all sufficiently large. Then, for sufficiently large H, there exists a positive number c depending only on d and m such that*

$$|\{\alpha \in \mathcal{A}_d(H) \ : \ \gcd(\alpha, m) = 1\}| \geq cH^{d^2}.$$

Proof. Let the prime factors of m be $\ell_1, \ldots, \ell_t$. Given $\alpha \in \mathcal{A}_d(H)$, let $x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$ be the minimal polynomial of α over $\mathbb{Z}$. By (2.2), we have

$$|a_i| \leq (2H)^d, \qquad i = 0, 1, \ldots, d-1.$$

If $\gcd(\alpha, m) \neq 1$, then there exists a prime factor, say ℓ_j, of m such that $\ell_j \mid a_0$. So, counting related minimal polynomials we obtain

$$\begin{aligned} &|\{\alpha \in \mathcal{A}_d(H) \ : \ \gcd(\alpha, m) \neq 1\}| \\ &\qquad \leq \sum_{j=1}^{t} d\left(2(2H)^d + 1\right)^{d-1} \cdot 2(2H)^d/\ell_j. \end{aligned} \tag{5.1}$$

Note that when $\ell_1, \ldots, \ell_t$ are all sufficiently large, the coefficient of H^{d^2} in the right-hand side of (5.1) is less than $C_5(d)$ defined in (1.12). Combining (5.1) with (1.12) completes the proof. □

Now, we are ready to get a partial comparison for (1.22).

Theorem 5.6. *Let $n = 2k$, where k is an integer with $k > 1$, and let d be a positive integer. Then, for sufficiently large H, there exists a positive number c depending on n and d such that*

$$M_{n,d,n-1}(H) \geq cH^{d^2k}.$$

Proof. Following the strategy in the proof of Theorem 5.2 and letting $a_i = b_i \in \mathcal{A}_d(H)$ for each $1 \leq i \leq k$, we can obtain the desired result by choosing sufficiently large primes p_i, q_i and using Lemma 5.5. □

Notice that from (1.22) and under the assumption in Theorem 5.6, we have

$$M_{n,d,n-1}(H) \leq H^{d^2(2k-1)-d(k-1)+o(1)},$$

which is still much larger than the lower bound in Theorem 5.6. This suggests that the optimal exponent of H in (1.22) might be

$$d^2(n-1) - d^2(\lceil (s+1)/2 \rceil - 1),$$

which would show that the lower bound in Theorem 5.6 is sharp up to a factor $H^{o(1)}$.

6. Comments

It might be of interest to investigate in more detail how tight our bounds are in Propositions 1.3 and 1.5. In Section 5 we have taken an initial step in this direction.

It would be interesting to study multiplicatively dependent vectors of polynomials over finite fields. In this case the degree plays the role of the height. While we expect that most of our results can be translated to this case, many tools need to be developed, and this should be of independent interest.

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Dipartimento di Matematica e Fisica, Università Roma Tre, Roma, I–00146, Italy
Email address: pappa@mat.uniroma3.it

School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia
Email address: shamin2010@gmail.com

School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia
Email address: igor.shparlinski@unsw.edu.au

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada
Email address: cstewart@uwaterloo.ca