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ABELIAN SPIDERS AND REAL CYCLOTOMIC INTEGERS

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ABSTRACT. If Γ is a finite graph, then the largest eigenvalue λ of the adjacency matrix of Γ is a totally real algebraic integer (λ is the Perron-Frobenius eigenvalue of Γ). We say that Γ is abelian if the field generated by λ^2 is abelian. Given a fixed graph Γ and a fixed set of vertices of Γ, we define a spider graph to be a graph obtained by attaching to each of the chosen vertices of Γ some 2-valent trees of finite length. The main result is that only finitely many of the corresponding spider graphs are both abelian and not Dynkin diagrams, and that all such spiders can be effectively enumerated; this generalizes a previous result of Calegari, Morrison, and Snyder. The main theorem has applications to the classification of finite index subfactors. We also prove that the set of Salem numbers of "abelian type" is discrete.

1. Introduction

Let Γ be a connected finite graph. Fix an integer k, and let $v_1, \ldots, v_k$ be a collection of k (not necessarily distinct) vertices of Γ. We say a graph Γ is **abelian** if $\mathbf{Q}(\lambda^2)$ is an abelian extension, where λ is the Perron–Frobenius eigenvalue of Γ (the unique largest real eigenvalue of the adjacency matrix M_Γ of Γ). If Γ is either one of the Dynkin diagrams (A_n, D_n, E_6, E_7, E_8) or the simply laced affine Dynkin diagrams ($\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$), then Γ is abelian, and $\lambda^2 = 4\cos^2(2\pi/N)$ for some integer N. Conversely, if $\lambda \leq 2$, then Γ is such a diagram.

For any k-tuple $\underline{r} = (r_1, \ldots, r_k)$ of non-negative integers, we define a (k-)**spider graph** $\Gamma_{\underline{r}}$ on Γ to be the graph obtained by adjoining a path (2-valent tree) with r_i edges to Γ at v_i.

Theorem 1.1. *Fix Γ and k. There are only finitely many abelian k-spiders $\Gamma_{\underline{r}}$ which are not Dynkin diagrams. There is an effective algorithm for determining all such spiders.*

Remark 1.2. *If Γ is not already of the form A_n or D_n, then only finitely many of the spiders $\Gamma_{\underline{r}}$ will be Dynkin diagrams.*

One motivation for this paper is the application to subfactors, as in [CMS11]. One of the main results (Theorem 1.0.3) of [CMS11] was a version of Theorem 1.1 for 1-spiders. The paper [CMS11] also contained a weaker result (Theorem 1.0.6) which was sufficient for the application to subfactors but had the advantage that the effective constants could be made explicit. In contrast, Theorem 1.1 already

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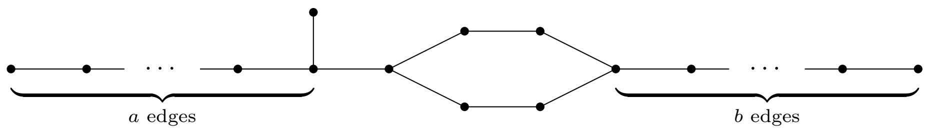
comes with computable effective constants, and, moreover, these constants will be small enough that our results are "effectively effective" in many cases (although there is certainly some combinatorial explosion as k increases). In order for this to be so, we have worked hard in this paper to make our results as tight as possible, even when weaker estimates would certainly suffice to prove the main theorem. As an application of Theorem 1.1 to the theory of subfactors, we prove the following result, brought to our attention by S. Morrison [Mor]. This result is used as an ingredient for the classification of subfactors with index at most $5\frac{1}{4}$; see the paper of Afzaly, Morrison, and Penneys [AMP]. Let $\Gamma_{a,b}$ denote the "AMP (Afzaly–Morrison–Penneys) spider", given as follows:



Theorem 1.3. *The spider $\Gamma_{a,b}$ above is abelian only when $(a,b) = (0,0)$ or $(1,1)$.*

1.1. **A guide to the paper.** We begin in §2 by giving some refined estimates for a certain normalized trace $\mathscr{M}$ on the set of totally real algebraic integers. In §3, we study the properties of the Perron–Frobenius eigenvalues of abelian spiders, and then use the estimates in §2 to prove a weak form of Theorem 1.1 (Corollary 3.7), namely, that any sufficiently large abelian spider (for fixed Γ and k) must have a Perron–Frobenius eigenvalue λ satisfying $\mathscr{M}(\lambda^2 - 2) < 14/5$. In §4, we classify totally real cyclotomic integers β with $\mathscr{M}(\beta) < 14/5$, following the arguments of [CMS11, Cas69]. We use these results of the previous two sections in §5 to complete the proof of Theorem 1.1. In §6 and §7, we give two detailed examples where the exceptional set of Theorem 1.1 is determined explicitly, which in particular allows us to prove Theorem 1.3. Finally, in §8, we give a different application of the bounds in §2 to Salem numbers.

2. Estimates for real cyclotomic integers

The first technical ingredient is the following inequality below, which is similar to (but quite a bit more complicated than) Lemma 4.2.3 of [CMS11]. In fact, it turns out that inequalities of a similar shape were first considered by Smyth in 1981 [Smy81, Smy84], where one intended application was the generalizations of Siegel's theorem on lower bounds for the trace of totally positive integers. The creation of such inequalities seems to be part science and part art. Let $\mathrm{Ch}_N(x)$ denote the minimal polynomial of $(\zeta_N + \zeta_N^{-1})^2$. The table below contains explicit expressions for the $\mathrm{Ch}_N(x)$ together with the value of $\mathscr{M}(\zeta_N + \zeta_N^{-1})$, where $\mathscr{M}(\beta) := \frac{\mathrm{Tr}_{K/\mathbf{Q}}(\beta^2)}{[K:\mathbf{Q}]}$ is the normalized trace of β^2. The coefficient a_N is used below in the definition of $B(x)$. The optimization of the coefficients a_N in the definition of $B(x)$ was performed by simulated annealing.

If N is not on this list, set $a_N = 0$. This list of polynomials includes every N where the inequality $\mathscr{M}(\zeta_N + \zeta_N^{-1}) > 13/6$ is satisfied, as well as a complete list of all such polynomials for $N < 11$.

N	$\mathrm{Ch}_N(x)$	$\mathscr{M}(\zeta_N + \zeta_N^{-1})$	a_N
1	$x - 4$	4	673
3	$x - 1$	1	6
4	x	0	4
5	$x^2 - 3x + 1$	3/2	2
7	$x^3 - 5x^2 + 6x - 1$	5/3	5
8	$x - 2$	2	157
9	$x^3 - 6x^2 + 9x - 1$	2	13
12	$x - 3$	3	578
15	$x^4 - 9x^3 + 26x^2 - 24x + 1$	9/4	43
16	$x^2 - 4x + 2$	2	49
20	$x^2 - 5x + 5$	5/2	215
21	$x^6 - 13x^5 + 64x^4 - 146x^3 + 148x^2 - 48x + 1$	13/6	10
24	$x^2 - 4x + 1$	2	25
28	$x^3 - 7x^2 + 14x - 7$	7/3	80
44	$x^5 - 11x^4 + 44x^3 - 77x^2 + 55x - 11$	11/5	24
52	$x^6 - 13x^5 + 65x^4 - 156x^3 + 182x^2 - 91x + 13$	13/6	1

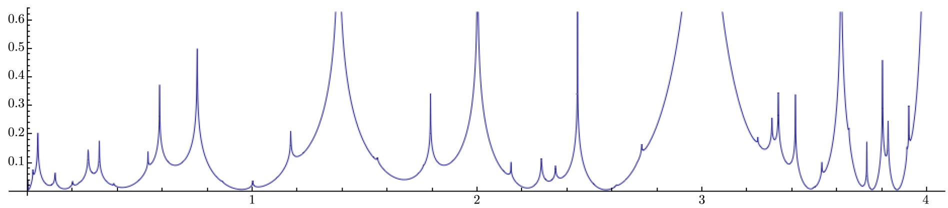


FIGURE 1. The graph of $B(x)$ in $[0, 4]$.

Definition 2.1. Define the function $B(x)$ as follows:

$$B(x) = \frac{9}{4} - x - \frac{1}{1000} \sum a_N \log |\mathrm{Ch}_N(x)|.$$

The key property of $B(x)$ is the following estimate:

Lemma 2.2. *For $x \in [0, 4]$ where $B(x)$ is defined, $B(x) > 0$. For $x > 4$, $B(x)$ is decreasing.*

The derivative of $B(x)$ lies in $\mathbf{Q}(x)$. The minimum value of $B(x)$ in $[0, 4]$ occurs at an algebraic number $\alpha \sim 0.00209304$ of degree 40, with $B(\alpha) \sim 0.00599001$. For $x > 4$ (where there are no singularities), $B(x)$ is decreasing. One has the estimate $\lim_{x\to\infty} B(x)/x = -1$.

Recall that the **house** $\overline{|\beta|}$ of β is the largest absolute value of all conjugates of β. Note that if $\overline{|\beta|} < L$, then $\overline{|\beta|}^2 < L^2$.

Theorem 2.3. *Let L be a non-negative real number, and let β be a totally real algebraic integer with $K = \mathbf{Q}(\beta^2)$ such that:*

1. *(1) β^2 is not a singularity of $B(x)$.*
2. *(2) There is an inequality $\overline{|\beta|} < L$.*
3. *(3) At most M conjugates of β^2 lie outside the interval $[0, 4]$.*

Then,

$$\mathscr{M}(\beta) = \frac{\mathrm{Tr}_{K/\mathbf{Q}}(\beta^2)}{[K:\mathbf{Q}]} < \frac{14}{5}$$

if either $B(L^2) > 0$ *or* $D := [K:\mathbf{Q}] \geq \frac{20}{11} \cdot M \cdot |B(L^2)|$.

Proof. Consider the sum $\sum B(\sigma\beta^2)$. If $\sigma\beta^2$ is not a singularity of B, then the sum of each logarithmic term is a negative rational number times the logarithm of the norm of an algebraic integer, and is hence negative. If $D = [K : \mathbf{Q}]$, it follows that β^2 has D conjugates and

$$\sum B(\sigma\beta^2) \leq \frac{9}{4} \cdot D - D\mathscr{M}(\beta).$$

On the other hand, we have the estimate $B(x) > 0$ for $x \in [0,4]$, and that $B(x)$ is decreasing otherwise. Hence, accounting for the (at most) M conjugates of β^2 outside $[0,4]$, we have

$$\sum B(\sigma\beta^2) > \min(0, M \cdot B(L^2)).$$

(Note that $B(L^2) \leq 0$ for $L > 2.01106$.) If $B(L^2) \geq 0$, we deduce from these two estimates that $9/4 - \mathscr{M}(\beta) > 0$ which proves the required inequality with room to spare. If $B(L^2) < 0$, then combining the two estimates yields

$$\frac{9}{4} - \mathscr{M}(\beta) - \frac{M \cdot B(L^2)}{D} = \left(\frac{14}{5} - \mathscr{M}(\beta)\right) + \left(\frac{M \cdot |B(L^2)|}{D} - \frac{11}{20}\right) > 0.$$

If D satisfies the inequality in the statement of the theorem, then the second term is non-negative and hence the first term is positive, as desired. □

3. The spectrum of $\Gamma_{\underline{r}}$

We begin by recording some basic properties of eigenvalues of graphs. A reference for this section is [MS05]. The following lemma is essentially Lemma 12 of [MS05]:

Lemma 3.1. *If* $r_i \geq 2$ *for all* i, *then the characteristic polynomial* $P_{\underline{r}}(x)$ *of* $\Gamma_{\underline{r}}$ *has the form:*

$$\left(t - \frac{1}{t}\right)^k P_{\underline{r}}(x) = \sum_{\underline{\epsilon}} t^{\sum \epsilon_i r_i} F_{\underline{\epsilon}}(x),$$

where $x = t + t^{-1}$, *the index* $\underline{\epsilon}$ *runs over* k*-tuples* $(\epsilon_1, \ldots, \epsilon_k)$ *with* $\epsilon_i \in \{1, -1\}$, *and where the polynomials* $F_{\underline{\epsilon}} \in \mathbf{Z}[x]$ *do not depend on* $\underline{r}$.

Let $Q(x) = F_\epsilon(x)$ where $\epsilon = (1, 1, \ldots, 1)$. Let S denote the set of real roots of $Q(x)$ in $(2, \infty)$, counted with multiplicity. Say that a vector $\underline{r}$ is large if *all* the entries r_i are large.

Lemma 3.2. *We have the following:*

1. *If* $\underline{r}' > \underline{r}$ *in the partial ordering, then the Perron Frobenius eigenvalue* λ' *of* $\Gamma_{\underline{r}'}$ *is strictly larger than* λ.
2. $P_{\underline{r}}(x)$ *has* $|S|$ *real roots* > 2 *for sufficiently large* $\underline{r}$, *and they converge from below to* S.

Proof. The first claim follows from the interlacing theorem (this is Theorem 9.1.1 of [GR01], but see also Lemma 2 of [MS05]) . The second claim is proved in [MS05]. The main point is that any root $> 1+\varepsilon$ of $P_{\underline{r}}(t+t^{-1})$ will continue (by interlacing) to be $> 1+\varepsilon$ as $\underline{r}$ grows. Then, for sufficiently large $\underline{r}$, Rouché's theorem will show that the number of real roots $> 1+\varepsilon$ of $P_{\underline{r}}(t+t^{-1})$ will be equal to the number of real roots of $Q(t+t^{-1})$. □

We immediately deduce:

Lemma 3.3. *There exists constants $M = M_\Gamma$ and $L = L_\Gamma$ such that:*

(1) *If λ is the Perron–Frobenius eigenvalue of $\Gamma_{\underline{r}}$, then $\lambda^2 - 2 \leq L$.*
(2) *At most M of the conjugates of $\lambda^2 - 2$ lie outside the interval $[-2, 2]$.*

In practice, these constants are often small and computable (indeed, often M is equal to one, as it will be in our examples). We have, moreover, the following easy upper bound for λ:

Lemma 3.4. *Suppose that the largest valence of any vertex of $\Gamma_{\underline{r}}$ is v. Then $\lambda \leq v$.*

Combined with Theorem 2.3 above, we deduce:

Corollary 3.5. *For all sufficiently large $\underline{r}$, the largest eigenvalue λ of $\Gamma_{\underline{r}}$ satisfies $\mathscr{M}(\lambda^2 - 2) < 14/5$.*

Proof. Since $\lambda^2 - 2$ is strictly increasing as $\underline{r}$ increases, it suffices to show that the degree of λ^2 is not bounded. Yet all the conjugates of $\lambda^2 - 2$ are bounded by L, and there are only a finite number of algebraic integers of fixed degree with this property by a well-known argument of Kronecker [Kro57]. □

We shall prove in Proposition 4.3 that if $\mathbf{Q}(\lambda^2)$ is abelian, then $\mathscr{M}(\lambda^2-2) < 14/5$ implies either that $\lambda \leq 2$ or $\lambda^2 - 2$ is one of a finite set of algebraic integers. This is enough to prove that there are only finitely many abelian spiders which are not Dynkin diagrams for sufficiently large $\underline{r}$. On the other hand, if one of the r_i is bounded by a constant B, then we can proceed by induction and consider the $k-1$ spiders of the finitely many graphs where a 2-valent tree of length $r_i \leq B$ is attached to Γ at v_i. This leads to a proof of Theorem 1.1. The problem is that Kronecker's argument, although "explicit", is not really so explicit in practice (since it involves checking a super-exponential set of polynomials). Instead, we shall give a different argument which can be used in practice.

Proposition 3.6. *Suppose that each element of $\underline{r}$ is at least $n \geq 2$. There is a bound:*

$$D := [\mathbf{Q}(\lambda^2) : \mathbf{Q}] \gg n,$$

where the implied constant depends only on Γ and is explicitly computable.

Proof. We may assume that n is large (in practice, what counts as "large" is usually not prohibitive). Write $\lambda = \rho + \rho^{-1}$. Certainly $[\mathbf{Q}(\lambda^2) : \mathbf{Q}(\rho)] \leq 4$, so it suffices to give a linear lower bound on the degree of ρ. Let ρ_∞ denote the largest root of $Q(t + t^{-1})$. We know that the values ρ are converging to ρ_∞; the basic idea is to show that this convergence is exponentially fast, which, together with the fact that the conjugates of ρ are constrained in absolute value, is enough to give the requisite bound on the degree of ρ. Write $\rho_\infty - \rho = \varepsilon$. Since $P_{\underline{r}}(\rho + \rho^{-1}) = 0$, we deduce that:

$$0 = P_{\underline{r}}\left(\rho + \rho^{-1}\right) = \sum_{\underline{\epsilon}} \rho^{\sum \epsilon_i r_i} F_{\underline{\epsilon}}(\rho + \rho^{-1}).$$

Taking absolute values and applying the triangle inequality, we deduce that

$$|Q(\rho + \rho^{-1})| \ll \rho^{-2n},$$

where the constants can easily be made effective in any particular case (they involve the supremum of the polynomials $F_{\underline{\epsilon}}(t + t^{-1})$ for t in a neighbourhood of ρ_∞). On the other hand, suppose that the root ρ_∞ of $Q(x)$ has multiplicity exactly m. Then there is an inequality $|Q(\rho + \rho^{-1})| > A\varepsilon^m$ for some explicitly computable constant $A > 0$ depending on the mth derivative of Q at $\rho_\infty + \rho_\infty^{-1}$. Since ρ is converging to $\rho_\infty > 1$, it satisfies $\rho^{2/m} > \theta$ for some explicit $\theta > 1$ which does not depend on n. It follows that, where (as above) the implicit constants can easily be evaluated explicitly, we have the following inequality:

$$|\rho_\infty - \rho| \ll \frac{1}{\theta^n}.$$

Let $R(t)$ be the minimal polynomial of ρ_∞. The polynomial $R(t)$ does not vanish on any conjugate of ρ because $\rho_\infty > |\sigma\rho|$ for all conjugates of ρ and $R(t)$ is irreducible. The polynomial $R(t)$ is bounded on the ball $|t| \le \rho_\infty$ by some absolute constant C. Let $D = [K : \mathbf{Q}]$ with $K = \mathbf{Q}(\lambda^2)$, and let $L = \mathbf{Q}(\rho)$. Since $[L : K] \le 4$, the degree of $L/\mathbf{Q}$ is at most $4D$. Since $R(\rho) \ne 0$, we have

$$1 \le N_{L/\mathbf{Q}}(R(\rho)) \le C^{4D-1} \cdot |\rho - \rho_\infty| \le \frac{C^{4D-1}}{\theta^n}.$$

Taking logarithms and using the fact that $\theta > 1$ leads to a linear lower bound in D, as desired. $\square$

Combining this result with Theorem 2.3 above, we deduce:

Corollary 3.7. *There exists an effectively computable constant m such that for all $r_i \ge m$, either the largest eigenvalue λ of $\Gamma_{\underline{r}}$ satisfies $\mathscr{M}(\lambda^2 - 2) < 14/5$, or $\Gamma_{\underline{r}}$ is the Dynkin diagram A_n or D_n.*

Proof. The previous lemma shows that we may find an explicit m so that the degree of λ^2 is large. The result then follows from Theorem 2.3 and Lemma 3.3 once we have an effective bound on m so that $\lambda^2 - 2$ is not conjugate to a singularity of $B(x)$. Note, however, that all the singularities of $B(x)$ are algebraic integers all of whose conjugates lie in $[0, 4]$. If $\lambda^2 - 2$ is such an integer, then $|\lambda| \le 2$ and $\Gamma_{\underline{r}}$ is either a Dynkin diagram or a simply laced affine Dynkin diagram. It is easy to see that only A_n and D_n can occur for sufficiently large m. $\square$

4. Totally real cyclotomic integers with small $\mathscr{M}$

In this section, we shall improve on some estimates from [CMS11]. We make, however, the following preliminary remark. Modifying the proof of Theorem 2.3 slightly, we see that there exists a lower bound on D (depending on L and any $\epsilon > 0$) that guarantees the inequality $\mathscr{M}(\beta) \le 9/4 + \epsilon$. However, this can be improved further. In the proof of Lemma 2.2, one could replace $B(x)$ by $B(x) - \delta$ for small but non-zero $\delta > 0$. (As mentioned directly after the statement of Lemma 2.2, one could take δ to be anything less than the minimum of $B(x)$ on $[0, 4]$, which is approximately 0.00599001.) This would allow us to modify the proof of Theorem 2.3 to give an explicit lower bound on D (in terms only of $B(L)$) which would guarantee that $\mathscr{M}(\beta) < 9/4$. We could then dispense with Proposition 4.3 below entirely and use Lemma 9.0.1 of [CMS11], which classifies those β with $\mathscr{M}(\beta) < 9/4$. However, such an argument would lead to (significantly) worse bounds.

We shall freely use many of the concepts from Cassels' paper [Cas69] and also from [CMS11]. The conductor of a cyclotomic integer β is the smallest integer N such that $\mathbf{Q}(\beta) \subset \mathbf{Q}(\zeta_N)$. Recall that two algebraic cyclotomic integers are called **equivalent** if their ratio is a root of unity, and that a cyclotomic integer β is **minimal** if it has the smallest conductor amongst all its equivalent forms. If β is totally real, it is not always the case that a minimal equivalent cyclotomic integer is also totally real, but this is almost true:

Lemma 4.1. *If β is a minimal cyclotomic integer of odd conductor N which is equivalent to a cyclotomic integer, then, up to a root of unity in $\mathbf{Q}(\zeta_N)$, either β or $\beta \cdot \sqrt{-1}$ is totally real.*

Proof. Suppose that β is minimal of odd conductor N. Write $\gamma = \zeta\beta$, where γ is totally real. If $\zeta \in \mathbf{Q}(\zeta_{2N}) = \mathbf{Q}(\zeta_N)$, then the result is trivial. Hence there exists a prime p such that $2N$ is exactly divisible by p^m and the order of ζ is exactly divisible by p^n for some $n > m$. Let ξ denote a primitive p^n root of unity. There exists a Galois automorphism σ fixing $\mathbf{Q}(\zeta_N)$ and hence fixing β such that

$$\gamma/\sigma\gamma = \sigma\zeta/\zeta = \xi^{p^m}.$$

Since γ is totally real, the latter element must also be real, which forces $p = 2$ and $\xi^4 = 1$ (noting that $p^m = 2$ if $p = 2$, since N is odd). The result follows. $\square$

Remark 4.2. Let α be a cyclotomic integer. Let $\mathscr{N}(\alpha)$ denote the minimum number of roots of unity required to express α. If $\alpha \in K = \mathbf{Q}(\zeta_N)$, let $\mathscr{N}_K(\alpha)$ denote the minimum number of roots of unity in K required to express α. We recall the following facts from [Cas69, CMS11] for cyclotomic integers α:

(1) If $\mathscr{N}(\alpha) \geq 3$, then $\mathscr{M}(\alpha) \geq 2$ ([Cas69], Lemma 3).
(2) If $\mathscr{N}(\alpha) > 1$, then $\mathscr{M}(\alpha) \geq 3/2$ ([Cas69], Lemma 2).
(3) If $\mathscr{N}(\alpha) > 1$, and α is not a root of unity times a conjugate of $1 + \zeta_5$, then $\mathscr{M}(\alpha) \geq 5/3$.

To prove the third claim, it follows in light of the previous two claims that we may assume $\mathscr{N}(\alpha) = 2$, so that α is a root of unity times $1 + \zeta_n$ for some n. The result then follows from Remark 9.0.2 of [CMS11].

Proposition 4.3. *Suppose that $\mathscr{M}(\beta) < 14/5$ and β is a totally real cyclotomic integer. Suppose, moreover, that β is not the sum of at most two roots of unity. Then $\boxed{\beta}$ is one of the following numbers:*

Proof. We may assume that $\mathscr{N}(\beta) \geq 3$. Consider the case $\mathscr{N}(\beta) = 3$. By Theorem 4.0.3 of [CMS11], we may assume that, up to conjugation and sign, either $\beta = 1 + \zeta + \zeta^{-i}$ for some root of unity ζ, or $\beta = \zeta_{12} + \zeta_{20} + \zeta_{20}^{17}$. The latter element is included on the list, the former elements satisfy $\mathscr{M}(\beta) \leq 14/5$ if and only if they are included in the statement of the theorem. Hence we may assume that $\mathscr{N}(\beta) \geq 4$.

$\boxed{\beta}$	$\mathscr{M}(\beta)$	$[\mathbf{Q}(\beta):\mathbf{Q}]$
$\dfrac{\sqrt{3}+\sqrt{7}}{2} = 2.188901\ldots$	$5/2$	4
$1+2\cos(2\pi/7) = 2.246979\ldots$	2	3
$\zeta_{12}+\zeta_{20}+\zeta_{20}^{17} = 2.404867\ldots$	2	8
$2\cos(11\pi/42)+2\cos(13\pi/42) = 2.486985\ldots$	$8/3$	12
$1+2\cos(2\pi/11) = 2.682507\ldots$	$12/5$	5
$1+2\cos(2\pi/13) = 2.770912\ldots$	$5/2$	6
$1+2\cos(2\pi/17) = 2.864944\ldots$	$21/8$	8
$1+2\cos(2\pi/19) = 2.891634\ldots$	$8/3$	9
$2\cos(2\pi/35)+2\cos(12\pi/35) = 4\cos(\pi/7)\cos(\pi/5) = 2.915596\ldots$	$5/2$	6
$1+2\cos(2\pi/23) = 2.925834\ldots$	$30/11$	11
$1+2\cos(2\pi/29) = 2.953241\ldots$	$39/14$	14
$1+2\cos(2\pi/30) = 2.956295\ldots$	$11/4$	4
$1+2\cos(2\pi/60) = 2.989043\ldots$	$11/4$	8
$\zeta_{84}^{-9}+\zeta_{84}^{-7}+\zeta_{84}^{3}+\zeta_{84}^{15} = 3.056668\ldots$	$5/2$	12
$2\cos(6\pi/55)+2\cos(16\pi/55) = 4\cos(\pi/11)\cos(\pi/5) = 3.104984\ldots$	$27/10$	10
$2\cos(8\pi/65)+2\cos(18\pi/65) = 4\cos(\pi/13)\cos(\pi/5) = 3.142033\ldots$	$11/4$	12
$2\cos(11\pi/70)+2\cos(17\pi/70) = 3.206780\ldots$	$8/3$	24
$2\cos(37\pi/210)+2\cos(47\pi/210) = 3.227019\ldots$	$11/4$	24
$2\cos(\pi/42)+2\cos(11\pi/42) = 3.354753\ldots$	$8/3$	12

Let us now weaken the assumption on β to assume merely that it is equivalent to a totally real cyclotomic integer, and that $\mathscr{N}(\beta) \ge 4$. This allows us to also assume that β is minimal, that is, it lives in $\mathbf{Q}(\zeta_N)$ where N is the conductor of β, and no multiple of β by a root of unity lives in a smaller cyclotomic field. Recall (following [Cas69, CMS11]) that we can write

$$\beta = \sum_S \alpha_i \zeta^i,$$

where $p^k \| N$, where ζ is a primitive p^kth root of unity, where $\alpha_i \in \mathbf{Q}(\zeta_M)$, where $pM = N$, and where S is a subset of $\{0,1,\ldots,p-1\}$ whose order we denote by X. Note that when $p\|N$, this expression is unique only up to translating each α_i by the same constant.

Assume that $p^2 \mid N$ for some p. Then $\mathscr{M}(\beta) = \sum \mathscr{M}(\alpha_i)$ ([CMS11], Lemma 5.2.1). If $|S| = X \ge 3$, then $\mathscr{M}(\beta) \ge 3$. If $X = 1$, then $\beta = \alpha\zeta$, and we could divide by ζ, contradicting the minimality of β. If $X = 2$, then $\mathscr{M}(\beta) = \mathscr{M}(\alpha_1) + \mathscr{M}(\alpha_2)$. The assumption $\mathscr{N}(\alpha_1) + \mathscr{N}(\alpha_2) > 3$ implies that $\mathscr{M}(\beta) \ge 3/2 + 3/2 = 3$ or $\mathscr{M}(\beta) \ge 1 + 2 = 3$. This also contradicts our assumptions, and so N is squarefree. Recall that this implies the equality (Eq. 3.9 of [Cas69]):

$$\mathscr{M}(\beta) = (p-X)\sum \mathscr{M}(\alpha_i) + \sum \mathscr{M}(\alpha_i - \alpha_j),$$

where we assume that exactly X of the α_i are non-zero.

Suppose that $p \mid N$ for some $p > 7$. Since $\mathscr{M}(\beta) < 7/2 \le (p+3)/4$, then by Lemma 1 of [Cas69] (as used in [CMS11]), we may assume that there are exactly $X \le (p-1)/2$ non-zero terms α_i in the expansion of β above. If $X \ge 4$, then we deduce that

$$(p-1)\mathscr{M}(\beta) \ge (p-X)X \ge 4(p-4).$$

This implies (for $p > 7$) that $\mathscr{M}(\beta) \ge 14/5$. Suppose that $X = 3$. If α_i is a root of unity for each i, then $\mathscr{N}(\beta) \le 3$, a contradiction. Hence at least one α_i is not a root of unity. If all the α_i are not roots of unity, then $(p-1)\mathscr{M}(\beta) \ge (p-3)(3/2)$

which directly leads to a contradiction. Otherwise, there must be at least two pairs which are non-zero, and so

$$(p-1)\mathscr{M}(\beta) \geq (p-3)(1+1+3/2)+2,$$

from which $\mathscr{M}(\beta) \geq 3$. Hence we may assume that $X = 2$, and in particular that

$$\beta = \alpha + \zeta\gamma,$$

where ζ is a primitive pth root of unity, α and γ are cyclotomic integers in $\mathbf{Q}(\zeta_M)$ for M dividing N and prime to p. Since $\mathscr{N}(\beta) > 3$, either α is a root of unity and $\mathscr{N}(\gamma) \geq 3$, or α and γ are both not roots of unity. In the first case, $\mathscr{N}(\gamma - \alpha) \geq 2$ so $\mathscr{M}(\gamma - \alpha) \geq 2$. Hence

$$(p-1)\mathscr{M}(\beta) \geq (p-2)(1+2)+2,$$

and so $\mathscr{M}(\beta) \geq 29/10$. In the second case, if $\alpha \neq \gamma$, then

$$(p-1)\mathscr{M}(\beta) \geq (p-2)(3/2+3/2)+1,$$

and $\mathscr{M}(\beta) \geq 14/5$. If $\alpha = \gamma$ and $\mathscr{M}(\alpha) \geq 5/3$, then $\mathscr{M}(\beta) \geq 3$. So, after conjugation, we must have:

$$\beta = (1+\zeta)(1+\zeta_5).$$

In this case, we have $\overline{|\beta|} = \overline{|1+\zeta|} \cdot \overline{|1+\zeta_5|}$. Note that $\overline{|1+\zeta|} = 2\cos(\pi/p)$. If $p > 13$, then we have $\mathscr{M}(\beta) \geq 45/32$, so this leaves only $p = 11$ and $p = 13$, and these cases are covered in the statement of the theorem. This portion of the argument is the one which most strongly requires the bound $\mathscr{M}(\beta) < 14/5$ rather than $\mathscr{M}(\beta) < 3$. In particular, all the integers $4\cos(\pi/p)\cos(\pi/5)$ for a prime $p > 5$ will satisfy this bound.

Lemma 4.4. *If $\beta \in K = \mathbf{Q}(\zeta_{105})$ is a sum of 4 or 5 roots of unity in K, and β is equivalent to a totally real cyclotomic integer, then either $\overline{|\beta|}$ is one of the exceptions listed in the statement of the theorem, or $\mathscr{M}(\beta) \geq 14/5$.*

Proof. One proceeds by enumeration, after noting by Lemma 4.1 that $\beta \in K$ is equivalent to a totally real integer if and only if β times some 420th root of unity is real. □

We let $p = 5$, and write $\beta = \sum \alpha_i \zeta^i$, where $\zeta^5 = 1$. We have the following by Lemmas 7.0.1 and 7.0.3 of [CMS11]:

Lemma 4.5. *If $\alpha \in L = \mathbf{Q}(\zeta_{21})$,*

1. *if $\mathscr{N}_L(\alpha) \geq 2$, then $\mathscr{M}(\alpha) \geq 5/3$,*
2. *if $\mathscr{N}_L(\alpha) \geq 3$, then $\mathscr{M}(\alpha) \geq 2$,*
3. *if $\mathscr{N}_L(\alpha) \geq 4$, then $\mathscr{M}(\alpha) \geq 5/2$,*
4. *if $\mathscr{N}_L(\alpha) \geq 5$, then $\mathscr{M}(\alpha) \geq 23/6$.*

Since we are not assuming that N is divisible by 5, we have to allow the possibility that $X = 1$.

We consider various cases:

1. If $X = 1$, then $\beta \in \mathbf{Q}(\zeta_{21})$. By Lemma 4.5, we may assume that $\mathscr{N}_L(\beta) > 5$. Hence $\mathscr{M}(\alpha) \geq 23/6$, which is a contradiction.

(2) If $X = 2$, then we may write $\beta = \alpha + \gamma\zeta$ with $\alpha, \gamma \in \mathbf{Q}(\zeta_{21})$, and we have the equality:

$$4\mathscr{M}(\beta) = 3\mathscr{M}(\alpha) + 3\mathscr{M}(\gamma) + \mathscr{M}(\alpha - \gamma).$$

Since $\mathscr{N}_K(\beta) > 5$, we may assume that either $\mathscr{N}_L(\alpha), \mathscr{N}_L(\gamma) \geq 3$, or $\mathscr{N}_L(\alpha) = 2$ and $\mathscr{N}_L(\gamma) \geq 4$, or $\mathscr{N}_L(\alpha) = 1$ and $\mathscr{N}_L(\gamma) \geq 5$. Using Lemma 4.5, and the fact that $\mathscr{N}_L(\alpha - \gamma) \geq \mathscr{N}_L(\gamma) - \mathscr{N}_L(\alpha)$, we find in each case that:

$$\mathscr{M}(\beta) \geq \frac{1}{4}(3 \cdot 2 + 3 \cdot 2) = 3,$$

$$\mathscr{M}(\beta) \geq \frac{1}{4}(3 \cdot 5/3 + 3 \cdot 5/2 + 5/3) = 85/24,$$

$$\mathscr{M}(\beta) \geq \frac{1}{4}(3 + 3 \cdot 5/2 + 5/2) = 13/4,$$

which all yield contradictions.

(3) If $X = 3$, then, as in the proof of the similar step in Lemma 9.0.1 of [CMS11], not all the α_i can be the same (since otherwise we could reduce to the case $X = 2$), and hence at least two of the $\alpha_i - \alpha_j$ are non-zero. More generally, we have

$$4\mathscr{M}(\beta) \geq 2\sum \mathscr{M}(\alpha_i) + \sum \mathscr{M}(\alpha_i - \alpha_j).$$

The values $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 2)$, $(1, 1, 3)$, are ruled out as values of $\{\mathscr{N}_L(\alpha_i)\}$ by Lemma 4.4. This leaves the possibilities:

$$(1, 1, > 3), (1, 2, > 2), (1, > 2, > 2), (> 1, > 1, > 1).$$

Considering each in turn and using Lemma 4.5, along with the fact that not all the α_i are equal in the final case, we have the four estimates:

$$\mathscr{M}(\beta) \geq \frac{1}{4}(2 \cdot 1 + 2 \cdot 1 + 2 \cdot 5/2 + 2 + 2) = 13/4,$$

$$\mathscr{M}(\beta) \geq \frac{1}{4}(2 \cdot 1 + 2 \cdot 5/3 + 2 \cdot 2 + 1 + 1 + 5/3) = 13/4,$$

$$\mathscr{M}(\beta) \geq \frac{1}{4}(2 \cdot 1 + 2 \cdot 2 + 2 \cdot 2 + 5/3 + 5/3) = 10/3,$$

$$\mathscr{M}(\beta) \geq \frac{1}{4}(2 \cdot 5/3 + 2 \cdot 5/3 + 2 \cdot 5/3 + 1 + 1) = 3,$$

which all lead to a contradiction.

(4) If $X = 4$ or $X = 5$, we may reduce to $X \leq 3$ exactly as in the proof of Lemma 9.0.1 of [CMS11].

□

5. Proof of Theorem 1.1

Proof of Theorem 1.1. Recall that the Perron–Frobenius eigenvalue λ of a graph Γ is < 2 (respectively, ≤ 2) if and only if Γ is a Dynkin diagram (respectively, simply laced affine Dynkin diagram). For topological reasons, only finitely many of the spiders $\Gamma_{\underline{r}}$ are affine Dynkin diagrams.

Assume that infinitely many of the $\Gamma_{\underline{r}}$ are abelian. We proceed by induction on k, the result for $k = 0$ being trivial. If there exist infinitely many such graphs with $r_1 \leq M$, then we may reduce the problem to $k - 1$ replacing Γ by the finitely many 1-spiders on Γ with a 2-valent tree of length $\leq M$ attached to Γ at v_1. Hence we may assume that all the r_i are tending to infinity. If the limit of the λ as $\underline{r}$ increases is ≤ 2, then all the $\Gamma_{\underline{r}}$ are Dynkin diagrams. Hence we may assume that the limit of the largest eigenvalue λ is > 2. By Proosition 3.6, we obtain a lower bound on $[\mathbf{Q}(\lambda)^2 : \mathbf{Q}]$ which allows us (for sufficiently large n) to deduce as in Corollary 3.7 that $\mathscr{M}(\lambda^2 - 2) < 14/5$. Since $\lambda > 2$, it is not the sum of two roots of unity. It follows that $\lambda > 2$ must be one of the finitely many exceptional numbers occurring in Proposition 4.3. Yet these numbers have (explicitly) bounded degree, and so using the lower bounds on $[\mathbf{Q}(\lambda^2) : \mathbf{Q}]$ in the proof of Proposition 3.6, these eigenvalues can occur as $\lambda^2 - 2$ for only finitely many $\Gamma_{\underline{r}}$. Hence we may explicitly compute such an n such that $\Gamma_{\underline{r}}$ is not abelian when each $r_i \geq n$. $\square$

6. Example I: The AMP spider

We shall consider two examples. Let $\Gamma_{a,b}$ be the AMP spider given in the introduction. Let $P_{a,b}(x)$ denote the characteristic polynomial of $\Gamma_{a,b}$. We find:

Lemma 6.1. *We have*

$$\left(t - \frac{1}{t}\right)^2 P_{a,b}\left(t + \frac{1}{t}\right) = F_{a,b}(t) + F_{a,b}(1/t),$$

where

$$F_{a,b}(t) = t^{a+b}(t^{-2} + 2 + 2t^2 - 2t^4 - 2t^6 - 2t^8 + t^{10}) + t^{a-b}(t^{-6} - 2 + t^6).$$

Let $\rho_\infty = 1.6826\ldots$ be the largest real root of $t^6 - 2t^4 - 2t^2 - 1 = 0$, which is also a root of

$$t^{-2} + 2 + 2t^2 - 2t^4 - 2t^6 - 2t^8 + t^{10} = 0$$

(the other roots of this polynomial are cyclotomic). Let $\gamma = \left(\rho_\infty + \rho_\infty^{-1}\right)^2 = 5.18438\ldots$ denote the largest real root of

$$x^3 - 6x^2 + 5x - 4 = 0.$$

The following is the specialization of Lemma 3.2:

Lemma 6.2. *The polynomial $P_{a,b}(x)$ has a unique pair of roots $(\lambda, -\lambda)$ of absolute value > 2. As a and b strictly increase, the value of λ strictly increases. The limit of λ^2 as $a, b \to \infty$ is γ.*

We now find an explicit exponential bound relating λ to γ.

Lemma 6.3. *Let $\rho \in [3/2, \rho_\infty)$ denote the largest root of $P_{a,b}(t + t^{-1})$, and assume $a, b \geq n \geq 10$. Then*

$$|\rho - \rho_\infty| < \frac{1}{6}(1.682)^{-2n}.$$

Remark 6.4. When we write a real number as a finite decimal, we refer to an exact element of $\mathbf{Z}[1/10]$. Although the inequalities below are quite tight, they still hold by some comfortable margin of error. Certain numbers are chosen to make various ratios integral, purely for presentational purposes.

Proof. Write $\rho = \rho_\infty - \varepsilon$. For $a, b \geq n \geq 10$, we have the estimate $\rho \in [1.682, \rho_\infty)$. In this range, the following inequalities hold:

$$|\rho^{-2} + 2 + 2\rho^2 - 2\rho^4 - 2\rho^6 - 2\rho^8 + \rho^{10}| > 270 \cdot \varepsilon,$$
$$|\rho^2 + 2 + 2\rho^{-2} - 2\rho^{-4} - 2\rho^{-6} - 2\rho^{-8} + \rho^{-10}| < 6,$$
$$|\rho^6 - 2 + \rho^{-6}| < 21.$$

The first inequality is obtained by looking at the derivative of this rational function in the interval $[1.682, \rho_\infty]$; the other inequalities are easy. Using the equality $P_{a,b}(\rho+\rho^{-1}) = 0$ together with the triangle inequality, we find that

$$\begin{aligned}
270 \cdot \varepsilon \cdot \rho^{2n} &\leq |\rho^{-2} + 2 + 2\rho^2 - 2\rho^4 - 2\rho^6 - 2\rho^8 + \rho^{10}| \cdot \rho^{2n} \\
&\leq |\rho^6 - 2 + \rho^{-6}| + |\rho^2 + 2 + 2\rho^{-2} - 2\rho^{-4} - 2\rho^{-6} - 2\rho^{-8} + \rho^{-10}| \cdot \rho^{-2n} \\
&\leq 42 + 6 \cdot \rho^{-2n} \\
&\leq 45.
\end{aligned}$$

The result follows. □

Lemma 6.5. *If $a, b \geq n \geq 10$, and λ is the Perron-Frobenius eigenvalue of $\Gamma_{a,b}$, then*

$$\begin{aligned}
|\lambda^6 - 6\lambda^4 + 5\lambda^2 - 4| \ &\cdot |\lambda^2|^{29/1000} \cdot |\lambda^2 - 2|^{14/100} \cdot |\lambda^2 - 3|^{471/1000} \\
&\cdot |\lambda^2 - 4|^{362/1000} \cdot |\lambda^6 - 6\lambda^4 + 9\lambda^2 - 1|^{8/625}
\end{aligned}$$

is bounded above by $23 \cdot (1.682)^{-2n}$.

Proof. The function is decreasing on the interval $[1.618, \rho_\infty]$. Hence, by interlacing, it suffices to consider the case $a = b = n$. The result is then an elementary calculus exercise from Lemma 6.3. The main point is that if one replaces λ in the above expression by $t + 1/t$, the resulting expression has derivative $< 138 = 6 \times 23$ in $[1.618, \rho_\infty]$ (for comparison, the exact value at ρ_∞ is approximately ~ 136.12). □

We can now give a lower bound on the degree of λ^2, following the argument of Proposition 3.6.

Proposition 6.6. *Suppose that $a, b \geq n \geq 10$. Then*

$$D = [\mathbf{Q}(\lambda^2) : \mathbf{Q}] > \frac{11}{25} \cdot n - \frac{1}{3}.$$

Proof. All proper conjugates $\sigma\lambda^2 \neq \lambda^2$ satisfy $0 < \sigma\lambda^2 < 4$. Hence (by calculus)

$$\begin{aligned}
|\lambda^6 - 6\lambda^4 + 5\lambda^2 - 4| \ &\cdot |\lambda^2|^{29/1000} \cdot |\lambda^2 - 2|^{14/100} \cdot |\lambda^2 - 3|^{471/1000} \\
&\cdot |\lambda^2 - 4|^{362/1000} \cdot |\lambda^6 - 6\lambda^4 + 9\lambda^2 - 1|^{8/625}
\end{aligned}$$

is bounded above in this interval by 10.56. (In contrast to the proof of Proposition 3.6, we include here some extra factors of $\lambda^2 - m$ for small m to mollify the first factor as much as possible.) On the other hand, since $\lambda \neq \rho_\infty$ is an algebraic integer, if $K = \mathbf{Q}(\lambda^2)$, the product of the expression above over all conjugates of λ (assuming it is non-zero) is a product of positive rational powers of norms, and is thus ≥ 1. Using the inequality above for $\sigma\lambda^2 \neq \lambda^2$ and Lemma 6.5 for $\sigma\lambda^2 = \lambda^2$, it follows that

$$1 < 23 \cdot (1.682)^{-2n} \cdot (10.56)^{D-1}.$$

If the degree D is less than the value in the theorem, the RHS is less than one. □

We deduce:

Proposition 6.7. *Suppose that $a, b \geq 56$. Then $\Gamma_{a,b}$ is not abelian.*

Proof. In the context of Theorem 2.3, with $\beta = \lambda^2 - 2$ we have $M = 1$ and $L = \gamma - 2$, where $B((\gamma-2)^2) \sim -13.1241\ldots$. This yields the upper bound $\mathscr{M}(\lambda^2 - 2) < 14/5$ as soon as $D \geq 20|B|/11$, or when $D \geq 24$. By Proposition 6.6, we have $D > 24$ as soon as $n \geq 56$. Hence, in this range, $\lambda^2 - 2$ must be one of the exceptions listed in Proposition 4.3. On the other hand, for n in this range, we also have the estimate $3.17438\ldots < \gamma - 2 - 1/100 < \lambda^2 - 2 < \gamma - 2 = 3.18438\ldots$, which certainly rules out all such exceptions. $\square$

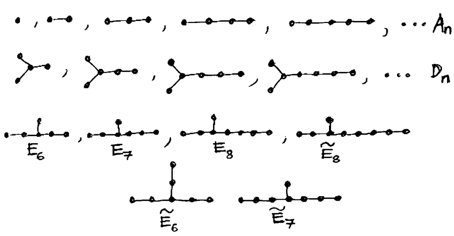
To complete the proof of Theorem 1.3, it suffices to consider the case when $a \leq 56$ or $b \leq 56$ (since the polynomial $P_{a,b}(x)$ is symmetric in a and b, we may assume the former). However, we can now apply the algorithm of [CMS11] to rule out the remaining cases (we thank Scott Morrison for carrying out this computation). We could also rule out the cases using the methods in this paper; however, we omit the details for reasons of space, and because we include the relevant details in the case of 3-spiders below.

7. Example II: 3-spiders

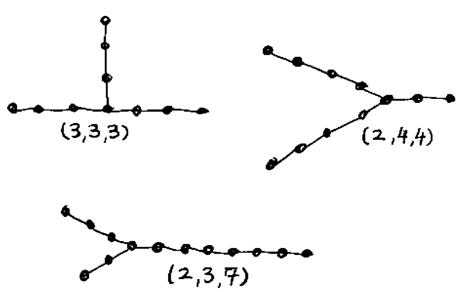
We consider the case when $k = 3$ and Γ is a single point. Let the resulting 3-spider be denoted $\Gamma_{a,b,c}$.

Theorem 7.1. *The complete set of abelian 3-spiders is as follows:*

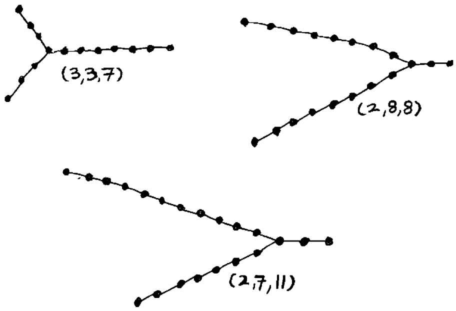
(1) *Those that are Dynkin diagrams, equivalently, those with $\lambda^2 \leq 4$:*



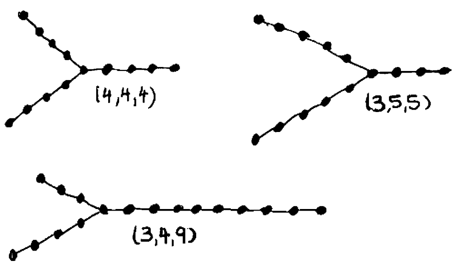
(2) *Exactly three spiders with $\lambda^2 = \frac{5+\sqrt{13}}{2} = 4.302775\ldots$*



(3) *Exactly three spiders with* $\lambda^2 = \zeta^{11} + \zeta^{10} + \zeta^3 + \zeta^2 + 2 = 4.377202\ldots$, *where* $\zeta = \exp(2\pi i/13)$*:*



(4) *Exactly three spiders with* $\lambda^2 = 3 + \sqrt{2} = 4.414213\ldots$, *namely:*



Remark 7.2. The first two non-Dynkin diagrams include graphs which correspond to the Haagerup and extended Haagerup subfactors, respectively (namely, the $(3,3,3)$ and $(3,3,7)$ spiders). However, none of the final class of graphs correspond to any subfactors, because the index is < 5 and does not occur as an index of any possible subfactor in the classification [JMS14].

By symmetry, we may assume that $a \le b \le c$. Let $P_{a,b,c}(x)$ denote the characteristic polynomial of $\Gamma_{a,b,c}$. Using Lemma 11 of [MS05], one easily establishes the following equality:

Lemma 7.3. *For* $x = t + t^{-1}$, *there is an equality*

$$\begin{aligned} P_{a,b,c}(x)\left(t - t^{-1}\right)^3 (-1)^{a+b+c-1} = {} & t^{a+b+c+4} - 2t^{a+b+c+2} + t^{a+b-c} + t^{a+c-b} + t^{b+c-a} \\ & - t^{a-b-c} - t^{b-a-c} - t^{c-a-b} + 2t^{-a-b-c-2} \\ & - t^{-a-b-c-4}. \end{aligned}$$

It is easy to identify the triples (a,b,c) such that $\Gamma_{a,b,c}$ is a Dynkin diagram, so we assume that the Perron–Frobenius eigenvalue λ of Γ is always strictly larger than 2. From Lemma 3.2, we deduce that the polynomial $P_{a,b,c}(x)$ has a unique pair of roots $(\lambda, -\lambda)$ of absolute value > 2, and that the limit as $a,b,c \to \infty$ of λ is $3/\sqrt{2}$.

Proposition 7.4. *Let* $D = [\mathbf{Q}(\lambda^2) : \mathbf{Q}]$. *Then either* $\mathscr{M}(\lambda^2 - 2) < 14/5$ *or* $D \le 12$.

Proof. It suffices to note that (taking $\beta = \lambda^2 - 2$) that at most one conjugate of β lies outside $[-2, 2]$, and so we deduce the inequality on $\mathscr{M}(\lambda^2 - 2)$ from Theorem 2.3 providing

$$D \geq 2 \cdot \left| B\left(\frac{25}{4}\right) \right| = 12.904524\ldots$$

□

Let us now make the running assumption that $\mathscr{M}(\lambda^2 - 2) \geq 14/5$; we shall deal with the alternative below. It follows that we may assume that $D = [\mathbf{Q}(\lambda^2) : \mathbf{Q}]$ is at most 12.

Lemma 7.5. *With $a \leq b \leq c$, we have $a \leq 30$.*

Proof. Since it is useful to have as tight a bound as possible, instead of using the trivial bound $|2x - 9| \leq 9$ on $[0, 4]$ we note that

$$|2x - 9| \cdot |x|^{52/100} \cdot |x - 1|^{337/1000} \cdot |x - 2|^{3/10} \cdot |x - 3|^{13/100} < 5.58$$

for $x \in [0, 4]$, a fact which is tedious but elementary to prove by calculus. By giving a lower estimate for the derivative of this function in a neighbourhood of $9/2$, we also find that

$$|2\lambda^2 - 9| \cdot |\lambda^2|^{52/100} \cdot |\lambda^2 - 1|^{337/1000} \cdot |\lambda^2 - 2|^{3/10} \cdot |\lambda^2 - 3|^{13/100} < 4.63 \cdot |(2\lambda^2 - 9)|$$

for all λ. Taken together, we deduce that

$$1 \leq (5.58)^{D-1} \cdot 4.63 \cdot |(2\lambda^2 - 9)|,$$

and hence, since $D \leq 12$,

$$\left| \lambda^2 - \frac{9}{2} \right| > 6.6132\ldots \times 10^{-10}.$$

This inequality is violated as soon as $a > 30$. □

7.1. **Fixed a, and varying b and c.** In this section, we effectively consider the 2-spiders on the Dynkin diagram $\Gamma = A_n$ with $n = a + 1$ and $v_1 = v_2$ a terminal point of Γ. Hence, for this section, the values of ρ_∞ reflects the appropriate root of the new polynomial $Q(t + t^{-1})$ in this setting. Note, however, that we still know that λ^2 has a unique conjugate outside the range $[0, 4]$.

Suppose that a is fixed, and let c and b with $c \geq b \geq a$ vary without bound. If one writes $\lambda^2 - 2 = \rho^2 + \rho^{-2}$, then ρ^2 is a Salem number, that is, all the conjugates of ρ^2 besides ρ^{-2} have modulus one. Since we are assuming $D = [\mathbf{Q}(\lambda^2) : \mathbf{Q}] \leq 12$, we also have the inequality $[\mathbf{Q}(\rho^2) : \mathbf{Q}] \leq 24$. As b and c tend to infinity, ρ tends towards the (unique) largest root ρ_∞ of the polynomial $1 - 2t^{2a+2} + t^{2a+4}$, which is the polynomial $Q(t + t^{-1})$ (up to powers of $t^{\pm 1}$) of §3.

Lemma 7.6. *We have an inequality:*

$$|1 - 2\rho^{2a+2} + \rho^{2a+4}| > \frac{1}{4^{23}}.$$

Proof. Since $\rho_\infty > \rho$ is the only real root of this polynomial greater than one, it follows that neither ρ nor any of its conjugates is a root of this polynomial. For any non-trivial conjugate of ρ^2, we have the easy estimate $|1 - 2\sigma\rho^{2a+2} + \sigma\rho^{2a+4}| \leq 4$, with a strict inequality for the real root. Hence the result follows from the fact that the norm of $1 - 2\rho^{2a+2} + \rho^{2a+4}$ from $\mathbf{Q}(\rho^2)$ to $\mathbf{Q}$ has absolute value at least one, and that the degree of ρ^2 is at most 24. □

By interlacing, the root ρ increases with b and c. Hence, by checking for suitable choices of b and c, we immediately deduce:

Lemma 7.7. *For each a, we have the following upper bound on $b = \min(b, c)$:*

a	$\min(b,c)$	a	$\min(b,c)$	a	$\min(b,c)$
1	67	11	59	21	69
2	55	12	60	22	70
3	53	13	61	23	71
4	53	14	62	24	72
5	53	15	63	25	73
6	54	16	64	26	74
7	55	17	65	27	75
8	56	18	66	28	76
9	57	19	67	29	77
10	58	20	68	30	78

7.2. **Fixed a and b, and varying c.** We have reduced to a finite number of pairs (a, b), and we could finish with an appeal to [CMS11]. Instead, however, we give a a treatment similar to the case when a is fixed and b and c are varying. As in the previous section, we assume $c \geq b \geq a$, and redefine the polynomials $Q(t+t^{-1})$ and ρ_∞ (for each (a, b)) to be the corresponding values for these 1-spiders.

Lemma 7.8. *We have an inequality:*

$$|\rho^{2a+2b+4} - 2\rho^{2a+2b+2} + \rho^{2b} + \rho^{2a} - 1| > \frac{1}{6^{23}}.$$

Proof. The proof is the same as the proof of Lemma 7.6; the polynomial above has ρ_∞ as a root. $\square$

By interlacing and computing the values of ρ for various triples (a, b, c), we deduce:

Lemma 7.9. *If $\Gamma_{a,b,c}$ is abelian, then one of the following holds:*

(1) *There are bounds $a \leq 30$, $b \leq 78$, and $c \leq 170$.*
(2) $\mathscr{M}(\lambda^2 - 2) < 14/5$.

We now complete the proof of Theorem 7.1. Suppose that $\mathscr{M}(\lambda^2 - 2) < 14/5$. Then by Proposition 4.3, we deduce that either $\lambda^2 - 2$ is a sum of two roots of unity or less (from which it follows immediately that $\Gamma_{a,b,c}$ is either a Dynkin diagram or a simply laced affine Dynkin diagram) or $\lambda^2 - 2$ is one of the following numbers:

$$\alpha = \frac{\sqrt{3} + \sqrt{7}}{2},$$
$$\beta = 1 + 2\cos(2\pi/7),$$
$$\gamma = \zeta_{12} + \zeta_{20} + \zeta_{20}^{17},$$
$$\delta = 2\cos(11\pi/42) + 2\cos(13\pi/42),$$

where we use the fact that $\beta^2 < 9/2$. The algebraic numbers α and δ have conjugates < 2, yet λ is totally real, so $\lambda^2 - 2$ has no such conjugate. In the second and third cases, we have

$$\beta \sim 2.060820\ldots \text{ or } \gamma \sim 2.098777\ldots.$$

For (a,b,c) outside the range in part (1) of Lemma 7.9, we dispense with these possibilities occurring as a value of λ as follows:

(1) if $a \geq 3$, then $\lambda > 2.074313\ldots > 2.060820\ldots$,
(2) if $a = 2$ and $b \geq 4$, then $\lambda > 2.074313\ldots > 2.060820\ldots$,
(3) if $a = 2$ and $b = 2$, then $\lambda < \sqrt{2+\sqrt{5}} < 2.060820\ldots$,
(4) if $a = 2$, $b = 3$, and $c \geq 5$, then $\lambda > 2.069782\ldots > 2.060820\ldots$,
(5) if $a = 1$, then $\lambda < \sqrt{2+\sqrt{5}} < 2.060820\ldots$

for β, and

(1) if $a \geq 4$, then $\lambda > 2.101002\ldots > 2.098777\ldots$,
(2) if $a = 2$ and $b \geq 5$, then $\lambda > 2.101002\ldots > 2.098777\ldots$,
(3) if $a = 2$ and $b \leq 4$, then $\lambda < 2.084868\ldots < 2.060820,\ldots$,
(4) if $a \leq 2$, then $\lambda < 2.093555\ldots < 2.098777\ldots$

for γ. Finally, we check all the remaining polynomials to see which give rise to abelian extensions. We say a few words about this computation. The first step consists of looping through the polynomials (which have root ρ) and dividing through by the cyclotomic factors. If the remaining polynomial is irreducible and of degree ≥ 48, then we are done. Degree considerations eliminated all polynomials with $a \geq 12$ except some of the form $(a,b,c) = (a, a+1, 2a+3)$, $(a, a+2, a+2)$ or (a,a,a). The polynomial was irreducible except in a few cases, namely, $(a,b,c) = (2,6,20)$, and the triple of graphs $(4,8,14)$, $(4,9,9)$, and $(5,5,8)$. The latter triple is somewhat interesting — the value of $\lambda^2 - 2$ in each case is the largest real root of $\theta^3 - 2\theta^2 - 4\theta + 7 = 0$, whose splitting field is the Hilbert class field of $\mathbf{Q}(\sqrt{229})$. The second check consisted of computing the corresponding minimal polynomial of $\lambda^2 - 2$, and then checking (using `polcompositum` in gp/pari) whether the field was Galois or not. Finally, it was checked whether any of the fields thus obtained were abelian or not (there were no false positives).

8. Salem numbers of abelian type

All Salem numbers ρ are reciprocal. If $\mathbf{Q}(\rho)$ is abelian, then since ρ is real, it must be totally real, yet ρ (by definition) has a conjugate of absolute value 1. Thus no Salem number can generate an abelian extension. In light of this, the following definition is perhaps not too confusing.

Definition 8.1. A Salem number ρ is of **abelian type** if $\mathbf{Q}(\rho + \rho^{-1})$ is an abelian extension.

If K is any totally real field, then, because the image of the units $\mathcal{O}_K^\times \otimes \mathbf{R}$ in $K \otimes \mathbf{R}$ has co-dimension one (by the proof of Dirichlet's unit theorem), there exists a totally positive unit $\alpha \in \mathcal{O}_K$ such that $\alpha > 1$ in one real embedding and < 1 in all other real embeddings. Replacing α by a suitable power so that it is > 2 and letting $\rho + \rho^{-1} = \alpha$, we find that ρ is a Salem number of abelian type if K is a (totally) real abelian field. Hence there exist infinitely many Salem numbers of abelian type in each such K. However, we prove the following:

Proposition 8.2. *The set of Salem numbers of abelian type is discrete in* $\mathbf{R}$.

Proof. It suffices to show that the Salem numbers of abelian type less than a given bound L is finite. Since the number of Salem numbers of bounded degree less than a given bound is finite, it suffices to prove that the Salem numbers of abelian type less than L have bounded degree. However, from Theorem 2.3, for all such Salem numbers of sufficiently large degree (depending on L) we have the bound $\mathscr{M}(\rho+\rho^{-1}) < 14/5$. If ρ is of abelian type, then by Proposition 4.3, the element ρ lives in some finite set (if ρ is Salem, then $\rho+\rho^{-1} > 2$ is not a sum of two roots of unity). □

Note that, from the classification of the smallest totally real cyclotomic integers [CMS11], one sees that the smallest Salem number of abelian type is $\theta = 1.635573\ldots$, the root of $\theta^6 - 2\theta^5 + 2\theta^4 - 3\theta^3 + 2\theta^2 - 2\theta + 1 = 0$.

One can make the previous proposition effective. Namely, suppose that $\rho > \rho'$ are two Salem numbers of abelian type. There is a bound $B(x^2+x^{-2}+2) > -11/10x^2$ for all $x > \theta$. Hence

$$\frac{20}{11} \cdot B(\rho^2+\rho^{-2}+2) \geq -2\rho^2,$$

and so either the degrees of ρ and ρ' are either bounded by $4\rho^2$, or the corresponding Salem numbers lie on the list in Proposition 4.3, in which case one can check that the bound still holds. In the former case, by estimating the norm of $\rho-\rho'$, which has degree at most $16\rho^4$ and each conjugate has absolute value at most 2ρ, we deduce that

Proposition 8.3. *Let $\rho > \rho'$ be two Salem numbers of abelian type. Then*

$$\rho - \rho' > \frac{1}{(2\rho)^{16\rho^4}}.$$

Naturally enough, a result essentially identical to Proposition 8.2 holds (with the same proof) if one replaces Salem numbers by numbers ρ conjugate to ρ^{-1} with a uniformly bounded number of real roots > 1.

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